CHARACTERIZATION OF RRMF CURVES

Recall: $\mathcal{A}(\xi) = u(\xi) + v(\xi)\mathbf{i} + p(\xi)\mathbf{j} + q(\xi)\mathbf{k}$ generates an RRMF curve through $\mathbf{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)$ iff polynomials $a(\xi), b(\xi)$ exist, such that

$$\frac{uv' - u'v - pq' + p'q}{u^2+v^2+p^2+q^2} = \frac{ab' - a'b}{a^2+b^2} \qquad (*)$$

$\mathbf{r}(\xi)$ is of degree $2n+1$ if $\deg(u,v,p,q) = n$

Let $m = \deg(a,b)$. Equation $(*)$ may be satisfied with $m<n$, $m=n$, $m>n$.

ERF $(\mathbf{e}_1(\xi), \mathbf{e}_2(\xi), \mathbf{e}_3(\xi))$ is of degree $2n$.

rational RMF $(\mathbf{t}(\xi), \mathbf{u}(\xi), \mathbf{v}(\xi))$: degree $2(m+n)$

$\mathbf{t}(\xi) = \mathbf{e}_1(\xi)$

$$\begin{bmatrix} \mathbf{u}(\xi) \\ \mathbf{v}(\xi) \end{bmatrix} = \frac{1}{a^2(\xi)+b^2(\xi)} \begin{bmatrix} a^2(\xi)-b^2(\xi) & -2a(\xi)b(\xi) \\ 2a(\xi)b(\xi) & a^2(\xi)-b^2(\xi) \end{bmatrix} \begin{bmatrix} \mathbf{e}_2(\xi) \\ \mathbf{e}_3(\xi) \end{bmatrix}$$

Some low-degree cases:

<u>$n=1$ (RRMF cubics)</u>: only trivial (planar curve) instances exist.

$n=2$ (RRMF quintics): non-trivial cases with $m=1$, $m=2$, and $m>2$ exist.

$m=2$ (generic) case has simple characterization

$$\text{vect}(A_2 \,\underset{\sim}{i}\, A_0^*) = A_1 \,\underset{\sim}{i}\, A_1^*$$

in terms of coefficients of $A(\xi)$.

Characterizations of $m=1$ & $m>2$ cases are much more involved & less intuitive.

NOTE: rational RMF is degree 6 in $m=1$ case, but degree 8 in $m=2$ case.

$n=3$ (degree 7 RRMF curves)

non-trivial cases exist for all $m \geq 0$.

$m=0$ case: lowest-order RRMF curves for which the ERF is an RMF.

Characterization in terms of coefficients of

$$A(\xi) = A_0(1-\xi)^3 + A_1 3(1-\xi)^2\xi + A_2 3(1-\xi)\xi^2 + A_3\xi^3$$

$$\text{scal}(A_0 \,\underset{\sim}{i}\, A_1^*) = \text{scal}(A_0 \,\underset{\sim}{i}\, A_2^*) = 0$$

$$3\,\text{scal}(A_1 \,\underset{\sim}{i}\, A_2^*) + \text{scal}(A_0 \,\underset{\sim}{i}\, A_3^*) = 0$$

$$\text{scal}(A_1 \,\underset{\sim}{i}\, A_3^*) = \text{scal}(A_2 \,\underset{\sim}{i}\, A_3^*) = 0$$

No known "simple" characterizations in terms of A_0, A_1, A_2, A_3 for cases with $m \geq 1$

NOTE: cases $n=m=2$ & $n=3, m=0$ admit solutions to problem of constructing RRMF motions that interpolate initial and final positions & frames $\mathbf{p}_i$ & $(\mathbf{t}_i, \mathbf{u}_i, \mathbf{v}_i)$ and $\mathbf{p}_f$ & $(\mathbf{t}_f, \mathbf{u}_f, \mathbf{v}_f)$.

$n=m=2$ case: some restrictions on initial/final data for existence of solutions.

$n=3, m=0$ case: solutions believed (but not proved) to exist for arbitrary data – two free parameters in solution procedure.

For $n=m=2$, rational RMF is degree 8, but for $n=3, m=0$ it is degree 6.

<u>Alternative characterization</u> of generic case $m=n$ (holds for any n).

With $\mathcal{A}(\xi) = u(\xi) + v(\xi)\mathbf{i} + p(\xi)\mathbf{j} + q(\xi)\mathbf{k}$

define:
$$\eta = (uu' + vv' + pp' + qq')^2 + (uv' - u'v - pq' + p'q)^2$$
$$\rho = (up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2$$

NOTE: $\eta + \rho = (u^2+v^2+p^2+q^2)(u'^2+v'^2+p'^2+q'^2) = |\mathcal{A}|^2 |\mathcal{A}'|^2$

$\sigma(\xi) = |A(\xi)|^2$ = parametric speed.

$A(\xi)$ generates an RRMF curve

$\Leftrightarrow$ $\eta(\xi)$ and $\rho(\xi)$ are divisible by $\sigma(\xi)$

Open problems

* Derive simple sufficient-and-necessary conditions on coefficients of $A(\xi)$ for an RRMF curve, analogous to that for $n=m=2$, for the cases:
  (i) $n=2$ and $m=1$ or $m>2$;
  (ii) $n=3$ and $m>0$ (especially the generic case, $n=m=3$).
* Elucidate geometrical meaning of the divisibility condition for the generic case $n=m$.