SPATIAL PH QUINTIC HERMITE INTERPOLANTS

Find coefficients of quaternion polynomial

$$\mathcal{A}(\xi) = \mathcal{A}_0(1-\xi)^2 + \mathcal{A}_1 2(1-\xi)\xi + \mathcal{A}_2\xi^2$$

such that $\mathbf{r}(\xi) = \int \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)\,d\xi$ interpolates given initial/final points & derivatives:

$$\mathbf{r}(0) = \mathbf{p}_i, \quad \mathbf{r}'(0) = \mathbf{d}_i \quad \& \quad \mathbf{r}(1) = \mathbf{p}_f, \quad \mathbf{r}'(1) = \mathbf{d}_f .$$

Set $\boldsymbol{\delta}_i = \mathbf{d}_i/|\mathbf{d}_i|$ & $\boldsymbol{\delta}_f = \mathbf{d}_f/|\mathbf{d}_f|$.

Derivatives: $\mathbf{r}'(0) = \mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^* = \mathbf{d}_i$, $\mathbf{r}'(1) = \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^* = \mathbf{d}_f$

Solutions are:

$$\mathcal{A}_0 = \sqrt{|\mathbf{d}_i|}\,\mathbf{n}_i \exp(\phi_0 \mathbf{i}), \quad \mathcal{A}_2 = \sqrt{|\mathbf{d}_f|}\,\mathbf{n}_f \exp(\phi_2 \mathbf{i})$$

where $\mathbf{n}_i = \dfrac{\mathbf{i}+\boldsymbol{\delta}_i}{|\mathbf{i}+\boldsymbol{\delta}_i|}$ & $\mathbf{n}_f = \dfrac{\mathbf{i}+\boldsymbol{\delta}_f}{|\mathbf{i}+\boldsymbol{\delta}_f|}$

are unit bisectors of $\mathbf{i}, \boldsymbol{\delta}_i$ & $\mathbf{i}, \boldsymbol{\delta}_f$ and ϕ_0, ϕ_2 are free angular parameters.

Notation: $\exp(\phi\mathbf{i}) = (\cos\phi, \sin\phi\,\mathbf{i})$

End points:

$$\underset{\sim}{P}_f - \underset{\sim}{P}_i = \int_0^1 \underset{\sim}{r}'(\xi)\, d\xi = \int_0^1 A(\xi)\, \underset{\sim}{i}\, A^*(\xi)\, d\xi$$

$$= \frac{1}{5} A_0 \underset{\sim}{i} A_0^* + \frac{1}{10}\left(A_0 \underset{\sim}{i} A_1^* + A_1 \underset{\sim}{i} A_0^*\right)$$

$$+ \frac{1}{30}\left(A_0 \underset{\sim}{i} A_2^* + 4 A_1 \underset{\sim}{i} A_1^* + A_2 \underset{\sim}{i} A_0^*\right)$$

$$+ \frac{1}{10}\left(A_1 \underset{\sim}{i} A_2^* + A_2 \underset{\sim}{i} A_1^*\right) + \frac{1}{5} A_2 \underset{\sim}{i} A_2^*$$

Using $A_0 \underset{\sim}{i} A_0^* = \underset{\sim}{d}_i$ & $A_2 \underset{\sim}{i} A_2^* = \underset{\sim}{d}_f$, can write as:

$$(3A_0 + 4A_1 + 3A_2)\, \underset{\sim}{i}\, (3A_0 + 4A_1 + 3A_2)^* = \underset{\sim}{d}$$

where $\underset{\sim}{d} = 120(\underset{\sim}{P}_f - \underset{\sim}{P}_i) - 15(\underset{\sim}{d}_i + \underset{\sim}{d}_f)$

$\qquad + 5\left(A_0 \underset{\sim}{i} A_2^* + A_2 \underset{\sim}{i} A_0^*\right)$: depends on ϕ_0, ϕ_2

Set $\underset{\sim}{\delta} = \underset{\sim}{d} / |\underset{\sim}{d}|$ and $\underset{\sim}{n} = \dfrac{\underset{\sim}{i} + \underset{\sim}{\delta}}{|\underset{\sim}{i} + \underset{\sim}{\delta}|}$, obtain A_1 as:

$$A_1 = -\frac{3}{4}(A_0 + A_2) + \frac{1}{4}\sqrt{|\underset{\sim}{d}|}\ \underset{\sim}{n}\, \exp(\phi_1 \underset{\sim}{i})$$

where $|\underset{\sim}{d}|$ & $\underset{\sim}{n}$ depend on ϕ_0, ϕ_2

& ϕ_1 = new free angular parameter

NOTE: since products $\mathcal{A}_r \, \mathbf{i} \, \mathcal{A}_s^*$ $(0 \le r, s \le 2)$ in expansion of $\mathbf{r}'(\xi) = \mathcal{A}(\xi) \, \mathbf{i} \, \mathcal{A}^*(\xi)$ depend only on <u>differences</u> $\phi_r - \phi_s$ of angular parameters, can choose $\phi_1 = 0$ without loss of generality.

$\Rightarrow$ There exists a 2-parameter family of spatial PH quintic interpolants to initial/final points & derivatives $\mathbf{p}_i, \mathbf{d}_i$ & $\mathbf{p}_f, \mathbf{d}_f$. Each pair (ϕ_0, ϕ_2) defines a different interpolant.

Replace ϕ_0, ϕ_2 by $\phi_m = \frac{1}{2}(\phi_0 + \phi_2)$, $\Delta\phi = \phi_2 - \phi_0$

Arc length of interpolant:

$$S = \int_0^1 \sigma(\xi)\, d\xi = \int_0^1 |\mathcal{A}(\xi)|^2 d\xi$$

$$= \frac{1}{120}\left[\, |3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2|^2 + 15(|\mathcal{A}_0|^2 + |\mathcal{A}_2|^2) - 5(\mathcal{A}_0\mathcal{A}_2^* + \mathcal{A}_2\mathcal{A}_0^*)\,\right]$$

S depends only on $\Delta\phi$, not ϕ_0 & ϕ_2 individually.

Varying ϕ_m with $\Delta\phi$ fixed generates interpolants of different shape, but identical arc length S.

Varying $\Delta\phi$ gives interpolants with arc lengths S in a finite range, $S \in [S_{min}, S_{max}]$.

Extremal cases $S = S_{min}$ & $S = S_{max}$ identify general helices.

Methods for fixing ϕ_m & $\Delta\phi$ based on minimization of various functions, e.g.,

$$F(\phi_m, \Delta\phi) = |A_1 - \tfrac{1}{2}(A_0 + A_2)|^2$$

identifies interpolant "closest" to a PH cubic (satisfying $A_0 - 2A_1 + A_2 = 0$).