HELICAL POLYNOMIAL CURVES

A helical space curve $\underset{\sim}{r}(\xi)$, also called a "curve of constant slope," may be characterized by any of the following properties:

(1) The tangent $\underset{\sim}{t}(\xi) = \underset{\sim}{r}'(\xi)/|\underset{\sim}{r}'(\xi)|$ maintains a constant angle ψ with a fixed unit vector $\underset{\sim}{a}$:

$$\underset{\sim}{a}\cdot\underset{\sim}{t}(\xi) = \cos\psi = \text{constant}, \quad \underset{\sim}{a} = \text{axis of helix.}$$

(2) The curvature $\kappa(\xi)$ and torsion $\tau(\xi)$ have a constant ratio: $\dfrac{\kappa(\xi)}{\tau(\xi)} = \tan\psi$

(3) The tangent indicatrix, i.e., the locus traced on the unit sphere S by the tangent $\underset{\sim}{t}(\xi)$, is a small circle C ($\underset{\sim}{a}$ = center of C on S).

C = great circle $\Rightarrow$ $\underset{\sim}{r}(\xi)$ planar

(4) $(\underset{\sim}{r}'' \times \underset{\sim}{r}''')\cdot\underset{\sim}{r}'''' \equiv 0.$

Any polynomial helix is a PH curve:

$$\underset{\sim}{a}\cdot\underset{\sim}{t}(\xi) = \cos\psi \Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{r}'(\xi) = \cos\psi\,|\underset{\sim}{r}'(\xi)|$$

only satisfied if $|\underset{\sim}{r}'(\xi)| = \sigma(\xi)$, a polynomial.

NOTE: circular helix $\underset{\sim}{r}(\theta) = (R\cos\theta, R\sin\theta, k\theta)$ is a transcendental curve: # of intersections with a plane may be ∞.

Cubic helical curves $\quad \underset{\sim}{r}(\xi) = \sum_{k=0}^{3} \underset{\sim}{P}_k \binom{3}{k} (1-\xi)^{3-k} \xi^k$

derivative: $\quad \underset{\sim}{r}'(\xi) = \sum_{k=0}^{2} 3\Delta \underset{\sim}{P}_k \binom{2}{k} (1-\xi)^{2-k} \xi^k$

$\Delta \underset{\sim}{P}_k = \underset{\sim}{P}_{k+1} - \underset{\sim}{P}_k$: choose coordinates such that

$\Delta \underset{\sim}{P}_0 = L_0(\sin\theta, 0, \cos\theta)$, $\quad \Delta \underset{\sim}{P}_1 = L_1(0,0,1)$

$\Delta \underset{\sim}{P}_2 = L_2(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

Proposition $\quad |\underset{\sim}{r}'(\xi)|^2 = [\sigma_0(1-\xi)^2 + \sigma_1 2(1-\xi)\xi + \sigma_2 \xi^2]^2$

$$\Longleftrightarrow \quad \cos\phi = 1 - \frac{2L_1^2}{L_0 L_2} \qquad (*)$$

NOTE: must have $L_1 \leqslant \sqrt{L_0 L_2}$

If $(*)$ is satisfied, $\sigma(\xi) = 3[L_0(1-\xi)^2 + L_1\cos\theta\, 2(1-\xi)\xi + L_2\xi^2]$

$$\kappa(\xi) = \frac{|\underset{\sim}{r}'(\xi) \times \underset{\sim}{r}''(\xi)|}{|\underset{\sim}{r}'(\xi)|^3} = \frac{6L_1|\sin\theta|}{\sigma^2(\xi)}$$

$$\tau(\xi) = \frac{[\underset{\sim}{r}'(\xi) \times \underset{\sim}{r}''(\xi)] \cdot \underset{\sim}{r}'''(\xi)}{|\underset{\sim}{r}'(\xi) \times \underset{\sim}{r}''(\xi)|^2} = \frac{-3L_0L_2\sin\phi}{L_1\sigma^2(\xi)}$$

Hence, $\dfrac{\kappa(\xi)}{\tau(\xi)} = \dfrac{-2L_1^2 |\sin\theta|}{L_0 L_2 \sin\phi} = \text{constant}$

$\Rightarrow$ All spatial PH cubics are helical.

Higher-order helical PH curves

With $\mathcal{A}(\xi) = u(\xi) + v(\xi)\,\mathbf{i} + p(\xi)\,\mathbf{j} + q(\xi)\,\mathbf{k}$

set $\mathbf{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)$ and $\sigma(\xi) = |\mathcal{A}(\xi)|^2$

Can show that $|\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)|^2 = \sigma^2(\xi)\,\rho(\xi)$

where $\rho(\xi) = |\mathbf{r}''(\xi)|^2 - \sigma'^2(\xi)$

$$= 4\,[\,(up'-u'p)^2 + (uq'-u'q)^2 + (vp'-v'p)^2 + (vq'-v'q)^2 + 2(uv'-u'v)(pq'-p'q)\,]$$

$$= 4\,[\,(uv'-u'v+pq'-p'q)^2 + (up'-u'p-vq'+v'q)^2 + (uq'-u'q+vp'-v'p)^2 - (uv'-u'v-pq'+p'q)^2\,]$$

$$= 4\,[\,(up'-u'p+vq'-v'q)^2 + (uq'-u'q-vp'+v'p)^2\,]$$

Helix condition $\kappa(\xi)/\tau(\xi) = \tan\psi$

becomes $\rho^{3/2}(\xi) = \tan\psi\,[\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)] \cdot \mathbf{r}'''(\xi)$

Hence $\mathbf{r}(\xi)$ helical $\Rightarrow \rho(\xi) = w^2(\xi)$, $w(\xi)$ = polynomial

Equivalently, $|\underset{\sim}{r}'(\xi)\times\underset{\sim}{r}''(\xi)|^2 = [\sigma(\xi)\omega(\xi)]^2$

<u>Definition</u>: A polynomial curve for which $|\underset{\sim}{r}'(\xi)|$ and $|\underset{\sim}{r}'(\xi)\times\underset{\sim}{r}''(\xi)|$ are both polynomials is called a "double" PH curve.

DPH curves = { polynomial curves with rational Frenet frames, curvature, torsion }

Every polynomial helix is a DPH curve.
All degree 5 DPH curves are helical, but there exist DPH curves of degree ≥ 7 that are not helical.

<u>Classification of quintic helical curves</u>

$\underset{\sim}{r}' = x'\underset{\sim}{i} + y'\underset{\sim}{j} + z'\underset{\sim}{k} = \mathcal{A}\,\underset{\sim}{i}\,\mathcal{A}^*$

$= (u^2+v^2-p^2-q^2)\underset{\sim}{i} + 2(uq+vp)\underset{\sim}{j} + 2(vq-up)\underset{\sim}{k}$

$\underset{\sim}{r}'$ is "primitive" if $\gcd(x',y',z') = 1$

but $\gcd(u,v,p,q) = 1 \not\Rightarrow \gcd(x',y',z') = 1$

$|\gcd(u+iv,\, p-iq)|^2$ is a real factor of $\gcd(x',y',z')$.

Montone helical quintics

$|\gcd(u+iv, p-iq)|^2$ is quadratic

$\underset{\sim}{t}(\xi)$ maintains fixed sense of rotation about $\underset{\sim}{a}$.

General helical quintics

$\gcd(x', y', z') = 1$

$\underset{\sim}{t}(\xi)$ may exhibit reversal in sense of rotation about $\underset{\sim}{a}$.