QUATERNION FORM OF SPATIAL PH CURVES

Polynomial curve in $\mathbb{R}^3$: $\underset{\sim}{r}(\xi) = (x(\xi), y(\xi), z(\xi))$

Sufficient & necessary condition for satisfaction of

$$x'^2(\xi) + y'^2(\xi) + z'^2(\xi) = \sigma^2(\xi)$$

involves 4 polynomials $u(\xi), v(\xi), p(\xi), q(\xi)$:

$$x'(\xi) = u^2(\xi) + v^2(\xi) - p^2(\xi) - q^2(\xi)$$
$$y'(\xi) = 2[u(\xi)q(\xi) + v(\xi)p(\xi)]$$
$$z'(\xi) = 2[v(\xi)q(\xi) - u(\xi)p(\xi)]$$
$$\sigma(\xi) = u^2(\xi) + v^2(\xi) + p^2(\xi) + q^2(\xi)$$

Consider as components of a quaternion polynomial

$$A(\xi) = u(\xi) + v(\xi)\underset{\sim}{i} + p(\xi)\underset{\sim}{j} + q(\xi)\underset{\sim}{k}$$

Then $\underset{\sim}{r}'(\xi) = A(\xi)\underset{\sim}{i}A^*(\xi)$, $\sigma(\xi) = |\underset{\sim}{r}'(\xi)| = |A(\xi)|$

Let $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\underset{\sim}{n})$ be unit quaternion defining rotation by angle θ about axis $\underset{\sim}{n}$.

Rotation of $\underset{\sim}{r}'(\xi)$:

$$\underset{\sim}{r}'(\xi) \to \mathcal{U}\underset{\sim}{r}'(\xi)\mathcal{U}^* = \mathcal{U}A(\xi)\underset{\sim}{i}A^*(\xi)\mathcal{U}^*$$

$= \tilde{A}(\xi)\, \underset{\sim}{i}\, \tilde{A}^*(\xi)$, where $\tilde{A}(\xi) = \mathcal{U} A(\xi)$

This expresses rotation invariance property of quaternion representation.

Components of $\tilde{A}(\xi)$ with $\underset{\sim}{n} = (n_x, n_y, n_z)$:

$$\tilde{u} = \cos\tfrac{1}{2}\theta\, u - \sin\tfrac{1}{2}\theta\, [n_x v + n_y p + n_z q]$$
$$\tilde{v} = \cos\tfrac{1}{2}\theta\, v + \sin\tfrac{1}{2}\theta\, [n_x u + n_y q - n_z p]$$
$$\tilde{p} = \cos\tfrac{1}{2}\theta\, p + \sin\tfrac{1}{2}\theta\, [n_y u + n_z v - n_x q]$$
$$\tilde{q} = \cos\tfrac{1}{2}\theta\, q + \sin\tfrac{1}{2}\theta\, [n_z u + n_x p - n_y v]$$

NOTE: any unit vector $\underset{\sim}{v}$ can be used in place of $\underset{\sim}{i}$ in $\underset{\sim}{r}'(\xi) = A(\xi)\, \underset{\sim}{i}\, A^*(\xi)$. This product always defines a pure vector quaternion.

Geometrical interpretation:

Write $A(\xi) = |A(\xi)|\, \mathcal{U}(\xi)$ where $\mathcal{U}(\xi) = \dfrac{A(\xi)}{|A(\xi)|}$

Then $\underset{\sim}{r}'(\xi) = |A(\xi)|^2\, \mathcal{U}(\xi)\, \underset{\sim}{i}\, \mathcal{U}^*(\xi)$

$|A(\xi)|^2$ & $\mathcal{U}(\xi)\, \underset{\sim}{i}\, \mathcal{U}^*(\xi)$ define continuous families of scalings & rotations of $\underset{\sim}{i}$ that generate $\underset{\sim}{r}'(\xi)$.

NOTE: For any given PH curve $\underset{\sim}{r}(\xi)$, quaternion polynomial $\mathcal{A}(\xi)$ in $\underset{\sim}{r}'(\xi) = \mathcal{A}(\xi)\,\underset{\sim}{i}\,\mathcal{A}^*(\xi)$ is <u>not unique</u>.

If $\mathcal{Q}$ is any quaternion satisfying $\mathcal{Q}\,\underset{\sim}{i}\,\mathcal{Q}^* = \underset{\sim}{i}$,

then $\underset{\sim}{r}'(\xi) = \tilde{\mathcal{A}}(\xi)\,\underset{\sim}{i}\,\tilde{\mathcal{A}}^*(\xi)$ where $\tilde{\mathcal{A}}(\xi) = \mathcal{A}(\xi)\mathcal{Q}$

since $[\mathcal{A}(\xi)\mathcal{Q}]\,\underset{\sim}{i}\,[\mathcal{A}(\xi)\mathcal{Q}]^* = \mathcal{A}(\xi)\,(\mathcal{Q}\,\underset{\sim}{i}\,\mathcal{Q}^*)\,\mathcal{A}^*(\xi)$

$$= \mathcal{A}(\xi)\,\underset{\sim}{i}\,\mathcal{A}^*(\xi)$$

Solutions of $\mathcal{Q}\,\underset{\sim}{i}\,\mathcal{Q}^* = \underset{\sim}{i}$ are:

$$\mathcal{Q} = (\cos\phi, \sin\phi\,\underset{\sim}{i}) \text{ for any } \phi.$$

There exists a one-parameter family of quaternion polynomials that generate a given spatial PH curve $\underset{\sim}{r}(\xi)$.

<u>Hopf map representation</u>

Identifying imaginary unit i with quaternion basis element $\underset{\sim}{i}$, we can write:

$$\mathcal{A}(\xi) = u(\xi) + v(\xi)\underset{\sim}{i} + [p(\xi) + q(\xi)\underset{\sim}{i}]\,\underset{\sim}{j}$$

$$= \alpha(\xi) + \beta(\xi)\,\underset{\sim}{j}$$

where $\alpha(\xi) = u(\xi) + v(\xi)\underset{\sim}{i}$, $\beta(\xi) = p(\xi) + q(\xi)\underset{\sim}{i}$

Can use two complex polynomials in place of one quaternion polynomial.

Hopf map $H: \mathbb{C}^2 = \mathbb{R}^4 \to \mathbb{R}^3$

$$\underset{\sim}{r}'(\xi) = H(\alpha(\xi), \beta(\xi))$$

$$= \left(|\alpha(\xi)|^2 - |\beta(\xi)|^2,\ 2\,\mathrm{Re}(\alpha(\xi)\bar{\beta}(\xi)),\ 2\,\mathrm{Im}(\alpha(\xi)\bar{\beta}(\xi)) \right).$$

Alternative (equivalent) to quaternion form.

NOTE: If $|A|^2 = |\alpha|^2 + |\beta|^2 = 1$, then $H(\alpha, \beta)$ defines map from 3-sphere $u^2+v^2+p^2+q^2=1$ in $\mathbb{R}^4$ to 2-sphere $x^2+y^2+z^2=1$ in $\mathbb{R}^3$.

Great circles on S^3 map to points on S^2.