SCALAR-VECTOR FORM OF QUATERNIONS

$A = (a, \underset{\sim}{a})$ $\quad a = \mathrm{scal}(A)$, $\underset{\sim}{a} = \mathrm{vect}(A)$

short-hand: write "pure scalar" quaternion $(a, \underset{\sim}{0}) = a$ & "pure vector" quaternion $(0, \underset{\sim}{a}) = \underset{\sim}{a}$

sum & product of $A = (a, \underset{\sim}{a})$ & $B = (b, \underset{\sim}{b})$:

$$A + B = (a + b, \underset{\sim}{a} + \underset{\sim}{b})$$

$$AB = (ab - \underset{\sim}{a} \cdot \underset{\sim}{b}, a\underset{\sim}{b} + b\underset{\sim}{a} + \underset{\sim}{a} \times \underset{\sim}{b})$$

$AB \neq BA$ since $\underset{\sim}{a} \times \underset{\sim}{b} = -\underset{\sim}{b} \times \underset{\sim}{a}$

Conjugate of A : $A^* = (a, -\underset{\sim}{a})$

Conjugate of product : $(AB)^* = B^* A^*$

inverse of A : $A^{-1} = \dfrac{A^*}{|A|^2}$ $\qquad AA^{-1} = A^{-1}A = 1$

modulus of A : $|A|^2 = AA^* = A^*A = a^2 + |\underset{\sim}{a}|^2$

Unit quaternions & rotations

unit quaternion $U = (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\, \underset{\sim}{n})$, $|\underset{\sim}{n}| = 1$

map $\underset{\sim}{v} \to U\underset{\sim}{v}U^*$ defines rotation of vector $\underset{\sim}{v}$ by angle θ about axis $\underset{\sim}{n}$

$$\mathcal{U}\underset{\sim}{v}\mathcal{U}^* = (0, (\underset{\sim}{n}\cdot\underset{\sim}{v})\underset{\sim}{n} + \sin\theta\, \underset{\sim}{n}\times\underset{\sim}{v} + \cos\theta(\underset{\sim}{n}\times\underset{\sim}{v})\times\underset{\sim}{n})$$

before rotation: $\underset{\sim}{v} = \underset{\sim}{v}_{\parallel} + \underset{\sim}{v}_{\perp} = (\underset{\sim}{n}\cdot\underset{\sim}{v})\underset{\sim}{n} + (\underset{\sim}{n}\times\underset{\sim}{v})\times\underset{\sim}{n}$

after rotation: $\underset{\sim}{v}_{\parallel} = (\underset{\sim}{n}\cdot\underset{\sim}{v})\underset{\sim}{n}$ unchanged

$$\underset{\sim}{v}_{\perp} = (\underset{\sim}{n}\times\underset{\sim}{v})\times\underset{\sim}{n} \rightarrow \sin\theta\, \underset{\sim}{n}\times\underset{\sim}{v} + \cos\theta(\underset{\sim}{n}\times\underset{\sim}{v})\times\underset{\sim}{n}$$

NOTE: Since $|\mathcal{U}| = 1$, $\mathcal{U} = (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\, \underset{\sim}{n})$ has 3 degrees of freedom. Compare 3×3 rotation matrix $M \in SO(3)$: 9 elements.

Compounding spatial rotations

1. rotation by $\mathcal{U}_1 = (\cos\tfrac{1}{2}\theta_1, \sin\tfrac{1}{2}\theta_1 \underset{\sim}{n}_1)$
2. rotation by $\mathcal{U}_2 = (\cos\tfrac{1}{2}\theta_2, \sin\tfrac{1}{2}\theta_2 \underset{\sim}{n}_2)$

$$\underset{\sim}{v} \xrightarrow{1} \mathcal{U}_1\underset{\sim}{v}\mathcal{U}_1^* \xrightarrow{2} \mathcal{U}_2\mathcal{U}_1\underset{\sim}{v}\,\mathcal{U}_1^*\mathcal{U}_2^*$$

$$= (\mathcal{U}_2\mathcal{U}_1)\underset{\sim}{v}\,(\mathcal{U}_2\mathcal{U}_1)^* = \mathcal{U}\underset{\sim}{v}\mathcal{U}^*$$

where $\mathcal{U} = \mathcal{U}_2\mathcal{U}_1$ defines compounded rotation

Quaternion algebra captures non-commutative nature of compounded spatial rotations.

Solutions u of equation $u\underset{\sim}{i}u^* = \underset{\sim}{v}$

Find unit quaternions u that rotate $\underset{\sim}{i}$ onto a given unit vector $\underset{\sim}{v}$.

Set $\underset{\sim}{i} \cdot \underset{\sim}{v} = \cos\alpha$

$$\underset{\sim}{e}_0 = \frac{\underset{\sim}{i}+\underset{\sim}{v}}{|\underset{\sim}{i}+\underset{\sim}{v}|} = \frac{\underset{\sim}{i}+\underset{\sim}{v}}{\sqrt{2(1+\cos\alpha)}} = \text{unit bisector of } \underset{\sim}{i} \,\&\, \underset{\sim}{v}$$

$$\underset{\sim}{e}_\perp = \frac{\underset{\sim}{i}\times\underset{\sim}{v}}{|\underset{\sim}{i}\times\underset{\sim}{v}|} = \frac{\underset{\sim}{i}\times\underset{\sim}{v}}{|\sin\alpha|} = \text{unit vector} \perp \text{to plane of } \underset{\sim}{i} \,\&\, \underset{\sim}{v}$$

One-parameter family of solutions:

$$u = \underset{\sim}{e}_0 \exp(\phi\underset{\sim}{i}), \quad \exp(\phi\underset{\sim}{i}) = (\cos\phi, \sin\phi\,\underset{\sim}{i})$$

Verify: $u\underset{\sim}{i}u^* = (0,\underset{\sim}{e}_0)\exp(\phi\underset{\sim}{i})(0,\underset{\sim}{i})\exp(-\phi\underset{\sim}{i})(0,-\underset{\sim}{e}_0)$

$$= (0,\underset{\sim}{e}_0)(0,\underset{\sim}{i})(0,-\underset{\sim}{e}_0) = 2(\underset{\sim}{i}\cdot\underset{\sim}{e}_0)\underset{\sim}{e}_0 - \underset{\sim}{i}$$

$$= \frac{2(1+\cos\alpha)}{\sqrt{2(1+\cos\alpha)}} \cdot \frac{\underset{\sim}{i}+\underset{\sim}{v}}{\sqrt{2(1+\cos\alpha)}} - \underset{\sim}{i} = \underset{\sim}{v}$$

There is a one-parameter family of spatial rotations that map $\underset{\sim}{i}$ onto a given unit vector $\underset{\sim}{v}$.

Write $u = \underset{\sim}{e}_0 \exp(\phi \underset{\sim}{i}) = (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\, \underset{\sim}{n})$

$\underset{\sim}{n}$ = axis of rotation, θ = angle of rotation

Can show that $\phi = -\sin^{-1} \dfrac{\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\alpha}$

$\phi \in [-\tfrac{1}{2}\pi, +\tfrac{1}{2}\pi]$ for $\theta \in [\alpha, 2\pi - \alpha]$

$\underset{\sim}{n}$ is given in terms of θ by:

$$\underset{\sim}{n}(\theta) = \frac{\sin\frac{1}{2}\alpha \cos\frac{1}{2}\theta\, \underset{\sim}{e}_\perp \pm \sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta}\ \underset{\sim}{e}_0}{\cos\frac{1}{2}\alpha \sin\frac{1}{2}\theta}$$

Special cases:

(1) if $\theta = \alpha$, $\underset{\sim}{n}(\theta) = \underset{\sim}{e}_\perp$ & rotation is along great circle between $\underset{\sim}{i}$ & $\underset{\sim}{v}$

(2) If $\theta = \pi$, $\underset{\sim}{n}(\theta) = \underset{\sim}{e}_0$ & $\underset{\sim}{i}$ makes clockwise or anti-clockwise half-turn about $\underset{\sim}{e}_0$ onto $\underset{\sim}{v}$

(3) if $\theta = 2\pi - \alpha$, $\underset{\sim}{n}(\theta) = -\underset{\sim}{e}_\perp$: rotation is along great circle, in opposite sense to (1)

Rotations in $\mathbb{R}^4$

$A = u + v\underset{\sim}{i} + p\underset{\sim}{j} + q\underset{\sim}{k}$ = vector in $\mathbb{R}^4$

Most general form of rotation in $\mathbb{R}^4$ is:

$$A \to u_1 A u_2^* \quad \text{with } |u_1| = |u_2| = 1$$

In $\mathbb{R}^2$, stationary set = point (center of rotation)
In $\mathbb{R}^3$, stationary set = line (axis of rotation)

Two possibilities in $\mathbb{R}^4$:

"simple" rotation: stationary set = plane (2D subset of $\mathbb{R}^4$). All points follow circular paths

"double" rotation: stationary set = point (intersection of two "absolutely orthogonal" planes in $\mathbb{R}^4$). In general, orbits of points are <u>not</u> closed curves.