RATIONAL ARC LENGTH PARAMETERIZATION?

Consider plane parametric curve:

$$\mathbf{r}(\xi) = (x(\xi), y(\xi))$$

result extends to $\mathbb{R}^n,\ n \geqslant 3$

ALGEBRA: desire "simple" parameterization, ie $x(\xi), y(\xi)$ are rational functions (admit evaluation by finite sequence of arithmetic operations). In general,

$$x(\xi) = \frac{X(\xi)}{W(\xi)}\ , \quad y(\xi) = \frac{Y(\xi)}{W(\xi)}$$

$W(\xi), X(\xi), Y(\xi)$ polynomials with $\gcd(W,X,Y) = 1$

GEOMETRY: prefer "natural" parameterization

parameter ξ = arc length s along curve

$$(ds)^2 = (dx)^2 + (dy)^2 = [x'^2 + y'^2]\,(ds)^2$$

$$\Rightarrow \quad x'^2 + y'^2 \equiv 1$$

TRIVIAL CASE: $X(\xi) = \lambda_1\xi + \lambda_0,\ Y(\xi) = \mu_1\xi + \xi_0,\ W(\xi)$

$$x'^2 + y'^2 \equiv 1 \iff \lambda_1^2 + \mu_1^2 = 1 \ : \text{straight line.}$$

Theorem It is impossible to parameterize any plane curve, other than a straight line, by rational functions of its arc length.

Proof by contradiction: For $\deg(W,X,Y) \geq 2$

$$x'^2 + y'^2 \equiv 1 \Rightarrow \left(\frac{WX'-W'X}{W^2}\right)^2 + \left(\frac{WY'-W'Y}{W^2}\right)^2 \equiv 1$$

or $(A,B,C) = (WX'-W'X,\ WY'-W'Y,\ W^2) =$

Pythagorean triple of polynomials: $A^2+B^2=C^2$

NOTE: can ignore case $W(\xi)=1$, since $X'^2+Y'^2 \neq \text{const}$ for polynomials $X(\xi), Y(\xi)$ with $\deg(X,Y) \geq 2$

$$A^2+B^2=C^2 \Leftrightarrow \begin{cases} A = w(u^2-v^2) \\ B = 2wuv \\ C = w(u^2+v^2) \end{cases}$$

for polynomials $w(\xi), u(\xi), v(\xi)$, $\gcd(u,v)=1$

$$x = \int \frac{WX'-W'X}{W^2}\, d\xi = \int \frac{u^2-v^2}{u^2+v^2}\, d\xi$$

$$y = \int \frac{WY'-W'Y}{W^2}\, d\xi = \int \frac{2uv}{u^2+v^2}\, d\xi$$

Can both integrals be rational if $u(\xi), v(\xi)$ not both constants & $\gcd(u,v)=1$?

Partial fraction expansion of Integrands:

$$\frac{f(\xi)}{g(\xi)} = \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{c_{ij}}{(\xi - z_i)^j} + \frac{\overline{c_{ij}}}{(\xi - \overline{z_i})^j}$$

where $f = u^2 - v^2$ or $2uv$, $g = u^2 + v^2$ with roots $z_i, \overline{z_i}$ of multiplicity m_i for $i = 1, \dots, N$.

$$c_{i1} = \underset{\xi = z_i}{\text{residue}} \frac{f(\xi)}{g(\xi)} = \frac{1}{(m_i - 1)!} \frac{d^{m_i - 1}}{d\xi^{m_i - 1}} (\xi - z_i)^{m_i} \frac{f(\xi)}{g(\xi)} \Big|_{\xi = z_i}$$

inverse linear terms in partial fraction expansion incur transcendental (logarithm or arctangent) functions in $\int \frac{f(\xi)}{g(\xi)} d\xi$

$$\int \frac{f(\xi)}{g(\xi)} d\xi = \text{rational function} \iff c_{i1} = \overline{c_{i1}} = 0, \quad i = 1, \dots, N$$

Also, by "calculus of residues"

$$\int_{-\infty}^{+\infty} \frac{f(\xi)}{g(\xi)} d\xi = 2\pi i \sum_{\operatorname{Im}(z_i) > 0} \underset{\xi = z_i}{\text{residue}} \frac{f(\xi)}{g(\xi)}$$

Provided that $\deg(f) \leq \deg(g) - 2$

CONTRADICTION : suppose

$$x(\xi) = \int \frac{u^2(\xi)-v^2(\xi)}{u^2(\xi)+v^2(\xi)}\, d\xi, \quad y(\xi) = \int \frac{2u(\xi)v(\xi)}{u^2(\xi)+v^2(\xi)}\, d\xi$$

are both rational, with $\deg(u,v) \geqslant 1$, $\gcd(u,v)=1$

Can choose constants α, β so that

$$\deg(\alpha u+\beta v)^2 < \deg(u^2+v^2)$$

Expand :

$$(\alpha u+\beta v)^2 = \tfrac{1}{2}(\alpha^2-\beta^2)(u^2-v^2) + \alpha\beta\, 2uv + \tfrac{1}{2}(\alpha^2+\beta^2)(u^2+v^2)$$

Then

$$\int \frac{(\alpha u+\beta v)^2}{u^2+v^2}\, d\xi = \tfrac{1}{2}(\alpha^2-\beta^2)x(\xi) + \alpha\beta\, y(\xi) + \tfrac{1}{2}(\alpha^2+\beta^2)\xi$$

is rational.

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{(\alpha u+\beta v)^2}{u^2+v^2}\, d\xi = 0 \quad \Rightarrow \quad \frac{(\alpha u+\beta v)^2}{u^2+v^2} \equiv 0$$

But this contradicts $\gcd(u,v)=1$.