A 4,000-year tour of algebra and geometry motivated by the investigation of Pythagorean-hodograph curves

Rida T. Farouki

Department of Mechanical & Aerospace Engineering, University of California, Davis

— chronology —

$\sim$ 1800 BC Larsa, Mesopotamia
Plimpton 322 — "Pythagorean triples" cuneiform tablet

$\sim$ 540 BC Crotone, Magna Graecia
the Pythagorean school — "theorem of Pythagoras"

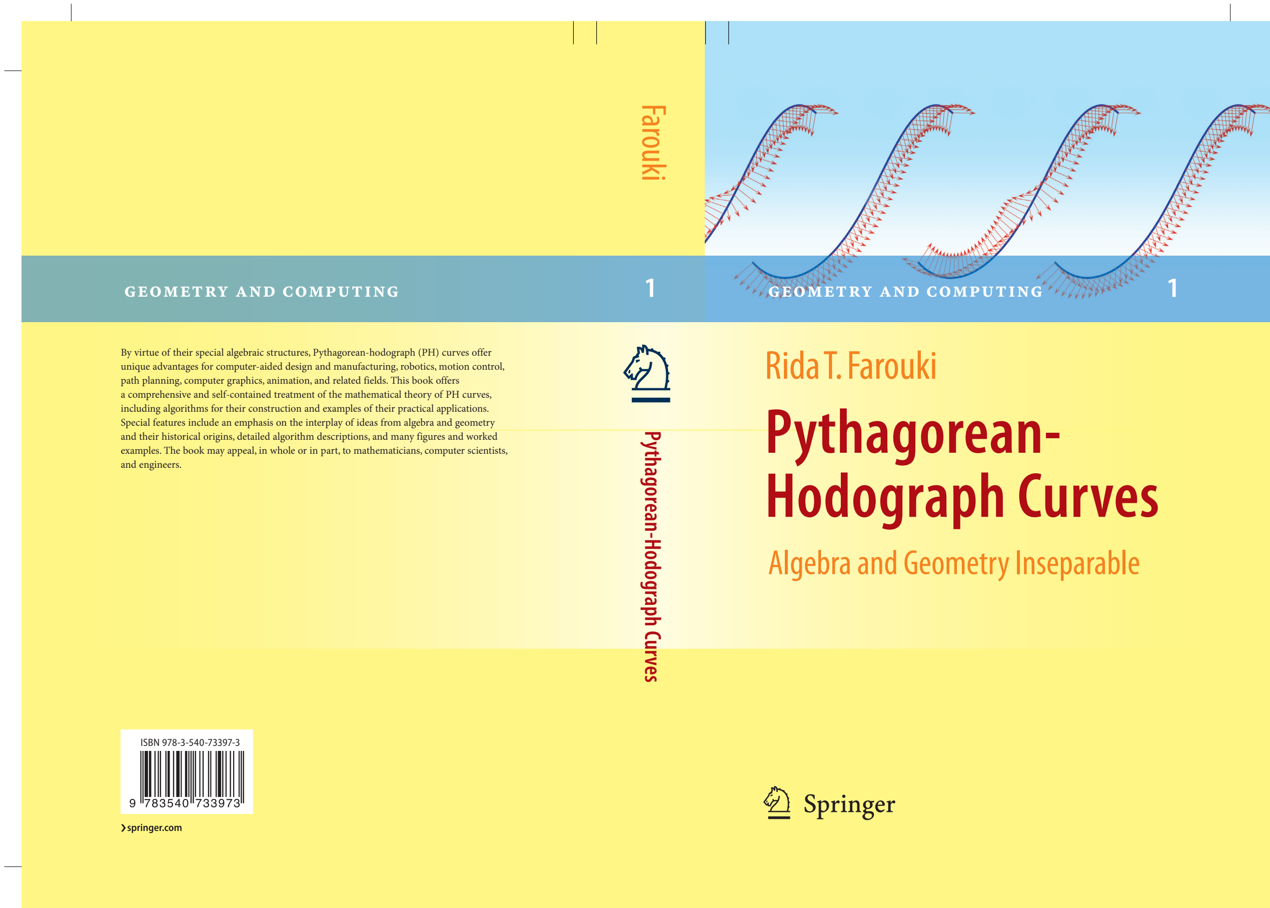
$\sim$ 825 AD Muhammad al-Khwarizmi, Baghdad
Kitab al–mukhtasar fi hisab al–jabr wa'l–muqabalah
rules of algebra; solutions of specific cubic equations

16th Century Italy – Tartaglia, Cardano, Ferrari
"solution by radicals" for cubic and quartic equations

1651–1708 Ehrenfried Walther von Tschirnhaus, Dresden
Tschirnhausen's cubic; reduction of algebraic equations;
caustics by reflection; manufacture of hard–fired porcelain

1745–1818 Caspar Wessel, Copenhagen
Om directionens analytiske betegning — geometry of complex numbers

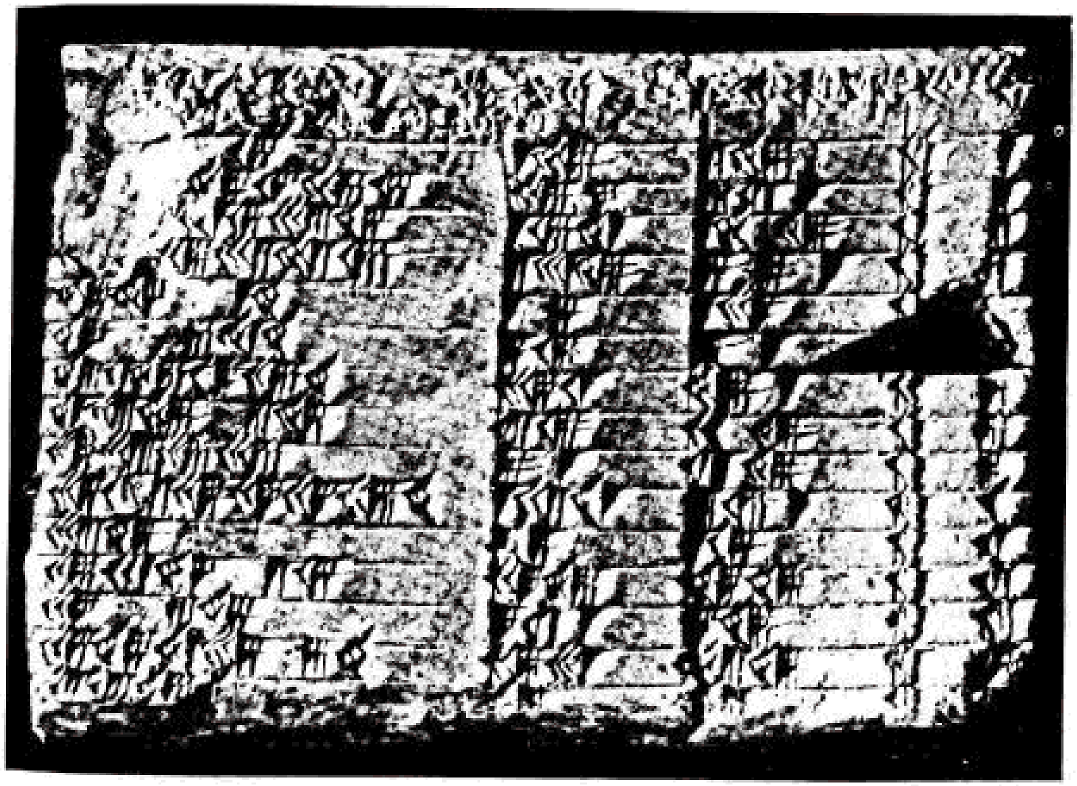
1805–1865 Sir William Rowan Hamilton, Dublin
algebra of quaternions; spatial rotations; origins of vector analysis



As long as algebra and geometry were separated, their progress was slow and their uses limited; but once these sciences were united, they lent each other mutual support and advanced rapidly together towards perfection.

Joseph-Louis Lagrange (1736-1813)

Plimpton 322



origin — Larsa (Tell Senkereh) in Mesopotamia $\sim$ 1820–1762 BC

discovered in 1920s — bought in market by dealer Edgar A. Banks — sold to collector George A. Plimpton for $10 — donated to Columbia University

deciphered in 1945 by Otto Neugebauer and Abraham Sachs — but significance, meaning, or "purpose" still the subject of great controversy

sketch of Plimpton 322 by Eleanor Robson

fifteen rows of sexagecimal numbers in four columns

$$3, 31, 49 \;\rightarrow\; 3 \times (60)^2 + 31 \times 60 + 49$$

$$1; 48, 54, 1, 40 \;\rightarrow\; 1 + \frac{48}{60} + \frac{54}{(60)^2} + \frac{1}{(60)^3} + \frac{40}{(60)^4}$$

first three columns generated by integers p & q through formulae

$$\left[\frac{p^2+q^2}{2pq}\right]^2, \quad p^2-q^2, \quad p^2+q^2$$

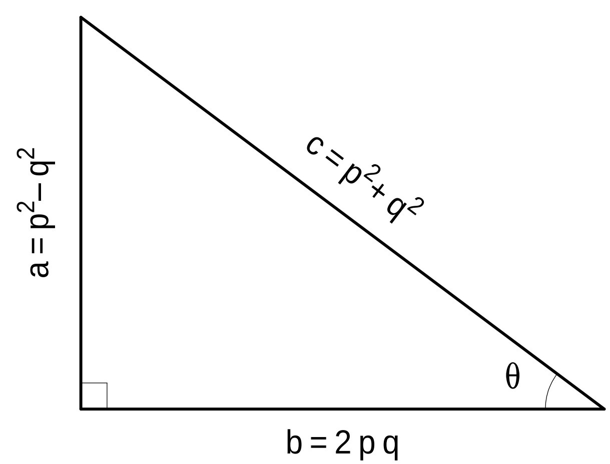
with $1 < q < 60$, $q < p$, p/q steadily decreasing

$f = [\,(p^2+q^2)/2pq\,]^2$	$a = p^2 - q^2$	$c = p^2 + q^2$	#
[1;59,0,]15	1,59	2,49	1
[1;56,56,]58,14,50,6,15	56,7	1,20,25	2
[1;55,7,]41,15,33,45	1,16,41	1,50,49	3
[1;]5[3,1]0,29,32,52,16	3,31,49	5,9,1	4
[1;]48,54,1,40	1,5	1,37	5
[1;]47,6,41,40	5,19	8,1	6
[1;]43,11,56,28,26,40	38,11	59,1	7
[1;]41,33,59,3,45	13,19	20,49	8
[1;]38,33,36,36	8,1	12,49	9
1;35,10,2,28,27,24,26,40	1,22,41	2,16,1	10
1;33,45	45,0	1,15,0	11
1;29,21,54,2,15	27,59	48,49	12
[1;]27,0,3,45	2,41	4,49	13
1;25,48,51,35,6,40	29,31	53,49	14
[1;]23,13,46,40	56	1,46	15

p	q
12	5
1,4	27
1,15	32
2,5	54
9	4
20	9
54	25
32	15
25	12
1,2 1	40
1,0	30
48	25
15	8
50	27
9	5

Pythagorean triples of integers

$$a^2 + b^2 = c^2 \iff \left\{ \begin{array}{lcl} a & = & p^2 - q^2 \\ b & = & 2\,p\,q \\ c & = & p^2 + q^2 \end{array} \right.$$



what is the "meaning" of Plimpton 322 ?

R. C. Buck (1980), *Sherlock Holmes in Babylon*, Amer. Math. Monthly **87**, 335-345

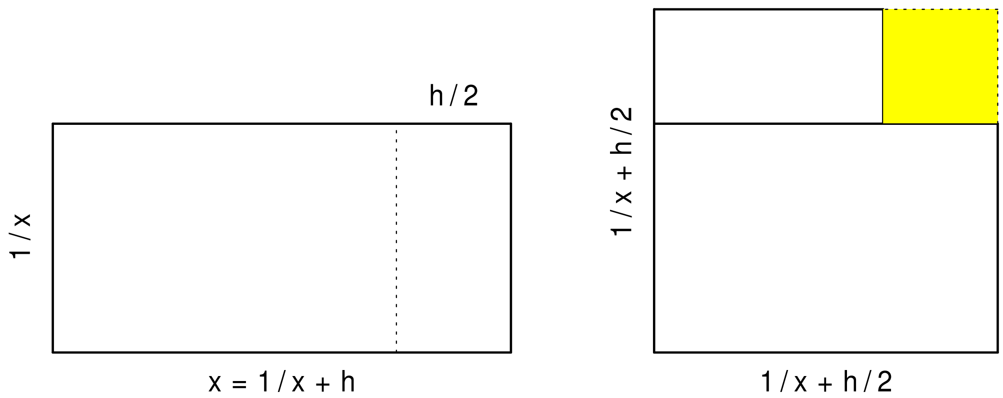
- investigate in isolation as a "mathematical detective story"
- an exercise in number theory $(a, b, c) = (p^2 - q^2, 2pq, p^2 + q^2)$?
- construction of a trigonometric table — $\sec^2\theta = [(p^2 + q^2)/2pq]^2$?

Eleanor Robson (2001), *Neither Sherlock Holmes nor Babylon — A Reassessment of Plimpton 322*, Historia Mathematica **28**, 167-206

- studied mathematics, then Akkadian and Sumerian at Oxford
- linguistic, cultural, historical context critical to a proper interpretation
- number theory & trigonometry interpretations improbable — more likely a set of "cut–and–paste geometry" exercises for the training of scribes

“cut-and-paste geometry” problem

find regular reciprocals $x, \frac{1}{x}$ satisfying $x = \frac{1}{x} + h$ for integer h



“cut-and-paste geometry” problem : $1 = \left(\frac{1}{x} + \frac{h}{2}\right)^2 - \left(\frac{h}{2}\right)^2$

writing $x = \frac{p}{q}$, $\frac{1}{x} = \frac{q}{p}$ gives $\frac{1}{x} + \frac{h}{2} = \frac{1}{2}\left(\frac{p}{q} + \frac{q}{p}\right)$, $\frac{h}{2} = \frac{1}{2}\left(\frac{p}{q} - \frac{q}{p}\right)$

scaling by $2pq$ yields $d = p^2 + q^2$, $s = p^2 - q^2$

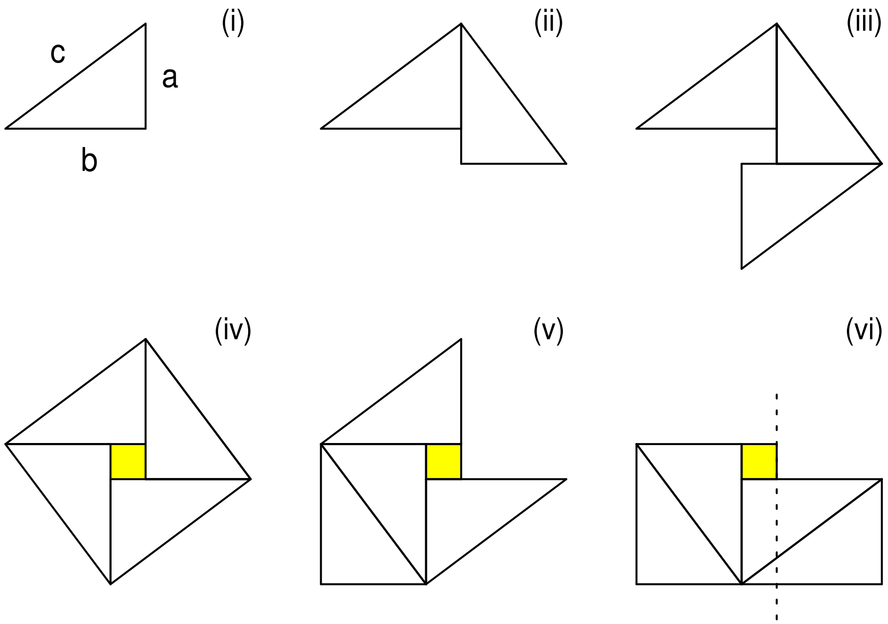
f represents (unscaled) area of large square

Pythagoras of Samos $\sim$ 580–500 BC

travelled to Egypt (possibly Mesopotamia), founded *Pythagorean School* in Crotone, S. E. Italy — no written records, no contemporary biography

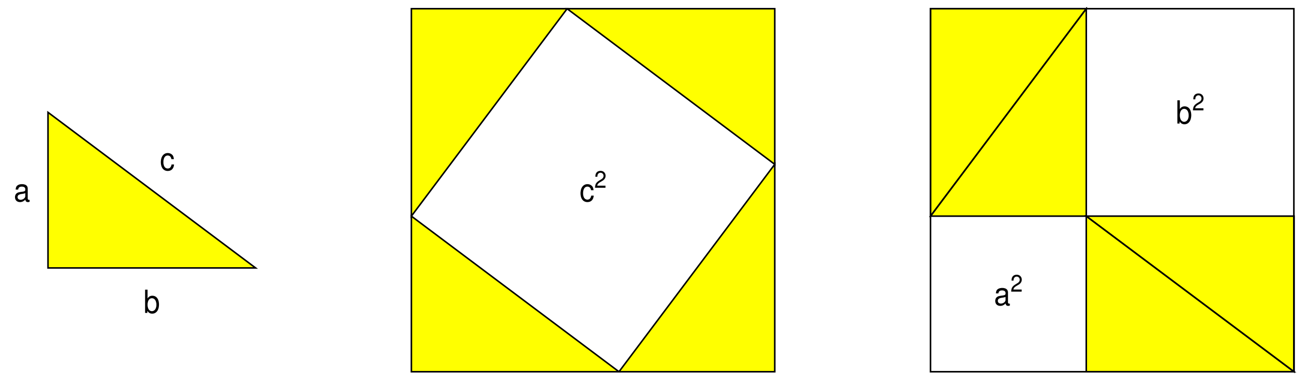
philosophy $=$ "love of wisdom," *mathematics* $=$ "that which is learned"

secretive and elitist practices incurred suspicions —
Pythagorean school destroyed, Pythagoras killed in Metapontum



proof of Pythagorean theorem, $a^2 + b^2 = c^2$

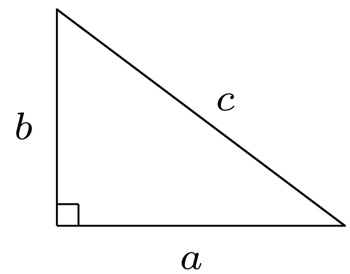
Pythagoras as legend



It is hard to let go of Pythagoras. He has meant so much to so many for so long. I can with confidence say to readers of this essay: most of what you believe, or think you know, about Pythagoras is fiction, much of it deliberately contrived.

M. F. Burnyeat, *London Review of Books* (2007)

W. Burkert (1972), *Lore and Science in Ancient Pythagoreanism*, Harvard University Press (translated by E. L. Minar, Jr.)



$$\boxed{a, b, c \;=\; \textbf{real numbers}}$$

choose any $a, b \;\longrightarrow\; c = \sqrt{a^2 + b^2}$

$$\boxed{a, b, c \;=\; \textbf{integers}}$$

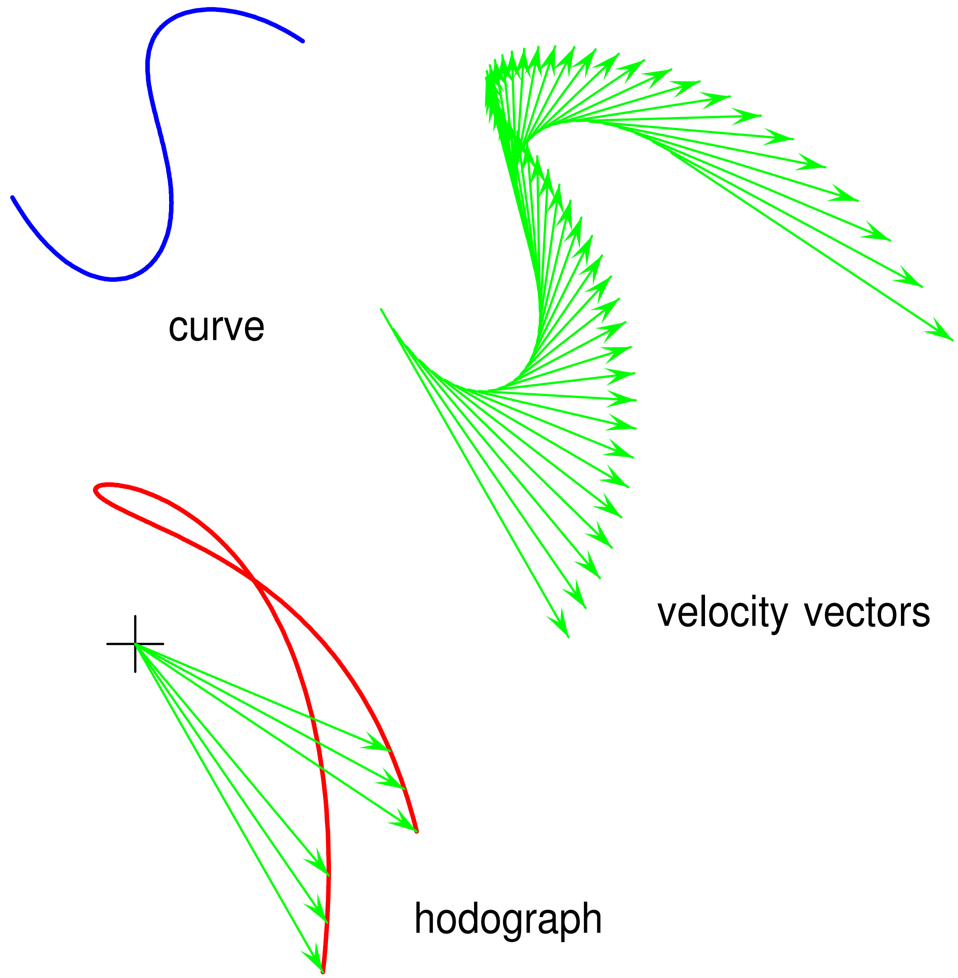
$$a^2 + b^2 = c^2 \iff \begin{cases} a = (u^2 - v^2)w \\ b = 2uvw \\ c = (u^2 + v^2)w \end{cases}$$

$$\boxed{a(t), b(t), c(t) \;=\; \textbf{polynomials}}$$

$$a^2(t) + b^2(t) \equiv c^2(t) \iff \begin{cases} a(t) = [\, u^2(t) - v^2(t) \,]\, w(t) \\ b(t) = 2\, u(t)v(t)w(t) \\ c(t) = [\, u^2(t) + v^2(t) \,]\, w(t) \end{cases}$$

K. K. Kubota, *Amer. Math. Monthly* **79**, 503 (1972)

hodograph $=$ **curve derivative,** $\mathbf{r}'(t)$



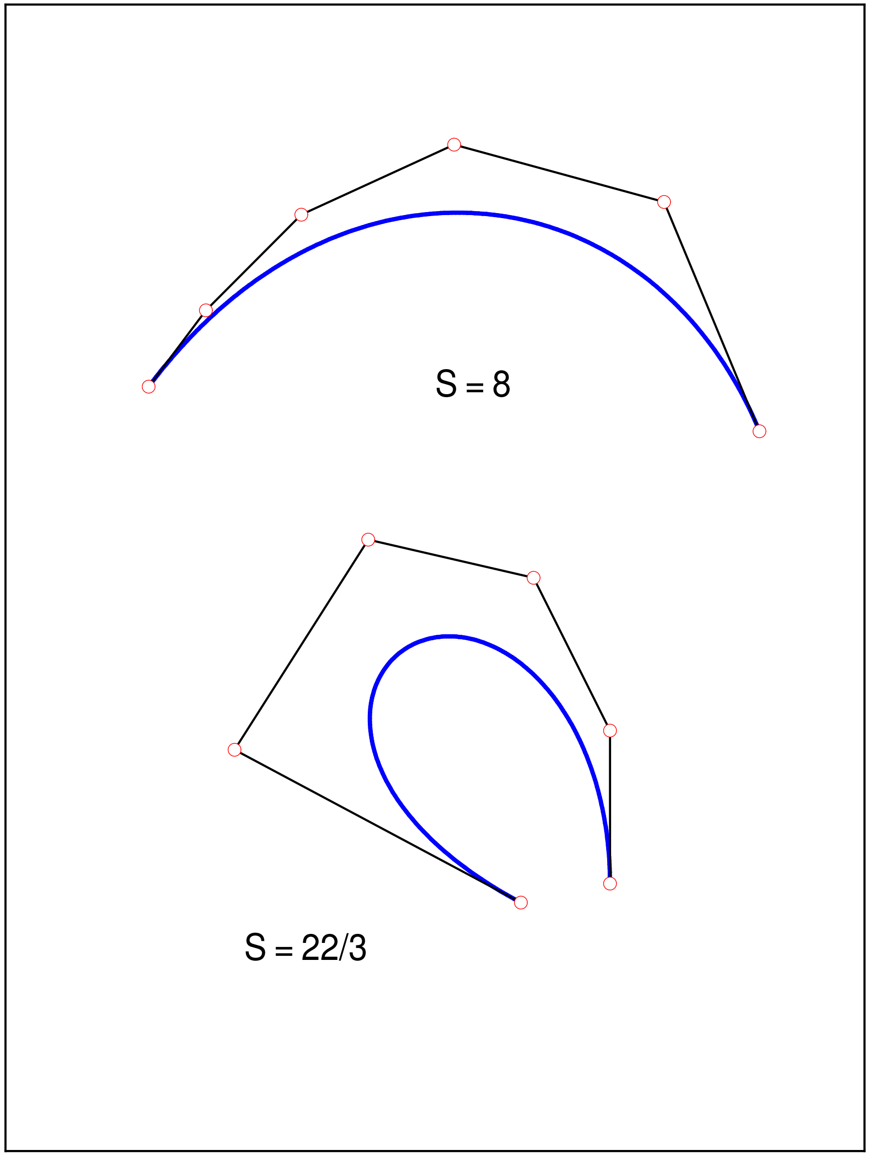
Pythagorean-hodograph (PH) curves

$\mathbf{r}(t) =$ PH curve in $\mathbb{R}^n \iff$ coordinate components of $\mathbf{r}'(t)$

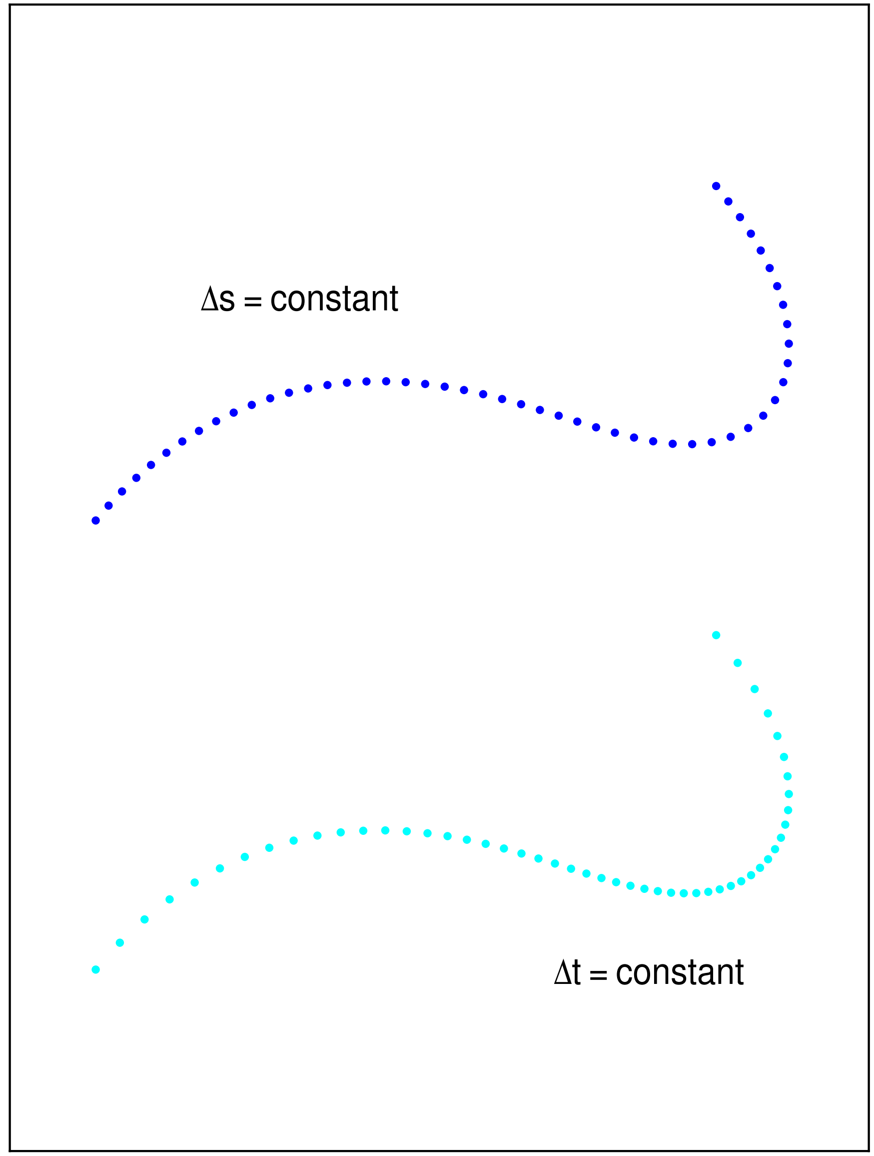
are elements of a "Pythagorean $(n+1)$-tuple of polynomials"

PH curves exhibit special algebraic structures in their hodographs

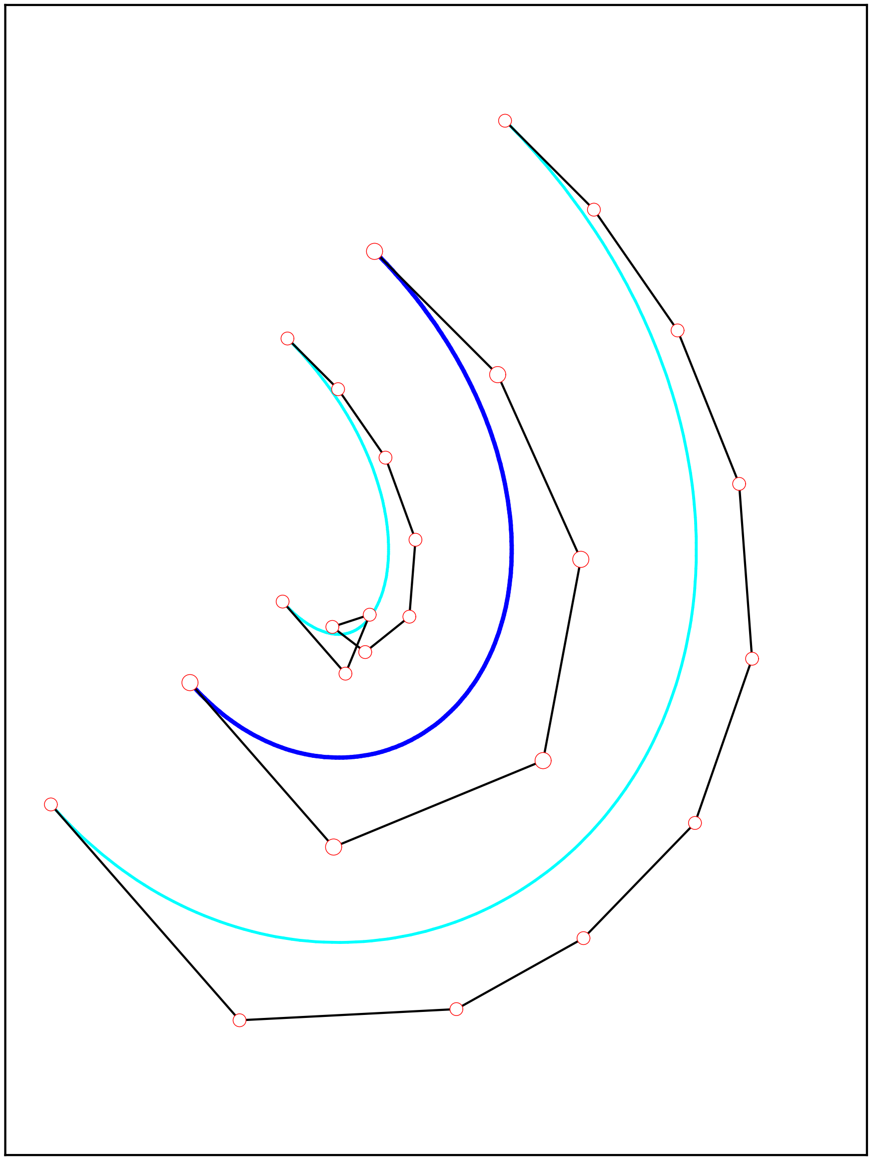
- rational offset curves $\mathbf{r}_d(t) = \mathbf{r}(t) + d\,\mathbf{n}(t)$
- polynomial arc-length function $s(t) = \displaystyle\int_0^t |\mathbf{r}'(\tau)|\,\mathrm{d}\tau$
- closed-form evaluation of energy integral $E = \displaystyle\int_0^1 \kappa^2\,\mathrm{d}s$
- real–time CNC interpolators, rotation-minimizing frames, etc.



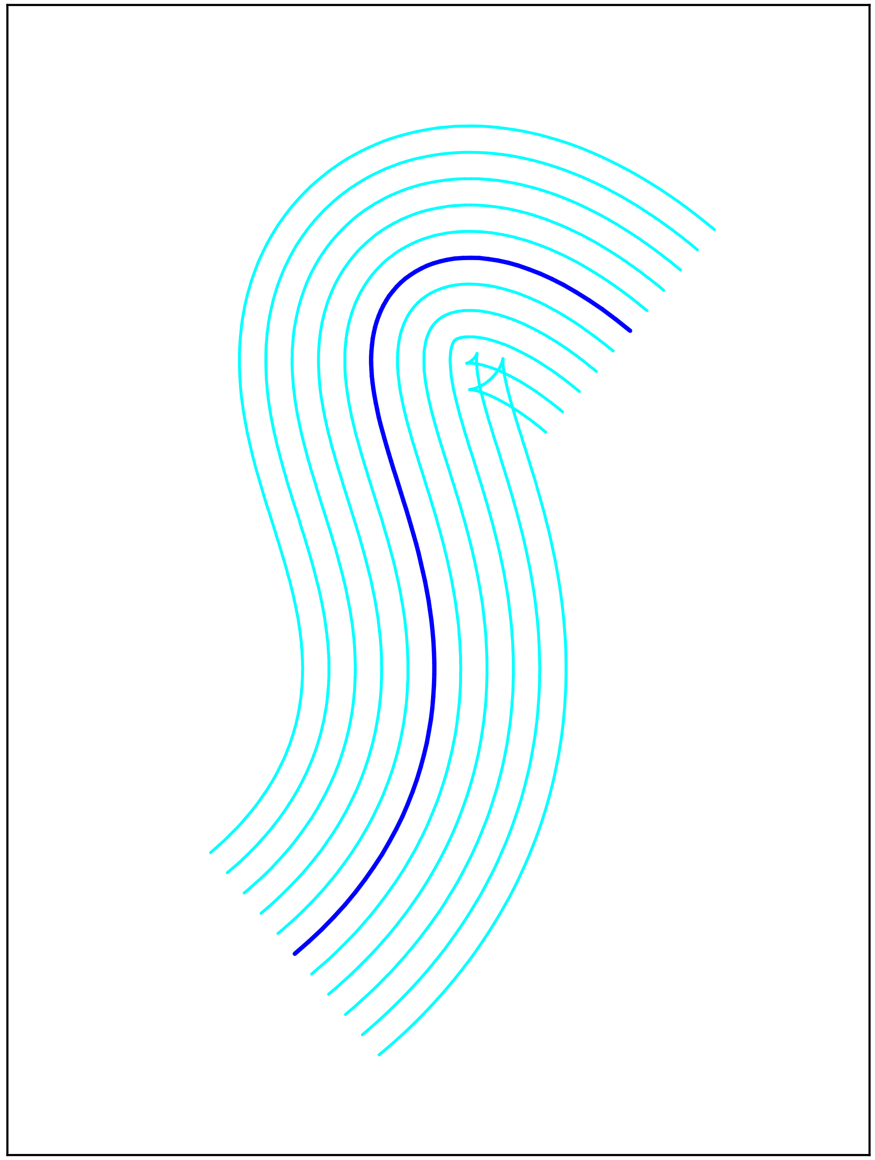
exact arc lengths



uniform arc–length rendering



Bezier control polygons of rational offsets



offsets exact at any distance

al-jabr wa’l-muqabalah

etymology of **algebra** and **algorithm**

Muhammad ibn Musa al–Khwarizmi (c. 825 AD),
Kitab al mukhtasar fi hisab al-jabr wa’l-muqabalah

al-jabr wa’l-muqabalah $=$ “restoration and balancing”
(rearranging terms in an equation to obtain solution)

translated into Latin as *Liber algebrae et almucabola*
by Englishman Robert of Chester (c. 1125 AD, Segovia)

---

another treatise translated by Adelhard of Bath (c. 1130 AD) as
Algoritmi de numero Indorum (al–Khwarizimi on the Hindu numeral system) — discovered in Cambridge by B. Boncompagni, 1857

Omar Khayyam (1048–1131)

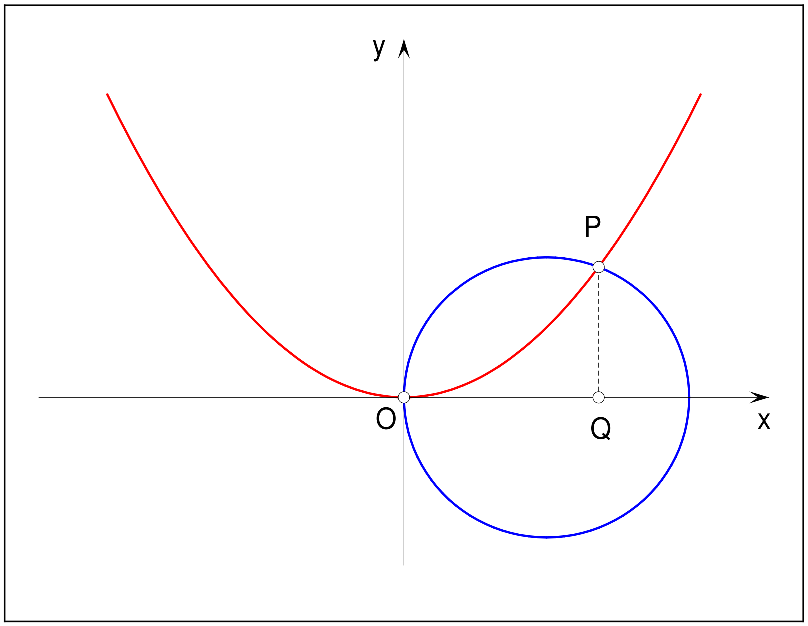
— astronomer, poet, mathematician —

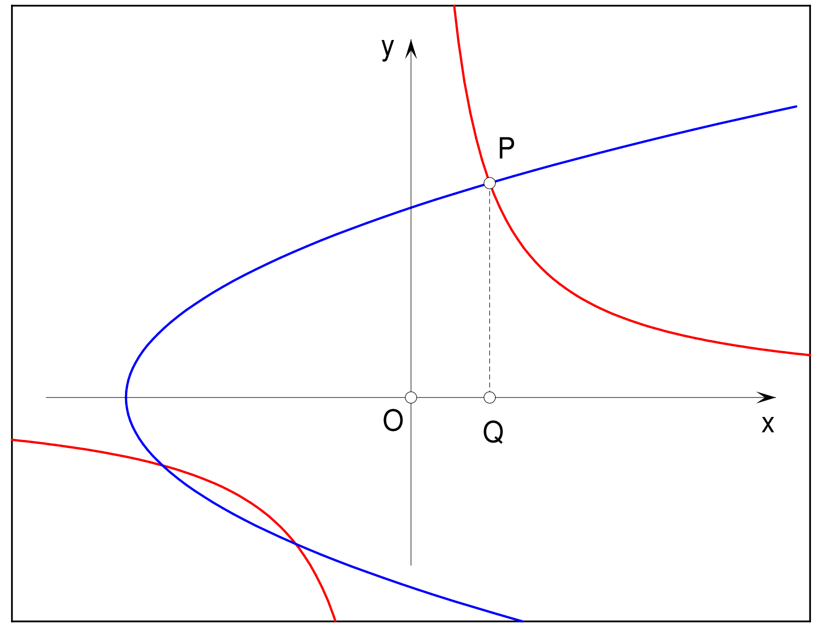
I say, with God's help and good guidance, that the art of al-jabr and al-muqabalah is a mathematical art, whose subject is pure number and mensurable quantitites in as far as they are unknown, added to a known thing with the help of which they may be found; and that thing is either a quantity or a ratio, so that no other is like it, and the thing is revealed to you by thinking about it. And what is required in it are the coefficients which are attached to its subject matter in the manner stated above. And the perfection of the art is knowing the mathematical methods by which one is led to the manner of extracting the numerical and mensurable unknowns.

Risala fi'l-barahin 'ala masa'il al-jabr wa'l-muqabalah

Omar Khayyam's solution of cubics

(i) $x^3 + a^2x = a^2b$ (ii) $x^3 + ax^2 = b^3$





(i) intersect parabola $x^2 = ay$ & circle $x^2 + y^2 - bx = 0$

(ii) intersect parabola $y^2 = b(x + a)$ & hyperbola $xy = b^2$

in both cases, positive root $=$ length OQ

Ruba’iyat (quatrains) of Omar Khayyam

Khayyam better known in the West as a poet: *Ruba’iyat* popularized by Edward FitzGerald (1859) — also musical score by Alan Hovhaness

The moving finger writes, and, having writ,
Moves on: nor all thy piety nor wit
Shall lure it back to cancel half a line,
Nor all thy tears wash out a word of it.

Khayyam realized that some cubics have more than one real root — sought a method for solving general cubics, but lacked knowledge of complex numbers

Pythagorean triples of polynomials

$$x'^2(t) + y'^2(t) = \sigma^2(t) \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{l} x'(t) = u^2(t) - v^2(t) \\ y'(t) = 2\,u(t)v(t) \\ \sigma(t) = u^2(t) + v^2(t) \end{array} \right.$$

K. Kubota, Pythagorean triples in unique factorization domains, *Amer. Math. Monthly* **79**, 503–505 (1972)

R. T. Farouki and T. Sakkalis, Pythagorean hodographs, *IBM J. Res. Develop.* **34** 736–752 (1990)

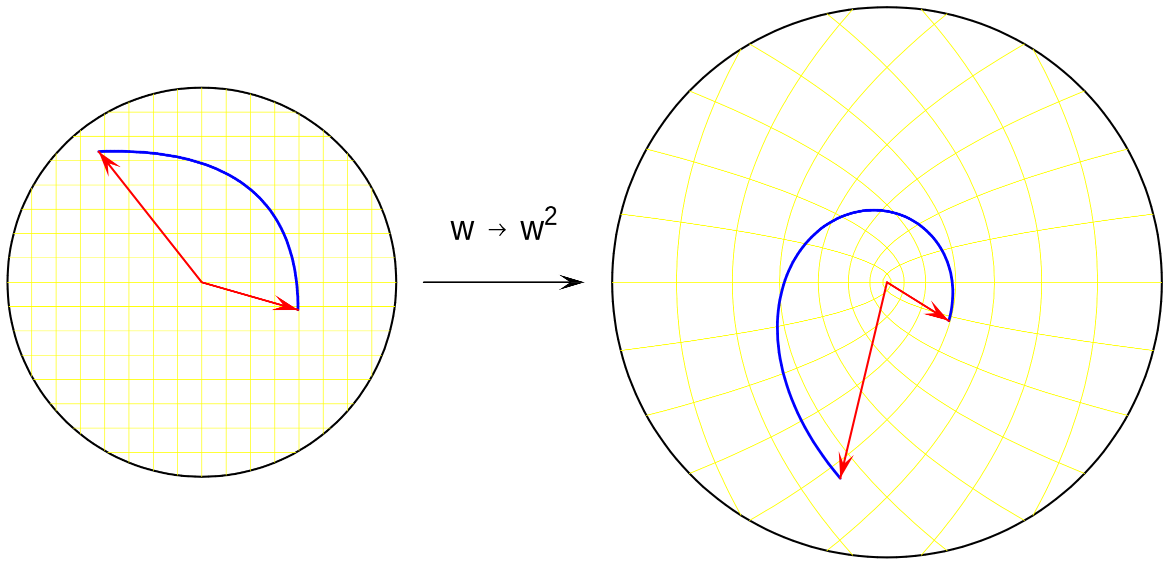
R. T. Farouki, The conformal map $z \to z^2$ of the hodograph plane, *Comput. Aided Geom. Design* **11**, 363–390 (1994)

$($**complex polynomial**$)^2 \to$ **planar Pythagorean hodograph**

choose complex polynomial $\quad \mathbf{w}(t) = u(t) + \mathrm{i}\,v(t)$

$\to$ planar Pythagorean hodograph $\quad \mathbf{r}'(t) = (x'(t), y'(t)) = \mathbf{w}^2(t)$

complex number model for planar PH curves



$$\mathbf{w}(t) = u(t) + \mathrm{i}\,v(t) \;\text{ maps to }\; \mathbf{r}'(t) = \mathbf{w}^2(t) = u^2(t) - v^2(t) + \mathrm{i}\,2\,u(t)v(t)$$

rotation invariance of planar PH form: rotate by θ, $\mathbf{r}'(t) \;\longrightarrow\; \tilde{\mathbf{r}}'(t)$

then $\tilde{\mathbf{r}}'(t) = \tilde{\mathbf{w}}^2(t)$ where $\tilde{\mathbf{w}}(t) = \tilde{u}(t) + \mathrm{i}\,\tilde{v}(t) = \exp(\mathrm{i}\,\frac{1}{2}\theta)\,\mathbf{w}(t)$

in other words, $$\begin{bmatrix} \tilde{u}(t) \\ \tilde{v}(t) \end{bmatrix} = \begin{bmatrix} \cos\frac{1}{2}\theta & -\sin\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$$

PH quintic Hermite interpolants

$$\mathbf{w}(t) \;=\; \mathbf{w}_0(1-t)^2 + \mathbf{w}_1 2(1-t)t + \mathbf{w}_2 t^2$$

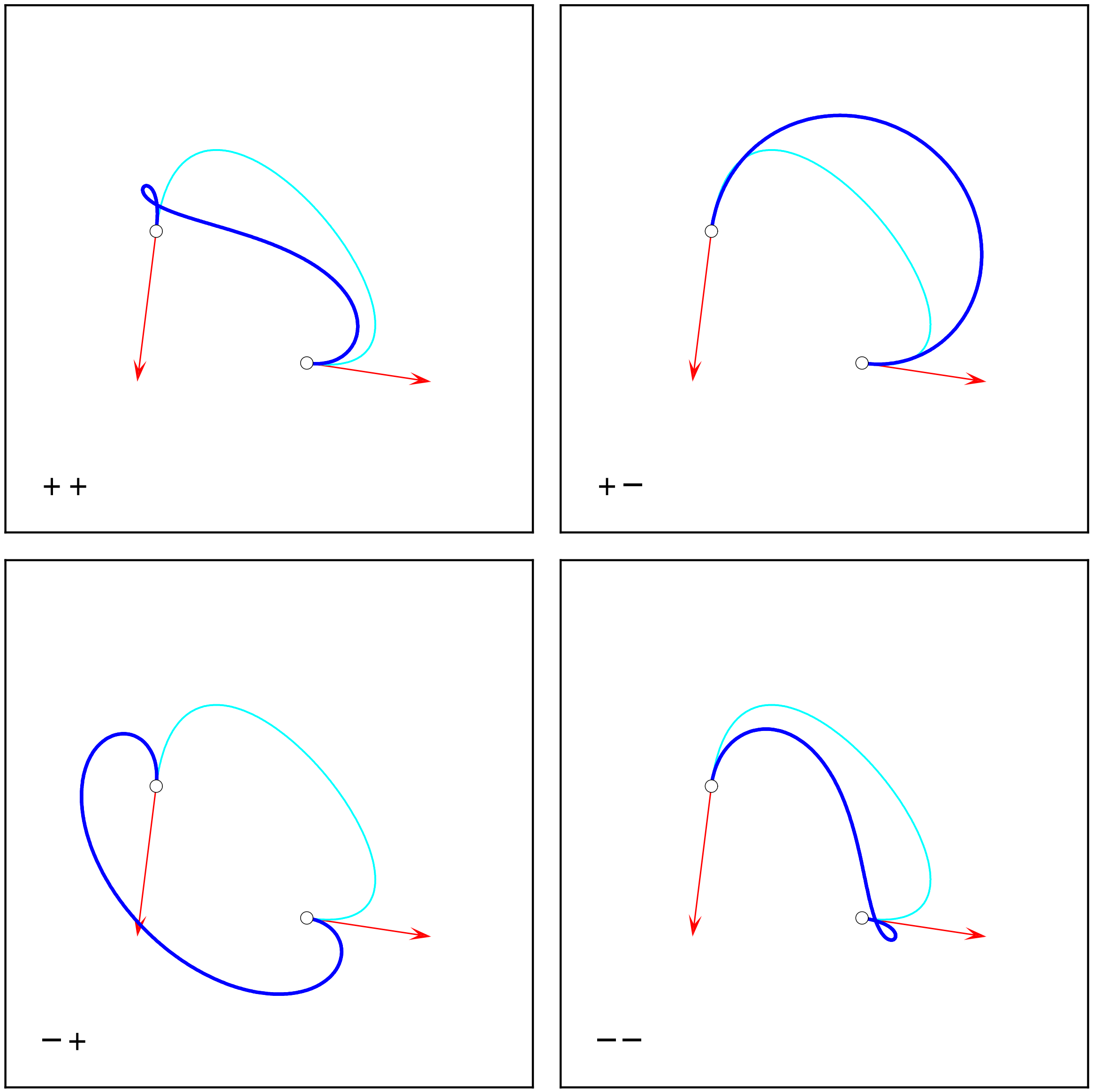
$$\mathbf{z}(t) \;=\; \int \mathbf{w}^2(t)\,\mathrm{d}t$$

$$\begin{aligned}
\mathbf{z}_1 &= \mathbf{z}_0 + \mathbf{w}_0^2/5\,, \\
\mathbf{z}_2 &= \mathbf{z}_1 + \mathbf{w}_0\mathbf{w}_1/5\,, \\
\mathbf{z}_3 &= \mathbf{z}_2 + (2\mathbf{w}_1^2 + \mathbf{w}_0\mathbf{w}_2)/15\,, \\
\mathbf{z}_4 &= \mathbf{z}_3 + \mathbf{w}_1\mathbf{w}_2/5\,, \\
\mathbf{z}_5 &= \mathbf{z}_4 + \mathbf{w}_2^2/5\,.
\end{aligned}$$

problem: find complex values $\mathbf{w}_0$, $\mathbf{w}_1$, $\mathbf{w}_2$ given $\mathbf{z}(0)$, $\mathbf{z}(1)$ and $\mathbf{z}'(0)$, $\mathbf{z}'(1)$

solution: nested pair of quadratic equations $\rightarrow$ four distinct interpolants!

four distinct PH quintic Hermite interpolants



choosing the "good" interpolant

absolute rotation index: $R_{\rm abs} \,=\, \dfrac{1}{2\pi}\displaystyle\int |\kappa|\,{\rm d}s$

w.l.o.g. take $\mathbf{z}(0)=0$ and $\mathbf{z}(1)=1$ (shift+scale of Hermite data)

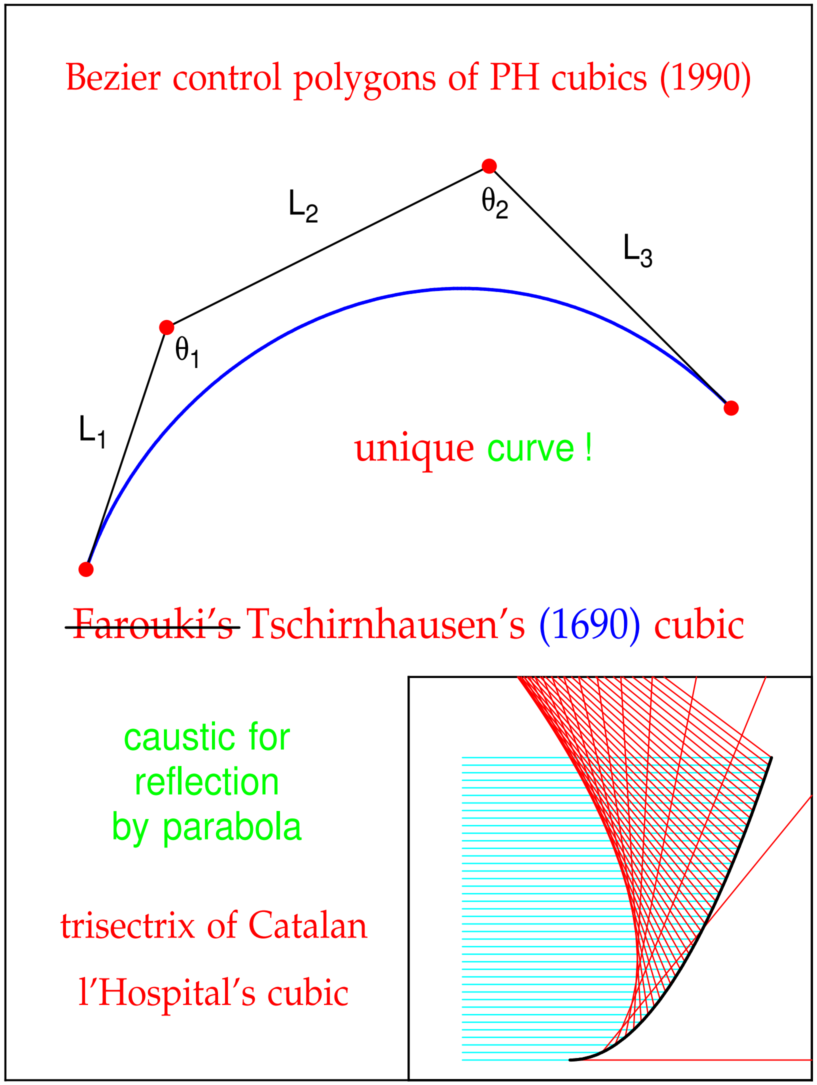
$$\mathbf{z}'(t) \,=\, \mathbf{k}\,[\,(t-\mathbf{a})(t-\mathbf{b})\,]^2$$

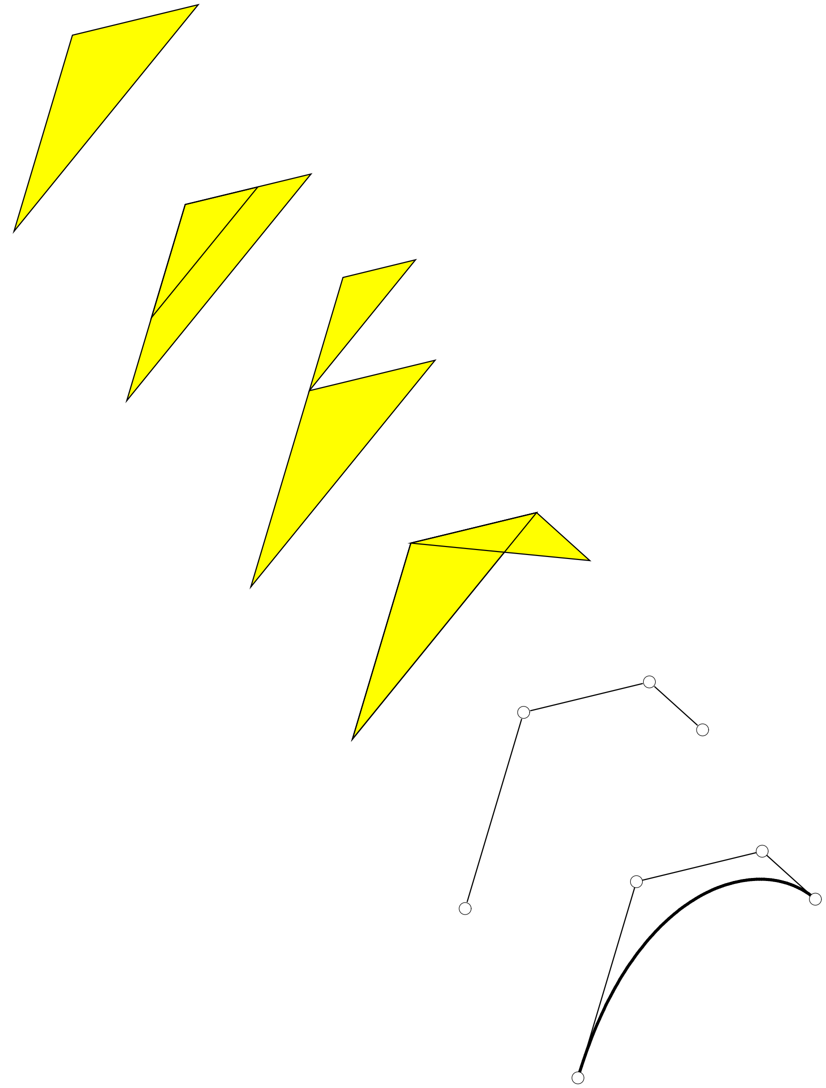
solve for $\mathbf{k},\,\mathbf{a},\,\mathbf{b}$ instead of $\mathbf{w}_0,\,\mathbf{w}_1,\,\mathbf{w}_2$

locations of $\mathbf{a},\,\mathbf{b}$ relative to $[\,0,1\,]$ gives $R_{\rm abs}$:

$$R_{\rm abs} \,=\, \frac{\angle\,0\,\mathbf{a}\,1+\angle\,0\,\mathbf{b}\,1}{\pi} \qquad \text{(no inflections)}$$

$$R_{\rm abs} \,=\, \frac{1}{\pi}\sum_{k=0}^{N}|\,\angle\,t_k\,\mathbf{a}\,t_{k+1}-\angle\,t_k\,\mathbf{b}\,t_{k+1}\,|$$





cubic PH curve $\Longleftrightarrow$ Bézier polygon satisfies $L_2 = \sqrt{L_1 L_3}$ and $\theta_2 = \theta_1$

Ehrenfried Walther von Tschirnhaus 1651–1708

- ◦ contemporary of Huygens, Leibniz, and Newton
- ◦ visited London and Paris after studying in Leiden
- ◦ investigated burning mirrors in Milan and Rome

- *Tschirnhaus transform* "A method for eliminating all intermediate terms from a given equation" — Acta Eruditorum, May 1683
- empirical & analytical investigations of *caustics by reflection*
- *Tschirnhausen's cubic* $=$ *unique cubic Pythagorean-hodograph curve*
- developed manufacture of hard–fired porcelain in Dresden

Tschirnhaus transform of cubic equation

$$t^3 + a_2 t^2 + a_1 t + a_0 = 0$$

Descartes: $t \to t - \frac{1}{3}a_2$ eliminates t^2 term

Tschirnhaus considers cubics of the form $t^3 = q\,t + r$

and defines transformation $t \to \tau$ by $t = \dfrac{2qa - 3r + 3a\tau}{q - 3a^2 - 3\tau}$,

where a is a root of the quadratic $3q\,a^2 - 9r\,a + q^2 = 0$

simplification gives $\tau^3 = \dfrac{(27r^2 - 4q^3)(2q^2 - 9ra)}{27q^2}$

Bing–Jerrard "reduced form" of quintic: $t^5 = q\,t + r$

explanation of Tschirnhaus transform

a Möbius transform (or fractional linear transform) of the form

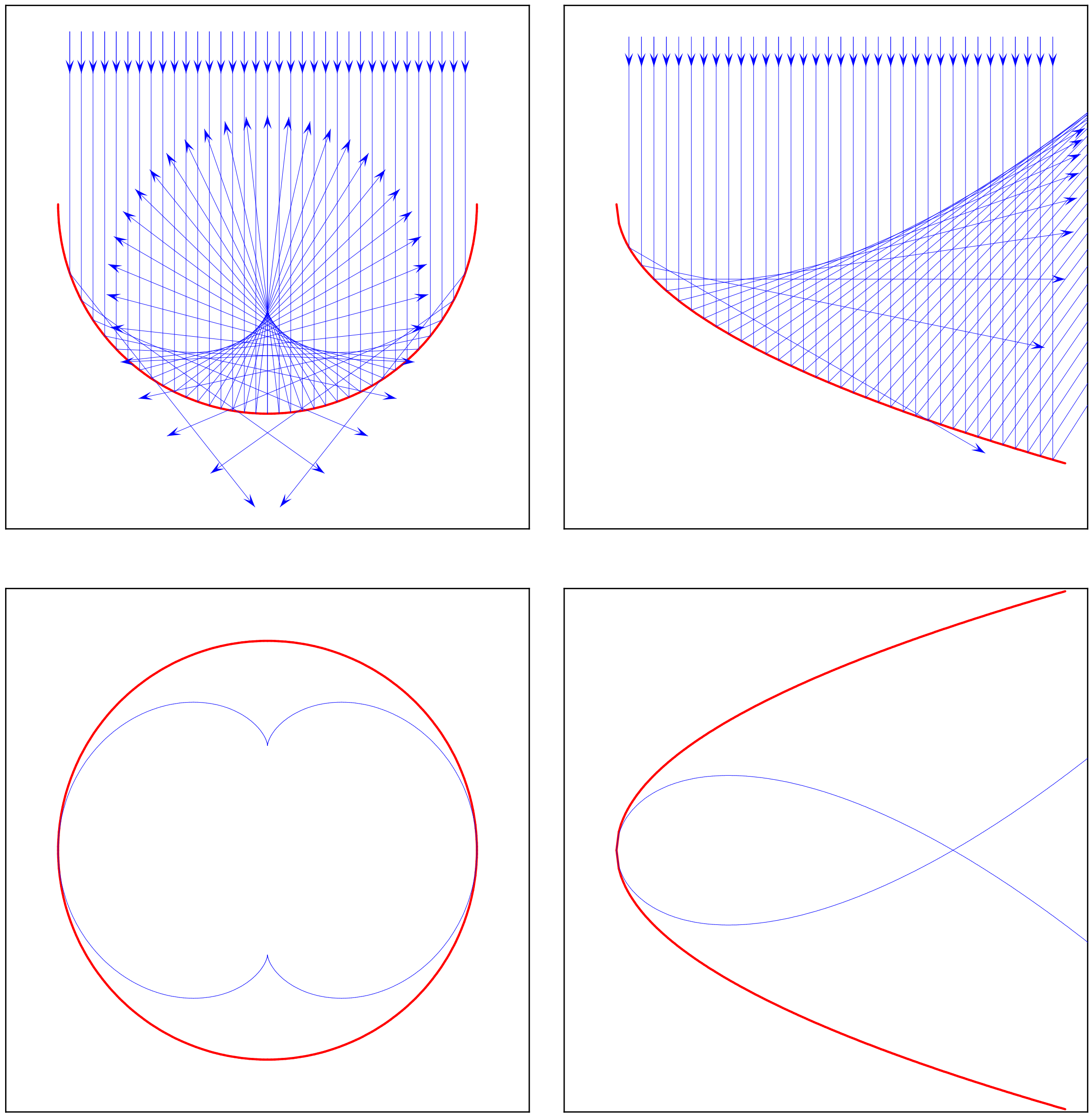
$$\mathbf{w} \;=\; \frac{\mathbf{a}\,\mathbf{z}+\mathbf{b}}{\mathbf{c}\,\mathbf{z}+\mathbf{d}}\,, \qquad \mathbf{ad}-\mathbf{bc} \;\neq\; 0$$

maps three given points $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ to three target points $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$

Tschirnhaus chooses the coefficients $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ so that the roots of the transformed cubic are symmetrically located about the origin

— i.e., the transformed cubic has the “simple” form $\mathbf{w}^3 = \mathbf{k}$

caustics for reflection by a circle and a parabola



left: *epicycloid* right: *Tschirnhausen's cubic*

slow acceptance of complex numbers

"solution by radicals" for cubics & quartics: **Niccolo Fontana** (1499-1557)
Girolamo Cardano (1501-1576), and **Lodovico Ferrari** (1522-1565)

complex arithmetic is required in the solution procedure
— *even when all the roots are real*

"We have shown the symbol $\sqrt{-1}$ to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility."

Augustus De Morgan (1806–1871)

geometrical interpretation of arithmetic operations on complex numbers was the key to their widespread acceptance — first propounded by the little–known Norwegian surveyor Caspar Wessel

Caspar Wessel (1745–1818)

- Norwegian surveyor gives first clear geometrical definitions of vector addition and multiplication of complex numbers

- *Om directionens analytiske betegning, et forsøg anvendt fornemmelig til plane og sphaeriske polygoners opløsning*

  (On the analytical representation of direction: an attempt, applied chiefly to solution of plane and spherical polygons)

- presented to Royal Danish Academy in 1797 by J. N. Tetens, Professor of Mathematics and Philosophy in Copenhagen, and published in the *Mémoires* for 1799

- precedes (published) work of Argand and Gauss, but remains largely unknown for 100 years

- republished by Sophus Lie in 1895, translated to French in 1897

- first complete English translation appeared only in 1999

Om
Directionens analytiske Betegning,
et Forsøg,
anvendt fornemmelig

til

plane og sphæriske Polygoners Opløsning.

Af

Caspar Wessel,

Landmaaler.

Kiøbenhavn 1798.

Trykt hos Johan Rudolph Thiele.

Wessel's algebra of line segments

How may we represent direction analytically: that is, how shall we express right lines so that in a single equation involving one unknown line and others known, both the length and direction of the unknown line may be expressed?

sums of directed line segments

Two right lines are added if we unite them in such a way that the second line begins where the first one ends, and then pass a right line from the first to the last point of the united lines.

products of directed line segments

As regards length, the product shall be to one factor as the other factor is to the unit. As regards direction, it shall diverge from the one factor as many degrees, and on the same side, as the other factor diverges from the unit, so that the direction angle of the product is the sum of the direction angles of the factors.

identification with complex numbers

Let $+1$ *be the positive unit, and* $+\epsilon$ *a unit perpendicular to it. Then the direction angle of* $+1$ *is* 0°*, that of* -1 *is* 180°*, that of* $+\epsilon$ *is* 90°*, and that of* $-\epsilon$ *is* 270°*. By the rule that the angle of a product is the sum of the angles of the factors, we have* $(+1)(+1) = +1$*,* $(+1)(-1) = -1, \ldots$*,* $(+\epsilon)(+\epsilon) = -1, \ldots$ *From this, it is seen that* $\epsilon = \sqrt{-1}$*.*

construction of C^2 PH quintic splines

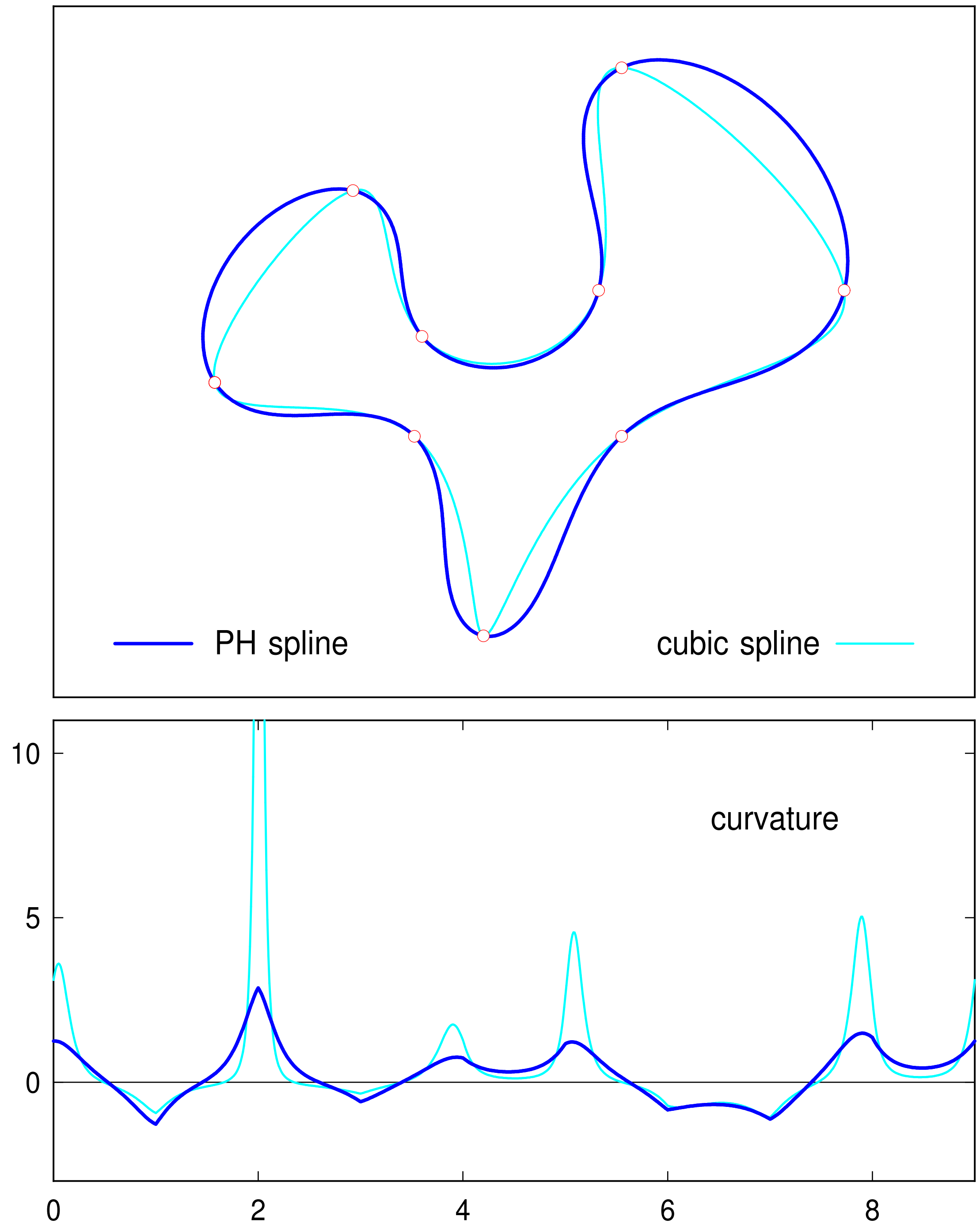
“tridiagonal” system of N quadratic equations in N complex unknowns

$$
\begin{aligned}
\mathbf{f}_1(\mathbf{z}_1,\ldots,\mathbf{z}_N) \;&=\; 17\,\mathbf{z}_1^2 \,+\, 3\,\mathbf{z}_2^2 \,+\, 12\,\mathbf{z}_1\mathbf{z}_2 \\
&\quad+\; 14\,\mathbf{a}_0\mathbf{z}_1 \,+\, 2\mathbf{a}_0\mathbf{z}_2 \,+\, 12\,\mathbf{a}_0^2 \,-\, 60\,\Delta\mathbf{p}_1 \;=\; 0\,, \\
\mathbf{f}_k(\mathbf{z}_1,\ldots,\mathbf{z}_N) \;&=\; 3\,\mathbf{z}_{k-1}^2 \,+\, 27\,\mathbf{z}_k^2 \,+\, 3\,\mathbf{z}_{k+1}^2 \,+\, 13\,\mathbf{z}_k\,(\mathbf{z}_{k-1}+\mathbf{z}_{k+1}) \\
&\quad+\; \mathbf{z}_{k-1}\mathbf{z}_{k+1} \,-\, 60\,\Delta\mathbf{p}_k \;=\; 0 \qquad \text{for}\;\; k=2,\ldots,N-1\,, \\
\mathbf{f}_N(\mathbf{z}_1,\ldots,\mathbf{z}_N) \;&=\; 17\,\mathbf{z}_N^2 \,+\, 3\,\mathbf{z}_{N-1}^2 \,+\, 12\,\mathbf{z}_N\mathbf{z}_{N-1} \\
&\quad+\; 14\,\mathbf{a}_N\mathbf{z}_N \,+\, 2\mathbf{a}_N\mathbf{z}_{N-1} \,+\, 12\,\mathbf{a}_N^2 \,-\, 60\,\Delta\mathbf{p}_N \;=\; 0\,.
\end{aligned}
$$

2^{N+m} distinct solutions — just one “good” solution among them

$m \in \{-1,0,+1\}$ depends on the adopted end conditions —
cubic end spans, periodic end condition, specified end-derivatives

compute all solutions by homotopy method (slow for $N \geq 10$)
use Newton-Raphson iteration for just the “good” solution (efficient)



PH spline
cubic spline
curvature
10
5
0
0
2
4
6
8

Welcome to the Spline Zoo

cardinal spline

B spline

spline–in–tension

conic spline

Wilson–Fowler spline

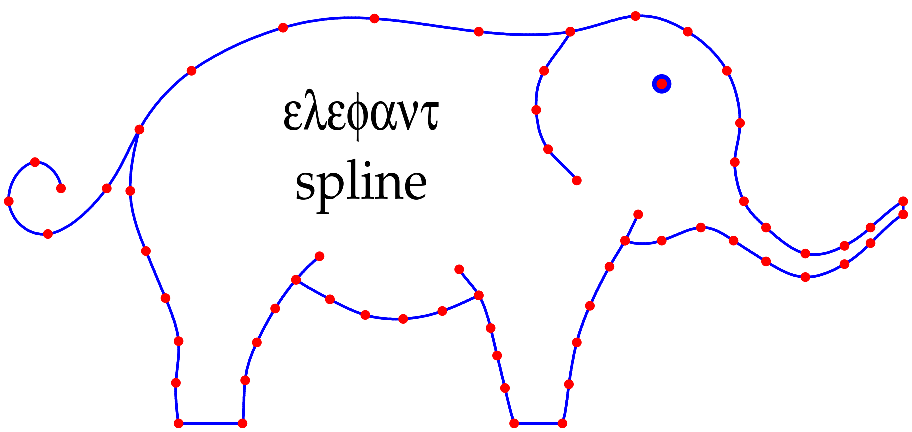
Q spline

β spline

γ spline

ν spline

τ spline

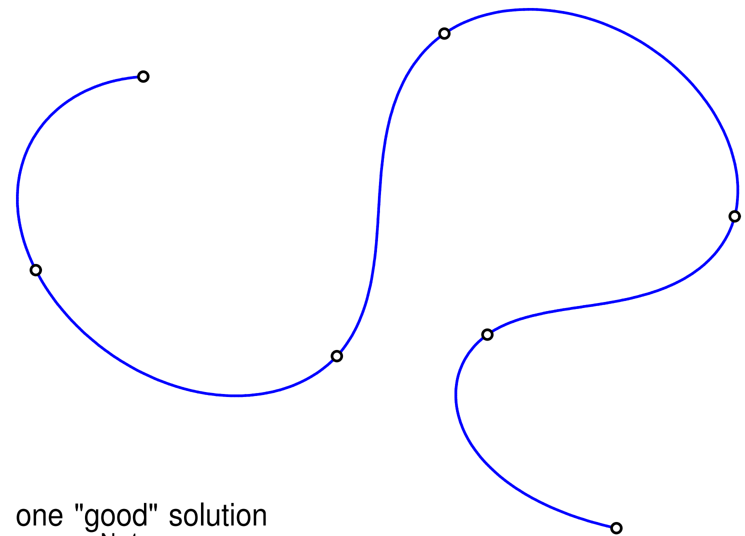


Catmull–Rom spline

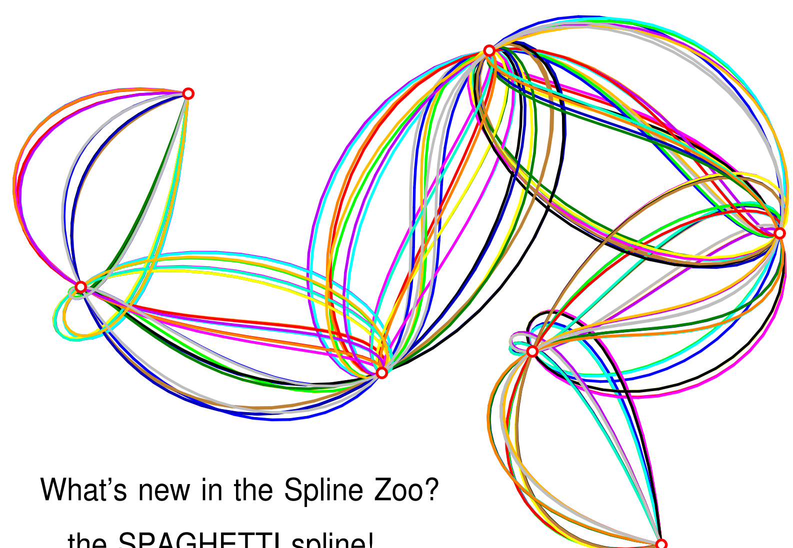
rational spline

"shape–preserving" splines

"There's always more room in the Zoo"
. . . Tom Lyche (1989)



one "good" solution
among 2^{N-1} possibilities



What's new in the Spline Zoo?

the SPAGHETTI spline!

Pythagorean quartuples of polynomials

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\ y'(t) = 2\,[\,u(t)q(t) + v(t)p(t)\,] \\ z'(t) = 2\,[\,v(t)q(t) - u(t)p(t)\,] \\ \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{array} \right.$$

R. Dietz, J. Hoschek, and B. Jüttler, An algebraic approach to curves and surfaces on the sphere and on other quadrics, *Computer Aided Geometric Design* **10**, 211–229 (1993)

H. I. Choi, D. S. Lee, and H. P. Moon, Clifford algebra, spin representation, and rational parameterization of curves and surfaces, *Advances in Computational Mathematics* **17**, 5-48 (2002)

choose quaternion polynomial $\quad \mathcal{A}(t) \;=\; u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

$\rightarrow$ spatial Pythagorean hodograph $\quad \mathbf{r}'(t) \;=\; (x'(t), y'(t), z'(t)) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$

fundamentals of quaternion algebra

quaternions are four-dimensional numbers of the form

$$\mathcal{A} \;=\; a + a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} \qquad \text{and} \qquad \mathcal{B} \;=\; b + b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$$

that obey the sum and (non–commutative) product rules

$$\mathcal{A} + \mathcal{B} \;=\; (a+b) \;+\; (a_x + b_x)\,\mathbf{i} \;+\; (a_y + b_y)\,\mathbf{j} \;+\; (a_z + b_z)\,\mathbf{k}$$

$$\begin{aligned}
\mathcal{A}\,\mathcal{B} \;=\;& (ab - a_x b_x - a_y b_y - a_z b_z) \\
+\;& (a b_x + b a_x + a_y b_z - a_z b_y)\,\mathbf{i} \\
+\;& (a b_y + b a_y + a_z b_x - a_x b_z)\,\mathbf{j} \\
+\;& (a b_z + b a_z + a_x b_y - a_y b_x)\,\mathbf{k}
\end{aligned}$$

basis elements $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$

equivalently, $\mathbf{i}\,\mathbf{j} = -\,\mathbf{j}\,\mathbf{i} = \mathbf{k}\,, \quad \mathbf{j}\,\mathbf{k} = -\,\mathbf{k}\,\mathbf{j} = \mathbf{i}\,, \quad \mathbf{k}\,\mathbf{i} = -\,\mathbf{i}\,\mathbf{k} = \mathbf{j}$

Hurwitz's theorem (1898) on composition algebras

key property — norm of product $=$ product of norms: $|\mathcal{A}\,\mathcal{B}| \;=\; |\mathcal{A}|\,|\mathcal{B}|$

commutative law — $\mathcal{A}\,\mathcal{B} = \mathcal{B}\,\mathcal{A}\,,$ associative law — $(\mathcal{A}\,\mathcal{B})\,\mathcal{C} = \mathcal{A}\,(\mathcal{B}\,\mathcal{C})$

there are four possible composition algebras, of dimension $n = 1$, 2, 4, 8

- $\mathbb{R}$ $(n = 1)$, real numbers — product is commutative & associative
- $\mathbb{C}$ $(n = 2)$, complex numbers — product is commutative & associative
- $\mathbb{H}$ $(n = 4)$, quaternions — product is associative, but not commutative
- $\mathbb{O}$ $(n = 8)$, octonions — product neither commutative nor associative

Sir William Rowan Hamilton (1805–1865)

- now most famous for contributions to optics & mechanics, but devoted most of his life to developing theory of quaternions

- complex numbers = "algebraic couples" ... no algebra of triples, but algebra of quartuples possible with non–commutative product

- terms scalar and vector first introduced by Hamilton in an article on quaternions (*Philosophical Magazine*, 1846)

- monumental works: *Lectures on Quaternions* (1853), *Elements of Quaternions* (1866) ... would "take any man a twelvemonth to read, and near a lifetime to digest" (Sir John Herschel)

- Hamilton's dream of revolutionizing mathematics & physics unrealized E. T. Bell, *Men of Mathematics* — Hamilton = "An Irish Tragedy"

- M. J. Crowe, *A History of Vector Analysis* — prevailing ignorance of the debt of vector analysis to quaternions is the real tragedy

the “troubled origins” of vector analysis

The algebraically *real* part may receive . . . all values contained on the one *scale* of progression of number from negative to positive infinity; we shall call it therefore the *scalar part*, or simply the *scalar*. On the other hand, the algebraically *imaginary* part, being constructed geometrically by a straight line or radius, which has, in general, for each determined quaternion, a determined length and determined direction in space, may be called the *vector part*, or simply the *vector* . . .

William Rowan Hamilton, *Philosophical Magazine* (1846)

A school of “quaternionists” developed, which was led after Hamilton’s death by Peter Tait of Edinburgh and Benjamin Pierce of Harvard. Tait wrote eight books on the quaternions, emphasizing their applications to physics. When Gibbs invented the modern notation for the dot and cross product, Tait condemned it as a “hermaphrodite monstrosity.” A war of polemics ensued, with luminaries such as Kelvin and Heaviside writing devastating invective against quaternions. Ultimately the quaternions lost, and acquired a taint of disgrace from which they never fully recovered.

John C. Baez, *The Octonions* (2002)

M. J. Crowe, *A History of Vector Analysis* (1967)

A high level of intensity and a certain fierceness characterized much of the debate, and must have led many readers to follow it with interest.

. . . Gibbs and Heaviside must have appeared to the quaternionists as unwelcome intruders who had burst in upon the developing dialogue between the quaternionists and the scientists of the day to arrive at a moment when success seemed not far distant. Charging forth, these two vectorists, the one brash and sarcastic, the other spouting historical irrelevancies, had promised a bright new day for any who would accept their overtly pragmatic arguments for an algebraically crude and highly arbitrary system. And worst of all, the system they recommended was, not some new system . . . but only a perverted version of the quaternion system. Heretics are always more hated than infidels, and these two heretics had, with little understanding and less acknowledgement, wrenched major portions from the Hamiltonian system and then claimed that these parts surpassed the whole.

the sad demise of quaternions

E. T. Bell, *Men of Mathematics*, Hamilton = “An Irish Tragedy”

Hamilton’s *Lectures on Quaternions* (1853) “would take any man a twelve-month to read, and near a lifetime to digest . . .”
– Sir John Herschel, discoverer of the planet Uranus

Hamilton’s vision of quaternions as the “universal language” of mathematical and physical sciences was never realized — this role is now occupied by vector analysis, distilled from the quaternion algebra by the physicists James Clerk Maxwell (1831-1879) and Josiah Willard Gibbs (1839-1903), and the engineer Oliver Heaviside (1850-1925)

some "dirty secrets" of vector analysis

- there are two *fundamentally different* types of vector in $\mathbb{R}^3$ — polar vectors and axial vectors
- a polar vector $\mathbf{v} = (v_x, v_y, v_z)$ becomes $(-v_x, -v_y, -v_z)$ under a transformation $(x, y, z) \rightarrow (-x, -y, -z)$ between right–handed and left–handed coordinate systems — also called a true vector
- an axial vector, such as the cross product $\mathbf{a} \times \mathbf{b}$, is unchanged under transformation $(x, y, z) \rightarrow (-x, -y, -z)$ — also called a pseudovector
- similar distinction exists between true scalars and pseudoscalars, e.g., $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is a pseudoscalar if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are true (polar) vectors
- vector analysis in $\mathbb{R}^3$ does not have a natural specialization to $\mathbb{R}^2$ or generalization to $\mathbb{R}^n$ for $n \geq 4$

quaternions and spatial rotations

set $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$ — a, b and $\mathbf{a}$, $\mathbf{b}$ are scalar and vector parts

(a, b and $\mathbf{a}, \mathbf{b}$ also called the real and imaginary parts of $\mathcal{A}, \mathcal{B}$)

$$\mathcal{A} + \mathcal{B} = (\, a + b \,,\, \mathbf{a} + \mathbf{b} \,)$$

$$\mathcal{A}\,\mathcal{B} = (\, ab - \mathbf{a} \cdot \mathbf{b} \,,\, a\,\mathbf{b} + b\,\mathbf{a} + \mathbf{a} \times \mathbf{b})$$

any unit quaternion has the form $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$

describes a spatial rotation by angle θ about unit vector $\mathbf{n}$

for any vector $\mathbf{v}$ the quaternion product $\boxed{\mathbf{v}' = \mathcal{U}\,\mathbf{v}\,\mathcal{U}^*}$

yields the vector $\mathbf{v}'$ corresponding to a rotation of $\mathbf{v}$ by θ about $\mathbf{n}$

unit quaternions $=$ (non-commutative) group under multiplication

quaternion model for spatial PH curves

quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

maps to
$$\begin{aligned}\mathbf{r}'(t) \;&=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \;=\; [\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\mathbf{i} \\ &\quad + 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mathbf{j} \,+\, 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\mathbf{k}\end{aligned}$$

rotation invariance of spatial PH form: rotate by θ about $\mathbf{n} = (n_x, n_y, n_z)$

define $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ — then $\mathbf{r}'(t) \;\to\; \tilde{\mathbf{r}}'(t) = \tilde{\mathcal{A}}(t)\,\mathbf{i}\,\tilde{\mathcal{A}}^*(t)$

where $\tilde{\mathcal{A}}(t) \;=\; \mathcal{U}\,\mathcal{A}(t)$ (can interpret as rotation in $\mathbb{R}^4$)

spatial PH quintics can be constructed as first–order Hermite interpolants

solve using quaternion representation $\to$ 2-parameter family of solutions

“knowledge and humility”

If I have seen further, it is by standing on the shoulders of giants.

Sir Isaac Newton, letter to Robert Hooke (1675)

Trace science then, with modesty thy guide;
First strip off all her equipage of pride,
Deduct what is but vanity, or dress,
Or learning’s luxury, or idleness;
Or tricks to show the stretch of human brain,
Mere curious pleasure, or ingenious pain:
Expunge the whole, or lop th’excrescent parts
Of all, our vices have created arts:
Then see how little the remaining sum,
Which served the past, and must the times to come!

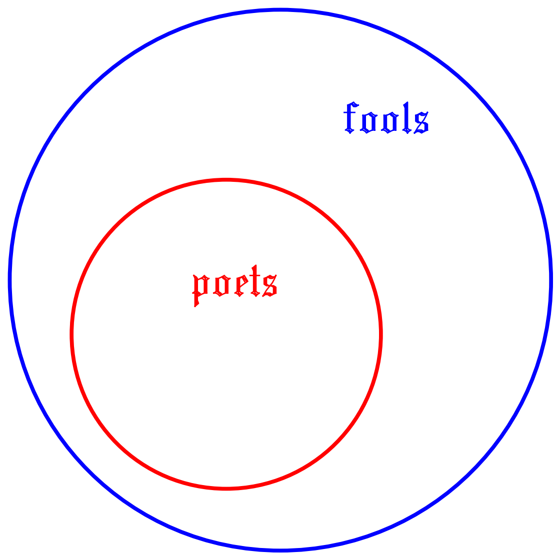
Alexander Pope (1688–1744), *Essay on Man*

Boolean algebra of poets & fools

Sir, I admit your general rule,
That every poet is a fool.

But you yourself may serve to show it,
That every fool is not a poet!

Alexander Pope (1688-1744)



all poets are fools, but not all fools are poets

(tutti i poeti sono sciocchi, ma non tutti gli sciocchi sono poeti)