Affine subspaces of matrices with constant rank

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Abstract

For every $m, n \in \mathbb{N}$ and every field K, let $M(m \times n, K)$ be the vector space of the $(m \times n)$-matrices over K and let $S(n, K)$ be the vector space of the symmetric $(n \times n)$-matrices over K. We say that an affine subspace S of $M(m \times n, K)$ or of $S(n, K)$ has constant rank r if every matrix of S has rank r. Define

$$\mathcal{A}^K(m \times n; r) = \{S \mid S \text{ affine subsapce of } M(m \times n, K) \text{ of constant rank } r\}$$

$$\mathcal{A}^K_{sym}(n; r) = \{S \mid S \text{ affine subsapce of } S(n, K) \text{ of constant rank } r\}$$

$$a^K(m \times n; r) = \max\{\dim S \mid S \in \mathcal{A}^K(m \times n; r)\}.$$

$$a^K_{sym}(n; r) = \max\{\dim S \mid S \in \mathcal{A}^K_{sym}(n, r)\}.$$

In this paper we prove the following two formulas for $r \leq m \leq n$:

$$a^{\mathbb{R}}_{sym}(n; r) \leq \left\lfloor \frac{r}{2} \right\rfloor \left(n - \left\lfloor \frac{r}{2} \right\rfloor \right)$$

$$a^{\mathbb{R}}(m \times n; r) = r(n - r) + \frac{r(r-1)}{2}.$$

1 Introduction

For every $m, n \in \mathbb{N}$ and every field K, let $M(m \times n, K)$ be the vector space of the $(m \times n)$-matrices over K and let $S(n, K)$ be the vector space of the symmetric $(n \times n)$-matrices over K. Moreover, denote the $\mathbb{R}$-vector space of the hermitian $(n \times n)$-matrices by $H(n)$.
We say that an affine subspace S of $M(m \times n, K)$ or of $S(n, K)$ (or of $H(n)$) has constant rank r if every matrix of S has rank r and we say that a linear subspace S of $M(m \times n, K)$ or of $S(n, K)$ has constant rank r if every nonzero matrix of S has rank r.

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Define

$$\mathcal{A}^K(m \times n; r) = \{S \mid S \text{ affine subsapce of } M(m \times n, K) \text{ of constant rank } r\}$$

$$\mathcal{A}^K_{sym}(n; r) = \{S \mid S \text{ affine subsapce of } S(n, K) \text{ of constant rank } r\}$$

$$\mathcal{A}_{herm}(n; r) = \{S \mid S \text{ affine subsapce of } H(n) \text{ of constant rank } r\}$$

$$\mathcal{A}^{\mathbb{R}}_{sym}(n; p, \nu) = \{S \mid S \text{ affine subsapce of } S(n, \mathbb{R}) \text{ s.t. each } A \in S \text{ has signature } (p, \nu)\}$$

$$\mathcal{A}_{herm}(n; p, \nu) = \{S \mid S \text{ affine subsapce of } H(n) \text{ s.t. each } A \in S \text{ has signature } (p, \nu)\}$$

$$\mathcal{L}^K(m \times n; r) = \{S \mid S \text{ linear subsapce of } M(m \times n, K) \text{ of constant rank } r\}$$

$$\mathcal{L}^K_{sym}(n; r) = \{S \mid S \text{ linear subsapce of } S(n, K) \text{ of constant rank } r\}$$

Let

$$a^K(m \times n; r) = \max\{\dim S \mid S \in \mathcal{A}^K(m \times n; r)\}$$

$$a^K_{sym}(n; r) = \max\{\dim S \mid S \in \mathcal{A}^K_{sym}(n; r)\}$$

$$a_{herm}(n; r) = \max\{\dim S \mid S \in \mathcal{A}_{herm}(n; r)\}$$

$$a^{\mathbb{R}}_{sym}(n; p, \nu) = \max\{\dim S \mid S \in \mathcal{A}^{\mathbb{R}}_{sym}(n; p, \nu)\}$$

$$a_{herm}(n; p, \nu) = \max\{\dim S \mid S \in \mathcal{A}_{herm}(n; p, \nu)\}$$

$$l^K(m \times n; r) = \max\{\dim S \mid S \in \mathcal{L}^K(m \times n; r)\}$$

$$l^K_{sym}(n; r) = \max\{\dim S \mid S \in \mathcal{L}^K_{sym}(n, r)\}.$$

There is a wide literature on linear subspaces of constant rank. In particular we quote the following theorems:

Theorem 1. ***(Westwick, [6])*** *For $2 \leq r \leq m \leq n$, we have:*

$$n - r + 1 \leq l^{\mathbb{C}}(m \times n; r) \leq m + n - 2r + 1$$

Theorem 2. ***(Ilic-Landsberg, [5])*** *If r is even and greater than or equal to 2, then*

$$l^{\mathbb{C}}_{sym}(n; r) = n - r + 1$$

In case r odd, the following result holds, see [5], [2], [3]:

Theorem 3. *If r is odd, then*

$$l^{\mathbb{C}}_{sym}(n; r) = 1$$

We mention also that, in [1], Flanders proved that, if $r \leq m \leq n$, a linear subspace of $M(m \times n, \mathbb{C})$ such that every of its elements has rank less than or equal to r has dimension less than or equal to rn.

In this paper we investigate on the maximal dimension of affine subspaces of constant rank. The main theorems we prove are the following.

Theorem 4. *Let $n, r \in \mathbb{N}$ with $r \leq n$. Then*

$$a^{\mathbb{R}}_{sym}(n; r) \leq \left\lfloor \frac{r}{2} \right\rfloor \left(n - \left\lfloor \frac{r}{2} \right\rfloor \right).$$

Theorem 5. *Let $m, n, r \in \mathbb{N}$ with $r \leq m \leq n$. Then*

$$a^{\mathbb{R}}(m \times n; r) = rn - \frac{r(r+1)}{2}.$$

We prove also a statement on the maximal dimension of affine subspaces with constant signature in the space of symmetric real matrices, see Theorem 11, and one on the maximal dimension of affine subspaces of constant rank in the space of the hermitian matrices, see Theorem 12.

2 Proofs of the theorems

Notation 6. *Let $m, n \in \mathbb{N} - \{0\}$ and K be a field.*
We denote the $n \times n$ identity matrix over K by I_n^K (or by I_n when the field is clear from the context).
We denote $E_{i,j}^{K,n}$ the $n \times n$ matrix over K such that

$$(E_{i,j}^{K,n})_{x,y} = \begin{cases} 1 & \textit{if } (x,y) = (i,j) \\ 0 & \textit{otherwise} \end{cases}$$

We omit the superscript when it is clear from the context.
For any $A \in M(m \times n, K)$ we denote the submatrix of A given by the rows $i_1, , \dots, i_k$ and the columns $j_1, \dots, j_s$ by $A_{(i_1,\dots,i_k)}^{(j_1,\dots,j_s)}$.

Lemma 7. *Let $n \in \mathbb{N} - \{0\}$ and let $A \in S(n, \mathbb{R})$. Then there exists $s \in \mathbb{R}$ such that $\det(I_n + sA) = 0$ if and only if $A \neq 0$.*

Proof. $\Rightarrow$ This implication is obvious.
$\Leftarrow$ Suppose $A \neq 0$. Then A has a nonzero eigenvalue λ. Let $s = -\frac{1}{\lambda}$. Then

$$\det(I_n + sA) = s^n \det\left(\frac{1}{s} I_n + A\right) = s^n \det(A - \lambda I_n) = 0.$$

□

Lemma 8. *Let $r \in \mathbb{N} - \{0\}$. Let K be a field such that, if $x \in K^r - \{0\}$, then $x_1^2 + \dots + x_r^2 \neq 0$.*
Then, for any $A \in M(r \times r, K)$ and $x \in K^r - \{0\}$, we have that

$$\det\begin{pmatrix} I_r + sA & sx \\ s\,{}^t x & 0 \end{pmatrix}$$

is a nonzero polynomial in s.

Proof. The statement follows immediately from the fact that the coefficient of s^2 in $\det\begin{pmatrix} I_r + sA & sx \\ s\,{}^t x & 0 \end{pmatrix}$ is $-(x_1^2 + \dots + x_r^2)$. □

Lemma 9. *Let $r \in \mathbb{N} - \{0\}$. Let $A \in H(r)$ be positive-definite or negative-definite and $x \in \mathbb{C}^r - \{0\}$. Then the matrix*

$$\begin{pmatrix} A & x \\ {}^t\overline{x} & 0 \end{pmatrix}$$

is invertible.

Proof. If A is positive-definite, up to elementary row operations and the same elementary column operations on the first r rows and the first r columns, we can suppose that $A = I_r$. If $\begin{pmatrix} x \\ 0 \end{pmatrix}$ were linear combination of the first r columns of $\begin{pmatrix} I & x \\ {}^t\overline{x} & 0 \end{pmatrix}$, we would have that $0 = |x_1|^2 + \ldots + |x_r|^2$, which is absurd. Analogously if A is negative-definite. □

Remark 10. Let $a, b, n \in \mathbb{N}$ with $a + b \leq n$. If $b \geq a$, then $(n-b)b \geq (n-a)a$.

Proof. Observe that $(n-b)b \geq (n-a)a$ if and only if $b^2 - a^2 \leq n(b-a)$, which is equivalent to $b + a \leq n$ (since $b - a \geq 0$), which is true by assumption. □

Proof of Theorem 4. Let $R \in \mathcal{A}^{\mathbb{R}}_{sym}(n; r)$. We want to prove that $\dim(R) \leq \lfloor \frac{r}{2} \rfloor \left(n - \lfloor \frac{r}{2} \rfloor\right)$. We can write R as $M + L$ where $M \in S(n, \mathbb{R})$ and L is a linear subspace of $S(n, \mathbb{R})$. Let Q be an invertible matrix such that ${}^tQMQ$ is a diagonal matrix D whose diagonal is $(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0)$, where 1 is repeated p times for some p and -1 is repeated q times with $p + q = r$. Let $V = {}^tQLQ$ and $S = {}^tQRQ = D + V$. Obviously $S \in \mathcal{A}^{\mathbb{R}}_{sym}(n; r)$; moreover, $\dim(S) = \dim(R)$, so to prove that $\dim(R) \leq \lfloor \frac{r}{2} \rfloor \left(n - \lfloor \frac{r}{2} \rfloor\right)$ it is sufficient to prove that $\dim(S) \leq \lfloor \frac{r}{2} \rfloor \left(n - \lfloor \frac{r}{2} \rfloor\right)$.
Let Z be the vector subspace of $M(n \times n, \mathbb{R})$ generated by the matrices $E_{i,j} + E_{j,i}$ for $i, j \in \{1, \ldots, p\}$ with $i \leq j$.
Let U be the vector subspace of $M(n \times n, \mathbb{R})$ generated by the matrices $E_{i,j} + E_{j,i}$ for $i, j \in \{p+1, \ldots, r\}$ with $i \leq j$.
Let W be the vector subspace of $M(n \times n, \mathbb{R})$ generated by the matrices $E_{i,j} + E_{j,i}$ for $i, j \in \{r+1, \ldots, n\}$ with $i \leq j$.
Let G be the vector subspace of $M(n \times n, \mathbb{R})$ generated by the matrices $E_{i,j} + E_{j,i}$ for $i \in \{r+1, \ldots, n\}$, $j \in \{p+1, \ldots, r\}$.
We want to prove that

$$V \cap (Z + U + W + G) = \{0\}$$

Let $A \in Z, B \in U, C \in W$, $H \in G$ such that $A + B + C + H \in V$.

- If there existed $h \in \{r+1, \ldots, n\}$ such that $C_{h,h} = 0$ and $H_{(h)} \neq 0$, take $s \in \mathbb{R} - \{0\}$ such that $\det(I_p + sA^{(1,\ldots,p)}_{(1,\ldots,p)}) \neq 0$ and $-I_q + sB^{(p+1,\ldots,r)}_{(p+1,\ldots,r)}$ is negative-definite; then, by Lemma 9, the matrix $\begin{pmatrix} -I_q + sB^{(p+1,\ldots,r)}_{(p+1,\ldots,r)} & s\,{}^t(H^{(p+1,\ldots,r)}_{(h)}) \\ sH^{(p+1,\ldots,r)}_{(h)} & 0 \end{pmatrix}$ would be invertible, so $D + s(A + B + C + H)$ would have rank greater than r, so S would not be of constant rank r, which is contrary to our assumption.

- Suppose there exists $h \in \{r+1, \dots, n\}$ such that $C_{h,h} \neq 0$ and $H_{(h)} \neq 0$; then $\det \begin{pmatrix} -I_q + sB^{(p+1,\dots,r)}_{(p+1,\dots,r)} & s\,{}^t\!(H^{(p+1,\dots,r)}_{(h)}) \\ sH^{(p+1,\dots,r)}_{(h)} & s\,C_{h,h} \end{pmatrix}$ is a polynomial in s with term of degree 1 equal to $\pm C_{h,h}$, so a nonconstant polynomial. Hence, for s different from a finite number of real numbers, such a determinant is nonzero and then we can find s such that $\det \begin{pmatrix} -I_q + sB^{(p+1,\dots,r)}_{(p+1,\dots,r)} & s\,{}^t\!(H^{(p+1,\dots,r)}_{(h)}) \\ sH^{(p+1,\dots,r)}_{(h)} & s\,C_{h,h} \end{pmatrix} \neq 0$ and $\det(I_p + sA^{(1,\dots,p)}_{(1,\dots,p)}) \neq 0$. So $\mathrm{rk}(D + s(A+B+C+H))$ would be greater than r, so S would not be of constant rank r, which is contrary to our assumption.

  Hence we can conclude that $H = 0$.

- If C were nonzero, take $s \in \mathbb{R}$ such that $\det(I_p + sA^{(1,\dots,p)}_{(1,\dots,p)}) \neq 0$ and $\det(-I_q + sB^{(p+1,\dots,r)}_{(p+1,\dots,r)}) \neq 0$; then $D + s(A+B+C+H)$, that is $D + s(A+B+C)$, would have rank greater than r, so S would not be of constant rank r, which is contrary to our assumption. So C must be zero.

- If at least one of A and B were nonzero, take $s \in \mathbb{R}$ such that

  $$\det(I_p + sA^{(1,\dots,p)}_{(1,\dots,p)}) = 0$$

  or

  $$\det(-I_q + sB^{(p+1,\dots,r)}_{(p+1,\dots,r)}) = 0$$

  (there exists by Lemma 7); then $D+s(A+B+C+H)$, that is $D+s(A+B)$, has rank less than r, so S would not be of constant rank r, which is contrary to our assumption. So also A and B must be zero.

So we have proved that $V \cap (Z+U+W+G) = \{0\}$. Hence

$$\dim(S) = \dim(V) \leq \dim(S(n, \mathbb{R})) - \dim(Z+U+W+G) = p(n-p)$$

In an analogous way we can prove that

$$\dim(S) = \dim(V) \leq q(n-q).$$

So

$$\dim(S) \leq \min\{p(n-p), q(n-q)\}. \tag{1}$$

Observe that

$$\min\{p(n-p), q(n-q)\} \leq \left\lfloor \frac{r}{2} \right\rfloor \left(n - \left\lfloor \frac{r}{2} \right\rfloor\right), \tag{2}$$

in fact: suppose for instance that $p \leq q$, then, by Remark 10, we have that

$$\min\{p(n-p), q(n-q)\} = p(n-p); \tag{3}$$

moreover, observe that $p \leq q$ ad $p+q=r$ imply that $p \leq \lfloor \frac{r}{2} \rfloor$; by applying again Remark 10 with $(a,b) = \left(p, \lfloor \frac{r}{2} \rfloor\right)$, we get

$$p(n-p) \leq \left\lfloor \frac{r}{2} \right\rfloor \left(n - \left\lfloor \frac{r}{2} \right\rfloor\right); \tag{4}$$

from (3) and (4), we get (2). From (1) and (2), we obtain that

$$\dim(S) \leq \left\lfloor \frac{r}{2} \right\rfloor \left(n - \left\lfloor \frac{r}{2} \right\rfloor \right).$$

□

Observe that in an analogous way we can prove the following two theorems:

Theorem 11. *Let $p, q, n \in \mathbb{N}$ such that $p + q \leq n$; then*

$$a^{\mathbb{R}}_{sym}(n; p, q) \leq \min\{p, q\}(n - \min\{p, q\}).$$

Sketch of the proof. Consider $S \in \mathcal{A}^{\mathbb{R}}_{sym}(n; p \times q)$. We can suppose $S = D + V$ where D is the diagonal matrix whose diagonal is $(1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$ where 1 is repeated p times and -1 is repeated q times and V is a linear subspace of $S(n, \mathbb{R})$ and then argue as in the proof of Theorem 4. □

Theorem 12. *(i) Let $n, r \in \mathbb{N}$ with $r \leq n$. Then*

$$a_{herm}(n; r) \leq 2 \left\lfloor \frac{r}{2} \right\rfloor (n - \left\lfloor \frac{r}{2} \right\rfloor).$$

(ii) Let $n, p, q \in \mathbb{N}$ with $p + q \leq n$. Then

$$a_{herm}(n; p, q) \leq 2 \min\{p, q\}(n - \min\{p, q\}).$$

Sketch of the proof. (i) Let $R \in \mathcal{A}_{herm}(n; r)$. We can write R as $M + L$ where $M \in H(n)$ and L is a linear subspace of $H(n)$. There exists a unitary matrix U and a diagonal real matrix P such that $P\,{}^t\overline{U}MUP$ is the diagonal matrix D whose diagonal is $(1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$, where 1 is repeated p times for some p and -1 is repeated q times with $p + q = r$. Consider $S = P\,{}^t\overline{U}RUP$; it is equal to $D + V$, where V is a vector subspace of $H(n)$.
Let Z be the vector subspace of $H(n)$ generated by the matrices $E_{l,j} + E_{j,l}$ and $iE_{l,j} - iE_{j,l}$ for $l, j \in \{1, \dots, p\}$ with $l < j$ and by the matrices $E_{l,l}$ for $l \in \{1, \dots, p\}$.
Let U be the vector subspace of $H(n)$ generated by the matrices $E_{l,j} + E_{j,l}$ and $iE_{l,j} - iE_{j,l}$ for $l, j \in \{p + 1, \dots, r\}$ with $l < j$ and by the matrices $E_{l,l}$ for $l \in \{p + 1, \dots, r\}$.
Let W be the vector subspace of $H(n)$ generated by the matrices $E_{l,j} + E_{j,l}$ and $iE_{l,j} - iE_{j,l}$ for $l, j \in \{r + 1, \dots, n\}$ with $l < j$ and by the matrices $E_{l,l}$ for $l \in \{r + 1, \dots, n\}$.
Let G be the vector subspace of $H(n)$ generated by the matrices $E_{l,j} + E_{j,l}$ and $iE_{l,j} - iE_{j,l}$ for $l \in \{r + 1, \dots, n\}$, $j \in \{p + 1, \dots, r\}$.
As in the proof of Theorem 4, we can prove that $V \cap (Z + U + W + G) = \{0\}$. Hence

$$\dim(R) = \dim(S) = \dim(V) \leq \dim(H(n)) - \dim(Z + U + W + G) = 2p(n - p).$$

In an analogous way we can prove that $\dim(R) \leq 2q(n - q)$.
As in the proof of Theorem 4 we can deduce that $\dim(R) \leq 2 \left\lfloor \frac{r}{2} \right\rfloor (n - \left\lfloor \frac{r}{2} \right\rfloor)$.
The proof of (ii) is analogous. □

Proof of Theorem 5. In order to prove that $a^{\mathbb{R}}(m \times n, r)$ is greater than or equal to $r(n-r) + \frac{r(r-1)}{2}$, i.e. greater than or equal to $rn - \frac{r(r+1)}{2}$, consider the following affine subspace of $M(m \times n, \mathbb{R})$:

$$S = \{A \in M(m \times n, \mathbb{R}) \mid A_{i,i} = 1 \ \forall i = 1, \dots, r, \ A_{i,j} = 0 \ \forall (i,j) \text{ with } i > j \text{ or } i > r\}.$$

The dimension of S is clearly $r(n-r) + \frac{r(r-1)}{2}$ and $S \in \mathcal{A}^{\mathbb{R}}(m \times n; r)$, so we get our inequality.
Now let us prove the other inequality.
Let $C \in \mathcal{A}^{\mathbb{R}}(m \times n; r)$. We want to prove that $\dim(C) \leq r(n-r) + \frac{r(r-1)}{2}$. We can write C as $A + W$ where $A \in M(m \times n, \mathbb{R})$ and W is a linear subspace of $M(m \times n, \mathbb{R})$. Let Q and R be invertible matrices such that, if we denote $Q^{-1}AR$ by J, we have that $J_{i,i} = 1$ for $i = 1, \dots, r$ and the other entries of J are equal to zero.
Let $V = Q^{-1}WR$ and $S = Q^{-1}CR = J+V$. Obviously $S \in \mathcal{A}^{\mathbb{R}}(m \times n; r)$; moreover, $\dim(S) = \dim(C)$, so to prove that $\dim(C) \leq r(n-r) + \frac{r(r-1)}{2}$ it is sufficient to prove that $\dim(S) \leq r(n-r) + \frac{r(r-1)}{2}$.
Consider now the following subspaces of $M(m \times n, \mathbb{R})$:

$$Z = \{A \in M(m \times n, \mathbb{R}) \mid A_{i,j} = 0 \ \forall (i,j) \text{ such that } i \neq j \text{ and } (i \leq r \text{ or } j \leq r)\}$$

$$T = \left\{ A \in M(m \times n, \mathbb{R}) \mid \begin{array}{l} A_{i,j} = 0 \ \forall (i,j) \text{ such that } i = j \text{ or } (i > r \text{ and } j > r) \text{ or } j > m; \\ A_{i,j} = A_{j,i} \ \forall (i,j) \text{ such that } j \leq m \end{array} \right\}.$$

We want to prove that

$$V \cap (Z + T) = \{0\}.$$

Let $\zeta \in Z$ and $\tau \in T$ such that $\zeta + \tau \in V$; we want to show that $\zeta = \tau = 0$.
We denote $\zeta_{(r+1,\dots,m)}^{(r+1,\dots,n)}$ by ζ' and $\tau_{(r+1,\dots,m)}^{(1,\dots,r)}$ by τ'.
We consider four cases:
<u>Case 1: $\tau' = 0$, $\zeta' = 0$.</u>
Since $\tau + \zeta \in V$ we have that $J + s(\tau + \zeta)$ must have rank r for every $s \in \mathbb{R}$; observe that $(\tau + \zeta)_{i,j} = 0 \ \forall (i,j)$ with $i > r$ or $j > r$ (by the definition of Z and T and the fact that $\tau' = 0$ and $\zeta' = 0$) and that $(\tau + \zeta)_{(1,\dots,r)}^{(1,\dots,r)}$ is symmetric; hence, by Lemma 7, we can conclude that $(\tau + \zeta)_{(1,\dots,r)}^{(1,\dots,r)} = 0$ and then that $\tau + \zeta = 0$.
<u>Case 2: $\tau' = 0$, $\zeta' \neq 0$.</u>
Take $s \in \mathbb{R} - \{0\}$ such that $\det\left(I_r + s(\tau + \zeta)_{(1,\dots,r)}^{(1,\dots,r)}\right)$ is nonzero and $h \in \{r+1, \dots, m\}$ and $l \in \{r+1, \dots, n\}$ such that $\zeta_{h,l} \neq 0$. Then, obviously, the matrix $(J + s(\tau + \zeta))_{(1,\dots,r,h)}^{(1,\dots,r,l)}$ is invertible, , which is impossible since $J + s(\tau + \zeta) \in S \in \mathcal{A}^{\mathbb{R}}(m \times n; r)$.
<u>Case 3: there exists $h \in \{r+1, \dots, m\}$ such that $\tau^{(h)} \neq 0$ and $\zeta_{(h,h)} = 0$.</u>
By Lemma 8, we have that $\det\left((J + s(\tau + \zeta))_{(1,\dots,r,h)}^{(1,\dots,r,h)}\right)$ is a nonzero polynomial in s, so we can find s such that $\det\left((J + s(\tau + \zeta))_{(1,\dots,r,h)}^{(1,\dots,r,h)}\right) \neq 0$, which is absurd since $J + s(\tau + \zeta) \in S \in \mathcal{A}^{\mathbb{R}}(m \times n; r)$.
<u>Case 4: there exists $h \in \{r+1, \dots, m\}$ such that $\tau^{(h)} \neq 0$ and $\zeta_{(h,h)} \neq 0$.</u>

Observe that $\det\left((J+s(\tau+\zeta)^{(1,\dots,r,h)}_{(1,\dots,r,h)}\right)$ is a polynomial in s with the term of degree 0 equal to 0 and the coefficient of the term of degree 1 equal to $\zeta_{h,h}$, which is nonzero; then there exists $s\in\mathbb{R}-\{0\}$ such that $\det\left((J+s(\tau+\zeta))^{(1,\dots,r,h)}_{(1,\dots,r,h)}\right)$ is nonzero; but this is impossible since $J+s(\tau+\zeta)\in S\in\mathcal{A}^{\mathbb{R}}(m\times n;r)$.
Observe that the four cases we have considered are the only possible ones because when τ' is nonzero we have one among Case 3 and Case 4. Thus we have proved that $V\cap(Z+T)=\{0\}$. Hence we have:

$$\begin{aligned}\dim(S) &= \dim(V)\le\dim M(m\times n,\mathbb{R})-\dim(Z+T)=\\ &= mn-\dim(Z)-\dim(T)=r(n-r)+\tfrac{r(r-1)}{2}\end{aligned}$$

and we can conclude. □

Remark 13. Let $F[x_1,\dots,x_k]$ denote the set of the polynomials in the indeterminates $x_1,\dots,x_k$ with coefficients on a field F. A matrix over $F[x_1,\dots,x_k]$ is said an Affine Column Indipendent matrix, or ACI-matrix, if its entries are polynomials of degree at most one and no indeterminate appears in two different columns. A completion of an ACI-matrix is an assignment of values in F to the indeterminates $x_1,\dots,x_k$; for instance, let us consider the matrix over $\mathbb{R}[x_1,\dots,x_5]$

$$A=\begin{pmatrix} x_1 & x_3 & x_4+x_5\\ 2x_1+x_2 & -x_3-1 & x_4-x_5\\ x_2+1 & 0 & 2x_4\end{pmatrix};$$

it is an ACI matrix; if we assign the values $1,1,2,5,7$ respectively to $x_1,\dots,x_5$, we get the completion of A

$$\begin{pmatrix}1&2&12\\3&-3&-2\\2&0&10\end{pmatrix}.$$

In [4] Huang and Zhan proved that all the completions of an $m\times n$ ACI-matrix A over a field F with $|F|\ge\max\{m,n+1\}$ have rank r if and only if there exists a nonsingular constant $m\times m$ matrix T and a permutation $n\times n$ matrix Q such that TAQ is equal to a matrix of the kind

$$\begin{pmatrix}B&*&*\\0&0&*\\0&0&C\end{pmatrix}$$

for some ACI-matrices B and C which are square upper triangular with nonzero constant diagonal entries and whose orders sum to r. Observe that the affine subspace given by the matrices $\begin{pmatrix}B&*&*\\0&0&*\\0&0&C\end{pmatrix}$ with B and C square upper triangular ACI-matrices with nonzero constant diagonal entries and whose orders are respectively k and $r-k$ is equal to the following number:

$$\frac{k(k-1)}{2}+\frac{(r-k)(r-k-1)}{2}+k(n-k)+(r-k)(m-k-r+k)=$$

$$= -\frac{r^2}{2} - \frac{r}{2} + k(n-m) + rm,$$

which obviously attains the maximum, i.e. $rn - \frac{r^2+r}{2}$, when $k = r$. Let M be a matrix over $F[x_1, \ldots, x_k]$ with the degree of every entry at most one. Define $\tilde{M}$ to be the ACI-matrix obtained from M in the following way: if an indeterminate x_i appears in more than one column, say in the columns $j_1, \ldots, j_s$, replace it with new indeterminates $x_i^{j_1}, \ldots, x_i^{j_s}$, precisely replace x_i in the j_l-th column with $x_i^{j_l}$ for $l = 1, \ldots s$.

Observe that an affine subspace of $M(m \times n, F)$ corresponds to an $m \times n$ matrix over $F[x_1, \ldots, x_l]$ for some $l \in \mathbb{N}$ with the degree of every entry at most one and so an $m \times n$ ACI matrix such that all the completions have rank r corresponds to an affine subspace of $M(m \times n, F)$ of constant rank r.

One might think that it is possible to deduce Theorem 5, in particular the inequality $a^{\mathbb{R}}(m \times n, r) \leq rn - \frac{r(r+1)}{2}$, from Huang-Zhan‘s result in the following way: let $S \in \mathcal{A}^{\mathbb{R}}(m \times n; r)$ and consider the affine subspace $\tilde{S}$ given by the “ACImade” matrices of S, that is, given by the matrices $\tilde{M}$ for $M \in S$; if $\tilde{S}$ were of constant rank, then it would correspond to an ACI matrix such that all the completions have rank r; so by Huang-Zhan‘s result we would have $\dim(\tilde{S}) \leq rn - \frac{r(r+1)}{2}$ and then $\dim(S) \leq rn - \frac{r(r+1)}{2}$, but it is not true that the affine subspace $\tilde{S}$ is of constant rank for any affine subspace S of constant rank, as the following example shows: let

$$S = \left\{ \begin{pmatrix} 1 & s \\ s & -1 \end{pmatrix} \mid s \in \mathbb{R} \right\};$$

We have that $\tilde{S} = \left\{ \begin{pmatrix} 1 & s \\ t & -1 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$, which is not of constant rank.

Remark 14. Observe that Theorem 5 does not hold on every field K, as the following example shows: consider the field $\mathbb{Z}/2$, $m = n = 2$ and $r = 1$; let

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle;$$

the affine subspace S is obviously of dimension 2 and constant rank 1, so for $m = n = 2$, $r = 1$, we have that $a^{\mathbb{Z}/2}(m \times n, r)$ is different from $rn - \frac{r(r+1)}{2} = 1$. Anyway, the main problem to extend Theorem 5 to other fields seems Lemma 7, which we use in Case 1 of the proof. Precisely, observe that the argument to prove the inequality $a^K(m \times n, r) \geq rn - \frac{r(r+1)}{2}$ works on any field K; as to the other inequality, it is easy to see that the argument in Cases 2,3,4 works for any field with cardinality greater than $r + 2$ (in fact, such condition guarantees for any nonzero polynomial p over K in one variable of degree less than or equal to $r + 1$ the existence of an element $s \in K - \{0\}$ such that $p(s) \neq 0$, in particular there exists $s \in K - \{0\}$ such that $\det \left(I_r + s(\tau + \zeta)^{(1,\ldots,r)}_{(1,\ldots,r)} \right) \neq 0$ in Case 2 and there exists $s \in K - \{0\}$ such that $\det \left((J + s(\tau + \zeta))^{(1,\ldots,r,h)}_{(1,\ldots,r,h)} \right) \neq 0$ in Cases 3 and 4); so the main problem to extend the theorem seems in Case 1, because we use Lemma 7.

Also for Theorem 4, the main obstacle to extend the statement to other fields seems the necessity to extend Lemma 7.

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