Interval matrices: realization of ranks by rational matrices

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Abstract

Let $\boldsymbol{\alpha}$ be a $p \times q$ interval matrix with $p \geq q$ and with the endpoints of all its entries in $\mathbb{Q}$. We prove that, if $\boldsymbol{\alpha}$ contains a rank-r real matrix with $r \in \{2, q-2, q-1, q\}$, then it contains a rank-r rational matrix.

1 Introduction

Let $p, q \in \mathbb{N} \setminus \{0\}$; a $p \times q$ interval matrix $\boldsymbol{\alpha}$ is a $p \times q$ matrix whose entries are intervals in $\mathbb{R}$; we usually denote the entry i, j, $\boldsymbol{\alpha}_{i,j}$, by $[\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}]$ with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ and we point out that we denote every interval matrix in bold. A $p \times q$ matrix A with entries in $\mathbb{R}$ is said contained in a $p \times q$ interval matrix $\boldsymbol{\alpha}$ if $a_{i,j} \in \boldsymbol{\alpha}_{i,j}$ for any i, j. There is a wide literature about interval matrices and the rank of the matrices they contain. In this paper we consider the following problem: let $\boldsymbol{\alpha}$ be an interval matrix whose entries have rational endpoints; for which r can we deduce that, if $\boldsymbol{\alpha}$ contains a rank-r real matrix, then $\boldsymbol{\alpha}$ contains a rank-r rational matrix?

Before sketching our results, we illustrate shortly some of the literature on interval matrices, and the rank of the contained matrices, on partial matrices and on the matrices with a given sign pattern; these last two research fields are connected with the theory of interval matrices.

Two of the most famous theorems on interval matrices are Rohn's theorems on full-rank interval matrices. We say that a $p \times q$ interval matrix $\boldsymbol{\alpha}$ has full rank if and only if all the matrices contained in $\boldsymbol{\alpha}$ have rank equal to $\min\{p, q\}$. For any $p \times q$ interval matrix $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$, let $\mathrm{mid}(\boldsymbol{\alpha})$, $\mathrm{rad}(\boldsymbol{\alpha})$ and $|\boldsymbol{\alpha}|$ be respectively the midpoint, the radius and the modulus of $\boldsymbol{\alpha}$, that is the $p \times q$ matrices such that

$$\mathrm{mid}(\boldsymbol{\alpha})_{i,j} = \frac{\underline{\alpha}_{i,j} + \overline{\alpha}_{i,j}}{2}, \qquad \mathrm{rad}(\boldsymbol{\alpha})_{i,j} = \frac{\overline{\alpha}_{i,j} - \underline{\alpha}_{i,j}}{2},$$

$$|\boldsymbol{\alpha}|_{i,j} = \max\{|\underline{\alpha}_{i,j}|, |\overline{\alpha}_{i,j}|\}$$

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for any i, j. The following theorems characterize respectively full-rank *square* interval matrices and full-rank $p \times q$ interval matrices, see [14], [16], [17], [21]; see [14] and [15] for other characterizations.

Theorem 1. ***(Rohn)*** *Let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times p$ interval matrix, where $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ for any i, j. Let $Y_p = \{-1, 1\}^p$ and, for any $x \in Y_p$, denote by T_x the diagonal matrix whose diagonal is x. Then $\boldsymbol{\alpha}$ is a full-rank interval matrix if and only if, for each $x, y \in Y_p$,*

$$\det\Big(\mathrm{mid}(\boldsymbol{\alpha})\Big) \det\Big(\mathrm{mid}(\boldsymbol{\alpha}) - T_x\, \mathrm{rad}(\boldsymbol{\alpha})\, T_y\Big) > 0.$$

Theorem 2. ***(Rohn)*** *A $p \times q$ interval matrix $\boldsymbol{\alpha}$ with $p \geq q$ has full rank if and only if the system of inequalities*

$$|\mathrm{mid}(\boldsymbol{\alpha})\, x| \leq \mathrm{rad}(\boldsymbol{\alpha})\, |x|, \qquad x \in \mathbb{R}^q$$

has only the trivial solution $x = 0$.

A research area which can be connected with the theory of interval matrices is the one of the *partial matrices*: let K be a field; a partial matrix over K is a matrix where only some of the entries are given and they are elements of K; a completion of a partial matrix is a specification of the unspecified entries. In [5], Cohen, Johnson, Rodman and Woerdeman determined the maximal rank of the completions of a partial matrix in terms of the ranks and the sizes of its maximal specified submatrices; see also [4] for the proof. The problem of a theoretical characterization of the minimal rank of the completions of a partial matrix seems more difficult and it has been solved only in some particular cases. We quote also the papers [13] and [23] about the NP-hardness of the problem and the paper [7] for rank-1 completions.
In [19] we generalized Theorem 1 to matrices whose entries are closed connected nonempty subsets of $\mathbb{R}$, i.e. the so-called matrices in Kahan arithmetic.
In [18] we determined the maximum rank of the matrices contained in a given interval matrix and we gave a theoretical characterization of interval matrices containing at least a matrix of rank 1. In the previous paper [6], the authors studied the complexity of an algorithm to decide if an interval matrix contains a rank-one matrix and proved that the problem is NP-complete.
Finally we quote another research area which can be related to partial matrices, to interval matrices and, more generally, to general interval matrices: the one of the *matrices with a given sign pattern*; let Q be a $p \times q$ matrix with entries in $\{+, -, 0\}$; we say that $A \in M(p \times q, \mathbb{R})$ has sign pattern Q if, for any i, j, we have that $a_{i,j}$ is positive (respectively negative, zero) if and only if $Q_{i,j}$ is $+$ (respectively $-, 0$). Obviously the set of the matrices with a given sign pattern can be thought as a matrix whose entries are in $\{(0, +\infty), (-\infty, 0), [0]\}$. There are several papers studying the minimal and maximal rank of the matrices with a given sign pattern, see for instance [1], [2], [9], [22]. In particular, in [1] and [2] the authors proved that the minimum rank of the real matrices with a given sign pattern is realizable by a rational matrix in case this minimum is at most 2 or at least $\min\{p, q\} - 2$.

Obviously the three theories we have quoted, that is the theory of interval matrices, the theory of partial matrices, and the theory of matrices with a given sign pattern can be seen as parts of the same theory: the one of *subset matrices*, i.e. matrices whose entries are subsets of a given field; we denote also subset matrices in bold.
In [20] we proved the following theorems:

Theorem 3. *Let $p \geq q$ and let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ and $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ for any i, j. If there exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) < q$, then there exists $B \in \boldsymbol{\alpha} \cap M(p \times q, \mathbb{Q})$ with $\mathrm{rk}(B) < q$.*

Theorem 4. *Let $p \geq q$ and let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ and $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ for any i, j. If there exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = 1$, then there exists $B \in \boldsymbol{\alpha} \cap M(p \times q, \mathbb{Q})$ with $\mathrm{rk}(B) = 1$.*

Moreover, in [20] we observed (see in Remark 13 there) that from the papers [3], [22] and [8] we can deduce that it is not true that, for any r, if a $p \times q$ interval matrix with the endpoints of all its entries in $\mathbb{Q}$ contains a rank-r real matrix, then it contains a rank-r rational matrix. In particular this is not true for $r = 3, \min\{p, q\} - 3$.
In this paper we prove that, if a $p \times q$ interval matrix with $p \geq q$ and with the endpoints of all its entries in $\mathbb{Q}$ contains a rank-r real matrix, then it contains a rank-r rational matrix for $r = 2, q - 2, q - 1, q$, see Theorem 8, Theorem 13 and Remark 9. Summarizing we get the following result; observe that the behaviour of interval matrices is similar to the one of the matrices with a given sign pattern showed in [1] and [2], even if, to prove it, we have to use a technique which is different from the one in [1] and [2].

Theorem 5. *Let $p \geq q$ and let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ and $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ for any i, j. If $r \in \{0, 1, 2, q - 2, q - 1, q\}$ and there exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = r$, then there exists $B \in \boldsymbol{\alpha} \cap M(p \times q, \mathbb{Q})$ with $\mathrm{rk}(B) = r$.*

2 Notation and first remarks

• Let $\mathbb{R}_{>0}$ be the set $\{x \in \mathbb{R} | \; x > 0\}$ and let $\mathbb{R}_{\geq 0}$ be the set $\{x \in \mathbb{R} | \; x \geq 0\}$; we define analogously $\mathbb{R}_{<0}$ and $\mathbb{R}_{\leq 0}$. We denote by $\mathbb{I}$ the set $\mathbb{R} - \mathbb{Q}$.
• Throughout the paper let $p, q \in \mathbb{N} \setminus \{0\}$.
• For any set X, let $|X|$ be the cardinality of X.
• For any field K, let $M(p \times q, K)$ denote the set of the $p \times q$ matrices with entries in K. For any $A \in M(p \times q, K)$, let $\mathrm{rk}(A)$ be the rank of A, let $A^{(j)}$ be the j-th column of A and, more generally, let $A^{(j_1,\ldots,j_r)}_{(i_1,\ldots,i_s)}$ be the submatrix of A given by the columns $j_1, \ldots, j_r$ and the rows $i_1, \ldots, i_s$ of A with the orders, respectively, $j_1, \ldots, j_r$ and $i_1, \ldots, i_s$.
• For any vector space V over a field K and any $v_1, \ldots, v_k \in V$, let $\langle v_1, \ldots, v_k \rangle$ denote the span of $v_1, \ldots, v_k$.
• Let $\boldsymbol{\alpha}$ be a $p \times q$ subset matrix over a field K. Given a matrix $A \in M(p \times q, K)$, we say that $A \in \boldsymbol{\alpha}$ if and only if $a_{i,j} \in \boldsymbol{\alpha}_{i,j}$ for any i, j.
We say that an entry of $\boldsymbol{\alpha}$ is **degenerate** if its cardinality is 1.

• Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ be two $p \times q$ interval matrices. We say that

$$\boldsymbol{\alpha}' \subset \boldsymbol{\alpha}$$

if $\boldsymbol{\alpha}'_{i,j} \subset \boldsymbol{\alpha}_{i,j}$ for every i, j.
We defer to some classical books on interval analysis, such as [10], [12] and [11] for the definition of sum and multiplication of two intervals. In particular, for any interval α in $\mathbb{R}$ and any interval β either in $\mathbb{R}_{>0}$ or in $\mathbb{R}_{<0}$, we define $\frac{\alpha}{\beta}$ to be the set $\left\{\frac{a}{b} \middle| \, a \in \alpha, \; b \in \beta\right\}$.

3 Rational realization of the rank 2

Lemma 6. *Let K be a field and let $k, n \in \mathbb{N} \setminus \{0\}$. Let $A \in M(k \times n, K)$ with $n > k$. If $A^{(1,\dots,k)}$ is invertible, then a basis of the kernel of A is given by the following vectors in K^n for $j = k+1, \dots, n$:*

$$v_j := \begin{pmatrix} \det(A^{(2,\dots,k,j)}) \\ -\det(A^{(1,3,\dots,k,j)}) \\ \vdots \\ (-1)^k \det(A^{(1,\dots,k-1,j)}) \\ 0 \\ \vdots \\ 0 \\ (-1)^{k+1} \det(A^{(1,\dots,k)}) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $(-1)^{k+1} \det(A^{(1,\dots,k)})$ is the j-th entry.

Proof. The vectors $v_{k+1}, \dots, v_n$ are obviously linearly independent, they are $n-k$ and we can easily see that they are in the kernel of A, so we conclude. □

Corollary 7. *(1) Let K be a field and let $k, n \in \mathbb{N} \setminus \{0\}$. Let $A \in M(k \times n, K)$ with $n > k$ and $\mathrm{rk}(A) = k$. For any $j_1, \dots, j_{k+1}$ in $\{1, \dots, n\}$ with $j_1 < \dots < j_{k+1}$, let $v_{j_1,\dots,j_{k+1}}$ be the vector such that*

- *the i-th entry is equal to 0 for every $i \neq j_1, \dots, j_{k+1}$,*
- *the j_l-entry, for $l = 1, \dots, k+1$, is equal to*

$$(-1)^l \det(A^{(j_1,\dots,\hat{j}_l,\dots,j_{k+1})}).$$

Then the kernel of A is generated by the vectors $v_{j_1,\dots,j_{k+1}}$ for $j_1, \dots, j_{k+1}$ elements of $\{1, \dots, n\}$ with $j_1 < \dots < j_{k+1}$.
(2) Let K be a field and let $m, n, k \in \mathbb{N} \setminus \{0\}$ with $n > k$. Let $A \in M(m \times n, K)$ with $n > \mathrm{rk}(A) \geq k$. For any $i_1, \dots, i_s$ in $\{1, \dots, m\}$ with $i_1 < \dots < i_s$, for any $j_1, \dots, j_{s+1}$ in $\{1, \dots, n\}$ with $j_1 < \dots < j_{s+1}$, let $v^{i_1,\dots,i_s}_{j_1,\dots,j_{s+1}}$ be the vector such that

- *the i-th entry is equal to* 0 *for every* $i \neq j_1, \dots, j_{s+1}$,
- *the* j_l*-entry, for* $l = 1, \dots, s+1$, *is equal to*

$$(-1)^l \det(A^{(j_1,\dots,\hat{j}_l,\dots,j_{s+1})}_{(i_1,\dots,i_s)}).$$

Then the kernel of A *is generated by the vectors* $v^{i_1,\dots,i_s}_{j_1,\dots,j_{s+1}}$ *for* $s \in \{k, \dots, \min\{m, n-1\}\}$, $i_1, \dots, i_s$ *in* $\{1, \dots, m\}$ *with* $i_1 < \dots < i_s$, $j_1, \dots, j_{s+1}$ *in* $\{1, \dots, n\}$ *with* $j_1 < \dots < j_{s+1}$.

Theorem 8. *Let* $p \geq q$ *and let* $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ *be a* $p \times q$ *interval matrix with* $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ *and* $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ *for any* i, j. *If there exists* $R \in \boldsymbol{\alpha}$ *with* $\mathrm{rk}(R) = 2$, *then there exists* $Q \in \boldsymbol{\alpha} \cap M(p \times q, \mathbb{Q})$ *with* $\mathrm{rk}(Q) = 2$.

Proof. Let $a_i, b_i, c_j, d_j \in \mathbb{R}$ for $i = 1, \dots, p$ and $j = 1, \dots, q$ such that

$$r_{i,j} = a_i c_j + b_i d_j$$

for any i, j. Observe that we can easily suppose that, for any $j \in \{1, \dots, q\}$, at least one of c_j and d_j is nonzero (call this assumption "assumption $(*)$").
For any $i \in \{1, \dots, p\}$ such that $\boldsymbol{\alpha}_{i,j}$ is nondegenerate for at least one $j \in \{1, \dots, q\}$ and for any $j \in \{1, \dots, q\}$ such that $\boldsymbol{\alpha}_{i,j}$ is nondegenerate for at least one $i \in \{1, \dots, p\}$, let A_i, B_i, C_j, D_j be open neighbours respectively of a_i, b_i, c_j, d_j such that

$$A_i C_j + B_i D_j \subset \boldsymbol{\alpha}_{i,j}$$

for any $(i, j) \in \{1, \dots, p\} \times \{1, \dots, q\}$ such that $\boldsymbol{\alpha}_{i,j}$ is nondegenerate.
Define

$$T = \{(i, j) \in \{1, \dots, p\} \times \{1, \dots, q\} |\ \boldsymbol{\alpha}_{i,j} \text{ is degenerate}\},$$
$$T_1 = \{i \in \{1, \dots, p\} |\ \exists\, j \in \{1, \dots, q\} \text{ s.t. } (i, j) \in T\},$$
$$T_2 = \{j \in \{1, \dots, q\} |\ \exists\, i \in \{1, \dots, p\} \text{ s.t. } (i, j) \in T\},$$
$$t_1 = |T_1|, \qquad\qquad t_2 = |T_2|;$$

for any $i \in T_1$, let

$$T(i, \cdot) = \{j \in \{1, \dots, q\} | (i, j) \in T\}$$

and, for any $j \in T_2$, let

$$T(\cdot, j) = \{i \in \{1, \dots, p\} | (i, j) \in T\}.$$

We can easily suppose that T is nonempty, hence $t_1 > 0$ and $t_2 > 0$, and that

$$T_1 = \{1, \dots, t_1\}, \qquad\qquad T_2 = \{1, \dots, t_2\}.$$

Obviously, for any $(i, j) \in T$

$$a_i c_j + b_i d_j = \boldsymbol{\alpha}_{i,j}, \tag{1}$$

where here $\boldsymbol{\alpha}_{i,j}$ denotes one of the two (equal) endpoints of $\boldsymbol{\alpha}_{i,j}$. So, if (i, j) and (i, h) are in T, we have:

$$c_h(\boldsymbol{\alpha}_{i,j} - b_i\, d_j) = c_j(\boldsymbol{\alpha}_{i,h} - b_i\, d_h),$$

thus

$$b_i(c_j\,d_h - c_h\,d_j) = c_j\,\boldsymbol{\alpha}_{i,h} - c_h\,\boldsymbol{\alpha}_{i,j}. \tag{2}$$

By (1), equation (2) holds also if $(i,h) \in T$ and $c_h = 0$, for any $j = 1,\ldots,q$.
From (1), we can deduce also that, if (i,j) and (i,h) are in T, then

$$a_i(c_h\,d_j - c_j\,d_h) = d_j\,\boldsymbol{\alpha}_{i,h} - d_h\,\boldsymbol{\alpha}_{i,j}. \tag{3}$$

Moreover, by (2), if (i,j), (i,h) and (i,k) are in T, then

$$(c_jd_k - c_kd_j)(c_j\,\boldsymbol{\alpha}_{i,h} - c_h\,\boldsymbol{\alpha}_{i,j}) = (c_jd_h - c_hd_j)(c_j\,\boldsymbol{\alpha}_{i,k} - c_k\,\boldsymbol{\alpha}_{i,j}),$$

that is

$$\begin{aligned} &c_j(c_j\,\boldsymbol{\alpha}_{i,h} - c_h\,\boldsymbol{\alpha}_{i,j})d_k + \\ &-c_j(c_j\,\boldsymbol{\alpha}_{i,k} - c_k\,\boldsymbol{\alpha}_{i,j})d_h + \\ &+[c_h(c_j\,\boldsymbol{\alpha}_{i,k} - c_k\,\boldsymbol{\alpha}_{i,j}) - c_k(c_j\,\boldsymbol{\alpha}_{i,h} - c_h\,\boldsymbol{\alpha}_{i,j})]d_j = 0 \end{aligned} \tag{4}$$

Let us consider the homogeneous linear system (S) in the unknowns δ_j for $j = 1,\ldots,t_2$ given by the equations

$$\begin{aligned} &\gamma_j(\gamma_j\,\boldsymbol{\alpha}_{i,h} - \gamma_h\,\boldsymbol{\alpha}_{i,j})\delta_k + \\ &-\gamma_j(\gamma_j\,\boldsymbol{\alpha}_{i,k} - \gamma_k\,\boldsymbol{\alpha}_{i,j})\delta_h + \\ &+[\gamma_h(\gamma_j\,\boldsymbol{\alpha}_{i,k} - \gamma_k\,\boldsymbol{\alpha}_{i,j}) - \gamma_k(\gamma_j\,\boldsymbol{\alpha}_{i,h} - \gamma_h\,\boldsymbol{\alpha}_{i,j})]\delta_j = 0 \end{aligned} \tag{5}$$

for any i,j,k,h such that (i,j), (i,h) and (i,k) are in T and the equations

$$\boldsymbol{\alpha}_{i,h}\delta_k - \boldsymbol{\alpha}_{i,k}\delta_h = 0 \tag{6}$$

for any i,h,k such that (i,h) and (i,k) are in T and $c_h = c_k = 0$ (observe that the first equations are obtained from (4) by replacing c_i with γ_i and d_i with the unknown δ_i).
Let us denote by $G_{(\boldsymbol{\alpha}_{r,s})_{(r,s)\in T},(\gamma_j)_{j\in T_2}}$, or by $G_{(\boldsymbol{\alpha}_{r,s}),(\gamma_j)}$ for short, the associated matrix, which has obviously t_2 columns.
If $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) \geq 1$, let $\overline{C} = \overline{C}_1 \times \ldots \times \overline{C}_{t_2}$ be a neighbourhood of $(c_1,\ldots,c_{t_2})$ contained in $C_1 \times \ldots \times C_{t_2}$ such that $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(\gamma_j)}) \geq 1$ for every $(\gamma_j)_j \in \overline{C}$.
By Corollary 7, if $1 \leq \mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(\gamma_j)}) \leq t_2 - 1$, we can see the kernel of $G_{(\boldsymbol{\alpha}_{r,s}),(\gamma_j)}$ for $(\gamma_j)_j \in \overline{C}$ as generated by some vectors

$$v_f\left((\gamma_j)_{j\in T_2}\right) \quad \text{for } f = 1,\ldots,g$$

(for some g) whose entries are polynomials in $\boldsymbol{\alpha}_{r,s}$ for $(r,s) \in T$ (which are fixed) and γ_j for $j \in T_2$. By (4) and (3), $(d_1,\ldots,d_{t_2})$ satisfies both the equations (5) and the equations (6) with c_j instead of γ_j; moreover also $(c_1,\ldots,c_{t_2})$ satisfies both the equations (5) and the equations (6) with c_j instead of γ_j; hence $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) \leq t_2 - 1$, because at least one of $(c_1,\ldots,c_{t_2})$ and $(d_1,\ldots,d_{t_2})$ must be nonzero by assumption $(*)$; moreover, since $(d_1,\ldots,d_{t_2})$ satisfies both the equations (5) and the equations (6) with c_j instead of γ_j, we have that, if $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) \geq 1$, then

$$\begin{pmatrix} d_1 \\ \vdots \\ d_{t_2} \end{pmatrix} = \lambda_1 v_1\left((c_j)_{j\in T_2}\right) + \ldots + \lambda_g v_g\left((c_j)_{j\in T_2}\right),$$

for some $\lambda_1, \ldots, \lambda_g \in \mathbb{R}$.
<u>If $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) \geq 1$, choose</u>
(i) $\tilde{c}_j \in \overline{C}_j \cap \mathbb{Q}$ for any $j \in T_2$,
(ii) $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_g)$ in a neighbourhood of $(\lambda_1, \ldots, \lambda_g)$ and in $\mathbb{Q}^g$,
in such a way that, if we define

$$\begin{pmatrix} \tilde{d}_1 \\ \vdots \\ \tilde{d}_{t_2} \end{pmatrix} = \tilde{\lambda}_1 v_1 \left((\tilde{c}_j)_{j\in T_2}\right) + \ldots + \tilde{\lambda}_g v_g \left((\tilde{c}_j)_{j\in T_2}\right), \tag{7}$$

we have that:
(a) $\tilde{c}_j = 0$ if and only if $c_j = 0$;
(b) if i, j, h are such that $(i, j), (i.h) \in T$, $c_j, c_h, \boldsymbol{\alpha}_{i,j}, \boldsymbol{\alpha}_{i,h}$ are nonzero and $\det \begin{pmatrix} c_j & \boldsymbol{\alpha}_{i,j} \\ c_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$, then $\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$;
(c) if $\det \begin{pmatrix} c_j & d_j \\ c_h & d_h \end{pmatrix} \neq 0$, then $\det \begin{pmatrix} \tilde{c}_j & \tilde{d}_j \\ \tilde{c}_h & \tilde{d}_h \end{pmatrix} \neq 0$;
(d) for any $j \in T_2$, if $d_j \neq 0$, then $\tilde{d}_j \neq 0$;
(e) $\tilde{d}_j \in D_j$ for any $j \in T_2$.
Observe that, by the choice of $(\tilde{c}_1, \ldots, \tilde{c}_{t_2})$ we have done, the rank of $G_{(\boldsymbol{\alpha}_{r,s}),(\tilde{c}_j)}$ is less than or equal to $t_2 - 1$: if $(\tilde{c}_1, \ldots, \tilde{c}_{t_2}) = 0$, then the equations (5) with $\tilde{c}_j$ instead of γ_j become trivial; moreover, by (a), we have that $(c_1, \ldots, c_{t_2}) = 0$, hence, by $(*)$, $(d_1, \ldots, d_{t_2}) \neq 0$; by (3), we have that $(d_1, \ldots, d_{t_2})$ satisfies equations (6) and then the system (S); if $(\tilde{c}_1, \ldots, \tilde{c}_{t_2}) \neq 0$, the statement follows from the fact that the transpose of $(\tilde{c}_1, \ldots, \tilde{c}_{t_2})$ is in the kernel of $G_{(\boldsymbol{\alpha}_{r,s}),(\tilde{c}_j)}$.
Observe also that, obviously, $\tilde{d}_j \in \mathbb{Q}$ for any $j \in T_2$, since the $\tilde{c}_j$ for $j \in T_2$ and the $\tilde{\lambda}_f$ for $f = 1, \ldots, g$ are in $\mathbb{Q}$.
<u>If $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) = 0$, choose</u>
(i) $\tilde{c}_j \in C_j \cap \mathbb{Q}$ for any $j \in T_2$,
(ii) $\tilde{d}_j \in D_j \cap \mathbb{Q}$ for any $j \in T_2$,
in such way that
(a) $\tilde{c}_j = 0$ if and only if $c_j = 0$;
(b) if i, j, h are such that $(i, j), (i, h) \in T$, $c_j, c_h, \boldsymbol{\alpha}_{i,j}, \boldsymbol{\alpha}_{i,h}$ are nonzero and $\det \begin{pmatrix} c_j & \boldsymbol{\alpha}_{i,j} \\ c_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$, then $\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$;
(c) if $\det \begin{pmatrix} c_j & d_j \\ c_h & d_h \end{pmatrix} \neq 0$, then $\det \begin{pmatrix} \tilde{c}_j & \tilde{d}_j \\ \tilde{c}_h & \tilde{d}_h \end{pmatrix} \neq 0$;
(d) for any $j \in T_2$, if $d_j \neq 0$, then $\tilde{d}_j \neq 0$.

Remark A. *Observe that, by our choices, if* $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) = 0$*, then* $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(\tilde{c}_j)}) = 0$. This follows from the following remarks.

- Equations (6) do not depend on γ_j.
- Let us consider equations (5);

  - if, for any i, j, h, k with $j, h, k \in T(i, \cdot)$, we have that $c_j = c_h = c_k = 0$, then, by condition (a), we have that $\tilde{c}_j = \tilde{c}_h = \tilde{c}_k = 0$, hence equations (5) with $\tilde{c}_l$ instead of γ_l for every l are trivial;

  - if there exist i, j, h, k with $j, h, k \in T(i, \cdot)$ and $c_j \neq 0$, then, since $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) = 0$, we must have

$$\det \begin{pmatrix} c_j & \boldsymbol{\alpha}_{i,j} \\ c_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = \det \begin{pmatrix} c_j & \boldsymbol{\alpha}_{i,j} \\ c_k & \boldsymbol{\alpha}_{i,k} \end{pmatrix} = 0;$$

hence

$$\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = \det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_k & \boldsymbol{\alpha}_{i,k} \end{pmatrix} = 0, \tag{8}$$

in fact: let us prove for example that $\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$:

· if $c_h = 0$ then $\boldsymbol{\alpha}_{i,h}$ must be zero (from $\det \begin{pmatrix} c_j & \boldsymbol{\alpha}_{i,j} \\ c_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$ and $c_j \neq 0$) and $\tilde{c}_h$ must be zero (by condition (a)), thus $\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$;

· if $c_h \neq 0$, then, from $\det \begin{pmatrix} c_j & \boldsymbol{\alpha}_{i,j} \\ c_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$ and $c_j \neq 0$ we get that either $\boldsymbol{\alpha}_{i,h} = \boldsymbol{\alpha}_{i,j} = 0$ or both $\boldsymbol{\alpha}_{i,h}$ and $\boldsymbol{\alpha}_{i,j}$ are nonzero; if $\boldsymbol{\alpha}_{i,h} = \boldsymbol{\alpha}_{i,j} = 0$, then obviously $\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$; if both $\boldsymbol{\alpha}_{i,h}$ and $\boldsymbol{\alpha}_{i,j}$ are nonzero, then $\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$ by condition (b).

Moreover observe that (8) and the fact that $\tilde{c}_j \neq 0$ imply that $\det \begin{pmatrix} \tilde{c}_k & \boldsymbol{\alpha}_{i,k} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$.
Therefore, in every case, equations (5) with $\tilde{c}_l$ instead of γ_l for every l are trivial.

Remark B. *If, for some i, j, h with $j, h \in T(i, \cdot)$ we have that* $\det \begin{pmatrix} c_j & d_j \\ c_h & d_h \end{pmatrix} = 0$ *(in particular, by (c), this holds if* $\det \begin{pmatrix} \tilde{c}_j & \tilde{d}_j \\ \tilde{c}_h & \tilde{d}_h \end{pmatrix} = 0$*), then* $\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$.

In fact: $\det \begin{pmatrix} c_j & d_j \\ c_h & d_h \end{pmatrix} = 0$ implies, by (2), that $\det \begin{pmatrix} c_j & \boldsymbol{\alpha}_{i,j} \\ c_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$; thus $\det \begin{pmatrix} \tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix} = 0$ (let z be the cardinality of the nonzero entries of $\begin{pmatrix} c_j & \boldsymbol{\alpha}_{i,j} \\ c_h & \boldsymbol{\alpha}_{i,h} \end{pmatrix}$; if $z = 4$ our statement is true by (b), if $z \leq 2$, it is obviously true; observe that z cannot be 3).
We have defined $\tilde{c}_j$ and $\tilde{d}_j$ for any $j \in T_2$. We want now to define $\tilde{a}_i$ and $\tilde{b}_i$ for any $i \in T_1$. Let $i \in T_1$.

- If $|T(i, \cdot)| = 1$, let $T(i, \cdot) = \{\bar{\jmath}(i)\}$; choose $\tilde{a}_i$ and $\tilde{b}_i$ in $\mathbb{Q}$ such that

$$\tilde{a}_i \tilde{c}_{\bar{\jmath}(i)} + \tilde{b}_i \tilde{d}_{\bar{\jmath}(i)} = \boldsymbol{\alpha}_{i,\bar{\jmath}(i)}. \tag{9}$$

- Suppose $|T(i,\cdot)| \geq 2$.

  CASE 1: if $\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j$ is nonzero for some distinct $j,h \in T(i,\cdot)$ (hence, in particular, there exists $j \in T(i,\cdot)$ with $\tilde{c}_j \neq 0$), define $\tilde{b}_i$ (by analogy with (2)) as follows:

$$\tilde{b}_i = \frac{\tilde{c}_j\,\boldsymbol{\alpha}_{i,h} - \tilde{c}_h\,\boldsymbol{\alpha}_{i,j}}{\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j} \tag{10}$$

  (observe that the denominator is nonzero by our assumption; moreover, it is a good definition, i.e. it does not depend on the choice of $j,h \in T(i,\cdot)$ such that $\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j \neq 0$, because the transpose of $(\tilde{d}_1,\ldots,\tilde{d}_{t_2})$ is in $Ker(G_{(\boldsymbol{\alpha}_{r,s}),(\tilde{c}_j)})$ (by (7) in case $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) \geq 1$ and by Remark A in case $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) = 0$), hence it satisfies the equations (5) with $\tilde{c}_j$ instead of γ_j); then define $\tilde{a}_i$ for $i = 1,\ldots,t_1$ by

$$\tilde{a}_i = \frac{\boldsymbol{\alpha}_{i,j} - \tilde{b}_i\tilde{d}_j}{\tilde{c}_j} \tag{11}$$

  for any $j \in T(i,\cdot)$ with $\tilde{c}_j \neq 0$; it is a good definition by our definition of $\tilde{b}_i$ and by Remark B, in fact: let $j,h \in T(i,\cdot)$ such that $\tilde{c}_j \neq 0$ and $\tilde{c}_h \neq 0$; we have to prove that $\frac{\boldsymbol{\alpha}_{i,j}-\tilde{b}_i\tilde{d}_j}{\tilde{c}_j} = \frac{\boldsymbol{\alpha}_{i,h}-\tilde{b}_i\tilde{d}_h}{\tilde{c}_h}$; this is equivalent to $(\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j)\tilde{b}_i = \tilde{c}_j\,\boldsymbol{\alpha}_{i,h} - \tilde{c}_h\,\boldsymbol{\alpha}_{i,j}$, which is true by the definition of $\tilde{b}_i$ in case $\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j \neq 0$ and by Remark B in case $\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j = 0$.

  CASE 2: if $\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j = 0$ for every $j,h \in T(i,\cdot)$, then, by Remark B, we have that $\det\begin{pmatrix}\tilde{c}_j & \boldsymbol{\alpha}_{i,j} \\ \tilde{c}_h & \boldsymbol{\alpha}_{i,h}\end{pmatrix} = 0$ for every $j,h \in T(i,\cdot)$.

  Case 2.1. If there exists $j \in T(i,\cdot)$ such that $\tilde{c}_j \neq 0$, define $\tilde{b}_i$ to be any element of $B_i \cap \mathbb{Q}$ and $\tilde{a}_i$ as in (11) for any $j \in T(i,\cdot)$ such that $\tilde{c}_j \neq 0$ (it is well defined because $\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j$ and $\tilde{c}_j\boldsymbol{\alpha}_{i,h} - \tilde{c}_h\boldsymbol{\alpha}_{i,j}$ are zero hence $\tilde{b}_i(\tilde{c}_j\tilde{d}_h - \tilde{c}_h\tilde{d}_j) = \tilde{c}_j\boldsymbol{\alpha}_{i,h} - \tilde{c}_h\boldsymbol{\alpha}_{i,j}$ for any $j,h \in T(i,\cdot)$).

  Case 2.2. If $\tilde{c}_j = 0$ for any $j \in T(i,\cdot)$, then, by (a), $c_j = 0$ for any $j \in T(i,\cdot)$; hence $d_j \neq 0$ for any $j \in T(i,\cdot)$ by assumption $(*)$; therefore, by condition (d), we have that $\tilde{d}_j \neq 0$ for any $j \in T(i,\cdot)$; define

$$\tilde{b}_i = \frac{\boldsymbol{\alpha}_{i,j}}{\tilde{d}_j} \tag{12}$$

  for any $j \in T(i,\cdot)$ (it is a good definition because, as we have already said, $(\tilde{d}_1,\ldots,\tilde{d}_{t_2})$ satisfies the system (S) with $\tilde{c}_j$ instead of γ_j, in particular satisfies equations (6)). Moreover define $\tilde{a}_i$ to be any element of $A_i \cap \mathbb{Q}$.

By continuity, we can do the choices of the $\tilde{c}_j$ and of the $\tilde{\lambda}_f$ in case $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) \geq 1$, of the $\tilde{c}_j$ and of the $\tilde{d}_j$ in case $\mathrm{rk}(G_{(\boldsymbol{\alpha}_{r,s}),(c_j)}) = 0$ in such way that:
- if $|T(i,\cdot)| = 1$ we can choose the $\tilde{a}_i$ and the $\tilde{b}_i$ satisfying (9) respectively in A_i and B_i,
- if $|T(i,\cdot)| \geq 2$, the $\tilde{a}_i$ and the $\tilde{b}_i$ we have defined are respectively in A_i and B_i.

Finally, for any $i \in \{1, \ldots, p\} \setminus T_1$, define

$$\tilde{a}_i = \begin{cases} a_i & \text{if } a_i \in \mathbb{Q} \\ \text{a point of } A_i \cap \mathbb{Q} & \text{if } a_i \in \mathbb{I}, \end{cases}$$

$$\tilde{b}_i = \begin{cases} b_i & \text{if } b_i \in \mathbb{Q} \\ \text{a point of } B_i \cap \mathbb{Q} & \text{if } b_i \in \mathbb{I}; \end{cases}$$

and, for any $j \in \{1, \ldots, q\} \setminus T_2$, define

$$\tilde{c}_j = \begin{cases} c_j & \text{if } c_j \in \mathbb{Q} \\ \text{a point of } C_j \cap \mathbb{Q} & \text{if } c_j \in \mathbb{I}, \end{cases}$$

$$\tilde{d}_j = \begin{cases} d_j & \text{if } d_j \in \mathbb{Q} \\ \text{a point of } D_j \cap \mathbb{Q} & \text{if } d_j \in \mathbb{I}. \end{cases}$$

We have

$$\tilde{a}_i \tilde{c}_j + \tilde{b}_i \tilde{d}_j \in \boldsymbol{\alpha}_{i,j}$$

for any $(i,j) \notin T$, since $\tilde{a}_i \in A_i$, $\tilde{b}_i \in B_i$, $\tilde{c}_j \in C_j$, $\tilde{d}_j \in D_j$ for any $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Moreover $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_j$, $\tilde{d}_j$ are in $\mathbb{Q}$ for any $i = 1, \ldots, p$ and $j = 1, \ldots, q$.
Finally we prove that

$$\tilde{a}_i \tilde{c}_j + \tilde{b}_i \tilde{d}_j = \boldsymbol{\alpha}_{i,j}$$

for any $(i,j) \in T$.
In case $|T(i,\cdot)| = 1$ the statement is true by (9).
Suppose $|T(i,\cdot)| \geq 2$.
• If $\tilde{c}_j \neq 0$ we can be either in Case 1 or in Case 2.1; in both cases the statement is true by our definition of $\tilde{a}_i$ (see (11)).
• Suppose $\tilde{c}_j = 0$ and there exist $h, k \in T(i,\cdot)$ such that $\tilde{c}_h \tilde{d}_k - \tilde{c}_k \tilde{d}_h \neq 0$; hence there exists $h \in T(i,\cdot)$ with $\tilde{c}_h \neq 0$ and we are in Case 1. From the fact that $\tilde{c}_j = 0$ we have, by (a), that $c_j = 0$; therefore, by assumption $(*)$, we have that $d_j \neq 0$; hence, by (d), we have that $\tilde{d}_j \neq 0$; hence $\tilde{c}_j \tilde{d}_h - \tilde{c}_h \tilde{d}_j = -\tilde{c}_h \tilde{d}_j \neq 0$ and the statement holds by our definition of $\tilde{b}_i$ (see (10)).
• Finally, suppose $\tilde{c}_j = 0$ and $\tilde{c}_h \tilde{d}_k - \tilde{c}_k \tilde{d}_h = 0$ for any $h, k \in T(i,\cdot)$. As before, this implies $c_j = 0$ (by (a)) and then $d_j \neq 0$ by assumption $(*)$, and finally, by (d), $\tilde{d}_j \neq 0$. From the fact that $\tilde{c}_h \tilde{d}_j - \tilde{c}_j \tilde{d}_h = 0$ for any $h \in T(i,\cdot)$, we get that $\tilde{c}_h \tilde{d}_j = 0$ for any $h \in T(i,\cdot)$, hence $\tilde{c}_h = 0$ for any $h \in T(i,\cdot)$ and in this case (Case 2.2) we have defined $\tilde{b}_i$ by (12). Then the statement is true by (12).

Hence the $(p \times q)$-matrix Q whose entry (i,j) is $\tilde{a}_i \tilde{c}_j + \tilde{b}_i \tilde{d}_j$, for any $i = 1, \ldots, p$ and $j = 1, \ldots, q$, is a rational matrix of rank less than or equal to 2 contained in $\boldsymbol{\alpha}$. If $\mathrm{rk}(Q) = 1$, by changing an entry of Q in an appropriate way, we can get a rational matrix of rank 2 contained in $\boldsymbol{\alpha}$. □

4 Rational realizations of the ranks $q-2$, $q-1$, q

Remark 9. Let $p \geq q$ and let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p \times q$ interval matrix with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ and $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ for any i,j. Suppose there exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = q$; then obviously there exists $B \in \boldsymbol{\alpha} \cap M(p \times q, \mathbb{Q})$ with $\mathrm{rk}(B) = q$.

Notation 10. *In the proof of the following theorem we will use the following notation for any* $b \in \mathbb{R}$*:*

$$b \wr v = \begin{cases} b & \text{if } b \in \mathbb{Q}, \\ v & \text{if } b \in \mathbb{I}, \end{cases}$$

where v *can be a real number or an interval.*
Moreover, if we have two systems of equations, (P) *and* (M)*, in the same unknowns, we call* (PM) *the system "union" of the two systems, i.e. the system given both by the equations of* (P) *and the equations of* (M)*.*

Remark 11. (i) If a linear system with rational entries has a solution c and V is a neighbourhood of c, then there is a solution of the system contained in V and with rational entries.
(ii) Let (S_t) be a linear system whose entries depend linearly on a parameter $t \in \mathbb{R}^n$. Let $\overline{t} \in \mathbb{R}^n$. If the system $S_{\overline{t}}$ has a solution b, then for every neighbourhood U of b, there exists a neighbourhood V of $\overline{t}$ such that if $t \in V$, S_t is solvable and the dimension of the solution space of S_t is equal to the dimension of the solution space of $S_{\overline{t}}$, then there is a solution of S_t in U.

Theorem 12. *Let* $p \geq q$ *and let* $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ *be a* $p \times q$ *interval matrix with* $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ *and* $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ *for any* i, j*. If there exists* $A \in \boldsymbol{\alpha}$ *with* $\mathrm{rk}(A) \leq q-2$*, then there exists* $B \in \boldsymbol{\alpha} \cap M(p \times q, \mathbb{Q})$ *with* $\mathrm{rk}(B) \leq q-2$*.*

Proof. We can suppose that $A^{(q-1)}, A^{(q)} \in \langle A^{(1)}, \dots A^{(q-2)} \rangle$; let

$$A^{(q-1)} = b_1 A^{(1)} + \dots + b_{q-2} A^{(q-2)} \tag{13}$$

for some $b_1, \dots, b_{q-2} \in \mathbb{R}$ and let

$$A^{(q)} = c_1 A^{(1)} + \dots + c_{q-2} A^{(q-2)} \tag{14}$$

for some $c_1, \dots, c_{q-2} \in \mathbb{R}$. We can suppose that, for any $j = 1, \dots, q-2$, either $b_j \neq 0$ or $c_j \neq 0$ (call this assumption $(**)$).
Up to swapping rows and columns, we can also suppose that $\boldsymbol{\alpha}_{i,q-1}, \boldsymbol{\alpha}_{i,q}$ are nondegenerate for $i = 1, \dots k$, while, for $i = k+1, \dots p$, at least one of $\boldsymbol{\alpha}_{i,q-1}$ and $\boldsymbol{\alpha}_{i,q}$ is degenerate.
For any $i = 1, \dots, p$, we define:

$$N_i = \{j \in \{1, \dots, q-2\} |\ a_{i,j} \in \mathbb{I}\}.$$

Moreover, let us define

$$B = \{j \in \{1, \dots, q-2\} |\ b_j \in \mathbb{I}\},$$

$$C = \{j \in \{1, \dots, q-2\} |\ c_j \in \mathbb{I}\}.$$

Finally we can suppose that $a_{i,j} \in \mathbb{Q}$ for any $i = 1, \dots, k$ and $j = 1, \dots, q-2$; in fact: if for some $i \in \{1, \dots, k\}$ the set N_i is nonempty, we have that for any $j \in N_i$ the entry $\boldsymbol{\alpha}_{i,j}$ is nondegenerate (since has rational endpoints and contains $a_{i,j}$, which is irrational); so there exist neighbourhoods $U_{i,j}$ of $a_{i,j}$ contained in $\boldsymbol{\alpha}_{i,j}$ for any $j \in N_i$ such that

$$\sum_{j\in\{1,\dots,q-2\}} b_j \cdot (a_{i,j} \wr U_{i,j}) \subset \boldsymbol{\alpha}_{i,q-1}$$

and

$$\sum_{j\in\{1,\dots,q-2\}} c_j \cdot (a_{i,j} \wr U_{i,j}) \subset \boldsymbol{\alpha}_{i,q};$$

hence, for any $j \in N_i$ we can change the entry $a_{i,j}$ into an element $\tilde{a}_{i,j}$ of $U_{i,j} \cap \mathbb{Q}$ and the entries $a_{i,q-1}$ and $a_{i,q}$ respectively into

$$\sum_{j\in\{1,\dots,q-2\}} b_j \cdot (a_{i,j} \wr \tilde{a}_{i,j}), \qquad \text{and} \qquad \sum_{j\in\{1,\dots,q-2\}} c_j \cdot (a_{i,j} \wr \tilde{a}_{i,j});$$

in this way we get again a matrix with the last two columns in the span of the first $q-2$ columns and such that the first $q-2$ entries of each of the first k rows of the matrix are rational. So we can suppose that $a_{i,j} \in \mathbb{Q}$ for any $i = 1, \dots, k$ and $j = 1, \dots, q-2$.
Moreover define

$$X^b = \{i \in \{k+1,\dots,p\} \,|\, \boldsymbol{\alpha}_{i,q-1} \text{ is degenerate, } \boldsymbol{\alpha}_{i,q} \text{ is nondegenerate and either } N_i = \emptyset \text{ or } b_j = 0 \;\forall j \in N_i\}.$$

$$X^c = \{i \in \{k+1,\dots,p\} \,|\, \boldsymbol{\alpha}_{i,q-1} \text{ is nondegenerate, } \boldsymbol{\alpha}_{i,q} \text{ is degenerate and either } N_i = \emptyset \text{ or } c_j = 0 \;\forall j \in N_i\}.$$

$$Y^b = \{i \in \{k+1,\dots,p\} \,|\, \boldsymbol{\alpha}_{i,q-1} \text{ is degenerate, } \boldsymbol{\alpha}_{i,q} \text{ is degenerate and either } N_i = \emptyset \text{ or } b_j = 0 \;\forall j \in N_i\}.$$

$$Y^c = \{i \in \{k+1,\dots,p\} \,|\, \boldsymbol{\alpha}_{i,q-1} \text{ is degenerate, } \boldsymbol{\alpha}_{i,q} \text{ is degenerate and either } N_i = \emptyset \text{ or } c_j = 0 \;\forall j \in N_i\}.$$

We want now to define some neighbours Z^i_j of b_j, V^i_j of c_j and $U_{i,j}$ of $a_{i,j}$ for some $j \in \{1,\dots,q-2\}$ and $i \in \{1,\dots,p\}$.

CASE 0: Let $i \in \{1,\dots,k\}$. Hence $\boldsymbol{\alpha}_{i,q-1}$ and $\boldsymbol{\alpha}_{i,q}$ are nondegenerate.
Choose
- for any $j \in B$, a neighbourhood Z^i_j of b_j,
- for any $j \in C$, a neighbourhood V^i_j of c_j,

such that the following two conditions hold:

$$\sum_{j\in\{1,\dots,q-2\}} (b_j \wr Z^i_j) \cdot a_{i,j} \subset \boldsymbol{\alpha}_{i,q-1},$$

$$\sum_{j\in\{1,\dots,q-2\}} (c_j \wr V^i_j) \cdot a_{i,j} \subset \boldsymbol{\alpha}_{i,q}.$$

CASE 1: Let $i \in \{k+1,\dots,p\}$ such that $\boldsymbol{\alpha}_{i,q-1}$ is degenerate and $\boldsymbol{\alpha}_{i,q}$ is nondegenerate.

• SUBCASE 1.1: $i \notin X^b$.
Hence $N_i \neq \emptyset$ and there exists $\overline{\jmath}(i) \in N_i$ such that $b_{\overline{\jmath}(i)} \neq 0$.
Choose
- for any $j \in B$, a neighbourhood Z^i_j of b_j,
- for any $j \in C$, a neighbourhood V^i_j of c_j,
- for any $j \in N_i$, a neighbourhood $U_{i,j}$ of $a_{i,j}$ contained in $\boldsymbol{\alpha}_{i,j}$,
such that the following three conditions hold:
if $b_{\overline{\jmath}(i)} \in \mathbb{I}$, we have that $Z^i_{\overline{\jmath}(i)}$ is contained either in $\mathbb{R}_{<0}$ or in $\mathbb{R}_{>0}$;

$$\sum_{j\in\{1,\ldots,q-2\}} (c_j \wr V^i_j) \cdot (a_{i,j} \wr U_{i,j}) \subset \boldsymbol{\alpha}_{i,q}; \tag{15}$$

$$-\frac{1}{(b_{\overline{\jmath}(i)} \wr Z^i_{\overline{\jmath}(i)})}\left[\sum_{j\in\{1,\ldots,q-2\}\setminus\{\overline{\jmath}(i)\}} (b_j \wr Z^i_j) \cdot (a_{i,j} \wr U_{i,j}) - a_{i,q-1}\right] \subset U_{i,\overline{\jmath}(i)}. \tag{16}$$

• SUBCASE 1.2: $i \in X^b$ and $N_i \neq \emptyset$.
Choose
- for any $j \in C$, a neighbourhood V^i_j of c_j,
- for any $j \in N_i$, a neighbourhood $U_{i,j}$ of $a_{i,j}$ contained in $\boldsymbol{\alpha}_{i,j}$,
such that (15) holds.
• SUBCASE 1.3: $i \in X^b$ and $N_i = \emptyset$.
Choose
- for any $j \in C$, a neighbourhood V^i_j of c_j,
such that (15) holds with $a_{i,j}$ instead of $(a_{i,j} \wr U_{i,j})$ (observe that $a_{i,j} \in \mathbb{Q}$ for any $j \in \{1, \ldots, q-2\}$).

CASE 2: Let $i \in \{k+1, \ldots, p\}$ such that $\boldsymbol{\alpha}_{i,q-1}$ is nondegenerate and $\boldsymbol{\alpha}_{i,q}$ is degenerate.
Analogous to Case 1 by swapping $q-1$ with q and b with c.

CASE 3: Let $i \in \{k+1, \ldots, p\}$ be such that $\boldsymbol{\alpha}_{i,q-1}$ and $\boldsymbol{\alpha}_{i,q}$ are degenerate.

SUBCASE 3.1: $i \notin Y^b \cup Y^c$ and there does not exist $j \in N_i$ such that $b_j \neq 0$ and $c_j \neq 0$; by the assumption $(**)$, there exist $\overline{\jmath}(i)$ and $\hat{\jmath}(i)$ in N_i such that

$$b_{\overline{\jmath}(i)} \neq 0, \quad c_{\overline{\jmath}(i)} = 0, \qquad c_{\hat{\jmath}(i)} \neq 0, \quad b_{\hat{\jmath}(i)} = 0.$$

We consider:
for any $j \in B$, neighbourhoods Z^i_j of b_j,
for any $j \in C$, neighbourhoods V^i_j of c_j,
for any $j \in N_i \setminus \{\overline{\jmath}(i), \hat{\jmath}(i)\}$, neighbourhoods $U_{i,j}$ of $a_{i,j}$ contained in $\boldsymbol{\alpha}_{i,j}$
such that the following four conditions hold:
if $b_{\overline{\jmath}(i)} \in \mathbb{I}$, we have that $Z^i_{\overline{\jmath}(i)}$ is contained either in $\mathbb{R}_{<0}$ or in $\mathbb{R}_{>0}$;
if $c_{\hat{\jmath}(i)} \in \mathbb{I}$, we have that $V^i_{\hat{\jmath}(i)}$ is contained either in $\mathbb{R}_{<0}$ or in $\mathbb{R}_{>0}$;

$$-\frac{1}{b_{\overline{\jmath}(i)} \wr Z^i_{\overline{\jmath}(i)}}\left[\sum_{j\in\{1,\ldots,q-2\}\setminus\{\hat{\jmath}(i)\overline{\jmath}(i)\}} (b_j \wr Z^i_j) \cdot (a_{i,j} \wr U_{i,j}) - a_{i,q-1}\right] \subset \boldsymbol{\alpha}_{i,\overline{\jmath}(i)}; \tag{17}$$

$$-\frac{1}{c_{\hat{\jmath}(i)} \wr V^i_{\hat{\jmath}(i)}} \left[\sum_{j\in\{1,\dots,q-2\}\setminus\{\hat{\jmath}(i)\bar{\jmath}(i)\}} (c_j \wr V^i_j) \cdot (a_{i,j} \wr U_{i,j}) - a_{i,q} \right] \subset \boldsymbol{\alpha}_{i,\hat{\jmath}(i)}; \tag{18}$$

SUBCASE 3.2: $i \notin Y^b \cup Y^c$ and there exists $\bar{\jmath}(i) \in N_i$ such that $b_{\bar{\jmath}(i)} \neq 0$ and $c_{\bar{\jmath}(i)} \neq 0$.
We consider:
for any $j \in B$, neighbourhoods Z^i_j of b_j,
for any $j \in C$, neighbourhoods V^i_j of c_j,
for any $j \in N_i \setminus \{\bar{\jmath}(i)\}$, neighbourhoods $U_{i,j}$ of $a_{i,j}$ contained in $\boldsymbol{\alpha}_{i,j}$
such that the following four conditions hold:
if $b_{\bar{\jmath}(i)} \in \mathbb{I}$, we have that $Z^i_{\bar{\jmath}(i)}$ is contained either in $\mathbb{R}_{<0}$ or in $\mathbb{R}_{>0}$;
if $c_{\bar{\jmath}(i)} \in \mathbb{I}$, we have that $V^i_{\bar{\jmath}(i)}$ is contained either in $\mathbb{R}_{<0}$ or in $\mathbb{R}_{>0}$;

$$-\frac{1}{b_{\bar{\jmath}(i)} \wr Z^i_{\bar{\jmath}(i)}} \left[\sum_{j\in\{1,\dots,q-2\}\setminus\{\bar{\jmath}(i)\}} (b_j \wr Z^i_j) \cdot (a_{i,j} \wr U_{i,j}) - a_{i,q-1} \right] \subset \boldsymbol{\alpha}_{i,\bar{\jmath}(i)}; \tag{19}$$

$$-\frac{1}{c_{\bar{\jmath}(i)} \wr V^i_{\bar{\jmath}(i)}} \left[\sum_{j\in\{1,\dots,q-2\}\setminus\{\bar{\jmath}(i)\}} (c_j \wr V^i_j) \cdot (a_{i,j} \wr U_{i,j}) - a_{i,q} \right] \subset \boldsymbol{\alpha}_{i,\bar{\jmath}(i)}; \tag{20}$$

SUBCASE 3.3: $i \in Y^c \setminus Y^b$.
In this case we must have: $N_i \neq \emptyset$, $c_j = 0$ $\forall j \in N_i$ and there exists $\bar{\jmath}(i) \in N_i$ such that $b_{\bar{\jmath}(i)} \neq 0$.
We consider:
for any $j \in B$, neighbourhoods Z^i_j of b_j,
for any $j \in N_i \setminus \{\bar{\jmath}(i)\}$, neighbourhoods $U_{i,j}$ of $a_{i,j}$ contained in $\boldsymbol{\alpha}_{i,j}$
such that:
if $b_{\bar{\jmath}(i)} \in \mathbb{I}$, we have that $Z^i_{\bar{\jmath}(i)}$ is contained either in $\mathbb{R}_{<0}$ or in $\mathbb{R}_{>0}$ and (19) holds.

SUBCASE 3.4: $i \in Y^b \setminus Y^c$.
Analogous to the previous subcase.

Finally observe that if $i \in Y^b \cap Y^c$ we must have $N_i = \emptyset$ since, by assumption $(**)$, there does not exist j such that $b_j = c_j = 0$. In this case we do not give at the moment any definition.

Definiition. • For any $j \in B$, let $\beta(j)$ be the set of the $i \in \{1, \dots, p\}$ such that we have chosen Z^i_j.
• For any $j \in C$, let $\gamma(j)$ be the set of the $i \in \{1, \dots, p\}$ such that we have chosen V^i_j.

Choice of the $\tilde{b}_j$ for any $j \in B$ and of the $\tilde{c}_j$ for any $j \in C$ in case $B \cup C \neq \emptyset$.

If $X^b \cup Y^b = \emptyset$, then, for any $j \in B$, choose $\tilde{b}_j$ in the set

$$\left(\cap_{i\in\beta(j)} Z^i_j\right) \cap \mathbb{Q}$$

(observe that in this case we have $\beta(j) \neq \emptyset$).

If $X^c \cup Y^c = \emptyset$, then, for any $j \in C$, choose $\tilde{c}_j$ in the set

$$\left(\cap_{i\in\gamma(j)} V_j^i\right) \cap \mathbb{Q}$$

(observe that in this case we have $\gamma(j) \neq \emptyset$).
If $X^b \cup Y^b \neq \emptyset$, define (M) to be the linear system given by the equations

$$\sum_{j\in\{1,\dots,q-2\}} (b_j \wr \tilde{b}_j) \cdot a_{i,j} = a_{i,q-1}, \tag{21}$$

for $i \in X^b \cup Y^b$, in the unknowns $\tilde{b}_j$ for $j \in B$. Observe that the linear system has rational coefficients and it is certainly solvable since $b := (b_j)_{j\in B}$ is a solution.
If $X^c \cup Y^c \neq \emptyset$, define (N) to be the linear system given by the equations

$$\sum_{j\in\{1,\dots,q-2\}} (c_j \wr \tilde{c}_j) \cdot a_{i,j} = a_{i,q}, \tag{22}$$

for $i \in X^c \cup Y^c$, in the unknowns $\tilde{c}_j$ for $j \in C$. Observe that the linear system has rational coefficients and it is certainly solvable since $c := (c_j)_{j\in C}$ is a solution.
Moreover, define W to be the set

$$\{i \in \{1,\dots,p\} | \, \boldsymbol{\alpha}_{i,q-1}, \boldsymbol{\alpha}_{i,q} \text{ are degenerate, } i \notin Y^b \cup Y^c, \\ \exists \bar{\jmath}(i) \in N_i \text{ s.t. } b_{\bar{\jmath}(i)} c_{\bar{\jmath}(i)} \neq 0\}$$

(that is W is the set of the i in Case 3.2) and, if $B \cup C \neq \emptyset$, consider, for $i \in W$, the following equation:

$$\begin{aligned} -\tfrac{1}{b_{\bar{\jmath}(i)} \wr \tilde{b}_{\bar{\jmath}(i)}} \left[\textstyle\sum_{j\in\{1,\dots,q-2\}\setminus\{\bar{\jmath}(i)\}} (b_j \wr \tilde{b}_j) \cdot (a_{i,j} \wr \tilde{a}_{i,j}) - a_{i,q-1}\right] = \\ -\tfrac{1}{c_{\bar{\jmath}(i)} \wr \tilde{c}_{\bar{\jmath}(i)}} \left[\textstyle\sum_{j\in\{1,\dots,q-2\}\setminus\{\bar{\jmath}(i)\}} (c_j \wr \tilde{c}_j) \cdot (a_{i,j} \wr \tilde{a}_{i,j}) - a_{i,q}\right], \end{aligned} \tag{23}$$

where the unknowns are the $\tilde{a}_{i,j}$ for $j \in N_i \setminus \{\bar{\jmath}(i)\}$. It is obviously equivalent to

$$\begin{aligned} \textstyle\sum_{j\in N_i\setminus\{\bar{\jmath}(i)\}} \left(\tfrac{b_j \wr \tilde{b}_j}{b_{\bar{\jmath}(i)} \wr \tilde{b}_{\bar{\jmath}(i)}} - \tfrac{c_j \wr \tilde{c}_j}{c_{\bar{\jmath}(i)} \wr \tilde{c}_{\bar{\jmath}(i)}}\right) \tilde{a}_{i,j} = \\ -\textstyle\sum_{j\notin N_i} \left(\tfrac{b_j \wr \tilde{b}_j}{b_{\bar{\jmath}(i)} \wr \tilde{b}_{\bar{\jmath}(i)}} - \tfrac{c_j \wr \tilde{c}_j}{c_{\bar{\jmath}(i)} \wr \tilde{c}_{\bar{\jmath}(i)}}\right) a_{i,j} + \tfrac{a_{i,q-1}}{b_{\bar{\jmath}(i)} \wr \tilde{b}_{\bar{\jmath}(i)}} - \tfrac{a_{i,q}}{c_{\bar{\jmath}(i)} \wr \tilde{c}_{\bar{\jmath}(i)}} \end{aligned} \tag{24}$$

We would like to find $b := (\tilde{b}_j)_{j\in B}$ and $c := (\tilde{c}_j)_{j\in C}$ such that they satisfy (M) and (N), the equation (24) in the unknowns $\tilde{a}_{i,j}$ for $j \in N_i \setminus \{\bar{\jmath}(i)\}$ is solvable for any $i \in W$ and, finally, we can find a solution of (24) "arbitrarily near" to $(a_{i,j})_{j\in N_i\setminus\{\bar{\jmath}(i)\}}$.
Let $(24)_{b,c}$ be the equation we get from (24) by replacing $\tilde{b}_j$ with b_j for every $j \in B$ and $\tilde{c}_j$ with c_j for every $j \in C$; observe that $(24)_{b,c}$ is true, because, by replacing in (24) $\tilde{b}_j$ with b_j for every $j \in B$ and $\tilde{c}_j$ with c_j for every $j \in C$, both members of (24) become equal to $a_{i,\bar{\jmath}(i)}$.
Let G be the set of the $i \in W$ such that at least one of the coefficients of $\tilde{a}_{i,j}$ for $j \in N_i \setminus \{\bar{\jmath}(i)\}$ in $(24)_{b,c}$ is nonzero. Let Z and V be neighbours respectively of b and c such that such coefficients are nonzero for any $(\tilde{b}, \tilde{c}) \in Z \times V$ and any $i \in G$.

Let us call (P) the linear system in the unknowns $\tilde{b}_j$ for $j \in B$ with entries depending on $\tilde{c}_j$ for $j \in C$ given by imposing the second member of (24) equal to 0 for any $i \in W \setminus G$, i.e. given by the equations

$$(b_{\overline{\jmath}} \wr \tilde{b}_{\overline{\jmath}}) \left(a_{i,q} - \sum_{j \notin N_i} (c_j \wr \tilde{c}_j) a_{i,j} \right) + \sum_{j \notin N_i} (b_j \wr \tilde{b}_j)(c_{\overline{\jmath}} \wr \tilde{c}_{\overline{\jmath}}) a_{i,j} - (c_{\overline{\jmath}} \wr \tilde{c}_{\overline{\jmath}}) a_{i,q-1} = 0$$

for any $i \in W \setminus G$, and given by the equations

$$\frac{b_j \wr \tilde{b}_j}{b_{\overline{\jmath}(i)} \wr \tilde{b}_{\overline{\jmath}(i)}} - \frac{c_j \wr \tilde{c}_j}{c_{\overline{\jmath}(i)} \wr \tilde{c}_{\overline{\jmath}(i)}} = 0$$

for $i \in W \setminus G$ and $j \in N_i \setminus \{\overline{\jmath}(i)\}$.
Let (P_c) be the system we get from (P) by replacing $\tilde{c}$ with c.
Let us call (X) the linear system in the unknowns $\tilde{c}_j$ given by the equality :

"rank of the incomplete matrix of (PM)=
= rank of the incomplete matrix of (P_cM)=
= rank of the complete matrix of (PM)".

REMARK A. *Observe that c is a solution of (X), in fact, b is a solution (P_c, M). Moreover c is a solution of (N). So c is a solution of (XN).*

REMARK B. *Observe also that, if $b_{\overline{\jmath}} \in \mathbb{I}$, then all the columns of the complete matrix associated to the linear system (P), apart from the column corresponding to $\tilde{b}_{\overline{\jmath}}$ are multiple of $c_{\overline{\jmath}} \wr \tilde{c}_{\overline{\jmath}}$ and not depending on the other $c_j \wr \tilde{c}_j$'s and it is easy to see that the system (X) is linear in the $\tilde{c}_j$.*
Also if $b_{\overline{\jmath}} \in \mathbb{Q}$, we can conclude easily that the system (X) is linear in the $\tilde{c}_j$.

Suppose $C \neq \emptyset$. By Remarks A and B, the system (XN) is a linear system in the unknowns $\tilde{c}_j$ with rational entries and c is in its solution set. Hence, by (i) of Remark 11, we can find a rational solution $\hat{c} := (\hat{c}_j)_{j \in C}$ with $\hat{c}_j \in \cap_{i \in \gamma(j)} V_j^i$ for every j, $\hat{c}_{\overline{\jmath}(i)} \neq 0$ for every $i \in W$, and $\hat{c} \in V$ if $G \neq \emptyset$.
Let $(P_{\hat{c}})$ be the system we get from (P) by replacing $\tilde{c}$ with $\hat{c}$. The linear system $(P_{\hat{c}}, M)$ in the unknowns $\tilde{b}_j$ for $j \in B$ is solvable by our choice of $\hat{c}$ and the dimension of its solution set is equal to the dimension of the solution set of (P_c, M) (in fact, $\hat{c}$ is a solution of (X)); moreover it has rational entries, so it has a rational solution $\hat{b} := (\hat{b}_j)_{j \in B}$ by (i) of Remark 11. Moreover, by (ii) of Remark 11, we can choose $\hat{c}$ so that $\hat{b} \in \cap_{i \in \beta(j)} Z_j^i$, $\hat{b}_{\overline{\jmath}(i)} \neq 0$ for every $i \in W$, and $\hat{b} \in Z$ if $G \neq \emptyset$.
If $C = \emptyset$, take $\hat{c} = c$ and argue analogously.
The couple $(\hat{b}, \hat{c})$ satisfies (P) (precisely $\hat{b}$ satisfies $(P_{\hat{c}})$). So, if we replace $(\tilde{b}, \tilde{c})$ with $(\hat{b}, \hat{c})$ in (23), we get a solvable equation in the $\tilde{a}_{i,j}$. Moreover $\hat{b}$ satisfies (M) and $\hat{c}$ satisfies (N). We choose $(\hat{b}, \hat{c})$ for $(\tilde{b}, \tilde{c})$.
If $C \neq \emptyset$ and $B = \emptyset$, we argue analogously.

Choice of the $\tilde{a}_{i,j}$ for $i \in \{k+1, \ldots, p\}$ and $j \in N_i$.

CASE 1, i.e. $\boldsymbol{\alpha}_{i,q-1}$ is degenerate and $\boldsymbol{\alpha}_{i,q}$ is nondegenerate.

SUBCASE 1.1: $i \notin X^b$ (hence $N_i \neq \emptyset$ and there exists $\overline{\jmath}(i) \in N_i$ such that $b_{\overline{\jmath}(i)} \neq 0$).

Choose $\tilde{a}_{i,j} \in U_{i,j} \cap \mathbb{Q}$ for any $j \in N_i \setminus \{\overline{\jmath}(i)\}$ and define $\tilde{a}_{i,\overline{\jmath}(i)}$ to be

$$-\frac{1}{b_{\overline{\jmath}(i)} \wr \tilde{b}_{\overline{\jmath}(i)}} \left[\sum_{j \in \{1,\dots,q-2\} \setminus \{\overline{\jmath}(i)\}} (b_j \wr \tilde{b}_j) \cdot (a_{i,j} \wr \tilde{a}_{i,j}) - a_{i,q-1} \right]. \tag{25}$$

By (16), we have that $\tilde{a}_{i,\overline{\jmath}(i)} \in \mathbb{Q} \cap U_{i,\overline{\jmath}(i)}$.
SUBCASE 1.2: $i \in X^b$ and $N_i \neq \emptyset$.
For any $j \in N_i$ choose $\tilde{a}_{i,j} \in U_{i,j} \cap \mathbb{Q}$.
SUBCASE 1.3: $i \in X^b$ and $N_i = \emptyset$.
In this case we have that $a_{i,j} \in \mathbb{Q}$ for any $j \in \{1, \dots, q-2\}$, so we have to choose no $\tilde{a}_{i,j}$.

CASE 2, i.e. $\boldsymbol{\alpha}_{i,q-1}$ is nondegenerate and $\boldsymbol{\alpha}_{i,q}$ is degenerate.
Analogous to Case 1.

CASE 3, i.e. $\boldsymbol{\alpha}_{i,q-1}$ and $\boldsymbol{\alpha}_{i,q}$ are degenerate.
SUBCASE 3.1: $i \notin Y^b \cup Y^c$ and there does not exist $j \in N_i$ such that $b_j \neq 0$ and $c_j \neq 0$.
Choose $\tilde{a}_{i,j} \in U_{i,j} \cap \mathbb{Q}$ for any $j \in N_i \setminus \{\overline{\jmath}(i), \hat{\jmath}(i)\}$ and define $\tilde{a}_{i,\overline{\jmath}(i)}$ to be

$$-\frac{1}{b_{\overline{\jmath}(i)} \wr \tilde{b}_{\overline{\jmath}(i)}} \left[\sum_{j \in \{1,\dots,q-2\} \setminus \{\hat{\jmath}(i), \overline{\jmath}(i)\}} (b_j \wr \tilde{b}_j) \cdot (a_{i,j} \wr \tilde{a}_{i,j}) - a_{i,q-1} \right]. \tag{26}$$

Moreover define $\tilde{a}_{i,\hat{\jmath}(i)}$ to be

$$-\frac{1}{c_{\hat{\jmath}(i)} \wr \tilde{c}_{\hat{\jmath}(i)}} \left[\sum_{j \in \{1,\dots,q-2\} \setminus \{\hat{\jmath}(i), \overline{\jmath}(i)\}} (c_j \wr \tilde{c}_j) \cdot (a_{i,j} \wr \tilde{a}_{i,j}) - a_{i,q} \right]. \tag{27}$$

By (17) and (18), we have that $\tilde{a}_{i,\overline{\jmath}(i)} \in \boldsymbol{\alpha}_{i,\overline{\jmath}(i)}$ and $\tilde{a}_{i,\hat{\jmath}(i)} \in \boldsymbol{\alpha}_{i,\hat{\jmath}(i)}$.
SUBCASE 3.2: $i \notin Y^b \cup Y^c$ and there exists $\overline{\jmath}(i) \in N_i$ such that $b_{\overline{\jmath}(i)} \neq 0$ and $c_{\overline{\jmath}(i)} \neq 0$.
Choose $\tilde{a}_{i,j} \in U_{i,j} \cap \mathbb{Q}$ for any $j \in N_i \setminus \{\overline{\jmath}(i)\}$ in such way that (23) holds and define $\tilde{a}_{i,\overline{\jmath}(i)}$ to be one of the members of (23). Observe that, if $B \cup C \neq \emptyset$, this is possible by the way we have chosen the $\tilde{b}_j$ and the $\tilde{c}_j$; if $B \cup C = \emptyset$, the equation (23) has rational coefficients and is solvable, because the $a_{i,j}$ for $i \in W$, $j \in N_i \setminus \{\overline{\jmath}(i)\}$ give a solution; so it has a rational solution in $\times_{j \in N_i \setminus \{\overline{\jmath}(i)\}} U_{i,j}$.
SUBCASE 3.3: $i \in Y^c \setminus Y^b$ (hence $N_i \neq \emptyset$, $c_j = 0$ $\forall j \in N_i$ and there exists $\overline{\jmath}(i) \in N_i$ such that $b_{\overline{\jmath}(i)} \neq 0$).
Choose $\tilde{a}_{i,j} \in U_{i,j} \cap \mathbb{Q}$ for any $j \in N_i \setminus \{\overline{\jmath}(i)\}$ and define $\tilde{a}_{i,\overline{\jmath}(i)}$ as in (25).
SUBCASE 3.4: $i \in Y^b \setminus Y^c$ (hence $N_i \neq \emptyset$, $b_j = 0$ $\forall j \in N_i$ and there exists $\overline{\jmath}(i) \in N_i$ such that $c_{\overline{\jmath}(i)} \neq 0$).
Analogous to the previous subcase.

Finally, let H be the $p \times q$ matrix such that, for every $i = 1, \dots, p$ and $j = 1, \dots, q-2$,

$$H_{i,j} = \begin{cases} \tilde{a}_{i,j} & \text{if } a_{i,j} \in \mathbb{I} \\ a_{i,j} & \text{if } a_{i,j} \in \mathbb{Q} \end{cases}$$

and such that

$$H^{(q-1)} = \sum_{j\in\{1,\dots,q-2\}|\, b_j\in\mathbb{Q}} b_j H^{(j)} + \sum_{j\in\{1,\dots,q-2\}|\, b_j\in\mathbb{I}} \tilde{b}_j H^{(j)},$$

$$H^{(q)} = \sum_{j\in\{1,\dots,q-2\}|\, c_j\in\mathbb{Q}} c_j H^{(j)} + \sum_{j\in\{1,\dots,q-2\}|\, c_j\in\mathbb{I}} \tilde{c}_j H^{(j)}.$$

By the choice of $\tilde{b}_j$ for $j \in B$ and of $\tilde{c}_j$ for $j \in C$, and the choice of $\tilde{a}_{i,j}$ for $i \in \{k+1,\dots,p\}$, $j \in N_i$, we have that $h_{i,q-1} = a_{i,q-1}$ when $a_{i,q-1}$ is rational, $h_{i,q} = a_{i,q}$ when $a_{i,q}$ is rational, $h_{i,q-1} \in \boldsymbol{\alpha}_{i,q-1}$ when $a_{i,q-1} \in \mathbb{I}$ and $h_{i,q} \in \boldsymbol{\alpha}_{i,q}$ when $a_{i,q} \in \mathbb{I}$. So the matrix H, whose rank is obviously less than or equal to $q-2$, is contained in $\boldsymbol{\alpha} \cap M(p\times q, \mathbb{Q})$. □

From Theorems 3 and 12 we can easily deduce the following result:

Theorem 13. *Let $p \geq q$ and let $\boldsymbol{\alpha} = ([\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j}])_{i,j}$ be a $p\times q$ interval matrix with $\underline{\alpha}_{i,j} \leq \overline{\alpha}_{i,j}$ and $\underline{\alpha}_{i,j}, \overline{\alpha}_{i,j} \in \mathbb{Q}$ for any i,j.*
(a) Suppose there exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = q-1$; then there exists $B \in \boldsymbol{\alpha}\cap M(p\times q, \mathbb{Q})$ with $\mathrm{rk}(B) = q-1$.
(b) Suppose there exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = q-2$; then there exists $B \in \boldsymbol{\alpha}\cap M(p\times q, \mathbb{Q})$ with $\mathrm{rk}(B) = q-2$.

Proof. (a) Since there exists $A \in \boldsymbol{\alpha}$ with $\mathrm{rk}(A) = q-1$, we can find a $p\times q$ interval matrix $\boldsymbol{\alpha}' = ([\underline{\alpha'}_{i,j}, \overline{\alpha'}_{i,j}])_{i,j}$ with $\underline{\alpha'}_{i,j} \leq \overline{\alpha'}_{i,j}$ and $\underline{\alpha'}_{i,j}, \overline{\alpha'}_{i,j} \in \mathbb{Q}$ for any i,j such that

$$A \in \boldsymbol{\alpha}' \subset \boldsymbol{\alpha}$$

and

$$\mathrm{rk}(X) \geq q-1 \qquad \forall X \in \boldsymbol{\alpha}'. \tag{28}$$

By applying Theorem 3 to the interval matrix $\boldsymbol{\alpha}'$, we get that there exists $B \in M(p\times q, \mathbb{Q}) \cap \boldsymbol{\alpha}'$ with $\mathrm{rk}(B) \leq q-1$; but, by (28), we have that $\mathrm{rk}(B) = q-1$ and we conclude.
(b) Analogous to (a), but we have to use Theorem 12 instead of Theorem 3. □

Remark 14. We point out that the result above can have some applications: for instance suppose to have a linear subspace L of dimension 2 or $q-2$ in $\mathbb{R}^q$ given as solution set of a linear system S:

$$L = \{x \in \mathbb{R}^q |\ Ax = 0\},$$

where A is a $p\times q$ matrix (of rank respectively $q-2$ or 2 obviously); we may want to find a linear subspace L' in $\mathbb{R}^q$ with the same dimension as L and given by a linear system $A'x = 0$ such that, for every i and j, the entry $A'_{i,j}$ is equal to $A_{i,j}$ if $A_{i,j}$ is rational and the entry $A'_{i,j}$ is rational and in a given interval containing $A_{i,j}$ if $A_{i,j}$ is irrational; the results of this paper allow us to say that this is possible and the proofs describe also algorithms to find A'. We observe that the most "expensive" step of the algorithm described in the proof of Theorem 12 is to calculate the linear system

(X) (before Remarks A and B), which requires $O(pq^4)$ elementary operations, and to solve the system (XN), which requires $O(p^2q^3)$ elementary operations, while the other steps are less "expensive", so we need in all $O(p^2q^3)$ elementary operations.

Remark 15. In [8], the authors exhibited a 12×12 sign pattern matrix Q such that there exists a real matrix B with $\mathrm{rk}(B) = 3$ and sign pattern Q and there does not exist a rational matrix A with $\mathrm{rk}(A) = 3$ and sign pattern Q.
In [22] the author showed that there exists a $p \times q$ sign pattern matrix Q such that there exists a real matrix B with $\mathrm{rk}(B) = q - 3$ and sign pattern Q and there does not exist a rational matrix A with $\mathrm{rk}(A) = q - 3$ and sign pattern Q.
Analogous results are in [3].
The same examples show that it is not true for any r, that, if an interval matrix contains a rank-r real matrix, then it contains a rank-r rational matrix. In fact let $\boldsymbol{\alpha}$ be an interval matrix containing B of rank r as above and such that, for any i, j, we have:
$\boldsymbol{\alpha}_{i,j} = \{0\}$ if and only if $b_{i,j} = 0$,
$\boldsymbol{\alpha}_{i,j} \subset \mathbb{R}_{>0}$ if and only if $b_{i,j} > 0$,
$\boldsymbol{\alpha}_{i,j} \subset \mathbb{R}_{<0}$ if and only if $b_{i,j} < 0$.
Obviously, since there does not exist a rational matrix with sign pattern Q and rank r, there does not exist a rational matrix in $\boldsymbol{\alpha}$ with rank r. So Theorem 5 is not generalizable to any rank, that is, it is not true for any r, that, if an interval matrix contains a rank-r real matrix, then it contains a rank-r rational matrix.

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