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Dottorando
Luca Sodomaco

Tutore
Prof. Giorgio Ottaviani

Coordinatore
Prof. Graziano Gentili

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Introduction

Best rank-k approximation for matrices

This thesis is motivated by questions as the following classical one:

Question 1. *Among all real $n_1 \times n_2$ matrices of rank at most k, which one is closest to an assigned real matrix U?*

If not specified, in this thesis we adopt a coordinate-free notation, since most of the objects and properties presented are invariant with respect to a certain group action. For example, any real $n_1 \times n_2$ matrix may be thought as an element of the tensor product $V^{\mathbb{R}} := V_1^{\mathbb{R}} \otimes V_2^{\mathbb{R}}$ of two real vector spaces $V_1^{\mathbb{R}}$ and $V_2^{\mathbb{R}}$ of dimensions n_1 and n_2, respectively.

The *rank* of a matrix $U \in V^{\mathbb{R}}$, denoted by $\mathrm{rk}(U)$, is the minimum positive integer k such that U can be written as the sum of k *decomposable* matrices in $V^{\mathbb{R}}$, namely matrices of the form $v_1 \otimes v_2$ for some vectors $v_j \in V_j^{\mathbb{R}}$, $j \in \{1, 2\}$. Having fixed coordinate systems in $V_1^{\mathbb{R}}$ and $V_2^{\mathbb{R}}$, the number $\mathrm{rk}(U)$ coincides with the more classical notion of rank, namely the maximum number of linearly independent rows or columns of the array of format $n_1 \times n_2$ associated to U. In particular, the array associated to a rank-one matrix is of the form $v_1 v_2^T$ for some column arrays $v_j \in \mathbb{R}^{n_j}$, $j \in \{1, 2\}$.

For the moment, $X^{\mathbb{R}}$ denotes the set of matrices in $V^{\mathbb{R}}$ of rank at most one. Note that $X^{\mathbb{R}}$ is a real cone through the origin in $V^{\mathbb{R}}$. In general, a subset $Z \subset \mathbb{K}^n$ (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is a *cone* when $z \in Z$ implies $\lambda z \in Z$ for all $\lambda \in \mathbb{K}$. For this reason, $X^{\mathbb{R}}$ may be regarded as a subset of the projective space $\mathbb{P}(V^{\mathbb{R}})$ as well. Hopefully without leading to confusion, we often use the same notation for a projective subset $Z \subset \mathbb{P}(\mathbb{K}^n)$ and for its associated cone $Z \subset \mathbb{K}^n$.

The set $X^{\mathbb{R}}$ is among the most deeply known varieties in Algebraic Geometry. Indeed, it is the affine cone of the image of the *Segre embedding*

$$\mathrm{Seg}\colon \mathbb{P}(V_1^{\mathbb{R}}) \times \mathbb{P}(V_2^{\mathbb{R}}) \to \mathbb{P}(V^{\mathbb{R}}), \quad \mathrm{Seg}([v_1], [v_2]) := [v_1 \otimes v_2]. \tag{0.0.1}$$

Instead the set of matrices in $V^{\mathbb{R}}$ of rank at most $1 \leq k \leq \min\{n_1, n_2\}$ is the affine cone of the subset

$$\bigcup_{P_1,\dots,P_k \in X^{\mathbb{R}}} \langle P_1, \dots, P_k \rangle \subset V^{\mathbb{R}}, \tag{0.0.2}$$

where $\langle P_1, \dots, P_k \rangle$ denotes the projective subspace generated by the points $P_1 = [v_1]$,…,$P_k = [v_k]$. The Zariski closure of the set introduced in (0.0.2) coincides with the known *k-th secant variety* of the projective variety $X^{\mathbb{R}}$ and is denoted by $\sigma_k(X^{\mathbb{R}})$. Actually, in this case the set in (0.0.2) is already closed, hence it is equal to $\sigma_k(X^{\mathbb{R}})$. Topologically speaking, this means that if a matrix $U \in V^{\mathbb{R}}$ is the limit of a sequence of matrices of rank k, then U has rank at most k as well. In the next section, we provide an example in which this property is not true any more for tensors.

Note that there is a sequence of inclusions (assuming $n_1 \leq n_2$)

$$\{0\} \subset X^{\mathbb{R}} = \sigma_1(X^{\mathbb{R}}) \subset \sigma_2(X^{\mathbb{R}}) \subset \cdots \subset \sigma_{n_1}(X^{\mathbb{R}}) = V^{\mathbb{R}}.$$

In particular, for all $1 \leq k \leq n_1 - 1$, the singular locus of $\sigma_k(X^{\mathbb{R}})$ is precisely $\sigma_{k-1}(X^{\mathbb{R}})$. Since $\sigma_k(X^{\mathbb{R}})$ are all projective varieties, they are cut out by homogeneous polynomials. The way for obtaining them is suggested by the intuition: indeed, the condition for a matrix $U = (u_{ij}) \in V^{\mathbb{R}}$ to have rank at most k is expressed algebraically by requiring all its minors of order $k+1$ to vanish. Moreover, the projective codimension and the degree of $\sigma_k(X^{\mathbb{R}})$ are respectively (see [HT])

$$\operatorname{codim}(\sigma_k(X^{\mathbb{R}})) = (n_1 - k)(n_2 - k), \quad \deg(\sigma_k(X^{\mathbb{R}})) = \prod_{j=0}^{n_1-1-k} \frac{\binom{n_2+j}{k}}{\binom{k+j}{k}}.$$

Let us go back to Question 1, which can be reformulated as follows.

Question 2. *Which are the critical points $A \in \sigma_k(X^{\mathbb{R}})$ of the distance function from a fixed matrix $U \in V^{\mathbb{R}}$ to the affine variety $\sigma_k(X^{\mathbb{R}})$?*

The expression “distance function” in Question 2 establishes that a distance in $V^{\mathbb{R}}$ has been already defined. In other words, we need to fix an inner product $q^{\mathbb{R}}$ on $V^{\mathbb{R}}$. The associated quadratic form, denoted by $q^{\mathbb{R}}$ as well, induces a *squared distance function*

$$\delta^{\mathbb{R}} \colon V^{\mathbb{R}} \times V^{\mathbb{R}} \to \mathbb{R}, \quad \delta^{\mathbb{R}}(A, B) := q^{\mathbb{R}}(A - B). \tag{0.0.3}$$

In addition, if we fix a matrix $U \in V^{\mathbb{R}}$ in one of the two arguments of $\delta^{\mathbb{R}}$, then we obtain the squared distance function from U defined by $\delta^{\mathbb{R}}_U(A) := \delta^{\mathbb{R}}(U, A)$ for all

$A \in V^{\mathbb{R}}$. A classical inner product over $V^{\mathbb{R}}$ is defined as follows. First, we assume that both $V_1^{\mathbb{R}}$ and $V_2^{\mathbb{R}}$ are equipped with inner products $q_1^{\mathbb{R}}$ and $q_2^{\mathbb{R}}$, respectively. Afterwards, we define the inner product of two decomposable matrices $x_1 \otimes x_2$ and $y_1 \otimes y_2$ as

$$q_F^{\mathbb{R}}(x_1 \otimes x_2, y_1 \otimes y_2) \coloneqq q_1^{\mathbb{R}}(x_1, y_1) q_2^{\mathbb{R}}(x_2, y_2) \tag{0.0.4}$$

and then we extend the definition to the whole vector space $V^{\mathbb{R}}$, using the fact that decomposable matrices span $V^{\mathbb{R}}$. The most remarkable property of the squared distance $\delta_F^{\mathbb{R}}$ induced by the inner product $q_F^{\mathbb{R}}$ already defined is its invariance with respect to the group $\mathrm{SO}(V_1^{\mathbb{R}}) \times \mathrm{SO}(V_2^{\mathbb{R}})$. Postponing a rigorous definition to the next chapters, this means that the distance $\delta_F^{\mathbb{R}}$ is compatible with the action by rotations on $V^{\mathbb{R}}$ of the subgroup $\mathrm{SO}(V_1^{\mathbb{R}}) \times \mathrm{SO}(V_2^{\mathbb{R}}) \subset \mathrm{SO}(V^{\mathbb{R}})$.

Given a matrix $U \in V^{\mathbb{R}}$, a critical point of $\delta_{F,U}^{\mathbb{R}}$ is a smooth point $A \in \sigma_k(X^{\mathbb{R}})$ such that

$$q_F^{\mathbb{R}}(U - A, B) = 0 \quad \forall B \in T_A(\sigma_k(X^{\mathbb{R}})), \tag{0.0.5}$$

where $T_A(\sigma_k(X^{\mathbb{R}}))$ denotes the tangent space of $\sigma_k(X^{\mathbb{R}})$ at A. In general, tangent spaces of secant varieties of projective varieties are described through the

Theorem 0.0.1 (Terracini Lemma). *Let $Z \subset \mathbb{P}^n_{\mathbb{K}}$ be an irreducible variety. If* $\mathrm{char}(\mathbb{K}) = 0$, *then*

$$T_z(\sigma_k(Z)) = \langle T_{z_1}(Z), \ldots, T_{z_k}(Z) \rangle$$

for all z in an open subset of $\sigma_k(Z)$, $z_1,\ldots,z_k \in Z$, $z \in \langle z_1, \ldots, z_k \rangle$.

Thanks to the previous theorem, we need to compute only the tangent spaces of the smooth variety $X^{\mathbb{R}}$. More precisely, given a rank-one matrix $A = \xi \otimes \eta \in X^{\mathbb{R}}$, we have the identity

$$T_A(X^{\mathbb{R}}) = \langle v_1 \otimes \eta + \xi \otimes v_2 \mid v_1 \in V_1^{\mathbb{R}}, v_2 \in V_2^{\mathbb{R}} \rangle. \tag{0.0.6}$$

Indeed, any curve $A(t) \coloneqq \xi(t) \otimes \eta(t)$ over $X^{\mathbb{R}}$ with $A(0) \coloneqq A$ has derivative at $t = 0$ defined by $A'(0) = \xi'(0) \otimes \eta + \xi \otimes \eta'(0)$. Since the vectors $\xi'(0)$ and $\eta'(0)$ are arbitrary, we get the identity (0.0.6). Therefore, by Theorem 0.0.1, the tangent space of $\sigma_k(X^{\mathbb{R}})$ at the smooth point $A = \xi_1 \otimes \eta_1 + \cdots + \xi_k \otimes \eta_k \in \sigma_k(X^{\mathbb{R}})$ is

$$T_A(\sigma_k(X^{\mathbb{R}})) = \left\langle \sum_{j=1}^k v_j^{(1)} \otimes \eta_j + \xi_j \otimes v_j^{(2)} \;\middle|\; v_j^{(l)} \in V_l^{\mathbb{R}} \ \forall j \right\rangle. \tag{0.0.7}$$

Hence, the condition for $A = \xi_1 \otimes \eta_1 + \cdots + \xi_k \otimes \eta_k$ to be critical for $\delta_{F,U}^{\mathbb{R}}$ restricted to $\sigma_k(X^{\mathbb{R}})$ is

$$q_F^{\mathbb{R}}\left(U - A, \sum_{j=1}^k v_j^{(1)} \otimes \eta_j + \xi_j \otimes v_j^{(2)}\right) = 0 \tag{0.0.8}$$

for all $v_j^{(l)} \in V_l^{\mathbb{R}}$, and all $1 \leq j \leq k$. Throughout the thesis, we indicate with $[n]$ the set $\{1, \dots, n\}$ for all integers $n \geq 1$.

Theorem 0.0.2 (Singular Value Decomposition). *Any matrix $U \in V^{\mathbb{R}}$ may be written as*

$$U = \sum_{j=1}^{n_1} \sigma_j \left(u_j^{(1)} \otimes u_j^{(2)}\right), \tag{0.0.9}$$

where $\sigma_1 \geq \cdots \geq \sigma_{n_1}$ are nonnegative real numbers, while $\{u_1^{(1)}, \dots, u_{n_1}^{(1)}\}$ and $\{u_1^{(2)}, \dots, u_{n_2}^{(2)}\}$ are orthonormal bases of $V_1^{\mathbb{R}}$ and $V_2^{\mathbb{R}}$, respectively. The decomposition (0.0.9) is called Singular Value Decomposition *(SVD) of U. The numbers σ_j are called* singular values *of U, while for all $j \in [n_1]$, the pair $(u_j^{(1)}, u_j^{(2)})$ is a* singular vector pair *of U. If the singular values are all distinct, then all singular vector pairs are unique up to a simultaneous change of sign.*

Now assume that (0.0.9) is a SVD of the matrix U. Define

$$U_J \coloneqq \sum_{j \in J} \sigma_j (u_j^{(1)} \otimes u_j^{(2)}) \tag{0.0.10}$$

for all $J \subset [n_1]$. Then

$$\begin{aligned} & q_F^{\mathbb{R}}\left(U - U_J, \sum_{s \in J} v_s^{(1)} \otimes u_s^{(2)} + u_s^{(1)} \otimes v_s^{(2)}\right) = \\ & = q_F^{\mathbb{R}}\left(\sum_{j \notin J} \sigma_j (u_j^{(1)} \otimes u_j^{(2)}), \sum_{s \in J} v_s^{(1)} \otimes u_s^{(2)} + u_s^{(1)} \otimes v_s^{(2)}\right) = \\ & = \sum_{j \notin J} \sum_{s \in J} \sigma_j \left[q_1^{\mathbb{R}}(u_j^{(1)}, v_s^{(1)}) q_2^{\mathbb{R}}(u_j^{(2)}, u_s^{(2)}) + q_1^{\mathbb{R}}(u_j^{(1)}, u_s^{(1)}) q_2^{\mathbb{R}}(u_j^{(2)}, v_s^{(2)})\right], \end{aligned}$$

and the last expression is zero since $q_1^{\mathbb{R}}(u_j^{(1)}, u_s^{(1)}) = q_2^{\mathbb{R}}(u_j^{(2)}, u_s^{(2)}) = 0$ for all $j \notin J$ and $s \in J$. This means that the matrix U_J defined above is a critical point of the distance function $\delta_{F,U}^{\mathbb{R}}$ on $\sigma_k(X^{\mathbb{R}})$. Moreover, one can prove that every critical point must be of this form. Indeed, we have the following result.

Theorem 0.0.3 (Eckart-Young). *Consider a matrix $U \in V^{\mathbb{R}}$ and its SVD as in (0.0.9). Let $1 \leq k \leq \mathrm{rk}(U)$. Then all the critical points on $\sigma_k(X^{\mathbb{R}})$ of the distance function $\delta_{F,U}^{\mathbb{R}}$ (induced by the orthogonally invariant quadratic form $q_F^{\mathbb{R}}$ in $V^{\mathbb{R}}$) are of the form U_J in (0.0.10) for all subsets $J \subset [\mathrm{rk}(U)]$ of cardinality k. If the nonzero singular values of U are distinct then the number of critical points is $\binom{\mathrm{rk}(U)}{k}$.*

As a corollary of Theorem 0.0.3, one may verify from the definition of SVD that the the distance between U and a critical point $U_J \in \sigma_k(X^{\mathbb{R}})$ is

$$q_F^{\mathbb{R}}(U - U_J) = \sum_{j \notin J} \sigma_j^2. \tag{0.0.11}$$

In particular, a global minimizer of $\delta_{F,U}^{\mathbb{R}}$ is obtained choosing $J = [k]$, and the corresponding minimum is $\sigma_{k+1}^2 + \cdots + \sigma_{n_1}^2$. This is often called a *best rank-k approximation* of the matrix U.

Theorem 0.0.3 gives a complete answer to Question 2, at least when the chosen inner product over $V^{\mathbb{R}}$ is the one introduced in (0.0.4). In the next chapters, we will see that, choosing a "sufficiently good" inner product $q^{\mathbb{R}}$ on $V^{\mathbb{R}}$ (in a sense made clearer throughout the thesis), the number of critical points of the associated function $\delta_U^{\mathbb{R}}$ could be still determined a priori, even though critical points are much harder to compute.

In our applications and examples, we always assume that $V_j = \mathbb{R}^{n_j}$ and that $q_j^{\mathbb{R}}$ is the standard Euclidean inner product, for all $j \in \{1, 2\}$. Then, the orthogonally invariant inner product $q_F^{\mathbb{R}}$ introduced via the identities in (0.0.4) has the more familiar expression

$$q_F^{\mathbb{R}}(U, A) = \operatorname{tr}(U^T A) \quad \forall U = (u_{ij}), A = (a_{ij}) \in \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}.$$

In particular, it coincides with the well-known *Frobenius inner product* on $\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$, which justifies the notation $q_F^{\mathbb{R}}$. The number $\operatorname{tr}(U^T A)$ is the trace of $U^T A$.

In this coordinate-based setting, the SVD of U written in (0.0.9) has the matricial expression

$$U = U_1 \Sigma U_2^T. \tag{0.0.12}$$

On one hand, U_1 and U_2 are the orthogonal matrices whose columns are formed by vectors $u_j^{(1)}$ and $u_j^{(2)}$, respectively. On the other hand, the matrix Σ has zero values at the entries (i, j) with $i \neq j$ and its diagonal entries are the singular values of U.

In addition, all the critical points on $\sigma_k(X^{\mathbb{R}})$ of $\delta_{F,U}^{\mathbb{R}}$ are given by

$$U_1 \Sigma_J U_2^T, \quad \Sigma_J := \sum_{j \in J} \Sigma_j \quad \forall J \subset [\operatorname{rk}(U)], \tag{0.0.13}$$

where Σ_j is the $n_1 \times n_2$ matrix whose (j, j)-th element is σ_j, while all the other entries of Σ_j are zero.

The Eckart-Joung Theorem does not provide only a method for solving Question 2. Starting from the definition of singular value and identity (0.0.12), one verifies that

$$U u_j^{(2)} = \sigma_j u_j^{(1)} \text{ and } U^T u_j^{(1)} = \sigma_j u_j^{(2)} \quad \forall j \in [n_1].$$

In particular, the squared singular values σ_i^2 are the eigenvalues of both the symmetric matrices UU^T and U^TU. That is, the singular values of U are the roots of the characteristic polynomial of UU^T

$$\det(UU^T - \varepsilon^2 I_{n_1}) = 0, \tag{0.0.14}$$

where I_{n_1} is the identity matrix of order n_1. Note that the polynomial at the left-hand side of equation (0.0.14) is homogeneous of degree $2n_1$ in the ring of polynomials in the entries of U and ε and with real coefficients.

Changing our perspective, if we fix $\varepsilon \geq 0$, equation (0.0.14) describes an affine hypersurface in $V^{\mathbb{R}}$. Can we describe its points?

The answer can be found again looking at the Eckart-Joung Theorem. Motivated by identity (0.0.11) when $k+1 = \mathrm{rk}(U) = n_1$, one may interpret the hypersurface defined by equation (0.0.14) as the locus of matrices $U \in V^{\mathbb{R}}$ admitting a best rank-(n_1-1) approximation of length ε. In other words, we are describing the so-called ***ε-offset*** of the variety $\sigma_{n_1-1}(X^{\mathbb{R}})$.

More in general, one might describe the ε-offset of $\sigma_k(X^{\mathbb{R}})$ for all $k \in [n_1]$ via the singular values of U, but there are only two cases in which we get a "nice" formula: the above-mentioned case $k = n_1 - 1$, and the case $k = 1$, which we are going to explain.

To this aim, we need to make another observation. Let $U \in V^{\mathbb{R}}$ be a matrix of full rank and let $U_{\{1\}} + \cdots + U_{\{n_1\}}$ be its SVD. Applying identity (0.0.11) for some $J \subset [n_1]$ with $|J| = k$ and for its complement $[n_1] \setminus J$, we can derive a simple, but substantial, identity involving two squared distances: one between U and a critical point U_J of $\delta^{\mathbb{R}}_{F,U}$ on $\sigma_k(X^{\mathbb{R}})$, the other between U and the associated critical point $U_{[n_1]\setminus J}$ of $\delta^{\mathbb{R}}_{F,U}$ on $\sigma_{n_1-k}(X^{\mathbb{R}})$, namely

$$q^{\mathbb{R}}_F(U - U_J) + q^{\mathbb{R}}_F(U_J) = q^{\mathbb{R}}_F(U_{[n_1]\setminus J}) + q^{\mathbb{R}}_F(U_J) = q^{\mathbb{R}}_F(U). \tag{0.0.15}$$

When $k = 1$, $J = \{j\}$ for some $j \in [n_1]$ and $q^{\mathbb{R}}_F(U_J) = \sigma_j^2$. On one hand, we have that $q^{\mathbb{R}}_F(U_J)$ is a root of the characteristic polynomial introduced in (0.0.14). On the other hand, $q^{\mathbb{R}}_F(U - U_J)$ is a root of the polynomial

$$\det\left[UU^T - (q^{\mathbb{R}}_F(U) - \lambda^2) I_{n_1}\right] = 0. \tag{0.0.16}$$

In a similar fashion of equation (0.0.14), equation (0.0.16) describes the λ-offset of the variety $\sigma_1(X^{\mathbb{R}}) = X^{\mathbb{R}}$. Thus we found an easy way to switch between the family of offsets of either $\sigma_{n_1-1}(X^{\mathbb{R}})$ or $X^{\mathbb{R}}$. More in general, simply from identity (0.0.15) one could determine the equation of any offset of $\sigma_k(X^{\mathbb{R}})$ from the corresponding equations of the offsets of $\sigma_{n_1-k}(X^{\mathbb{R}})$.

In fact, this nice relation is not just a coincidence. Indeed, it turns out that $\sigma_{n_1-k}(X^{\mathbb{R}})$ is the *dual affine cone* of $\sigma_k(X^{\mathbb{R}})$ (see Definition 1.3.1).

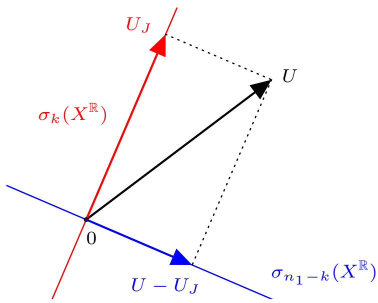

Figure 1: The matrices U, $U_J \in \sigma_k(X^{\mathbb{R}})$ and $U - U_J \in \sigma_{n_1-k}(X^{\mathbb{R}})$.

Summing up, in Questions 1 and 2 we considered the problem of computing the best rank-k approximation of a real rectangular matrix. The answer comes from the celebrated Eckart-Joung Theorem, which uses the Singular Value Decomposition of a matrix. In a slightly wider perspective, this theorem allows us to compute the family of offset hypersurfaces of all determinantal varieties, and to derive a nice duality property.

From matrices to tensors

Motivated by the results stated above, mathematicians tried to understand if the *best rank-k approximation problem* for matrices might be restated and solved for tensors of any format as well.

Given nonnegative integers $1 \le n_1 \le \cdots \le n_d$, a real tensor of format $n_1 \times \cdots \times n_d$ is any element of the tensor product

$$V^{\mathbb{R}} := V_1^{\mathbb{R}} \otimes \cdots \otimes V_d^{\mathbb{R}}$$

of d real vector spaces $V_1^{\mathbb{R}}, \dots, V_d^{\mathbb{R}}$ of dimensions $n_1, \dots, n_d$, respectively. Our choice is to omit the dependence from $\underline{n} := (n_1, \dots, n_d)$ in the notation of $V^{\mathbb{R}}$. This dependence is made clearer from the context in the single chapters and examples. In coordinates, a tensor $T \in V^{\mathbb{R}}$ is indicated by an $\underline{n}$-dimensional array of real numbers $(T_{i_1 \cdots i_d})$, where $i_j \in [n_j]$ for all $j \in [d]$.

A tensor $T \in V^{\mathbb{R}}$ is *decomposable* (or is a *rank-one tensor*) if $T = v_1 \otimes \cdots \otimes v_d$ for some vectors $v_j \in V_j^{\mathbb{R}}$ for all $j \in [d]$. Keeping in mind the matrix intuition, the *rank* of a tensor $T \in V^{\mathbb{R}}$, denoted by $\mathrm{rk}(T)$, is the minimum positive integer k such that T can be written as the sum of k decomposable tensors of $V^{\mathbb{R}}$.

In this section, we denote by $X_d^{\mathbb{R}}$ the affine cone of tensors in $V^{\mathbb{R}}$ of rank at most one. This is known as the affine cone of the image of the Segre embedding

$$\mathrm{Seg}_d\colon \prod_{j=1}^{d} \mathbb{P}(V_j^{\mathbb{R}}) \to \mathbb{P}(V^{\mathbb{R}}), \quad \mathrm{Seg}_d([v_1],\dots,[v_d]) \coloneqq [v_1\otimes\cdots\otimes v_d]. \tag{0.0.17}$$

For brevity, we do not indicate explicitly the dependence from $\underline{n}$ in the notation for both Seg_d and $X_d^{\mathbb{R}}$ as well. For $d=2$, we have that $\mathrm{Seg}_2 = \mathrm{Seg}$ and $X_2^{\mathbb{R}} = X^{\mathbb{R}}$ as in the previous section.

Mimicking (0.0.2) in the matrix case, the set of tensors of rank at most k is the union of all linear spans of k points of the projective variety $X^{\mathbb{R}}$. We show through an example extracted from [Lan, §2.4.5] that in this case this union is not closed. Suppose that $d=3$ and consider the tensor

$$T = v_1\otimes v_2\otimes w_3 + v_1\otimes w_2\otimes v_3 + w_1\otimes v_2\otimes v_3 \in V^{\mathbb{R}}.$$

As the presentation of T suggests, $\mathrm{rk}(T)\le 3$. One can verify that actually $\mathrm{rk}(T)=3$, but T can be written as the limit of a sequence of tensors of rank at most two:

$$T = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[(v_1+\varepsilon w_1)\otimes(v_2+\varepsilon w_2)\otimes(v_3+\varepsilon w_3) - v_1\otimes v_2\otimes v_3].$$

This example suggests to introduce the *border rank* of a tensor $T\in V^{\mathbb{R}}$, denoted by $\underline{\mathrm{rk}}(T)$, as the minimum k such that T belongs to the k-secant variety $\sigma_k(X_d^{\mathbb{R}})$. Note that $\mathrm{rk}(T)\ge\underline{\mathrm{rk}}(T)$. Equality holds in the case of matrices ($d=2$), but the above example tells us that the inequality might be strict even for $d=3$.

Unlike the case of matrices, finding the generators for the ideal of $\sigma_k(X_d^{\mathbb{R}})$ is a very hard task. When $k=1$, the usual way to compute equations for $\sigma_1(X_d^{\mathbb{R}}) = X_d^{\mathbb{R}}$ is by following this idea: a tensor $T\in V^{\mathbb{R}}$ is decomposable if and only if it can be written as

$$T = v_j\otimes T_j,\quad v_j\in V_j^{\mathbb{R}},\ T_j\in\otimes_{i\neq j}V_j^{\mathbb{R}} \tag{0.0.18}$$

for all $j\in[d]$. Now T written in (0.0.18) is a rank-one matrix, hence all its 2×2 minors must vanish. In this way, one obtains all the equations for $X_d^{\mathbb{R}}$ and observes that the ideal of $X^{\mathbb{R}}$ is generated by quadrics. This approach is no longer useful for computing the equations of the higher order secant varieties of $X_d^{\mathbb{R}}$.

Let us equip the tensor space $V^{\mathbb{R}}$ with the most natural inner product. First, suppose that $q_j^{\mathbb{R}}$ is an arbitrary inner product on $V_j^{\mathbb{R}}$, for all $j\in[d]$. The *Frobenius inner product* of two decomposable tensors $x_1\otimes\cdots\otimes x_d$ and $y_1\otimes\cdots\otimes y_d$ is defined as

$$q_F^{\mathbb{R}}(x_1\otimes\cdots\otimes x_d, y_1\otimes\cdots\otimes y_d) \coloneqq q_1^{\mathbb{R}}(x_1,y_1)\cdots q_d^{\mathbb{R}}(x_d,y_d), \tag{0.0.19}$$

and is extended to $V^{\mathbb{R}}$ by linearity. If we assume for all $j \in [d]$ that $V_j^{\mathbb{R}} = \mathbb{R}^{n_j}$ and that $q_j^{\mathbb{R}}$ is the standard Euclidean inner product, then

$$q_F^{\mathbb{R}}(T, U) = \sum_{i_1,\ldots,i_d} T_{i_1,\ldots,i_d} U_{i_1,\ldots,i_d} \tag{0.0.20}$$

for all $T = (T_{i_1,\ldots,i_d})$ and $U = (U_{i_1,\ldots,i_d})$ in $\mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d}$. The induced *squared distance function* is again $\delta_F^{\mathbb{R}}(U, T) := q_F^{\mathbb{R}}(U - T)$ for all $U, T \in V^{\mathbb{R}}$. Moreover, the *squared distance function from a tensor* $U \in V^{\mathbb{R}}$ is $\delta_{F,U}^{\mathbb{R}}(A) := \delta_F^{\mathbb{R}}(U, A)$ for all $A \in V^{\mathbb{R}}$.

Consider the following restatement of Question 2 in this generalized setting, at least in the case $k = 1$.

Question 3. *Given a tensor $U \in V^{\mathbb{R}}$, find all critical points of the distance function $\delta_{F,U}^{\mathbb{R}}$ over the Segre product $X_d^{\mathbb{R}}$.*

In the previous section, we naively did not care much about possible non-real solutions of Question 2: indeed, every real matrix admits only real singular values and singular vector pairs. This is no longer true for higher format real tensors. In fact, the following result includes a generalization of singular values and singular vector pairs for real tensors, which might be non-real. If so, they are discarded when trying to find a real solution of Question 3. We refer to critical points of $\delta_{F,U}^{\mathbb{R}}$ on $X_d^{\mathbb{R}}$ simply as *decomposable critical tensors*.

In the statement below, we consider the complexifications $V_j := V_j^{\mathbb{R}} \otimes \mathbb{C}$, $V := V^{\mathbb{R}} \otimes \mathbb{C}$ and the complex algebraic variety $X_d := X_d^{\mathbb{C}}$ (defined by the common complex zeros of the generators of the ideal of $X_d^{\mathbb{R}}$). In addition, the distance function $\delta_{F,U}^{\mathbb{R}}$ is extended to a complex-valued function $\delta_{F,U} \colon X_d \to \mathbb{C}$, which is not a Hermitian inner product. This approach is followed in general throughout the thesis: indeed, even if $\delta_{F,U}$ has the meaning of a distance function only over the reals, the complex critical points of $\delta_{F,U}$ are necessary to depict all the metric information about the real variety $X_d^{\mathbb{R}}$. For example, an important is being played by the *isotropic tensors* of V, which are all tensors $T \in V$ such that $q_F(T) = 0$.

Theorem 0.0.4 (Lim, Qi). *Given a real tensor $U \in V^{\mathbb{R}}$, the non-isotropic decomposable critical tensors for U correspond to tensors $\sigma(x_1 \otimes \cdots \otimes x_d) \in \prod_{j\in[d]} V_j$ such that $q_j(x_j) = 1$ for all $j \in [d]$ and*

$$q_F(U, x_1 \otimes \cdots \otimes x_{j-1} \otimes \_ \otimes x_{j+1} \otimes \cdots \otimes x_d) = \sigma \cdot q_j(x_j, \_) \quad \forall j \in [d] \tag{0.0.21}$$

for some $\sigma \in \mathbb{C}$, called singular value *of U. The corresponding d-ple $(x_1, \ldots, x_d)$ is called* singular vector d-ple *for U.*

Note that both sides of relation (0.0.21) correspond to linear operators on V_j. In applications, more precisely when q_F is expressed as in (0.0.20), the system (0.0.21) may be rewritten as

$$U \cdot (x_1 \otimes \cdots \otimes x_{j-1} \otimes x_{j+1} \otimes \cdots \otimes x_d) = \sigma x_j \quad \forall j \in [d], \tag{0.0.22}$$

where at the left-hand side the dot means tensor contraction.

The best rank-one approximation problem stated in Question 3 is solved by the following result.

Theorem 0.0.5 (Lim)**.** *Let $U \in V^{\mathbb{R}}$. Then U always admits real singular values and real singular vector d-ples. In particular, if $\tilde{\sigma}$ is a singular value of U such that $\tilde{\sigma}^2$ is maximum, and if $(\tilde{x}_1, \ldots, \tilde{x}_d)$ is a singular d-ple associated with $\tilde{\sigma}$, then $\tilde{\sigma}(\tilde{x}_1 \otimes \ldots \otimes \tilde{x}_d)$ is the best rank-one approximation of U.*

Anyway, there are various considerations left about the singular values and the singular vector d-ples of a tensor. The first question one could formulate is: *how many possible distinct singular values can a tensor have? How many of them can be real?*

We already know that every matrix $U \in V^{\mathbb{R}} = \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$ admits n_1 real nonnegative singular values, counted with multiplicity. For higher dimensional tensors, we need to distinguish between the problems of either computing all the singular values of a tensor, or restricting only to the real ones.

If we are in the first case, there is a positive result which actually counts the number of critical decomposable tensors for an (eventually non-real) $U \in V$. There is a reasonable price to pay, as well as many results in Algebraic Geometry: indeed, this number is constant for "almost all choices" of $U \in V$, or for a *general* $U \in V$, meaning out of some Zariski closed set of V.

Theorem 0.0.6 (Friedland, Ottaviani)**.** *Let $U \in V^{\mathbb{R}}$ be general. Then U admits $c(\underline{n})$ distinct singular values. The integer $c(\underline{n})$ is the coefficient in the monomial $h_1^{n_1-1} \cdots h_d^{n_d-1}$ in the polynomial*

$$\prod_{i=1}^{d} \frac{\hat{h}_i^{n_i} - h_i^{n_i}}{\hat{h}_i - h_i}, \quad \hat{h}_i := \sum_{j \neq i} h_j. \tag{0.0.23}$$

With a little effort, one might verify that amazingly $c(n_1, n_2) = n_1$, thus recovering the matrix case. We explore further details of formula (0.0.23) and some other related formulas in Chapters 2 and 5.

The second problem is even more complicated than the first. Geometrically speaking, there exists a certain hypersurface of $V^{\mathbb{R}}$ which divides the space $V^{\mathbb{R}}$ into "chambers". In each of these regions, the number of real singular values, counted with multiplicity, is constant. In this thesis, we deal with the problem of counting the number of real critical points of the distance function from a real algebraic variety in relatively easier cases.

Next, we might discuss how to compute the ε-offset of the Segre product X_d, and of its corresponding dual affine cone $X_d^\vee \subset V$. In the case of matrices, the solution was quite straightforward thanks to the "characteristic polynomial trick" of equation (0.0.14).

For higher dimensional tensors, in general what we can do is replacing the characteristic polynomial in (0.0.14) with the unique generator (up to scalars) of the ideal

$$I_U \cap \mathbb{C}[U_{i_1,\ldots,i_d}, \sigma], \tag{0.0.24}$$

where

$$I_U \subset \mathbb{C}[x_{j,1}, \ldots, x_{j,n_j}, U_{i_1,\ldots,i_d}, \sigma] \tag{0.0.25}$$

is the ideal generated by the equations in (0.0.21) together with the conditions $q_j(x_j) = 1$ for all $j \in [d]$. The fact that the ideal in (0.0.24) is principal is explained in a more general setting in Section 4.1.

Example 0.0.7. Let us examine an example for $d = 3$ and $n_1 = n_2 = n_3 = 2$. For a tensor $U = (U_{ijk}) \in \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$, the singular vector triple system in (0.0.22) becomes

$$\begin{cases} U \cdot x_2 \otimes x_3 = \sigma x_1 \\ U \cdot x_1 \otimes x_3 = \sigma x_2 \\ U \cdot x_1 \otimes x_2 = \sigma x_3 \ , \end{cases}$$

together with the conditions $x_{j,1}^2 + x_{j,2}^2 = 1$ for all $j \in [3]$. By the formula (0.0.23), a general tensor U admits six critical decomposable tensors $\sigma(x_1 \otimes x_2 \otimes x_3)$. They are described by the ideal I_U in (0.0.25). Eliminating the variables $x_{j,1}, x_{j,2}$ ($1 \le j \le 3$) from I_U produces a principal ideal. Its unique generator, up to sign, has the form

$$f(U_{ijk}, \sigma^2) = \sum_{j=0}^{6} c_j(U) \sigma^{2j}, \tag{0.0.26}$$

where $c_j(U)$ is a homogeneous polynomial in the entries U_{ijk} of degree $20 - 2j$ for all $0 \le j \le 6$. It turns out that, for every nonnegative value $\sigma \ge 0$, the equation $f(U_{ijk}, \sigma^2) = 0$ defines the σ-offset of $X_3^\vee$. Again we might replace the variable σ^2 with the expression $q_F(U) - \varepsilon^2$. Then the equation $f(U_{ijk}, q_F(U) - \varepsilon^2) = 0$ defines the ε-offset of X_3, as we wanted.

Let us look more closely at the polynomial $f(U_{ijk}, \sigma^2)$. The first thing to observe is that $\deg_{\varepsilon^2}[f(U_{ijk}, \sigma^2)] = 6$, namely the expected number of singular values of U. In particular, the highest coefficient $c_6(U)$ has degree eight and is the product of the following four quadratic polynomials:

$$
\begin{aligned}
\theta_1(U) &= (U_{111} - U_{122} - U_{212} - U_{221})^2 + (U_{222} - U_{211} - U_{121} - U_{112})^2,\\
\theta_2(U) &= (U_{111} + U_{122} + U_{212} - U_{221})^2 + (U_{222} + U_{211} + U_{121} - U_{112})^2,\\
\theta_3(U) &= (U_{111} + U_{122} - U_{212} + U_{221})^2 + (U_{222} + U_{211} - U_{121} + U_{112})^2,\\
\theta_4(U) &= (U_{111} - U_{122} + U_{212} + U_{221})^2 + (U_{222} - U_{211} + U_{121} + U_{112})^2.
\end{aligned}
$$

Moreover, the variety $\mathcal{V}(\theta_1 \cdots \theta_4) \subset \mathbb{P}(V) \cong \mathbb{P}^7_{\mathbb{C}}$ coincides with the dual Segre embedding of the product $Q_1 \times Q_2 \times Q_3$ of the isotropic quadrics

$$Q_1 = Q_2 = Q_3 = \{[1, \sqrt{-1}], [1, -\sqrt{-1}]\} \subset \mathbb{P}^1_{\mathbb{C}}.$$

Geometrically speaking, $\mathrm{Seg}_3(Q_1 \times Q_2 \times Q_3) \subset \mathbb{P}(V)$ may be thought as the six vertex "cube" of decomposable tensors of the form $x_1 \otimes x_2 \otimes x_3$, where $x_j = (1, \pm\sqrt{-1})$ for all $j \in [3]$. Each polynomial $\theta_j(U)$ defines a pair of conjugate hyperplanes dual to a pair of conjugate vertices of $[\mathrm{Seg}_3(Q_1 \times Q_2 \times Q_3)]^\vee$.

This fact marks a first important difference between matrices and tensors: the existence of a positive degree leading coefficient in the equation of the ε-offset of $X_d^\vee$, or in other words, the non-integrality of the function $\delta_{F,U}$ over $X_d^\vee$. Instead, the lowest coefficient $c_0(U)$ can be written in the form

$$c_0(U) = \mathrm{Det}(U)^2 \cdot g(U). \tag{0.0.27}$$

On one hand, the polynomial $\mathrm{Det}(U)$ defines the dual Segre product $X_3^\vee$ and is called the *hyperdeterminant* of U, somehow generalizing the notion of determinant of a square matrix. Its development is

$$
\begin{aligned}
\mathrm{Det} = &\left[\det\begin{pmatrix} U_{111} & U_{122} \\ U_{211} & U_{222}\end{pmatrix} + \det\begin{pmatrix} U_{121} & U_{112} \\ U_{221} & U_{212}\end{pmatrix}\right]^2\\
&- 4\det\begin{pmatrix} U_{111} & U_{112} \\ U_{211} & U_{212}\end{pmatrix}\det\begin{pmatrix} U_{121} & U_{122} \\ U_{221} & U_{222}\end{pmatrix}.
\end{aligned}
\tag{0.0.28}
$$

On the other hand, the polynomial $g(U)$ is the product of the following three quartic forms:

$$
\begin{aligned}
g_1(U) &= \det\left[\left(U_1^{(1)} + \sqrt{-1}\,U_1^{(2)}\right)\left(U_1^{(1)} - \sqrt{-1}\,U_1^{(2)}\right)\right],\\
g_2(U) &= \det\left[\left(U_2^{(1)} + \sqrt{-1}\,U_2^{(2)}\right)\left(U_2^{(1)} - \sqrt{-1}\,U_2^{(2)}\right)\right],\\
g_3(U) &= \det\left[\left(U_3^{(1)} + \sqrt{-1}\,U_3^{(2)}\right)\left(U_3^{(1)} - \sqrt{-1}\,U_3^{(2)}\right)\right].
\end{aligned}
\tag{0.0.29}
$$

The three factors $g_1(U)$, $g_2(U)$ and $g_3(U)$ of $g(U)$ represent three quartic hypersurfaces in $\mathbb{P}^7_{\mathbb{C}}$. Each of them is the union of two conjugate quadric hypersurfaces.

In turn, the singular locus of each of these quadric hypersurfaces has dimension three and meets the Segre product X_3 in a quadric surface. Finally, these six quadric surfaces may be interpreted as the two dimensional "faces" of the cube $\mathrm{Seg}_3(Q_1 \times Q_2 \times Q_3)$ of totally isotropic rank-one tensors, as Figure 2 suggests.

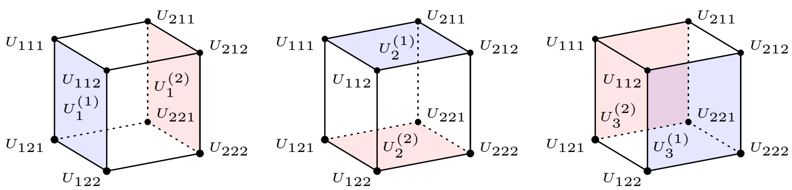

Figure 2: The slices $U_j^{(1)}$ and $U_j^{(2)}$ appearing in $g_j(U)$.

The results of Example 0.0.7 outline the core of this thesis. We investigate more on the "non-integrality" of the distance function $\delta_{F,U}$ for tensors, using the Frobenius inner product. In particular, we determine the zero loci of the lowest and the highest coefficient of the generator of the ideal in (0.0.24). Moreover, in the case of a "hypercube format" tensor, namely for $n_1 = \cdots = n_d = n$ for some $n \geq 1$, we compute all the exponents of the factors appearing in the above-mentioned coefficient. This leads to the following closed formula for the product of the singular values of a general tensor $U \in V$.

Theorem 0.0.8. *Let $n \geq 1$, $d \geq 2$, and $\underline{n} = (n, \ldots, n)$. For all subsets $J \subset [d]$, we define*

$$X_{d,J} \coloneqq \mathrm{Seg}_d(Y_1 \times \cdots \times Y_d) \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_d), \tag{0.0.30}$$

where $Y_j \coloneqq Q_j$ if $j \in J$ and $Y_j \coloneqq \mathbb{P}(V_j)$ otherwise. Moreover, we define $f_{d,J}$ to be the equation of $X_{d,J}^\vee$, when it is a hypersurface, otherwise $f_{d,J} \coloneqq 1$. For $J = \emptyset$ the polynomial $f_d \coloneqq f_{d,\emptyset}$ coincides with the hyperdeterminant *of a tensor.*

1. *(Theorem 5.0.5) Assume that the linear system $\mathcal{S}_d$ defined in Remark 5.3.19 has maximal rank. Pick a tensor $U \in V_1 \otimes \cdots \otimes V_d$. If U admits the maximum number $c(\underline{n})$ (see Theorem 0.0.6) of singular values, counted with multiplicity (hypothesis verified for a general U), their squared product has the following rational expression:*

$$(\sigma_1 \cdots \sigma_{c(\underline{n})})^2 = \prod_{j=0}^{d} g_j(U)^{2-j}, \quad g_j \coloneqq \prod_{|J|=j} f_{d,J} \quad \forall\ 0 \leq j \leq d. \tag{0.0.31}$$

2. *(Proposition 5.3.18) When $d \in \{2,3,4\}$, the system $\mathcal{S}_d$ has maximal rank and the product formula (0.0.31) is true for all $n \geq 2$:*

$$(\sigma_1 \cdots \sigma_{c(\underline{n})})^2 = \begin{cases} g_0(U)^2 = \det(U)^2 & \textit{for } d = 2 \\ \frac{g_0(U)^2 g_1(U)}{g_3(U)} & \textit{for } d = 3 \\ \frac{g_0(U)^2 g_1(U)}{g_3(U) g_4(U)^2} & \textit{for } d = 4\ . \end{cases}$$

3. *(Proposition 5.4.4) In the binary case $n = 2$, the linear system $\mathcal{S}_d$ has maximal rank $d-1$ and the formula (0.0.31) is true for any $d \geq 1$.*

The proof of Theorem 0.0.8 in the case $n = 2$ is the core of the single-authored paper [Sod]. When $n \geq 3$, so far the assumption on $\mathcal{S}_d$ has been checked for all positive integers n and d less than 100. The right-hand side of (0.0.31) should be interpreted as the ratio between the lowest and the highest coefficient of the analogous of the polynomial in (0.0.26), which for all $\sigma \geq 0$ describes the σ-offset of $X_d^\vee$. Note that the exponents on the right-hand side of (0.0.31) may be positive or negative so that the product of the singular values has a rational expression. When $d = 2$, formula (0.0.31) simply recovers the fact that the product of the singular values of a square matrix is equal to its determinant.

The distance from a real algebraic variety

Questions 1, 2 and 3 and all its related problems are all instances of a very general question:

Given a data point u and an algebraic variety $X^{\mathbb{R}}$ in a real n-dimensional Euclidean space $(V^{\mathbb{R}}, q^{\mathbb{R}})$, which is the closest point $x \in X^{\mathbb{R}}$ to u? How many critical points does $\delta_u^{\mathbb{R}}$ admit for a general data point u? What are the main properties of the ε-offset of $X^{\mathbb{R}}$ with respect to the metric $q^{\mathbb{R}}$?

This problem is very natural in Real Algebraic Geometry. In the first two sections, we concentrated on the distance from the varieties of matrices of rank at most k as well as tensors of rank one.

The road map to study the distance from a real algebraic variety is essentially the same that we took previously. The squared distance function in exam is again

$$\delta_u^{\mathbb{R}} \colon X^{\mathbb{R}} \to \mathbb{R}, \quad \delta_u^{\mathbb{R}}(x) := q^{\mathbb{R}}(u - x)$$

and the critical values of $\delta_u^{\mathbb{R}}$ are attained at the smooth points $x \in X^{\mathbb{R}}$ such that the gradient $\nabla q^{\mathbb{R}}(u-x)$ is orthogonal to the tangent space $T_x(X^{\mathbb{R}})$. For the standard Euclidean distance, $\nabla q^{\mathbb{R}}(u-x) = 2(u-x)$. Of course, even if the original problem is set over the reals, we need to consider the complex variety $X = X^{\mathbb{C}}$ and extend the quadratic form $q^{\mathbb{R}}$ to a complex-valued function $q\colon X \to \mathbb{C}$, which is not a Hermitian inner product. Algebraically speaking, critical points of δ_u are obtained requiring the rank of the augmented Jacobian matrix

$$\begin{pmatrix} \nabla q(u-x) \\ \mathrm{Jac}(f)(x) \end{pmatrix}$$

to be strictly lower than $c+1$, where c is the codimension of X and $(f) = (f_1,\dots,f_s)$ is the ideal of X. In other words, all minors of size $c+1$ of the above matrix must vanish simultaneously.

Clearly, we have to get rid of the singular points of X, where the previous condition is satisfied trivially. Algebraically, this is achieved by saturating the previously obtained ideal with respect to the ideal of the singular locus of X. We may call the resulting ideal as the *critical ideal* of X, and we denote it by $I_{\mathrm{crit}}(X)$.

As we show in detail in Chapter 4, for a general choice of data point $u \in V$, the ideal $I(X)$ in $\mathbb{C}[x_1,\dots,x_n]$ is zero-dimensional and consists of all (complex) critical points $x \in X$ of the function δ_u. Their number, namely the degree of $I_{\mathrm{crit}}(X)$, is constant on a Zariski open subset of V and is called the *Euclidean distance degree* of X. It is denoted by EDdegree(X). It was introduced by Draisma, Horobeţ, Ottaviani, Sturmfels and Thomas in [DHOST].

For example, the ED degree of the determinantal variety $\sigma_k(X)$ of matrices of rank at most k is $\binom{n_1}{k}$ by Theorem 0.0.3. Or the ED degree of the variety X_d of rank-one tensors is expressed by the Friedland-Ottaviani formula (0.0.23).

On the other hand, if we restrict to the real part $V^{\mathbb{R}}$, there exists a hypersurface, called *Euclidean Distance discriminant* (ED discriminant) of X and denoted by Σ_X, that divides the space $V^{\mathbb{R}}$ into chambers. If we let $u \in V^{\mathbb{R}}$ vary in each one of these chambers, the number of critical points of $\delta_u^{\mathbb{R}}$ is constant. The first example of this phenomenon is provided by the ED discriminant of the ellipse $X^{\mathbb{R}}\colon 4x_1^2 + x_2^2 - 1 = 0 \subset \mathbb{R}^2$:

$$\begin{aligned}\Sigma_X\colon 64x_1^6 &+ 48x_1^4x_2^2 + 12x_1^2x_2^4 + x_2^6 - 432x_1^4 \\ &+ 756x_1^2x_2^2 - 27x_2^4 + 972x_1^2 + 243x_2^2 - 729 = 0.\end{aligned}$$

For plane curves, the ED discriminant Σ_X is usually called the *evolute* of $X^{\mathbb{R}}$ and has the property that the tangent lines to Σ_X correspond to the normal lines to $X^{\mathbb{R}}$.

It turns out that the ideal

$$I_{\mathrm{crit}}(X) + (\varepsilon^2 - q(u-x)) \subset \mathbb{C}[x_1,\dots,x_n,u_1,\dots,u_n,\varepsilon]$$

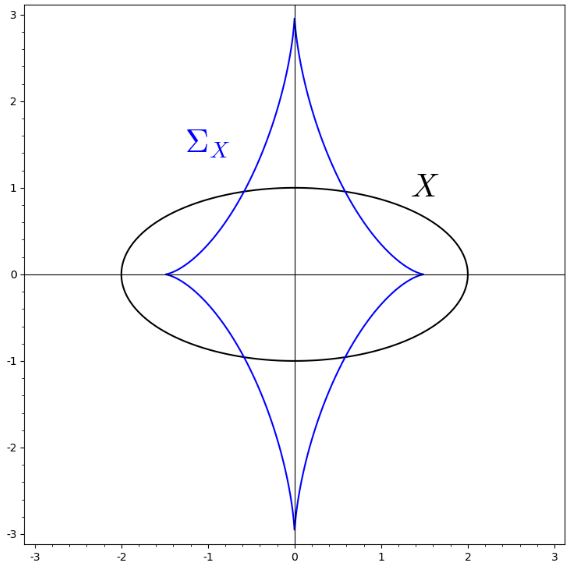

Figure 3: The evolute Σ_X of the ellipse $X^{\mathbb{R}}$ divides $\mathbb{R}^2$ into two chambers. If the data point u is picked in the internal chamber, then the number of critical points of $\delta_u^{\mathbb{R}}$ is four. Otherwise, it is two.

defines a variety of dimension n. Its projection (by elimination of the variables x_i, in a similar fashion of Example 0.0.7 where the x_i's correspond to the entries of the given tensor U) in $\mathbb{C}[u_1, \ldots, u_n, \varepsilon]$ is generated by a single polynomial in ε^2. In a joint work with Ottaviani [OS], we denote this generator (defined up to a scalar factor) by $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ and we call it *Euclidean Distance polynomial* (ED polynomial) of X at u (see Definition 4.1.3). For any $u \in V^{\mathbb{R}}$, $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ has among its roots the distance from u to $X^{\mathbb{R}}$. In addition, for any fixed real value $\varepsilon \geq 0$, the equation

$$\mathrm{EDpoly}_{X,u}(\varepsilon^2) = 0$$

defines the hypersurface of all data $u \in V^{\mathbb{R}}$ having distance ε from $X^{\mathbb{R}}$, namely the ε-offset of $X^{\mathbb{R}}$. The ε^2-degree of $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ coincides with the Euclidean Distance degree of X (see Theorem 4.2.2).

The ED polynomial of a real algebraic variety is undoubtedly the most important tool of this thesis. Indeed, a relevant part of this thesis is devoted to the study of its main properties. Here we briefly outline the main ones.

First, we show that the ED polynomial of an affine cone X and of its dual affine cone $X^\vee$ are linked by the following formula which enhances [DHOST, Theorem 5.2]:

Theorem (Theorem 4.2.8). *Let $X \subset V$ be an affine cone and $X^\vee$ its dual in V. Then*
$$\mathrm{EDpoly}_{X,u}(\varepsilon^2) = \mathrm{EDpoly}_{X^\vee,u}(q(u) - \varepsilon^2).$$

The last identity was essentially applied in (0.0.15) and for rank-one tensors, and its meaning is that projective duality corresponds to variable reflection for the ED polynomial.

Next, we focus on the extreme coefficients of $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$. Here, transversality between X and the isotropic quadric Q plays a crucial role. In particular, it is sufficient to prove that the highest coefficient of the ED polynomial is a scalar, as the following result states.

Theorem (Corollary 4.3.7). *Let $X \subset \mathbb{P}(V)$ be a projective variety. If X is transversal to Q, according to Definition 4.3.3 then for any data point $u \in V$*
$$\mathrm{EDpoly}_{X,u}(\varepsilon^2) = \sum_{j=0}^{d} p_j(u)\varepsilon^{2j},$$
where $d = \mathrm{EDdegree}(X)$ and $p_j(u)$ is a homogeneous polynomial in the coordinates of u of degree $2d-2j$. In particular, $p_d(u) = p_d \in \mathbb{C}$, $\deg(p_0) = 2d$ and the ED polynomial of X is an integral algebraic function.

In the other direction, the lowest coefficient of the ED polynomial describes the data points $u \in V$ having "distance zero" from X. If we restrict to the real points, we essentially recover the variety $X^{\mathbb{R}}$, plus another possible locus of real data points. But if we allow non-real solutions, there is always a hypersurface of isotropic vectors having "distance zero" from X.

Theorem (Theorem 4.4.6). *Let $X \subset V$ be an affine cone such that $X \cup X^\vee \not\subset Q$. Then the locus of zeros $u \in V$ of $\mathrm{EDpoly}_{X,u}(0)$ is*
$$X \cup (X^\vee \cap Q)^\vee.$$
In particular, at least one between X and $(X^\vee \cap Q)^\vee$ is a hypersurface.

The last result describes set-theoretically the vanishing locus of the lowest coefficient $\mathrm{EDpoly}_{X,u}(0)$. With a stronger assumption, we can determine also "scheme-theoretically" the condition for a data point $u \in V$ to have distance zero from X. In other words, we give a more precise description of the lowest term of the ED polynomial, with reasonable transversality assumptions.

Theorem (Theorem 4.4.12). *Let $X \subset \mathbb{P}(V)$ be an irreducible variety and suppose that X and $X^\vee$ are transversal to Q. Let $u \in V$ be a data point.*

1. *If* $\operatorname{codim}(X) \geq 2$, *then* $(X^\vee \cap Q)^\vee$ *is a hypersurface and*

$$\mathrm{EDpoly}_{X,u}(0) = g$$

up to a scalar factor, where g *is the equation of* $(X^\vee \cap Q)^\vee$. *Moreover* $X \subset (X^\vee \cap Q)^\vee$.

2. *If* X *is a hypersurface, then*

$$\mathrm{EDpoly}_{X,u}(0) = f^2 g$$

up to a scalar factor, where f *is the equation of* X *and* g *is either the constant* 1 *if* X *is a hyperplane, or the equation of* $(X^\vee \cap Q)^\vee$.

Structure of the thesis

In this introduction, we have chosen to regard the best rank-k approximation problem only for nonsymmetric tensors. Actually, in this thesis we consider tensors with partial symmetry as well, starting from *symmetric tensors*.

After the preliminary Chapter 1, in Chapter 2 we introduce the basic tools of Spectral Theory of symmetric tensors, which is useful for the problem of symmetric tensor approximation. On one hand, we recall the definition of E-eigenvector and E-eigenvalue of a symmetric tensor. On the other hand, we introduce the notion of *E-characteristic polynomial* of a symmetric tensor, which is defined as the multivariate resultant of a certain homogeneous polynomial system. The rest of the chapter is devoted to the study of the extreme coefficients of the E-characteristic polynomial of a symmetric tensor. The results of this chapter are based on the single-authored paper [Sod18]. In particular, the main result is Theorem 2.0.2, which provides a rational formula for the product of the E-eigenvalues of a symmetric tensor. This formula is generalized later in Theorem 5.0.5. In the last part of the chapter, we investigate binary symmetric tensors, called also *binary forms*, and we derive a formula for the E-characteristic polynomial of a *harmonic* binary form, which points out that the E-eigenvalues of a real harmonic binary form are all real.

Nevertheless, we want to separate the study of symmetric tensors from the general case of partially symmetric tensors because the proof of Theorem 2.0.2 uses resultant theory, which we do not consider for the rest of the thesis.

In Chapter 3 we remain in the context of symmetric tensors, and we focus on *harmonic symmetric tensors*. The main results in this direction are Proposition 3.0.3 and Theorem 3.0.4, which describe completely all the elements of the space $H^d\mathbb{C}^3$ of harmonic ternary forms of degree d which are stable or semistable under the action of the complex orthogonal group $\mathrm{SO}(3,\mathbb{C})$. Thanks to the $\mathrm{SO}(3,\mathbb{C})$-equivariant harmonic decomposition of the space $S^d\mathbb{C}^3$, these results allow us to give sufficient conditions of stability and semistability in the whole space $S^d\mathbb{C}^3$. Afterwards, we produce an example of a real harmonic ternary form with non-real E-eigenvalues.

In Chapter 4 we change our perspective. This chapter intends to frame the previous ones in the general setting of *Real Algebraic Geometry of Data* as well as setting up the necessary tools used in the final Chapter 5. After stating the problem of computing the distance from a real algebraic variety, the core of this chapter is devoted to the main properties of the *ED polynomial* of a real algebraic variety. The main results are based on the joint paper with Ottaviani [OS].

Finally, we come back to tensors again. The main results of this thesis are collected in Chapter 5. There we study the ED polynomial of a Segre-Veronese variety, whose roots are the *singular values* of a partially symmetric tensor. First, we describe completely the vanishing loci of the highest and lowest coefficient of the ED polynomial of such a variety. Second, when restricting to tensors of hypercube format, we compute the multiplicities of the factors appearing in the lowest and highest coefficients. The main result is the above-cited Theorem 5.0.5. Part of this chapter achieves the single-authored paper [Sod], where the product formula is proved for binary tensors.

Chapter 1
Preliminaries

This chapter might be viewed as a toolkit for the development of the thesis. In what follows, we aim at giving a compact exposition of the main definitions and already known results that will be applied in the next chapters.

In the first part, we set up the notation and stress the notion of orthogonality that we adopt. Then we recall the definition of the most important invariant for this thesis, the *Euclidean Distance degree* of an algebraic variety X in any complexified Euclidean space (V, q), starting from irreducible affine varieties.

When X is an affine cone, we may read it projectively. In this context, we are allowed to introduce a notion of *dual affine cone* $X^\vee$ of X. Armed with this definition, we recall a result in [DHOST] which states that the ED degrees of a projective variety and its dual variety coincide.

If additionally, the variety considered is transversal to the isotropic quadric Q associated with the quadratic form q, then its ED degree is expressed as the sum of the *polar classes* of the variety. For this reason and for their applications in Chapters 2, 4 and 5, we dedicate Section 1.5 to the construction of the polar classes of a projective variety.

Also, in Section 1.7 we furnish a basic introduction to the *Chern classes* of a projective variety: indeed, Chern classes may be computed via polar classes, and vice-versa. Moreover, the ED degree formula in Theorem 5.1.1, stated in the nonsymmetric case in Theorem 0.0.6, is derived basically from the computation of the top Chern class of a certain vector bundle over a Segre-Veronese product of projective spaces.

1.1 Orthogonality vs polarity in Euclidean space

As suggested in the introduction, up to different notation, our ambient space is a real n-dimensional Euclidean space $(V^{\mathbb{R}}, q^{\mathbb{R}})$. In most cases, we suppose that

$x_1, \ldots, x_n$ is a system of coordinates of $V^{\mathbb{R}}$. Moreover, the quadratic form $q^{\mathbb{R}}$ has the expression $q^{\mathbb{R}}(x) = x^T A x$ for some positive definite real $n \times n$ matrix $A = A(q^{\mathbb{R}})$. Up to a linear change of coordinates, we assume that A is the identity matrix I_n, namely $q^{\mathbb{R}}(x) = \sum_{i=1}^n x_i^2$. By abuse of notation, we denote with the same letter the quadratic form $q^{\mathbb{R}}$ and its associated bilinear form, in other words we have $q^{\mathbb{R}}(x,x) = q^{\mathbb{R}}(x)$.

Finally, we define $V := V^{\mathbb{R}} \otimes \mathbb{C}$ and the function $q\colon V \to \mathbb{C}$ by $q(x) := x^T A x$, where A is the symmetric matrix associated to the quadratic form $q^{\mathbb{R}}$. In particular, q is not induced by a Hermitian inner product on V. The wider space V admits nonzero vectors x such that $q(x) = 0$, called *isotropic* vectors. The quadratic equation $q(x) = 0$ defines the *isotropic quadric* $Q \subset V$, whose only real point is the origin. The quadric Q has no real points in the projective setting.

Now let us look at V as an affine chart of $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}(V \oplus \mathbb{C}) = V \cup H_\infty$, where $H_\infty \cong \mathbb{P}(V)$ is the *hyperplane at infinity* of V. In coordinates, we introduce a new variable x_0 and hence H_∞ has equation $x_0 = 0$. Every vector $x = (x_1, \ldots, x_n) \in V$ corresponds to the point $[(1,x)] = [(1, x_1, \ldots, x_n)] \in \mathbb{P}^n_{\mathbb{C}}$. In the following, we indicate by $\langle x \rangle$ the point $[(0,x)] \in H_\infty$. For each affine variety $X \subset V$, we denote by $\overline{X}$ the closure of X in the Zariski topology of $\mathbb{P}^n_{\mathbb{C}}$. Moreover, we define $X_\infty := \overline{X} \cap H_\infty$. In particular, the points at infinity of Q fill the smooth quadric $Q_\infty \subset H_\infty$.

Summing up, from the lines above we see that every choice of a quadratic form $q^{\mathbb{R}}$ in $V^{\mathbb{R}}$ produces a unique projective quadric $Q_\infty = \mathcal{V}(q) \subset H_\infty$. This approach is very useful for understanding better the notion of orthogonality in V. Indeed, orthogonality in V corresponds essentially to polarity in H_∞, with respect to the definition below (see [Pie15, §4]).

Definition 1.1.1. Let Q_∞ be a smooth quadric in H_∞. Then it induces a *polarity*, classically called a *reciprocity*, between points and hyperplanes in H_∞. The polar hyperplane $P^\perp$ of a point $P \in H_\infty$ is the linear span of the points on Q_∞ such that the tangent hyperplane to Q at that point contains P. In coordinates, $q = q(x_1, \ldots, x_n)$ induces an isomorphism

$$\begin{array}{cccc} \varphi_q\colon & H_\infty & \to & (H_\infty)^* = \{\text{hyperplanes of } H_\infty\} \\ & P = [p_1, \ldots, p_n] & \mapsto & P^\perp\colon \sum_{i=1}^n p_i \frac{\partial q}{\partial x_i} = 0 \end{array}$$

If $W \subset H_\infty$ is a subspace of dimension r, then we can associate to W the dual space of $\varphi_q(W)$, which is a subspace of H_∞ of dimension $n - r - 1$. We denote this subspace by $W^\perp$.

Therefore, given an affine space $L \subset V$ and a point $u \in V$, the orthogonal space to L passing through u is constructed as follows: we consider $L_\infty \subset H_\infty$

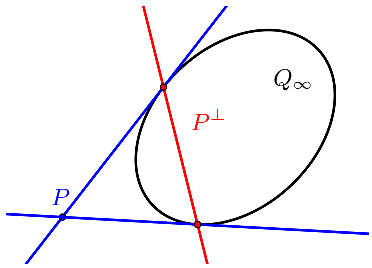

Figure 1.1: A point $P \in H_\infty$ and its polar hyperplane $P^\perp$.

and we associate to it the subspace $(L_\infty)^\perp$ via the polarity φ_q of Definition 1.1.1. Hence the desired orthogonal space is

$$u + L^\perp := \langle [(u,1)], (L_\infty)^\perp \rangle \cap V.$$

1.2 The ED degree of an algebraic variety

Now we have all the necessary metric information to proceed. We fix a radical ideal $I(X^{\mathbb{R}}) := \langle f_1, \ldots, f_s \rangle \subset \mathbb{R}[x_1, \ldots, x_n]$ and we let $X^{\mathbb{R}} \subset V^{\mathbb{R}}$ be the real zero locus of $I(X^{\mathbb{R}})$. In other words, $X^{\mathbb{R}}$ is a real algebraic variety in $V^{\mathbb{R}}$. We denote by $X \subset V$ the complex zero locus of $I(X) := \langle f_1, \ldots, f_s \rangle \subset \mathbb{C}[x_1, \ldots, x_n]$. We indicate by $\mathrm{Jac}(f) = \mathrm{Jac}(f_1, \ldots, f_s)$ the $s \times n$ Jacobian matrix, whose entry in row i and column j is the partial derivative $\partial f_i / \partial x_j$. The ideal of the singular locus X_{sing} of X is defined by

$$I(X_{\mathrm{sing}}) := I(X) + \langle c \times c \text{ minors of } \mathrm{Jac}(f) \rangle,$$

where c is the codimension of X.

Our aim is to pick a data point $u \in V$ (usually in $V^{\mathbb{R}}$) and look for the critical points $x \in X$ of the squared distance function $\delta_u \colon X \to \mathbb{C}$. They are attained at smooth points $x \in X$ such that the vector $u - x$ is in the normal space $N_x X := (T_x X)^\perp$, according to Definition 1.1.1. We stress that both $N_x X$ and $T_x X$ are linear spaces in V. The corresponding affine spaces passing through x are denoted by $x + N_x X$ and $x + T_x X$, respectively.

Algebraically speaking, $x \in X_{\mathrm{sm}}$ is a critical point of δ_u if the gradient ∇q at $u - x$ lives in the row space of the jacobian matrix $\mathrm{Jac}(f)$ of X at x. When q is the standard Euclidean metric, $\nabla q(u - x) = 2(u - x)$. An indicative example is depicted in Figure 1.2. This constructions suggests to define an ideal, called the *critical ideal* of X, as (see the formula (2.1) [DHOST])

$$I_{\mathrm{crit}}(X) := \left(I(X) + \left\langle (c+1)\times(c+1) - \text{minors of } \begin{pmatrix} \nabla q(u-x) \\ \mathrm{Jac}(f)(x) \end{pmatrix} \right\rangle\right) : I(X_{\mathrm{sing}})^{\infty}. \tag{1.2.1}$$

The critical ideal $I_{\mathrm{crit}}(X)$ lives in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_n]$ and defines an affine variety in $V \times V$. The geometrical idea behind this property is that we are letting the data point u vary in V, hence its coordinate should be treated as indeterminates independent from the x_i's.

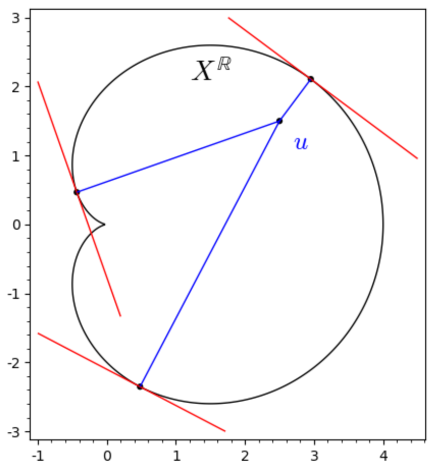

Figure 1.2: The cardioid $X^{\mathbb{R}}$: $(x_1^2+x_2^2-2x_1)^2-4(x_1^2+x_2^2)=0$ and a point $u\in\mathbb{R}^2$ admitting three normal lines to $X^{\mathbb{R}}$.

Definition 1.2.1. The variety in $V \times V$ defined by the critical ideal $I_{\mathrm{crit}}(X)$ is called *ED correspondence* of X and is denoted by $\mathcal{E}(X)$.

More precisely, the ED correspondence may be described as

$$\mathcal{E}(X) = \overline{\{(x,u) \in X \times V \mid x \in X_{\mathrm{sm}},\ x \text{ critical point of } \delta_u\}}.$$

Now consider the diagram below:

$$\begin{tikzcd} & \mathcal{E}(X) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & V \end{tikzcd} \tag{1.2.2}$$

The two maps π_1 and π_2 are the projections onto the two factors of $X \times V$.

Theorem 1.2.2. *[DHOST, Theorem 4.1] The ED correspondence $\mathcal{E}(X)$ of an irreducible variety $X \subset V$ of codimension c is an irreducible variety of dimension n inside $V \times V$. The projection π_1 is an affine vector bundle of rank c over X_{sm}. Over general data points $u \in V$, the second projection π_2 has finite fibers $\pi_2^{-1}(u)$ of constant cardinality.*

Given a data point $u \in V$, we have that

$$\pi_2^{-1}(u) \cong \{x \in X_{\mathrm{sm}} \mid x \text{ critical point of } \delta_u\},$$

hence the cardinality $|\pi_2^{-1}(u)|$ counts the number of critical points of δ_u.

Definition 1.2.3. Let $X \subset V$ be an irreducible variety. The cardinality of the general fiber of π_2 is called *ED degree* of X and is denoted by $\mathrm{EDdegree}(X)$.

Remark 1.2.4. On one hand, the singular points of X are not taken into account when defining the critical ideal of X, namely the invariant $\mathrm{EDdegree}(X)$ counts only smooth local minima, maxima and saddle points on X of the squared distance function δ_u. Indeed, the squared distance function $\delta_u \colon X \to \mathbb{C}$ cannot be differentiated at the singular points of X.

On the other hand, the singular points of X might be local minima, maxima or saddle points of δ_u. For example, let X be the cuspidal cubic of equation $x_1^3 - x_2^2 = 0$. It turns out that the cusp $(0,0)$ is a global minimum of δ_u for any data point u contained in the light blue region showed in Figure 1.3. This region is known as the *Voronoi cell* of X at $(0,0)$. Its boundary is the quartic curve

$$\partial\mathrm{Vor}_{(0,0)}X \colon 27y^4 + 128x^3 + 72xy^2 + 32x^2 + y^2 + 2x = 0.$$

More generally, the Voronoi cell of X at the point $x \in X$ is the set of all $u \in V$ such that x is the closest point to u on X. A detailed study of the Voronoi cells of algebraic varieties and of their algebraic boundaries is done in [CRSW].

By Theorem 1.2.2, the ED degree of an algebraic variety is, in fact, an algebraic invariant of X, and somehow measures the algebraic complexity of computing the distance from X. Note that we started by considering a real affine variety $X^{\mathbb{R}}$, but the notion of ED degree is related to complex algebraic varieties. In particular, we can compute the ED degree of the isotropic quadric Q.

Proposition 1.2.5. *Let $Q \subset V$ be the isotropic quadric defined by the quadratic form q. Then* $\mathrm{EDdegree}(Q) = 0$.

Proof. By construction, $\mathrm{EDdegree}(Q)$ counts the number of critical points of δ_u on Q for a general point $u \in V$, or rather, the number of normal spaces to Q that can be drawn from u.

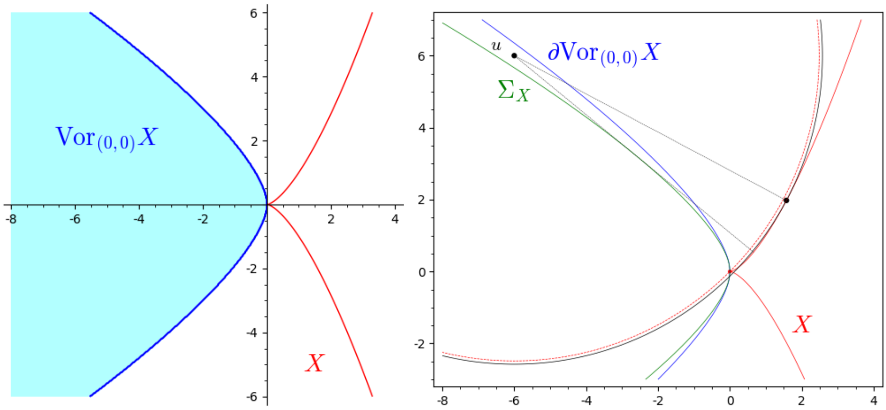

Figure 1.3: The Voronoi cell $\mathrm{Vor}_{(0,0)}X$ of the cuspidal cubic $X\colon x_1^3 - x_2^2 = 0$ at the origin. As shown on the right, the real locus of $\mathrm{Vor}_{(0,0)}X$ contains the real locus of the evolute Σ_X of X. If we draw a circumference centered at a point u between the green and the blue curve, we see that the origin is closer to u than any critical point of $\delta_u^{\mathbb{R}}$.

Note that, if $x \in Q$ and $x \neq 0$, then x is a smooth point of Q and T_xQ is the subspace of vectors $v \in V$ such that $q(x,v) = 0$. At infinity, we have that $(T_xQ)_\infty$ coincides with $T_{\langle x\rangle}(Q_\infty) \subset H_\infty$. By Definition 1.1.1, the polar subspace to $T_{\langle x\rangle}(Q_\infty)$ is the point $\langle x\rangle$. This means that, moving back in V, the normal space N_xQ is the line spanned by the vector x. In particular, $N_xQ \subset T_xQ$ or, affinely, $x + N_xQ \subset x + T_xQ$.

Summing up, it is immediate that, if x is a critical point of δ_u, then necessarily the vector $u - x$ must be proportional to x. In particular, $u - x \in Q$, hence $u \in Q$. This contradicts the fact that u is general in V. From this we conclude that $\mathrm{EDdegree}(Q) = 0$. $\square$

The following lemma is an immediate property coming from the definition of ED degree.

Lemma 1.2.6. *Let $X \subset V$ be an irreducible affine variety and let $v \in V$. Then* $\mathrm{EDdegree}(v + X) = \mathrm{EDdegree}(X)$*, namely the ED degree is invariant under translation in V.*

The simplest affine varieties to consider are the affine subspaces of V. We compute their ED degree in the following proposition.

Proposition 1.2.7. *Let $L \subset V$ be an affine subspace. If L is transversal to Q out of the origin, then* $\mathrm{EDdegree}(L) = 1$. *Otherwise* $\mathrm{EDdegree}(L) = 0$.

Proof. Applying Lemma 1.2.6, we can assume that $L \subset V$ is a linear space.

Assume first that L is not transversal to Q away from the origin. Then there exists $y \in Q$, $y \neq 0$ such that the linear span between T_yQ and L does not generate V, namely $L \subset T_yQ$. Going at infinity, we have the inclusion $L_\infty \subset (T_yQ)_\infty = T_{\langle y\rangle}(Q_\infty)$.

Consider the linear system $\mathcal{L}$ of hyperplanes in H_∞ supported by L_∞. For each element $W \in \mathcal{L}$, $W^\perp$ is a point in $T_{\langle y\rangle}(Q_\infty)$. The union of all these points generates the polar subspace $(L_\infty)^\perp$, which is contained in $T_{\langle y\rangle}(Q_\infty)$. Going back to V, from the above argument we conclude that the orthogonal space $L^\perp$ to L must be contained in T_yQ as L itself.

Now pick a general point $u \in V$. If there exists a critical point $x \in L$ for δ_u, then $u - x \in L^\perp \subset T_yQ$. In particular, $q(y, u - x) = 0$. Since $x \in L \subset T_yQ$, $q(y, x) = 0$ as well. Therefore $u \in T_yQ$, but this contradicts the generality assumption on u. The contradiction comes from the assumption that there exists such a critical point $x \in L$. From this we conclude that $\mathrm{EDdegree}(L) = 0$.

Otherwise L is transversal to Q. In this case, the orthogonal space $L^\perp$ is such that $V = L \oplus L^\perp$. In particular, $L^\perp$ defines the orthogonal projection $\pi_{L^\perp}\colon V \to L$ onto L. For each data point $u \in V$, the point $\pi_{L^\perp}(u)$ is the unique critical point for δ_u on L. In particular, $\mathrm{EDdegree}(L) = 1$. ☐

Example 1.2.8. In the vector space $\mathbb{C}^2$ equipped with the Euclidean quadratic form $q = x_1^2 + x_2^2$, we let L be the line of equation $x_1 + \sqrt{-1}\,x_2 = 0$. In particular, L is generated by the vector $(1, \sqrt{-1})$ and $L^\perp = L$. Consider a point $u = (u_1, u_2) \in \mathbb{C}^2$ and assume that there exist $x \in L$, $x = \lambda(1, \sqrt{-1})$, such that $u - x$ is orthogonal to $(1, \sqrt{-1})$. The equation $q((u_1 - \lambda, u_2 - \sqrt{-1}), (1, \sqrt{-1})) = 0$ simplifies to $u_1 + \sqrt{-1}\,u_2 = 0$, namely $u \in L^\perp$. This implies that $\mathrm{EDdegree}(L) = 0$. The result is the one expected since L is contained in the isotropic quadric $Q \subset \mathbb{C}^2$.

Remark 1.2.9. If $L \subset V$ is the complexification of a real affine space $L^{\mathbb{R}} \subset V^{\mathbb{R}}$ such that $L^{\mathbb{R}} \neq \{0\}$, then automatically L is transversal to Q out of the origin. In particular, $\mathrm{EDdegree}(L) = 1$ by Proposition 1.2.7.

In the following, we always consider an affine variety X not contained in the isotropic quadric Q. This is related to the following fact, which is the second part of [DHOST, Theorem 4.1].

Proposition 1.2.10. *Given an irreducible variety $X \subset V$, if $T_xX \cap N_xX = \{0\}$ at some point $x \in X_{\mathrm{sm}}$, then π_2 is a dominant map and* $\mathrm{EDdegree}(X)$ *is positive. In particular, if X admits at least one real smooth point, then* $\mathrm{EDdegree}(X) > 0$.

Example 1.2.11. In the following `M2` code [GS], the ED degree of a plane curve in $X \subset \mathbb{C}^2$ is displayed by the command `degree IsatX`. When X is the cardioid whose real points are depicted in Figure 1.2, one may verify with the code below that $\mathrm{EDdegree}(X) = 3$. Note that all the three critical points in Figure 1.2 are real. We show in Figure 1.4 that some of the critical points might be non-real depending on the data point chosen.

```
R = QQ[x_1,x_2,u_1,u_2];
IX = ideal((x_1^2+x_2^2-2*x_1)^2-4*(x_1^2+x_2^2));
ISingX = IX+minors(codim IX,compress transpose jacobian IX);
jacX = matrix{{u_1-x_1,u_2-x_2}}||(compress transpose jacobian IX);
IcritX = minors((codim IX)+1, jacX);
IsatX = saturate(IX+J,ISingX);
codim IsatX, degree IsatX
```

Let us look again at the projection π_2 in (1.2.2). The branch locus of π_2 corresponds to the closure of the set of data points $u \in V$ for which there are fewer than $\mathrm{EDdegree}(X)$ complex critical points of δ_u. This fact leads to the following definition.

Definition 1.2.12. The branch locus of π_2 defines an affine variety $\Sigma_X \subset V$, which we call the *ED discriminant* of X and we indicate with Σ_X.

In most cases, the ED discriminant is a hypersurface of V, by the Nagata-Zariski Purity Theorem (see [Zar, Nag]). The reason for this name appears when restricting to the real space $V^{\mathbb{R}}$. In this case, the real part of the ED discriminant $\Sigma_{X^{\mathbb{R}}} := (\Sigma_X)^{\mathbb{R}}$ divides the vector space $V^{\mathbb{R}}$ into chambers. If we let the real data point u vary in one of these chambers, then the number of real critical points of $\delta_u^{\mathbb{R}}$ is constant. In particular, when u is approaching $\Sigma_{X^{\mathbb{R}}}$, either two real critical points of $\delta_u^{\mathbb{R}}$ or two non-real conjugate critical points of δ_u are collapsing together. This is visualized in Figure 1.4 in the case of the ED discriminant of the ellipse $X^{\mathbb{R}}\colon x_1^2 + 4x_2^2 - 4 = 0$. Or, looking back at Remark 1.2.4, the ED discriminant of the cuspidal cubic $X\colon x_1^3 - x_2^2 = 0$ is the quartic curve

$$\Sigma_X\colon 6561y^4 + 18432x^3 + 15552xy^2 + 6144x^2 + 288y^2 + 512x = 0$$

which separates the region of real data points providing two real critical points (to the right of Σ_X) from the region of real data points providing no real critical points (to the left of Σ_X) (see the right image of Figure 1.3).

In the next section, we focus on affine cones on V, namely projective varieties $X \subset \mathbb{P}(V)$. As a quick preview, we mention a formula by Trifogli for the degree of Σ_X in a special case.

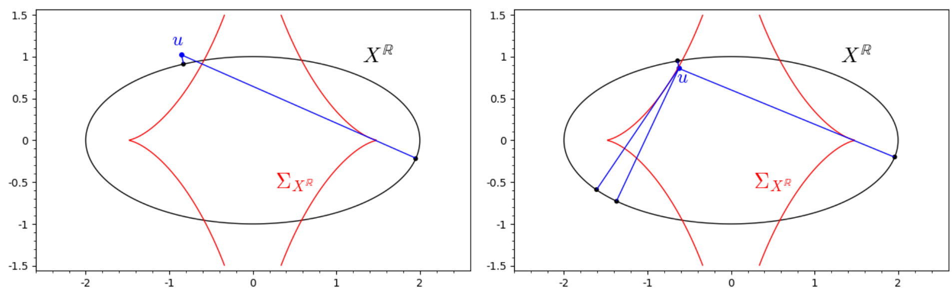

Figure 1.4: On the left, a data point u admitting two critical points for $\delta_u^{\mathbb{R}}$, while the other two are complex-conjugate. On the right, u goes in the internal region of the evolute $\Sigma_{X^{\mathbb{R}}}$ and acquires two more critical points for $\delta_u^{\mathbb{R}}$.

Theorem 1.2.13 (Trifogli)**.** *If X is a general hypersurface of degree d in $\mathbb{P}(V)$ then*

$$\deg(\Sigma_X) = d(n-1)(d-1)^{n-1} + 2d(d-1)\frac{(d-1)^{n-1}-1}{d-2}.$$

For instance, when $n = 2$ the above formula tells us that the degree of the ED discriminant of a general projective plane curve C of degree d is

$$\deg(\Sigma_C) = 3d(d-1).$$

The ED discriminant $\Sigma_{C^{\mathbb{R}}}$ of a real affine plane curve $C^{\mathbb{R}} \subset \mathbb{R}^2$ coincides with the curve classically known as the *evolute* of $C^{\mathbb{R}}$. To see this, we recall that the evolute of $C^{\mathbb{R}}$ is by definition the envelope of the normal lines to $C^{\mathbb{R}}$. More generally, the evolute is an instance of what is called an *envelope* of a family of curves in $\mathbb{R}^2$: in our case, the family of all normal lines to $C^{\mathbb{R}}$. By construction, the envelope is tangent to each element of the family at some point. More specifically, the evolute $\Sigma_{C^{\mathbb{R}}}$ is tangent to each normal line to $C^{\mathbb{R}}$ at some point. In other words, if we pick a point $u \in \mathbb{R}^2$ and we want to draw all normal lines to $C^{\mathbb{R}}$ passing through u, then it is equivalent to drawing all tangent lines to $\Sigma_{C^{\mathbb{R}}}$ passing through u. Since the number of normal lines exiting from u is equal to the number of real critical points of $\delta_u^{\mathbb{R}}$, and two normal lines collapse together when $u \in \Sigma_{C^{\mathbb{R}}}$, then the evolute $\Sigma_{C^{\mathbb{R}}}$ coincides with the real part of the ED discriminant of C.

Keeping in mind the above description, it is straightforward to notice that if two critical points of $\delta_u^{\mathbb{R}}$ coincide, then their corresponding normal lines to $C^{\mathbb{R}}$ coincide as well. The converse is not true since there exist distinct pairs of critical points (x_1, x_2) of $\delta_u^{\mathbb{R}}$ such that their associated normal lines to $C^{\mathbb{R}}$ coincide. This happens when the curve $C^{\mathbb{R}}$ has a *bottleneck* between the two points x_1 and x_2.

More generally, a bottleneck of a smooth algebraic variety $X \subset V$ is a pair of distinct points $(x_1, x_2) \in X$ such that the normal spaces at x_1 and x_2 contain the line spanned by x and y. As the previous definition suggests, the determination of all bottlenecks of a variety X is closely related to the computation of the ED degree of X. It turns out that the number of (complex) bottlenecks of a variety X is, in fact, another invariant of X and is called the *bottleneck degree* of X. In practice, the bottleneck degree of X measures the complexity of computing all bottlenecks of X. In this thesis, we do not enter into the details of this important invariant. An excellent reference for the bottlenecks and the bottleneck degree of an algebraic variety is [DEW].

Remark 1.2.14. The only property that we want to observe and that descends almost immediately from the definition of bottleneck of a plane curve is the following: given a curve $C^{\mathbb{R}} \subset \mathbb{R}^2$, a pair $(x_1, x_2) \in C^{\mathbb{R}}$ is a bottleneck of $C^{\mathbb{R}}$ if and only if the common normal line joining x_1 and x_2 is bitangent to the envelope $\Sigma_{C^{\mathbb{R}}}$. Figure 1.5 confirms this fact when $C^{\mathbb{R}}$ is the lemniscata of equation

$$(x_1^2 + x_2^2)^2 - 2(x_1^2 - x_2^2) = 0$$

and its evolute is the sextic curve

$$\begin{aligned}\Sigma_{C^{\mathbb{R}}} : 729x_1^6 + 729x_1^4x_2^2 - 729x_1^2x_2^4 - 729x_2^6 - 1944x_1^4 + \\ + 1944x_1^2x_2^2 - 1944x_2^4 + 1728x_1^2 - 1728x_2^2 - 512 = 0.\end{aligned}$$

1.3 ED degree for affine cones and duality

In this section, X represents an affine cone (through the origin) in V, meaning that if $x \in X$, then $\lambda x \in X$ for all $\lambda \in \mathbb{C}$. As we anticipated in the introduction, for affine cones we consider the following notion of duality.

Definition 1.3.1. Let X be an irreducible affine cone in a Euclidean space (V, q). The *dual affine cone* of X is

$$X^\vee \coloneqq \overline{\bigcup_{x \in X_{\mathrm{sm}}} N_x X} \subset V,$$

where $N_x X \coloneqq (T_x X)^\perp$ is the normal space at $x \in X$ defined via the polarity map φ_q of Definition 1.1.1. If X is a non-reduced variety, then $X^\vee$ is the empty set.

The last relation requires a few clarifications. Given a real Euclidean space $(V^{\mathbb{R}}, q^{\mathbb{R}})$, the inner product $q^{\mathbb{R}}$ induces a natural isomorphism between $V^{\mathbb{R}}$ and the

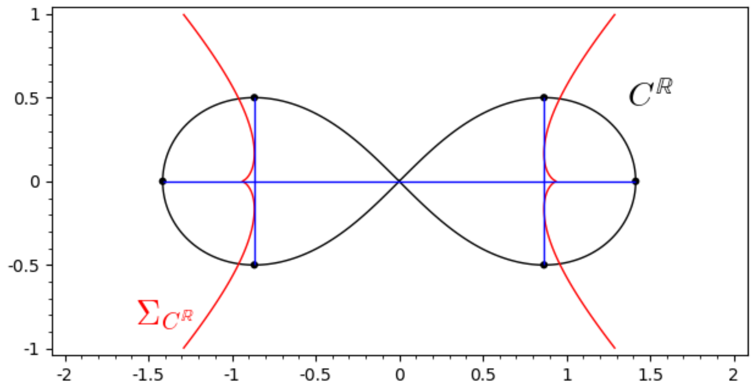

Figure 1.5: For the lemniscata $C \subset \mathbb{C}^2$ whose real part is depicted above, we get that there are six real bottlenecks, which correspond to three pairs of real points with coinciding normal lines. This number has been computed with a `M2` code inspired by [DEW, §1.1], where we have removed the singular locus of C.

dual vector space $V^{\mathbb{R}^*} := \{l \colon V^{\mathbb{R}} \to \mathbb{R} \mid l \text{ is linear}\}$, which takes a vector $v \in V^{\mathbb{R}}$ and associates it to the linear map $w \mapsto q(v, w)$. This isomorphism is natural, namely, it does not depend on the orthonormal basis of $V^{\mathbb{R}}$ chosen. Therefore, in the following, we always identify any vector space with its dual.

We remark that Definition 1.3.1 is equivalent to the standard definition of the dual of a projective variety as in [Tev, Definition 1.1]. The variety $X^\vee$ is an irreducible affine cone as well. Note that if X is a linear subspace, then $X^\vee = X^\perp$, the orthogonal subspace with respect to q.

Following the assumptions on X of the preceding section, we always consider affine cones X such their respective dual affine cones $X^\vee$ are not contained in the isotropic quadric Q.

A slightly different approach to define the dual affine cone of X comes with the following definition.

Definition 1.3.2. Let $X \subset V$ be an irreducible affine cone. The *conormal variety* of X is the correspondence

$$\mathcal{N}(X) := \overline{\{(x, y) \in V \times V \mid x \in X_{\mathrm{sm}} \text{ and } y \in N_x X\}}.$$

The ideal of the conormal variety in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is

$$N(X) := \left(I(X) + \left\langle (c+1) \times (c+1) - \text{minors of} \begin{pmatrix} y \\ J(f) \end{pmatrix} \right\rangle \right) : I(X_{\mathrm{sing}})^\infty.$$

It is known that, if $X \subset V$ is an irreducible affine cone, then $\mathcal{N}(X)$ is irreducible in $V \times V$ and has dimension $n-1$. One might consider the diagram

$$\begin{tikzcd} & \mathcal{N}(X) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & V \end{tikzcd} \tag{1.3.1}$$

and verify that the projection of $\mathcal{N}(X)$ onto the second factor V is precisely the dual affine cone $X^\vee$. The well-known *Biduality Theorem* (see [GKZ, Chapter 1]) states that $\mathcal{N}(X)$ equals $\mathcal{N}(X^\vee)$ up to swapping the two factors. The consequence is that $(X^\vee)^\vee = X$. For this reason, we write $\mathcal{N}(X, X^\vee)$ for $\mathcal{N}(X)$ and $N(X, X^\vee)$ for $N(X)$.

The conormal variety $\mathcal{N}(X, X^\vee)$ plays a crucial role in the proof of the following important result, which we will apply several times in the next chapters.

Theorem 1.3.3. *[DHOST, Theorem 5.2] Let $X \subset V$ be an irreducible affine cone and $u \in V$ a general data point. The map*

$$\begin{array}{rccc} \varphi_u\colon & V & \to & V \\ & x & \mapsto & u-x \end{array} \tag{1.3.2}$$

gives a bijection from the critical points of δ_u on X to the critical points of δ_u on $X^\vee$. Consequently, $\mathrm{EDdegree}(X) = \mathrm{EDdegree}(X^\vee)$*. If we restrict to the real points, the map is proximity-reversing: the closer a real critical point $x \in X$ is to the data point $u \in V^{\mathbb{R}}$, the further $u - x \in X^\vee$ is from u.*

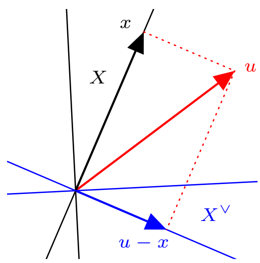

Figure 1.6: The bijection between critical points of δ_u on X and on its dual $X^\vee$.

Proof. Let $u \in V$ be a general point. If $x \in X$ is a critical point of δ_u, then $u - x \in N_xX$. In turn, this implies that the pair $(x, u-x)$ belongs to the conormal variety $\mathcal{N}(X, X^\vee)$. By the generality of u, $u-x$ is a smooth point on $X^\vee$. In addition, the Biduality Theorem implies that $x = u-(u-x) \in N_{u-x}(X^\vee)$, namely $u - x$ is a critical point of δ_u on $X^\vee$. In particular, the map φ_u defined in (1.3.2) sends critical points of δ_u on X into critical points of δ_u on $X^\vee$. Since the above argument may be repeated with $X^\vee$ in place of X, this map φ_u is in fact a bijection between the two mentioned sets of critical points.

Now observe that, since X and $X^\vee$ are affine cones, the vectors x and $u - x$ above are orthogonal, hence the Pythagorean Theorem tells us that $q(u - x) + q(x) = q(u)$. In particular, this means that φ_u is proximity-reversing. $\square$

It is clear that the conormal variety $\mathcal{N}(X, X^\vee)$ and the ED correspondences $\mathcal{E}(X)$ and $\mathcal{E}(X^\vee)$ are closely related. Indeed, there is another correspondence that somehow rules all three. It is introduced in the next definition.

Definition 1.3.4. Let $X \subset V$ be an irreducible affine cone. The *joint ED correspondence* of X and $X^\vee$ is

$$\begin{aligned}\mathcal{E}(X, X^\vee) &:= \overline{\{(x, u - x, u) \in V_x \times V_y \times V_u \mid x \in X_{\mathrm{sm}} \text{ and } u - x \in N_x X\}} \\ &= \overline{\{(u - y, y, u) \in V_x \times V_y \times V_u \mid y \in X^\vee_{\mathrm{sm}} \text{ and } u - y \in N_x X^\vee\}}\end{aligned}$$

Now we might consider the three projections of $\mathcal{E}(X, X^\vee)$ onto either one of the three product spaces $V_x \times V_u$, $V_y \times V_u$ and $V_x \times V_y$. In the first two cases, we recover the ED correspondence $\mathcal{E}(X)$ and $\mathcal{E}(X^\vee)$, respectively. In the last case, we get the conormal variety $\mathcal{N}(X, X^\vee)$. The affine variety $\mathcal{E}(X, X^\vee)$ is irreduible of dimension n, since $\mathcal{E}(X)$ has these properties. The projection $\mathcal{E}(X, X^\vee) \to \mathcal{E}(X)$ is birational with inverse $(x, u) \mapsto (x, u - x, u)$.

Every affine cone $X \subset V$ may be associated to a projective variety in $\mathbb{P}(V)$. With a little abuse of notation, we denote this variety by X as well. Moreover, we define the *ED degree* of a projective variety in $\mathbb{P}(V)$ as the ED degree of the corresponding affine cone in V.

It turns out that the notions of ED correspondence, joint ED correspondence have a projective counterpart. We state their definition and an analogue of Theorem 1.2.2 for completeness. Consider the map

$$\begin{array}{rccc}\varphi\colon & (V \setminus \{0\}) \times V & \to & \mathbb{P}(V) \times V \\ & (x, u) & \mapsto & ([x], u)\end{array}$$

Definition 1.3.5. The closure of $\varphi(\mathcal{E}_X \cap [(V \setminus \{0\}) \times V])$ is called the *projective ED correspondence of* X, and it is denoted by $\mathcal{PE}(X)$.

Similarly to (1.2.2) we might consider the following diagram of projections involving the ED correspondence $\mathcal{PE}(X)$:

$$\begin{tikzcd} & \mathcal{PE}(X) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & V \end{tikzcd} \tag{1.3.3}$$

Theorem 1.3.6. *[DHOST, Theorem 4.4] Let $X \subset \mathbb{P}(V)$ be an irreducible variety not contained in the isotropic quadric Q. Then the projective ED correspondence $\mathcal{PE}(X)$ of X is an n-dimensional irreducible variety in $\mathbb{P}(V) \times V$. Its projection onto X is a vector bundle over $X_{\mathrm{sm}} \setminus Q$ of rank $c+1$. The fiber over general data points u of its projection onto V are finite of cardinality equal to* $\mathrm{EDdegree}(X)$.

Mimicking to the case of the projective ED correspondence, we define

$$\begin{array}{rccc} \varphi\colon & (V \setminus \{0\}) \times (V \setminus \{0\}) \times V & \to & \mathbb{P}(V) \times \mathbb{P}(V) \times V \\ & (x, y, u) & \mapsto & ([x], [y], u) \end{array}$$

Definition 1.3.7. The closure of

$$\varphi\left\{\mathcal{E}(X, X^\vee) \cap [(V \setminus \{0\}) \times (V \setminus \{0\}) \times V]\right\}$$

is called the *projective joint ED correspondence of* X, and it is denoted by $\mathcal{PE}(X, X^\vee)$.

When $X \subset V$ is an affine cone, we could derive nice formulas for $\mathrm{EDdegree}(X)$ involving several important intrinsic invariants of Algebraic Geometry like *polar classes*, *Chern classes* (for smooth varieties) and *Chern-Mather classes* (for singular varieties). We recall a very short summary of these invariants in the next sections, together with the main results which include them in the ED degree philosophy.

1.4 The Chow ring of a projective variety

In this brief section, we outline the definition and some basic facts related to the *Chow groups* and the *Chow ring* of a projective variety. It is a necessary step towards the definition of the polar classes and the Chern classes of a projective variety. All the necessary material is taken from [Ful, Chapter 8]. Following the idea of the previous sections, we adopt a coordinate-free approach and consider a projective space $\mathbb{P}(V)$ for some complex n-dimensional vector space V.

Let $X \subset \mathbb{P}(V)$ be a smooth, irreducible projective variety of dimension $\dim(X) = m$. For all $d \geq 0$, we denote by $Z^d(X)$ the free abelian group generated by *d-cocycles* of X (or *$(m-d)$-cycles* of X), namely formal linear combinations

$$\alpha = \sum \alpha_i V_i$$

with integer coefficients of d-codimensional irreducible subvarieties $V_i \subset X$. A d-cocycle $\alpha \in Z^d(X)$ is *principal* if it is of the form $\alpha = \operatorname{div}(f)$, where f is a rational function on a subvariety Y of codimension $d-1$ in X. Two d-cocycles α and β are *rationally equivalent* if $\alpha - \beta$ is a sum of principal d-cocycles. Rational equivalence is an equivalence relation in $Z^d(X)$ and is denoted by $\sim$.

Definition 1.4.1. The *d-th Chow group* $A^d(X)$ is the quotient group $Z^d(X)/\sim$.

For example, it turns out that the first Chow group $A^1(X)$ coincides with the *Picard group* of X (see [Ful, Chapter 2, §1]). In particular, consider the case $X = \mathbb{P}(V)$. Then $A^1(\mathbb{P}(V)) = \operatorname{Pic}(\mathbb{P}(V)) \cong \mathbb{Z}$, and its generator is the class of a hyperplane, denoted by H.

Consider the direct sum $A^*(X) := \bigoplus_{i \geq 0} A^i(X)$, which is clearly an abelian group with respect to the addition of rational equivalence classes of cycles. One might turn $A^*(X)$ into a graded ring by defining a suitable product

$$A^d(X) \times A^e(X) \to A^{d+e}(X).$$

Let V and W be two irreducible subvarieties of codimensions d and e in X, respectively. If V and W intersect transversally (for a discussion of the precise meaning of *transversality* we refer, for example, to [Sch]), their intersection $V \cap W$ is a subvariety of codimension $d+e$ in X. Otherwise, if V and W do not intersect transversally, the smoothness of X allows us to replace V with another cocycle $\alpha = \sum \alpha_i V_i$ on X which is rationally equivalent to V and which intersects W transversally. This property is classically known as the Moving Lemma (see [Ful, Chapter 11, §4]). Moreover, one may verify that the two subvarieties $V \cap W$ and $\alpha \cap W$ are rationally equivalent as well.

Definition 1.4.2. The *intersection product* of two rational equivalence classes $[V] \in A^d(X)$ and $[W] \in A^e(X)$ of irreducible subvarieties V and W in X is

$$[V][W] := [\alpha \cap W] = \left[\sum \alpha_i (V_i \cap W)\right] \in A^{d+e}(X),$$

where α is any d-cocycle in X which is rationally equivalent to V and which meets W transversally.

Proposition 1.4.3. *[Ful, Proposition 8.3] The group $A^*(X)$ equipped with the intersection product is a graded ring and is called the* Chow ring *(or* intersection ring*) of X.*

Example 1.4.4. Going back to the example $X = \mathbb{P}(V)$, for all $d \geq 0$, the d-th Chow group of $\mathbb{P}(V)$ is $A^d(\mathbb{P}(V)) = (H^d) \cong \mathbb{Z}$, where H^d is the class of a linear space of codimension d, namely the product of d 1-codimensional general hyperplanes of $\mathbb{P}(V)$. In particular, H^{n-1} denotes the class of a point, so it has empty intersection with a general hyperplane. In other words, $H^n = 0$ and therefore the n-th Chow goup of $\mathbb{P}(V)$ is $A^n(\mathbb{P}(V)) = 0$. Moreover, $H^n = 0$ is the only relation involving H. Summing up, the Chow ring of $\mathbb{P}(V)$ is $A^*(\mathbb{P}(V)) \cong \mathbb{Z}[H]/(H^n)$. In particular, if $Y \subset \mathbb{P}(V)$ is an irreducible subvariety of codimension d, then its rational equivalence class $[Y] \in A^d(\mathbb{P}(V))$ is given by cH^d. The coefficient c corresponds to the number of points of intersection between Y and a general linear subspace of $\mathbb{P}(V)$ of dimension d. In other words, c is equal to the *degree* of Y.

Example 1.4.5. Let $X \subset \mathbb{P}(V) \cong \mathbb{P}^{m-1}$ and $Y \subset \mathbb{P}(W) \cong \mathbb{P}^{n-1}$ be two smooth projective varieties and consider their product $X \times Y$ embedded in $\mathbb{P}(V \otimes W)$. There exists a homomorphism of rings $A^*(X) \otimes A^*(Y) \to A^*(X \times Y)$ which preserves the grading of the Chow rings involved. When $X = \mathbb{P}(V)$ and $Y = \mathbb{P}(W)$, then $A^*(\mathbb{P}(V)) \otimes A^*(\mathbb{P}(W)) \cong A^*(\mathbb{P}(V) \times \mathbb{P}(W))$. From this fact and Example 1.4.4, one can derive that

$$A^*(\mathbb{P}(V) \times \mathbb{P}(W)) \cong \frac{\mathbb{Z}[s,t]}{(s^m, t^n)},$$

where s and t correspond to the pullbacks of the hyperplane classes of $\mathbb{P}(V)$ and $\mathbb{P}(W)$ via the two projections of $\mathbb{P}(V) \times \mathbb{P}(W)$ onto the factors $\mathbb{P}(V)$ and $\mathbb{P}(W)$, respectively. In particular, the rational equivalence class $[Z] \in A^d(X \times Y)$ of an irreducible subvariety $Z \subset X \times Y$ of codimension d may be written as

$$[Z] = c_0 s^d + c_1 s^{d-1} t + \cdots + c_d t^d,$$

where the vector of coefficients $(c_0, \ldots, c_d)$ is known as the *multidegree* of Z in $X \times Y$.

1.5 Polar classes of projective varieties

Let V be an n-dimensional complex vector space and let $X \subset V$ be an irreducible affine cone, namely $X \subset \mathbb{P}(V)$ as an irreducible projective variety. The main tool of this section is the conormal variety $\mathcal{N}(X, X^\vee)$ introduced in Definition 1.3.2. One may verify that $\mathcal{N}(X, X^\vee)$ is an irreducible subvariety of codimension n in $\mathbb{P}(V) \times \mathbb{P}(V)$. Applying the facts outlined in Section 1.4, we may consider the rational equivalence class $[\mathcal{N}(X, X^\vee)] \in A^*(\mathbb{P}(V) \times \mathbb{P}(V))$. Following Example 1.4.5, we have that

$$A^*(\mathbb{P}(V) \times \mathbb{P}(V)) \cong \frac{\mathbb{Z}[s,t]}{(s^n, t^n)}$$

and

$$[\mathcal{N}(X, X^\vee)] = \delta_0 s^{n-1}t + \cdots + \delta_{n-2} s t^{n-1} \tag{1.5.1}$$

for some nonnegative integers $\delta_i = \delta_i(X)$. Geometrically speaking, $\delta_j(X)$ counts the number of intersections between $\mathcal{N}(X, X^\vee)$ and a subvariety of the form $L \times M \subset \mathbb{P}(V) \times \mathbb{P}(V)$, where L and M are linear subspaces of $\mathbb{P}(V)$ of dimensions $n-1-j$ and $j+1$, respectively.

Actually, the coefficients $\delta_j(X)$ may be introduced in a more formal way using the Gauss map of a projective variety. For completeness, we briefly outline this second approach, which is nicely explained in [Pie15].

We denote by $G(k+1, V) = \mathbb{G}(k, \mathbb{P}(V))$ the *Grassmannian* of $(k+1)$-dimensional vector subspaces of V, namely of k-dimensional projective subspaces of $\mathbb{P}(V)$. Let $j \geq 0$ and $L_j \subset \mathbb{P}(V)$ be a linear space of codimension $k-(j-2)$. Define

$$\Sigma(L_j) \coloneqq \{M \in G(k+1, V) \mid \dim(M \cap L_j) \geq j-1\}.$$

Actually, $\Sigma(L_j)$ is an example of a *Schubert variety* (see [GH, Chapter 1, §5]).

Example 1.5.1. Assume that $V \cong \mathbb{C}^3$ and that $k = 1$. Then $G(2, V)$ is the dual space $\mathbb{P}(V)^\vee$. If we let L_1 be a general point of $\mathbb{P}(V)$, then necessarily $j = 1$ and

$$\Sigma(L_1) = \{M \in \mathbb{P}(V)^\vee \mid \dim(M \cap L_1) \geq 0\} = \{\text{lines in } \mathbb{P}(V) \text{ containing } L_1\}.$$

Having fixed k, if L and L' are two linear spaces of codimension $k-(j-2)$ in $\mathbb{P}(V)$, it turns out that the corresponding subvarieties $\Sigma(L)$ and $\Sigma(L')$ are projectively equivalent, hence their rational equivalence classes are equal. So we denote simply by Σ_j the variety $\Sigma(L_j)$.

Now consider a projective variety X of dimension m in $\mathbb{P}(V)$. The *Gauss map* associated to X is the rational map

$$\begin{array}{rccc} \gamma_X\colon & X & \dashrightarrow & G(m+1, V) \\ & x & \longmapsto & T_x X \end{array} \tag{1.5.2}$$

which is defined over the smooth points of X.

Definition 1.5.2. The j-th *polar variety* associated to $X \subset \mathbb{P}(V)$ is

$$\mathcal{P}_j(X) \coloneqq \gamma_X^{-1}(\Sigma_j) = \{x \in X \mid \dim(T_x X \cap L_j) \geq j-1\}.$$

It turns out that the degree of $\mathcal{P}_j(X)$ is precisely the coefficient $\delta_{n-2-j}(X)$ introduced in (1.5.1). In the literature, the integers $\delta_{n-2-j}(X)$ are ofter regarded as the polar classes (or *polar ranks*) of X, representing the whole rational equivalence classes of the $\mathcal{P}_j(X)$'s.

Example 1.5.3. For example, let $V \cong \mathbb{C}^3$ and X be a smooth conic in $\mathbb{P}(V)$. For $j = 0$, one could see easily from Definition 1.5.2 that $\mathcal{P}_0(X) = X$, hence $\delta_1(X) = \deg(X) = 2$. Instead for j we have

$$\mathcal{P}_1(X) = \{x \in X \mid T_x X \text{ passes through a general point } u \in \mathbb{P}(V)\} = \{x_1, x_2\},$$

where x_1 and x_2 are depicted in Figure 1.7. Hence, $\delta_0(X) = 2$. Note that, dually

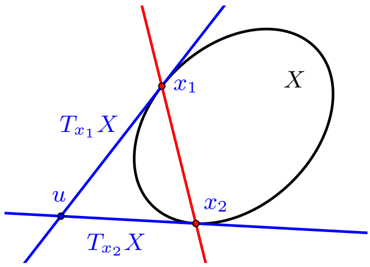

Figure 1.7: The first polar variety of a plane conic.

speaking, the general point u corresponds to a general line which intersects $X^\vee$ in the two points representing the tangent lines $T_{x_1}X$ and $T_{x_2}X$. In other words, $\delta_0(X) = 2$ corresponds to the degree of the conic $X^\vee$.

Actually, what happens in Example 1.5.3 is true for any irreducible projective variety $X \subset \mathbb{P}(V)$. On one hand, $\delta_0(X) = \deg(X^\vee)$ if $X^\vee$ is a hypersurface. On the other hand, $\delta_{n-2}(X) = \deg(X)$. This is confirmed by the interpretation given after relation (1.5.1): the invariant $\delta_0(X)$ counts the points of intersection between $\mathcal{N}(X, X^\vee)$ and a general subvariety $\mathbb{P}(V) \times M$, where M is a line. This, in turn, is the number of points of intersections between $X^\vee$ and the line M, namely $\deg(X^\vee)$. More in general, we have following result.

Theorem 1.5.4. *[Hol, Theorem 3.4] If $\delta_j(X) = 0$ for all $j < l$ and $\delta_l(X) \neq 0$, then $\dim(X^\vee) = n - 2 - l$ and $\delta_l(X) = \deg(X^\vee)$.*

One of the reasons why the polar classes of a projective variety $X \subset \mathbb{P}(V)$ are important for our upcoming computations lies in the following important result.

Theorem 1.5.5. *[DHOST, Theorem 5.4] If $\mathcal{N}(X, X^\vee)$ does not intersect the diagonal $\Delta(\mathbb{P}(V)) \subset \mathbb{P}(V) \times \mathbb{P}(V)$, then*

$$\mathrm{EDdegree}(X) = \delta_0(X) + \cdots + \delta_{n-2}(X).$$

Hence, for example the ED degree of the smooth conic X of Example 1.5.3 (with respect to an isotropic quadric Q such that $\mathcal{N}(X, X^\vee)$ satisfies the hypothesis of Theorem 1.5.5) is $\mathrm{EDdegree}(X) = \delta_0(X) + \delta_1(X) = 4$, somehow confirming the right picture in Figure 1.4.

In Proposition 4.4.8 we furnish a sufficient condition for $\mathcal{N}(X, X^\vee)$ not to intersect $\Delta(\mathbb{P}(V))$, in terms of Whitney stratifications of X (see Definition 4.3.3). For example, the hypothesis of Theorem 1.5.5 is satisfied when $X \cap Q$ is a smooth variety disjoint from the singular locus of X.

The polar classes of X are projective invariants of X, and are closely related to another family of invariants associated to vector bundles on X, the so-called *Chern classes* of smooth projective varieties. When we say "Chern classes of X" we mean the Chern classes of the *tangent bundle* $\mathcal{T}X \to X$ (see Definition 1.6.6). In Section 1.7, we provide a rough introduction to Chern classes of holomorphic vector bundles in the easiest possible way.

1.6 Overview of holomorphic vector bundles

Before introducing Chern classes, we recall the notion of a holomorphic vector bundle and state some useful properties. The main references used are [EH, Chapter 5] and [GH, Chapter 0, §5]. For brevity, after the next definition, we avoid repeating the word "holomorphic" to address a vector bundle.

Definition 1.6.1. Let E and M be complex manifolds. A *holomorphic vector bundle* $\pi\colon E \to M$ consists of a family $\{E_x\}_{x\in M}$ of complex vector spaces parametrized by M such that

1. $E := \bigcup_{x\in M} E_x$,
2. The projection $\pi\colon E \to M$ taking E_x to x is holomorphic, and
3. For every $x_0 \in M$, there exists an open set U in M containing x_0 and a biholomorphic map
$$\varphi_U\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^k$$
taking the vector space E_x isomorphically onto $\{x\} \times \mathbb{C}^k$ for each $x \in U$. The map φ_U is called a *(holomorphic) trivialization* of E over U.

The dimension of the fibers E_x of E is called the *rank* of E. In particular, a rank-one vector bundle is called a *line bundle*. For any pair of trivializations φ_{U_α} and φ_{U_β}, the map

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(\mathbb{C}^k), \quad g_{\alpha\beta}(x) := (\varphi_{U_\alpha} \circ \varphi_{U_\beta}^{-1})|_{\{x\}\times\mathbb{C}^k}$$

is holomorphic. The maps $g_{\alpha\beta}$ are called *transition functions* for E relative to the trivializations φ_{U_α}, φ_{U_β}. The transition functions of E satisfy the relations

$$\begin{aligned} g_{\alpha\beta}(x) \cdot g_{\beta\alpha}(x) = Id \quad &\text{for all } x \in U_\alpha \cap U_\beta \\ g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = Id \quad &\text{for all } x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}$$

Conversely, every open cover $\mathcal{U} = \{U_\alpha\}$ of M with holomorphic maps $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{C}^k)$ satisfying the two above identities yields a unique vector bundle $E \to M$ with transition functions $\{g_{\alpha\beta}\}$.

Given vector bundles $E \to M$ and $F \to M$, F is a *subbundle* of E (and we write $F \subset E$) if $F_x \subset E_x$ for all $x \in M$ and F is a submanifold of E. If $F \subset E$, we can define the *quotient bundle* E/F given by $(E/F)_x \coloneqq E_x/F_x$.

Given a holomorphic map $f \colon M \to N$ between complex manifolds and a vector bundle $E \to N$, we define the *pullback bundle* f^*E by setting $(f^*E)_x = E_{f(x)}$.

A *(holomorphic) morphism* of vector bundles $E \to M$ and $F \to M$ is a holomorphic map $f \colon E \to F$ such that $f(E_x) \subset F_x$ and $f_x \coloneqq f|_{E_x} \colon E_x \to F_x$ is linear.

If $f \colon E \to F$ is a morphism of vector bundles over M, then

$$\mathrm{Ker}(f) \coloneqq \bigcup_{x \in M} \mathrm{Ker}(f_x) \subset E, \quad \mathrm{Im}(f) \coloneqq \bigcup_{x \in M} \mathrm{Im}(f_x) \subset F$$

are subbundles of E and F, respectively if and only if the maps f_x all have the same rank.

Two vector bundles $E \to M$ and $F \to M$ are *isomorphic* if there exists a morphism of vector bundles $f \colon E \to F$ such that f_x is an isomorphism for all $x \in M$.

The first example of a vector bundle over M of rank k is the product bundle $M \times \mathbb{C}^k$, also called *trivial bundle*. More in general, a vector bundle $E \to M$ of rank k is *trivial* if it is isomorphic to $M \times \mathbb{C}^k$.

If $(E, g_{\alpha\beta})$ and $(F, h_{\alpha\beta})$ are vector bundles over M of rank k and l respectively, we can construct many interesting vector bundles over M like

1. the *direct sum* (or *Whitney sum*) $(E \oplus F, j_{\alpha\beta})$, where $(E \oplus F)_x \coloneqq E_x \oplus F_x$ and
$$j_{\alpha\beta}(x) \coloneqq \begin{pmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & h_{\alpha\beta}(x) \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^k \oplus \mathbb{C}^l),$$

2. the *tensor product* $(E \otimes F, j_{\alpha\beta})$, where $(E \otimes F)_x \coloneqq E_x \otimes F_x$ and
$$j_{\alpha\beta}(x) \coloneqq g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x) \in \mathrm{GL}(\mathbb{C}^k \otimes \mathbb{C}^l),$$

3. the *dual* $(E^\vee, j_{\alpha\beta})$, where $E^\vee_x \coloneqq (E_x)^\vee$ and $j_{\alpha\beta}(x) \coloneqq [g^T_{\alpha\beta}(x)]^{-1} \in \mathrm{GL}(\mathbb{C}^k)$,

4. the *r-th exterior power* $(\wedge^r E, j_{\alpha\beta})$, where $(\wedge^r E)_x \coloneqq \wedge^r(E_x)$ and
$$j_{\alpha\beta}(x) \coloneqq \wedge^r g_{\alpha\beta}(x) \in \mathrm{GL}(\wedge^r \mathbb{C}^k).$$

In particular, $(\wedge^k E, j_{\alpha\beta})$ is a line bundle with transition functions

$$j_{\alpha\beta}(x) \coloneqq \det(g_{\alpha\beta}(x)) \in \mathbb{C}^*$$

and is called the *determinant* bundle of E.

From the constructions listed above and the basic operations of inclusion, quotient, and pullback, one may build other useful vector bundles. For example, given vector bundles $E \to M$ and $F \to M$, it turns out that the vector bundle $F \otimes E^\vee$ is isomorphic to another interesting vector bundle over M, namely the *Hom-bundle* $\mathrm{Hom}(E,F)$, whose fibers are $(\mathrm{Hom}(E,F))_x \coloneqq \mathrm{Hom}(E_x, F_x)$ for all $x \in M$.

Definition 1.6.2. A *(holomorphic) section* of a vector bundle $E \to M$ over $U \subset M$ is a holomorphic map $\sigma\colon U \to E$ such that $\sigma(x) \in E_x$ for all $x \in U$. A *global section* of a vector bundle $E \to M$ is a section defined over all of M. A vector bundle $E \to M$ is *globally generated* if there exist global sections $\sigma_1, \ldots, \sigma_r$ such that the vectors $\sigma_1(x), \ldots, \sigma_r(x)$ span E_x for all $x \in M$.

In the following, we denote by $\mathcal{O}_M$ the sheaf of holomorphic functions of the complex manifold M. For every open subset $U \subset M$, $\mathcal{O}_M(U) = \Gamma(U, \mathcal{O}_M)$ is the ring of regular functions on U. For a detailed introduction to sheaves on manifolds, we refer to [GH, Chapter 0, §3].

Definition 1.6.3. For a vector bundle $E \to M$ and an open subset $U \subset M$, the sections of M over U form an $\mathcal{O}_M(U)$-module which defines locally a presheaf over M. One may verify that this presheaf is, in fact, a sheaf of $\mathcal{O}_M$-modules and is called the *sheaf of sections* of $E \to M$.

In particular, $\mathcal{O}_M$ corresponds to the sheaf of sections of the trivial line bundle $M \times \mathbb{C} \to M$. For a positive integer k, the direct sum of k copies of $\mathcal{O}_M$ is defined as $\mathcal{O}_M(U)^{\oplus k}$ for all $U \subset M$. It is indicated by $\mathcal{O}_M^{\oplus k}$. More generally, an $\mathcal{O}_M$-module is *free* (*locally free*) of rank k if it is isomorphic (locally isomorphic) to $\mathcal{O}_M^{\oplus k}$.

The following result yields a useful dictionary between vector bundles over a complex manifold M and locally free $\mathcal{O}_M$-modules.

Proposition 1.6.4. *The set of vector bundles of rank k over a complex manifold M and the set of locally free $\mathcal{O}_M$-modules of rank k are in bijection. The bijection is defined by sending a vector bundle $E \to M$ to its sheaf of sections.*

Motivated by the previous result, in the following we denote with the same symbol E both a vector bundle over M and its associated sheaf of sections.

We make a further consideration before introducing the degeneracy loci of a vector bundle morphism.

Remark 1.6.5. Let M be a compact complex manifold and let $\mathcal{F}$ be an $\mathcal{O}_M$-module. Then there is a one to one correspondence between morphisms of sheaves $\varphi\colon \mathcal{O}_M \to \mathcal{F}$ and global sections of $\mathcal{F}$, namely elements of $\Gamma(M, \mathcal{F})$.

In fact, let $\varphi\colon \mathcal{O}_M \to \mathcal{F}$ be a morphism of sheaves. Then for every open subset $U \subset M$ we have a homomorphism of $\mathcal{O}_M(U)$-modules $\varphi_U\colon \mathcal{O}_M(U) \to \mathcal{F}(U)$. In particular for $U = M$ we have the homomorphism $\varphi_M\colon \Gamma(M, \mathcal{O}_M) \to \Gamma(M, \mathcal{F})$. Since M is compact, every holomorphic function $M \to \mathbb{C}$ is constant. In other words, $\Gamma(M, \mathcal{O}_M) \cong \mathbb{C}$. This implies that φ_M determines and is determined by the image of $1 \in \mathbb{C}$ in $\Gamma(M, \mathcal{F})$.

Conversely, let $\sigma \in \Gamma(M, \mathcal{F})$ be a global section of $\mathcal{F}$ and define the ring homomorphism $\varphi_M\colon \mathbb{C} \to \Gamma(M, \mathcal{F})$ such that $\varphi_M(1) := \sigma$. Then for every open subset $U \subset M$ we can define a ring homomorphism $\varphi_U\colon \Gamma(U, \mathcal{O}_M) \to \Gamma(U, \mathcal{F})$ in order to make the following diagram commute:

$$\begin{tikzcd}
\Gamma(M, \mathcal{O}_M) \arrow[r] \arrow[d] & \Gamma(M, \mathcal{F}) \arrow[d] \\
\Gamma(U, \mathcal{O}_M) \arrow[r] & \Gamma(U, \mathcal{F}).
\end{tikzcd}$$

Here the vertical arrows are the restriction maps respectively of $\mathcal{O}_M$ and $\mathcal{F}$. This gives rise to a morphism of sheaves $\varphi : \mathcal{O}_M \to \mathcal{F}$.

As a corollary of last remark, we obtain that there is a one to one correspondence between morphisms $\varphi : \mathcal{O}_M^{\oplus r} \to \mathcal{F}$ and sets $\{\sigma_1, \ldots, \sigma_r\}$ of global sections of $\mathcal{F}$.

Our prototype for a smooth manifold is the projective space $\mathbb{P}(V)$ associated to an n-dimensional complex vector space V. We briefly list the first mostly used vector bundles on $\mathbb{P}(V)$:

1. The trivial line bundle $\mathbb{P}(V) \times L$, where L is a one dimensional complex vector space, is indicated by $\mathcal{O} := \mathcal{O}_{\mathbb{P}(V)}$. Therefore, any trivial bundle $E \to \mathbb{P}(V)$ of rank r is of the form $E = \mathcal{O}^{\oplus r} = \mathcal{O} \otimes \mathbb{C}^r$.

2. the *tautological line bundle* on $\mathbb{P}(V)$ is the subbundle of $\mathcal{F}$, usually indicated by $\mathcal{O}(-1) := \mathcal{O}_{\mathbb{P}(V)}(-1)$, such that, for each point $x = [v] \in \mathbb{P}(V)$, the fiber $\mathcal{O}(-1)_x$ is the line spanned by v.

3. The *hyperplane bundle* of $\mathbb{P}(V)$ is the dual $\mathcal{O}(-1)^\vee$ of the tautological bundle and is usually denoted by $\mathcal{O}(1)$.

4. For $d > 0$, we define the line bundle $\mathcal{O}(d) := \mathcal{O}(1)^{\otimes d}$. Analogously, for $d < 0$ we define $\mathcal{O}(d) := \mathcal{O}(-d)^\vee$.

Analogously, one might consider a smooth projective variety $X \subset \mathbb{P}(V)$ and define the vector bundles $X \times L$ and $\mathcal{O}_X(d)$ for all $d \in \mathbb{Z}$. Clearly one might consider the above vector bundles as "bricks" for constructing other vector bundles on X. Besides the above-mentioned vector bundles, it is not easy to construct higher-rank vector bundles on X.

Actually, there is another vector bundle which we can associate naturally to X, the *tangent bundle* $\mathcal{T}X \to X$. Let $\{U_i\}_i$ be a finite open covering of X given by charts $\varphi_i \colon U \to \varphi_i(U_i) \subset \mathbb{C}^m$, where $m = \dim(X)$. For example, when $X = \mathbb{P}(V)$, one might consider the standard open covering $\mathbb{P}(V) = \bigcup_{i=1}^n U_i$, where

$$U_i \coloneqq \{[x_1, \dots, x_n] \in \mathbb{P}(V) \mid x_i \neq 0\} \quad \forall i \in [n]$$

and the charts $\varphi_i \colon U_i \to \mathbb{C}^n$ defined as

$$\varphi_i\left([x_1, \dots, x_n]\right) \coloneqq \left(\frac{x_1}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right).$$

Definition 1.6.6. The *Jacobian* of the transition maps $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \cong \varphi_i(U_i \cap U_j)$ is the matrix

$$J(\varphi_{ij})(\varphi_j(x)) \doteq \left(\frac{\partial^k \varphi_{ij}}{\partial x_l}(\varphi_j(x))\right)_{k,l}.$$

The *tangent bundle* of X is the vector bundle $\mathcal{T}X$ of rank $m = \dim(X)$ given by the transition functions

$$g_{ij}(x) \doteq J(\varphi_{ij})(\varphi_j(x)).$$

The *cotangent bundle* of X is the dual of $\mathcal{T}X$ and is denoted by Ω_X.

The above definition is independent of the open covering $\{U_i\}_i$ and the charts $\{\varphi_i\}_i$ chosen.

Proposition 1.6.7 (Euler sequence)**.** *On $\mathbb{P}(V)$ there exists a natural short exact sequence of vector bundles*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \otimes V \longrightarrow \mathcal{T}\mathbb{P}(V) \longrightarrow 0. \tag{1.6.1}$$

Proof. From the definition of tautological line bundle $\mathcal{O}(-1)$ we get a short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \otimes V \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the quotient bundle of rank n. Tensoring with $\mathcal{O}(1)$ we get the sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1) \otimes V \to \mathcal{Q}(1) \to 0.$$

Now let us examine the action $\sigma\colon \mathrm{GL}(V) \times \mathbb{P}(V) \to \mathbb{P}(V)$ of $\mathrm{GL}(V)$ on $\mathbb{P}(V)$ defined by $\sigma(g, [v]) := [g(v)]$. This action induces, for every $[v] \in \mathbb{P}(V)$, the map

$$\sigma_{[v]}\colon \mathrm{GL}(V) \to \mathbb{P}(V), \quad \sigma_{[v]}(g) := \sigma(g, [v]) = [g(v)]\ \forall g \in \mathrm{GL}(V).$$

The differential at $g = Id$ of $\sigma_{[v]}$ is the surjective linear map

$$D\sigma_{[v]}\colon T_{Id}\mathrm{GL}(V) = \mathfrak{gl}(V) \to T_{[v]}\mathbb{P}(V), \quad D\sigma_{[v]}(g) := g(v)\ \forall g \in \mathfrak{gl}(V).$$

Then we have

$$\begin{aligned} T_{[v]}\mathbb{P}(V) &\cong \frac{\mathfrak{gl}(V)}{\ker(D\sigma_{[v]})} = \frac{\mathfrak{gl}(V)}{\{g \mid g(v) \in \langle v\rangle\}} \cong \mathrm{Hom}\left(\langle v\rangle, \frac{V}{\langle v\rangle}\right) = \\ &= \mathrm{Hom}(\mathcal{O}(-1)_v, \mathcal{Q}_v) \cong \mathcal{Q}_v \otimes \mathcal{O}(1)_v = [\mathcal{Q} \otimes \mathcal{O}(1)]_v = \mathcal{Q}(1)_v. \end{aligned}$$

Therefore $\mathcal{Q}(1) \cong \mathcal{T}\mathbb{P}(V)$, concluding the proof. □

It turns out that, if Y is a projective subvariety of X, then there is a canonical injection $\mathcal{T}Y \subset \mathcal{T}X\,|_Y$, where $\mathcal{T}X\,|_Y$ denotes the restriction to Y of the tangent bundle of X. This gives rise to the definition of another important vector bundle on $Y \subset X$.

Definition 1.6.8. Let Y be a projective subvariety of X. The *normal bundle* of Y in X, denoted by $\mathcal{N}_{Y/X} \to Y$, is the cokernel of the natural injection $\mathcal{T}Y \subset \mathcal{T}X\,|_Y$. Moreover, the following is a short exact sequence of vector bundles on Y, called *normal bundle sequence* of Y:

$$0 \longrightarrow \mathcal{T}Y \longrightarrow \mathcal{T}X\,|_Y \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0. \tag{1.6.2}$$

The above short exact sequence is used in Example 1.7.4 in the case of a hypersurface $X \subset \mathbb{P}(V)$.

1.7 Chern classes of smooth projective varieties

There are various motivations and ways to define Chern classes. The original approach to Chern as well as other characteristic classes came from algebraic topology in the 1930s. Nevertheless, Chern's approach in the influential paper [Che] was different and used differential geometry. During his valuable collaboration with Weil in the 1940s, they proved that the differential and the topological approaches are equivalent.

A more axiomatic approach to Chern classes was given by Grothendieck in [Gro]. The description of Chern classes that we follow is based on degeneracy loci of holomorphic vector bundle morphisms and requires further reasonable assumptions that are satisfied in all our forthcoming applications.

Let X be a smooth, irreducible projective variety in $\mathbb{P}(V)$ and let $E \to X$ be a vector bundle of rank e. One could associate to E a list of classes $c_j \in A^j(X)$, for all $1 \leq j \leq e$, the so called *Chern classes* of E. Of particular relevance is the *top Chern class* $c_{\text{top}}(E) = c_e(E)$. When $e = \dim(X)$, then $c_{\text{top}}(E) \in A^{\dim(X)}(X)$, so it is a 0-cycle, or a sum of classes of points. Since rational equivalence preserves the sum of the cofficients of a 0-cycle, we can associate to $c_{\text{top}}(E)$ an integer, called the *top Chern number* of E. The *total Chern class* of E is $c(E) := 1 + c_1(E) + \cdots + c_{\text{top}}(E)$.

Above all the properties of Chern classes, basically, there are two formal rules (or rather "axioms", following Grothendieck's approach) that Chern classes must satisfy and which allow us to start with some basic computations. On one hand, if $L \to X$ is a line bundle and D is a Cartier divisor such that $L = \mathcal{O}(D)$, then $c(L) = 1 + D$, namely $c_1(L)$ is the rational equivalence class of D. On the other hand, if we consider any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of vector bundles on X, the *Whitney sum* property states that $c(B) = c(A)c(C)$. In particular, the total Chern class of a direct sum of bundles is the product of the total Chern classes of the summands.

Example 1.7.1 (Chern classes of the tangent bundle to $\mathbb{P}(V)$)**.** It turns out that every line bundle on $\mathbb{P}(V)$ is of the form $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}(V)}(d)$ for some $d \in \mathbb{Z}$. The Cartier divisor associated to this line bundle is dH, where H is again the class of a hyperplane in $\mathbb{P}(V)$. In particular, we have that $c_1(\mathcal{O}(d)) = dH$ and $c(\mathcal{O}(d)) = 1 + dH$. More in general, considering the direct sum $\mathcal{O}(d) \oplus \mathcal{O}(e)$, we have

$$c(\mathcal{O}(d) \oplus \mathcal{O}(e)) = c(\mathcal{O}(d))c(\mathcal{O}(e)) = (1 + dH)(1 + eH) = 1 + (d + e)H + deH^2.$$

These facts, joint with the Euler sequence in (1.6.1), allow us to compute the total Chern class of $\mathcal{T}\mathbb{P}(V)$. Indeed, from (1.6.1) we get

$$c(\mathcal{O}_{\mathbb{P}(V)})c(\mathcal{T}\mathbb{P}(V)) = c(\mathcal{O}_{\mathbb{P}(V)}(1) \otimes V).$$

On the left-hand side we have $c(\mathcal{O}_{\mathbb{P}(V)}) = c(\mathcal{O}_{\mathbb{P}(V)}(0)) = 1 + 0H = 1$, whereas on the right-hand side $c(\mathcal{O}_{\mathbb{P}(V)}(1) \otimes V) = (1 + H)^n$. Therefore, we conclude that $c(\mathcal{T}\mathbb{P}(V)) = (1 + H)^n$. In particular, since $\operatorname{rk}(\mathcal{T}\mathbb{P}(V)) = n - 1$, the top Chern number of $\mathcal{T}\mathbb{P}(V)$ is n.

Actually, Chern classes can be constructed geometrically using degeneracy loci of vector bundle morphisms associated to E, when E is globally generated (this hypothesis is satisfied by the vector bundles considered in this thesis).

So let $E \to X$ be a globally generated vector bundle of rank $e \leq \dim(X)$. Fix r general global sections $\sigma_1, \dots, \sigma_r \in \Gamma(X, E)$, where $r \leq e$. Equivalently, by Remark 1.6.5, let $\varphi\colon \mathcal{O}_X^{\oplus r} \to E$ be a general morphism, where here we are considering E as the sheaf of sections of the vector bundle $E \to X$.

Observe that for every $x \in X$ the map φ_x is, in fact, a linear map between vector spaces. Therefore, φ can be interpreted as a family of linear maps between vector spaces. In other words, for every $x \in X$ we get vectors $\sigma_1(x), \dots, \sigma_r(x)$ in the fiber E_x. Since we are assuming that the φ is general, we might expect that the $\sigma_i(x)$'s to be linearly independent for almost all $x \in M$.

Definition 1.7.2. For all $j \leq r$, the *j-th degeneracy locus* of φ is

$$D_j(\varphi) \coloneqq \{x \in X \mid \operatorname{rk}(\varphi_x) \leq j\}.$$

In particular, the $(r-1)$-th degeneracy locus $D(\varphi) \coloneqq D_{r-1}(\varphi)$ is the locus of points x at which the linear map φ_x does not have full rank.

Note that the set $D_i(\varphi)$ has a natural structure of subscheme of X defined by the ideal generated by the minors of φ of order $i+1$. From this fact one may conclude for example that $D(\varphi)$ has codimension $e-r+1$ in X, namely $D(\varphi) \in A^{e-r+1}(X)$.

For our purposes, the next Theorem may be regarded as a definition of the Chern classes of a vector bundle $E \to X$, provided that E is globally generated.

Theorem 1.7.3. *[Ful, Example 14.4.3] For a globally generated vector bundle $E \to X$ of rank e, and a general morphism $\varphi\colon \mathcal{O}_X^{\oplus r} \to E$ with $r \leq e$, the cycle $D(\varphi)$ is rationally equivalent to $c_{e-r+1}(E)$.*

In particular, if $\varphi'\colon \mathcal{O}_X^{\oplus r} \to E$ is another such general morphism, then the cycles $D(\varphi)$ and $D(\varphi')$ are rationally equivalent.

Example 1.7.4 (Chern classes of projective hypersurfaces)**.** Let X be a projective hypersurface of degree d in $\mathbb{P}(V)$. Then X may be interpreted as the degeneracy locus $D(\varphi)$ of a morphism $\varphi\colon \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(d)$, namely of a global section s of the line bundle $\mathcal{O}_{\mathbb{P}(V)}(d)$. In particular, X belongs to the class $c_1(\mathcal{O}_{\mathbb{P}(V)}(d)) = dH$, where H is the class of a hyperplane in $\mathbb{P}(V)$. We denote by h the restriction of H to X, namely $h = c_1(\mathcal{O}_X(1))$.

In this case, the short exact sequence in (1.6.2) becomes

$$0 \longrightarrow \mathcal{T}X \longrightarrow \mathcal{T}\mathbb{P}(V)|_X \longrightarrow \mathcal{N}_{X/\mathbb{P}(V)} \longrightarrow 0.$$

Note that $\mathcal{N}_{X/\mathbb{P}(V)}$ is in fact a line bundle on X, since X is a hypersurface. Moreover, it turns out that $\mathcal{N}_{X/\mathbb{P}(V)} = \mathcal{O}_X(d)$ (see for example [Huy, Proposition 2.4.7]). Then, from the Whitney sum property we have that

$$c(\mathcal{T}X)c(\mathcal{O}_X(d)) = c(\mathcal{T}\mathbb{P}(V)|_X),$$

where $c(\mathcal{T}\mathbb{P}(V)|_X) = (1+h)^n$ following Example 1.7.1 and $c(\mathcal{O}_X(d)) = 1 + dh$. Therefore the total Chern class of X (that is, the total Chern class of $\mathcal{T}X$) is

$$c(\mathcal{T}X) = \frac{(1+h)^n}{1+dh} = \sum_{i,j=0}^{n-2}\binom{n}{i}(-d)^j h^{i+j} = \sum_{s=0}^{n-2}\left[\sum_{i=0}^{s}\binom{n}{i}(-d)^{s-i}\right]h^s. \quad (1.7.1)$$

In particular, since $\deg(h^{n-2}) = \deg(X) = d$, the degree of $c_{\mathrm{top}}(\mathcal{T}X)$ is

$$\deg(c_{\mathrm{top}}(\mathcal{T}X)) = \deg(c_{n-2}(\mathcal{T}X)) = d\sum_{i=0}^{n-2}\binom{n}{i}(-d)^{n-2-i}$$

and coincides with the *topological Euler characteristic* of X, by the *Gauss-Bonnet formula* (see for example [Huy, p. 235]).

1.8 ED degree formulas in terms of Chern and Chern-Mather classes

The main result recovered in Theorem 1.5.5 is a formula, based on polar classes, for the ED degree of a projective variety $X \subset \mathbb{P}(V)$ which is somehow transversal to the isotropic quadric Q.

In the previous section, we recovered the computation of the Chern classes $c_i(X) = c_i(\mathcal{T}X)$ of a smooth projective variety $X \subset \mathbb{P}(V)$ of dimension m. It turns out that these invariants are related with the polar classes $\delta_i(X)$ by the following formula (see [Hol]):

$$\delta_i(X) = \sum_{j=0}^{m-i}(-1)^j\binom{m+1-j}{i+1}c_j(X)\cdot h^{m-j} \quad \forall\, 0 \le i \le n-2, \qquad (1.8.1)$$

where $h = c_1(\mathcal{O}_X(1))$ is the hyperplane class.

The last identities lead to the following important result, which is an alternative formulation of the *Catanese-Trifogli formula* for the ED degree of X (see [CT, p. 6026]).

Theorem 1.8.1. *[DHOST, Theorem 5.8] Let X be a smooth irreducible variety of dimension m in $\mathbb{P}(V)$, and suppose that X is transversal to the isotropic quadric Q. Then*

$$\mathrm{EDdegree}(X) = \sum_{i=0}^{m} (-1)^i (2^{m+1-i} - 1) c_i(X) \cdot h^{m-i},$$

where $h = c_1(\mathcal{O}_X(1))$ is the hyperplane class.

The last result or its original formulation by Catanese and Trifogli is useful to determine the EDdegree of a smooth variety $X \subset \mathbb{P}(V)$, at least when its Chern classes are enough easy to compute. For example, one might derive the ED degree of a general hypersurface in $\mathbb{P}(V)$.

Proposition 1.8.2. *For all $n \geq 2$ define the integer*

$$N := \begin{cases} 2(n-1) & \text{if } d = 2 \\ d\frac{(d-1)^{n-1}-1}{d-2} & \text{if } d \geq 3. \end{cases} \tag{1.8.2}$$

If the hypersurface $X \subset \mathbb{P}(V)$ of degree d is general, then $\mathrm{EDdegree}(X) = N$.

The above result is a special case of [DHOST, Corollary 2.10]. For example, a general plane curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ has $\mathrm{EDdegree}(C) = d^2$ and a general surface $S \subset \mathbb{P}^3_{\mathbb{C}}$ has $\mathrm{EDdegree}(S) = d(d^2 - d + 1)$.

Remaining in the case of a projective hypersurface $X \subset \mathbb{P}(V)$, things get more difficult if we allow X to have isolated singularities. Recalling from [Dol, §1.2.3] that $\mu(X, x)$ is the *Milnor number* of an isolated singularity $x \in X$ (more in general, of a complete intersection variety $X \subset \mathbb{P}(V)$), we have the following ED degree formula.

Proposition 1.8.3. *[Pie15, p. 146] Let $X \subset \mathbb{P}(V)$ be a hypersurface of degree d with only isolated singularities. For any point $x \in X_{\mathrm{sing}}$, let*

$$e(X, x) := \mu(X, x) + \mu(H \cap X, x),$$

where H is a general hyperplane section of X containing x. Then

$$\mathrm{EDdegree}(X) = N - \sum_{x \in X_{\mathrm{sing}}} e(X, x), \tag{1.8.3}$$

where the integer N was defined in (1.8.2).

It is known (see [Dol, Example 1.2.3]) that if $x \in X$ is a singular point of type A_k, then

$$\mu(X, x) = k, \quad \mu(H \cap X, x) = 1.$$

This gives the formula of the ED degree of a hypersurface with s singularities of type $A_{k_1}, \ldots, A_{k_s}$

$$\mathrm{EDdegree}(X) = N - (k_1 + 1) - \cdots - (k_s + 1).$$

In particular, if we consider a plane curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d with with δ ordinary nodes and κ ordinary cusps, the ED degree of C is

$$\mathrm{EDdegree}(C) = d^2 - 2\delta - 3\kappa. \tag{1.8.4}$$

Proposition 1.8.3 is the first example recalled of an ED degree formula related to non-smooth projective varieties. In this wider perspective, the Chern classes of smooth projective varieties are replaced by the so-called *Chern-Mather* classes, introduced by MacPherson in [Mac74]. Another excellent introduction to Chern-Mather classes is furnished by Aluffi in [Alu18].

Roughly speaking, Chern-Mather classes are constructed as follows. Let $X \subset \mathbb{P}(V)$ be a projective variety of dimension m and consider the Gauss map γ_X defined in (1.5.2). The *Nash blow-up* of X is the closure $\widetilde{X}$ of the image of γ_X. It comes equipped with a proper map $\nu\colon \widetilde{X} \to X$.

Now let $\mathcal{U} \to G(m+1, V)$ be the *universal bundle* over the Grassmannian $G(m+1, V)$, where $\mathcal{U} := \{(v, W) \in V \times G(m+1, V) \mid v \in W\}$. In particular, $\mathrm{rk}(\mathcal{U}) = m + 1$. Similarly as in the proof of Proposition 1.6.7, the vector bundle $\mathcal{U}$ gives a short exact sequence

$$0 \to \mathcal{U} \to \mathcal{O} \otimes V \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ denotes again the quotient bundle. Then $\mathrm{rk}(\mathcal{Q}) = n - k - 1$ and $\mathcal{Q} \otimes \mathcal{U}^\vee$ is isomorphic to the tangent bundle $\mathcal{T}G(m+1, V)$.

Since $\widetilde{X}$ is a smooth variety, we can consider the Chern classes of the restriction to $\widetilde{X}$ of the universal bundle $\mathcal{U}$ and, in turn, their push-forward to X with respect to the map ν. The resulting classes are called the *Chern-Mather classes* of X and are denoted as $c_i^M(X)$. They agree with Chern classes if X is smooth.

Aluffi proves in [Alu18] the following generalization of Theorem 1.8.1. Below we use a slightly different convention than in [Alu18]. Indeed, for us $c_i^M(X)$ is the component of dimension $m - i$ (as with standard Chern classes), while in Aluffi's paper it is the component of dimension i.

Theorem 1.8.4. *[Alu18, Proposition 2.9] Let X be an irreducible variety of dimension m in $\mathbb{P}(V)$, and suppose that X is transversal to the isotropic quadric Q. Then*

$$\mathrm{EDdegree}(Y) = \sum_{i=0}^{m} (-1)^i \left(2^{m+1-i} - 1\right) c_i^M(Y) \cdot h^{m-i}. \tag{1.8.5}$$

The polar classes $\delta_i(X)$ of a non-smooth projective variety $X \subset \mathbb{P}(V)$ may be written in terms of Chern-Mather classes as well, thus generalizing the classical formula in (1.8.1). This generalization is due to Piene ([Pie88, Theorem 3] and [Pie78]), see also [Alu18, Proposition 3.13]:

$$\delta_i(X) = \sum_{j=0}^{m-i} (-1)^j \binom{m+1-j}{i+1} c_j^M(X) \cdot h^{m-j} \quad \forall\, 0 \leq i \leq n-2. \tag{1.8.6}$$

The integer at the right-hand side of (1.8.6) is always nonnegative. For example, when $\delta_0(X) \neq 0$, then $X^\vee$ is a hypersurface (see Theorem 1.5.4) of degree

$$\deg(X^\vee) = \delta_0(X) = \sum_{j=0}^{m} (-1)^j\, (m+1-j)\, c_j^M(X) \cdot h^{m-j}. \tag{1.8.7}$$

Chapter 2

The distance from the variety of rank-one symmetric tensors

In the introduction, we focused on the *best rank-one approximation problem* for rectangular matrices as well as higher-order format tensors, with respect to the *Frobenius inner product*. The key result, for matrices, is the Eckart-Joung Theorem which uses the SVD of a matrix. The tensor counterpart is due to Lim and is reported in Theorem 0.0.5. It is based on singular values and singular vector tuples of tensors. In particular, we intentionally did not consider tensors with (partial) symmetry.

Actually, we may think of the Eckart-Joung Theorem as a generalization of the classical Spectral Theorem, which is related to real symmetric matrices. This important result tells us that, given a symmetric bilinear operator $f \in S^2 V^{\mathbb{R}}$, then f admits n real eigenvectors $u_1, \ldots, u_n \in V^{\mathbb{R}}$ forming an orthonormal basis of $V^{\mathbb{R}}$, and n associated real eigenvalues $\lambda_1, \ldots, \lambda_n$, not necessarily distinct. Each eigenvector-eigenvalue pair (u_j, λ_j) is a solution of the equation

$$f(u) = \lambda u \Longleftrightarrow (f - \lambda Id)(u) = 0, \tag{2.0.1}$$

where Id denotes the identity operator in $S^2 V^{\mathbb{R}}$. In particular, each eigenvalue λ_j is such that the operator $f - \lambda_j Id$ is not invertible. If we call A_f and I the symmetric matrices associated to the operators f and Id, respectively, we have that each λ_j is a root of the *characteristic polynomial*

$$\psi_{A_f}(\lambda) \coloneqq \det(A_f - \lambda I). \tag{2.0.2}$$

The Spectral Theorem tells us more. If we let $U = (u_1 | \cdots | u_n)$ be the orthogonal matrix formed by the normalized eigenvectors of f, and we let $D =$

$\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ be the matrix whose diagonal elements are the eigenvalues of f, then A_f admits the *spectral decomposition*

$$A_f = UDU^T = \lambda_1 u_1^2 + \cdots + \lambda_n u_n^2. \tag{2.0.3}$$

Moreover, assume that $S^2V^{\mathbb{R}}$ is equipped with the restriction of the Frobenius inner product $q_F^{\mathbb{R}}$ introduced in (0.0.4) for matrices and later generalized to higher format tensors in (0.0.19). This makes $(S^2V^{\mathbb{R}}, q_F^{\mathbb{R}})$ a real Euclidean space. Then all rank-one symmetric matrices which are critical points of the function $\delta_{A_f}^{\mathbb{R}}$ defined as in (0.0.3) with respect to the inner product $q_F^{\mathbb{R}}$ are of the form

$$X = UD_iU^T = \lambda_i u_i^2, \quad D_i := \mathrm{diag}(0, \ldots, \lambda_i, \ldots, 0). \tag{2.0.4}$$

What is more, the closest rank-one symmetric matrix to A_f corresponds to the largest eigenvalue of f in absolute value, as the following relation suggests:

$$q_F^{\mathbb{R}}(A_f - \lambda_i u_i^2) - q_F^{\mathbb{R}}(A_f - \lambda_j u_j^2) = \lambda_i^2 - \lambda_j^2 \quad \forall 1 \le i, j \le n.$$

The goal of this chapter is basically to replay the above itinerary in the space of real symmetric tensors of degree $d \ge 3$. If $(V^{\mathbb{R}}, q^{\mathbb{R}})$ is, as usual, an n-dimensional real Euclidean space, the d-th symmetric power $S^dV^{\mathbb{R}} \subset V^{\mathbb{R}\otimes d}$ describes all the symmetric tensors of degree d over $V^{\mathbb{R}}$. The set of all real symmetric tensors of (symmetric) rank one is the affine cone of the d-th Veronese embedding of $V^{\mathbb{R}}$. In this chapter, we indicate this affine cone with $X_{(d)}^{\mathbb{R}}$. We equip the space $S^dV^{\mathbb{R}}$ with the restriction $q_F^{\mathbb{R}}$ of the Frobenius inner product over $V^{\mathbb{R}\otimes d}$ induced by $q^{\mathbb{R}}$. As pointed out in the introduction, for our investigations we need to extend the Euclidean space $(V^{\mathbb{R}}, q^{\mathbb{R}})$ to its complexification (V, q). This, in turn, yields an extension from the real Euclidean space $(S^dV^{\mathbb{R}}, q_F^{\mathbb{R}})$ to the complex space (S^dV, q_F). A precise definition of q_F is furnished in Section 2.1. The variety defined by the common complex zeros of the elements in $I(X_{(d)}^{\mathbb{R}})$ is denoted by $X_{(d)}$.

In the assumption given above, for a given real symmetric tensor $f \in S^dV^{\mathbb{R}}$, one could study the critical points on $X_{(d)}^{\mathbb{R}}$ of the squared distance function $\delta_{F,f}^{\mathbb{R}}(g) := q_F^{\mathbb{R}}(f - g)$. By a result of Lim and Qi, these, in turn, are defined in terms of the so-called *E-eigenvectors* and *E-eigenvalues* of f. E-eigenvectors and E-eigenvalues of symmetric tensors may be regarded as the cornerstone of the *Spectral Theory* of symmetric tensors. Their definition is clearly inspired by the normalized eigenvalues and eigenvectors of a symmetric matrix ($d = 2$). As a matter of fact, one might frame E-eigenvectors and E-eigenvalues as the symmetric counterpart of the singular vector d-ples and singular values of a nonsymmetric tensor in $V^{\mathbb{R}\otimes d}$, which we presented in Theorem 0.0.4. In Section 2.1 we recall the definition of E-eigenvectors and E-eigenvalues and their main properties.

The other main character of this chapter is the *E-characteristic polynomial* $\psi_f(\lambda)$ of a symmetric tensor $f \in S^dV$, which generalizes well the notion of characteristic polynomial $\psi_f(\lambda) = \det(f - \lambda I_n)$ of a symmetric matrix $f \in S^2V$. It is defined in Section 2.2. The E-eigenvalues of $f \in S^dV$ are roots of ψ_f, but the converse is true only for *regular* symmetric tensors (see Definition 2.2.6 and [Qi07, Theorem 4]). In particular, for a general symmetric tensor $f \in S^dV$, the degree of $\psi_f(\lambda)$ is equal to the number of (distinct) E-eigenvalues of f, and corresponds to the ED degree of $X \subset S^dV$ (introduced in general in Definition 1.2.3) with respect to the isotropic quadric $Q_F := \mathcal{V}(q_F)$. The computation of this number is due to Cartwright and Sturmfels and is reported below.

Theorem 2.0.1. *[CS, Theorem 5.5] For all $n \geq 2$, define the integer $N := n$ for $d = 2$, whereas $N := ((d-1)^n - 1)/(d-2)$ for $d \geq 3$. Every symmetric tensor $f \in S^dV$ has at most N distinct E-eigenvalues when d is even, and at most N pairs $(\lambda, -\lambda)$ of distinct E-eigenvalues when d is odd. This bound is attained for general symmetric tensors. In particular,* $\mathrm{EDdegree}(X) = N$ *with respect to the isotropic quadric Q_F.*

This fact was previously conjectured in [NQWW] and is confirmed by Friedland-Ottaviani formula in Theorem 5.1.1 (see also [OO]). However, this result had already essentially been known in complex dynamics due to Fornæss and Sibony, who in [FS] discuss global questions of iteration of rational maps in higher dimension.

Our main contributions in this chapter are related especially to the study of the coefficients of $\psi_f(\lambda)$. In the following, $\widetilde{Q}$ denotes the Veronese embedding in $\mathbb{P}(S^dV)$ of the isotropic quadric $Q := \mathcal{V}(q) \subset \mathbb{P}(V)$, whereas $\Delta_{\widetilde{Q}}(f)$ is the *$\widetilde{Q}$-discriminant* of f, namely the equation of $\widetilde{Q}^\vee$, when it is a hypersurface (see Section 2.1). We show that the highest coefficient of $\psi_f(\lambda)$, when it has maximum degree, is the $(d-2)$-th power (respectively the $((d-2)/2)$-th power) when d is odd (respectively when d is even) of $\Delta_{\widetilde{Q}}(f)$. This fact, together with a known formula for the lowest coefficient of $\psi_f(\lambda)$, leads to a closed formula for the product of the E-eigenvalues of f when ψ_f has maximum degree, which generalizes the fact that the determinant of a symmetric matrix is equal to the product of its eigenvalues.

Theorem 2.0.2. *[Sod18, Main Theorem] Let $f \in S^dV^{\mathbb{R}}$ be a real symmetric tensor of degree $d \geq 2$. If f admits the maximum number $N =$* $\mathrm{EDdegree}(X)$ *of E-eigenvalues (counted with multiplicity) defined in Theorem 2.0.1, then their product is*

$$\lambda_1 \cdots \lambda_N = \pm \frac{\mathrm{Res}\left(\frac{1}{d}\nabla f\right)}{\Delta_{\widetilde{Q}}(f)^{\frac{d-2}{2}}}. \tag{2.0.5}$$

Theorem 2.0.2 is the main result of this chapter and its proof is developed in Section 2.3. We note (see Lemma 2.3.10) that the assumption of Theorem 2.0.2 is satisfied for a general f, and it corresponds geometrically to the fact that the hypersurface $\mathcal{V}(f) \subset \mathbb{P}(V)$ is transversal to the isotropic quadric Q (see Remark 2.3.11). We stress that the polynomial $\mathrm{Res}\left(\frac{1}{d}\nabla f\right)$ appearing in the numerator of (2.0.5) is equal to the classical *discriminant* $\Delta_d(f)$ of f times a constant factor. For the definition of discriminant of a homogeneous polynomial and a relation between $\mathrm{Res}\left(\frac{1}{d}\nabla f\right)$ and $\Delta_d(f)$ we refer to Section 2.2. Moreover, the product of the E-eigenvalues of f is *a priori* equal to the right-hand side of (2.0.5) times a constant factor depending only on n and d. Using the definitions of resultant and $\widetilde{Q}$-discriminant, we prove via Lemma 2.3.1 that this constant factor is (in absolute value) 1 by specializing to the family of *scaled Fermat polynomials*. However, the identity (2.0.5) is given up to sign since the definition of E-eigenvalue has this sign ambiguity.

The combination of Theorem 2.0.1 and Theorem 2.0.2 shows that the degree of the E-characteristic polynomial ψ_f is equal to N (or $2N$, depending on d even or odd), whereas it is smaller than the "expected" one exactly when f admits at least an isotropic eigenvector. This in particular motivated our research on the geometric meaning of the vanishing of the leading coefficient of ψ_f. Therefore Theorem 2.0.2 describes that, if the coefficients of f annihilate the polynomial $\Delta_{\widetilde{Q}}(f)$, then some of the E-eigenvalues of f have gone "to infinity": in practice, f admits at least an isotropic eigenvector whose corresponding eigenvalue does not appear as a root of the E-characteristic polynomial ψ_f. We stress that both numerator and denominator in (2.0.5) are orthogonal invariants of f, namely polynomials in the coefficients of f that are invariant under the orthonormal linear changes of coordinates in f.

The results presented in this chapter will be generalized later in Chapter 5 in the context of partially symmetric tensors, including the case of nonsymmetric tensors presented in Theorem 0.0.8. Nevertheless, we chose to separate the case of symmetric tensors from all the other ones. The main reason is that, unlike all the other cases, some of the proofs are based on the theory of resultants of a homogeneous polynomial system, which we summarize in Section 2.2.

2.1 E-eigenvalues and E-eigenvectors of symmetric tensors

In this section we recall the main properties of E-eigenvectors and E- eigenvalues of symmetric tensors. We start by setting our notation more in detail. Let $V^{\mathbb{R}}$ be an n-dimensional real vector space. Given an integer $d \geq 2$, the tensor product

$V^{\mathbb{R}\otimes d}$ of d copies of $V^{\mathbb{R}}$ describes all real tensors of format $n^{\times d}$, or tensors of order d on $V^{\mathbb{R}}$. An excellent reference for spaces of tensors and the algebraic geometry related to them is [Lan]. Consider the projection operator $\pi_S \colon V^{\mathbb{R}\otimes d} \to V^{\mathbb{R}\otimes d}$ defined on decomposable elements by

$$\pi_S(x_1 \otimes \cdots \otimes x_d) = \frac{1}{d!} \sum_{\sigma \in S_d} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(d)},$$

where S_d is the group of permutations of d elements. The map π_S is in fact a projection, since $\pi_S^2(T) = \pi_S(T)$ for all $T \in V^{\mathbb{R}\otimes d}$.

Definition 2.1.1. The *symmetric product* of d vectors $x_1,\ldots,x_d$ is the vector $x_1 \cdots x_d := \pi_S(x_1 \otimes \cdots \otimes x_d) \in V^{\mathbb{R}\otimes d}$. The image $\pi_S(V^{\mathbb{R}\otimes d})$ is called the *d-th symmetric power* of $V^{\mathbb{R}}$ and it is denoted by $S^d V^{\mathbb{R}}$. Its elements are called *symmetric tensors of order* d on $V^{\mathbb{R}}$.

Actually, if we want to consider real tensors of order d on $V^{\mathbb{R}}$ as d-linear maps $V^{\mathbb{R}\times d} \to \mathbb{R}$, we should consider the tensor product $(V^{\mathbb{R}*})^{\otimes d}$, where $V^{\mathbb{R}*} := \{f \colon V^{\mathbb{R}} \to \mathbb{R} \mid f \text{ is linear}\}$ is the dual vector space of $V^{\mathbb{R}}$. In particular, the space $S^d V^{\mathbb{R}*}$ corresponds to the space of symmetric d-linear maps on $V^{\mathbb{R}}$, namely d-linear maps $f \colon V^{\mathbb{R}\times d} \to \mathbb{R}$ such that, for all $x_1, \ldots, x_d \in V^{\mathbb{R}}$,

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) = f(x_1, \ldots, x_d)$$

for every permutation $\sigma \in S_d$.

In fact, there is another interpretation of $S^d V^{\mathbb{R}^*}$ as the space of degree d homogeneous polynomial functions on $V^{\mathbb{R}}$. In other words, given a multilinear map $f \in S^d V^{\mathbb{R}^*}$, the map sending $x \in V^{\mathbb{R}}$ to $f(x, \ldots, x) \in \mathbb{R}$ is polynomial and homogeneous of degree d. Thus, having fixed a basis $(x_1, \ldots, x_n)$ of $V^{\mathbb{R}}$, a symmetric tensor $f \in S^d V^{\mathbb{R}^*}$ may be seen simply as a d-dimensional array $f = (c_{i_1 \cdots i_d})_{1 \le i_j \le n}$ of real numbers which are symmetric under permutations of the indices. The corresponding homogeneous polynomial of degree d is

$$f = \sum_{i_1, \ldots, i_d = 1}^{n} c_{i_1 \cdots i_d} x_{i_1} \cdots x_{i_d}.$$

Actually, we adopt a different notation and write f as

$$f = \sum_{|\alpha| = d} \binom{d}{\alpha} f_\alpha x^\alpha, \tag{2.1.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\binom{d}{\alpha} := \frac{d!}{\alpha_1! \cdots \alpha_n!}$ is the multinomial coefficient. In particular, we suppose that $(f_\alpha)_\alpha$ is a system of coordinates for $S^d V^{\mathbb{R}}$.

As we show throughout this chapter, the advantage of treating symmetric tensors as homogeneous polynomials (or *forms*) is that we can associate with every $f \in S^d V^{\mathbb{R}^*}$ a projective hypersurface $X_f := \mathcal{V}(f) \subset \mathbb{P}(V^{\mathbb{R}})$ of degree d. Geometrically speaking, it is obtained as a hyperplane section of $X^{\mathbb{R}}_{(d)}$, here viewed as a projective variety in $\mathbb{P}(S^d V^{\mathbb{R}^*})$.

Remark 2.1.2. In the following, we fix a positive definite symmetric bilinear form $q^{\mathbb{R}}$ on $V^{\mathbb{R}}$. This assumption allows us to identify the two vector spaces $V^{\mathbb{R}}$ and $V^{\mathbb{R}^*}$. Therefore, if not specified, we always deal with $V^{\mathbb{R}\otimes d}$ as well as $S^d V^{\mathbb{R}}$ even if its elements are viewed as (symmetric) multilinear maps or, in the symmetric case, homogeneous polynomials. Indeed, our aim is to focus more on the metric properties of $S^d V^{\mathbb{R}^*}$.

Definition 2.1.3. A symmetric tensor $f \in S^d V^{\mathbb{R}}$ has *(real) symmetric rank one* if $f = l^d$ for some linear form $l = \xi_1 x_1 + \cdots + \xi_n x_n \in V^{\mathbb{R}}$. Symmetric tensors of rank one fill the affine cone $X^{\mathbb{R}}_{(d)}$ of the image of the *d-th Veronese embedding* of $\mathbb{P}(V^{\mathbb{R}})$

$$v_d \colon \mathbb{P}(V^{\mathbb{R}}) \hookrightarrow \mathbb{P}(S^d V^{\mathbb{R}}), \quad v_d([l]) := [l^d].$$

More in general, the *(real) symmetric rank* of $f \in S^d V^{\mathbb{R}}$ is the smallest positive integer r such that $f = l_1^d + \cdots + l_r^d$ for some linear forms $l_j \in V^{\mathbb{R}}$.

We remark that, in a slightly wider perspective, the variety $v_d(\mathbb{P}(V^{\mathbb{R}}))$ is the restriction to $\mathbb{P}(S^d V^{\mathbb{R}})$ of the Segre embedding of d copies of $\mathbb{P}(V^{\mathbb{R}})$ introduced in its full generality in (0.0.17).

In the introduction of this chapter, we set the Frobenius inner product as the Euclidean structure over the space $S^d V^{\mathbb{R}}$. How does it work explicitly?

In a coordinate-free way, the Frobenius inner product $q^{\mathbb{R}}_F$ of two symmetric tensors of rank one l^d and $\tilde{l}^d$ is

$$q^{\mathbb{R}}_F(l^d, \tilde{l}^d) := q^{\mathbb{R}}(l, \tilde{l})^d, \tag{2.1.2}$$

where $q^{\mathbb{R}}$ is a fixed inner product on $V^{\mathbb{R}}$. If we consider two symmetric products of linear forms $f = l_1 \cdots l_d$ and $g = \tilde{l}_1 \cdots \tilde{l}_d$, then

$$q^{\mathbb{R}}_F(f, g) = \frac{1}{d!} \sum_{\sigma \in S_d} q^{\mathbb{R}}(l_1, \tilde{l}_{\sigma(1)}) \cdots q^{\mathbb{R}}(l_d, \tilde{l}_{\sigma(d)}).$$

More generally, if we consider two symmetric tensors $f = (f_\alpha)_\alpha$ and $g = (g_\alpha)_\alpha$ written as in (2.1.1), and if we assume that the basis $(x_1, \ldots, x_n)$ of $V^{\mathbb{R}}$ is orthonormal, it turns out that

$$q^{\mathbb{R}}_F(f, g) = \sum_{|\alpha| = d} \binom{d}{\alpha} f_\alpha g_\alpha \ .$$

Thus in particular the squared Frobenius norm of $f = (f_\alpha)_\alpha \in S^d V^{\mathbb{R}}$ is

$$q_F^{\mathbb{R}}(f) = q_F^{\mathbb{R}}(f, f) = \sum_{|\alpha|=d} \binom{d}{\alpha} f_\alpha^2 \, . \tag{2.1.3}$$

Definition 2.1.4. Let $f \in S^d V^{\mathbb{R}}$ and consider the function $\delta_{F,f}^{\mathbb{R}} \colon X_{(d)}^{\mathbb{R}} \to \mathbb{R}$ defined by $\delta_{F,f}^{\mathbb{R}}(x^d) := q_F^{\mathbb{R}}(f - x^d)$ for all $x \in V^{\mathbb{R}}$. Then a *critical rank-one symmetric tensor* for f is a critical point $x^d \in X_{(d)}^{\mathbb{R}}$ of $\delta_{F,f}^{\mathbb{R}}$.

The notions of *E-eigenvalue* and *E-eigenvector* of a symmetric tensor were proposed independently by Lek-Heng Lim and Liqun Qi in [Lim, Qi05] in the more general setting of n-dimensional tensors of order d on V, namely elements of $V^{\otimes d}$. There are different types of eigenvectors and eigenvalues in the literature, see [CS, HHLQ, NQWW, Qi07, QL].

Definition 2.1.5. Given a symmetric tensor $f \in S^d V$, a nonzero vector $x \in V$ such that $q(x) = 1$ is called an *E-eigenvector* of f (where the "E" stands for "Euclidean") if there exists $\lambda \in \mathbb{C}$ such that x is a solution of the equation

$$q_F(f, x^{d-1} \cdot \_) = \lambda q(x, \_) \tag{2.1.4}$$

The scalar λ corresponding to x is called an *E-eigenvalue* of f, while the pair (λ, x) is called an *E-eigenpair* of f. The corresponding power $x^d \in S^d V$ is called an *E-eigentensor* of f. In particular, for even order d, (λ, x) is an E-eigenpair of f if and only if $(\lambda, -x)$ is so; for odd order d, (λ, x) is an E-eigenpair of f if and only if $(-\lambda, -x)$ is so.

If $x \in V$ is a solution of (2.1.4) such that $q(x) = 0$, we call x an *isotropic eigenvector* of f.

An E-eigenvalue λ of $f \in S^d V^{\mathbb{R}}$ is called a *Z-eigenvalue* of f if it has a real E-eigenvector x. In this case, the corresponding E-eigenvector x is called a *Z-eigenvector* of f associated with λ.

By an *eigenvector* of f we mean any solution of equation (2.1.4), whether it has unit norm or not.

Note that both sides of relation (2.1.4) correspond to linear operators on V. If we assume that q is defined as $q(x) = x_1^2 + \cdots + x_n^2$, we can rewrite the system (2.1.4) as

$$\frac{1}{d} \nabla f(x) = \lambda x, \tag{2.1.5}$$

where ∇f is the gradient of the corresponding homogeneous polynomial. The factor $1/d$ appearing in (2.1.5) follows the notation in [Qi05] conformed to the symmetric case.

Observe that, if (λ, x) satisfies (2.1.4), then $(\alpha^{d-2}\lambda, \alpha x)$ satisfies (2.1.4) for any nonzero $\alpha \in \mathbb{C}$. This is why we impose the additional quadratic equation $q(x) = 1$ in Definition 2.1.5.

At this point, it is important to stress how different is the behavior of symmetric tensors of order $d \geq 3$ compared with symmetric matrices ($d = 2$). First of all, Definition 2.1.5 agrees with the standard definition of normalized eigenvalue and normalized eigenvector of a symmetric matrix. Instead, if we consider nonsymmetric square matrices, then the notion of E-eigenvector excludes the nonzero complex eigenvectors x such that $q(x) = 0$. In addition, despite the case of symmetric matrices, there exist real symmetric tensors of order $d \geq 3$ admitting non-real E-eigenvalues. An example is shown below.

Example 2.1.6. Assume that $n = 2$, $d = 5$ and consider the symmetric tensor

$$f(x_1, x_2) = x_1^5 + x_2^5,$$

written as a binary form of degree five. Assuming that $q(x_1, x_2) = x_1^2 + x_2^2$, the system analogous to (2.1.5), together with the normalization condition, is

$$\begin{cases} x_1^4 = \lambda x_1 \\ x_2^4 = \lambda x_2 \\ x_1^2 + x_2^2 = 1 \end{cases}$$

Let $\theta = e^{\frac{2}{3}\pi\sqrt{-1}}$ be a third root of unity. It is not difficult to verify that $x = (\theta/\sqrt{1+\theta^2}, 1/\sqrt{1+\theta^2})$ is a non-real E-eigenvector of f with corresponding E-eigenvalue $\lambda = \sqrt{-1}$. The binary form f is an instance of a Fermat polynomial, which are treated in detail in Lemma 2.3.1.

The absence of non-real eigenvalues and eigenvectors is just one of the properties of symmetric matrices that fail for symmetric tensors of higher degree. Nevertheless, a positive result is the following:

Theorem 2.1.7. *[QL, Theorem 2.18] Consider a symmetric tensor $f \in S^d V^{\mathbb{R}}$. Then f always has Z-eigenvalues.*

Z-eigenvalues are important since they are the fundamental tool for determining the best rank-one approximation of a real symmetric tensor. A remarkable fact observed in [Lim, Qi05] is that the Z-eigenvectors of $f \in S^d V^{\mathbb{R}}$ correspond to the critical points of the function $f\colon V^{\mathbb{R}} \to \mathbb{R}$ restricted on the affine variety $\{x \in V^{\mathbb{R}} \mid q(x) = 1\}$. Hence, if $q(x) = x_1^2 + \cdots + x_n^2$, the Z-eigenvectors of f are the normalized real solutions x, in orthonormal coordinates, of:

$$\operatorname{rank}\begin{pmatrix} \nabla f(x) \\ x \end{pmatrix} \leq 1.$$

Theorem 2.1.8 (Lim, variational principle). *Suppose that $f \in S^d V^{\mathbb{R}}$. The critical rank-one symmetric tensors for f are exactly of the form x^d, where x is an eigenvector of f. Moreover, if λ_* is a Z-eigenvalue of f such that $|\lambda_*|$ is maximum, and if x_* is a Z-eigenvector associated with λ_*, then $\lambda_* x_*^d$ is a best rank-one approximation of f.*

This interpretation is used by Draisma, Ottaviani and Tocino in [DOT], where they deal more in general with the *best rank-k approximation problem* for tensors.

Let us examine again Definition 2.1.5. Another consequence is the following property, where we are assuming (without loss of generality) that the complex form q is induced by the standard inner product $q^{\mathbb{R}}(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ on $V^{\mathbb{R}}$.

Proposition 2.1.9. *Let $f \in S^d V$. If (λ, x) is an E-eigenpair of f, then $\lambda = f(x)$.*

Proof. Apply the operator $q(\_, x)$ on both sides of equation (2.1.4). Then we have

$$q\left(\frac{1}{d}\nabla f(x), x\right) = q(\lambda x, x).$$

Using Euler's identity, the left hand side of last identity is equal to $f(x)$, whereas by linearity and the fact that $q(x) = 1$ the right-hand side is equal to λ. □

Note that, as a consequence of the previous result, every Z-eigenvalue of $f \in S^d V^{\mathbb{R}}$ is a real E-eigenvalue, but a real E-eigenvalue is not necessarily a Z-eigenvalue.

So far we did not mention another important property of E-eigenvalues, which somehow motivated Definition 2.1.5. We are talking about their invariance with respect to an orthonormal linear change of coordinates in V. More precisely, the set of automorphisms $A \in \mathrm{Aut}(V)$ that preserve the bilinear product $q(\_, \_)$, i.e., such that $q(Ax, Ay) = q(x, y)$ for all $x, y \in V$, forms the orthogonal group $\mathrm{O}(V)$ and is a subgroup of $\mathrm{Aut}(V)$. The special orthogonal group $\mathrm{SO}(V)$ is defined as the set of all A in $\mathrm{O}(V)$ with determinant 1. An $\mathrm{SO}(V)$-*invariant* (or *orthogonal invariant*) for $f \in S^d V$ is a polynomial in the coefficients of f that does not vary under the action of $\mathrm{SO}(V)$ on the coefficients of f, where the above-mentioned action is the one induced by the linear action of $\mathrm{SO}(V)$ on the coordinates of V. We will treat in more detail this topic in Chapter 3. The main result is the following.

Theorem 2.1.10. *[Qi07, Theorem 1] Given $f \in S^d V$, the set of the E-eigenvalues of f is a $\mathrm{SO}(V)$-invariant of f.*

2.2 The E-characteristic polynomial of a symmetric tensor

In this section, we study a fundamental tool for computing the E-eigenvalues of a symmetric tensor, namely its *E-characteristic polynomial.* Actually, the definition of E-characteristic polynomial needs the knowledge of the *resultant* of a set of m homogeneous polynomials in m variables (see [CLO, GKZ]). It is introduced via the following proposition.

Proposition 2.2.1. *Let $f_1, \ldots, f_m$ be m homogeneous polynomials of positive degrees $d_1, \ldots, d_m$ respectively in the variables $z_1, \ldots, z_m$. Then there is a unique polynomial $\operatorname{Res}(f_1, \ldots, f_m)$ over $\mathbb{Z}$ in the coefficients of $f_1, \ldots, f_m$ such that*

i) $\operatorname{Res}(f_1, \ldots, f_m) = 0$ *if and only if the system $f_1 = \cdots = f_m = 0$ has a solution in $\mathbb{P}^{m-1}_{\mathbb{C}}$.*

ii) $\operatorname{Res}\left(\frac{1}{d}\nabla f\right) = (a_1 \cdots a_m)^{(d-1)^{m-1}}$, *where $f(z_1, \ldots, z_m) = a_1 z_1^d + \cdots + a_m z_m^d$, $a_1, \ldots, a_m \in \mathbb{C}$, is the* scaled Fermat polynomial.

iii) $\operatorname{Res}(f_1, \ldots, f_m)$ *is irreducible, even when regarded as a polynomial over $\mathbb{C}$ in the coefficients of $f_1, \ldots, f_m$.*

The normalization assumption of *ii)* coincides with the classical definition made in [CLO, Chapter 3, Theorem 2.3 and Theorem 3.5] and in [GKZ, p. 427].

The degree of the resultant is known in general.

Proposition 2.2.2. $\operatorname{Res}(f_1, \ldots, f_m)$ *is a homogeneous polynomial of degree $\prod_{j \neq i} d_j$ with respect to the coefficients of f_i for all $i \in [m]$. Hence the total degree of* $\operatorname{Res}(f_1, \ldots, f_m)$ *is*

$$\deg \operatorname{Res}(f_1, \ldots, f_m) = \sum_{i=1}^{m} d_1 \cdots d_{i-1} d_{i+1} \cdots d_m.$$

In particular, when all the forms $f_1, \ldots, f_m$ have the same degree d, the resultant has degree d^{m-1} in the coefficients of each f_i, namely $\deg \operatorname{Res}(f_1, \ldots, f_m) = md^{m-1}$.

The notion of resultant is closely related to the classical notion of *discriminant* of a homogeneous polynomial of degree d in m variables. An excellent reference for the theory of resultants and discriminants is [GKZ]. The problem of computing the discriminant of a homogeneous polynomial is a particular case of a more general geometric problem, that is, finding the equations of the dual $X^\vee$ of an irreducible projective variety $X \subset \mathbb{P}(V)$ (see [GKZ, Hol, Tev]), which in this thesis was introduced in Definition 1.3.1.

As we outlined in (1.3.1), the dual variety $X^\vee$ is obtained as the image of the conormal variety $\mathcal{N}(X, X^\vee)$ via the projection onto $V = V^*$ and is an irreducible variety. Moreover, since $\dim(\mathcal{N}(X, X^\vee)) = n-2$, it follows that $\dim(X^\vee) \le n-2$ and we expect that in "typical" cases $X^\vee$ is a hypersurface.

Definition 2.2.3. Let $X \subset \mathbb{P}(V)$ be a projective variety. If $X^\vee$ is a hypersurface, then it is defined by the vanishing of a homogeneous polynomial, denoted by Δ_X and called the X-*discriminant*. We assume the X-discriminant to have relatively prime integer coefficients: in this way, Δ_X is defined up to sign. If $X^\vee$ is not a hypersurface, then we set $\Delta_X := 1$.

If $X \subset \mathbb{P}(V)$ is an irreducible variety such that $X^\vee$ is a hypersurface, then Δ_X is an irreducible homogeneous polynomial over the complex numbers. When X is the Veronese variety $X_{(d)}$ introduced in Definition 2.1.3, then it is known that, for all $d > 1$, $X_{(d)}^\vee$ is a hypersurface and its equation coincides, up to a constant factor, with the discriminant $\Delta_d(h)$ of a homogeneous polynomial h of degree d in n variables.

Now we have all the necessary tools to introduce the E-characteristic polynomial of a symmetric tensor. If not otherwise specified, we assume that $q(x) = x_1^2 + \cdots + x_n^2$ is the complex form associated with the standard Euclidean inner product in $V^{\mathbb{R}}$.

Definition 2.2.4. Given $f \in S^dV$, when d is even the *E-characteristic polynomial* ψ_f of f is defined by $\psi_f(\lambda) := \operatorname{Res}(F_\lambda)$, where $\lambda \in \mathbb{C}$ and $\operatorname{Res}(F_\lambda)$ is the resultant of the n-dimensional vector

$$F_\lambda(x) := \frac{1}{d}\nabla f(x) - \lambda q(x)^{\frac{d-2}{2}} x. \tag{2.2.1}$$

When d is odd, the E-characteristic polynomial is defined as $\psi_f(\lambda) := \operatorname{Res}(G_\lambda)$, where $\operatorname{Res}(G_\lambda)$ is the resultant of the $(n+1)$-dimensional vector

$$G_\lambda(x_0, x) := \begin{pmatrix} x_0^2 - q(x) \\ \frac{1}{d}\nabla f(x) - \lambda x_0^{d-2} x \end{pmatrix}. \tag{2.2.2}$$

For $d = 2$, the E-characteristic polynomial agrees with the characteristic polynomial $\psi_{A_f}(\lambda)$ of a symmetric matrix A_f viewed in (2.0.2). In this case, the roots of ψ_{A_f} are all the eigenvalues of f, and if the coefficients of f are real, then the roots of ψ_{A_f} are all real by the Spectral Theorem. Moreover, the leading coefficient of ψ_{A_f} is 1, implying that its constant term is equal to the product of the eigenvalues of f, that is the determinant of A_f.

The interesting fact is that this happens only for $d = 2$: as we show throughout this chapter, given $f \in S^dV$ with $d > 2$, then some of the roots of the E-characteristic polynomial ψ_f may not be real even though the coefficients of f

are real. However, there exist symmetric tensors with only real E-eigenvalues, as shown by Maccioni in [Mac] and Kozhasov in [Koz]. Moreover, the leading coefficient of ψ_f is a homogeneous polynomial over $\mathbb{Z}$ in the coefficients of f with a positive degree for $d > 2$.

Theorem 2.1.10 states that the symmetric functions of the E-eigenvalues of f are orthogonal invariants of f, giving rise to the following corollary (see [LQZ, Theorem 3.3]):

Corollary 2.2.5. *Given $f \in S^dV$, all the coefficients of the E-characteristic polynomial ψ_f are $\mathrm{SO}(V)$-invariants of f.*

Given $f \in S^dV$, we observe that if d is even there exists a nonzero constant $c \in \mathbb{Z}$ such that

$$\psi_f(\lambda) := \mathrm{Res}(F_\lambda(x)) = c \cdot \Delta_d\left(f(x) - \lambda q(x)^{\frac{d}{2}}\right), \tag{2.2.3}$$

where the n-dimensional vector $F_\lambda(x)$ was introduced in (2.2.1) (see again [GKZ, Proposition XIII, 1.7]). On the other hand, a relation equivalent to (2.2.3) is no longer possible for odd d: in (2.2.2), an additional variable x_0 is required to make the polynomial ψ_f well-defined.

In the study of the E-characteristic polynomial ψ_f, a crucial role is played by a family of particular symmetric tensors, the ones admitting at least a singular point on the isotropic quadric Q.

Definition 2.2.6. A symmetric tensor $f \in S^dV$ is *irregular* if there exists a nonzero vector $x \in V$ such that $q(x) = 0$ and $\nabla f(x) = 0$. Otherwise f is called *regular*.

Clearly, any symmetric matrix ($d = 2$) corresponds to a regular symmetric tensor. Instead for $d \geq 3$ there exists symmetric tensors that are not regular.

Example 2.2.7. Assume that $n = 2$, $d = 4$ and consider the quartic binary form $f(x_1, x_2) = (x_1^2 + x_2^2)^2$. Then the condition $\nabla f(x) = 0$ becomes

$$\begin{cases} x_1(x_1^2 + x_2^2) = 0 \\ x_2(x_1^2 + x_2^2) = 0 \ . \end{cases}$$

Since the vectors $(1, \sqrt{-1})$ and $(1, -\sqrt{-1})$ are isotropic solutions of the previous system, then f is not regular. Note that, in this example, every vector $x \in V$ such that $q(x) = 1$ is an E-eigenvector of f with eigenvalue $\lambda = 1$ (see [OT, Lemma 3.7]).

The first general property about irregular symmetric tensors is that, when $d > 2$, their E-characteristic polynomial is identically zero.

Proposition 2.2.8. *Given $f \in S^d V$ with $d > 2$, if f is irregular then ψ_f is the zero polynomial.*

Proof. Suppose that f is irregular. Then, by Definition 2.2.6 there exists a nonzero vector $x \in V$ such that $q(x) = 0$ and $\nabla f(x) = 0$. Looking at Definition 2.2.4, this implies that, for even $d > 2$, x is a solution of the system $F_\lambda(x) = 0$ for all $\lambda \in \mathbb{C}$, whereas for odd $d > 2$, the pair $(0, x)$ is a solution of the system $G_\lambda(x_0, x) = 0$ for all $\lambda \in \mathbb{C}$. By resultant theory, this means that $\psi_f(\lambda) = 0$ for all $\lambda \in \mathbb{C}$, namely ψ_f is identically zero. □

Remark 2.2.9. The statement of Proposition 2.2.8 is no longer true for $d = 2$. In fact, for $d = 2$ and any $n \geq 1$ there exist irregular symmetric tensors $f \in S^d V$ such that ψ_f is not identically zero. For example, the polynomial $f(x) = (x_1 + \sqrt{-1}x_2)^2 + x_3^2 + \cdots + x_n^2$ is irregular because the vector $(1, \sqrt{-1}, 0, \ldots, 0)$ is a solution of $\nabla f(x) = 0$, whereas one can easily check that $\psi_f(\lambda) = \lambda^2(1 - \lambda)^{n-2}$, hence it is not identically zero.

The notion of regularity of a symmetric tensor plays a crucial role in the following result.

Theorem 2.2.10. *Suppose that $d \geq 3$. Given $f \in S^d V$, every E-eigenvalue of f is a root of the E-characteristic polynomial ψ_f. If f is regular, then every root of ψ_f is an E-eigenvalue of f.*

Proof. For completeness we recover and adapt the proofs in [Qi07, Theorem 4] and in [QL, Theorem 2.23]. Suppose that $x \in V$ is an E-eigenvector of f and $\lambda \in \mathbb{C}$ is the E-eigenvalue associated with λ. Then looking at Definition 2.2.4, when d is even we get that x and $-x$ are nonzero solutions of the system $F_\lambda(x) = 0$; when d is odd, $(1, x)$ and $(-1, -x)$ are nonzero solutions of the system $G_\lambda(x_0, x) = 0$. Therefore λ is a root of ψ_f by Proposition 2.2.1.

On the other hand, suppose that f is regular and let $\lambda \in \mathbb{C}$ be a root of ψ_f. By Definition 2.2.4 and Proposition 2.2.1, when d is even there exists a nonzero vector $x \in V$ such that $F_\lambda(x) = 0$ for that λ; when d is odd, there exists a nonzero vector $x \in V$ and $x_0 \in \mathbb{C}$ such that $G_\lambda(x_0, x) = 0$ for that λ. If $q(x) = 0$, both $F_\lambda(x) = 0$ and $G_\lambda(x_0, x) = 0$ yield the condition $\nabla f(x) = 0$, which cannot be satisfied because of the regularity of f. Hence $q(x) \neq 0$ and we define $\tilde{x} := x/q(x)$. Therefore, when d is even the equation (2.1.4) is satisfied by $(\lambda, \tilde{x})$ and $(\lambda, -\tilde{x})$, while for odd d it is satisfied by $(\lambda, \tilde{x})$ and $(-\lambda, -\tilde{x})$. This implies that λ is an E-eigenvalue of f. □

Example 2.2.11. Let us consider the case in which d is even and $f = q(x)^{\frac{d}{2}}$. Then equation (2.1.4) becomes $q(x)^{\frac{d-2}{2}}x = \lambda x$: this means, if $d = 2$, that every nonzero vector $x \in V$ such that $q(x) = 1$ is an E-eigenvector of f with E-eigenvalue $\lambda = 1$ (and in fact the E-characteristic polynomial of f is $\psi_f(\lambda) = (\lambda - 1)^n$). Instead for $d > 2$ every nonzero vector $x \in V$ such that $q(x) = 1$ is an E-eigenvector of f with corresponding E-eigenvalue $\lambda = 1$, and every nonzero vector $x \in V$ such that $q(x) = 0$ is an isotropic eigenvector of f. In particular f is irregular for $d > 2$, and in fact in this case the E-characteristic polynomial of f is identically zero by Proposition 2.2.8.

The greatest difference among eigenvectors of a symmetric matrix and eigenvectors of a symmetric tensor of degree $d > 2$ is related to the presence or not of isotropic eigenvectors. Suppose that $f \in S^dV$ admits an isotropic eigenvector x and let $P := [x]$ be the corresponding point of the isotropic quadric $Q \subset \mathbb{P}(V)$. In the same fashion of Proposition 2.1.9, this time we have that $f(x) = 0$, that is, $P \in X_f$. As we show in Section 2.3, equation (2.1.4) acquires a new interesting meaning: the isotropic eigenvectors of f are all the nonzero vectors x such that $[x] =: P \in X_f \cap Q$ and P is singular for X_f (and hence f is irregular) or P is smooth for X_f and X_f is tangent to Q at P.

We study more in detail the coefficients of the E-characteristic polynomial of a symmetric tensor. Given a general $f \in S^dV$, from Theorem 2.0.1 and Theorem 2.2.10 we have that $\deg(\psi_f) \leq N$ for even d, where N is the integer defined in Theorem 2.0.1. Thus ψ_f can be written as

$$\psi_f(\lambda) = \sum_{j=0}^{N} c_j \lambda^j, \tag{2.2.4}$$

where for all $0 \leq j \leq N$ the coefficient $c_j = c_j(n, d)$ is a homogeneous polynomial in the coefficents of f. Otherwise if d is odd and (λ, x) is an E-eigenpair of f, then $(-\lambda, -x)$ is an E-eigenpair of f as well. This means that for odd d the E-characteristic polynomial ψ_f has maximum degree N in λ^2 and in particular it contains only even power terms of λ. Hence ψ_f can be written explicitly as

$$\psi_f(\lambda) = \sum_{j=0}^{N} c_{2j} \lambda^{2j}. \tag{2.2.5}$$

Now we focus on the constant term of the E-characteristic polynomial ψ_f. In particular we recover the fact that, when nonzero, the constant term of ψ_f is a power of $\operatorname{Res}\left(\frac{1}{d}\nabla f\right)$ times a constant factor.

Theorem 2.2.12. *[LQZ, Theorem 3.5] Let $f \in S^dV$. Then for even d we have that*

$$c_0 = c \cdot \operatorname{Res}\left(\frac{1}{d}\nabla f\right), \tag{2.2.6}$$

while for odd d we have that

$$c_0 = c \cdot \operatorname{Res}\left(\frac{1}{d}\nabla f\right)^2 \tag{2.2.7}$$

for some constant $c \in \mathbb{Z}$ depending on n and d.

Proof. The relations (2.2.6) and (2.2.7) are trivially satisfied when f is irregular (compare with Proposition 2.2.8), so we can assume f regular. When d is even, from relation (2.2.1) we have that

$$c_0 = \psi_f(0) = \operatorname{Res}(F_\lambda)|_{\{\lambda=0\}} = c \cdot \operatorname{Res}(F_0) = c \cdot \operatorname{Res}\left(\frac{1}{d}\nabla f\right)$$

for some constant $c = c(n,d) \in \mathbb{Z}$.

Now suppose that d is odd. From relation (2.2.2) we have that

$$c_0 = \psi_f(0) = \operatorname{Res}(G_\lambda)|_{\{\lambda=0\}} = c \cdot \operatorname{Res}(G_0), \quad G_0(x_0, x) = \begin{pmatrix} x_0^2 - q(x) \\ \frac{1}{d}\nabla f(x) \end{pmatrix}$$

for some constant $c = c(n,d) \in \mathbb{Z}$. In order to prove relation (2.2.7), it is sufficient to prove that

$$\operatorname{Res}(G_0) = \operatorname{Res}\left(\frac{1}{d}\nabla f\right)^2. \tag{2.2.8}$$

First of all, we prove that the system

$$\begin{cases} x_0^2 - q(x) = 0 \\ \frac{1}{d}\nabla f(x) = 0 \end{cases} \tag{2.2.9}$$

has a nonzero solution if and only if $\operatorname{Res}\left(\frac{1}{d}\nabla f\right) = 0$. Let (x_0, x) be a nonzero solution of (2.2.9). In particular, x is a nonzero solution of $\nabla f(x) = 0$. Thus, $\operatorname{Res}\left(\frac{1}{d}\nabla f\right) = 0$. On the other hand, suppose that $\operatorname{Res}\left(\frac{1}{d}\nabla f\right) = 0$. Then $\nabla f(x) = 0$ admits a nonzero solution x and $(q(x)^{\frac{1}{2}}, x)$ is a nonzero solution of (2.2.9).

Hence the equations $\operatorname{Res}(G_0) = 0$ and $\operatorname{Res}\left(\frac{1}{d}\nabla f\right) = 0$ define the same variety. By definition $\operatorname{Res}\left(\frac{1}{d}\nabla f\right)$ is an irreducible polynomial over $\mathbb{Z}$ in the coefficients of f. Therefore

$$\operatorname{Res}(G_0) = \operatorname{Res}\left(\frac{1}{d}\nabla f\right)^k$$

for some positive integer k. Since the polynomial $x_0^2 - q(x)$ is quadratic, from Proposition 2.2.2 we have that $\mathrm{Res}(G_0)$ is a homogeneous polynomial in the coefficients of $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ of degree $2(d-1)^{n-1}$. On the other hand, the degree of $\mathrm{Res}\left(\frac{1}{d}\nabla f\right)^k$ is $k(d-1)^{n-1}$. Therefore relation (2.2.9) is satisfied only if $k = 2$. This completes the proof. □

We apply the following result when we study the degree of the leading coefficient and of the constant term of ψ_f, viewed as polynomials in the coefficients of f (see [LQZ, Proposition 3.6]).

Proposition 2.2.13. *Consider $f \in S^d V$ and let ψ_f be its E-characteristic polynomial written as in (2.2.4), (2.2.5).*

i) When d is even, c_i is a homogeneous polynomial in the coefficients of f with degree $n(d-1)^{n-1} - i$. In particular $\deg(c_N) = n(d-1)^{n-1} - N =: \varphi_n(d)$, where N is the integer defined in Theorem 2.0.1. In particular $\varphi_n(2) = 0$ for all $n \geq 2$.

ii) When d is odd, c_{2i} is a homogeneous polynomial in the entries of f with degree $2n(d-1)^{n-1} - 2i$. In particular $\deg(c_{2N}) = 2n(d-1)^{n-1} - 2N = 2\varphi_n(d)$.

Remark 2.2.14. It can be easily shown that the polynomial $\varphi_n(d)$ defined in Proposition 2.2.13 is a strictly increasing function in the variable d. This fact, together with Proposition 2.2.13, implies that c_N (respectively c_{2N}) has positive degree in the coefficients of f for all $n \geq 2$ and $d > 2$.

We have this natural question: is there a geometric meaning for the vanishing of the leading coefficient of ψ_f? The answer is positive and is stated in Proposition 2.3.9.

2.3 The product of the E-eigenvalues of a symmetric tensor

In this section we give the proof of Theorem 2.0.2. The proof starts with an example: in fact, the next lemma studies the product of the E-eigenvalues of a particular class of symmetric tensors, the scaled Fermat polynomials

$$f(x_1, \dots, x_n) = a_1 x_1^d + \cdots + a_n x_n^d, \quad a_1, \dots, a_n \in \mathbb{C}. \tag{2.3.1}$$

This result is important to prove the identity (2.0.5) up to sign in the statement of Theorem 2.0.2.

Lemma 2.3.1. *Let $d \geq 2$ and suppose that f is the scaled Fermat polynomial in (2.3.1). The product $\lambda_1 \cdots \lambda_N$ of the E-eigenvalues of f, where N is the number defined in Theorem 2.0.1, can be written as*

$$\lambda_1 \cdots \lambda_N = \frac{\operatorname{Res}\left(\frac{1}{d}\nabla f\right)}{h^{\frac{d-2}{2}}}, \tag{2.3.2}$$

where $h = h(a_1, \ldots, a_n)$ is a homogeneous polynomial of degree $2\varphi_n(d)/(d-2)$ and the polynomial $\varphi_n(d)$ was defined in Proposition 2.2.13. Moreover, the leading term of h with respect to the lexicographic term order is monic and it is equal to

$$LT_{Lex}(h) = \prod_{s=1}^{n} a_s^{2\frac{(d-1)^{n-1}-(d-1)^{s-1}}{d-2}}.$$

Proof. In this case, rewriting the number N of E-eigenvalues as

$$N = \sum_{j=1}^{n} \binom{n}{j} (d-2)^{j-1},$$

the binomial $\binom{n}{j}$ denotes the number of E-eigenvalues for f whose corresponding E-eigenvectors have exactly j nonzero coordinates, while the factor $(d-2)^{j-1}$ corresponds to the number of $(j-1)$-arrangements (allowing repetitions) of the elements of $\{0, 1, \ldots, d-3\}$, for all $j \in [n]$.

Let $x = (x_1, \ldots, x_n)$, with $q(x) = 1$ be an E-eigenvector of f. We have

$$a_i x_i^{d-1} = \lambda x_i \quad \forall\, i \in [n]. \tag{2.3.3}$$

Suppose that exactly j coordinates of x are nonzero, call them $x_{k_1}, \ldots, x_{k_j}$ with indices $1 \leq k_1 < \cdots < k_j \leq n$. Moreover, we write $a_i = \xi_i^{d-2}$ for all $i \in [n]$.

Looking at (2.3.3), if $x_i \neq 0$ we obtain that $\lambda = a_i x_i^{d-2} = (\xi_i x_i)^{d-2}$ for all $i \in [n]$. Moreover, considering (2.3.3) with respect to the indices $i_1 < i_2$, we get the relations $a_{i_1} x_{i_1}^{d-1} = \lambda x_{i_1}$, $a_{i_2} x_{i_2}^{d-1} = \lambda x_{i_2}$, from which we obtain the equation

$$x_{i_1} x_{i_2} \prod_{k=0}^{d-3} (\xi_{i_1} x_{i_1} - \varepsilon^k \xi_{i_2} x_{i_2}) = 0,$$

where ε is a $(d-2)$-th root of unity. This means that, for any indices $i_1 < i_2$ it could be that $x_{i_1} = 0$, $x_{i_2} = 0$ or $\xi_{i_1} x_{i_1} = \varepsilon^k \xi_{i_2} x_{i_2}$ for some $k \in \{0, 1, \ldots, d-3\}$. Therefore the coordinates of x, when nonzero, can be always written as

$$x_{k_l} = \left(\frac{1}{q(x)}\right)^{\frac{1}{2}} \xi_{k_1} \cdots \widehat{\xi_{k_l}} \cdots \xi_{k_j} \varepsilon^{\alpha_{k_l}},$$

where $\alpha_{k_l} \in \{0, 1, \ldots, d-3\}$ for all $l \in [j]$. Since $|\varepsilon| = 1$, we can assume $\alpha_{k_1} = 0$. In addition to this, the squared norm of x can be written as

$$q(x) = \xi_{k_2}^2 \cdots \xi_{k_j}^2 + \sum_{l=2}^{j} \xi_{k_1}^2 \cdots \widehat{\xi_{k_l}^2} \cdots \xi_{k_j}^2 \varepsilon^{2\alpha_{k_l}}$$

and the E-eigenvalue corresponding to x is

$$\lambda = (\xi_{k_l} x_{k_l})^{d-2} = \left(\frac{1}{q(x)}\right)^{\frac{d-2}{2}} a_{k_1} \cdots a_{k_j}.$$

From this argument we obtain that the product of the E-eigenvalues of the scaled Fermat polynomial f is equal to

$$\lambda_1 \cdots \lambda_N = \frac{g}{h^{\frac{d-2}{2}}},$$

where $g = g(a_1, \ldots, a_n)$ and $h = h(a_1, \ldots, a_n)$ are equal respectively to

$$g \coloneqq \prod_{j=1}^{n} \prod_{1 \le k_1 < \cdots < k_j \le n} \prod_{\alpha_{k_2}, \ldots, \alpha_{k_j} = 0}^{d-3} a_{k_1} \cdots a_{k_j}, \tag{2.3.4}$$

$$h \coloneqq \prod_{j=1}^{n} \prod_{1 \le k_1 < \cdots < k_j \le n} \prod_{\alpha_{k_2}, \ldots, \alpha_{k_j} = 0}^{d-3} \left(\xi_{k_2}^2 \cdots \xi_{k_j}^2 + \sum_{l=2}^{j} \xi_{k_1}^2 \cdots \widehat{\xi_{k_l}^2} \cdots \xi_{k_j}^2 \varepsilon^{2\alpha_{k_l}} \right). \tag{2.3.5}$$

Now consider in particular the polynomial g defined in (2.3.4). We have that

$$\begin{aligned} g &= \prod_{j=1}^{n} \prod_{1 \le k_1 < \cdots < k_j \le n} (a_{k_1} \cdots a_{k_j})^{(d-2)^{j-1}} \\ &= \prod_{j=1}^{n} (a_1 \cdots a_n)^{\binom{n-1}{j-1}(d-2)^{j-1}} \\ &= (a_1 \cdots a_n)^{(d-1)^{n-1}}, \end{aligned}$$

where the last polynomial coincides exactly with $\mathrm{Res}\left(\frac{1}{d}\nabla f\right)$ by Proposition 2.2.1. On the other hand, having fixed *Lex* as term order in $\mathbb{Z}[a_1, \ldots, a_n]$, the leading term of h is equal to

$$\begin{aligned} LT_{Lex}(h) &= \prod_{j=2}^{n} \prod_{1 \le k_1 < \cdots < k_j \le n} \prod_{\alpha_{k_2}, \ldots, \alpha_{k_j} = 0}^{d-3} \xi_{k_1}^2 \cdots \xi_{k_{j-1}}^2 \varepsilon^{2\alpha_{k_j}} \\ &= \prod_{j=2}^{n} \prod_{1 \le k_1 < \cdots < k_j \le n} (\xi_{k_1}^2 \cdots \xi_{k_{j-1}}^2)^{(d-2)^{j-1}}. \end{aligned}$$

Observe that in the last product (with j fixed) the factors $\xi_1^{2(d-2)^{j-1}},\ldots,\xi_{j-1}^{2(d-2)^{j-1}}$ appear $\binom{n-1}{j-1}$ times, while $\xi_s^{2(d-2)^{j-1}}$ appears $\binom{n-1}{j-1}-\binom{s-1}{j-1}$ times for $j\le s\le n$. Hence (assuming $\binom{a}{b}=0$ for $a<b$)

$$\begin{aligned}
LT_{Lex}(h) &= \prod_{j=2}^{n}\prod_{s=1}^{n}\xi_s^{2\left[\binom{n-1}{j-1}-\binom{s-1}{j-1}\right](d-2)^{j-1}}\\
&= \prod_{s=1}^{n}\xi_s^{2\left[(d-1)^{n-1}-(d-1)^{s-1}\right]}\\
&= \prod_{s=1}^{n}a_s^{2\frac{(d-1)^{n-1}-(d-1)^{s-1}}{d-2}}.
\end{aligned}$$

□

Remark 2.3.2. Observe that in Lemma 2.3.1 the degree of $LT_{Lex}(h)$, that is the total degree of h, is

$$\begin{aligned}
\frac{2}{d-2}\sum_{s=1}^{n}[(d-1)^{n-1}-(d-1)^{s-1}] &= \frac{2}{d-2}\left[n(d-1)^{n-1}-\frac{(d-1)^n-1}{d-2}\right]\\
&= \frac{2}{d-2}\varphi_n(d),
\end{aligned}$$

where $\varphi_n(d)$ was introduced in Proposition 2.2.13 and $d\ge 3$. This value is the one expected as shown in the sequel.

Now consider the variety $\widetilde{Q}=v_d(Q)$, namely the Veronese embedding in $\mathbb{P}(S^dV)$ of the isotropic quadric $Q\subset\mathbb{P}(V)$. In particular, $\widetilde{Q}$ is a smooth projective variety, hence we can introduce the Chern classes of $\widetilde{Q}$ in order to compute its polar classes $\delta_i(\widetilde{Q})$, according to the relations (1.8.1).

Lemma 2.3.3. *In the hypotheses above, $\delta_0(\widetilde{Q})=2\sum_{k=0}^{n-2}\alpha_k d^k$, where*

$$\alpha_k := (k+1)\sum_{j=0}^{n-2-k}\binom{n}{j}(-1)^j 2^{n-2-k-j}. \tag{2.3.6}$$

Proof. First of all we compute the Chern polynomial of the tangent bundle $\mathcal{T}Q$, for brevity indicated with $c(Q)$ (this computation was performed in Example 1.7.4):

$$c(Q)=\frac{(1+h)^n}{1+2h}=\sum_{i,j=0}^{n-2}\binom{n}{i}(-2)^j h^{i+j}=\sum_{s=0}^{n-2}\left(\sum_{i=0}^{s}\binom{n}{i}(-2)^{s-i}\right)h^s,$$

where $h = c_1(\mathcal{O}_Q(1))$. Then we compute the polar class $\delta_0(\widetilde{Q})$ using an equivalent formulation of (1.8.1) with $m = n-2$ and [Tev, Theorem 7.2], taking into account that $\widetilde{Q} = v_d(Q)$.

$$\begin{aligned}
\delta_0(\widetilde{Q}) &= \sum_{k=0}^{n-2}(-1)^{n-2-k}(k+1)\deg(c_{n-2-k}(\widetilde{Q})) \\
&= \sum_{k=0}^{n-2}(-1)^{n-1-k}(k+1)\deg\left[\left(\sum_{j=0}^{n-2-k}\binom{n}{j}(-2)^{n-2-k-j}\right)h^{n-2-k}(dh)^k\right] \\
&= \sum_{k=0}^{n-2}(k+1)\deg\left[\left(\sum_{j=0}^{n-2-k}\binom{n}{j}(-1)^j 2^{n-2-k-j}\right)d^k h^{n-2}\right] \\
&= 2\sum_{k=0}^{n-2}(k+1)\left[\sum_{j=0}^{n-2-k}\binom{n}{j}(-1)^j 2^{n-2-k-j}\right]d^k.
\end{aligned}$$

□

In the following technical Lemma, we rewrite the polynomial $\varphi_n(d)$ defined in Proposition 2.2.13 in a useful way for the sequel.

Lemma 2.3.4. *Let $\varphi_n(d)$ be the polynomial defined in Proposition 2.2.13. Then $\varphi_n(d) = (d-2)\sum_{k=0}^{n-2}\beta_k d^k$, where*

$$\beta_k := (k+1)\sum_{l=0}^{n-2-k}\binom{k+l+1}{l}(-1)^l. \tag{2.3.7}$$

Proof. With a bit of work, the polynomial $\varphi_n(d)$ can be rewritten as

$$\varphi_n(d) = (d-2)\sum_{k=0}^{n-2}(k+1)(d-1)^k = (d-2)\sum_{k=0}^{n-2}\beta_k d^k,$$

where

$$\begin{aligned}
\beta_k &= \sum_{i=k}^{n-2}(i+1)\binom{i}{k}(-1)^{i-k} \\
&= \sum_{l=0}^{n-2-k}(l+k+1)\binom{l+k}{k}(-1)^l \\
&= \sum_{l=0}^{n-2-k}\left[\frac{l(l+k)!}{l!k!} + \frac{(k+1)(l+k)!}{l!k!}\right](-1)^l
\end{aligned}$$

$$= (k+1) \sum_{l=0}^{n-2-k} \left[\binom{k+l}{k+1} + \binom{k+l}{k} \right] (-1)^l$$
$$= (k+1) \sum_{l=0}^{n-2-k} \binom{k+l+1}{l} (-1)^l. \qquad \square$$

Now we prove that the degree of the leading coefficient of ψ_f is a multiple of the polar class $\delta_0(\widetilde{Q})$ computed in Lemma 2.3.3.

Proposition 2.3.5. *For any $n \geq 2$ and for any $0 \leq k \leq n-2$, $\alpha_k = \beta_k$. In particular,*

$$\varphi_n(d) = \frac{d-2}{2}\delta_0(\widetilde{Q}). \tag{2.3.8}$$

Proof. From the identities (2.3.6) and (2.3.7) we see that both α_k and β_k are multiples of $k+1$. In particular, we have to prove that

$$\sum_{j=0}^{n-2-k} \binom{n}{j} (-1)^j 2^{n-2-k-j} = \sum_{j=0}^{n-2-k} \binom{k+j+1}{j} (-1)^j. \tag{2.3.9}$$

The proof is by induction on n. If $n=2$, both the sides of the equality are equal to 1. Suppose now that the equality is true at the n-th step. At the $(n+1)$-th step, the right-hand side of the equality is

$$\sum_{j=0}^{n-1-k} \binom{k+j+1}{j} (-1)^j = \sum_{j=0}^{n-2-k} \binom{k+j+1}{j} (-1)^j + (-1)^{n-1-k} \binom{n}{n-k},$$

while the left-hand side at the $(n+1)$-th step is equal to

$$\sum_{j=0}^{n-1-k} \binom{n+1}{j} (-1)^j 2^{n-1-k-j} =$$
$$= 2^{n-1-k} + \sum_{j=1}^{n-1-k} \binom{n+1}{j} (-1)^j 2^{n-1-k-j}$$
$$= 2^{n-1-k} + \sum_{j=1}^{n-1-k} \left[\binom{n}{j} + \binom{n}{j-1} \right] (-1)^j 2^{n-1-k-j}$$
$$= \sum_{j=0}^{n-1-k} \binom{n}{j} (-1)^j 2^{n-1-k-j} + \sum_{j=1}^{n-1-k} \binom{n}{j-1} (-1)^j 2^{n-1-k-j}$$

$$
\begin{aligned}
&= 2\sum_{j=0}^{n-2-k}\binom{n}{j}(-1)^j 2^{n-2-k-j} + (-1)^{n-1-k}\binom{n}{n-k} + \\
&\quad + \sum_{j=0}^{n-2-k}\binom{n}{j}(-1)^{j+1}2^{n-2-k-j} \\
&= \sum_{j=0}^{n-2-k}\binom{n}{j}(-1)^j 2^{n-2-k-j} + (-1)^{n-1-k}\binom{n}{n-1-k}
\end{aligned}
$$

By applying the induction hypothesis we conclude the proof of (2.3.9). □

Remark 2.3.6. Matteo Gallet suggested an alternative proof of the identity (2.3.9), applying the so-called *"Zeilberger's Algorithm"* (see [Zei90, Zei91]). For example, using the Mathematica package `HolonomicFunctions`, developed by Cristoph Koutschan (see [Kou]), the code

```
Annihilator[Sum[Binomial[n,j]*(-1)^j*2^(n-2-k-j),{j,0,n-2-k}],{S[k],S[n]}]
Annihilator[Sum[Binomial[k+j+1,j]*(-1)^j,{j,0,n-2-k}],{S[k],S[n]}]
```

provides the operators that annihilate the left-hand and right-hand side in (2.3.9), respectively, thus showing that (2.3.9) holds true.

Corollary 2.3.7. *Consider the isotropic quadric $Q \subset \mathbb{P}(V)$ and its Veronese embedding $\widetilde{Q} \subset \mathbb{P}(S^dV)$ with the same notations as before. Then $\widetilde{Q}^\vee$ is a hypersurface of $\mathbb{P}(S^dV)$ of degree $\deg(\widetilde{Q}^\vee) = \delta_0(\widetilde{Q})$.*

Proof. From Remark 2.2.14 and Proposition 2.3.5 we have that $\delta_0(\widetilde{Q})$ is a positive integer for all $n \geq 2$ and $d > 2$. Applying Theorem 1.5.4 we conclude the proof. □

Summing up, there is an explicit formula for the degree of the leading coefficient of ψ_f in terms of the degree of the dual variety of Q embedded in $\mathbb{P}(S^dV)$ via the Veronese map, stated in the following corollary.

Corollary 2.3.8. *Given $f \in S^dV$, if f is general then*

$$deg(c_N) = \frac{d-2}{2}\deg(\widetilde{Q}^\vee)$$

when d is even, while

$$deg(c_{2N}) = (d-2)\deg(\widetilde{Q}^\vee)$$

when d is odd.

In the following, we prove that the leading coefficient of ψ_f is a power of the discriminant $\Delta_{\widetilde{Q}}(f)$, where the exponent was obtained in Corollary 2.3.8. The next two lemmas clarify the geometrical meaning of the vanishing of the polynomial c_N (respectively c_{2N}).

Lemma 2.3.9. *Assume that* $d > 2$ *and let* $f \in S^d V$*. Then the leading coefficient of* ψ_f *vanishes if and only if the system*

$$\begin{cases} \frac{1}{d}\nabla f(x) = \lambda x \\ q(x) = 0 \ , \end{cases} \tag{2.3.10}$$

called deficit system *in [LQZ], has a nontrivial solution.*

Proof. If f is irregular, from Definition 2.2.6 we have that the system (2.3.10) has a nontrivial solution when $\lambda = 0$, while from Proposition 2.2.8 we have that ψ_f is identically zero.

Suppose instead that f is regular. By Theorem 2.2.10 the roots of ψ_f are exactly the E-eigenvalues of f and for even d we have $\deg(\psi_f) \leq N$, whereas for odd d $\deg(\psi_f) \leq 2N$. However, we know by Theorem 2.0.1 that a general f has N distinct E-eigenvalues when d is even, and N pairs $(\lambda, -\lambda)$ of distinct E-eigenvalues when d is odd, which means that ψ_f would have exactly N distinct roots when d is even, and $2N$ distinct roots when d is odd. On the other hand, E-eigenvalues are the normalized solutions x of equation (2.1.4), and by definition ψ_f is the resultant of the homogeneization of the system whose equations are (2.1.4) and the condition $q(x) = 1$. The solutions at infinity of this system are precisely the solution of the system (2.3.10). Hence a symmetric tensor f such that ψ_f has not the maximum degree provides a nontrivial solution of the system (2.3.10), or equivalently admits an isotropic eigenvector. □

Lemma 2.3.10. *Given* $f \in S^d V$*, the system (2.3.10) has a nontrivial solution if and only if the coefficients of* f *annihilate the polynomial* $\Delta_{\widetilde{Q}}(f)$*, namely* f *is represented by a point of* $\widetilde{Q}^\vee$*.*

Proof. Suppose that x is a solution of (2.3.10). By regularity of f we have that $\lambda \neq 0$. Moreover, $P = [x]$ is a smooth point of f, and f is tangent to Q at P. This means that f, thought as a point of $\mathbb{P}(S^d V)$, belongs to $\widetilde{Q}^\vee$, namely its coefficients annihilate the polynomial $\Delta_{\widetilde{Q}}(f)$. The converse is true by reversing the implications. □

Remark 2.3.11. One could ask if the condition on f to have the maximum number of E-eigenvalues imposed in Theorem 2.0.2 has a geometric counterpart. For example, this condition is not the same as requiring X_f to be regular: although any symmetric tensor f having the maximum number of E-eigenvalues is necessarily regular, there exist regular symmetric tensors f admitting at least one isotropic eigenvector. The right property to consider is revealed by Lemma 2.3.10, which shows that $f \in S^dV$ admits an isotropic eigenvector if and only if the hypersurface X_f and the isotropic quadric Q are tangent. This means that the condition on f in Theorem 2.0.2 is satisfied if and only if X_f is transversal to Q.

Remark 2.3.11 is even more interesting when considering the following result (see [Alu00, Claim 3.2]):

Proposition 2.3.12. *If two smooth hypersurfaces of degree d_1, d_2 in projective space are tangent along a positive dimensional set, then $d_1 = d_2$.*

An immediate consequence of Proposition 2.3.12 is the following

Corollary 2.3.13. *Given $f \in S^dV$ with $d > 2$, if X_f is smooth then f has always a finite number of isotropic eigenvectors.*

Aiming at explaining better Lemma 2.3.10, below we give an example of a symmetric tensor f admitting an isotropic eigenvector, with a study of the tangency between the variety X_f and the isotropic quadric Q. Moreover, we compute explicitly the E-characteristic polynomial ψ_f and observe that $\deg(\psi_f) < N$.

Example 2.3.14 (A plane cubic admitting an isotropic eigenvector)**.** First of all, we recall that due to Theorem 2.0.1, a general ternary form has $N = d^2 - d + 1$ E-eigenvalues. Consider the cubic ternary form

$$\begin{aligned} f(x_1,x_2,x_3) = {} & 342\sqrt{-1}\,x_1^3 - 522\sqrt{-1}\,x_1x_2^2 - 389\sqrt{-1}\,x_1^2x_3 \\ & + 79\sqrt{-1}\,x_2^2x_3 - 474\sqrt{-1}\,x_1x_3^2 + 95\sqrt{-1}\,x_3^3 \\ & - 773x_1^2x_2 + 191x_2^3 - 48x_1x_2x_3 + 175x_2x_3^2. \end{aligned}$$

It can be easily verified that the vector $x = (0,1,-\sqrt{-1})$ is an isotropic eigenvector of f. In particular the projective curve X_f is tangent to the isotropic quadric Q at $[x] \in \mathbb{P}^2$, and the common tangent line has equation $x_2 - \sqrt{-1}\,x_3 = 0$. In order to represent graphically this situation, we consider the change of coordinates

$$z_1 = -\sqrt{-1}\,x_1, \quad z_2 = x_2 + \sqrt{-1}\,x_3, \quad z_3 = x_2 - \sqrt{-1}\,x_3.$$

In the z_i's the quadric Q (the red curve in the affine representation of Figure 2.1) has equation $z_1^2 - z_2z_3 = 0$. The image of the isotropic eigenvector x is $z = (0,2,0)$, while the image of the projective curve X_f (the blue curve in Figure 2.1) is the projective curve of equation

$$\begin{aligned} g(z_1,z_2,z_3) = {} & 342z_1^3 + 581z_1^2z_2 + 192z_1^2z_3 + 498z_1z_2z_3 \\ & + 139z_2^2z_3 + 24z_1z_3^2 + 48z_2z_3^2 + 4z_3^3. \end{aligned}$$

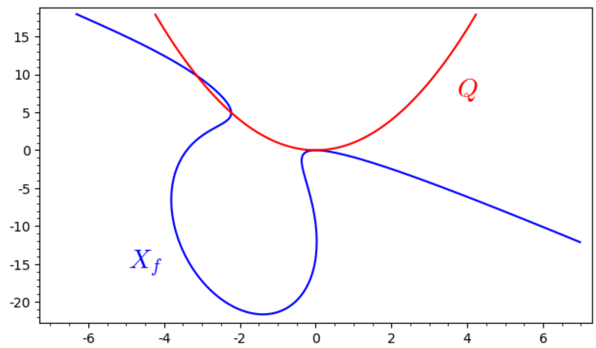

Figure 2.1: The isotropic quadric Q and the plane cubic X_f in the affine plane $z_2 = 2$. They are not transversal at the origin.

The presence of an isotropic eigenvector can be detected by computing explicitly the E-characteristic polynomial of f as well. In order to compute $\psi_f(\lambda)$ we used the following `Macaulay2` code [GS] (for the package `Resultants` see [Sta]), taking into account Definition 2.2.4 modified according to the given change of coordinates:

```
loadPackage "Resultants"; KK=QQ[t]; R=KK[z_0..z_3];
f=342*z_1^3+581*z_1^2*z_2+192*z_1^2*z_3+498*z_1*z_2*z_3+139*z_2^2*z_3+
24*z_1*z_3^2+48*z_2*z_3^2+4*z_3^3;
F_0=z_0^2-(-z_1^2+z_2*z_3);
F_1=diff(z_1,f)/3+t*z_0*z_1;
F_2=diff(z_2,f)/3+diff(z_3,f)/3-t*z_0*(z_2+z_3)/2;
F_3=diff(z_2,f)/3-diff(z_3,f)/3+t*z_0*(z_2-z_3)/2;
Echarpoly=resultant({F_0,F_1,F_2,F_3}, Algorithm=>"Macaulay");
factor Echarpoly
```

The output of `factor Echarpoly` is

$$\begin{aligned}\psi_g(\lambda) = {} & 2240537920394580000\lambda^{12} \\ & + 1737672597491537284396875\lambda^{10} \\ & + 456866094404925313122181875\lambda^{8} \\ & + 5386198710022212712472131345526 25\lambda^{6} \\ & + 27460315843205568529626477207835 48350\lambda^{4} \\ & + 21377525988865149579810902794140 43391031\lambda^{2} \\ & + 13843807659909379464027427753236 120270069196.\end{aligned}$$

Since a general cubic ternary form has seven E-eigenvalues, we expect that the degree of ψ_f is fourteen, but in this case $\deg(\psi_f) = 12$. This confirms that f has one isotropic eigenvector and six E-eigenvectors (counted with multiplicity) up to sign.

Returning to the proof of Theorem 2.0.2, an immediate consequence of Corollary 2.3.8 and Lemmas 2.3.9 and 2.3.10 is the following formula for the leading coefficient of the E-characteristic polynomial of a symmetric tensor.

Theorem 2.3.15. *Given $f \in S^dV$ and $d > 2$, if f does not admit isotropic eigenvectors, then*

$$c_N = e \cdot \Delta_{\widetilde{Q}}(f)^{\frac{d-2}{2}} \tag{2.3.11}$$

when d is even, while

$$c_{2N} = e \cdot \Delta_{\widetilde{Q}}(f)^{d-2} \tag{2.3.12}$$

when d is odd, for some integer constant $e = e(n, d)$.

Proof. Applying Lemma 2.3.9 and Lemma 2.3.10 we obtain that the varieties $\{c_N = 0\}$ and $\{\Delta_{\widetilde{Q}}(f) = 0\}$ coincide. The proof for the case $n = 2$ is postponed to Section 2.4, where we treat more in detail binary forms. If $n > 2$, then $\widetilde{Q}$ is an irreducible hypersurface and the variety $\widetilde{Q}^\vee$ is irreducible as well. Corollary 2.3.7 tells us that $\widetilde{Q}^\vee$ is in fact a hypersurface. Hence, for even d, $c_N = e \cdot \Delta_{\widetilde{Q}}(f)^j$, whereas for odd d we have $c_{2N} = e \cdot \Delta_{\widetilde{Q}}(f)^k$ for some integer constant $e = e(n, d)$ and positive integers j, k. Moreover, from Corollary 2.3.8 we have that $j = (d-2)/2$ and $k = d-2$. □

Proof of Theorem 2.0.2. Theorems 2.2.12 and 2.3.15 describe respectively the constant term c_0 and the leading coefficient c_N (or c_{2N}) of the E-characteristic polynomial ψ_f of a general symmetric tensor f, up to a constant integer factor. Moreover, the product of the E-eigenvalues of f is c_0/c_N (respectively c_0/c_{2N}). If we restrict to the class of scaled Fermat polynomials, as in Lemma 2.3.1, we notice that the integers c and e of Theorems 2.2.12 and 2.3.15 have to coincide, for the leading term of the denominator in (2.3.2) is monic and by definition $\Delta_{\widetilde{Q}}(f)$ has relatively prime integer coefficients. This concludes the proof. □

2.4 The case of binary symmetric tensors

In this final section, we focus on the case of binary forms ($n = 2$) and complete the proof of Theorem 2.0.2, recovering the results of Li, Qi and Zhang in [LQZ]. In particular, we show that in this particular case equation (2.0.5) can be rewritten more explicitly. Here any element of S^dV is represented by a binary form

$$f(x_1,x_2)=\sum_{j=0}^{d}\binom{d}{j}a_j x_1^{d-j}x_2^{j}\ , \quad a_0,\ldots,a_d\in\mathbb{C}. \tag{2.4.1}$$

According to Theorem 2.0.1, a general binary form f of degree d admits $N=d$ E-eigenvectors. As one can easily see from relation (2.1.4), the E-eigenvectors of f are the normalized solutions (x_1,x_2) of the equation $D(f)=0$, where the *discriminant operator* D is

$$D(f):=x_1\frac{\partial f}{\partial x_2}-x_2\frac{\partial f}{\partial x_1}. \tag{2.4.2}$$

The operator D is well-known and its properties are collected in [Mac].

We are interested in the E-characteristic polynomial ψ_f of a regular binary form f. We know that $\deg(\psi_f)=d$ in the even case, while $\deg(\psi_f)=2d$ in the odd case. A remarkable formula for the leading coefficient of the E-characteristic polynomial of a 2-dimensional tensor of order d is given in [LQZ]. We show that this formula can be simplified a lot in the symmetric case.

Following the argument used in [LQZ], the isotropic eigenvectors of f are the solutions of the following simplified version of the system (2.3.10):

$$\begin{cases}\sum_{j=1}^{d}\binom{d-1}{j-1}a_{j-1}x_1^{d-j}x_2^{j-1} & = & \lambda x_1\\ \sum_{j=1}^{d}\binom{d-1}{j-1}a_{j}x_1^{d-j}x_2^{j-1} & = & \lambda x_2\\ x_1^2+x_2^2 & = & 0\ .\end{cases} \tag{2.4.3}$$

We observe that all the nontrivial solutions (x_1,x_2) of (2.4.3) are nonzero multiples of $(1,\sqrt{-1})$ or $(1,-\sqrt{-1})$. Substituting $(1,\sqrt{-1})$ to (2.4.3) and eliminating λ we obtain the condition

$$\sum_{j=0}^{d}\binom{d}{j}a_j\sqrt{-1}^{\,j}=0. \tag{2.4.4}$$

In the same manner, considering instead the vector $(1,-\sqrt{-1})$ we obtain the condition

$$\sum_{j=0}^{d}\binom{d}{j}a_j(-\sqrt{-1})^{j}=0. \tag{2.4.5}$$

Therefore, if the binary form f has at least one isotropic eigenvector, then the product of the left-hand sides of equations (2.4.4) and (2.4.5) vanishes. On the other hand, if this product is zero, then $(1,\sqrt{-1})$ or $(1,-\sqrt{-1})$ is a solution of the system (2.4.3) and is in turn an isotropic eigenvector of f.

We observe that the left-hand sides in (2.4.4) and (2.4.5) have an interesting interpretation. Consider the linear change of coordinates defined by the equations

$$x_1 = \gamma_{11}z_1 + \gamma_{12}z_2, \quad x_2 = \gamma_{21}z_1 + \gamma_{22}z_2.$$

Applying this change of coordinates, the binary form $f(x_1, x_2)$ is transformed into the binary form $\widetilde{f}(z_1, z_2)$ in the new variables z_1, z_2 defined by

$$\widetilde{f}(z_1, z_2) = \sum_{j=0}^{d} \binom{d}{j} a_j (\gamma_{11}z_1 + \gamma_{12}z_2)^{d-j}(\gamma_{21}z_1 + \gamma_{22}z_2)^j = \sum_{j=0}^{d} \binom{d}{j} \widetilde{a}_j z_1^{d-j} z_2^j,$$

where (see [Stu, Proposition 3.6.1])

$$\widetilde{a}_j = \sum_{k=0}^{d} \left[\sum_{l=\max(0,k-j)}^{\min(k,d-j)} \binom{d-j}{l} \binom{j}{k-l} \gamma_{11}^l \gamma_{12}^{k-l} \gamma_{21}^{d-j-l} \gamma_{22}^{j-k+l} \right] a_k \tag{2.4.6}$$

for all $0 \le j \le d$. Let us introduce the new coordinates

$$z_1 = -\frac{\sqrt{-1}}{2}(x_1 + \sqrt{-1}\,x_2), \quad z_2 = -\frac{\sqrt{-1}}{2}(x_1 - \sqrt{-1}\,x_2).$$

The inverse change of coordinates has equations

$$x_1 = \sqrt{-1}(z_1 + z_2), \quad x_2 = z_1 - z_2.$$

With this choice, applying formula (2.4.6) the coefficients $\widetilde{a}_j$ of the transformed binary form $\widetilde{f}(z_1, z_2)$ are

$$\widetilde{a}_j = \sum_{k=0}^{d} \left[\sum_{l=\max(0,k-j)}^{\min(k,d-j)} \binom{d-j}{l} \binom{j}{k-l} \sqrt{-1}^{2(j+l)-k} \right] a_k$$

for all $0 \le j \le d$. In particular the extreme coefficients become

$$\widetilde{a}_0 = \sum_{j=0}^{d} \binom{d}{j} a_j \sqrt{-1}^j, \quad \widetilde{a}_d = (-1)^d \sum_{j=0}^{d} \binom{d}{j} a_j (-\sqrt{-1})^j.$$

Therefore, if we define $b_0 := \widetilde{a}_0$ and $b_d := (-1)^d \widetilde{a}_d$, then the left-hand sides of equations (2.4.4) and (2.4.5) are equal to b_0 and b_d, respectively. Moreover, we observe that the product $b_0 b_d$ has integer coefficients even though some of the coefficients of b_0 and b_d have nonzero imaginary part: in fact we see that

$$\begin{aligned} b_0 b_d &= \sum_{j,k=0}^{d} \binom{d}{j}\binom{d}{k} a_j a_k (-1)^j \sqrt{-1}^{j+k} \\ &= \sum_{s=0}^{2d} \left[\sum_{j=\min(0,s-d)}^{\max(s,d)} \binom{d}{j}\binom{d}{s-j} a_j a_{s-j} (-1)^j \right] \sqrt{-1}^{s}, \end{aligned} \tag{2.4.7}$$

where in the last relation all summands corresponding to odd indices s vanish. Since the coefficient of a_0^2 in the expression of $b_0 b_d$ is 1, we conclude that $b_0 b_d = \Delta_{\widetilde{Q}}(f)$ up to sign. In particular $\widetilde{Q}^\vee = \{b_0 b_d = 0\}$: in fact in this case $\widetilde{Q}$ is the union of two distinct points (more precisely, the classes of the rank-one symmetric tensors $(x_1 + \sqrt{-1}x_2)^d$ and $(x_1 - \sqrt{-1}x_2)^d$), while the variety $\widetilde{Q}^\vee$ is the quadric union of the hyperplanes $\{b_0 = 0\}$, $\{b_d = 0\}$. In particular, the hyperplane $\{b_0 = 0\}$ parametrizes the binary forms having $(1, \sqrt{-1})$ as isotropic eigenvector, while $\{b_d = 0\}$ parametrizes the binary forms having $(1, -\sqrt{-1})$ as isotropic eigenvector.

Regarding the leading coefficient of the E-characteristic polynomial ψ_f, the previous argument suggests that it must coincide with $c b_0^i b_d^j$ for some $c = c(d) \in \mathbb{Z}$. Since ψ_f is a polynomial in the indeterminates $a_0, \ldots, a_d$ with integer coefficients, it follows that $i = j$. Hence, for even d, $c_d = e\Delta_{\widetilde{Q}}(f)^p$, whereas for odd d we have $c_{2d} = e\Delta_{\widetilde{Q}}(f)^q$ for some $e = e(d) \in \mathbb{Z}$ and positive integers p, q. From Corollary 2.3.8 we have that $p = (d-2)/2$ and $q = d-2$, thus completing the proof of Theorem 2.3.15 in the case $n = 2$.

Remark 2.4.1. If we specialize to the class of scaled Fermat binary forms

$$f(x_1, x_2) = \alpha x_1^d + \beta x_2^d, \quad \alpha, \beta \in \mathbb{C},$$

from relation (2.4.7) we confirm the statement of Lemma 2.3.1 by observing that

$$\Delta_{\widetilde{Q}}(f) = \alpha^2 + (1 + (-1)^d)\sqrt{-1}^d \alpha\beta + \beta^2.$$

Example 2.4.2 (E-characteristic polynomial of a binary cubic form)**.** In this computational example we derive symbolically the E-characteristic polynomial of the cubic binary form

$$f(x_1, x_2) = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3.$$

The `Macaulay2` code used is

```
restart
loadPackage "Resultants"; d=3; KK=QQ[a_0..a_d,t]; R=KK[x_0,x_1,x_2];
f=sum(d+1,j->binomial(d,j)*a_j*x_1^(d-j)*x_2^j);
F_0=x_0^2-x_1^2-x_2^2;
F_1=diff(x_1,f)/d-t*x_0^(d-2)*x_1;
F_2=diff(x_2,f)/d-t*x_0^(d-2)*x_2;
Echarpoly=resultant({F_0,F_1,F_2}, Algorithm=>"Macaulay");
factor Echarpoly
```

The output obtained is

$$
\begin{aligned}
\psi_f(\lambda^2) = &-(a_0^2+9a_1^2-6a_0a_2+9a_2^2-6a_1a_3+a_3^2)\lambda^6\\
&+(a_0^4+12a_0^2a_1^2+24a_1^4-6a_0^3a_2-12a_0a_1^2a_2+9a_0^2a_2^2+45a_1^2a_2^2\\
&-8a_0a_2^3+24a_2^4-6a_0^2a_1a_3-8a_1^3a_3-12a_0a_1a_2a_3-12a_1a_2^2a_3\\
&+3a_0^2a_3^2+9a_1^2a_3^2-6a_0a_2a_3^2+12a_2^2a_3^2-6a_1a_3^3+a_3^4)\lambda^4\\
&-2(8a_1^6-24a_0a_1^4a_2+21a_0^2a_1^2a_2^2+6a_1^4a_2^2-4a_0^3a_2^3-15a_0a_1^2a_2^3+12a_0^2a_2^4\\
&+6a_1^2a_2^4+8a_2^6+4a_0^2a_1^3a_3-6a_0^3a_1a_2a_3-6a_0^2a_1a_2^2a_3-15a_1^3a_2^2a_3\\
&-24a_1a_2^4a_3+a_0^4a_3^2+6a_0^2a_1^2a_3^2+12a_1^4a_3^2-3a_0^3a_2a_3^2-6a_0a_1^2a_2a_3^2+6a_0^2a_2^2a_3^2\\
&+21a_1^2a_2^2a_3^2+4a_0a_2^3a_3^2-3a_0^2a_1a_3^3-4a_1^3a_3^3-6a_0a_1a_2a_3^3+a_0^2a_3^4)\lambda^2\\
&+(3a_1^2a_2^2-4a_0a_2^3-4a_1^3a_3+6a_0a_1a_2a_3-a_0^2a_3^2)^2
\end{aligned}
$$

Note in particular that the lowest coefficient of ψ_f is the square of the discriminant of f, whereas the highest coefficient agrees with formula (2.4.7). Moreover, the λ^2-discriminant of ψ_f may be written, up to scalars, as

$$\Delta_{\lambda^2}\left[\psi_f(\lambda^2)\right] = (a_0^2+9a_1^2-6a_0a_2+9a_2^2-6a_1a_3+a_3^2)\,g_1^2\,g_2^3\,,$$

On one hand, as pointed out in Proposition 4.2.4, the hypersurface cut out by the polynomial $g_1(f)$ corresponds to the bisector hypersurface $B(X_{(3)},X_{(3)})$, namely the locus of binary cubic forms admitting two distinct critical rank-one symmetric binary cubics at the same distance from f. The expression of g_1 in coordinates is the following (see also Figure 2.2):

$$
\begin{aligned}
g_1(f) = &\,2a_0a_1^3-3a_0^2a_1a_2+3a_1^3a_2-6a_0a_1a_2^2-3a_1a_2^3+a_0^3a_3+3a_0a_1^2a_3\\
&+6a_1^2a_2a_3-3a_0a_2^2a_3-2a_2^3a_3+3a_1a_2a_3^2-a_0a_3^3\,.
\end{aligned}
$$

On the other hand, the hypersurface cut out by the polynomial $g_2(f)$ corresponds to the ED discriminant $\Sigma_{X_{(3)}}$, namely the locus of binary cubic forms having two coinciding critical rank-one symmetric binary cubics. The expression of g_2 in

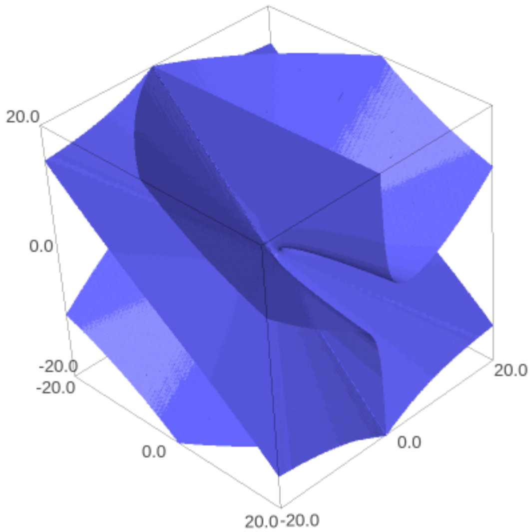

Figure 2.2: The bisector hypersurface $B(X_{(3)}, X_{(3)})$ of the rational normal curve $X_{(3)} \subset \mathbb{P}^3_{\mathbb{C}}$ in the affine chart $a_3 = 1$.

coordinates is the following (see also Figure 2.3):

$$\begin{aligned} g_2(f) = {} & 4a_0^2a_1^2 + 32a_1^4 - 4a_0^3a_2 - 52a_0a_1^2a_2 + 24a_0^2a_2^2 + 61a_1^2a_2^2 - 48a_0a_2^3 \\ & + 32a_2^4 - 4a_0^2a_1a_3 - 48a_1^3a_3 + 34a_0a_1a_2a_3 - 52a_1a_2^2a_3 + a_0^2a_3^2 \\ & + 24a_1^2a_3^2 - 4a_0a_2a_3^2 + 4a_2^2a_3^2 - 4a_1a_3^3 \,. \end{aligned}$$

We conclude this chapter focusing on complex *harmonic binary forms*. Let $\Delta\colon S^dV \to S^{d-2}V$ denote the *Laplace operator* defined in coordinates by

$$\Delta f(x_1, \ldots, x_n) \coloneqq \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

Definition 2.4.3. The subspace in S^dV of *complex harmonic forms* in n indeterminates is

$$H^dV \coloneqq \ker(\Delta) \subset S^dV.$$

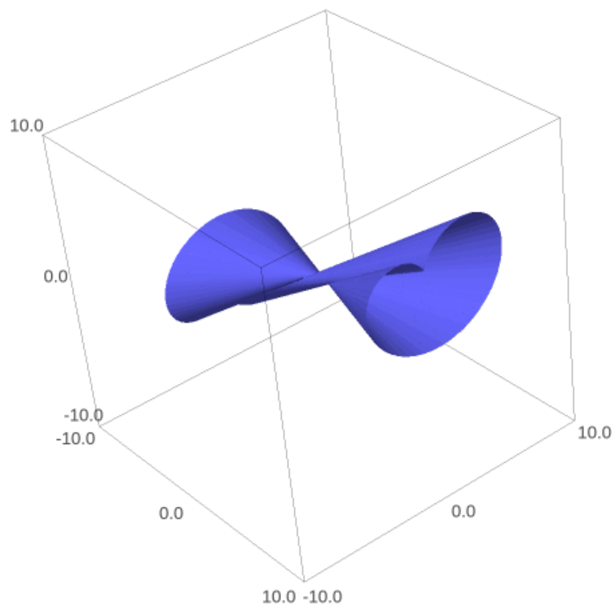

Figure 2.3: The ED discriminant $\Sigma_{X_{(3)}}$ of the rational normal curve $X_{(3)} \subset \mathbb{P}^3_{\mathbb{C}}$ in the affine chart $a_3 = 1$.

With an induction argument one may prove that the map Δ is surjective, thus implying that

$$\dim(H^dV) = \dim(S^dV) - \dim(S^{d-2}V) = \binom{d+n-1}{n-1} - \binom{d+n-3}{n-1}. \quad (2.4.8)$$

Hence, in our case $n = 2$ we have that $\dim(H^dV) = 2$. In particular, if x_1 and x_2 denote the linear forms associated to the vectors $(1,0)$ and $(0,1)$ of V, we define the linear forms $z_1 = x_1 + \sqrt{-1}\,x_2$ and $z_2 = x_1 - \sqrt{-1}\,x_2$. One may verify that z_1^d and z_2^d belong to H^dV and are linearly independent, hence they form a basis of H^dV. Then any complex harmonic binary form $g \in S^dV$ can be written as

$$g = az_1^d + bz_2^d, \quad [a,b] \in \mathbb{P}^1_{\mathbb{C}}. \quad (2.4.9)$$

We prove the following result.

Proposition 2.4.4. *Consider g as in (2.4.9). Then its characteristic polynomial is equal to (up to a scalar factor)*

$$\psi_f(\lambda) = (4ab - \lambda^2)^{\frac{d}{2}}$$

for even d and

$$\psi_f(\lambda) = (4ab - \lambda^2)^d$$

for odd d.

Proof. Developing the identity (2.4.9), we have that

$$g = az_1^d + bz_2^d = a(x_1 + \sqrt{-1}\,x_2)^d + b(x_1 - \sqrt{-1}\,x_2)^d$$

Then the gradient of g with respect to the variables x_1 and x_2 is

$$\frac{\partial g}{\partial x_1} = d(az_1^{d-1} + bz_2^{d-1}), \quad \frac{\partial g}{\partial x_2} = \sqrt{-1}\,d(az_1^{d-1} - bz_2^{d-1}).$$

We compute the E-eigenvectors $x = (x_1, x_2) \in V$ of g via the system (2.1.5). They correspond to the normalized solutions of the equation $D(g) = 0$, where D is the discriminant operator defined in (2.4.2):

$$\frac{1}{d}D(g) = x_2(az_1^{d-1} + bz_2^{d-1}) - \sqrt{-1}\,x_1(az_1^{d-1} - bz_2^{d-1}) = -\sqrt{-1}\,(az_1^d - bz_2^d).$$

More precisely, the equation $D(g) = 0$ simplifies to

$$a(x_1 + \sqrt{-1}\,x_2)^d - b(x_1 - \sqrt{-1}\,x_2)^d = 0.$$

The left hand side of the last equation can be factorized as

$$\begin{aligned} a(x_1 + \sqrt{-1}\,x_2)^d - b(x_1 - \sqrt{-1}\,x_2)^d &= \prod_{j=0}^{d-1}[\xi(x_1 + \sqrt{-1}\,x_2) - \varepsilon^j\eta(x_1 - \sqrt{-1}\,x_2)] \\ &= \prod_{j=0}^{d-1}[(\xi - \varepsilon^j\eta)x_1 + \sqrt{-1}\,(\xi + \varepsilon^j\eta)x_2] \end{aligned}$$

where $\xi^d = a$ and $\eta^d = b$. Hence the eigenvectors are

$$v_j = \left(-\sqrt{-1}\,(\xi + \varepsilon^j\eta), \xi - \varepsilon^j\eta\right), \quad j \in \{0, \ldots, d-1\}.$$

In addition, the norm of v_j is equal to

$$q(v_j)^{\frac{1}{2}} = \sqrt{-(\xi + \varepsilon^j\eta)^2 + (\xi - \varepsilon^j\eta)^2} = \sqrt{-4\varepsilon^j\xi\eta} = 2\sqrt{-1}\,\sqrt{\varepsilon^j\xi\eta}.$$

Summing up, the E-eigenvectors of g are (call them v_j again)

$$v_j = \frac{1}{2\sqrt{-1}\,\sqrt{\varepsilon^j\xi\eta}}\left(-\sqrt{-1}\,(\xi + \varepsilon^j\eta), \xi - \varepsilon^j\eta\right), \quad j \in \{0, \ldots, d-1\}. \tag{2.4.10}$$

In this case, the system (2.1.5) becomes

$$\frac{1}{d}\frac{\partial g}{\partial x} = \lambda x, \quad \frac{1}{d}\frac{\partial g}{\partial y} = \lambda y.$$

If we substitute the coordinates of v_j in (2.4.10) in the first equation (substituting in the second equation would lead to the same result), then we get the identity

$$a\left(\frac{-\varepsilon^j\eta}{\sqrt{\varepsilon^j\xi\eta}}\right)^{d-1} + b\left(\frac{-\xi}{\sqrt{\varepsilon^j\xi\eta}}\right)^{d-1} = -\frac{\xi+\varepsilon^j\eta}{2\sqrt{\varepsilon^j\xi\eta}}\lambda.$$

Solving for λ the last equation gives an expression for the E-eigenvalues of g:

$$\begin{aligned}\lambda &= (-1)^{d-2}2\frac{(a\varepsilon^{j(d-1)}\eta^{d-1}+b\xi^{d-1})\varepsilon^j\xi\eta}{\sqrt{ab}(\xi+\varepsilon^j\eta)}\\ &= (-1)^{d-2}2\frac{ab(\xi+\varepsilon^j\eta)}{\sqrt{ab}(\xi+\varepsilon^j\eta)}\\ &= (-1)^{d-2}2\sqrt{ab}\end{aligned}$$

□

In the original variables x_1 and x_2, a complex harmonic binary form $h \in H^dV$ may be written as

$$h(x_1,x_2) = \sum_{i=0}^{d}\binom{d}{i}a_i x_1^{d-i}x_2^i, \quad a_i = \begin{cases}(-1)^{\frac{i}{2}}\xi & \text{if } i \text{ is even}\\ (-1)^{\frac{i-1}{2}}\eta & \text{if } i \text{ is odd}\end{cases} \tag{2.4.11}$$

for some $[\xi,\eta] \in \mathbb{P}^1_{\mathbb{C}}$

Example 2.4.5. For $d = 4$, a general harmonic binary form is

$$f = \xi x_1^4 + 4\eta x_1^3x_2 - 6\xi x_1^2x_2^2 - 4\eta x_1x_2^3 + \xi x_2^4, \quad [\xi,\eta] \in \mathbb{P}^1_{\mathbb{C}}.$$

The analogous statement to Proposition 2.4.4 is the following.

Corollary 2.4.6. *Consider f as in (2.4.11). Then its E-characteristic polynomial is equal to (up to a scalar factor)*

$$\psi_f(\lambda) = (\xi^2+\eta^2-\lambda^2)^{\frac{d}{2}}$$

if d is even and

$$\psi_f(\lambda) = (\xi^2+\eta^2-\lambda^2)^{d}$$

if d is odd.

The last corollary yields another interesting fact.

Corollary 2.4.7. *Every E-eigenvalue of a real harmonic binary form* $g \in H^dV^{\mathbb{R}}$ *is a Z-eigenvalue.*

The last corollary does not generalize to real harmonic forms in more than two variables. Indeed, we show in Example 3.5.9 that there exist real harmonic ternary forms which admit non-real E-eigenvalues, starting from the case $d = 3$.

Chapter 3

On the orthogonal stability of binary and ternary forms

In Chapter 2 we outlined the main properties of the Euclidean eigenvalues and eigenvectors of a degree d symmetric tensor $f \in S^d V$ on some complex vector space $V = V^{\mathbb{R}} \otimes \mathbb{C}$, where $(V^{\mathbb{R}}, q^{\mathbb{R}})$ is our usual starting real Euclidean space of dimension n. Moreover, we stressed two facts

1. In Theorem 2.2.10, we reported that the roots of the E-characteristic polynomial $\psi_f(\lambda)$ of a regular symmetric tensor $f \in S^d V$ correspond to the E-eigenvalues of f.

2. In Remark 2.3.11 we observed that a symmetric tensor $f \in S^d V$ admits the maximum number of E-eigenvalues, counted with multiplicity, if and only if the hypersurface $X_f \subset \mathbb{P}(V)$ defined by f is transversal to the isotropic $Q \subset \mathbb{P}(V)$ defined via the quadratic function $q\colon V \to \mathbb{C}$ associated, in turn, with the quadratic form $q^{\mathbb{R}}$ on $V^{\mathbb{R}}$. Moreover, transversality between X_f and Q is not related to the regularity of f.

Transversality between a projective variety $X \subset \mathbb{P}(V)$ and the isotropic quadric Q plays an important role in this chapter as well as in the forthcoming ones. Moreover, here we want to stress another property shared by the E-eigenvalues of $f \in S^d V$ and the coefficients of the E-characteristic polynomial $\psi_f(\lambda)$, namely their invariance with respect to the *complex special orthogonal group* $\mathrm{SO}(V)$ (see Theorem 2.1.10 and Corollary 2.2.5).

We recall that the complex orthogonal group $\mathrm{O}(V)$ is the subgroup of $\mathrm{GL}(V)$ of invertible linear operators of V that preserve a fixed bilinear form: in our case, the bilinear form $q\colon V \times V \to \mathbb{C}$ associated to the inner product $q^{\mathbb{R}}$ on $V^{\mathbb{R}}$. For the rest of the chapter, we assume that $q(x, y) = \sum_{i=1}^{n} x_i y_i$ is the bilinear form associated to the standard Euclidean inner product in $V^{\mathbb{R}} = \mathbb{R}^n$. The group $\mathrm{O}(V)$

is, in fact, an example of a *matrix Lie group* and has two connected components. The component of $\mathrm{O}(V)$ containing the identity Id_V is precisely the complex special orthogonal group $\mathrm{SO}(V)$.

By our assumption, in the following we adopt the notations $\mathrm{GL}(V) = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{O}(V) = \mathrm{O}(n, \mathbb{C})$, $\mathrm{SO}(V) = \mathrm{SO}(n, \mathbb{C})$ and similarly for the other known matrix Lie groups on V. In particular, $\mathrm{SO}(n, \mathbb{C})$ is the subgroup of complex matrices A of size n such that $A^T A = I$ and $\det(A) = 1$. Note that the matrix Lie groups $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ are different from the *unitary group* $\mathrm{U}(n)$ and the *special unitary group* $\mathrm{SU}(n)$. The last two are defined analogously to $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$, but with respect to a Hermitian inner product on $\mathbb{C}^n$. Note that the bilinear form q fixed is symmetric rather than conjugate-symmetric. Our main references for the theory about matrix Lie groups, matrix Lie algebras and representation theory are [Hal, FH].

The space $S^d\mathbb{C}^n$ is a $\mathrm{GL}(n, \mathbb{C})$-module via the representation

$$\rho_d \colon \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(S^d\mathbb{C}^n), \quad \rho_d(g)(f(x)) \coloneqq f(g^{-1}(x)) \tag{3.0.1}$$

for all $f \in S^d\mathbb{C}^n$ and all $g \in \mathrm{GL}(n, \mathbb{C})$. In this chapter, the elements of $\mathbb{P}(S^d\mathbb{C}^n)$ are called forms of degree d in n variables. Our aim is to investigate the action of the subgroup $\mathrm{SO}(n, \mathbb{C})$ on $\mathbb{P}(S^d\mathbb{C}^n)$. In particular, a very hard task in Geometric Invariant Theory is the computation of all stable and semistable elements of $\mathbb{P}(S^d\mathbb{C}^n)$ for the action of $\mathrm{SO}(n, \mathbb{C})$. For the notion of *semistability* and *stability* with respect to the action of an algebraic group G on a projective variety X, we refer to Section 3.1. All the necessary material is provided essentially from [LeP].

The theory of stable and semistable binary forms ($n = 2$) with respect to the action of the *special linear group* $\mathrm{SL}(2, \mathbb{C})$ (that is, the subgroup of $\mathrm{GL}(2, \mathbb{C})$ of complex 2×2 matrices with determinant 1) as well as the group $\mathrm{SO}(2, \mathbb{C})$ has been extensively studied since the XIX century. In Section 3.2 we report the classical result by Hilbert in this direction, which uses the well-known *Hilbert-Mumford criterion* stated in Theorem 3.1.6.

The core of this chapter deals with the action of the complex orthogonal group $\mathrm{SO}(3, \mathbb{C})$ on the space $S^d\mathbb{C}^3$ of complex ternary forms. In order to give conditions for stability and semistability on $S^d\mathbb{C}^3$, a crucial role is played by the space of harmonic ternary forms $H^d\mathbb{C}^3 \subset S^d\mathbb{C}^3$, introduced in Definition 2.4.3. It has dimension $2d+1$ by formula (2.4.8). This space is important since every form in $S^d\mathbb{C}^3$ admits a unique *harmonic decomposition*, as stated below.

Theorem 3.0.1. *Let $q \in S^2\mathbb{C}^3$ be the quadratic form $q(x) = x_1^2 + x_2^2 + x_3^2$ fixed by the group $\mathrm{SO}(3, \mathbb{C})$. The space $S^d\mathbb{C}^3$ admits the following $\mathrm{SO}(3, \mathbb{C})$-equivariant* *decomposition*

$$S^d\mathbb{C}^3 = \bigoplus_{k=0}^{\lfloor \frac{d}{2} \rfloor} q^k H^{d-2k}\mathbb{C}^3,$$

that is, every polynomial $f \in S^d\mathbb{C}^3$ *admits a unique* harmonic decomposition

$$f = \begin{cases} f_d + qf_{d-2} + \cdots + q^{\frac{d}{2}} f_0 & \textit{for even } d \\ f_d + qf_{d-2} + \cdots + q^{\frac{d-1}{2}} f_1 & \textit{for odd } d \end{cases}$$

where we call $f_{d-2k} \in H^{d-2k}\mathbb{C}^3$ *the* harmonic form associated to f of degree $d-2k$*, for all* $0 \le k \le \lfloor \frac{d}{2} \rfloor$*.*

The harmonic decomposition stated above allows us to give the following stability condition.

Proposition 3.0.2. *Let* $f \in S^d\mathbb{C}^3$ *be a ternary form and consider its unique harmonic decomposition as in Theorem 3.0.1. Then* f *is stable (semistable) with respect to* $\mathrm{SO}(3,\mathbb{C})$ *if at least one of the harmonic forms* f_{d-2k} *associated to* f *is stable (semistable) with respect to* $\mathrm{SO}(3,\mathbb{C})$*.*

A relevant part of this chapter is devoted to the study of the $\mathrm{SO}(3,\mathbb{C})$-stability and semistability in the space $H^d\mathbb{C}^3$ of harmonic ternary forms. In this direction, we provide a necessary and sufficient stability condition in the following Proposition 3.0.3 and Theorem 3.0.4. In their proofs, we consider essentially three relevant facts:

1. There exists an isomorphism between the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(3,\mathbb{C})$ of the groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO}(3,\mathbb{C})$.

2. All the irreducible $\mathrm{SL}(2,\mathbb{C})$-modules are of the form $S^k\mathbb{C}^2$ for some $k \ge 1$.

3. The space $H^d\mathbb{C}^3 \subset S^d\mathbb{C}^3$ of complex harmonic ternary forms, introduced in Definition 2.4.3, is an irreducible $\mathrm{SO}(3,\mathbb{C})$-module of dimension $2d+1$.

Summing up all these pieces of information, there is an $\mathrm{SL}(2,\mathbb{C})$-equivariant isomorphism between the spaces $H^d\mathbb{C}^3$ and $S^{2d}\mathbb{C}^2$. Therefore, we can use the well-known semistability theory of binary forms of degree $2d$ to understand its counterpart in $H^d\mathbb{C}^3$.

The first characterization of stable and semistable complex harmonic ternary forms with respect to the action of $\mathrm{SO}(3,\mathbb{C})$ is furnished by the next proposition.

Proposition 3.0.3. *Let* $h \in H^d\mathbb{C}^3$ *be a harmonic ternary form and denote with* X_h *its associated curve in* $\mathbb{P}^2_{\mathbb{C}}$*. Denote with* Q *the quadric fixed by the group*

$\mathrm{SO}(3, \mathbb{C})$. *Then h is non-stable (non-semistable) with respect to* $\mathrm{SO}(3, \mathbb{C})$ *if and only if its associated divisor $X_h \cap Q$ on Q may be written in the form*

$$X_h \cap Q = kP + D$$

for some $k \geq d$ ($k \geq d+1$), where D is a divisor of degree $2d-k$ on Q.

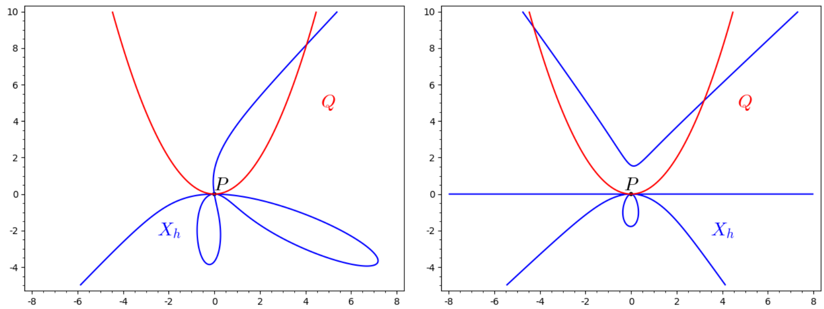

Figure 3.1: From left to right, an affine picture of a non-stable and a non-semistable harmonic plane quintic. We have $|X_h \cap Q| = 10$. In the first case, $X_h \cap Q = 5P + D$, whereas in the second case $X_h \cap Q = 6P + D$.

Last proposition leaves us the following problem: *which harmonic ternary forms $h \in H^d\mathbb{C}^3$ determine a plane curve X_h such that $X_h \cap Q$ is a divisor on Q of the form $kP + D$ for some $k \geq d$?*

The main result of this chapter answers this question. In the following statement, for any integer $m \geq 0$ and any curve $X \subset \mathbb{P}^2_{\mathbb{C}}$ with ideal $I(X) = (f)$, we indicate with mX the divisor on $\mathbb{P}^2_{\mathbb{C}}$ whose ideal is generated by the polynomial f^m. Moreover, we denote by CT_PX the *tangent cone* of X at the point $P \in X$. This notion and the related notion of multiplicity of a point are recalled in Section 3.5. When not specified, the stability and semistability of either an element of $H^d\mathbb{C}^3$ or an element of $S^{2d}\mathbb{C}^2$ are considered with respect to either $\mathrm{SO}(3, \mathbb{C})$ or $\mathrm{SL}(2, \mathbb{C})$.

Theorem 3.0.4. *Let $h \in H^d\mathbb{C}^3$ be a harmonic ternary form. Then*

1. *If d is even, h is non-stable if and only if there exists $P \in Q$ and an integer $k \geq \frac{d}{2}$ such that P is a k-ple point for X_h and $\frac{d}{2}T_PQ \subset CT_PX_h$. Moreover, h is non-semistable if and only if it is non-stable and $\varphi_{2d}(h) = 0$, where $\varphi_{2d} \in S^2H^d\mathbb{C}^3$ is the* $\mathrm{SO}(3, \mathbb{C})$*-invariant corresponding to the* $\mathrm{SL}(2, \mathbb{C})$*-invariant in $S^2S^{2d}\mathbb{C}^2$ defined in (3.5.7).*

2. *If d is odd, h is non-stable if and only if there exists $P \in Q$ and an integer $k \geq \frac{d+1}{2}$ such that P is a k-ple point for X_h. In particular $\frac{d-1}{2}T_PQ \subset CT_PX_h$. Moreover, h is non-semistable if and only if it is non-stable and $\varphi_{2d}(h) = 0$.*

In particular, if h is non-semistable, then X_h is reducible in the form $X_h = LX_{h'}$, where the line L and the curve $X_{h'}$ are tangent to Q in a common point, and $h' \in H^{d-1}\mathbb{C}^3$ is non-stable.

As we can observe from the statement of the last theorem, the multiplicity of roots of binary forms is somehow related to the multiplicity of the corresponding harmonic curve at a certain isotropic point. When this multiplicity is large enough, the curve turns out to be non-semistable and is forced to split into a line and a non-stable harmonic curve of one degree less. For example, the non-stable plane quintic X_h on the left in Figure 3.1 is such that P is a triple point for X_h and the double line $2T_PQ$ is contained in the tangent cone CT_PX_h. Moreover, the non-semistable quintic X_h on the right is the union of T_PQ and a non-stable plane quartic. After the proof of Theorem 3.0.4, we provide concrete examples of non-stable and non-semistable harmonic ternary forms in small degrees.

3.1 Semistability and stability criteria

Roughly speaking, the necessary ingredients needed to talk about stability and semistability in this chapter are basically two: a linear algebraic group G acting on a projective variety X.

The group $\mathrm{GL}(n, \mathbb{C})$ of complex invertible matrices of order n is naturally an affine algebraic variety. In the following, a *linear algebraic group* is any Zariski closed subgroup of $\mathrm{GL}(n, \mathbb{C})$. In particular, all the groups mentioned in the introduction of this chapter are linear algebraic groups. Their structure of reduced algebraic variety is the one induced by $\mathrm{GL}(n, \mathbb{C})$. If G denotes a linear algebraic group, the map $G \to G$ sending an element g to its inverse g^{-1} is an isomorphism of algebraic varieties. Moreover, since the variety G is non-reduced, the open subset of smooth points is nonempty and invariant under translation. In other words, G is smooth.

Definition 3.1.1. Consider a projective algebraic variety X. We say that G *acts* on X if there exists a morphism $\sigma\colon G \times X \to X$ such that

1. for all $g \in G$, the morphism $\sigma(g, \_)\colon X \to X$ sending any $x \in X$ to $g.x := \sigma(g, x) \in X$ is an automorphism of X,

2. the map sending any $g \in G$ to the automorphism $X \to X$ already defined is a group homomorphism.

Moreover, the image $G.x := \{\sigma(g,x) \mid g \in G\}$ is called the *orbit* of $x \in X$ under the action of G.

Let $\psi\colon G \times X \to X \times X$ be the morphism sending any pair $(g,x) \in G \times X$ to $(g.x, x) \in X \times X$. We say that the action of G on X is *proper* if the morphism ψ is proper (see [Har, Chapter 2, §4]). If G acts on the projective varieties X and Y and $\varphi\colon X \to Y$ is a morphism of projective varieties, then φ is G-*equivariant* if the following diagram commutes:

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ G \times Y & \longrightarrow & Y, \end{array}$$

where the horizontal arrows are the actions of G on X and Y, while the vertical arrows are equal to the product map $Id_G \times \varphi$.

Now we briefly recover the essential ingredients of representation theory needed for this chapter. For further details, we refer to [FH]. For our purposes, V is a finite-dimensional complex vector space. For a given linear algebraic group G, then V is a G-*module* if there exists a finite-dimensional representation $\rho\colon G \to \mathrm{GL}(V)$. A G-module V is *simple* if the representation ρ is irreducible, meaning that all proper G-invariant vector subspaces of V are trivial. Moreover, V is *semi-simple* if ρ is totally reducible, namely it is a direct sum of irreducible representations. All these notions are necessary to introduce the next important definition.

Definition 3.1.2. A linear algebraic group G is *linearly reductive* if all finite dimensional G-modules V are semi-simple.

Basically, all the examples of linear algebraic groups appearing in this chapter are linearly reductive. For further details we refer to [LeP, Section 6.2].

We move directly to the most important definitions used in this chapter. Let $X \subset \mathbb{P}(V)$ be a projective algebraic variety for some n-dimensional complex vector space V. Suppose that G is a linearly reductive group acting on X. We denote by $S = S(X)$ the homogeneous coordinate ring of X. Then $S \cong \mathbb{C}[x_1,\ldots,x_n]/I$ for some homogeneous ideal $I \subset \mathbb{C}[x_1,\ldots,x_n]$. One might consider the subring $S^G \subset S$ of the elements that are G-invariant. It turns out that A^G is finitely generated, and the inclusion $S^G \hookrightarrow S$ yields a rational map $X = \mathrm{Proj}(S) \dashrightarrow \mathrm{Proj}(S^G)$ of projective algebraic varieties.

Definition 3.1.3. A point $x \in X$ is *semistable* under the action of G if there exists a homogeneous G-invariant polynomial with strictly positive degree which does not vanish at x. Otherwise, such a point is said to be *non-semistable* for the action of G.

The locus of semistable points of X is indicated with X^{ss}. In particular, the variety $X \setminus X^{ss}$ corresponds to the locus of common zeros of the elements in S^G of positive degree. Therefore the restriction $X^{ss} \to \mathrm{Proj}(S^G)$ becomes a morphism.

For the moment, we call $C(X)$ the affine cone of the projective variety X. We recall the following result without proof.

Lemma 3.1.4. *A point $x \in X$ is non-semistable under the action of G if and only if it is the image of a point $p \in C(X) \setminus \{0\}$ such that $0 \in \overline{G.p}$.*

The second important notion to define is the following.

Definition 3.1.5. A point $x \in X$ is said to be *stable* under the action of G if it is semistable and if the orbit map $G \to X^{ss}$ sending $g \in G$ to $g.x$ is proper. The locus of semistable points of X is indicated with X^s.

The next tool we introduce is a fundamental criterion to compute the semistable or stable points of the action of a linearly reductive algebraic group G on a projective variety X. Until the end of this paragraph, the symbol $\mathbb{C}^*$ denotes a one-parameter subgroup of G. Of course, the action of a one-parameter subgroup $\mathbb{C}^*$ on X is defined by restricting the action of G.

Theorem 3.1.6 (Hilbert-Mumford)**.** *Let G be a linearly reductive group acting on a projective algebraic variety X. A point $x \in X$ is semistable (stable) if and only if it is semistable (stable) for the action induced by every one parameter subgroup $\mathbb{C}^* \subset G$.*

The above statement is essentially [LeP, Theorem 6.5.3]. We also mention a simplified proof of Theorem 3.1.6, valid only over $\mathbb{C}$, which uses elementary linear algebra results [Sur].

3.2 Stability criteria for binary forms

In this section we focus on the case of $V = \mathbb{C}^2$ ($n = 2$) and we choose $X = \mathbb{P}(S^d\mathbb{C}^2)$, with $d \geq 2$, as our projective variety. The next classical result by Hilbert determines all semistable and stable elements of X under the action of $\mathrm{SL}(2, \mathbb{C})$, using the already mentioned Hilbert-Mumford criterion stated in Theorem 3.1.6.

Theorem 3.2.1 (Hilbert)**.** *A binary form $f \in \mathbb{P}(S^d\mathbb{C}^2)$ is semistable (stable) for the action of $\mathrm{SL}(2, \mathbb{C})$ if and only if every root of f has multiplicity $\leq \frac{d}{2}$ ($< \frac{d}{2}$).*

Remark 3.2.2. Note the difference in the previous result between the even and the odd case: in the odd case, the conditions $\leq \frac{d}{2}$ and $< \frac{d}{2}$ coincide, namely a binary form of odd degree is semistable if and only if it is stable.

Sketch of the proof. Consider a one parameter subgroup $\mathbb{C}^* \subset \mathrm{SL}(2,\mathbb{C})$. We can choose coordinates (z_1, z_2) for $\mathbb{C}^2$ such that

$$\mathbb{C}^* = \left\{ \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \mid t \in \mathbb{C}^* \right\}.$$

Then $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by sending (z_1, z_2) to $(tz_1, \frac{1}{t}z_2)$. The induced action $\sigma\colon \mathbb{C}^* \times \mathbb{P}(S^d\mathbb{C}^2) \to \mathbb{P}(S^d\mathbb{C}^2)$ is defined by

$$\sigma(g, (a_0, \dots, a_d)) = (t^d a_0, t^{d-2} a_1, \dots, t^{2-d} a_{d-1}, t^{-d} a_d),$$

for all $g \in \mathbb{C}^*$, where the vector $a = (a_0, \dots, a_d)$ identifies the binary form

$$f = \sum_{j=0}^{d} \binom{d}{j} a_j z_1^{d-j} z_2^j \in S^d\mathbb{C}^2.$$

Then it is almost immediate to see that $0 \in \overline{\mathbb{C}^*.a}$ if and only if $a_k = \dots = a_{2k} = 0$ for $d = 2k$ or $a_{k+1} = \dots = a_{2k+1} = 0$ for $d = 2k+1$. Therefore, applying Theorem 3.1.6, the set of non-semistable binary forms is

$$X \setminus X^{ss} = \left\{ f = z^{k+1} \sum_{j=0}^{k-1} \binom{2k}{j} a_j z^{k-1-j} w^j \mid [a_0, \dots, a_{k-1}] \in \mathbb{P}^{k-1}_{\mathbb{C}} \right\}$$

for $d = 2k$, whereas

$$X \setminus X^{ss} = \left\{ f = z^{k+1} \sum_{j=0}^{k} \binom{2k+1}{j} a_j z^{k-j} w^j \mid [a_0, \dots, a_k] \in \mathbb{P}^{k}_{\mathbb{C}} \right\}$$

for $d = 2k+1$. In particular, for $d = 2k$ we have that the multiplicity of z is at least $k + 1 > k = \frac{d}{2}$, and similarly for $d = 2k+1$ the multiplicity of z is at least $k + 1 > k + \frac{1}{2} = \frac{d}{2}$. $\square$

Example 3.2.3. Consider for example the case $d = 3$. By Theorem 3.2.1, the semistable (or stable, by Remark 3.2.2) cubic binary forms are exactly the ones having three distinct roots. To determine geometrically which are these forms, we consider the projective space $X = \mathbb{P}(S^3\mathbb{C}^2) \cong \mathbb{P}^3_{\mathbb{C}}$ parametrizing all the cubic binary forms. The rational normal curve $C_3 \subset X$ parametrizes the cubic binary forms of rank one, namely the forms $f = (\lambda x_1 + \mu x_2)^3$. Moreover, the tangential surface of C_3, denoted by TC_3, is the locus of cubic binary forms having at least a double root. Hence the locus of semistable cubic binary forms is precisely $X \setminus TC_3$. In addition to this, we observe that $TC_3 = V(\Delta)$, where Δ is the cubic discriminant. We recall that Δ is the only one generator for the algebra of invariants for $\mathrm{SL}(V)$ acting on X.

Another linearly reductive group which acts on $\mathbb{P}(S^d\mathbb{C}^2)$ is the special orthogonal group $\mathrm{SO}(2,\mathbb{C})$. The analogous result to Theorem 3.2.1 is the following.

Theorem 3.2.4. *A binary form $f \in \mathbb{P}(S^d\mathbb{C}^2)$ is semistable (stable) for the action of $\mathrm{SO}(2,\mathbb{C})$ if and only if the isotropic roots of f have multiplicity $\leq \frac{d}{2}$ ($< \frac{d}{2}$).*

We consider two explanatory examples of the last result.

Example 3.2.5 (Cubic binary forms)**.** Consider the case $d = 3$, hence $X = \mathbb{P}(S^3\mathbb{C}^2)$. Denote with L_1, L_2 the lines tangent to C_3 in z_1^3 and z_2^3 respectively, where $z_1 = x_1 + \sqrt{-1}\,x_2$ and $z_2 = x_1 - \sqrt{-1}\,x_2$ are the two isotropic linear forms of $\mathbb{C}^2$ (see Figure 3.2). Then L_1 and L_2 are the loci of cubic forms having z_1^2 and z_2^2 as a factor, respectively. In particular we have that $z_1^2z_2 \in L_1$, $z_1z_2^2 \in L_2$ and additionally $L_1 \cap L_2 = \emptyset$. Hence the locus of semistable points with respect to the action of $\mathrm{SO}(2,\mathbb{C})$ is $X^{ss} = X \setminus (L_1 \cup L_2)$.

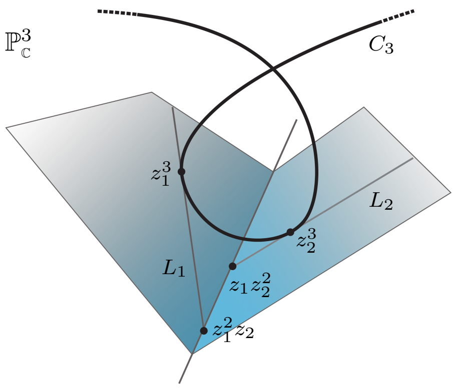

Figure 3.2: The locus of non-semistable cubics for $\mathrm{SO}(2,\mathbb{C})$.

Example 3.2.6 (Quartic binary forms)**.** Now consider the case $d = 4$. Denote with L_1, L_2 the lines tangent to the rational normal quartic C_4 at the points z_1^4 and z_2^4, respectively. In addition, let π_1 and π_2 be the osculating planes to C_4 at z_1^4 and z_2^4, respectively (see Figure 3.3). On one hand, L_1 and L_2 are the loci of quartic forms having z_1^3 and z_2^3 as a factor, respectively. On the other hand, π_1 and π_2 are the loci of quartic forms having z_1^2 and z_2^2 as a factor, respectively. In particular, $L_1 \subset \pi_1$ and $L_2 \subset \pi_2$. We remark that $z_1^3 z_2 \in \pi_1$, $z_1 z_2^3 \in \pi_2$, and $\pi_1 \cap \pi_2 = \{z_1^2 z_2^2\}$. Then the loci of semistable and stable points of X with respect to the action of $\mathrm{SO}(2, \mathbb{C})$ are respectively $X^{ss} = X \setminus (L_1 \cup L_2)$ and $X^s = X \setminus (\pi_1 \cup \pi_2)$.

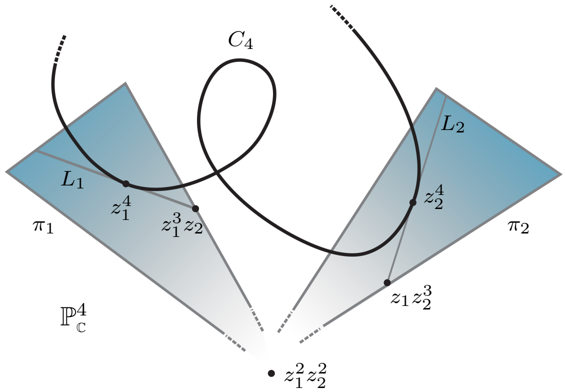

Figure 3.3: The loci of non-stable and non-semistable quartics for $\mathrm{SO}(2, \mathbb{C})$.

In general, we have the following result.

Proposition 3.2.7. *Let $X = \mathbb{P}(S^d\mathbb{C}^2)$ and denote by C_d the rational normal curve of degree d in X. Call z_1 and z_2 the two isotropic linear forms of $\mathbb{C}^2$.*

1. *Suppose that $d = 2k$. Let L_1 and L_2 be the projective subspaces of X that parametrize the binary forms of degree d having z^{k+1} and w^{k+1} as a factor, respectively. Moreover, denote by π_1 and π_2 the projective subspaces of X that parametrize the binary forms of degree d having z^k and w^k as a factor, respectively. Then $\dim(L_1) = \dim(L_2) = k-1$, $\dim(\pi_1) = \dim(\pi_2) = k$ and the loci of semistable and stable points of X under the action of $\mathrm{SO}(2, \mathbb{C})$ on X are respectively*

$$X^{ss} = X \setminus (L_1 \cup L_2), \quad X^s = X \setminus (\pi_1 \cup \pi_2).$$

2. *Suppose that $d = 2k+1$. Let L_1 and L_2 be the projective subspaces of X that parametrize the binary forms of degree d having z^{k+1} and w^{k+1} as a factor, respectively. Then $\dim(L_1) = \dim(L_2) = k$ and the locus of semistable (equivalently, stable) points of X under the action of $\mathrm{SO}(2, \mathbb{C})$ on X is*

$$X^{ss} = X^s = X \setminus (L_1 \cup L_2).$$

3.3 An orthogonal stability criterion for harmonic ternary forms

In the previous section we considered the action of the complex special linear group $\mathrm{SL}(2,\mathbb{C})$ as well as the complex special orthogonal group $\mathrm{SO}(2,\mathbb{C})$ on the projective space $\mathbb{P}(S^d\mathbb{C}^2)$ of (classes of) complex binary forms of degree d.

The aim of this section is to prove Proposition 3.0.3. First, we need to explain the three facts outlined in the preamble, before the statement of Proposition 3.0.3.

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of $\mathrm{SL}(2,\mathbb{C})$ corresponds to the space of 2×2 complex matrices with trace zero. It has dimension three with the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.3.1}$$

The basis elements have the following relations with respect to the Lie bracket:

$$[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = H.$$

On the other hand, the Lie algebra $\mathfrak{so}(3,\mathbb{C})$ of $\mathrm{SO}(3,\mathbb{C})$ is the space of 3×3 complex matrices A satisfying $A^T = -A$. It is three dimensional and a basis of $\mathfrak{so}(3,\mathbb{C})$ is given by

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; F_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The linear map $\phi\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(3,\mathbb{C})$ such that

$$\phi(H) = -2\sqrt{-1}F_1, \quad \phi(X) = F_2 - \sqrt{-1}F_3, \quad \phi(Y) = F_2 + \sqrt{-1}F_3 \tag{3.3.2}$$

is a Lie algebra isomorphism. At the level of Lie groups, we have the following result.

Lemma 3.3.1. *[Hal, Lemma 4.30] There exists a Lie group homomorphism*

$$\Phi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{C})$$

such that

1. *Φ is surjective,*
2. *$\ker(\Phi) = \{Id, -Id\}$, and*
3. *the associated Lie algebra homomorphism is the map $\phi\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(3,\mathbb{C})$ defined in (3.3.2).*

Now consider the space $S^k\mathbb{C}^2$ of (classes of) complex binary forms of degree k. It is a $(2k+1)$-dimensional $\mathrm{SL}(2,\mathbb{C})$-module via the restriction to $\mathrm{SL}(2,\mathbb{C})$ of the representation ρ_k defined in (3.0.1), which we call here Π_k. The associated Lie algebra representation $\pi_k\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(S^k\mathbb{C}^2)$ is such that for all $g \in \mathfrak{sl}(2,\mathbb{C})$

$$\pi_k(A) = \frac{d}{dt}\Pi_k(e^{tA})\bigg|_{t=0}. \tag{3.3.3}$$

Applying the above identity to the vector $z = (z_1, z_2) \in \mathbb{C}^2$ we obtain that

$$\pi_k(A)f(z) = \frac{d}{dt}f(e^{-tA}z)\bigg|_{t=0}.$$

Let $z(t)$ be a curve in $\mathbb{C}^2$ defined as $z(t) = e^{-tA}z$. In particular, $z(0) = z$. We can write $z(t) = (z_1(t), z_2(t))$ and by the chain rule

$$\pi_k(A)f = \frac{\partial f}{\partial z_1}\frac{dz_1}{dt}\bigg|_{t=0} + \frac{\partial f}{\partial z_2}\frac{dz_2}{dt}\bigg|_{t=0}.$$

However $\frac{dz}{dt}\big|_{t=0} = -Az$, so we obtain the formula

$$\pi_k(A)f = -(A_{11}z_1 + A_{12}z_2)\frac{\partial f}{\partial z_1} - (A_{21}z_1 + A_{22}z_2)\frac{\partial f}{\partial z_2}. \tag{3.3.4}$$

As one might foresee from the first lines of this section, the representations π_k are very important for the whole development of this chapter. Indeed, they are "good" representations of $\mathfrak{sl}(2,\mathbb{C})$ (see [Hal, Theorem 4.9]).

Theorem 3.3.2. *For each integer $k \geq 0$, the representation π_k of $\mathfrak{sl}(2,\mathbb{C})$ is irreducible. Moreover, if π is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ with dimension $k+1$, then π is equivalent to the representation π_k described before.*

At this point, we might carry all the information provided so far to the Lie algebra $\mathfrak{so}(3,\mathbb{C})$, via the isomorphism ϕ defined in (3.3.2).

The space $S^d\mathbb{C}^3$ of (classes of) complex ternary forms is an $\mathrm{SO}(3,\mathbb{C})$-module via the restriction to $\mathrm{SO}(3,\mathbb{C})$ of the representation ρ_d defined in (3.0.1), which we keep calling ρ_d.

Since $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(3,\mathbb{C})$ are isomorphic Lie algebras, they have essentially the same representations. More specifically, if π is a representation of $\mathfrak{sl}(2,\mathbb{C})$, then $\pi\circ\phi^{-1}$ is a representation for $\mathfrak{so}(3,\mathbb{C})$, and every representation of $\mathfrak{so}(3,\mathbb{C})$ is of this form. In particular, the irreducible representations of $\mathfrak{so}(3,\mathbb{C})$ are precisely of the form $\sigma_k := \pi_k \circ \phi^{-1}$. This fact holds true at the level of Lie groups as well, at least for even k.

Proposition 3.3.3. *[Hal, Proposition 4.29] Let $\sigma_k = \pi_k \circ \phi^{-1}$ be the irreducible representation of $\mathfrak{so}(3,\mathbb{C})$ for some $k \geq 1$. If k is even, then there is a representation Σ_k of $\mathrm{SO}(3,\mathbb{C})$ such that $\Sigma_k(e^A) = e^{\sigma_k(A)}$ for all $A \in \mathfrak{so}(3,\mathbb{C})$. If k is odd, then there is no such representation of $\mathrm{SO}(3,\mathbb{C})$.*

The third ingredient is the space of harmonic ternary forms $H^d\mathbb{C}^3 \subset S^d\mathbb{C}^3$, introduced in Definition 2.4.3 as well as in the preamble of this chapter. Consider the representation $(H^d\mathbb{C}^3, \rho_d)$ of $\mathrm{SO}(3,\mathbb{C})$ (see for example [Ste, Lemma 4.1, Section 4.4]).

Theorem 3.3.4. *For every $d \geq 1$, the representation $(H^d\mathbb{C}^3, \rho_d)$ of $\mathrm{SO}(3,\mathbb{C})$ is irreducible. Therefore, $H^d\mathbb{C}^3$ is a simple $\mathrm{SO}(3,\mathbb{C})$-module. In addition, every simple $\mathrm{SO}(3,\mathbb{C})$-module is equivalent to $H^d\mathbb{C}^3$ for some $d \geq 1$.*

Summing up all the properties given in this section, by Schur's Lemma we conclude that

Theorem 3.3.5. *There exists an $\mathrm{SL}(2,\mathbb{C})$-equivariant isomorphism*

$$\varphi\colon H^d\mathbb{C}^3 \longrightarrow S^{2d}\mathbb{C}^2.$$

This result implies, thanks to the results outlined in the previous lines of this section, that for each $d \geq 1$ we can read $H^d\mathbb{C}^3$ as an irreducible $\mathrm{SL}(2,\mathbb{C})$ module of dimension $2d+1$. Therefore, we can transfer all the $\mathrm{SL}(2,\mathbb{C})$-stability theory of $S^{2d}\mathbb{C}^2$ to describe all stable and semistable elements of $H^d\mathbb{C}^3$ with respect to the group $\mathrm{SO}(3,\mathbb{C})$.

Proof of Proposition 3.0.3. Let h be a complex harmonic ternary form. Since X_h is a curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$, it intersects the isotropic quadric Q in $2d$ points, counted with multiplicity. In particular, $X_h \cap Q$ is a divisor of degree $2d$ in Q.

Observe that, thanks to Theorem 3.0.1, if h_1 and h_2 are not proportional forms in $H^d\mathbb{C}^3$, they give different divisors on Q. This is not true if f_1 and f_2 are not proportional forms in $S^d\mathbb{C}^3$: indeed, given a harmonic form $h \in H^d\mathbb{C}^3$ and two not proportional forms g_1 and g_2 in $S^{d-2}\mathbb{C}^3$, the forms $f_1 = h_1 + qg_1$ and $f_2 = h_2 + qg_2$ are not proportional but yield the same divisor on Q, since

$$X_{f_1} \cap Q = X_h \cap Q = X_{f_2} \cap Q.$$

The divisor $X_h \cap Q$ corresponds to a divisor on the projective line $\mathbb{P}^1_{\mathbb{C}}$, namely to the equivalence class of a binary form $f \in S^{2d}\mathbb{C}^2$. The statement follows by considering the stability and semistability criterion of Theorem 3.2.1. $\square$

We make a couple of comments about Proposition 3.0.3. The case $d = 1$ is trivial: indeed, every line in $\mathbb{P}^1_{\mathbb{C}}$ is harmonic and is non-stable, while all lines tangent to Q are non-semistable.

The first nontrivial case is $d = 2$:

1. By Proposition 3.0.3, a harmonic ternary form $h \in S^2\mathbb{C}^3$ is non-stable if $X_h \cap Q = 2P + D$, with $|D| = 2$. How do we achieve such a divisor? If X_h is a smooth conic, then necessarily X_h is tangent to Q at some point. Now consider two linear forms a and b vanishing at the point $P = [p] \in Q$ and their corresponding lines L_a and L_b. The tangent line T_PQ is the locus of points $P' = [p']$ such that $q(p, p') = 0$. The singular conic $C_{ab} = L_a \cup L_b$ is harmonic if and only if $\Delta(ab) = 0$. By the relation

$$\Delta(ab) = \Delta a + 2q(\nabla a, \nabla b) + \Delta b,$$

this is equivalent to the requirement $q(\nabla a, \nabla b) = 0$. This condition, together with the condition $P \in L_a \cap L_b$, yields either $L_a = T_PQ$ or $L_b = T_PQ$. This can be verified easily by looking at the dual space $\mathbb{P}^2_{\mathbb{C}}$ (see Figure 3.4): on one hand, the lines L_a and L_b correspond to points in $\mathbb{P}^2_{\mathbb{C}}$, each one of them belongs to the polar line (with respect to $Q^\vee$) of the other. On the other hand, the line P contains the points L_a and L_b, and P is tangent to $Q^\vee$ at the point T_PQ. This forces either L_a or L_b to coincide with T_PQ.

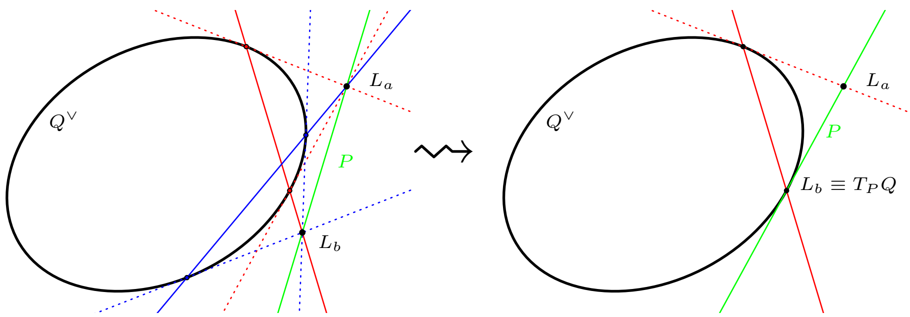

Figure 3.4: The non-stable singular harmonic plane conic $C_{ab} = L_a \cup L_b$ in $(\mathbb{P}^2_{\mathbb{C}})^\vee$.

2. Finally, a harmonic ternary form $h \in H^2\mathbb{C}^3$ is non-semistable if either $X_h \cap Q = 3P + P'$ for some points $P \neq P'$ or $X_h \cap Q = 4P$. The first case is achieved by the conic $L \cup T_PQ$, where L is a line passing through $P \in Q$ distinct from the tangent line T_PQ. The second case is achieved by the double line $2T_PQ$ for some $P \in Q$.

3.4 A dictionary between harmonic ternary forms and binary forms

As one might foresee, the geometrical description of Proposition 3.0.3 is difficult in general. To give a more detailed characterization of stable and semistable elements of $H^d\mathbb{C}^3$, we need to introduce suitable bases of $H^d\mathbb{C}^3$ and $S^{2d}\mathbb{C}^2$ and give a coordinate-based description of the $\mathrm{SL}(2,\mathbb{C})$-equivariant isomorphism φ of Theorem 3.3.5.

Consider the basis element $H \in \mathfrak{sl}(2,\mathbb{C})$ introduced in (3.3.1) and apply to it the representation π_k via the relation (3.3.4): we get

$$\pi_k(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}.$$

Applying $\pi_k(H)$ to the monomial $z_1^j z_2^{k-j} \in S^k\mathbb{C}^2$ we get the identity

$$\pi_k(H)(z_1^j z_2^{k-j}) = (k-2j)z_1^j z_2^{k-j}.$$

This means that $z_1^j z_2^{k-j}$ is an eigenvector of $\pi_k(H)$ with corresponding eigenvalue $k-2j$, for all $0 \le j \le k$. Therefore, the linear operator $\pi_k(H)$ is diagonalizable.

Applying again formula (3.3.4) to the other basis elements X and Y of $\mathfrak{sl}(2,\mathbb{C})$, we get the identities

$$\pi_k(X) = -z_2\frac{\partial}{\partial z_1}, \quad \pi_k(Y) = -z_1\frac{\partial}{\partial z_2}.$$

Applying $\pi_k(X)$ and $\pi_k(Y)$ to $z_1^j z_2^{k-j} \in S^k\mathbb{C}^2$ we obtain that

$$\pi_k(X)(z_1^j z_2^{k-j}) = -jz_1^{j-1}z_2^{k-j+1}, \quad \pi_k(Y)(z_1^j z_2^{k-j}) = (j-k)z_1^{j+1}z_2^{k-j-1}.$$

Hence the operators $\pi_k(X)$ and $\pi_k(Y)$ somehow move from one eigenvector of $\pi_k(H)$ to the previous or the next one. This is why they are also called *lowering* and *rising* operators.

Now assume that $k = 2d$ for some integer $d \ge 1$. Which are the linear operators in $\mathrm{GL}(H^d\mathbb{C}^3)$ corresponding respectively to $\pi_{2d}(H)$, $\pi_{2d}(X)$ and $\pi_{2d}(Y)$? By Proposition 3.3.3, every $(2d+1)$-dimensional irreducible representation of $\mathfrak{so}(3,\mathbb{C})$ is of the form $\sigma_{2d} = \pi_{2d} \circ \phi^{-1}$, where ϕ is the isomorphism between $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(3,\mathbb{C})$. Then looking at the relations in (3.3.2) we get

$$\begin{aligned}
\pi_{2d}(H) &= \sigma_{2d}(\phi(H)) = 2\sqrt{-1}\left(-x_3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3}\right),\\
\pi_{2d}(X) &= \sigma_{2d}(\phi(X)) = -(\sqrt{-1}\,x_2 + x_3)\frac{\partial}{\partial x_1} + \sqrt{-1}\,x_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3},\\
\pi_{2d}(Y) &= \sigma_{2d}(\phi(Y)) = (\sqrt{-1}\,x_2 - x_3)\frac{\partial}{\partial x_1} - \sqrt{-1}\,x_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}.
\end{aligned}$$

Therefore, we can construct a basis of eigenvectors for $H^d\mathbb{C}^3$ as for $S^{2d}\mathbb{C}^2$, by finding an eigenvector of $\pi_{2d}(H)$ and then by lowering or rising it using the operators $\pi_{2d}(Y)$ and $\pi_{2d}(X)$. In particular, we observe that applying the operator $\pi_{2d}(H)$ to the form $(x_2 - \sqrt{-1}\,x_3)^d \in H^d\mathbb{C}^3$ we get

$$\pi_{2d}(H)[(x_2 - \sqrt{-1}\,x_3)^d] = 2d(x_2 - \sqrt{-1}\,x_3)^d,$$

that is, $(x_2 - \sqrt{-1}\,x_3)^d$ is an eigenvector of $\pi_{2d}(H)$ with eigenvalue $2d$. Moreover,

$$\pi_{2d}(X)(x_2 - \sqrt{-1}\,x_3)^d = 0.$$

This means that the other eigenvectors of $\pi_{2d}(H)$ are obtained applying repeatedly to $(x_2 - \sqrt{-1}\,x_3)^d$ the lowering operator $\pi_{2d}(Y)$ until we obtain, at the $(2d+1)$-th iteration, the zero vector:

$$\begin{aligned}
\pi_{2d}(Y)(x_2 - \sqrt{-1}\,x_3)^d &= -2\sqrt{-1}\,dx_1(x_2 - \sqrt{-1}\,x_3)^{d-1}\\
\pi_{2d}(Y)^2(x_2 - \sqrt{-1}\,x_3)^d &= 2d(x_2 + \sqrt{-1}\,x_3)(x_2 - \sqrt{-1}\,x_3)^{d-1}\\
&\quad - 4d(d-1)x_1^2(x_2 - \sqrt{-1}\,x_3)^{d-2}\\
\pi_{2d}(Y)^3(x_2 - \sqrt{-1}\,x_3)^d &= \cdots
\end{aligned} \tag{3.4.1}$$

In this way, we obtained a concrete "dictionary" between the spaces $H^d\mathbb{C}^3$ and $S^{2d}\mathbb{C}^2$. The following proposition gives a concrete expression for the eigenvectors of $\pi_{2d}(H)$.

Proposition 3.4.1. *Define $v_j := \pi_{2d}(Y)^j(x_2 - \sqrt{-1}\,x_3)^d$ for all $0 \le j \le 2d$. Then*

$$v_{2k} = \sum_{s=0}^{k} \alpha_{k,s}\mathcal{D}_{d,k+s}A^{d-k-s}B^{k-s}C^{2s} \tag{3.4.2}$$

for all $1 \le k \le d$, whereas

$$v_{2k+1} = \sum_{s=0}^{k-1} \left[\alpha_{k,s} + 2(s+1)\alpha_{k,s+1}\right]\mathcal{D}_{d,k+1+s}A^{d-k-1-s}B^{k-s}C^{2s+1} \tag{3.4.3}$$

for all $1 \le k \le d-1$, where

$$\begin{aligned}
&A = (x_2 - \sqrt{-1}\,x_3), \quad B = 2(x_2 + \sqrt{-1}\,x_3), \quad C = -2\sqrt{-1}\,x_1,\\
&\mathcal{D}_{m,n} = \begin{cases} \frac{m!}{(m-n)!} & n \le m \\ 0 & n > m \end{cases}
\end{aligned} \tag{3.4.4}$$

and the coefficients $\alpha_{k,s}$ have the following recursive definition:

$$\alpha_{k,s} = \begin{cases} 0 & \textit{for } k < 0 \textit{ and } s > k \\ 1 & \textit{for } k = s = 0 \\ \alpha_{k-1,s-1} + (4s+1)\alpha_{k-1,s} + 2(s+1)(2s+1)\alpha_{k-1,s+1} & \textit{otherwise.} \end{cases}$$

Proof. The proof is by induction on k. Consider the three polynomials A, B and C defined in (3.4.4). From the definition of $\pi_{2d}(Y)$ in (3.4.1) we get that

$$v_0 = A^d, \quad v_1 = \pi_{2d}(Y)(A^d) = dA^{d-1}C,$$

hence the formulas (3.4.2) and (3.4.3) are true for $k = 0$.

Now consider the polynomial $A^pB^qC^r$ for $p, r \geq 1$ and $q \geq 0$ and apply to this polynomial the operator $\pi_{2d}(Y)$:

$$(x_2 + \sqrt{-1}\,x_3)\frac{\partial}{\partial x_1}(A^pB^qC^r) = -\sqrt{-1}\,rA^pB^{q+1}C^{r-1}$$
$$-x_1\left(\frac{\partial}{\partial x_2} + \sqrt{-1}\,\frac{\partial}{\partial x_3}\right)(A^pB^qC^r) = -\sqrt{-1}\,pA^{p-1}B^qC^{r+1}.$$

Hence

$$\pi_{2d}(Y)(A^pB^qC^r) = rA^pB^{q+1}C^{r-1} + pA^{p-1}B^qC^{r+1} \quad \forall p, r \geq 1,\ q \geq 0. \quad (3.4.5)$$

Assume that the relation (3.4.2) for v_{2k} is true. From relation (3.4.5) we observe that the summand of index s in the expression for v_{2k+1} depends on the summands indexed by s and $s+1$ in the expression for v_{2k}. So the idea is to apply the operator $\pi_{2d}(Y)$ to these two summands, using relation (3.4.5). We get that

$$\begin{aligned}
\pi_{2d}(Y)(&\alpha_{k,s}\mathcal{D}_{d,k+s}A^{d-k-s}B^{k-s}C^{2s}\\
&+ \alpha_{k,s+1}\mathcal{D}_{d,k+s+1}A^{d-k-(s+1)}B^{k-(s+1)}C^{2(s+1)}) =\\
&= 2s\alpha_{k,s}\mathcal{D}_{d,k+s}A^{d-k-s}B^{k-s+1}C^{2s-1}\\
&+ (d-k-s)\alpha_{k,s}\mathcal{D}_{d,k+s}A^{d-k-1-s}B^{k-s}C^{2s+1}\\
&+ 2(s+1)\alpha_{k,s+1}\mathcal{D}_{d,k+1+s}A^{d-k-1-s}B^{k-s}C^{2s+1}\\
&+ (d-k-s-1)\alpha_{k,s+1}\mathcal{D}_{d,k+1+s}A^{d-k-s-2}B^{k-s-1}C^{2s+3}\\
&= \alpha_{k,s}\mathcal{D}_{d,k+1+(s-1)}2[(s-1)+1]A^{d-k-1-(s-1)}B^{k-(s-1)}C^{2(s-1)+1}\\
&+ [\alpha_{k,s} + 2(s+1)\alpha_{k,s+1}]\mathcal{D}_{d,k+s+1}A^{d-k-1-s}B^{k-s}C^{2s+1}\\
&+ \alpha_{k,s+1}\mathcal{D}_{d,k+1+(s+1)}A^{d-k-1-(s+1)}B^{k-(s+1)}C^{2(s+1)+1}.
\end{aligned}$$

In particular the second summand of the last polynomial is precisely the summand of index s in the expression for v_{2k+1}. This argument can be repeated for every summand of v_{2k+1}, proving that formula (3.4.3) is true.

Now we prove that the formula (3.4.2) is true as well. Assume again that (3.4.2) is true at the step k. To compute the polynomial $v_{2(k+1)}$, observe that the summand of index s in the expression for $v_{2(k+1)}$ depends on the summands indexed by $s-1$, s and $s+1$ in the expression for v_{2k}. Hence we apply the operator $\pi_{2d}(Y)^2$ to these three summands, using relation (3.4.5). We get that for $s-1$:

$$\begin{aligned}
\pi_{2d}(Y)^2(\alpha_{k,s-1}\mathcal{D}_{d,k+s-1}A^{d-k-(s-1)}B^{k-(s-1)}C^{2(s-1)}) = \\
&= \pi_{2d}(Y)\left[2(s-1)\alpha_{k,s-1}\mathcal{D}_{d,k+s-1}A^{d-k-(s-1)}B^{k-s+2}C^{2s-3}\right] \\
&\quad + \pi_{2d}(Y)\left[\alpha_{k,s-1}\mathcal{D}_{d,k+s}A^{d-k-s}B^{k-(s-1)}C^{2s-1}\right] \\
&= 2(s-1)(2s-3)\alpha_{k,s-1}\mathcal{D}_{d,k+(s-1)}A^{d-k-(s-1)}B^{k-s+3}C^{2s-4} \\
&\quad + 2(s-1)(d-k-s+1)\alpha_{k,s-1}\mathcal{D}_{d,k+(s-1)}A^{d-k-s}B^{k-s+2}C^{2(s-1)} \\
&\quad + (2s-1)\alpha_{k,s-1}\mathcal{D}_{d,k+s}A^{d-k-s}B^{k-s+2}C^{2(s-1)} \\
&\quad + (d-k-s)\alpha_{k,s-1}\mathcal{D}_{d,k+s}A^{d-k-1-s}B^{k-(s-1)}C^{2s} \\
&= C_{s-1} + \text{ other terms}
\end{aligned}$$

for s:

$$\begin{aligned}
\pi_{2d}(Y)^2(\alpha_{k,s}\mathcal{D}_{d,k+s}A^{d-k-s}B^{k-s}C^{2s}) = \\
&= \pi_{2d}(Y)(2s\alpha_{k,s}\mathcal{D}_{d,k+s}A^{d-k-s}B^{k-s+1}C^{2s-1}) \\
&\quad + \pi_{2d}(Y)(\alpha_{k,s}\mathcal{D}_{d,k+s+1}A^{d-k-1-s}B^{k-s}C^{2s+1}) \\
&= 2s(2s-1)\alpha_{k,s}\mathcal{D}_{d,k+s}A^{d-k-s}B^{k-s+2}C^{2s-2} \\
&\quad + 2s(d-k-s)\alpha_{k,s}\mathcal{D}_{d,k+s}A^{d-k-s-1}B^{k-s+1}C^{2s} \\
&\quad + (2s+1)\alpha_{k,s}\mathcal{D}_{d,k+s+1}A^{d-k-1-s}B^{k-s+1}C^{2s} \\
&\quad + (d-k-1-s)\alpha_{k,s}\mathcal{D}_{d,k+s+1}A^{d-k-2-s}B^{k-s}C^{2s+2} \\
&= C_s + \text{ other terms}
\end{aligned}$$

for $s+1$:

$$\begin{aligned}
\pi_{2d}(Y)^2(\alpha_{k,s+1}\mathcal{D}_{d,k+s+1}A^{d-k-(s+1)}B^{k-(s+1)}C^{2(s+1)}) = \\
&= \pi_{2d}(Y)\left[2(s+1)\alpha_{k,s+1}\mathcal{D}_{d,k+s+1}A^{d-k-(s+1)}B^{k-s}C^{2s+1}\right] \\
&\quad + \pi_{2d}(Y)\left[\alpha_{k,s+1}\mathcal{D}_{d,k+s+2}A^{d-k-s-2}B^{k-(s+1)}C^{2s+3}\right] \\
&= 2(s+1)(2s+1)\alpha_{k,s+1}\mathcal{D}_{d,k+s+1}A^{d-k-(s+1)}B^{k-s+1}C^{2s} \\
&\quad + 2(s+1)(d-k-s-1)\alpha_{k,s+1}\mathcal{D}_{d,k+s+1}A^{d-k-s-2}B^{k-s}C^{2s+2} \\
&\quad + (2s+3)\alpha_{k,s+1}\mathcal{D}_{d,k+s+2}A^{d-k-s-2}B^{k-s}C^{2s+2} \\
&\quad + (d-k-s-2)\alpha_{k,s+1}\mathcal{D}_{d,k+s+2}A^{d-k-s-3}B^{k-(s+1)}C^{2s+4} \\
&= C_{s+1} + \text{ other terms,}
\end{aligned}$$

where

$$\begin{aligned} C_{s-1} &= \alpha_{k,s-1}\mathcal{D}_{d,k+1+s}A^{d-(k+1)-s}B^{k+1-s}C^{2s} \\ C_s &= (4s+1)\alpha_{k,s}\mathcal{D}_{d,k+1+s}A^{d-(k+1)-s}B^{k+1-s}C^{2s} \\ C_{s+1} &= 2(s+1)(2s+1)\alpha_{k,s+1}\mathcal{D}_{d,k+1+s}A^{d-(k+1)-s}B^{k+1-s}C^{2s}. \end{aligned}$$

Finally we obtain that

$$C_{s-1}+C_s+C_{s+1} = \alpha_{k+1,s}\mathcal{D}_{d,k+1+s}A^{d-(k+1)-s}B^{k+1-s}C^{2s},$$

where

$$\alpha_{k+1,s} = \alpha_{k,s-1}+(4s+1)\alpha_{k,s}+2(s+1)(2s+1)\alpha_{k,s+1}.$$

This completes the proof. □

Remark 3.4.2. We write now more explicitly which are the middle and the two last eigenvectors:

$$v_d = \begin{cases} \sum_{s=0}^{\frac{d}{2}} \alpha_{\frac{d}{2},s}\mathcal{D}_{d,\frac{d}{2}+s}(AB)^{\frac{d}{2}-s}C^{2s} & \text{for even } d \\ \sum_{s=0}^{\frac{d-3}{2}} \left[\alpha_{\frac{d-1}{2},s}+2(s+1)\alpha_{\frac{d-1}{2},s+1}\right]\mathcal{D}_{d,\frac{d+1}{2}+s}(AB)^{\frac{d-1}{2}-s}C^{2s+1} & \text{for odd } d \end{cases}$$

$$\begin{aligned} v_{2d-1} &= (\alpha_{d-1,0}+2\alpha_{d-1,1})d!B^{d-1}C \\ v_{2d} &= \alpha_{d,0}d!B^d \end{aligned}$$

In the following, we consider the bases

$$\{v_j \mid 0\le j\le 2d\}, \quad \left\{\binom{2d}{j}z_1^{2d-j}z_2^j \;\middle|\; j=0,\dots,2d\right\}$$

for $H^d\mathbb{C}^3$ and $S^{2d}\mathbb{C}^2$, respectively. With these choices of bases, the isomorphism $\varphi\colon S^{2d}\mathbb{C}^2\to H^d\mathbb{C}^3$ of Theorem 3.3.5 is represented by the diagonal matrix

$$\operatorname{diag}\left(\frac{(-1)^j}{j!} \;\middle|\; 0\le j\le 2d\right). \tag{3.4.6}$$

3.5 Geometrical description of the stability for harmonic ternary forms

In this section, we implement the machinery created in the previous sections. In particular, in Proposition 3.4.1 we derived explicitly a good basis $\{v_j\}_j$ of eigenvectors of the irreducible $\mathrm{SO}(3,\mathbb{C})$-module $H^d\mathbb{C}^3$. Using the equivalence with the irreducible $\mathrm{SL}(2,\mathbb{C})$-module $S^{2d}\mathbb{C}^2$ and Hilbert's Theorem 3.2.1, we are ready to describe geometrically all the stable and semistable harmonic plane cubics under the action of $\mathrm{SO}(3,\mathbb{C})$, thus proving Theorem 3.0.4.

Let $h \in H^d\mathbb{C}^3$ and consider its associated curve X_h in $\mathbb{P}^2_{\mathbb{C}}$. As anticipated by Proposition 3.0.3, the crucial part of the proof of Theorem 3.0.4 deals with the study of the intersection between X_h and the isotropic quadric $Q\colon x_1^2+x_2^2+x_3^2 = 0$ in $\mathbb{P}^2_{\mathbb{C}}$. Below we recall some basic facts about the multiplicity of a point on a hypersurface in $\mathbb{P}^n_{\mathbb{C}}$. A detailed treatise on this subject may be found for example in [Mum, Chapter 5].

For every form $h \in S^d\mathbb{C}^n \cong \mathbb{C}[x_1,\dots,x_n]_d$ and every multi-index $\beta = (\beta_1,\dots,\beta_n)$ with $\beta_j \geq 0$ for all $1 \leq jn$, we adopt the notations $|\beta| := \beta_1+\dots+\beta_n$, $\beta! := \beta_1!\cdots\beta_n!$, $x^\beta := x_1^{\beta_1}\cdots x_n^{\beta_n}$ and

$$\frac{\partial^{|\beta|}h}{\partial x^\beta}(P) := \frac{\partial^{|\beta|}h}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}}(P) \quad \forall P \in \mathbb{P}^n_{\mathbb{C}}.$$

Fix a point $P \in X_h$, say $P = [y_1,\dots,y_n]$. Any line passing through P is of the form $L = PR$ for some point $R = [z_1,\dots,z_n] \in \mathbb{P}^n_{\mathbb{C}}$. In particular

$$L = \left\{[x] \in \mathbb{P}^n_{\mathbb{C}} \mid x_i = \lambda y_i + \mu z_i \ \forall i \in \{1,\dots,n\} \text{ for some } [\lambda,\mu] \in \mathbb{P}^1_{\mathbb{C}}\right\}.$$

Consider the Taylor expansion of $\widetilde{h}(\lambda,\mu) := h(\lambda y_1 + \mu z_1,\dots,\lambda y_n + \mu z_n)$ in a neighbourhood of P, that is at $\mu = 0$ (the notation used below takes into account Schwarz' Theorem):

$$\begin{aligned}\widetilde{h}(\lambda,\mu) &= \sum_{m=0}^{d} \frac{1}{m!}\frac{\partial^m \widetilde{h}}{\partial \mu^m}(\lambda,0)\mu^m \\ &= \sum_{m=0}^{d} \frac{1}{m!}\left[\sum_{|\beta|=0}^{m} \frac{m!}{\beta!}\frac{\partial^m h}{\partial x^\beta}(\lambda P)z^\beta\right]\mu^m \\ &= \sum_{m=0}^{d} \frac{1}{m!}\left[\sum_{|\beta|=0}^{m} \frac{m!}{\beta!}\frac{\partial^m h}{\partial x^\beta}(P)z^\beta\right]\lambda^{d-m}\mu^m.\end{aligned} \tag{3.5.1}$$

From this Taylor expansion we obtain that

$$I_P(X_h,L) = m_* \iff \mu^{m_*} \mid_{\widetilde{h}} \quad \text{and} \quad \mu^{m_*+1} \nmid_{\widetilde{h}},$$

where $I_P(X_h,L)$ is the *intersection multiplicity* between X_h and L at P. Now suppose that the coordinates z_i of R are general and consider the polynomial

$$h_{m,P} = \frac{1}{m!}\sum_{|\beta|=0}^{m} \frac{m!}{\beta!}\frac{\partial^m h}{\partial x^\beta}(P)z^\beta, \tag{3.5.2}$$

namely the coefficient of $\lambda^{d-m}\mu^m$ in (3.5.1).

Definition 3.5.1. The *multiplicity* of P for X_h is

$$m_P(X_h) = \min\{I_P(X_h, L) \mid L \text{ is a line passing through } P\}.$$

In other words, the point P has *multiplicity* m for X_h (or is an *m-ple point* of X_h) if $h_{m,P}$ is not identically zero and every partial derivative of h of order smaller than m is zero. Moreover, the hypersurface defined by the equation $h_{m,P} = 0$ is called the *tangent cone* of X_h at P and is denoted by $CT_P X_h$. The tangent cone of X_h at P is the union of the lines tangent to X_h at P. If $m_P(X_h) = 1$, then X_h is smooth at P and $CT_P X_h$ coincides with the tangent space $T_P X_h$.

As pointed out in the end of Section 3.4, any harmonic form $h \in H^d\mathbb{C}^3$ may be written as

$$h = \sum_{j=0}^{2d} \frac{(-1)^j}{j!} a_j v_j \tag{3.5.3}$$

for some complex coefficients $a_0, \ldots, a_{2d}$. Fix the isotropic point $P = [0, 1, -\sqrt{-1}]$. From the expressions of the polynomials v_j in (3.4.2) and in (3.4.3), we obtain immediately that $v_j(P) = 0$ for $j \neq 2d$ and $v_{2d}(P) = 2^{2d}\alpha_{d,0}d!$. Hence, using (3.5.2) we get

$$h_{0,P} = h(P) = \frac{2^{2d}\alpha_{d,0}d!}{(2d)!} a_{2d}.$$

In particular $P \in X_h$ if and only if $a_{2d} = 0$.

Our goal is to obtain a closed expression of the polynomials $h_{m,P}$: as we show in a while, they are fundamental to study the stability and semistability of harmonic ternary forms. In the next technical lemma, we start by computing all the partial derivatives of h of order m, using the formulas obtained in Proposition 3.4.1.

Lemma 3.5.2. *Let h as in (3.5.3) and $P = [0, 1, -\sqrt{-1}]$. Then*

$$\frac{\partial^m h}{\partial x^\beta}(P) = (-1)^{\beta_3} \frac{d!\beta_1! 2^{\bar\gamma} \sqrt{-1}^{\,\beta_1+\beta_3}}{\bar\gamma!} \mathcal{A}_{m,\beta_1} a_{\bar\gamma}$$

for every multi-index $\beta = (\beta_1, \beta_2, \beta_3)$ with $|\beta| = m$, where $\bar\gamma = 2(d-m) + \beta_1$ and

$$\mathcal{A}_{m,\beta_1} = \begin{cases} \alpha_{\frac{\bar\gamma}{2}, \frac{\beta_1}{2}} & \text{if } \beta_1 \text{ is even} \\ \alpha_{\frac{\bar\gamma-1}{2}, \frac{\beta_1-1}{2}} + (\beta_1+1)\alpha_{\frac{\bar\gamma-1}{2}, \frac{\beta_1+1}{2}} & \text{if } \beta_1 \text{ is odd.} \end{cases}$$

Proof. From the definition of h we have that

$$\frac{\partial^m h}{\partial x^\beta}(P) = \sum_{j=0}^{2d} \frac{(-1)^j}{j!} a_j \frac{\partial^m v_j}{\partial x^\beta}(P).$$

Suppose that β_1 is even. Then $\frac{\partial^m v_j}{\partial x^\beta}(P) = 0$ for every odd index j, by the formula for v_j in (3.4.3). Moreover, looking at the formulas in (3.4.2), we see that the only possible nonzero value of $\frac{\partial^m v_j}{\partial x^\beta}(P)$, with even j, comes from the summand with $j = \frac{\beta_1}{2}$. Hence

$$\begin{aligned}\frac{\partial^m v_j}{\partial x^\beta}(P) &= \frac{\partial^m}{\partial x^\beta}\left(\alpha_{\frac{j}{2},\frac{\beta_1}{2}} \mathcal{D}_{d,\frac{\beta_1+j}{2}} A^{d-\frac{\beta_1+j}{2}} B^{\frac{j-\beta_1}{2}} C^{\beta_1}\right)\Big|_P \\ &= \beta_1!(-2\sqrt{-1}\,)^{\beta_1} \alpha_{\frac{j}{2},\frac{\beta_1}{2}} \mathcal{D}_{d,\frac{\beta_1+j}{2}} \frac{\partial^{m-\beta_1}}{\partial x_2^{m-\beta_1-\beta_3} \partial x_3^{\beta_3}} \left(A^{d-\frac{\beta_1+j}{2}} B^{\frac{j-\beta_1}{2}}\right)\Big|_P .\end{aligned}$$

From last equality and the fact that $A(P) = 0$ we see that

$$\frac{\partial^{m-\beta_1}}{\partial x_2^{m-\beta_1-\beta_3} \partial x_3^{\beta_3}} \left(A^{d-\frac{\beta_1+j}{2}} B^{\frac{j-\beta_1}{2}}\right)\Big|_P = 0$$

when the exponent of A, namely $d - \frac{\beta_1+j}{2}$, is not equal to $m - \beta_1$. In turn, this happens when $j \neq 2(d-m) + \beta_1 = \bar{\gamma}$. Otherwise if $j = \bar{\gamma}$, then

$$\begin{aligned}\frac{\partial^{m-\beta_1}}{\partial x_2^{m-\beta_1-\beta_3} \partial x_3^{\beta_3}} \left(A^{m-\beta_1} B^{d-m}\right)\big|_P &= 2^{2(d-m)} \frac{\partial^m}{\partial x_2^{m-\beta_1-\beta_3} \partial x_3^{\beta_3}} \left(A^{m-\beta_1}\right)\big|_P \\ &= (m-\beta_1)!(-\sqrt{-1}\,)^{\beta_3} 2^{2(d-m)}.\end{aligned}$$

Summing up, we obtain that

$$\begin{aligned}\frac{\partial^m h}{\partial x^\beta}(P) &= \frac{(-1)^{\bar{\gamma}}}{\bar{\gamma}!} a_{\bar{\gamma}} \frac{\partial^m v_{\bar{\gamma}}}{\partial x^\beta}(P) \\ &= \frac{(-1)^{\bar{\gamma}}}{\bar{\gamma}!} a_{\bar{\gamma}} \beta_1!(-2\sqrt{-1}\,)^{\beta_1} \alpha_{\frac{\bar{\gamma}}{2},\frac{\beta_1}{2}} \mathcal{D}_{d,d-m+\beta_1} (m-\beta_1)!(-\sqrt{-1}\,)^{\beta_3} 2^{2(d-m)} \\ &= (-1)^{\beta_3} \frac{d!\beta_1! 2^{\bar{\gamma}} \sqrt{-1}^{\,\beta_1+\beta_3}}{\bar{\gamma}!} \alpha_{\frac{\bar{\gamma}}{2},\frac{\beta_1}{2}} a_{\bar{\gamma}}.\end{aligned}$$

If β_1 is odd, a similar computation produces the following identity:

$$\frac{\partial^m h}{\partial x^\beta}(P) = (-1)^{\beta_3} \frac{d!\beta_1! 2^{\bar{\gamma}} \sqrt{-1}^{\,\beta_1+\beta_3}}{\bar{\gamma}!} \left[\alpha_{\frac{\bar{\gamma}-1}{2},\frac{\beta_1-1}{2}} + (\beta_1+1)\alpha_{\frac{\bar{\gamma}-1}{2},\frac{\beta_1+1}{2}}\right] a_{\bar{\gamma}}. \quad \square$$

Lemma 3.5.3. *Let h and P as before. Then for every $0 \leq m \leq d$,*

$$h_{m,P} = \frac{d!}{(2d-m)!} \sum_{\beta_1=0}^{m} \binom{2d-m}{\bar{\gamma}} 2^{\bar{\gamma}} \mathcal{A}_{m,\beta_1} a_{\bar{\gamma}} (\sqrt{-1}x_1)^{\beta_1} (x_2 - \sqrt{-1}\, x_3)^{m-\beta_1}.$$

Proof. Applying the definition of $h_{m,P}$ and using the relation $\frac{m!}{\beta!} = \binom{m}{\beta_1}\binom{m-\beta_1}{\beta_3}$, we have

$$\begin{aligned} h_{m,P} &= \frac{1}{m!} \sum_{|\beta|=m} \frac{m!}{\beta!} \frac{\partial^m h}{\partial x^\beta}(P) x^\beta \\ &= \frac{1}{m!} \sum_{\beta_1=0}^{m} \binom{m}{\beta_1} x_1^{\beta_1} \sum_{\beta_3=0}^{m-\beta_1} \binom{m-\beta_1}{\beta_3} \frac{\partial^m h}{\partial x^\beta}(P) x_2^{\beta_2} x_3^{\beta_3}. \end{aligned}$$

By Lemma 3.5.2, the last polynomial is equal to

$$\begin{aligned} &= \frac{1}{m!} \sum_{\beta_1=0}^{m} \binom{m}{\beta_1} x_1^{\beta_1} \sum_{\beta_3=0}^{m-\beta_1} \binom{m-\beta_1}{\beta_3} (-1)^{\beta_3} \frac{d!\beta_1! 2^{\bar{\gamma}} \sqrt{-1}^{\,\beta_1+\beta_3}}{\bar{\gamma}!} \mathcal{A}_{m,\beta_1} a_{\bar{\gamma}} x_2^{\beta_2} x_3^{\beta_3} \\ &= \frac{d!}{m!} \sum_{\beta_1=0}^{m} \binom{m}{\beta_1} \frac{\beta_1! 2^{\bar{\gamma}}}{\bar{\gamma}!} \mathcal{A}_{m,\beta_1} a_{\bar{\gamma}} (\sqrt{-1}\, x_1)^{\beta_1} \sum_{\beta_3=0}^{m-\beta_1} \binom{m-\beta_1}{\beta_3} x_2^{\beta_2} (-\sqrt{-1}\, x_3)^{\beta_3} \\ &= \frac{d!}{m!} \sum_{\beta_1=0}^{m} \binom{m}{\beta_1} \frac{\beta_1! 2^{\bar{\gamma}}}{\bar{\gamma}!} \mathcal{A}_{m,\beta_1} a_{\bar{\gamma}} (\sqrt{-1}\, x_1)^{\beta_1} (x_2 - \sqrt{-1} x_3)^{m-\beta_1} \\ &= d! \sum_{\beta_1=0}^{m} \frac{2^{\bar{\gamma}}}{(m-\beta_1)!\bar{\gamma}!} \mathcal{A}_{m,\beta_1} a_{\bar{\gamma}} (\sqrt{-1}\, x_1)^{\beta_1} (x_2 - \sqrt{-1}\, x_3)^{m-\beta_1} \\ &= \frac{d!}{(2d-m)!} \sum_{\beta_1=0}^{m} \binom{2d-m}{\bar{\gamma}} 2^{\bar{\gamma}} \mathcal{A}_{m,\beta_1} a_{\bar{\gamma}} (\sqrt{-1}\, x_1)^{\beta_1} (x_2 - \sqrt{-1}\, x_3)^{m-\beta_1}. \qquad \square \end{aligned}$$

The relevant fact from Lemma 3.5.3 is that the expression of $h_{m,P}$ involves the coefficients $a_{\bar{\gamma}} = a_{2(d-m)+\beta_1}$ for $\beta_1 = 0, \dots, m$, namely the coefficients in $\{a_{2d-2m}, \dots, a_{2d-m}\}$. In particular, for $m = 0$ the polynomial $h_{0,P} = H(P)$ involves only the coefficient m_{2d}. For $m = 1$, $h_{1,P}$ involves the coefficients m_{2d-2} and m_{2d-1}. For $m = 2$, $h_{2,P}$ involves the coefficients m_{2d-4}, m_{2d-3} and m_{2d-2}, and so on.

Remark 3.5.4. Imposing the conditions $a_{2(d-m+1)+\beta_1} = 0$ for every $0 \leq \beta_1 \leq m-1$, that is, imposing $h_{m-1,P}$ to be the zero polynomial, simplifies a lot the expression for $h_{m,P}$. Indeed, the only nonzero coefficients remaining in $h_{m,P}$ are

a_{2d-2m} and $a_{2d-2m+1}$. More explicitly, the expression for $h_{m,P}$ in Lemma 3.5.3 simplifies as

$$\begin{aligned} h_{m,P} = \frac{d!2^{2(d-m)}(x_2 - \sqrt{-1}\,x_3)^{m-1}}{(2(d-m))!(m-1)!} & \left[\frac{\mathcal{A}_{m,0}}{m} a_{2(d-m)}(x_2 - \sqrt{-1}\,x_3) \right. \\ & \left. + \frac{2\mathcal{A}_{m,1}}{2(d-m)+1} a_{2(d-m)+1}\sqrt{-1}\,x_1 \right]. \end{aligned} \tag{3.5.4}$$

In other words, if we have already supposed that $h_{m',P} = 0$ for every $m' < m$ (namely that P is a m-ple point of X_h), then the condition $h_{m,P} = 0$ is obtained by requiring only two more vanishing coefficients.

Carrying on this "degree by degree" computation of the Taylor expansion of h at P, we get the following corollary.

Corollary 3.5.5. *Let h be as before and $P = [0, 1, -\sqrt{-1}]$. Then*

1. *if d is even and P is a $\frac{d}{2}$-ple point of X_h, then*

$$h_{\frac{d}{2},P} = \frac{2^{d+1}}{\left(\frac{d-2}{2}\right)!}(x_2 - \sqrt{-1}\,x_3)^{\frac{d-2}{2}} \left[\frac{\mathcal{A}_{\frac{d}{2},0}}{d} a_d(x_2 - \sqrt{-1}\,x_3) + \frac{\mathcal{A}_{\frac{d}{2},1}}{d+1} a_{d+1}\sqrt{-1}\,x_1 \right].$$

2. *if d is odd and P is a $\frac{d+1}{2}$-ple point of X_h, then*

$$h_{\frac{d+1}{2},P} = \frac{2^d(x_2 - \sqrt{-1}\,x_3)^{\frac{d-1}{2}}}{(d-1)\left(\frac{d-1}{2}\right)!} \left[\frac{\mathcal{A}_{\frac{d+1}{2},0}}{d+1} a_{d-1}(x_2 - \sqrt{-1}\,x_3) + \frac{\mathcal{A}_{\frac{d+1}{2},1}}{d} a_d\sqrt{-1}\,x_1 \right].$$

The previous corollary allows us to proceed in the proof of Theorem 3.0.4.

Proof of Theorem 3.0.4. Consider a harmonic ternary form $h \in H^d\mathbb{C}^3$, written as in (3.5.3). In particular, the binary form

$$f = \sum_{j=0}^{2d} \binom{2d}{j} a_j z_1^{2d-j} z_2^j$$

is such that $\varphi(f) = h$, where φ is the equivariant isomorphism of Theorem 3.3.5 defined explicitly in (3.4.6).

Suppose that h is non-stable. This means, in turn, that f is non-stable, and by Theorem 3.2.1 a general non-stable binary form of degree $2d$ has a root of multiplicity $m \geq d$. We can assume that $z_1^d|_f$ or, equivalently, that

$$a_{d+1} = \cdots = a_{2d} = 0. \tag{3.5.5}$$

Consider the point $P = [0, 1, -\sqrt{-1}] \in \mathbb{P}^2_{\mathbb{C}}$. By Remark 3.5.4 and Corollary 3.5.5, we have that

1. if d is even, the conditions $a_{d+2} = \cdots = a_{2d} = 0$ in (3.5.5) imply that P is a $(d/2)$-ple point of X_h, while the condition $a_{d+1} = 0$ implies that $\frac{d}{2} T_P Q \subset CT_P X_h$.
2. if d is odd, the conditions (3.5.5) imply that P is a $[(d+1)/2]$-ple point of X_h, and in particular $\frac{d-1}{2} T_P Q \subset CT_P X_h$.

Now suppose that h is non-semistable. Then the corresponding binary form f is non-semistable, and again by Theorem 3.2.1 a general non-semistable binary form of degree $2d$ has a root of multiplicity $m \geq d+1$. We can assume that $z_1^{d+1}|_f$ or, equivalently, that

$$a_d = \cdots = a_{2d} = 0. \tag{3.5.6}$$

By Definition 3.1.3, every $\mathrm{SL}(2, \mathbb{C})$-invariant of f of positive degree vanishes. In particular, the quadratic $\mathrm{SL}(2, \mathbb{C})$-invariant

$$(f, f)_{2d} \coloneqq (2d!)^2 \sum_{j=0}^{2d} (-1)^j \binom{2d}{j} a_j a_{2d-j} \tag{3.5.7}$$

vanishes. We denote the corresponding $\mathrm{SO}(3, \mathbb{C})$-invariant by φ_{2d}. Therefore, by the conditions (3.5.6) and (3.5.7), any non-semistable harmonic ternary form h is non-stable and is such that $\varphi_{2d}(h) = 0$.

On the other hand, suppose that d is even and that there exists $P \in H \cap Q$ and an integer $m \geq d/2$ such that P is an m-ple point for X_h and $\frac{d}{2} T_P Q \subset CT_P X_h$. Then we can assume that $P = [0, 1, -\sqrt{-1}]$ and again applying Remark 3.5.4 and Corollary 3.5.5 we obtain the conditions (3.5.5). Using the ismorphism φ in (3.4.6), we see that $z_1^d|_f$, hence f is non-stable. In turn, this implies that h is non-stable. Moreover, looking at (3.5.7), in this case $\varphi_{2d}(h) = 0$ if and only if $a_d = 0$, and this implies that $z_1^{d+1}|_f$, namely the binary form f associated to h is non-semistable, and thus h is non-semistable as well.

Otherwise d is odd. Suppose that there exists $P \in X_h \cap Q$ and an integer $m \geq (d+1)/2$ such that P is an m-ple point for X_h. A similar argument implies that h is non-stable, and the additional condition $\varphi_{2d}(h) = 0$ provides the non-semistability of h.

For the last part of the statement, if we look at the expressions in (3.4.2) and (3.4.3) for the eigenvectors v_j, we get that any non-semistable harmonic ternary form h is reducible: in fact, we have that $a_d = \cdots = a_{2d} = 0$, while the polynomials $v_0, \ldots, v_{d-1}$ have $z_2 - \sqrt{-1}\, z_3$ as a common factor and $T_P Q \colon z_2 - \sqrt{-1}\, z_3 = 0$. In particular, $X_h = T_P Q \cup X_{h'}$, where h' is a harmonic ternary form of degree $d-1$.

Repeating the same argument to h', we can conclude that, for even d, P is a $(d/2)$-ple point for $X_{h'}$, whereas for odd d we have that P is a $[(d-1)/2]$-ple point for $X_{h'}$ and $\frac{d-1}{2}T_PQ \subset CT_PX_h$. Thus $X_{h'}$ is tangent to Q at P and h' is non-stable. $\square$

In the following, we consider two explanatory examples of Theorem 3.0.4 for $d \in \{3,4\}$.

Example 3.5.6 (Harmonic plane cubics, $d=3$)**.** A general harmonic ternary cubic can be written as

$$f(x_1,x_2,x_3) = \sum_{j=0}^{6} a_j v_j, \tag{3.5.8}$$

where the polynomials v_j are the eigenvectors obtained in (3.4.2) and in (3.4.3), with rescaled coefficients for simplicity:

$$\begin{aligned}
v_0 &= (x_2 - \sqrt{-1}\,x_3)^3\\
v_1 &= x_1(x_2 - \sqrt{-1}\,x_3)^2\\
v_2 &= (x_2 + \sqrt{-1}\,x_3)(x_2 - \sqrt{-1}\,x_3)^2 - 4x_1^2(x_2 - \sqrt{-1}\,x_3)\\
v_3 &= 2x_1^3 - 3x_1(x_2^2 + x_3^2)\\
v_4 &= (x_2 - \sqrt{-1}\,x_3)(x_2 + \sqrt{-1}\,x_3)^2 - 4x_1^2(x_2 + \sqrt{-1}\,x_3)\\
v_5 &= x_1(x_2 + \sqrt{-1}\,x_3)^2\\
v_6 &= (x_2 + \sqrt{-1}\,x_3)^3
\end{aligned} \tag{3.5.9}$$

To visualize some examples, we consider the change of coordinates

$$z_1 = -\sqrt{-1}\,x_1, \quad z_2 = x_2 + \sqrt{-1}\,x_3, \quad z_3 = x_2 - \sqrt{-1}\,x_3.$$

In the z_i's the isotropic quadric Q (the red curve in Figure 3.5) has equation $z_1^2 - z_2z_3 = 0$. In Figure 3.5 we considered the coordinates $z_1 = x$, $z_2 = 2$ and $z_3 = y$.

Starting from the picture at the top-left corner of Figure 3.5, we consider five different harmonic plane cubics $C_1, \ldots, C_5$ defined by the polynomials $h_1, \ldots, h_5$, respectively. They are written explicitly in (3.5.10). In particular, h_1 is a general linear combination as in (3.5.8). Then h_2 is obtained by setting $a_6 = 0$, while the other coefficients a_j are randomly chosen. In particular, the curve C_2 passes through the origin $(0,0)$ corresponding to $P = [0,1,-\sqrt{-1}]$. Going further, h_3 is obtained by setting $a_5 = a_6 = 0$, and the other coefficients a_j are randomly chosen. In particular, the curve C_3 is tangent to Q at $(0,0)$. The two last pictures of Figure 3.5 are the most interesting in relation to Theorem 3.0.4. Indeed, in the second to last picture, C_4 is singular at $(0,0)$. In particular, T_PQ

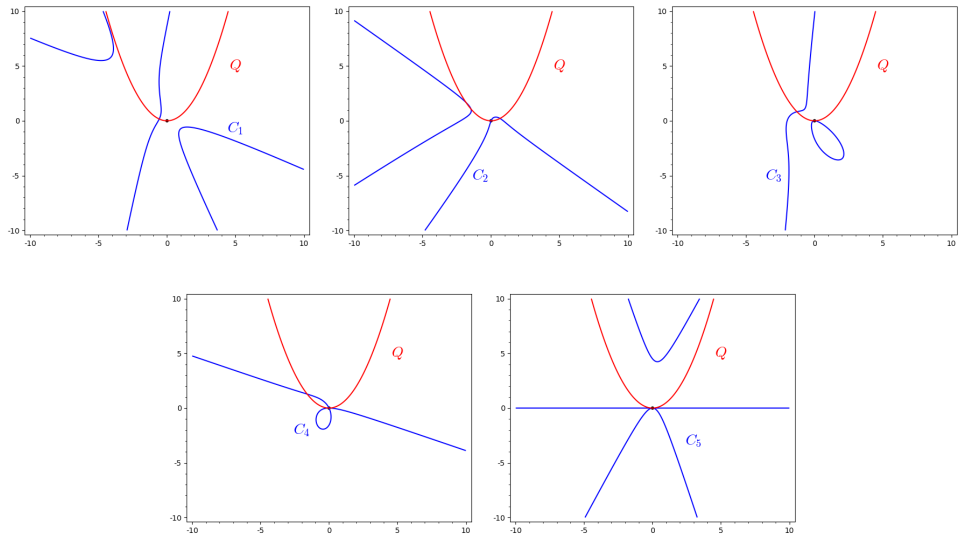

Figure 3.5: Examples of harmonic plane cubics in the affine plane $z_2 = 2$. The last two pictures represent a non-stable and a non-semistable harmonic plane cubic.

is tangent to C_4. By Theorem 3.0.4, C_4 is non-stable. Finally, in the last picture C_5 is reducible in the form $C_5 = LC$, where the line $L\colon y = 0$ and the conic $C\colon 18x^2 - 3xy - 2y^2 + 9y = 0$ are tangent to Q at the origin. By Theorem 3.0.4, this means that C_5 is non-semistable, namely it is non-stable and its norm is zero.

$$
\begin{aligned}
h_1 &= 9x^3 + 18x^2y + 3xy^2 - y^3 - 8x^2 + 27xy + 9y^2 + 10x - 4y + 12\\
h_2 &= 6x^3 - 4x^2y - 8xy^2 + 5y^3 + 8x^2 + 18xy - 2y^2 - 12x + 4y\\
h_3 &= 16x^3 + 16x^2y + 9xy^2 - y^3 + 32x^2 + 48xy + 8y^2 + 16y\\
h_4 &= 8x^3 - 32x^2y + 6xy^2 + y^3 + 24xy - 16y^2\\
h_5 &= y(18x^2 - 3xy - 2y^2 + 9y)
\end{aligned}
\tag{3.5.10}
$$

Example 3.5.7. Let us examine the first nontrivial case in even degree, that is

$d = 4$. The basis vectors v_j, $j = 0, \ldots, 8$ for $H^4\mathbb{C}^3$ are explicitly

$$\begin{aligned}
v_0 &= (x_2 - \sqrt{-1}\,x_3)^4\\
v_1 &= x_1(x_2 - \sqrt{-1}\,x_3)^3\\
v_2 &= (x_2 - \sqrt{-1}\,x_3)^2(6x_1^2 - x_2^2 - x_3^2)\\
v_3 &= x_1(x_2 - \sqrt{-1}\,x_3)(4x_1^2 - 3x_2^2 - 3x_3^2)\\
v_4 &= 8x_1^4 + 3(x_2^2 + x_3^2)(-8x_1^2 + x_2^2 + x_3^2)\\
v_5 &= x_1(x_2 + \sqrt{-1}\,x_3)(4x_1^2 - 3x_2^2 - 3x_3^2)\\
v_6 &= (x_2 + \sqrt{-1}\,x_3)^2(6x_1^2 - x_2^2 - x_3^2)\\
v_7 &= x_1(x_2 + \sqrt{-1}\,x_3)^3\\
v_8 &= (x_2 + \sqrt{-1}\,x_3)^4.
\end{aligned} \tag{3.5.11}$$

Starting from the picture at the top-left corner of Figure 3.6, we consider six different harmonic plane cubics $D_1, \ldots, D_5$ defined by the polynomials $p_1, \ldots, p_6$, respectively. They are written explicitly in (3.5.12). In particular, p_1 is a general linear combination as in (3.5.8). Then p_2 is obtained by setting $a_8 = 0$, while the other coefficients a_j are randomly chosen. In particular, the curve D_2 passes through the origin $(0, 0)$. Going further, p_3 is obtained by setting $a_7 = a_8 = 0$, and the other coefficients a_j are randomly chosen. In particular, the curve D_3 is tangent to Q at $(0, 0)$. The more coefficients a_j we set to zero, the more "singular" the curve D_j becomes at the origin. Eventually, we have the two last pictures of Figure 3.6. In the second to last picture, D_5 is singular at the origin and $2T_PQ \subset CT_PD_5$. By Theorem 3.0.4, D_5 is non-stable. Finally, in the last picture D_6 is reducible in the form $D_6 = LD$, where the line $L\colon y = 0$ and the cubic $D\colon 20x^3 + 24x^2y - 3xy^2 + 2y^3 + 30xy + 8y^2 = 0$ are tangent to Q at the origin, and D is non-stable. By Theorem 3.0.4, this means that D_6 is non-semistable.

$$\begin{aligned}
p_1 &= 8x^4 - 12x^3y - 54x^2y^2 - 2xy^3 + 5y^4 - 32x^3 + 48x^2y - 18xy^2\\
&\quad - 18y^3 + 192x^2 - 48xy + 12y^2 + 40x + 64y + 96\\
p_2 &= 40x^4 - 24x^3y + 5xy^3 + 8y^4 - 64x^3 + 240x^2y\\
&\quad - 36xy^2 + 24x^2 - 96xy + 60y^2 - 80x + 8y\\
p_3 &= 48x^4 + 40x^3y + 36x^2y^2 + 9xy^3 + 8y^4 + 288x^2y\\
&\quad + 60xy^2 + 12y^3 - 216x^2 + 72y^2 - 72y\\
p_4 &= 8x^4 - 6x^3y - 12x^2y^2 + 3xy^3 - 4y^4 - 20x^3\\
&\quad + 48x^2y - 9xy^2 - 4y^3 - 30xy + 12y^2\\
p_5 &= 72x^4 + 48x^2y^2 - 9xy^3 - 6y^4 + 432x^2y + 16y^3 + 108y^2\\
p_6 &= y(20x^3 + 24x^2y - 3xy^2 + 2y^3 + 30xy + 8y^2)
\end{aligned} \tag{3.5.12}$$

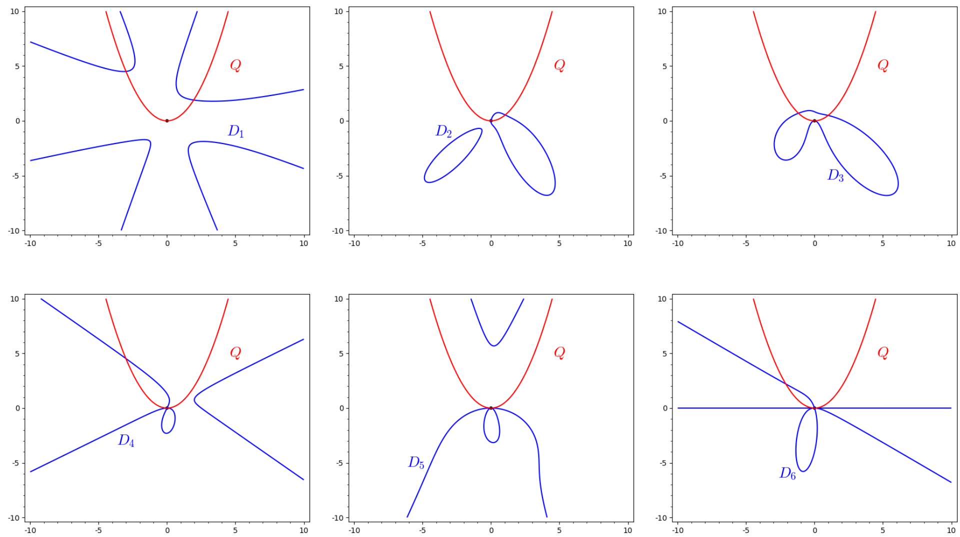

Figure 3.6: Examples of harmonic plane quartics in the affine plane $z_2 = 2$. The last two pictures represent a non-stable and a non-semistable harmonic plane quartic.

An immediate consequence of Theorem 3.0.4 finds an application in the theory of E-characteristic polynomials outlined in Chapter 2.

Corollary 3.5.8. *Let $h \in H^d\mathbb{C}^3$ be a complex harmonic ternary form of degree d written as*

$$h = \sum_{j=0}^{2d} b_j v_j,$$

where the polynomials v_j are written in (3.4.2) and in (3.4.3). Then

1. *if $b_{2d-1} = b_{2d} = 0$, then h admits an isotropic eigenvector, by Remark 2.3.11,*

2. *if additionally $b_{2d-2} = 0$, then h is irregular and therefore its E-characteristic polynomial $\psi_h(\lambda)$ is identically zero, by Proposition 2.2.8.*

Example 3.5.9 (A real harmonic ternary cubic with non-real E-eigenvalues)**.** Consider the space $H^d\mathbb{R}^3$ of real harmonic ternary cubics. In particular, the following linear combination

$$h = b_1(v_0 + v_6) + b_2(v_1 + v_5) + b_3(v_2 + v_4) + b_4 v_3, \quad b_j \in \mathbb{R},$$

where the polynomials v_j are written in (3.5.9), is a polynomial in $H^d\mathbb{R}^3$, since each polynomial v_j is the complex-conjugate of v_{6-j}.

Choosing random values for the coefficients b_j, and computing the E-characteristic polynomial of h, one might quickly check numerically that there exists real harmonic ternary cubics admitting non-real E-eigenvalues. Here is an example with $b_1 = 99$, $b_2 = 44$, $b_3 = 86$ and $b_4 = -30$:

$$h = -60x_1^3 + 344x_1^2x_2 + 134x_1x_2^2 + 13x_2^3 + 46x_1x_3^2 - 383x_2x_3^2$$

The E-characteristic polynomial $\psi_h(\lambda)$ has the expected degree $\deg(\psi_h(\lambda)) = 14$, namely degree seven in λ^2. Its roots are the following:

$$\begin{pmatrix} 2448.93 \\ -2448.93 \\ 270.306 \\ -270.306 \\ 14.0615\sqrt{-1} \\ -14.0615\sqrt{-1} \\ 20.0813\sqrt{-1} \\ -20.0813\sqrt{-1} \\ 102.249\sqrt{-1} \\ -102.249\sqrt{-1} \\ -291.514 + 135.21\sqrt{-1} \\ -291.514 - 135.21\sqrt{-1} \\ 291.514 + 135.21\sqrt{-1} \\ 291.514 - 135.21\sqrt{-1} \end{pmatrix}$$

Chapter 4
The ED polynomial of an algebraic variety

In Chapter 1 we introduced the *ED degree* of an algebraic variety X in a complex vector space $V = V^{\mathbb{R}} \otimes \mathbb{C}$. This invariant has been introduced for answering the following question.

What is the number of critical points on X of the squared distance function $\delta_u \colon X \to \mathbb{C}$ for a general point $u \in V$?

In problems like finding the *best rank-one approximation* of a symmetric tensor mentioned in Chapter 2, it is also important the value of the distance, and if the value $\delta_u(x)$ for a critical point $x \in X$ satisfies some algebraic relation depending on the coordinates of u. How can we relate the results in Chapter 2 to this slightly different approach? A partial answer was given in the Introduction for rectangular matrices as well as higher format tensors, but it may be replayed for symmetric tensors.

For a given real symmetric tensor $f \in S^d V^{\mathbb{R}}$, we assume that (λ, x) is a Z-eigenpair for f. From Chapter 2, we learned that the symmetric tensor λx^d is critical for $\delta_{F,f}$ when restricted to the affine cone $X_{(d)}$ of the image of the d-th Veronese embedding of $\mathbb{P}(V)$. The squared E-eigenvalue λ^2 is a root of the E-characteristic polynomial $\psi_f(\lambda)$, for odd d, or of the product $\psi_f(\lambda)\psi_f(-\lambda)$, for even d. In addition, the value $q_F(f - \lambda x^d)$ is critical for $\delta_{F,f}$ on $X_{(d)}$. An immediate consequence of Theorem 1.3.3 is the identity

$$q_F(f - \lambda x^d) + q_F(\lambda x^d) = q_F(f), \tag{4.0.1}$$

where $q_F(\lambda x^d) = \lambda^2$. Summing up these facts and taking into account that $f - \lambda x^d$ belongs to the dual affine cone $X_{(d)}^{\vee}$ of $X_{(d)}$, the conclusion is that

$\psi_f(\lambda) = 0$ or $\psi_f(\lambda)\psi_f(-\lambda) = 0$ may be interpreted as the equation of the λ-*offset* of $X_{(d)}^\vee$, namely the affine hypersurface of symmetric tensors in S^dV having distance λ form $X_{(d)}^\vee$.

And what about the λ-*offset* of $X_{(d)}$? By simply looking at the identity (4.0.1) and remembering Theorem 1.3.3, the equation of the ε-offset of $X_{(d)}$ is obtained setting to zero the E-characteristic polynomial $\psi_f(\lambda)$ (or the product $\psi_f(\lambda)\psi_f(-\lambda)$) evaluated at $\lambda^2 = q_F(f)-\varepsilon^2$. Actually, this "Pyhtagorean duality" is stated more generally in Theorem 4.2.8.

In this chapter, we deal exactly with these "ε-offset hypersurfaces" of algebraic varieties $X \subset V$. This topic is indeed classical. For example, Salmon studied in [Sal, §373, ex. 3] the offsets of affine conics and called them *parallel curves*. He considered the distance parameter ε as a variable and observed that the parallel curve of a parabola drops degree with respect to a general conic.

Salmon's computation uses invariant theory of pencils of conics. Here we give an analogous example in $\mathbb{R}^3$ to highlight the contents of this chapter. Let $C \subset \mathbb{P}^2_{\mathbb{R}}$ be the projective conic of equation $4x_1^2 + x_2^2 - x_3^2 = 0$, which we regard as an affine cone in $\mathbb{R}^3$. Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ be a data point. Denote by M_C the 4×4 symmetric matrix of C and by M_S the matrix of the sphere

$$S = S(u,\varepsilon)\colon (x_1-u_1)^2 + (x_2-u_2)^2 + (x_3-u_3)^2 - \varepsilon^2 = 0.$$

The point $(x_1, x_2, x_3) \in C$ is critical for δ_u if and only if S is tangent to C. By a well-known result of Cayley, $C \cap S$ is not smooth if and only if the determinant of the matrix $M_C + \mu M_S$ has at least two coincident roots. Therefore, the nontrivial factor of $\Delta_\mu\,[\det(M_C + \mu M_S)]$, where the operator Δ_μ computes the discriminant with respect to the variable μ, provides the equation of the ε-offset of C.

In this thesis, the alredy obtained polynomial is called the *ED polynomial* of C at u. We write this polynomial as $\mathrm{EDpoly}_{C,u}(\varepsilon^2)$, highlighting that the variable ε appears squared. This notation was introduced in the joint work with Ottaviani [OS], which is the core of this chapter. In our example, the expression for $\mathrm{EDpoly}_{C,u}(\varepsilon^2)$ is

$$\begin{aligned}
\mathrm{EDpoly}_{C,u}(\varepsilon^2) &= 900\varepsilon^8 - 60(64u_1^2 + 5u_2^2 + 21u_3^2)\varepsilon^6\\
&+ (6016u_1^4 + 2960u_1^2u_2^2 - 275u_2^4 + 3312u_1^2u_3^2 - 810u_2^2u_3^2 + 621u_3^4)\varepsilon^4\\
&- 2(2048u_1^6 + 2208u_1^4u_2^2 + 540u_1^2u_2^4 - 25u_2^6 + 1184u_1^4u_3^2 - 224u_1^2u_2^2u_3^2\\
&+ 185u_2^4u_3^2 + 324u_1^2u_3^4 - 207u_2^2u_3^4 + 63u_3^6)\varepsilon^2\\
&+ (4u_1^2 + u_2^2 - u_3^2)^2(64u_1^4 + 80u_1^2u_2^2 + 25u_2^4 + 48u_1^2u_3^2 - 30u_2^2u_3^2 + 9u_3^4).
\end{aligned}$$

More in general, the ED polynomial of an algebraic variety X is computed via elimination of variables from the ideal of the so-called *offset correspondence* of X, for this example see also Example 4.1.4. All the details are explained in Section 4.1, while its first properties are listed in Section 4.2. Note that $\mathrm{EDpoly}_{C,u}(\varepsilon^2)$ has degree four in ε^2. This agrees with the fact that C, wiewed in the complex space $\mathbb{C}^3$, is transversal to the isotropic quadric $Q\colon x_1^2 + x_2^2 + x_3^2 = 0$, implying $\mathrm{EDdegree}(C) = 4$ by Proposition 1.8.2. More in general, Horobeţ and Weinstein showed that, if X is a variety in V, then the ε^2-degree of $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ equals $\mathrm{EDdegree}(X)$ (see Theorem 4.2.2). Hence, they linked the subject about offset hypersurfaces to ED degree. The ε-offsets of an algebraic variety have several engineering applications, starting from geometric modeling techniques. They also establish an interesting link between Algebraic Geometry and Architectural Geometry: for instance, rational curves and surfaces with rational offsets possess various applications in Computer-Aided Manufacturing (see [PAHK, Chapter 10]). Farouki and Neff studied algebraic properties of the ε-offset in the setting of plane curves.

In Section 4.3, we analyze the highest coefficient of the ED polynomial of an algebraic variety X. Coming back to the ED polynomial of C, note that its highest coefficient does not depend on the data point u. In particular, the algebraic function $\delta_u\colon C \to \mathbb{C}$ is *integral*, meaning geometrically that no branch goes to infinity. This fact holds whenever the algebraic variety X is transversal to Q, as stated in Proposition 4.3.4. Note that in this chapter the transversality assumptions are considered with respect to Definition 4.3.3, which uses the notion of *Whitney stratification* of a variety.

Finally, in Section 4.4 we consider the lowest coefficient $\mathrm{EDpoly}_{X,u}(0)$, which somehow describes the variety of data points having distance zero to the variety X. In our example, observe that $\mathrm{EDpoly}_{C,u}(0)$ is the product of the square of the equation of C times a quartic polynomial. Indeed, this polynomial is the equation of $(C^\vee \cap Q)^\vee$, and its projective real locus is empty. In general, the interpretation is that, when restricting to $V^{\mathbb{R}}$, the points having distance zero from the real locus of X are essentially the points of $X^{\mathbb{R}}$. Anyway, the complex points $u \in V$ such that $q(u - x) = 0$ for some critical point $x \in X$ for δ_u fill an entire hypersurface in V. When X is not a hypersurface, then $(X^\vee \cap Q)^\vee$ is, and its equation is exactly $\mathrm{EDpoly}_{X,u}(0) = 0$. All these facts lead in general to Theorem 4.4.12, which can be applied for projective varieties. The core of the proof of this result applies the identity in Theorem 4.4.10, which in turn needs Aluffi's formula (1.8.4) for the ED degree as well as Piene's formulas (1.8.6) and (1.8.7) for the polar varieties of a singular projective variety.

4.1 Definition of the ED polynomial

The notation used in this chapter agrees with Section 1.1. In Section 1.2 we recalled the definition of the ED degree of an affine algebraic variety X in a complex vector space. This important definition uses the ED correspondence $\mathcal{E}(X)$ of X, introduced in Definition 1.2.1.

A nice geometrical description of the ε-offset hypersurface of X is furnished by Horobeţ and Weinstein in [HW]. For a complex number ε, we define the ε-*hyperball* centered at $u \in V$ as

$$\mathcal{V}(\delta_u - \varepsilon^2) \coloneqq \{y \in V \mid \delta_u(y) = \varepsilon^2\}. \tag{4.1.1}$$

Note that ε-*hyperball* centered at $u \in V$ is the usual sphere in $\mathbb{C}^n$ only when δ_u is defined via the standard Euclidean quadratic form $q^{\mathbb{R}}$ on $\mathbb{R}^n$. Taking into account our assumptions on $q^{\mathbb{R}}$, in general it is a smooth affine quadric hypersurface in V. For example, in the complex vector space S^dV of degree d symmetric tensors $f = (f_\alpha)_{|\alpha|=d}$ equipped with the Frobenius quadratic form q_F, the ε-hyperball centered at f is not a sphere (see relation (2.1.3)).

Suppose that $x \in X_{\mathrm{sm}}$ is a critical point of δ_u. Then trivially $x \in \mathcal{V}(\delta_u - \delta_u(x))$ and the $\pm\sqrt{\delta_u(x)}$-hyperball centered at u intersects X non-transversally. This leads to the following definition.

Definition 4.1.1. [HW, Definition 2.1] The ε-*offset hypersurface* of X is defined to be the union of the centers of ε-hyperballs that intersect the variety X non-transversally at some point $x \in X$. Equivalently the ε-offset hypersurface is the envelope of a family of ε-hyperballs centered on the variety. For a fixed ε we denote the ε-offset hypersurface by $\mathcal{O}_\varepsilon(X)$.

The geometrical description given above suggests also the following definition.

Definition 4.1.2. For a fixed $\varepsilon \in \mathbb{C}$, the ε-*offset correspondence* of X is

$$\mathcal{O}_\varepsilon\mathcal{E}(X) \coloneqq \mathcal{E}(X) \cap \mathcal{V}(\delta_{(\cdot)} - \varepsilon^2),$$

where the ED correspondence $\mathcal{E}(X)$ was introduced in Definition 1.2.1 and the hypersurface $\mathcal{V}(\delta_{(\cdot)} - \varepsilon^2) \subset V \times V$ consists of all pairs (x, u) such that $\delta_u(x) = \varepsilon^2$.

Analogously to (1.2.2), we might consider the diagram

$$\begin{tikzcd} & \mathcal{O}_\varepsilon\mathcal{E}(X) \arrow[dl, "\widetilde{\pi}_1"'] \arrow[dr, "\widetilde{\pi}_2"] & \\ X & & V \end{tikzcd} \tag{4.1.2}$$

Then the ε-offset hypersurface of X is

$$\mathcal{O}_\varepsilon(X) = \overline{\widetilde{\pi}_2(\mathcal{O}_\varepsilon\mathcal{E}(X))} \subset V.$$

We recall that the ED correspondence $\mathcal{E}(X)$ is an affine variety in $V \times V$ of dimension n. Since the general pair $(x, u) \in \mathcal{E}(X)$ is such that $\delta_u(x) \neq \varepsilon^2$, we

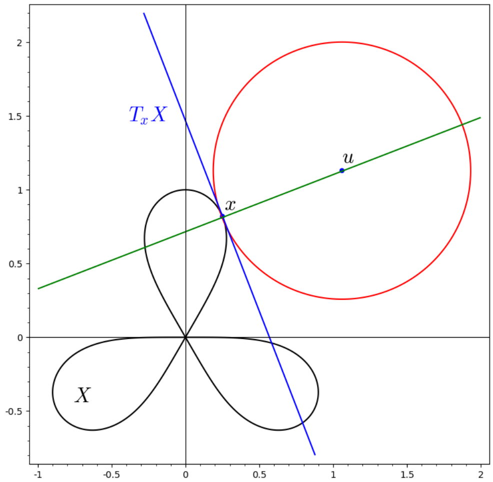

Figure 4.1: A critical point $x \in X$ for the distance function δ_u on the "three leaved clover" $X\colon (x_1^2+x_2^2)^3+3x_1^2x_2-x_2^3=0$ and the $\sqrt{\delta_u(x)}$-hyperball centered at u.

have that $\mathcal{O}_\varepsilon\mathcal{E}(X)$ has dimension $n-1$ in $V\times V$. Moreover, the second projection $\widetilde{\pi}_2$ is finite-to-one, hence the dimension of $\widetilde{\pi}_2(\mathcal{O}_\varepsilon\mathcal{E}(X))$ is $n-1$ as well. This fact essentially motivates the name ε-offset *hypersurface* of X.

How do we get the equation of $\mathcal{O}_\varepsilon(X)$? The ideal of $\mathcal{E}(X)$ is the critical ideal $I_{\mathrm{crit}}(X)$ introduced in (1.2.1). Therefore, the ideal of $\mathcal{O}_\varepsilon\mathcal{E}(X)$ is

$$I(\mathcal{O}_\varepsilon\mathcal{E}(X))=I_{\mathrm{crit}}(X)+(\delta_u-\varepsilon^2)\subset\mathbb{C}[x_1,\dots,x_n,u_1,\dots,u_n].$$

Finally, the projection onto the second factor π_2 corresponds algebraically to elimination of the variables of $x\in X$. Hence the ideal of $\mathcal{O}_\varepsilon(X)$ is

$$I(\mathcal{O}_\varepsilon(X))=I(\mathcal{O}_\varepsilon\mathcal{E}(X))\cap\mathbb{C}[u_1,\dots,u_n].$$

Actually, in this chapter we suppose that ε is a variable as well as the coordinates of u. Therefore, we assume that $I(\mathcal{O}_\varepsilon(X))$ lives in the larger ring $\mathbb{C}[u_1,\dots,u_n,\varepsilon]$. The unique generator of this ideal is the main object of this chapter.

Definition 4.1.3. Up to a scalar factor, we denote the generator of $I(\mathcal{O}_\varepsilon(X))$ by $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ and we call it the *Euclidean Distance polynomial (ED polynomial)* of X at u.

Note that, if the variety X is the complex zero locus of a real variety $X^{\mathbb{R}}$, then the elimination procedure that produces the ideal $I(\mathcal{O}_\varepsilon(X))$ gives $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ with real coefficients.

Example 4.1.4. According to Definition 4.1.3, the following M2 code computes the ED polynomial of the affine cone $C \subset \mathbb{R}^3$ discussed in the preamble of this chapter, with respect to $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. It may be adapted to any affine variety in a complex vector space V.

```
R = QQ[x_1,x_2,x_3,u_1,u_2,u_3,e];
IX = ideal(4*x_1^2+x_2^2-x_3^2);
ISingX = IX+minors(codim IX,compress transpose jacobian IX);
jacX = matrix{{u_1-x_1,u_2-x_2,u_3-x_3}}||
(compress transpose jacobian IX);
IcritX = saturate(IX+minors((codim IX)+1, jacX), ISingX);
IoffsetX = IcritX+ideal(sum(3,j->(u_(j+1)-x_(j+1))^2)-e^2);
EDpolyX = (eliminate({x_1,x_2,x_3},IoffsetX))_0
```

In Definition 1.2.12, we introduced the ED discriminant Σ_X as the branch locus of the second projection in (1.2.2). Analogously, here we consider the second projection $\widetilde{\pi}_2$ in (4.1.2). For a general $\varepsilon \in \mathbb{C}$, the branch locus $B_\varepsilon(X, X)$ of $\widetilde{\pi}_2$ is generically a hypersurface inside $\mathcal{O}_\varepsilon(X)$, by the Nagata-Zariski Purity Theorem [Zar, Nag]. Since $\mathcal{O}_\varepsilon(X)$ is in turn a hypersurface in V, then $B_\varepsilon(X, X)$ is a codimension two variety in V. More precisely, the variety $B_\varepsilon(X, X)$ is populated by all data points $u \in \mathcal{O}_\varepsilon(X)$ such that either $\{(x_1, u), (x_2, u)\} \subset \mathcal{O}_\varepsilon\mathcal{E}(X)$ for some smooth critical points $x_1 \neq x_2$ of δ_u on X, or $(x, u) \in \mathcal{O}_\varepsilon\mathcal{E}(X)$ with multiplicity two for some smooth critical point x of δ_u on X.

Definition 4.1.5. The *bisector hypersurface* of X is the closure of the union of all branch loci of $\widetilde{\pi}_2$:

$$B(X, X) := \overline{\bigcup_{\varepsilon \in \mathbb{C}} B_\varepsilon(X, X)}.$$

For instance, in Example 2.4.2 we computed the bisector hypersurface of the rational normal curve $X_{(3)}$. Moreover, the red curve in Figure 4.2 corresponds to the bisector curve of $X\colon x_1^3 - x_2 = 0$. Finally, we determine in (5.4.13) the equation of the bisector hypersurface of $X_3 \cong \mathrm{Seg}_3(\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}})$.

4.2 First properties of the ED polynomial

The following proposition states that the distance function is a root of the ED polynomial. It is a consequence of Lemma 1.2.2 and Definition 4.1.3.

Proposition 4.2.1. *For general $u \in V$, the roots of* $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ *are precisely of the form $\varepsilon^2 = \delta_u(x)$, where x is a critical point of the squared distance function δ_u on X_{sm}. In particular the distance ε from the real locus $X^{\mathbb{R}}$ to a data point $u \in V^{\mathbb{R}}$ is a root of* $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$*. Moreover* $\mathrm{EDpoly}_{X,u}(0) = 0$ *for all $u \in X^{\mathbb{R}}$.*

Introducing the ED polynomial $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$, we stressed that ε is seen as a variable as well as the coordinates of the data point u. Hence, a natural question is to compute the ε^2-degree of $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$.

Theorem 4.2.2. *[HW, Theorem 2.9] For a general $u \in V$, the ε^2-degree of* $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ *coincides with the ED degree of X.*

Proof. Observe that a fixed general point $u \in V$ is an element of the ε-offset hypersurface $\mathcal{O}_\varepsilon(X)$ for precisely two times ED degree many distinct ε. This is because u has $N = \mathrm{EDdegree}(X)$ many critical points to X, say $x_1, \ldots, x_N$ and then the corresponding offset hypersurfaces that include u are the ones where

$$\varepsilon \in \left\{\pm\sqrt{\delta_u(x_1)}, \ldots, \pm\sqrt{\delta_u(x_N)}\right\}.$$

This is equivalent to $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ having exactly two times N many roots, and the roots are precisely $\pm\sqrt{\delta_u(x_i)}$ for $i \in [N]$ by Proposition 4.2.1. □

The ED polynomial behaves well under the union of varieties, as shown by the following proposition.

Proposition 4.2.3. *Assume $X = X_1 \cup \cdots \cup X_r$ for some integer $r \geq 0$, where $X_i \subset V$ is a reduced variety for every $i \in \{1, \ldots, r\}$ and $X_i \neq X_j$ for every $i \neq j$. Then*

$$\mathrm{EDpoly}_{X,u}(\varepsilon^2) = \prod_{i=1}^{r} \mathrm{EDpoly}_{X_i,u}(\varepsilon^2).$$

Proof. For general $u \in V$, the variety of the critical ideal $I_{\mathrm{crit}}(X)$ in V with respect to X is the union of the varieties $I_{\mathrm{crit}}(X_i)$. The conclusion follows by Lemma 4.2.1. □

As we pointed out in Section 1.2, for an affine variety $X \subset V$, the number of complex-valued critical points of δ_u remains constant as the data point u varies in V, and this number is equal to $\mathrm{EDdegree}(X)$. On the other hand, the number of real-valued critical points of δ_u is constant on the connected components of the complement of the ED discriminant Σ_X (see Definition 1.2.12). In particular, if u is close to Σ_X, then two distinct real (or complex conjugate) roots of $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ tend to coincide. This fact implies the following natural result, which appears essentially in [HW, Proposition 2.13], where the discriminant of the ED polynomial was called *offset discriminant*.

Proposition 4.2.4. *Given an affine variety $X \subset V$, let f and g be the equations of the ED discriminant Σ_X and of the bisector hypersurface $B(X,X)$, respectively. Define*

$$\Delta_X(u) := \Delta_{\varepsilon^2}\mathrm{EDpoly}_{X,u}(\varepsilon^2)$$

to be the discriminant of the ED polynomial of X at u. Then $f(u)\,g(u)$ divides $\Delta_X(u)$. In particular, if X is symmetric to a finite number s of affine hyperplanes $L_1,\dots,L_s$ of equations $l_1,\dots,l_s$, then

$$L_1 \cup \cdots \cup L_s \subset B(X,X),$$

namely the product $l_1(u)\cdots l_s(u)$ divides $g(u)$.

Proof. By definitions 1.2.12 and 4.1.5 of ED discriminant and bisector hypersurface, any point $u \in \Sigma_X \cup B(X,X)$ satisfies $\Delta_X(u) = 0$, since two roots in ε^2 coincide, by Proposition 4.2.1. Let $u \in L_i$ and let $x \in X_{\mathrm{sm}} \setminus L_i$ be a critical point of the distance function δ_u (there exists at least one). Call y the reflection of x with respect to L_i. In particular, $y \in X$ as well and y is again a critical point of δ_u. Since $\delta_u(x) = \delta_u(y)$, we have that $u \in B(X,X)$, hence it is a zero of $\Delta_X(u)$. □

Remark 4.2.5. If $X \subset V$ is symmetric to an infinite number of affine hyperplanes of V, then there exist $p \in V$ and $r \in \mathbb{C}$ such that the hyperplanes of symmetry of X are exactly the ones containing p and X is the sphere centered in p of radius r. In this special case, $\Delta_X(u)$ coincides with the equation of Σ_X, which in turn is the sphere centered in p of radius zero.

Example 4.2.6. We provide a couple of examples related to Proposition 4.2.4. First, Salmon remarks in [Sal, §372 ex. 3] that when X is an ellipse with symmetry axes L_1, L_2, with the notation of Proposition 4.2.4, then

$$\Delta_X(u) = (l_1 l_2)^2 f^3,$$

of total degree 22. On one hand, the union $L_1 \cup L_2$ forms the bisector curve $B(X,X)$. On the other hand, the curve cut out by f is a sextic Lamé curve and corresponds to the evolute of the ellipse depicted in Figure 3.

A concrete example of an affine curve without symmetry axes is furnished in Figure 4.2, where $X\colon x_1^3 - x_2 = 0$ and

$$\Delta_X(u) = h^2 f^3.$$

Again, the curve cut out by f is the evolute

$$\Sigma_X\colon 8748x_1x_2^5 + 9375x_1^4 + 20250x_1^2x_2^2 - 729x_2^4 - 4800x_1x_2 + 256 = 0\,,$$

whereas the curve cut out by h is the bisector curve

$$B(X,X)\colon 216x_1^3x_2^5 + 3125x_1^6 + 1125x_1^4x_2^2 + 27x_1^2x_2^4 + 27x_2^6 \\ + 400x_1^3x_2 + 144x_1x_2^3 - 48x_1^2 - 16x_2^2 = 0\,.$$

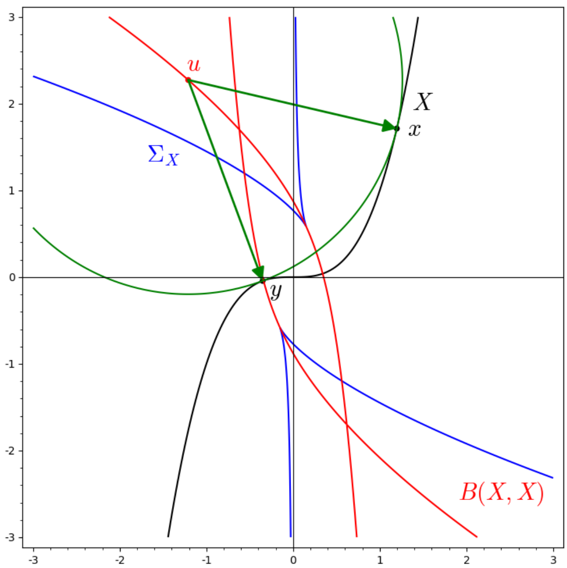

Figure 4.2: The two components Σ_X and $B(X,X)$ of the zero locus of $\Delta_{\varepsilon^2}\mathrm{EDpoly}_{X,u}(\varepsilon^2)$.

Note that in these computations as well as in Example 2.4.2 and in (5.4.12), the equations of $B(X,X)$ and Σ_X appear in the expression of $\Delta_X(u)$ with exponents two and three, respectively. It should be interesting to compute in general the exponents occurring in the factorization of $\Delta(u)$.

The principal results of this chapter regard affine cones X with the origin as the vertex. In this case $I(X)$ is homogeneous and moreover, all coefficients of $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$ are homogeneous in the variables of u.

Definition 4.2.7. Let $Z \subset \mathbb{P}(V)$ be a projective variety. The *ED polynomial* of Z is by definition the ED polynomial of its affine cone of Z in V.

As we underlined in Section 1.3, for affine cones there is a notion of duality. By Theorem 1.3.3, the ED degrees of an affine cone X and of its dual affine cone $X^\vee$, both interpreted as varieties in V, coincide. This nice geometric property has a natural algebraic counterpart in the study of the ED polynomial of X, stated in the following proposition.

Theorem 4.2.8. *Let $X \subset V$ be an affine cone and $X^\vee$ its dual in V. Then*

$$\mathrm{EDpoly}_{X,u}(\varepsilon^2) = \mathrm{EDpoly}_{X^\vee,u}(q(u) - \varepsilon^2).$$

Proof. Fix a data point $u \in V$ and let x be a critical point of δ_u on X. Then, by Proposition 4.2.1, $\varepsilon_1^2 = q(u-x)$ is a root of $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$. Applying Theorem 1.3.3, $u - x$ is a critical point of δ_u on $X^\vee$. Hence again by Proposition 4.2.1, $\varepsilon_2^2 = q(x)$ is a root of $\mathrm{EDpoly}_{X^\vee,u}$. Moreover, by the Pythagorean Theorem we have the equality $q(u) = \varepsilon_1^2 + \varepsilon_2^2$, thus giving the desired result. □

When X is a hypersurface transversal to Q (to be defined in Definition 4.3.3), after Theorem 4.4.12, the last result allows to write explicitly the equation of $X^\vee$ from $\mathrm{EDpoly}_{X,u}(\varepsilon^2)$.

An immediate consequence of Theorems 1.3.3 and 4.2.8 is the equality between the discriminants of the ED polynomials of an affine cone X and its associated dual affine cone $X^\vee$.

Corollary 4.2.9. *Let $X \subset V$ be an affine cone and $X^\vee$ its dual in V. Then*

$$\Delta_X(u) = \Delta_{X^\vee}(u),$$

where $\Delta_X(u)$ was defined in Proposition 4.2.4. In particular, the ED discriminants Σ_X and $\Sigma_{X^\vee}$ coincide, as well as the bisector hypersurfaces $B(X,X)$ and $B(X^\vee, X^\vee)$.

A special case occurs by adding a hyperplane "at infinity" at the affine space V, which in Section 1.1 was denoted by H_∞. In particular, $V \cup H_\infty \cong \mathbb{P}(\mathbb{C} \oplus V) \cong \mathbb{P}^n_{\mathbb{C}}$, with coordinates $x_0, x_1, \ldots, x_n$, and $H_\infty \colon x_0 = 0$.

We know from [DHOST, Section 6] that in general the ED degree of an affine variety $X \subset V$ is not preserved under the operation of projective closure. Here we stress that the ED degree of X is computed with respect to the quadratic form q, whereas the quadratic form $\overline{q}(x) := x_0^2 + x_1^2 + \cdots + x_n^2$ is the one used to compute the ED degree of the projective closure $\overline{X}$ in $V \cup H_\infty$. Note that there are infinitely many possible choices of quadratic forms on $\mathbb{P}^n_{\mathbb{C}}$ that restrict to q on V. This is one of the reasons why $\mathrm{EDdegree}(X)$ and $\mathrm{EDdegree}(\overline{X})$ are not related in general. Actually, this fact is even clearer when comparing the ED polynomial of X with respect to the ED polynomial of $\overline{X}$, as shown in the following example (see also Example 4.3.6).

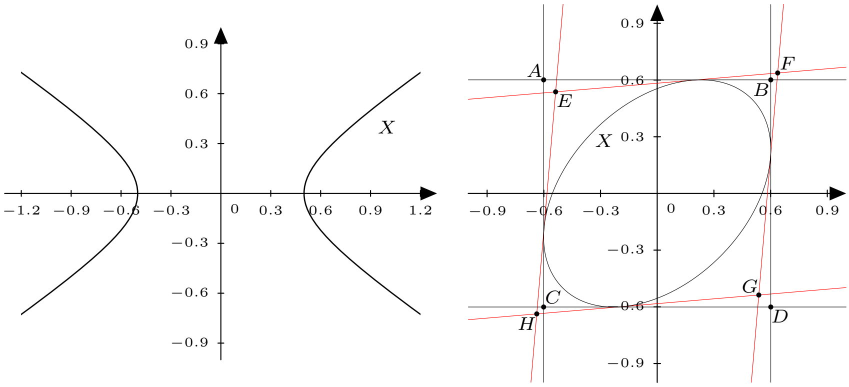

Figure 4.3: The example of the hyperbola X.

Example 4.2.10. Consider the hyperbola $X\colon 4x_1^2 - 9x_2^2 - 1 = 0$ in Figure 4.3. Given a data point $u = (u_1, u_2) \in \mathbb{C}^2$, one can verify that

$$\mathrm{EDpoly}_{X,u}(0) = (4u_1^2 - 9u_2^2 - 1)^2 (1296u_1^4 + 2592u_1^2u_2^2 + 1296u_2^4 - 936u_1^2 + 936u_2^2 + 169).$$

The second factor of the above polynomial is the product of the four pairwise conjugate lines tangent to X and meeting $Q_\infty = \{[0, 1, \sqrt{-1}], [0, 1, -\sqrt{-1}]\}$ at infinity. This fact is clarified after Proposition 4.4.2. On the other hand, we consider the projective closure $\overline{X}\colon 4x_1^2 - 9x_2^2 - x_0^2 = 0$ of X and we compute its ED polynomial with respect to the point $\overline{u} = [1, u_1, u_2]$. Now we obtain that

$$\mathrm{EDpoly}_{\overline{X},\overline{u}}(0) = (4u_1^2 - 9u_2^2 - 1)^2 (1024u_1^4 + 2880u_1^2u_2^2 + 2025u_2^4 - 832u_1^2 + 1170u_2^2 + 169).$$

Note that the second factor of $\mathrm{EDpoly}_{\overline{X},\overline{u}}(0)$ corresponds to the dual variety of $\overline{X}^\vee \cap \overline{Q}$ which is the union of four pairwise conjugate lines different from its corresponding ones in $\mathrm{EDpoly}_{X,u}(0)$ (see Corollary 4.4.6).

In order to display all these lines, we consider the change of coordinates of equations $z_1 = -\sqrt{-1}\,x_1, z_2 = x_2 + \sqrt{-1}\,x_3, z_3 = x_2 - \sqrt{-1}\,x_3$. Then the image of X is the ellipse of equation $13z_1^2 - 10z_1z_2 + 13z_2^2 - 4 = 0$. On one hand, the points A, B, C, D in Figure 4.3 generate the four lines corresponding to the second factor of $\mathrm{EDpoly}_{X,u}(0)$. Note that in the new coordinates the lines meeting the points $[0, 1, \sqrt{-1}]$ and $[0, 1, -\sqrt{-1}]$ at infinity are the horizontal and vertical lines respectively. On the other hand, the union of the four lines generated by E, F, G, H correspond to the second factor of $\mathrm{EDpoly}_{\overline{X},\overline{u}}(0)$.

A positive result is that, under some reasonable transversality assumptions, $\mathrm{EDdegree}(X)$ and $\mathrm{EDdegree}(\overline{X})$ can be related. In particular, in [DHOST, Lemma 6.7] it is shown a bijection between the critical points of the distance function from the origin $(0,\dots,0)$ on X and the critical points of the distance function from $u_0 := (1,0,\dots,0)$ on $\overline{X}$. This property has a natural interpretation in terms of ED polynomial, as the following result suggests.

Proposition 4.2.11. *Assume that* $\mathrm{EDdegree}(X) = \mathrm{EDdegree}(\overline{X}) = r$ *and that there are r critical points of δ_0 on X which satisfy $q(x) \neq -1$. Then, up to scalars,*

$$\mathrm{EDpoly}_{X,0}(\varepsilon^2) = (1+\varepsilon^2)^r \cdot \mathrm{EDpoly}_{\overline{X},u_0}\left(\frac{\varepsilon^2}{1+\varepsilon^2}\right).$$

Proof. By [DHOST, Lemma 6.7], the map

$$x \mapsto \left(\frac{1}{1+q(x)}, \frac{1}{1+q(x)}x\right)$$

is a bijection between the critical points of δ_0 on X and the critical points of δ_{u_0} on $\overline{X}\setminus X_\infty$. Define $r = \mathrm{EDdegree(X)}$ and let $x_1,\dots,x_r$ be the critical points of δ_0 on X. By Proposition 4.2.1, the roots of $\mathrm{EDpoly}_{X,0}(\varepsilon^2)$ are $\varepsilon_i^2 = q(x_i - 0) = q(x_i)$, $i \in \{1,\dots,r\}$. Let $\tilde{x}_i \in \overline{X}$ be the critical point of δ_{u_0} corresponding to x_i for all $i \in \{1,\dots,r\}$. Hence $\tilde{\varepsilon}_i^2 = \overline{q}(\tilde{x}_i - u_0)$ is a root of $\mathrm{EDpoly}_{\overline{X},u_0}(\tilde{\varepsilon}^2)$ for every i. Then by hypothesis we have

$$\mathrm{EDpoly}_{X,0}(\varepsilon^2) = c\cdot(\varepsilon^2-\varepsilon_1^2)\cdots(\varepsilon^2-\varepsilon_r^2), \quad \mathrm{EDpoly}_{\overline{X},u_0}(\tilde{\varepsilon}^2) = \tilde{c}\cdot(\tilde{\varepsilon}^2-\tilde{\varepsilon}_1^2)\cdots(\tilde{\varepsilon}^2-\tilde{\varepsilon}_r^2)$$

for some scalars c and $\tilde{c}$. Moreover, the following equalities hold true:

$$\tilde{\varepsilon}_i^2 = \overline{q}(\tilde{x}_i - u_0) = \overline{q}\left(\frac{1}{1+q(x_i)} - 1, \frac{1}{1+q(x_i)}x_i\right) = \frac{\varepsilon_i^2}{1+\varepsilon_i^2}, \quad i \in \{1,\dots,r\}.$$

From this it follows that

$$\begin{aligned}\prod_{i=1}^r(\tilde{\varepsilon}^2 - \tilde{\varepsilon}_i^2) &= \prod_{i=1}^r\left(\frac{\varepsilon^2}{1+\varepsilon^2} - \frac{\varepsilon_i^2}{1+\varepsilon_i^2}\right) = \\ &= \prod_{i=1}^r \frac{\varepsilon^2 - \varepsilon_i^2}{(1+\varepsilon^2)(1+\varepsilon_i^2)} = \frac{c'}{(1+\varepsilon^2)^r}\prod_{i=1}^r(\varepsilon^2 - \varepsilon_i^2),\end{aligned}$$

where $c' = \prod_{i=1}^r(1+\varepsilon_i^2)$. From the last chain of equalities, we obtain the desired identity. ☐

Another reasonable property of the ED polynomial is its behavior under the isometry group $\mathrm{Isom}(V)$, where an isometry is the composition of a translation and an element of the orthogonal group $O(V)$ introduced at the end of Section

2.1. Fix $g \in \mathrm{Isom}(V)$ and consider the transformed variety $gX := \{gx \mid x \in X\}$ of X. Then the identity

$$\mathrm{EDdegree}(gX) = \mathrm{EDdegree}(X)$$

holds true. For any data point $u \in V$ and any critical point x for δ_u we have that $g \cdot x \in gX$ is a critical point for $\delta_{g \cdot u}$. Moreover, we have the identity $q(g \cdot u - g \cdot x) = q(u - x)$. The immediate consequence in terms of ED polynomial is that

$$\mathrm{EDpoly}_{gX,u}(\varepsilon^2) = \mathrm{EDpoly}_{X,g^{-1}u}(\varepsilon^2).$$

In the projective setting, we can reduce to the subgroup of isometries that fix the origin, which is precisely $O(V)$.

Proposition 4.2.12.

1. *Let $X \subset V$ be an affine variety. Let $G \subset \mathrm{Isom}(V)$ be a group that leaves X invariant. Then the coefficients of* $\mathrm{EDpoly}_{X,u}$ *are G-invariant.*

2. *Let $X \subset \mathbb{P}(V)$ be a projective variety. Let $G \subset O(V)$ be a group that leaves X invariant. Then the coefficients of* $\mathrm{EDpoly}_{X,u}$ *are G-invariant.*

Proof. The proof is the same in both cases. Let $g \in G$. Since $q(u - x) = q(g \cdot u - g \cdot x)$, the critical values of δ_u coincide with the critical values of $\delta_{g \cdot u}$. □

Now consider the uniform scaling in V with scale factor $c \in C$. Calling cX the scaling of X, for any data point $u \in V$ and any critical point x for δ_u we have that $cx \in cX$ is a critical point for δ_{cu}. Moreover, $q(cu - cx) = c^2 q(u - x)$ and this implies that

$$\mathrm{EDpoly}_{cX,u}(c^2\varepsilon^2) = \mathrm{EDpoly}_{X,c^{-1}u}(\varepsilon^2).$$

Remark 4.2.13. There are many meaningful examples with such a G-action. If X is the affine cone of rank-one symmetric tensors in S^dV introduced in Definition 2.1.3, the group $G = O(V)$ works. If $X = X_\mu$ is the affine cone of decomposable partially symmetric tensors in $S^\mu V = S^{\mu_1}V_1 \otimes \cdots \otimes S^{\mu_s}V_r$ introduced in Chapter 5, the group $G = O(V_1) \times \ldots \times O(V_s)$ works. In these examples, X and the isotropic quadric Q are not transversal. It should be interesting to study the intersection between X and Q when a positive dimensional group $G \subset O(V)$ acts on X.

All the mentioned properties of the ED polynomial are general. Now we start considering specific types of varieties. The simplest varieties to consider are affine subspaces of V.

Proposition 4.2.14 (The ED polynomial of an affine subspace)**.** *Let $L \subset V^{\mathbb{R}}$ be an affine subspace and let $\pi_{L^\perp}$ be the orthogonal projection onto $L^\perp$. Then for any data point $u \in V$ the ED polynomial of L is*

$$\mathrm{EDpoly}_{L,u}(\varepsilon^2) = \varepsilon^2 - q\left(\pi_{L^\perp}(u)\right).$$

Proof. By Proposition 1.2.7 and Remark 1.2.9, the only critical point of δ_u from L is $\pi_L(u)$, hence $\mathrm{EDdegree}(L) = 1$. The statement follows by the identity $\pi_{L^\perp}(u) = u - \pi_L(u)$, see also [DHOST, Example 2.2]. □

Remark 4.2.15. Proposition 4.2.14 extends to complex subspaces L such that L_∞ is transversal to Q_∞.

The case of linear subspaces (that is, affine subspaces containing the origin) is simpler and it is contained in next Corollary. This is generalized to any variety in Theorem 4.4.12.

Corollary 4.2.16 (The ED polynomial of a linear subspace)**.** *Let $L \subset V$ be a linear subspace transversal to Q (this is always the case if L is the complexification of a real subspace).*

1. *If* $\mathrm{codim}(L) \geq 2$, *then the dual projective variety of $L^\perp \cap Q$ is the quadric hypersurface cut out by a polynomial g and*

$$\mathrm{EDpoly}_{L,u}(t^2) = t^2 - g(u).$$

2. *If L is the hyperplane cut out by a polynomial f, then*

$$\mathrm{EDpoly}_{L,u}(t^2) = t^2 - f^2(u).$$

Proof. In the following, we interpret L, $L^\perp$ and Q as projective varieties of $\mathbb{P}(V)$. It is straightforward to check that the dual variety of $L^\perp \cap Q$ is the join between L and $L^\perp \cap Q$ (see Definition 4.4.1), which is a quadric cone with vertex L, having equation $q\left(\pi_{L^\perp}(u)\right)$. If L is a hyperplane then $L^\perp \cap Q = \emptyset$ and the quadric cone has rank one. □

Example 4.2.17. For example, if $L \subset \mathbb{C}^2$ is a point, then the quadric Z_L cut out by $q\left(\pi_{L^\perp}(u)\right)$ is the circumference centered in L of radius zero, namely the union of the lines joining L and the points of Q_∞ (the dashed lines in Figure 4.4), where in this case $Q_\infty = I \cup J$, $I = [1, \sqrt{-1}, 0]$, $J = [1, -\sqrt{-1}, 0]$.

More in general, the quadric Z_L has a nontrivial description. We refer to Section 4.4 for a complete study of the lowest term of the ED polynomial of an affine variety. Anyway, when restricting to the real points of V, the quadric Z_L restricts to L.

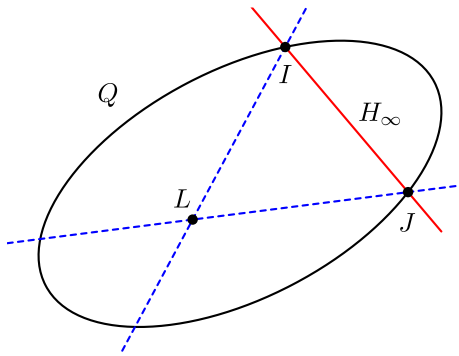

Figure 4.4: The dashed lines above form the quadric Z_L when $L \subset \mathbb{C}^2$ is a point.

Example 4.2.18 (The ED polynomial of an affine conic)**.** Let $u = (u_1, u_2) \in \mathbb{C}^2$ be a data point. A general affine conic $C \subset \mathbb{C}^2$ has equation

$$C\colon ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0.$$

The ED polynomial of C at u may be written in the form

$$\mathrm{EDpoly}_{C,u}(\varepsilon^2) = c_4\varepsilon^8 + c_3\varepsilon^6 + c_2\varepsilon^4 + c_1\varepsilon^2 + c_0.$$

Note that this polynomial contains only even powers of ε and has degree four in ε^2, according to the fact that the general plane conic C is such that EDdegree(C) $= 4$, by Proposition 1.8.2. In particular, we display the two extreme coefficients of $\mathrm{EDpoly}_{C,u}$:

$$\begin{aligned} c_4 &= (b^2 - 4ac)^2[(a-c)^2 + b^2],\\ c_0 &= (au_1^2 + bu_1u_2 + cu_2^2 + du_1 + eu_2 + f)^2 g(u_1, u_2). \end{aligned}$$

Note that the first factor of c_0 is the square of the equation of C. Let us concentrate on the factor $g(u_1, u_2)$. As explained in more generality in Proposition 4.4.2, the locus of zeros of g is the union of the four lines tangent to C and passing through the points of $Q_\infty = \{[0, 1, \sqrt{-1}], [0, 1, -\sqrt{-1}]\}$. We denote these lines by $L_1^+, L_2^+, L_1^-, L_2^-$, where

$$L_1^+ \cap L_2^+ = [0, 1, \sqrt{-1}], \quad L_1^- \cap L_2^- = [0, 1, -\sqrt{-1}].$$

The foci of the conic C are by definition the four pairwise intersections $L_1^+ \cap L_1^-$, $L_1^+ \cap L_2^-$, $L_2^+ \cap L_1^-$ and $L_2^+ \cap L_2^-$. If C is either an ellipse or a hyperbola, then C has two real foci. Otherwise if C is either a parabola or a circumference, then C has only one real focus. In the case of a circumference, the only real focus coincides with the center. In particular, there exist real solutions of the equation $c_0 = 0$ outside the real conic $C^{\mathbb{R}}$ (see also Remark 4.4.4).

Now consider the highest coefficients c_4. If both of the two factors in c_4 do not vanish, then either C is an ellipse or an hyperbola, respectively when $b^2 - 4ac$ is positive or negative. Moreover, $b^2 - 4ac = 0$ if and only if C is a parabola, whereas $(a-c)^2 + b^2 = 0$ if and only if (over $\mathbb{R}$) C is a circumference:

1. If C is a parabola then EDdegree(C) = 3. Rewriting the equation of C as $(ax_1 + bx_2)^2 + cx_1 + dx_2 + e = 0$, we get that $\mathrm{EDpoly}_{C,u}(\varepsilon^2)$ has degree 3 in ε^2 and its leading coefficient is $-16(bc - ad)^2(a^2 + b^2)^3$. In particular, the condition $bc - ad = 0$ forces C to be the union of two distinct parallel lines.

2. If C is a circumference, then EDdegree(C) = 2. Rewriting the equation of C as $(x_1 - a)^2 + (x_2 - b)^2 - r^2 = 0$, we get that $\mathrm{EDpoly}_{C,u}(\varepsilon^2)$ has degree 2 in ε^2. In particular, it factors as

   $$\mathrm{EDpoly}_{C,u}(\varepsilon^2) = [(x_1-a)^2+(x_2-b)^2-(\varepsilon+r)^2][(x_1-a)^2+(x_2-b)^2-(\varepsilon-r)^2].$$

4.3 The highest coefficient of the ED polynomial

In this section, we study the leading term of the ED polynomial of an affine (respectively, projective) variety X. We show that with transversality assumptions the leading term is scalar, in other terms the ED polynomial may be written as a monic polynomial, see Proposition 4.3.4 (resp. Corollary 4.3.7). In algebraic terms, this implies that the ED polynomial is an integral algebraic function.

We recall from [VdW, Section 100] that an algebraic function

$$f(t,u) = \sum_{k=0}^{d} t^k p_k(u)$$

is called *integral* if the leading coefficient $p_d(u)$ is constant. The branches of a integral algebraic function are well-defined everywhere. Otherwise, if $p_d(u) = 0$, then one branch goes to infinity.

To express transversality, we need to recall the *Whitney stratification* of an algebraic variety. The following definitions are recalled from [Nic, Section 4.2].

Definition 4.3.1. Let $X, Y \subset V$ be two disjoint smooth quasi-projective varieties. We say that the pair (X, Y) satisfies the *Whitney regularity condition (a)* at $x_0 \in X \cap \overline{Y}$ if, for any sequence $y_n \in Y$ such that

1. $y_n \to x_0$,

2. the sequence of tangent spaces $T_{y_n}Y$ converges to the subspace T,

we have $T_{x_0}X \subset T$. The pair (X, Y) is said to satisfy the *Whitney regularity condition (a) along* X, if it satisfies this condition at any $x \in X \cap \overline{Y}$.

Definition 4.3.2. Suppose X is a subset of V. A *stratification* of X is an increasing, finite filtration

$$F_{-1} = \emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_m = X$$

satisfying the following properties:

1. F_k is closed in X for all k.
2. For every $k \in \{1, \ldots, m\}$ the set $X_k = F_k \setminus F_{k-1}$ is a smooth manifold of dimension k with finitely many connected components called the k-dimensional *strata* of the stratification.
3. (The frontier condition) For every $k \in \{1, \ldots, m\}$ we have $\overline{X}_k \setminus X_k \subset F_{k-1}$.

The stratification is said to satisfy the Whitney condition (a) if, for every $0 \leq j < k \leq m$, the pair (X_j, X_k) satisfies Whitney's regularity condition (a) along X_j. Note that if X is smooth then the trivial stratification $\emptyset \subset X$ satisfies the Whitney condition (a).

We recall (see [PP]) that any affine (or projective) variety admits a Whitney stratification, satisfying condition (a) and a stronger condition (b) that we do not use explicitly in this chapter.

Definition 4.3.3. We say that a variety X is *transversal* to a smooth variety Y (in the applications we have $Y = Q$) when there exists a Whitney stratification of X such that each stratum is transversal to Y.

If X is smooth and the schematic intersection $X \cap Y$ is smooth, then X is transversal to Y according to Definition 4.3.3.

If each stratum of a Whitney stratification of X is transversal to a smooth variety Y, then by [PP, Lemma 1.2] this stratification induces a Whitney stratification of $X \cap Y$.

Proposition 4.3.4. *Let $X \subset V$ be an affine variety. If X_∞ is transversal to Q_∞ (according to Definition 4.3.3), then the ED polynomial of X is an integral algebraic function.*

Proof. This proof uses the interpretation of duality as polarity with respect to the isotropic quadric Q_∞, hence for more details about the subject and the notation we refer to Section 1.1.

If the ED polynomial is not an integral algebraic function, then there is a point $u \in V$ that annihilates the leading coefficient of $\mathrm{EDpoly}_{X,u}$. We get a sequence $\{u_k\} \subset V$ such that $u_k \to u$ and a corresponding sequence $\{x_k\} \subset X$ of critical points for δ_{u_k} such that $\mathrm{EDpoly}_{X,u_k}(\varepsilon_k^2) = 0$ when $\varepsilon_k^2 = q(x_k - u_k)$ diverges. In particular we have that $< u_k - x_k > \in \left[(T_{x_k}X)_\infty\right]^\perp$ for all k, where the dual is taken in the projective subspace H_∞. Up to subsequences, we may assume that

$$\lim_{k\to\infty} \langle x_k \rangle =: \langle x \rangle \in X_\infty, \text{ for some } x \in V. \tag{4.3.1}$$

We may assume there are two different strata X_1 and X_2 of $\overline{X}$ such that $x_k \in X_1$ for all k and $\langle x \rangle \in X_2$. In the topology of the compact space $\overline{X} = X \cup X_\infty$ we still have $x_k \to \langle x \rangle \in X_\infty$, more precisely in $\mathbb{P}(\mathbb{C} \oplus V)$ we have $[(1, x_k)] \to [(0, x)]$. We may assume (up to subsequences) that $\{T_{x_k}X_1\}$ has a limit. Hence by Whitney condition (a) we have $T_{\langle x \rangle}X_2 \subset \lim_{k\to\infty} T_{x_k}X_1$. From this and from (4.3.1) we have immediately that

$$\langle x \rangle \in T_{\langle x \rangle}(X_2)_\infty \subset \lim_{k\to\infty} (T_{x_k}X_1)_\infty. \tag{4.3.2}$$

Since $\{x_k\}$ diverges we get

$$\langle x \rangle = \lim_{k\to\infty} \langle u - x_k \rangle = \lim_{k\to\infty} \langle u_k - x_k \rangle.$$

By construction, we have that $\langle u_k - x_k \rangle \in [(T_{x_k}X_1)_\infty]^\perp$ for all k. This fact and relation (4.3.1) imply that

$$\langle x \rangle = \lim_{k\to\infty} \langle u_k - x_k \rangle \in \lim_{k\to\infty} [(T_{x_k}X_1)_\infty]^\perp \subset \left[\lim_{k\to\infty} (T_{x_k}X_1)_\infty\right]^\perp.$$

In particular $\langle x \rangle \in [\lim_{k\to\infty}(T_{x_k}X_1)_\infty] \cap [\lim_{k\to\infty}(T_{x_k}X_1)_\infty]^\perp$, hence $\langle x \rangle \in Q_\infty$.

We now show that $T_{\langle x \rangle}(X_2 \cap H_\infty) \subset T_{\langle x \rangle}Q_\infty$, where

$$T_{\langle x \rangle}Q_\infty = \{\langle y \rangle \in H_\infty \mid q(\langle x \rangle, \langle y \rangle) = 0\}.$$

Pick a nonzero vector $v \in V$ such that $\langle v \rangle \in T_{\langle x \rangle}(X_2 \cap H_\infty)$. We claim that $q(\langle x \rangle, \langle v \rangle) = 0$, where q is the quadratic form in $H_\infty = \mathbb{P}(V)$, which is the quadratic form defined on V.

Indeed, pick a sequence $\langle v_k \rangle \to \langle v \rangle$, where $\langle v_k \rangle \in (T_{x_k}X_1)_\infty$. Since $q(u_k - x_k, v_k) = 0$ for all k, at the limit we get $q(\langle x \rangle, \langle v \rangle) = 0$. This contradicts the transversality between X_∞ and Q_∞, as we wanted. $\square$

Remark 4.3.5. The transversality conditions stated in Proposition 4.3.4 are sufficient for the integrality of the distance function, but not necessary: for example, the parabola studied in Example 4.2.18 is in general transversal to Q_∞, but not transversal to H_∞. Nevertheless, its ED polynomial is monic.

On the other hand, the cardioid studied in Example 4.3.6 is singular at Q_∞ (and consequently is not transversal to H_∞), and its ED polynomial has a leading coefficient of positive degree. It should be interesting to find general necessary conditions for the integrality of the distance function.

Example 4.3.6. Let $C \subset \mathbb{R}^2$ be the real affine cardioid of equation $(x_1^2 + x_2^2 - 2x_1)^2 - 4(x_1^2 + x_2^2) = 0$. It has a cusp at the origin and at the isotropic points at H_∞. Hence, according to the formula (1.8.4), $\mathrm{EDdegree}(\overline{C}) = 16 - 3 \times 3 = 7$. On the other hand, $\mathrm{EDdegree}(C) = 3$: the drop is caused essentially by the non-transversality with Q_∞. Computing the ED polynomial of C, one may observe that its leading coefficient is $(x_1 - 1)^2 + x_2^2$, namely one branch of the distance function diverges when the chosen data point is $u = (1, 0)$. It is interesting to note that the projective embedding of $(1, 0)$ is the meeting point of the three tangent cones at the three cusps of $\overline{C}$. Moreover, the cardioid C is the trace left by a point, initially at the origin, on the perimeter of a circle of radius 1 that is rolling around the circle centered in $(1, 0)$ of the same radius. In Figure 4.5 we see that the ED discriminants Σ_C and $\Sigma_{\overline{C}}$ are dramatically different. While the real part of Σ_C is again a cardioid, the real part of $\Sigma_{\overline{C}}$ divides the plane into four connected components, one of them is shown in the detail on the right of Figure 4.5.

Corollary 4.3.7. *Let $X \subset \mathbb{P}(V)$ be a projective variety. If X is transversal to Q, according to Definition 4.3.3 then for any data point $u \in V$*

$$\mathrm{EDpoly}_{X,u}(\varepsilon^2) = \sum_{j=0}^{d} p_j(u)\varepsilon^{2j},$$

where $d = \mathrm{EDdegree}(X)$ and $p_j(u)$ is a homogeneous polynomial in the coordinates of u of degree $2d - 2j$. In particular, $p_d(u) = p_d \in \mathbb{C}$, $\deg(p_0) = 2d$ and the ED polynomial of X is an integral algebraic function.

Proof. For affine cones, the assumption that X is transversal to Q is equivalent to X_∞ transversal to Q_∞. □

Another consequence of Proposition 4.3.4 and Theorem 4.2.2, valid for projective varieties, is the following.

Corollary 4.3.8. *Let $X \subset \mathbb{P}(V)$ be a projective variety. If X is transversal to Q, according to Definition 4.3.3 then the degree of the ε-offset of X is*

$$\deg(\mathcal{O}_\varepsilon(X)) = 2\mathrm{EDdegree}(X).$$

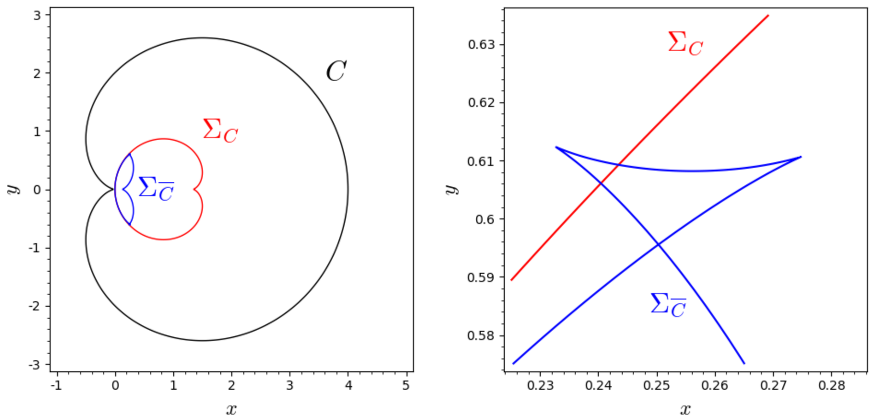

Figure 4.5: The cardioid C with its ED discriminant Σ_C and the restriction to the affine plane of the ED discriminant $\Sigma_{\overline{C}}$ of its projectivization $\overline{C}$. A detail of $\Sigma_{\overline{C}}$ on the right.

4.4 The lowest coefficient of the ED polynomial

In the last section, we gave some geometric conditions on X that affect the shape of its ED polynomial. In particular, we have obtained some useful pieces of information about the degrees of the coefficients of $\mathrm{EDpoly}_{X,u}$ for a general data point $u \in V$. In this section, we aim to describe completely the locus of points $u \in V$ such that the lowest term of the ED polynomial of X at u vanishes. First, we recall a definition.

Definition 4.4.1. Let $X, Y \in \mathbb{P}(V)$ be two disjoint projective varieties. The union

$$J(X,Y) = \bigcup_{x\in X,\ y\in Y} \langle x, y\rangle \subset \mathbb{P}(V)$$

of the lines joining points of X to points of Y is again a projective variety (see [Har, Example 6.17]) and is called the *join of X and Y. Note that $J(X,\emptyset) = X$.*

Proposition 4.4.2. *Consider an affine variety $X \subset V$. Given any point $p \in X_{\mathrm{sm}}$, we define $J_{X,p} := J(p, [(T_pX)_\infty]^\perp \cap Q_\infty) \subset V \cup H_\infty = \mathbb{P}(V \oplus \mathbb{C})$. Then the zero locus $u \in V$ of $\mathrm{EDpoly}_{X,u}(0)$ is $J_X \cap V$, where*

$$J_X := \overline{\bigcup_{p\in X_{\mathrm{sm}}} J_{X,p}} \subset V \cup H_\infty\,.$$

Proof. The variety $[(T_pX)_\infty]^\perp \cap Q_\infty \subset H_\infty$ parametrizes all the directions in V corresponding to isotropic vectors w such that $w \in N_pX$. In particular, for all $p \in X_{\mathrm{sm}}$, the variety $J_{X,p}$ is the union of the lines passing from p and generated by isotropic vectors w such that $w \in N_pX$.

Now pick a point $u \in J_X \cap V$. Then u is a limit of points u_k, such that there exist $p_k \in X_{\mathrm{sm}}$ with $u_k \in J_{X,p_k}$. Hence $u_k = p_k + w_k$ for some vector $w_k \in V$ whose direction belongs to $[(T_{p_k}X)_\infty]^\perp \cap Q_\infty$. In particular $q(w_k) = q(u_k - p_k) = 0$ and $w_k \in N_{p_k}X$, that is, p_k is a critical point on X of the squared distance function δ_{u_k}. In particular, u_k is a zero of $\mathrm{EDpoly}_{X,u}(0)$, hence u is a zero of $\mathrm{EDpoly}_{X,u}(0)$ as we wanted.

Conversely, let $u \in V$ be a general zero of $\mathrm{EDpoly}_{X,u}(0)$. Then there exist $p \in X_{\mathrm{sm}}$ such that $u - p \in N_pX$ and $0 = q(u-p)$. Define $w = u - p$. Then the direction corresponding to w, when nonzero, is represented by a point of $[(T_pX)_\infty]^\perp \cap Q_\infty$, namely $u \in J_{X,p} \subset J_X \cap V$. When $w = 0$, we have $u = p \in J_{X,p}$. Conclusion follows by taking closures. □

Example 4.4.3. The simplest case is when V is 2-dimensional. Let C be an affine plane curve which is transversal with the isotropic quadric at infinity. In this case, $Q_\infty = \{[0,1,\sqrt{-1}],[0,1,-\sqrt{-1}]\}$.

Looking at Proposition 4.4.2, for any $p \in C_{\mathrm{sm}}$, we have $J_{C,p} \neq \emptyset$ if and only if $T_pC = p + \langle v \rangle$ with $v \in \{(1,\sqrt{-1}),(1,-\sqrt{-1})\}$. In other words, the zero locus of $\mathrm{EDpoly}_{C,u}(0)$ is the union of C and of the tangent lines to C meeting Q_∞.

Remark 4.4.4. As showed in Example 4.2.18, the real foci of a real conic $C^{\mathbb{R}}$ are zeros of the lowest coefficient of the ED polynomial of C. Therefore, given a real affine variety $X^{\mathbb{R}} \subset V^{\mathbb{R}}$, the hypersurface $J_X \cap V$ might contain real points which do not necessarily belong to $X^{\mathbb{R}}$. The reason is that there could exist real data points u admitting (non real) critical points x for the function δ_u such that $q(u-x) = 0$. This is essentially the price to pay for describing the distance function with algebraic tools and, above all, for allowing non real solutions to the problem. For a concrete example, let $C^{\mathbb{R}}$ be the real ellipse of equation

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0, \quad a \geq b > 0.$$

The two foci of $C^{\mathbb{R}}$ are $u^\pm = (\pm c, 0)$, where $c^2 = a^2 - b^2$. On one hand, the only real critical points of δ_{u^+} and δ_{u^-} on $C^{\mathbb{R}}$ are the points $(\pm a, 0)$. On the other hand, with straightforward computations one verifies that the non real points $z^\pm = \left(\frac{a^2}{c}, \pm\sqrt{-1}\frac{b^2}{c}\right)$ are critical for δ_{u^+} on C, as well as $w^\pm = \left(-\frac{a^2}{c}, \pm\sqrt{-1}\frac{b^2}{c}\right)$ are critical for δ_{u^-} on C. Note that, for instance $T_{z^\pm}C\colon x_1 \pm \sqrt{-1}x_2 - c = 0$ and coincides with the normal space $N_{z^\pm}C$, with $u^+ - z^\pm \in N_{z^\pm}C$. In addition, we have that $q(u^+ - z^\pm) = q(u^- - w^\pm) = 0$, thus confirming the fact that $u^\pm$ are real points of $J_C \cap V$.

Proposition 4.4.5. *Let $X \subset V$ be an irreducible cone. Then*

$$J_X \cap V = X \cup (X^\vee \cap Q)^\vee.$$

Proof. For the proof of this statement, we stress the hidden assumption $X \cup X^\vee \not\subset Q$, which was given in Chapter 1.

Suppose that there exists a point $p \in X_{\mathrm{sm}}$ such that $u \in J_{X,p}$. If $u = p$, then in particular $u \in X$. If $u \neq p$, then $u - p$ is a nonzero element of $X^\vee \cap Q$. We show that necessarily $u \in (X^\vee \cap Q)^\vee$ when $(X^\vee \cap Q)^\vee$ is nonempty.

By definition, we have that $(X^\vee \cap Q)^\vee = \overline{S}$, where

$$S = \bigcup_{z \in (X^\vee \cap Q)_{\mathrm{sm}}} N_z(X^\vee \cap Q).$$

If $X^\vee \cap Q$ is non-reduced, then $(X^\vee \cap Q)^\vee = \emptyset$ and by taking closures we have trivially that $J_X \cap V \subset X \cup (X^\vee \cap Q)^\vee$. Hence suppose that $X^\vee \cap Q$ is a reduced variety. On one hand, by construction p is critical for δ_u on X. On the other hand, since u is general, $u - p$ is critical for δ_u on $X^\vee$ by Theorem 1.3.3.

In order to prove that $u \in (X^\vee \cap Q)^\vee$, it remains to show that $u \in N_{u-p}(X^\vee \cap Q)$. Indeed, pick a vector $y \in T_{u-p}(X^\vee)$ such that $\langle y, u - p \rangle = 0$. The condition that $u - p$ is critical for δ_u implies that p is orthogonal to any tangent vector to $X^\vee$ at $u - p$, so we have $\langle y, p \rangle = 0$. Then

$$\langle y, u \rangle = \langle y, u - p \rangle + \langle y, p \rangle = 0 + 0 = 0,$$

thus proving our claim. By taking closures we get

$$J_X \cap V \subset X \cup (X^\vee \cap Q)^\vee.$$

On the other hand, suppose that $u \in X \cup (X^\vee \cap Q)^\vee$. If $(X^\vee \cap Q)^\vee = \emptyset$, then $u \in X$ and clearly $u \in J_X$. Now assume that $(X^\vee \cap Q)^\vee$ is nonempty and that $u \in S$. Then there exists a smooth $z \in X^\vee \cap Q$ such that $u \in N_z(X^\vee \cap Q)$. In particular, z is critical for δ_u on $X^\vee$. By Theorem 1.3.3, $u - z$ is critical for δ_u on X. This means that z is an element of $[(T_{u-z}X)_\infty]^\perp \cap Q_\infty$. In particular, $u \in J_{X,u-z}$. By taking the Zariski closure, we have that $X \cup (X^\vee \cap Q)^\vee \subset J_X \cap V$. $\square$

Corollary 4.4.6. *Let $X \subset V$ be an affine cone such that $X \cup X^\vee \not\subset Q$. Then the locus of zeros $u \in V$ of* $\mathrm{EDpoly}_{X,u}(0)$ *is*

$$X \cup (X^\vee \cap Q)^\vee.$$

In particular, at least one between X and $(X^\vee \cap Q)^\vee$ is a hypersurface.

Corollary 4.4.7. *Let $X \subset V$ be an affine cone such that $X \cup X^\vee \not\subset Q$. If $X^\vee \cap Q$ is a non-reduced variety, then necessarily X is a hypersurface.*

In the following, we show an improvement of Corollary 4.4.6 applying the theory of Chern-Mather classes. The price to pay is an additional transversality assumption.

Let $Y \subset \mathbb{P}(V)$ be an irreducible projective variety and consider its conormal variety introduced in Definition 1.3.2. As Theorem 1.5.5 states, if $\mathcal{N}(Y, Y^\vee)$ does not intersect the diagonal $\Delta(\mathbb{P}(V)) \subset \mathbb{P}(V) \times \mathbb{P}(V)$, then the ED degree of Y is equal to the sum of its polar classes.

A sufficient condition for $\mathcal{N}(Y, Y^\vee)$ not to intersect $\Delta(\mathbb{P}(V))$ is furnished in the following result.

Proposition 4.4.8. *Assume that $Y^\vee$ (or Y) is transversal to Q, according to definition 4.3.3. Then $\mathcal{N}(Y, Y^\vee)$ does not intersect $\Delta(\mathbb{P}(V))$.*

Proof. By Biduality Theorem it is enough to prove the result for $Y^\vee$. Suppose that $(y, y) \in \mathcal{N}(Y, Y^\vee)$ for some $y \in Y$. By Definition 1.3.2 and by hypothesis, there exists a sequence of vectors (y_i, x_i) and pairs $(Y_1, Y_2), (Y_1', Y_2')$ satisfying the Whitney regularity condition (a) along Y and $Y^\vee$ respectively such that

1. $(y_i, x_i) \to (y, y)$,
2. $y_i \in Y_1$, $x_i \in Y_1' \cap N_{y_i} Y_1$ for all i, and
3. $y \in Y_2 \cap Y_2'$.

In particular, point (2) says that $q(y_i, x_i) = 0$ for all i, hence taking the limit we find $y \in Q$.

Now take a vector $v \in T_y Y_2'$. We show that $v \in T_y Q$, obtaining that $T_y Y_2' \subset T_y Q$, and thus contradicting the transversality assumption. By the Whitney condition (a) we have that $T_y Y_2' \subset \lim_{i \to \infty} T_{x_i} Y_1'$. This means that there exists a sequence $\{v_i\}$ with $v_i \in T_{x_i} Y_1'$ for all i such that $v_i \to v$. From point (2) we have $y_i \in N_{x_i} Y_1'$ for all i, hence $q(y_i, v_i) = 0$ for all i. Finally taking the limit we have $q(y, v) = 0$, that is $v \in T_y Q$. $\square$

In the following, we assume the hypothesis of Proposition 4.4.8. Theorem 1.5.5 allows us to express the ED degree of Y in terms of Chern-Mather classes $c_j^M(Y)$ via Aluffi's formula in Theorem 1.8.4. Indeed, the polar classes $\delta_j(Y)$ of Y are determined by the Chern-Mather classes of Y via the formulas (1.8.6).

In particular, we aim to apply (1.8.7) in the case when $Y = X^\vee \cap Q$, once we know the Chern-Mather classes of $X^\vee \cap Q$. Since $X^\vee \cap Q$ is a divisor in $X^\vee$ with normal bundle $\mathcal{O}(2)$, these can be computed by a result of Pragacz and Parusinski in [PP]. We need the assumption that $X^\vee$ is transversal to Q, according to Definition 4.3.3.

Denote by $c^M(Y) = \sum_i c_i^M(Y)$ the total Chern-Mather class of Y. The equation displayed in three lines just after [PP, Lemma 1.2] shows that

$$c^M(X^\vee \cap Q) = \frac{1}{1+2h} \sum_{i\geq 0} 2h \cdot c_i^M(X^\vee), \tag{4.4.1}$$

hence

$$c_j^M(X^\vee \cap Q) \cdot h^{m-1-j} = 2\sum_{i=0}^{j}(-1)^{j-i}2^{j-i}c_i^M(X^\vee) \cdot h^{m-i}. \tag{4.4.2}$$

Lemma 4.4.9. *The following identity holds true:*

$$2^{m+1-i} - 1 = (m+1-i) + \sum_{j=i}^{m}(m-j)2^{j-i}.$$

Proof. Using the identities ($r \neq 1$)

$$\sum_{k=s}^{m} r^k = \frac{r^s - r^{m+1}}{1-r} \quad \text{and} \quad \sum_{k=1}^{m} kr^{k-1} = \frac{1-r^{m+1}}{(1-r)^2} - \frac{(m+1)r^m}{1-r},$$

we have that

$$\begin{aligned}\sum_{j=i}^{m}(m-j)2^{j-i} &= \frac{1}{2^i}\left(m\sum_{j=i}^{m}2^j - 2\sum_{j=i}^{m}j2^{j-1}\right)\\ &= \frac{2^{m+1}-(m+2-i)2^i}{2^i}\\ &= 2^{m+1-i}-(m+2-i),\end{aligned}$$

thus getting the desired identity. □

Theorem 4.4.10. *Assume that $X^\vee$ is transversal to Q, according to definition 4.3.3. If X is not a hypersurface, then*

$$2\mathrm{EDdegree}(X) = 2\mathrm{EDdegree}(X^\vee) = \deg((X^\vee \cap Q)^\vee). \tag{4.4.3}$$

Otherwise if X is a hypersurface, then

$$2\mathrm{EDdegree}(X) = 2\mathrm{EDdegree}(X^\vee) = 2\deg(X) + \deg((X^\vee \cap Q)^\vee). \tag{4.4.4}$$

In particular, $(X^\vee \cap Q)^\vee$ has always even degree.

Proof. If X is not a hypersurface, then $(X^\vee \cap Q)^\vee$ is a hypersurface by Corollary 4.4.6. Hence we can apply the identity (1.8.7) and we obtain that (here $m = \dim(X^\vee)$)

$$
\begin{aligned}
\deg((X^\vee \cap Q)^\vee) &= \sum_{j=0}^{m-1} (-1)^j\,(m-j)\,c_j^M(X^\vee \cap Q)h^{m-1-j} \\
&= \sum_{j=0}^{m} (-1)^j\,(m-j)\,c_j^M(X^\vee \cap Q)h^{m-1-j} \\
(*) &= 2\sum_{j=0}^{m} (-1)^j\,(m-j)\left[\sum_{i=0}^{j} (-1)^{j-i}2^{j-i}c_i^M(X^\vee)\cdot h^{m-i}\right] \\
&= 2\sum_{i=0}^{m} (-1)^i c_i^M(X^\vee)h^{m-i}\left[\sum_{j=i}^{m}(m-j)2^{j-i}\right] \\
&= 2\sum_{i=0}^{m} (-1)^i(2^{m+1-i}-1)c_i^M(X^\vee)\cdot h^{m-i} \\
&\quad - 2\sum_{i=0}^{m}(-1)^i(m+1-i)c_i^M(X^\vee)\cdot h^{m-i},
\end{aligned}
\tag{4.4.5}
$$

where in $(*)$ we used (4.4.2) and in the last equality we applied Lemma 4.4.9. In the last expression obtained, since $X^\vee$ is transversal to Q, by (1.8.5) the first term coincides with $2\mathrm{EDdegree}(X^\vee) = 2\mathrm{EDdegree}(X)$, whereas by (1.8.7) the second term is equal to $2\delta_0(X^\vee)$, which vanishes because X is not a hypersurface. Hence the identity (4.4.3) is satisfied.

Now assume that X is a hypersurface. The expression in the first line of (4.4.5) is exactly the polar class $\delta_0(X^\vee \cap Q)$. The computation (4.4.5) shows that the same expression is equal to

$$
2\mathrm{EDdegree}(X) - 2\delta_0(X^\vee) = 2\sum_{j=1}^{n-2}\delta_j(X^\vee),
$$

by Theorem 1.5.5. If this expression vanishes we get $\delta_j(X^\vee) = 0$ for $j \geq 1$, which is equivalent to $\delta_j(X) = 0$ for $j \leq n-3$. Hence the defect $\mathrm{codim}(X^\vee) - 1$ is $n-2$ and $X^\vee$ is a point in $\mathbb{P}(V)$, namely X is a hyperplane in $\mathbb{P}(V)$. In particular, $X^\vee \cap Q$ is empty and therefore $(X^\vee \cap Q)^\vee = \mathbb{P}(V)$. In conclusion, the identity (4.4.4) is satisfied.

Otherwise if $\delta_0(X^\vee \cap Q) \neq 0$, since in this case $0 \neq \delta_0(X^\vee) = \deg(X)$, the identity (4.4.4) is satisfied as well. $\square$

One may wonder if Theorem 4.4.10 remains true without transversality assumptions. The case of symmetric tensors studied in Chapter 2, as well as the case of partially symmetric tensors in Chapter 5, answer in negative. Already the binary cubic case gives a counterexample, as shown in Example 2.4.2. Indeed, for (partially symmetric) tensors the degree of $X^\vee$ is greater than the ED degree of X, the opposite of the general case.

Corollary 4.4.11. *Let $X^\vee$ be a positive dimensional variety which is transversal to a smooth quadric Q, according to Definition 4.3.3. Then $(X^\vee \cap Q)^\vee$ is a hypersurface.*

Proof. The computation (4.4.5) shows that

$$\delta_0(X^\vee \cap Q) = 2\mathrm{EDdegree}(X) - 2\delta_0(X^\vee).$$

If $(X^\vee \cap Q)^\vee$ is not a hypersurface, we get $\delta_0(X^\vee \cap Q) = 0$, hence $\delta_j(X^\vee) = 0$ for all $j \geq 1$, namely $\delta_j(X) = 0$ for all $j \leq n-3$. Hence the defect $\mathrm{codim}(X^\vee) - 1$ is $n-3$ and $X^\vee$ is zero dimensional. □

In the projective case, Theorem 4.4.10 leads us to a more precise description of the lowest term of the ED polynomial, with reasonable transversality assumptions. In particular, the factor corresponding to $(X^\vee \cap Q)^\vee$ is always present when X is not a hyperplane. If X is a hypersurface, an additional factor corresponding to X appears.

Theorem 4.4.12. *Let $X \subset \mathbb{P}(V)$ be an irreducible variety and suppose that X and $X^\vee$ are transversal to Q. Let $u \in V$ be a data point.*

1. *If $\mathrm{codim}(X) \geq 2$, then $(X^\vee \cap Q)^\vee$ is a hypersurface and*

$$\mathrm{EDpoly}_{X,u}(0) = g$$

up to a scalar factor, where g is the equation of $(X^\vee \cap Q)^\vee$. Moreover $X \subset (X^\vee \cap Q)^\vee$.

2. *If X is a hypersurface, then*

$$\mathrm{EDpoly}_{X,u}(0) = f^2 g$$

up to a scalar factor, where f is the equation of X and g is either the constant 1 if X is a hyperplane, or the equation of $(X^\vee \cap Q)^\vee$.

Proof. First, assume that $\mathrm{codim}(X) \geq 2$. On one hand, by Corollary 4.4.6 we have that $(X^\vee \cap Q)^\vee$ is a hypersurface, hence $\mathrm{EDpoly}_{X,u}(0) = g^k$ for some positive integers k. In particular,

$$\deg(\mathrm{EDpoly}_{X,u}(0)) = k \deg(g).$$

On the other hand, comparing degrees by Theorem 4.4.10 we get $k = 1$. The inclusion follows again from Corollary 4.4.6.

Now assume that X is a hypersurface. If X is a hyperplane, the statement follows by Corollary 4.2.16. Otherwise $\deg(X) \geq 2$ and therefore $X^\vee$ is positive dimensional, thus implying that $(X^\vee \cap Q)^\vee$ is a hypersurface by Corollary 4.4.11.

On one hand, by Corollary 4.4.6 we have that $\mathrm{EDpoly}_{X,u}(0) = f^h g^k$ for some positive integers h and k, hence

$$\deg(\mathrm{EDpoly}_{X,u}(0)) = h\deg(f) + k\deg(g).$$

On the other hand, by Corollary 4.3.7 and Theorem 4.4.10 we have that $h \geq 2$ and $\deg(\mathrm{EDpoly}_{X,u}(0)) = 2\deg(f) + \deg(g)$. □

The hypotheses of Theorem 4.4.12 are reasonable and agree with the principal results in the ED degree-philosophy. Anyway, in the most important examples studied in this thesis, related to varieties of tensors with the Frobenius quadratic form, these hypotheses are not satisfied. A positive result is that we can relax the assumptions of transversality at least for computing the exact multiplicity of the equation of X in $\mathrm{EDpoly}_{X,u}(0)$, when X is a hypersurface.

Proposition 4.4.13. *Let $X \subset \mathbb{P}(V)$ be an irreducible projective hypersurface. Then the equation of X appears with multiplicity two in* $\mathrm{EDpoly}_{X,u}(0)$.

Proof. Quadric hypersurfaces of $\mathbb{P}(V)$ that are transversal to X and $X^\vee$ form a dense open subset $U \subset \mathbb{P}(S^2V)$. In particular, Q is the limit of a sequence $\{Q_j\} \subset U$. Let $\mathrm{EDpoly}^{(j)}_{X,u}(\varepsilon^2)$ be the ED polynomial of X at $u \in V$ with respect to te quadric Q_j, for all j. By Theorem 4.4.12, for all j we have

$$\mathrm{EDpoly}^{(j)}_{Y,y}(0) = f^2 \cdot g_j,$$

where g_j is the equation of $(X^\vee \cap Q_j)^\vee$. Moreover, by Corollary 4.4.6 we know that $\mathrm{EDpoly}_{X,u}(0) = f^\alpha \cdot g^\beta$ for some nonnegative integers α and β, where g is the equation of $(X^\vee \cap Q)^\vee$. In particular,

$$f^\alpha \cdot g^\beta \cdot h = \mathrm{EDpoly}_{X,u}(0) \cdot h = \lim_{j\to\infty} \mathrm{EDpoly}^{(j)}_{X,u}(0) = \lim_{j\to\infty} f^2 \cdot g_j = f^2 \cdot \lim_{j\to\infty} g_j,$$

for some homogeneous polynomial h, possibly a scalar. In particular, $\alpha \geq 2$.

We show that actually $\alpha = 2$. If $\alpha \geq 3$, then f divides $\lim_{j\to\infty} g_j$, that is, f divides g or f divides h. It remains to show that f cannot divide g. In particular, our claim is that $\mathrm{codim}_{\mathbb{R}}[(X^\vee \cap Q)^\vee] \geq 2$.

Consider a smooth point $z \in X^\vee \cap Q$ and the corresponding normal space $S_z := N_z(X^\vee \cap Q)$. Assume that $l_1, \ldots, l_r$ are the linear polynomials defining S_z.

We denote by $\overline{S}_z$ the variety defined by $\bar{l}_1, \ldots, \bar{l}_r$, where the bar means complex conjugation. If $z \in \overline{S}_z$, then $q(z-\bar{z}, y) = 0$ for all $y \in T_z(X^\vee \cap Q)$. In particular, $q(\bar{z}, z) = q(\bar{z}, z) - q(z, z) = q(\bar{z} - z, z) = 0$, contradiction. This implies that $S_z \neq \overline{S}_z$ and, in turn, that $\operatorname{codim}_{\mathbb{R}}(S_z) \geq 2$. The claim follows by Definition 1.3.1. $\square$

Chapter 5
The ED polynomial of a Segre-Veronese variety

In Chapter 2, we studied the E-characteristic polynomial $\psi_f(\lambda)$ of a symmetric tensor $f \in S^dV$. As pointed out in the preamble of Chapter 4, the polynomial $\psi_f(\lambda)$ is useful for determining the equation of the λ-offset hypersurface of the affine cone $X_{(d)}^\vee$, where $X_{(d)}$ is the affine cone of the image of the d-th Veronese embedding of $\mathbb{P}(V)$ of Definition 2.1.3. Now that we have in mind the general theory on ED polynomials of Chapter 4, we derive the following conclusion.

Corollary 5.0.1 (Lim, Qi). *Consider a symmetric tensor f in the space S^dV. The ED polynomial of $X_{(d)}^\vee$ with respect to the isotropic quadric Q_F is*

$$\mathrm{EDpoly}_{X_{(d)}^\vee, f}(\lambda^2) = \begin{cases} \psi_f(\lambda)\psi_f(-\lambda) & \textit{if } d \textit{ is even} \\ \psi_f(\lambda) & \textit{if } d \textit{ is odd.} \end{cases}$$

Note that in this case the variety $X_{(d)} \cap Q_F$ is non-reduced of multiplicity d. This non-transversality is confirmed by the fact that the identity of Theorem 4.4.10 cannot hold, since the integer $\deg(X_{(d)}^\vee) - \mathrm{EDdegree}(X_{(d)}^\vee)$ vanishes for $d = 2$ and is positive when $d > 2$ for all $n \geq 2$, as pointed out in the end of Section 2.2.14.

The case $d = 2$ deals with real symmetric matrices. For any symmetric matrix $u \in S^2V$, we have the identity

$$\mathrm{EDpoly}_{X_{(d)}^\vee, u}(\lambda^2) = \psi_u(\lambda)\psi_u(-\lambda) = \det(U - \lambda I_n)\det(U + \lambda I_n), \tag{5.0.1}$$

In particular, the lowest term of $\mathrm{EDpoly}_{X_{(d)}^\vee, u}$ is the square of $\det(u)$.

In the Introduction, we already considered a similar problem in the vector space $V_1 \otimes V_2$ of $n_1 \times n_2$ rectangular matrices ($n_1 \leq n_2$). Here $X = X_2$ is the

Segre product introduced in (0.0.1), and Q_F is the isotropic quadric defined by the Frobenius inner product defined via the relations in (0.0.4). In this setting, for any matrix $u \in V_1 \otimes V_2$ we have the identity

$$\mathrm{EDpoly}_{X^\vee,u}(\sigma^2) = \det(uu^T - \sigma^2 I_{n_1}). \tag{5.0.2}$$

Here, the roots σ^2 of $\mathrm{EDpoly}_{X^\vee,u}(\sigma^2)$ are the squared singular values of u.

The two cases resumed above are instances of a much harder problem: studying the ED polynomial of $X^\vee$, where X is the affine cone of rank-one partially symmetric tensors of a prescribed format.

We outline more rigorously our problem. Given a vector of dimensions $\underline{n} = (n_1, \ldots, n_s)$ and a vector of degrees $\mu = (\mu_1, \ldots, \mu_s)$, our ambient vector space is the tensor product

$$S^\mu V^{\mathbb{R}} := S^{\mu_1} V_1^{\mathbb{R}} \otimes \cdots \otimes S^{\mu_s} V_s^{\mathbb{R}}$$

where the symmetric power of a vector space was defined in 2.1.1. The vectors in $S^\mu V^{\mathbb{R}}$ are addressed simply as *(real) partially symmetric tensors*. Their format is clear from the context and it is specified only if necessary in the forthcoming examples.

Similarly to Chapter 2, we want to fix a reasonable square distance function $\delta^{\mathbb{R}}_{\mu,u} \colon S^\mu V^{\mathbb{R}} \to \mathbb{R}$ from $u \in S^\mu V^{\mathbb{R}}$ and restrict it to a certain "target variety" $Z_{\mathbb{R}}$, looking for the critical points of $\delta^{\mathbb{R}}_{\mu,u}$ on $Z_{\mathbb{R}}$. To this aim, we consider the complexification $S^\mu V := (S^\mu V^{\mathbb{R}}) \otimes \mathbb{C} = S^\mu(V^{\mathbb{R}} \otimes \mathbb{C})$, the complex variety $Z := Z_{\mathbb{C}}$ and the function $\delta_{\mu,u} \colon S^\mu V \to \mathbb{C}$ restricted to Z.

Our natural choice for $Z_{\mathbb{R}}$ is the affine cone $X^{\mathbb{R}}_\mu$ of partially symmetric tensors in $S^\mu V^{\mathbb{R}}$ of rank at most one, namely the affine cone of the image of the Segre -Veronese embedding

$$\mathrm{Seg}_\mu \colon \prod_{j=1}^{s} \mathbb{P}(V_j^{\mathbb{R}}) \to \mathbb{P}(S^\mu V^{\mathbb{R}}), \quad \mathrm{Seg}_\mu([v_1], \ldots, [v_s]) := [v_1^{\mu_1} \otimes \cdots \otimes v_s^{\mu_s}]. \tag{5.0.3}$$

For brevity, we do not indicate the dependence from $\underline{n}$ in the notations of $X^{\mathbb{R}}_\mu$ and Seg_μ. If $\mu = (1, \ldots, 1) =: 1^s$, we recover the Segre variety $X^{\mathbb{R}}_s := X^{\mathbb{R}}_{1^s}$ introduced in (0.0.17). If $s = 1$, we have $\mu = (d)$ and we get the *Veronese variety* $X^{\mathbb{R}}_{(d)}$ of Definition 2.1.3.

Now consider s arbitrary inner products $q_1^{\mathbb{R}}, \ldots, q_s^{\mathbb{R}}$ on the spaces $V_1^{\mathbb{R}}, \ldots, V_s^{\mathbb{R}}$. In applications, $q_j^{\mathbb{R}}$ is simply the standard Euclidean inner product on $V_j^{\mathbb{R}}$. The *Frobenius inner product* on $S^\mu V^{\mathbb{R}}$ is defined, for partially symmetric tensors of rank one, to be

$$q^{\mathbb{R}}_{F,\mu}(x_1^{\mu_1} \otimes \cdots \otimes x_s^{\mu_s}, y_1^{\mu_1} \otimes \cdots \otimes y_s^{\mu_s}) := q_1^{\mathbb{R}}(x_1, y_1)^{\mu_1} \cdots q_s^{\mathbb{R}}(x_s, y_s)^{\mu_s}$$

for all $x_j, y_j \in V_j^{\mathbb{R}}$ and all $j \in [s]$, and is extended to $S^\mu V^{\mathbb{R}}$ by linearity, similarly to (2.1.2). The squared Frobenius distance function from a fixed $u \in S^\mu V^{\mathbb{R}}$ is

$$\delta^{\mathbb{R}}_{F,\mu,u}(x) \coloneqq q^{\mathbb{R}}_{F,\mu}(u-x)$$

for all $x \in S^\mu V^{\mathbb{R}}$. Note that $\delta^{\mathbb{R}}_{F,\mu,u}$ is the only distance function on $S^\mu V^{\mathbb{R}}$ compatible with the group embedding $\mathrm{SO}(V_1^{\mathbb{R}}) \times \cdots \times \mathrm{SO}(V_s^{\mathbb{R}}) \subset \mathrm{SO}(S^\mu V^{\mathbb{R}})$.

Definition 5.0.2. A *critical (partially symmetric) tensor* for $u \in S^\mu V$ is defined to be any critical point $x \in X_\mu$ for the function $\delta_{F,\mu,u}$.

The isotropic quadic in $S^\mu V$ is $Q_{F,\mu} \coloneqq \mathcal{V}(q_{F,\mu})$. A partially symmetric tensor $u \in S^\mu V$ is *isotropic* if $u \in Q_{F,\mu}$.

Theorem 5.0.3 (Lim, Qi)**.** *Given a partially symmetric tensor $u \in S^\mu V$, the non-isotropic critical tensors for u correspond to tensors $\sigma x = \sigma(x_1^{\mu_1} \otimes \cdots \otimes x_s^{\mu_s}) \in S^\mu V$ such that $q_j(x_j) = 1$ for all $j \in [s]$ and*

$$q_{F,\mu}(u, x_1^{\mu_1} \otimes \cdots \otimes x_j^{\mu_j - 1} \cdot \_ \otimes \cdots \otimes x_s^{\mu_s}) = \sigma\, q_j(x_j, \_), \quad 1 \le j \le s, \tag{5.0.4}$$

for some $\sigma \in \mathbb{C}$, called (partially simmetric) singular value *of u. The corresponding s-ple $(x_1, \ldots, x_s)$ is called* (partially symmetric) singular vector s-ple *for u. Moreover, we call* (partially symmetric) singular tensor *for u any partially symmetric tensor written as $\sigma(x_1^{\mu_1} \otimes \cdots \otimes x_s^{\mu_s})$, where $(x_1, \ldots, x_s)$ and σ are a singular s-ple and a singular value for u, respectively.*

When $s = 1$, the system (5.0.4), together with the normalization assumptions, corresponds to the E-eigenpair system in (2.1.4).

Summing up, in this chapter we compute the ED polynomial of the dual affine cone $X_\mu^\vee \subset S^\mu V$, with respect to the isotropic quadric $Q_{F,\mu}$. As pointed out in Section 5.1, the roots of the ED polynomial of $X_\mu^\vee$ at $u \in S^\mu V$ are the squared singular values of u. Secondly, we report in Theorem 5.1.1 the Friedland-Ottaviani formula for the ED degree of X_μ with respect to $Q_{F,\mu}$, which corresponds, via Theorem 4.2.2, to the ε^2-degree of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$. An independent formulation of Theorem 4.2.2 was given by Aluffi and Harris in [AH]. We underline that their computation does not apply the Catanese-Trifogli formula of Theorem 1.8.1, since X_μ is not transversal to $Q_{F,\mu}$. Indeed, we show that the ED polynomial has a nontrivial highest coefficient.

Definition 5.0.4. For all $J \subset [s] \coloneqq \{1, \ldots, s\}$, we define

$$X_{\mu,J} \coloneqq \mathrm{Seg}_\mu(Y_1 \times \cdots \times Y_s) \subset \mathbb{P}(S^\mu V), \tag{5.0.5}$$

where $Y_j \coloneqq Q_j \coloneqq \mathcal{V}(q_j)$ if $j \in J$ and $Y_j \coloneqq \mathbb{P}(V)$ otherwise. Moreover, we define $f_{\mu,J}$ to be the equation of the dual variety of $X_{\mu,J}$, when it is a hypersurface, otherwise $f_{\mu,J} := 1$. When $\mu = 1^s$, we use the notation $f_{s,J} := f_{1^s,J}$. Generally, $f_\mu := f_{\mu,\emptyset}$ is the equation of the dual variety of $X_\mu = X_{\mu,\emptyset}$ when it is a hypersurface, usually called *μ-discriminant* of a partially symmetric tensor. For $\mu = 1^s$ the μ-discriminant is known as the *hyperdeterminant* of a tensor in $V_1 \otimes \cdots \otimes V_s$, whereas for $s = 1$ and $\mu = (d)$ the μ-discriminant is addressed simply as the *discriminant* of a symmetric tensor.

Section 5.2 keeps dealing with partially symmetric tensors of any format. First, we use Corollary 4.4.6 to determine the vanishing locus of $\mathrm{EDpoly}_{X_\mu^\vee,u}(0)$. Moreover, in Proposition 5.2.6 we describe the set of partially symmetric tensors that fail to have the maximum number of singular values, that is the vanishing locus of the highest coefficient of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$.

Afterwards, the results stated in Section 5.3 are related to the current research and encourage further investigations on this topic. Our main goal would be to generalize Theorem 2.0.2 in the context of partially symmetric tensors of any format. A first answer was given in [Sod] in the special case of partially symmetric binary tensors. Actually, the argument used in [Sod] may be generalized to partially symmetric tensors with hypercube format, that is when $n_1 = \cdots = n_s = n$ and $V_1 = \cdots = V_s$. Applying the results on the lowest coefficient of ED polynomials in Chapter 4 and the inspiring work by Oeding [Oed] on symmetrizations of the discriminant of a partially symmetric tensor, we determine an explicit expression, involving powers of the polynomials $f_{\mu,J}$, for the highest coefficient of the ED polynomial of the dual variety of X_μ. This leads to the following closed formula for the product of the singular values of a general partially symmetric tensor $t \in S^\mu V$, which generalizes Theorem 2.0.2 in the context of symmetric tensors.

Theorem 5.0.5. *1. Assume that the linear system $\mathcal{S}_d$ introduced in Remark 5.3.19 has maximal rank. Let $\mu = (\mu_1, \ldots, \mu_s)$ be a partition of an integer $d \geq 1$. If the partially symmetric tensor $u \in S^\mu V$ admits the maximum number $N = \mathrm{EDdegree}(X_\mu)$ of singular values, counted with multiplicity (hypothesis verified for a general u), their squared product is*

$$(\sigma_1 \cdots \sigma_N)^2 = \prod_{J \subset [s]} f_{\mu,J}(u)^{2-\sum_{k\in J}\mu_k}. \tag{5.0.6}$$

2. (Theorem 2.0.2) In the symmetric case $\mu = (d)$, the product formula (5.0.6) is true for any $n \geq 2$:

$$(\sigma_1 \cdots \sigma_N)^2 = \frac{f_{(d)}(u)^2}{f_{(d),\{1\}}(u)^{d-2}}\ .$$

3. *(Proposition 5.3.18) When $d \in \{2,3,4\}$, the system $\mathcal{S}_d$ has maximal rank and the product formula (5.0.6) is true for all $\mu \vdash d$ and all $n \geq 2$.*

4. *(Proposition 5.4.4) In the binary case $n = 2$, the system $\mathcal{S}_d$ has maximal rank $d-1$ and the product formula (5.0.6) is true for any $d \geq 1$.*

We stress that, for $n \geq 3$ and $\mu \neq (d)$, the previous theorem has been proved so far for small values of n and d. Indeed, we need to show that the linear system $\mathcal{S}_d = \mathcal{S}_d(n)$, defined in Remark 5.3.19, has the maximal possible rank. So far, this has been checked for all positive integers n and d less than 100, using the software `Macaulay2`. For arbitrary large n and d, this check depends on nontrivial binomial inequalities.

The right-hand side of (5.0.6) should be interpreted as the ratio between the lowest and the highest coefficient of the ED polynomial of $X_\mu^\vee$ at $u \in S^\mu V$. Depending on the sign of their exponent, the polynomials $f_{\mu,J}$ appear in the numerator or the denominator of this ratio, otherwise, they do not appear at all if their exponent is zero. The nonsymmetric case $\mu = 1^d$ was stated in Theorem 0.0.8.

The binary case ($n = 2$) considered in Section 5.4 reveals many interesting aspects. In Proposition 5.4.6 we show that for tensors of binary format, the polynomial $f_{\mu,J}$ admits a sum of squares (SOS) decomposition for every nonempty subset $J \subset [s]$. In particular, $f_{\mu,J}(u) > 0$ for any nonzero real partially symmetric binary tensor u. This fact confirms the following known result.

Proposition 5.0.6. *[Lim, Proposition 2] If the dual variety of X_μ is a hypersurface, then 0 is a singular value of $u \in S^\mu V^{\mathbb{R}}$ if and only if $f_\mu(u) = 0$.*

Finally, we pick a general $2 \times 2 \times 2$ tensor u and we compute simbolically all the coefficients of $\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2)$, and the equation of the ED discriminant $\Sigma_{X_3^\vee}$, in terms of $\mathrm{SO}(V)^3$-invariants. This is useful for studying more in detail the $6 = 3!$ singular values of u, even when u is partially symmetric. Note that in this case the formula in (5.0.6) or in (0.0.31) simplifies as

$$(\sigma_1 \cdots \sigma_6)^2 = \frac{g_0^2\, g_1}{g_3}$$

$$g_0 = f_3, \quad g_1 = f_{3,\{1\}}\, f_{3,\{2\}}\, f_{3,\{3\}}, \quad g_2 = 1, \quad g_3 = f_{3,\{1,2,3\}}.$$

5.1 The distance from a Segre-Veronese variety

We follow and generalize the notation adopted for symmetric tensors in (2.1.1). For each space V_j, we set an orthonormal system of coordinates $\{x_{j,1}, \ldots, x_{j,n_j}\}$. Every partially symmetric tensor $u \in S^\mu V$ may be written as an s-homogeneous polynomial of s-degree μ, namely as a polynomial which is homogeneous of degree μ_j in the variables of V_j for all $j \in [s]$:

$$u = \sum_{\substack{|\alpha_j|=\mu_j \\ j\in[s]}} \left[\prod_{j=1}^{s} \binom{\mu_j}{\alpha_j} x_j^{\alpha_j}\right] u_{\alpha_1\cdots\alpha_s},$$

where $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,n_j}) \in \mathbb{Z}_{\geq 0}^{n_j}$, $|\alpha_j| := \sum_{k=1}^{n_j} \alpha_{j,k}$, $x_j^{\alpha_j} := x_{j,1}^{\alpha_{j,1}} \cdots x_{j,n_j}^{\alpha_{j,n_j}}$ and $\binom{\mu_j}{\alpha_j} := \frac{\mu_j!}{\alpha_{j,1}!\cdots\alpha_{j,n_j}!}$ is the multinomial coefficient, for all $j \in [s]$. In particular, we suppose that $(u_{\alpha_1\cdots\alpha_s})$ is as a system of coordinates for $S^\mu V$.

Then, the squared Frobenius norm of $u = (u_{\alpha_1\cdots\alpha_s}) \in S^\mu V$ is

$$q_{F,\mu}(u) = \sum_{\substack{|\alpha_j|=\mu_j \\ j\in[s]}} \left[\prod_{j=1}^{s} \binom{\mu_j}{\alpha_j}\right] u_{\alpha_1\cdots\alpha_s}^2. \tag{5.1.1}$$

Note that the preceding identity gives also the equation of the isotropic quadric $Q_{F,\mu}$ as a smooth quadric hypersurface in $S^\mu V$.

In this section, we show two equivalent approaches for computing the ED polynomial of the dual variety of a Segre-Veronese variety X_μ at a given tensor $u \in S^\mu V$. The first way follows the original setting about the ED polynomial of an algebraic variety explained in Chapter 4, and uses the Pythagorean Theorem. The second one applies directly Theorem 5.0.3.

We recall that the polynomial $\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2)$ was defined in general in 4.1.3. For any fixed $\varepsilon \in \mathbb{C}$, the variety defined in $S^\mu V^{\mathbb{R}}$ by the vanishing of $\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2)$ coincides with the ε-offset hypersurface of $X_\mu^{\mathbb{R}}$. Here we suppose that $u \in S^\mu V$ is fixed and that $\varepsilon \in \mathbb{C}$ is a variable, hence we view $\mathrm{EDpoly}_{X_\mu,t}(\varepsilon^2)$ as a univariate polynomial.

The first property of $\mathrm{EDpoly}_{X_\mu,t}(\varepsilon^2)$ that we mention deals with its ε^2-degree. In Theorem 0.0.6 we recalled Friedland-Ottaviani formula for the ED degree of X_μ in the nonsymmetric case $\mu = 1^s$, where $X_s = X_{1^s}$. Below we furnish their formula in its full generality.

Theorem 5.1.1 (Friedland, Ottaviani). *The ED degree of $X_\mu \subset S^\mu V$ equals the coefficient of the monomial $h_1^{n_1-1}\cdots h_s^{n_s-1}$ in the polynomial*

$$\prod_{i=1}^{s} \frac{\hat{h}_i^{n_i} - h_i^{n_i}}{\hat{h}_i - h_i}, \quad \hat{h}_i := \left(\sum_{j=1}^{s} \mu_j h_j\right) - h_i.$$

Sketch of the proof. For simplicity, we give an idea of the proof in the nonsymmetric case $\mu = 1^s$. We denote for brevity by Π the product $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_s)$. In particular, $X_s = \mathrm{Seg}_s(\Pi)$. For every $j \in [s]$, we call π_j the projection of Π onto the j-th factor. Moreover, for all $j \in [s]$ we define the quotient bundle on $\mathbb{P}(V_j)$ as

$$\mathcal{Q}_j \coloneqq \frac{\mathcal{O}_j^{\oplus n_j}}{\mathcal{O}_j(-1)} = \frac{\mathcal{O}_{\mathbb{P}(V_j)}^{\oplus n_j}}{\mathcal{O}_{\mathbb{P}(V_j)}(-1)}.$$

Now consider the pullbacks

$$\pi_j^* \mathcal{O}_j^{\oplus n_j}, \quad \pi_j^* \mathcal{O}_j(-1), \quad \pi_j^* \mathcal{O}_j(1), \quad \pi_j^* \mathcal{Q}_j,$$

which represent vector bundles on Π. In particular, we have the short exact sequence

$$0 \to \pi_j^* \mathcal{O}_j(-1) \to \pi_j^* \mathcal{O}_j^{\oplus n_j} \to \pi_j^* \mathcal{Q}_j \to 0.$$

If we tensorize the above sequence with respect to the vector bundle $\bigotimes_{i \neq j} \pi_i^* \mathcal{O}_i(1)$, we get another short exact sequence as

$$0 \to \left(\bigotimes_{i \neq j} \pi_i^* \mathcal{O}_i(1) \right) \otimes \pi_j^* \mathcal{O}_j(-1) \to \left(\bigotimes_{i \neq j} \pi_i^* \mathcal{O}_i(1) \right) \otimes \pi_j^* \mathcal{O}_j^{\oplus n_j} \to R_j \to 0. \tag{5.1.2}$$

where we defined the vector bundle on X_μ

$$R_j \coloneqq \left(\bigotimes_{i \neq j} \pi_i^* \mathcal{O}_i(1) \right) \otimes \pi_j^* \mathcal{Q}_j.$$

Finally, we define

$$R \coloneqq \bigoplus_{j=1}^{s} R_j.$$

As explained in [FO], the vector bundle R on X_μ is the right one for studying the number of critical points of the squared distance function $\delta_{F,\mu,u}$ on X_μ. For every $i \in [s]$ we consider the hyperplane class $h_i \coloneqq c_1(\pi_i^* \mathcal{O}_i(1))$. In particular,

$$c(R) = \prod_{j=1}^{s} c(R_j) \in A^*(X_\mu) \cong \frac{\mathbb{Z}[h_1, \ldots, h_s]}{(h_1^{n_1}, \ldots, h_s^{n_s})}.$$

Since $\mathrm{rk}(R) = \dim(X_\mu)$, it follows that the top Chern class of R is of the form

$$c_{\mathrm{top}}(R) = c \prod_{i=1}^{s} h_i^{n_i - 1}$$

for some positive integer c, which turns out to be equal to EDdegree(X). In order to determine c, we consider the above sequence (5.1.2) and by the Whitney sum property (see Section 1.7) we have that

$$\begin{aligned} c(R_j) &= \frac{c\left[\left(\bigotimes_{i\neq j}\pi_i^*\mathcal{O}_i(1)\right)\otimes\pi_j^*\mathcal{O}_j^{\oplus n_j}\right]}{c\left[\left(\bigotimes_{i\neq j}\pi_i^*\mathcal{O}_i(1)\right)\otimes\pi_j^*\mathcal{O}_j(-1)\right]} \\ &= \frac{\left(1+\sum_{i\neq j}h_i\right)^{n_j}}{1-h_j+\sum_{i\neq j}h_i} \\ &= \left[\sum_{k=0}^{n_j}\binom{n_j}{k}\hat{h}_j^k\right]\left[\sum_{k\geq 0}(h_j-\hat{h}_j)^k\right], \end{aligned}$$

where we defined $\hat{h}_j := \sum_{i\neq j}h_i$. Expanding the last expression and multiplying over j, we thus obtain the desired coefficient c in $c_{\mathrm{top}}(R)$. □

Corollary 5.1.2. *Assume that $n_1 = \cdots = n_s = 2$ and consider the vector $\mu = (\mu_1, \dots, \mu_s)$. Then*

$$\mathrm{EDdegree}(X_\mu) = s!\mu_1\cdots\mu_s.$$

Proof. According to Theorem 5.1.1, the ED degree of X_μ equals the coefficient of the monomial $h_1\cdots h_s$ in the polynomial

$$\prod_{i=1}^s(\hat{h}_i+h_i) = \prod_{i=1}^s\left[\left(\sum_{j=1}^s\mu_jh_j\right)-h_i+h_i\right] = (\mu_1h_1+\cdots+\mu_sh_s)^s,$$

and the coefficient of $h_1\cdots h_s$ in the last expression is precisely $s!\mu_1\cdots\mu_s$. □

The following is an immediate consequence of Proposition 4.2.1 and describes which are the roots of the ED polynomial of X_μ.

Proposition 5.1.3. *For a general $u\in S^\mu V$, the roots of* $\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2)$ *are precisely of the form $\varepsilon^2 = q_{F,\mu}(u-z)$, where z is a critical tensor of rank one for u on X_μ. In particular, the distance ε from $X_\mu^{\mathbb{R}}$ to $u\in S^\mu V^{\mathbb{R}}$ is a root of* $\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2)$. *Moreover $u\in S^\mu V^{\mathbb{R}}$ satisfies* $\mathrm{EDpoly}_{X_\mu,u}(0)=0$ *if and only if $u\in X_\mu^{\mathbb{R}}$.*

Anyway, this is not the end of our construction. Indeed, as anticipated in the introduction, we consider the ED polynomial of the dual variety of X_μ, rather than the ED polynomial of the variety X_μ itself. The passage between the ED polynomial of X_μ and the ED polynomial of $X_\mu^\vee$ is just a variable reflection, by Theorem 4.2.8. Hence, for any $u \in S^\mu V$,

$$\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2) = \mathrm{EDpoly}_{X_\mu^\vee,u}(q_{F,\mu}(u) - \varepsilon^2). \tag{5.1.3}$$

The next result clarifies the reason why we concentrate on ED polynomials of dual varieties of Segre-Veronese varieties.

Proposition 5.1.4. *For any partially symmetric tensor $u \in S^\mu V$ and any singular tensor $\sigma x \in S^\mu V$ for u, we have* $\mathrm{EDpoly}_{X_\mu^\vee,u}(\sigma^2) = 0$.

Proof. By Proposition 5.1.3, the roots of $\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2)$ are of the form $\varepsilon^2 = q_{F,\mu}(u-z)$, where z is a critical binary tensor of rank one for u on X_μ. Moreover, by Theorem 5.0.3 the non-isotropic critical tensors of rank one for u correspond to the singular tensors for u. Then, consider a singular tensor σx for u. The root $q_{F,\mu}(u - \sigma x)$ of $\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2)$ corresponds, via Theorem 4.2.8, with the root

$$q_{F,\mu}(u) - q_{F,\mu}(u - \sigma x) = 2q_{F,\mu}(u, \sigma x) - q_{F,\mu}(\sigma x) = 2\sigma q_{F,\mu}(u, x) - \sigma^2 = \sigma^2,$$

of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ (see Figure 5.1), where we used the fact that $\sigma = q_{F,\mu}(u, x)$ for any singular tensor σx for u, which is a direct consequence of equation (5.0.4). $\square$

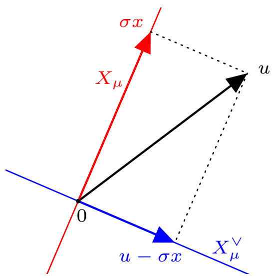

Figure 5.1: Singular tensors $\sigma x \in X_\mu$ and critical points $u - \sigma x \in X_\mu^\vee$ for the distance function $\delta_{F,\mu,u}$ on $X_\mu^\vee$ are in correspondence via the Pythagorean Theorem.

Remark 5.1.5. The converse of Proposition 5.1.4 is true only for general tensors. Indeed, there exist tensors $u \in S^\mu V$ such that some of the roots of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ do not correspond to singular values of u. In the symmetric case ($s = 1$, $\mu = (d)$), this phenomenon is studied in detail for example in [Qi07, Theorem 4] and in [LQZ].

Remark 5.1.6. On one hand, one may verify from (5.1.3) that

$$\begin{aligned}\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2) &= \mathrm{EDpoly}_{X_\mu^\vee,u}(q_{F,\mu}(u)-\varepsilon^2)\\ &= \sum_{k=0}^{N}(-1)^k\left[\sum_{j=k}^{N}\binom{j}{k}q_{F,\mu}(u)^{j-k}a_j(u)\right]\varepsilon^{2k}.\end{aligned} \tag{5.1.4}$$

Hence the highest terms of $\mathrm{EDpoly}_{X_\mu,u}(\varepsilon^2)$ and $\mathrm{EDpoly}_{X_\mu^\vee,u}(q_{F,\mu}(u)-\varepsilon^2)$ are equal to $a_N(u)$ up to sign, where $N = \mathrm{EDdegree}(X_\mu)$. On the other hand, the corresponding lowest terms are not proportional.

Summing up, a first way to compute the ED polynomial of $X_\mu^\vee$ at $u \in S^\mu V$ is by applying Definition 4.1.3 together with the identity (5.1.3). The following `Macaulay2` code computes the ED polynomial of $X_\mu^\vee$ in the case $s = 1$, $n_1 = 2$, $\mu = (d)$, namely when X_μ is the rational normal curve of degree $d \geq 2$.

```
R = QQ[z_0..z_d, c_0..c_d, e];
RatNormCurve = minors(2, matrix{toList(z_0..z_(d-1)), toList(z_1..z_d)});
Jac = compress transpose jacobian RatNormCurve;
M = matrix{apply(d+1, j-> binomial(d, j)*(z_j-c_j))};
It = saturate(RatNormCurve + minors(d, M||Jac), ideal(toList(z_0..z_d)));
Hyperball = ideal(sum(d+1, j-> binomial(d,j)*z_j^2)-e^2);
EDpoly = (eliminate(toList(z_0..z_d), It+Hyperball))_0;
```

The output `EDpoly` is the ED polynomial of the dual variety of `RatNormCurve` because we are taking into account (5.1.3) in the definition of `Hyperball`. Indeed, the usual relation $q_{F,\mu}(u-z)-\varepsilon^2$ is replaced by $q_{F,\mu}(z)-\varepsilon^2$ (see the definition in (4.1.1)). Moreover, we stress that the metric $q_{F,\mu}$ used in S^dV is the one defined in equation (5.1.1). With this choice, the ED polynomial `EDpoly` of $X_{(d)}^\vee$ has degree $d = \mathrm{EDdegree}(X_{(d)})$ in `e^2`.

Unfortunately, with this approach, the symbolic computation of the ED polynomial is very hard also in the symmetric case $\mu = (d)$ for small values of d. The main reason lies in the computation of the critical ideal $I_{\mathrm{crit}}(X_\mu)$. Actually, Theorem 5.0.3 and Proposition 5.1.4 provide a slightly more effective way for computing $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$, described in the following corollary.

Corollary 5.1.7. *Given a general partially symmetric tensor $u = (u_{\alpha_1\cdots\alpha_s}) \in S^\mu V$, define $J \subset \mathbb{C}[\{x_{j,1},x_{j,n_j}\},\{u_{\alpha_1\cdots\alpha_s}\},\varepsilon]$ to be the ideal generated by all the relations in equation (5.0.4), when restricted to $S^\mu V$. Then*

$$(J + \langle q_j(x_j)-1 \mid 1 \leq j \leq s\rangle)\cap\mathbb{C}[u_{\alpha_1\cdots\alpha_s},\varepsilon] = \left(\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)\right).$$

Below we give a second and more efficient Macaulay2 code for computing the ED polynomial of $X_\mu^\vee$ in the case $s=1$, $n_1=2$, $\mu=(d)$, with the alternative approach stated in Corollary 5.1.7 (note the similarity with the code in Example 2.4.2 for $d=3$):

```
R = QQ[x_0, x_1, a_0..a_d, e];
u = sum(d+1, j-> binomial(d,j)*a_j*x_0^(d-j)*x_1^j);
I = ideal(first entries((1/d)*diff(matrix{{x_0, x_1}}, u)-
e*matrix{{x_0, x_1}}));
EDpoly = (eliminate({x_0, x_1},I + ideal(x_0^2+x_1^2-1)))_0;
```

Note that in this case equations (5.0.4) simplify as in (2.1.4), which in turn correspond to the system (2.1.5), where we interpret t as a binary form of degree d. For more details about the output of the above code we refer to Section 5.4. The case of the ED polynomial of the dual of the rational normal curve of degree d is studied in Chapter 2 in the context of Veronese varieties.

5.2 On the zero loci of the extreme coefficients

In this and the following sections, we focus on the lowest and highest coefficients of the ED polynomial of $X_\mu^\vee$ at a given partially symmetric tensor $u \in S^\mu V$.

A crucial role is played by the varieties $X_{\mu,J}$, which were introduced in Definition 5.0.4. First, we investigate the dimension of their respective dual affine cones.

Lemma 5.2.1. *The variety $X_{\mu,J}^\vee$ is a hypersurface in $\mathbb{P}(S^\mu V)$ if and only if*

$$2(n_j-1) \le \dim(X_{\mu,J}) = n_1+\cdots+n_s-s-|J| \tag{5.2.1}$$

for all indices $j \notin J$ such that $\mu_j = 1$.

Proof. Considering a slight modification of [GKZ, Chapter 1, Corollary 5.10], the dual affine cone $X_{\mu,J}^\vee$ is a hypersurface if and only if the following system of inequalities is satisfied:

$$\begin{aligned} \dim(\widetilde{Q}_j) + \operatorname{codim}(\widetilde{Q}_j^\vee) - 1 &\le \dim(X_{\mu,J}) \quad \forall j \in J \\ \dim(X_{(\mu_j)}) + \operatorname{codim}(X_{(\mu_j)}^\vee) - 1 &\le \dim(X_{\mu,J}) \quad \forall j \notin J, \end{aligned} \tag{5.2.2}$$

On one hand, the first set of inequalities is related to the variety $\widetilde{Q}_j$, which is the μ_j-th Veronese embedding into $\mathbb{P}(S^{\mu_j}V_j)$ of the isotropic quadric $Q_j \subset \mathbb{P}(V_j)$. On the other hand, the second set of inequalities is related to the variety $X_{(\mu_j)}$, namely the μ_j-th Veronese embedding of $\mathbb{P}(V_j)$ into $\mathbb{P}(S^{\mu_j}V_j)$.

Through the computations made in Lemma 2.3.3 and Corollary 2.3.7, we verified that $\widetilde{Q}_j^\vee$ is a hypersurface for all $\mu_j \geq 1$. Hence for all $j \in J$ the corresponding inequality in (5.2.2) becomes

$$n_j - 2 \leq \dim(X_{\mu,J}) = n_1 + \cdots + n_s - s - |J|,$$

which is trivially satisfied.

Now consider the second set of inequalities in (5.2.2). Note that $\dim(X_{(\mu_j)}) = \dim(\mathbb{P}(V_j)) = n_j - 1$ for all j. If $\mu_j > 1$, then $X_{(\mu_j)}^\vee$ is a hypersurface in $\mathbb{P}(S^{\mu_j}V_j)$, hence the corresponding inequality in (5.2.2) becomes $n_j - 1 \leq \dim(X_{\mu,J})$, which is trivially satisfied. Otherwise $\mu_j = 1$ and $X_{(\mu_j)}^\vee = \mathbb{P}(V_j)^\vee = \emptyset$. Therefore, we have that $\operatorname{codim}(\mathbb{P}(V_j)^\vee) = n_j$ and the corresponding inequality in (5.2.2) coincides with (5.2.1). □

Lemma 5.2.2. *Assume $n_1 = \cdots = n_s = n$, $V_j = V$ and $Q_j = Q$ for all j. Consider a subset $J \subset [s]$. Then $X_{\mu,J}^\vee$ is a hypersurface unless*

1. *$n = 2$, $|J| = s - 1$ and $\mu_j = 1$ for all $j \notin J$,*
2. *$n \geq 3$, $(s, |J|) \in \{(1,0),(2,1)\}$ and $\mu_j = 1$ for all $j \notin J$.*

Proof. Without loss of generality, we may assume $J = [l] \subset [s]$ with $0 \leq l \leq s$. If $l = s$, then $J = [s]$ and the inequalities in (5.2.1) are vacuously satisfied, hence $X_{\mu,J}^\vee$ is a hypersurface. So in the following we assume $l < s$.

By Lemma 5.2.1, we need to verify that $l \leq (s-2)(n-1)$ for all $j \notin J$ such that $\mu_j = 1$. In particular, note that the preceding inequality is independent of j, hence in the following, we assume that $\mu_j = 1$ for some $j \notin J$.

If $n = 2$, we are left with the inequality $l \leq s - 2$. In particular, in this case $X_{\mu,J}^\vee$ is not a hypersurface if and only if $l = s - 1$. Note that this is the case $X_{\mu,[s-1]} = \operatorname{Seg}_\mu(Q^{\times(s-1)} \times \mathbb{P}(V))$ with $\mu = (\mu_1, \ldots, \mu_{s-1}, 1)$.

Now suppose that $n \geq 3$. If $s = 1$, then by our assumptions $J = \emptyset$ and the inequality $l \leq (s-2)(n-1)$ becomes $n \leq 1$, which is not satisfied. Indeed, this is the trivial case $X_{1,\emptyset} = \mathbb{P}(V)$, and $\mathbb{P}(V)^\vee = \emptyset$. If $s = 2$, then the inequality $l \leq (s-2)(n-1)$ is not satisfied for $(s,l) = (2,1)$. Finally if $s \geq 3$, the inequality $l \leq (s-2)(n-1)$ is satisfied for all $n \geq 3$. □

The vanishing locus of the lowest coefficient $a_0(u) = \operatorname{EDpoly}_{X_\mu^\vee,u}(0)$ is completely described in the following result, which descends immediately from Corollary 4.4.6 and Proposition 4.4.13.

Corollary 5.2.3. *The set of tensors $u \in S^\mu V$ which admit a partially symmetric critical tensor of rank one z such that $q_{F,\mu}(u - z) = 0$ is*

$$\mathcal{V}(a_0) = X_\mu^\vee \cup (X_\mu \cap Q_{F,\mu})^\vee. \tag{5.2.3}$$

Moreover, if $X_\mu^\vee$ is a hypersurface, its equation f_μ has multiplicity two in a_0.

Observe that an immediate consequence of Lemma 5.2.2 is that $X_\mu^\vee = X_{\mu,\emptyset}^\vee$ is always a hypersurface except for the trivial case $s = 1$, $\mu_1 = 1$.

Now we take a closer look at the other component $(X_\mu \cap Q_{F,\mu})^\vee$ in (5.2.3). By definition, the affine cone of the variety $X_\mu \cap Q_{F,\mu}$, which we keep calling $X_\mu \cap Q_{F,\mu}$, is isomorphic to

$$\{(x_1, \ldots, x_s) \in V_1 \times \cdots \times V_s \mid q_{F,\mu}(x_1^{\mu_1} \otimes \cdots \otimes x_s^{\mu_s}) = 0\}.$$

Define

$$Y_{\mu,j} := \{(x_1, \ldots, x_s) \in V_1 \times \cdots \times V_s \mid q_j(x_j) = 0\} \quad \forall j \in [s].$$

Then clearly $X_\mu \cap Q_{F,\mu} \cong Y_{\mu,1} \cup \cdots \cup Y_{\mu,s}$.

Lemma 5.2.4. *For all $j \in [s]$, we have $(Y_{\mu,j})_{red} \cong X_{\mu,\{j\}}$, where $(Y_{\mu,j})_{red}$ denotes the reduced locus of $Y_{\mu,j}$. Moreover if $\mu_j > 1$, then $Y_{\mu,j}^\vee = \emptyset$.*

Proof. It follows immediately by the definition of $Y_{\mu,j}$ that its reduced locus is isomorphic to $X_{\mu,\{j\}}$. Consider any $j \in [s]$ and $x = x_1^{\mu_1} \otimes \cdots \otimes x_s^{\mu_s} \in Y_{\mu,j}$. On one hand, $x \in Q_{F,\mu}$ and the tangent space $T_x Q_{F,\mu}$ is the hyperplane filled by all tensors u such that $q_{F,\mu}(x, u) = 0$. On the other hand, $x \in X_\mu$ and the tangent space of X_μ at x is

$$T_x X_\mu = \left\langle x, v_1 x_1^{\mu_1 - 1} \otimes \cdots \otimes x_s^{\mu_s}, \ldots, x_1^{\mu_1} \otimes \cdots \otimes v_s x_s^{\mu_s - 1} \mid v_k \in V_k \ \forall k \in [s] \right\rangle.$$

For any $k \in [s]$ we pick a nonzero vector $v_k \in V_k$ and we consider the partially symmetric tensor

$$x_1^{\mu_1} \otimes \cdots \otimes v_k x_k^{\mu_k - 1} \otimes \cdots \otimes x_s^{\mu_s} \in T_x X_\mu.$$

Then we get the relation

$$\begin{aligned} q_{F,\mu}(x, x_1^{\mu_1} \otimes \cdots \otimes v_k x_k^{\mu_k - 1} \otimes \cdots \otimes x_s^{\mu_s}) = \\ = q_1(x_1)^{\mu_1} \cdots q_k(v_k, x_k) \cdot q_k(x_k)^{\mu_k - 1} \cdots q_s(x_s)^{\mu_s} \end{aligned}$$

for all $k \in [s]$. In particular, $q_{F,\mu}(x, x_1^{\mu_1} \otimes \cdots \otimes v_k x_k^{\mu_k - 1} \otimes \cdots \otimes x_s^{\mu_s}) = 0$ for all $k \neq j$. Now assume $k = j$. If $\mu_j = 1$, then for a general v_j we get $q(x_j, v_j) \neq 0$ and in turn

$$q_{F,\mu}(x, x_1^{\mu_1} \otimes \cdots \otimes v_j x_j^{\mu_j - 1} \otimes \cdots \otimes x_s^{\mu_s}) \neq 0.$$

This implies that the general point $x \in Y_{\mu,j}$ is smooth if $\mu_j = 1$. Otherwise if $\mu_1 > 1$, then

$$q_{F,\mu}(x, x_1^{\mu_1} \otimes \cdots \otimes v_j x_j^{\mu_j - 1} \otimes \cdots x_s^{\mu_s}) = 0.$$

This means that $T_x X_\mu \subset T_x Q_{F,\mu}$, and every point $x \in Y_{\mu,j}$ is not smooth. Therefore, by the definition of dual variety we have that $Y_{\mu,j}^\vee = \emptyset$ if $\mu_j > 1$. $\square$

An immediate consequence of Lemma 5.2.4 is the identity

$$(X_\mu \cap Q_{F,\mu})^\vee = \bigcup_{j\in[s]\colon \mu_j=1} X^\vee_{\mu,\{j\}}. \tag{5.2.4}$$

Remark 5.2.5. Consider the identity (5.2.4) in the case $\mu = 1^s$. In particular, we have that (calling $Q_{F,s} \coloneqq Q_{F,1^s}$)

$$(X_s \cap Q_{F,s})^\vee = \bigcup_{j=1}^{s} X^\vee_{s,\{j\}},$$

where

$$X_{s,\{j\}} = \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_{j-1}) \times Q_j \times \mathbb{P}(V_{j+1}) \times \cdots \times \mathbb{P}(V_s)$$

for all $j \in [s]$. By Lemma 5.2.1, the dual affine cone $X^\vee_{s,\{j\}}$ is a hypersurface if and only if

$$2(n_k - 1) \le n_1 + \cdots + n_s - s - 1 \quad \forall k \in [s],\ k \neq j.$$

On the contrary, $X^\vee_{s,\{j\}}$ is not a hypersurface if and only if there exists $k \in [s]$, $k \neq j$ such that

$$2(n_k - 1) \ge n_1 + \cdots + n_s - s.$$

Without loss of generality, assume that $n_1 \ge \cdots \ge n_s$. Suppose that $(X_s \cap Q_{F,s})^\vee$ is not a hypersurface. In particular, $X^\vee_{s,\{1\}}$ and $X^\vee_{s,\{2\}}$ are not hypersurfaces. These facts give the inequalities

$$n_2 \ge n_1 + \sum_{j=3}^{s}(n_j - 1), \quad n_1 \ge n_2 + \sum_{j=3}^{s}(n_j - 1),$$

namely

$$n_1 \ge n_2 + \sum_{j=3}^{s}(n_j - 1) \ge n_1 + 2\sum_{j=3}^{s}(n_j - 1).$$

Therefore $\sum_{j=3}^{s}(n_j - 1) = 0$, namely $n_3 = \cdots = n_s = 1$ and $n_1 = n_2$.

The conclusion of this remark is the following: if $(X_s \cap Q_{F,s})^\vee$ is not a hypersurface, then $s = 2$ and $n_1 = n_2$, so we are in the square matrix case. Nevertheless, here $X_2^\vee$ is the hypersurface defined by the determinant. In particular, the polynomial $\mathrm{EDpoly}_{X_s^\vee,u}(0)$ admits an extra coefficient (different from f_s^2) unless we are in the matrix case. This marks another big difference between the matrix case and all the other possible tensor formats.

In the second part of this section, we are interested in giving a complete description of the vanishing locus of the highest coefficient $a_N(u)$ of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\sigma)$. In the following proof, we use the notations introduced in Section 1.1.

Proposition 5.2.6. *The following inclusion holds true:*

$$\mathcal{V}(a_N) \subset \bigcup_{j\,:\,\mu_j>1} X_{\mu,\{j\}}^\vee \cup \bigcup_{|J|>1} X_{\mu,J}^\vee.$$

Proof. For any nonzero partially symmetric binary tensor $u \in S^\mu V$, we indicate by $\langle u\rangle \in H_\infty$ the line spanned by u. Now assume that $a_N(u) = 0$. From this fact, from Corollary 4.3.7 and (5.1.4), there exists a sequence $\{u_k\} \subset S^\mu V$ such that $u_k \to u$ and two corresponding sequences $\{f_k\} \subset X_\mu^\vee$ and $\{u_k - f_k\} \subset X_\mu$ of critical points for δ_{F,μ,u_k} on $X_\mu^\vee$ and X_μ, respectively, such that $\mathrm{EDpoly}_{X_\mu^\vee,u_k}(\varepsilon_k^2) = 0$ and $\mathrm{EDpoly}_{X_\mu,u_k}(\eta_k^2) = 0$ when $\varepsilon_k^2 = q_{F,\mu}(f_k - u_k)$ and $\eta_k^2 = q_{F,\mu}(f_k)$ diverge simultaneously (see Figure 5.2). In particular, we have that

$$\langle u_k - f_k\rangle \in [(T_{f_k}X_\mu^\vee)_\infty]^\perp = (N_{f_k}X_\mu^\vee)_\infty, \quad \langle f_k\rangle \in [(T_{t_k-f_k}X_\mu)_\infty]^\perp = (N_{u_k-f_k}X_\mu)_\infty$$

for all k, where the external duals are taken in the projective subspace H_∞. Up

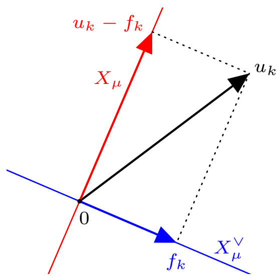

Figure 5.2: The sequences $\{f_k\} \subset X_\mu^\vee$ and $\{u_k - f_k\} \subset X_\mu$.

to subsequences, we may assume that

$$\lim_{k\to\infty} \langle f_k\rangle =: \langle f\rangle \in (X_\mu)_\infty^\vee, \text{ for some } f \in S^\mu V. \tag{5.2.5}$$

In the topology of the compact space $\overline{X_\mu} = X_\mu \cup (X_\mu)_\infty$ we still have $f_k \to \langle f\rangle \in (X_\mu)_\infty^\vee$, more precisely in $\mathbb{P}(\mathbb{C}\oplus V)$ we have $[(1, f_k)] \to [(0, f)]$. Consequently, we have that

$$\lim_{k\to\infty} \langle u_k - f_k\rangle = \langle u - f\rangle \in (X_\mu)_\infty.$$

Repeating the argument of Proposition 4.3.4, one verifies that (*i*) $q_{F,\mu}(\langle u-f\rangle)=0$, namely $\langle u-f\rangle\in(Q_{F,\mu})_\infty$, where in this case $q_{F,\mu}$ is the quadratic form defined in H_∞.

(*ii*) $T_{\langle u-f\rangle}(X_\mu)_\infty\subset T_{\langle u-f\rangle}(Q_{F,\mu})_\infty$.

Remembering that $u-f$ is a decomposable tensor, then $\langle u-f\rangle=[x_1^{\mu_1}\otimes\cdots\otimes x_s^{\mu_s}]$ for some vectors $x_i\in V_i$. By (*i*), necessarily $q_j(x_j)=0$ for some j, say $j=1$. Now there are two possible cases to study.

If $\mu_1=1$, then we may suppose that $q_1(x_1)=0$. The inclusion in (*ii*) implies that for any $\langle v\rangle\in T_{\langle u-f\rangle}(X_\mu)_\infty$, we have $q_{F,\mu}(\langle v\langle,\langle u-f\rangle)=0$, where in this case $q_{F,\mu}(\cdot,\cdot)$ is the bilinear form on $S^\mu V$ restricted to H_∞. More explicitly, we may write

$$v=\sum_{i=1}^s x_1\otimes\cdots\otimes x_{i-1}^{\mu_{i-1}}\otimes\xi_i\cdot x_i^{\mu_i-1}\otimes x_{i+1}^{\mu_{i+1}}\otimes\cdots\otimes x_s^{\mu_s}$$

for some $\xi_i\in V_i$ for all $i\in[s]$. Then

$$\begin{aligned}0&=q_{F,\mu}(\langle v\rangle,\langle u-f\rangle)\\&=\sum_{i=1}^s q_{F,\mu}([x_1\otimes\cdots\otimes x_{i-1}^{\mu_{i-1}}\otimes\xi_i\cdot x_i^{\mu_i-1}\otimes x_{i+1}^{\mu_{i+1}}\otimes\cdots\otimes x_s^{\mu_s}],[x_1^{\mu_1}\otimes\cdots\otimes x_s^{\mu_s}])\\&=\sum_{i=1}^s q_1(x_1)\cdots q_{i-1}(x_{i-1})^{\mu_{i-1}}\cdot q_i(\xi_i,x_i)\cdot q_i(x_i)^{\mu_i-1}\cdot q_{i+1}(x_{i+1})^{\mu_{i+1}}\cdots q_s(x_s)^{\mu_s}.\end{aligned}$$

By our assumption $q_1(x_1)=0$. Then necessarily we get the identity

$$q_1(\xi_1,x_1)\cdot q_2(x_2)^{\mu_2}\cdots q_s(x_s)^{\mu_s}=0.$$

Taking v sufficiently general, we may suppose that $q_1(\xi_1,x_1)\neq 0$. Therefore there exists at least one more index $i\neq 1$ such that $q_i(x_i)=0$. In particular $u-f\in X_{\mu,J}$, where $J=\{j\in[s]\mid q_j(x_j)=0\}$.

Otherwise $\mu_1>1$, then $u-f\in X_{\mu,\{1\}}$, otherwise $u-f\in X_{\mu,J}$ for some $J\subset[s]$ such that $|J|>1$.

Now assume that $\mu_1=1$ (the proof in the case $\mu_1>1$ is the same). We show that necessarily $t\in X_{\mu,J}^\vee$. By definition, $X_{\mu,J}^\vee=\overline{S_{\mu,J}}$, where

$$S_{\mu,J}=\bigcup_{x\in(X_{\mu,J})_{\mathrm{sm}}}N_xX_{\mu,J}.$$

Now suppose that $y\in T_{u-f}X_{\mu,J}$. By the previous claims, $q_{F,\mu}(y,u-f)=0$. On the other hand, we have $q_{F,\mu}(y,f)=0$, since X_μ and $X_\mu^\vee$ are affine cones. Then

$$q_{F,\mu}(y,u)=\widetilde{q}(y,u-f)+\widetilde{q}(y,f)=0+0=0.$$

This means that $u\in S_{\mu,J}$, hence $u\in X_{\mu,J}^\vee$. $\square$

Proposition 5.2.6 gives the following piece of information: if a partially symmetric tensor $u \in S^\mu V$ does not admit the expected number of critical points, then it must have a critical point with a precise isotropic structure.

5.3 The product of the singular values of a tensor

For all $J \subset [s]$, we recall that $f_{\mu,J}$ denotes the equation of $X_{\mu,J}^\vee$, when it is a hypersurface. Otherwise, we set $f_{\mu,J} := 1$. Moreover, we use the notation $f_\mu := f_{\mu,\emptyset}$. For brevity, we define $\theta_{\mu,J} := \deg(f_{\mu,J})$ and $\theta_\mu := \deg(f_\mu)$.

In Lemma 5.2.2 we determined the conditions for the variety $X_{\mu,J}^\vee$ to be a hypersurface or not. When $X_{\mu,J}^\vee$ is a hypersurface, its degree (that is, $\theta_{\mu,J}$) coincides with the polar class $\delta_0(X_{\mu,J})$ introduced in Section 1.5. In turn, since $X_{\mu,J}$ is a smooth variety, the invariant $\delta_0(X_{\mu,J})$ may be written in terms of the Chern classes of $X_{\mu,J}$ via the formulas in (1.8.1) for $i = 0$. Setting $m = m(\mu, J) = \dim(X_{\mu,J})$ for brevity, we have

$$\delta_0(X_{\mu,J}) = \sum_{k=0}^{m} (-1)^k (m + 1 - k) c_k(X_{\mu,J}) \cdot h^{m-k}, \tag{5.3.1}$$

where $h = c_1(\mathcal{O}_{X_{\mu,J}}(1))$ is the hyperplane class.

Our goal is to write explicitly the Chern classes $c_k(X_{\mu,J})$. Let $\tilde{H}_j$ be the hyperplane class of $\mathbb{P}(V_j)$ and denote with $\tilde{h}_j$ the restriction of $\tilde{H}_j$ to the variety Y_j. In particular, recalling that $X_{\mu,J} = \mathrm{Seg}_\mu(Y_1 \times \cdots \times Y_s)$ (see Definition 5.0.4) we have the relations

$$\tilde{h}_j^{n_j - \chi_J(j)} = 0 \quad \text{for all } j \in [s],$$

where χ_J is the characteristic function of the subset J. Keeping in mind all these relations, we have that

$$\begin{aligned} c(X_{\mu,J}) &= \prod_{j=1}^{s} c(Y_j) = \prod_{j=1}^{s} \left(\sum_{i=0}^{n_j - 1} c_i(Y_j) \right) = \prod_{j=1}^{s} \left(\sum_{i=0}^{n_j - 1} \gamma_i(Y_j) \tilde{h}_j^i \right) \\ &= \sum_{k=0}^{m} \left[\sum_{|\beta| = k} \left(\prod_{l=1}^{s} \gamma_{\beta_l}(Y_l) \right) \tilde{h}^\beta \right] = \sum_{k=0}^{m} \sum_{|\beta|=k} \gamma_\beta \tilde{h}^\beta, \end{aligned}$$

where $\tilde{h}^\beta = \tilde{h}_1^{\beta_1} \cdots \tilde{h}_s^{\beta_s}$ and using (1.7.1) we have

$$\gamma_{\beta_l}(Y_l) = \begin{cases} \sum_{i=0}^{\beta_l} \binom{n_l}{i} (-2)^{\beta_l - i} & \text{if } l \in J \\ \binom{n_l}{\beta_l} & \text{if } l \notin J. \end{cases}, \quad \gamma_\beta = \gamma_\beta(X_{\mu,J}) := \prod_{l=1}^{s} \gamma_{\beta_l}(Y_l).$$

Therefore, the k-th Chern class of $X_{\mu,J}$ is

$$c_k(X_{\mu,J}) = \sum_{|\beta|=k} \gamma_\beta \tilde{h}^\beta \quad \forall\ 0 \leq k \leq m.$$

Then, taking into account the Segre-Veronese embedding Seg_μ, we have that $h = \mu_1\tilde{h}_1 + \cdots + \mu_s\tilde{h}_s$, hence the power h^{m-k} in equation (5.3.1) becomes

$$h^{m-k} = (\mu_1\tilde{h}_1 + \cdots + \mu_s\tilde{h}_s)^{m-k} = \sum_{|\omega|=m-k} \binom{m-k}{\omega} \mu^\omega \tilde{h}^\omega.$$

Summing up, the product $c_k(X_{\mu,J}) \cdot h^{m-k}$ in (5.3.1) may be written as

$$\begin{aligned} c_k(X_{\mu,J}) \cdot h^{m-k} &= \sum_{|\beta|=k} \gamma_\beta \tilde{h}^\beta \cdot \sum_{|\omega|=m-k} \binom{m-k}{\omega} \mu^\omega \tilde{h}^\omega \\ &= (m-k)! \left[\sum_{|\beta|=k} \eta_\beta \gamma_\beta \mu^{\underline{n}-\beta-\chi_J} \right] \tilde{h}^m \end{aligned} \tag{5.3.2}$$

where we observed that, for every k, the product $\tilde{h}^\beta \cdot \tilde{h}^\omega$ is nonzero if and only if $\beta + \omega = \underline{n} - 1^s - \chi_J$ (namely $\beta_l + \omega_l = n_l - 1 - \chi_J(l)$ for all $l \in [s]$) and consequently that $|\beta + \omega| = |\underline{n} - 1^s - \chi_J| = m$. Moreover, we defined

$$\eta_\beta := \begin{cases} \frac{1}{(\underline{n}-1^s-\beta-\chi_J)!} = \prod_{l=1}^s \frac{1}{(n_l-1-\beta_l-\chi_J(l))!} & \text{if } n_l - \beta_l - \chi_J(l) \geq 1\ \forall\ l \in [s] \\ 0 & \text{otherwise.} \end{cases}$$

Finally we observe that, from the definition of $X_{\mu,J}$, we have $\deg(\tilde{h}^m) = 2^{|J|}$. Merging together formulas (5.3.1) and (5.3.2) we get the following result

Proposition 5.3.1. *If $X_{\mu,J}^\vee$ is a hypersurface, then its degree is*

$$\theta_{\mu,J} = 2^{|J|} \sum_{k=0}^{m} (-1)^k (m+1-k)! \sum_{|\beta|=k} \eta_\beta \gamma_\beta \mu^{\underline{n}-1^s-\beta-\chi_J}. \tag{5.3.3}$$

We consider two explicit examples of the above formula, which are useful for example in Remark 5.3.19. See also Corollary 5.4.1 for a simplified version of (5.3.3) in the binary case.

Example 5.3.2. Consider the case $s = 2$, $n_1 = n_2 = n$ and the vector of degrees $\mu = (\mu_1, \mu_2)$. The degree of the dual affine cone of $X_{\mu,\{1\}} = \mathrm{Seg}_\mu(Q \times \mathbb{P}(V))$ is

$$\theta_{\mu,\{1\}} = 2 \sum_{k=0}^{2n-3} (-1)^k (2n-2-k)! \sum_{\substack{\beta_1 \leq n-2 \\ \beta_2 \leq n-1 \\ |\beta|=k}} \frac{\left[\sum_{l=0}^{\beta_1} \binom{n}{l}(-2)^{\beta_1-l}\right] \binom{n}{\beta_2}}{(n-\beta_1-2)!(n-1-\beta_2)!} \mu_1^{n-\beta_1-2} \mu_2^{n-1-\beta_2}.$$

Example 5.3.3. Consider the case $s=3$, $n_1=n_2=n_3=n$ and the vector of degrees $\mu=(\mu_1,\mu_2,\mu_3)$. Below we compute the degrees of the dual affine cones of $X_{\mu,\{1\}}=\mathrm{Seg}_\mu(Q\times\mathbb{P}(V)\times\mathbb{P}(V))$ and $X_{\mu,\{1,2\}}=\mathrm{Seg}_\mu(Q\times Q\times\mathbb{P}(V))$, respectively:

$$\theta_{\mu,\{1\}}=2\cdot\sum_{s=0}^{3n-4}(-1)^s(3n-3-s)!\cdot$$
$$\cdot\sum_{\substack{\beta_1\le n-2\\ \beta_2,\beta_3\le n-1\\ |\beta|=k}}\frac{\sum_{l=0}^{\beta_1}\binom{n}{l}(-2)^{\beta_1-l}}{(n-\beta_1-2)!}\left[\prod_{i=2}^{3}\frac{\binom{n}{\beta_i}}{(n-\beta_i-1)!}\right]\mu^{\underline{n}-1^3-\beta-\chi_{\{1\}}},$$

$$\theta_{\mu,\{1,2\}}=4\cdot\sum_{s=0}^{3n-5}(-1)^s(3n-4-s)!\cdot$$
$$\cdot\sum_{\substack{\beta_1,\beta_2\le n-2\\ \beta_3\le n-1\\ |\beta|=k}}\left[\prod_{i=1}^{2}\frac{\sum_{l=0}^{\beta_i}\binom{n}{l}(-2)^{\beta_i-l}}{(n-\beta_i-2)!}\right]\frac{\binom{n}{\beta_3}\mu^{\underline{n}-1^3-\beta-\chi_{\{1,2\}}}}{(n-\beta_3-1)!}.$$

For the rest of the chapter, we assume that $n_1=\cdots=n_s=n$. Moreover, without loss of generality, we may identify all vector spaces $V_j=V$ as well as the quadratic forms $q_j=q$ and the corresponding isotropic quadrics $Q_j=Q$.

From now on, we consider an integer d and all degree vectors $\mu=(\mu_1,\ldots,\mu_s)$ such that $|\mu|=d$, namely all *partitions* μ of d. We use the symbol $\mu\vdash d$ to denote any partition of d. The two trivial partitions are $\mu=(1,\ldots,1)$ (d times), denoted by $\mu=1^d$, and the symmetric partition $\mu=(d)$.

Proposition 5.2.6 carries the following fact: if $a_N=g_1^{\beta_1}\cdots g_r^{\beta_r}$ is the irreducible factorization of the highest coefficient a_N of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$, then the g_k's are either proportional to $f_{\mu,\{j\}}$ for some $1\le j\le s$ such that $\mu_j>1$, or to $f_{\mu,J}$ for some $J\subset[s]$ such that $|J|>1$. In particular, we may write, up to scalars,

$$a_N(u)=\prod_{j\,:\,\mu_j>1}f_{\mu,\{j\}}(u)^{\alpha_{\mu,\{j\}}}\cdot\prod_{|J|>1}f_{\mu,J}(u)^{\alpha_{\mu,J}},\quad \alpha_{\mu,J}\ge 0. \tag{5.3.4}$$

Moreover, from Lemma 5.2.3 we have that, up to scalars,

$$a_0(u)=f_\mu(u)^2\cdot\prod_{j\,:\,\mu_j=1}f_{\mu,\{j\}}(u)^{-\alpha_{\mu,\{j\}}},\quad \alpha_{\mu,\{j\}}\le 0\quad\text{when}\quad \mu_j=1. \tag{5.3.5}$$

The reason for the negative sign in the notation of the integers $\alpha_{\mu,J}$ in (5.3.5) is clarified in the proof of Theorem 5.0.5. When $\mu=1^d$, we observe that

$\mathrm{EDpoly}_{X_d^\vee,u}(\varepsilon^2)$ is invariant under the action of the symmetric group Σ_d on the entries of $u = (u_{i_1\cdots i_d})$. More precisely, we use the notation

$$\alpha_{d,j} := \alpha_{1^d,J} \quad \text{for all} \quad J \subset [d]. \tag{5.3.6}$$

A nontrivial task is showing what are the exponents $\alpha_{\mu,J}$ appearing in the expressions of a_N and a_0. Actually, Corollary 5.3.13 simplifies a lot our problem, stating that $\alpha_{\mu,J} \in \{\alpha_{d,1}, \ldots, \alpha_{d,d}\}$ for all $\mu \vdash d$ and all $J \subset [s]$. In order to determine the exponents $\alpha_{\mu,J}$, we consider the following list of linear conditions whose coefficients are the degrees $\theta_{\mu,J}$, which descend from relations (5.3.4), (5.3.5) and the identity $\deg(a_N) + 2N = \deg(a_0)$, where we already know from Corollary 5.2.3 that $\alpha_{\mu,\emptyset} = \alpha_\mu = -2$:

$$\sum_{J\subset[s]} \alpha_{\mu,J}\theta_{\mu,J} + 2N = 0 \quad \text{for all} \quad \mu \vdash d. \tag{5.3.7}$$

The main idea of the proof of Corollary 5.3.13 is related to partial symmetrizations of the ED polynomial of $X_d^\vee$. To this aim, we recall some definitions and preliminary results.

Definition 5.3.4. Let $\mu = (\mu_1, \ldots, \mu_s)$ be a partition of d. A *symmetrization* of μ is any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of d with

$$\lambda_j = \mu_{i_{j,1}} + \cdots + \mu_{i_{j,l_j}} \quad \text{for all } 1 \le j \le r, \tag{5.3.8}$$

where $\mu = (\mu_{i_{1,1}}, \ldots, \mu_{i_{1,l_1}}, \ldots, \mu_{i_{r,1}}, \ldots, \mu_{i_{r,l_r}})$ after a possible permutation. We write $\lambda \prec \mu$ to indicate that λ is a symmetrization of μ. We stress that different choices of μ_i appearing in different sums (5.3.8) yield different symmetrizations of μ, even if some of the μ_i are equal.

Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ such that $\lambda \prec \mu$, we may consider the inclusion $S^\lambda V \subset S^\mu V$. Since the group $\mathrm{GL}(V)$ is linearly reductive (see Definition 3.1.2), $S^\mu V$ is a $\mathrm{GL}(V)$-module and $S^\lambda V$ is a submodule of $S^\mu V$, there exists a unique $\mathrm{GL}(V)$-invariant complement to $S^\lambda V$ in $S^\mu V$, denoted by $W^{\lambda,\mu}$ (see [LeP, Lemma 6.2.2]). We have a natural projection

$$\pi_{\lambda,\mu} : \mathbb{P}(S^\mu V) \dashrightarrow \mathbb{P}(S^\lambda V)$$

from $W^{\lambda,\mu}$, whose definition on decomposable elements is

$$\begin{aligned}\pi_{\lambda,\mu}\left([a_{i_{1,1}}^{\mu_{i_{1,1}}} \otimes \cdots \otimes a_{i_{1,t_1}}^{\mu_{i_{1,t_1}}} \otimes \cdots \otimes a_{i_{s,1}}^{\mu_{i_{s,1}}} \otimes \cdots \otimes a_{i_{t,t_s}}^{\mu_{i_{t,t_s}}}]\right) :=\\ := [a_{i_{1,1}}^{\mu_{i_{1,1}}} \cdots a_{i_{1,t_1}}^{\mu_{i_{1,t_1}}} \otimes \cdots \otimes a_{i_{s,1}}^{\mu_{i_{s,1}}} \cdots a_{i_{t,t_s}}^{\mu_{i_{t,t_s}}}].\end{aligned}$$

The projection $\pi_{\lambda,\mu}$ induces another projection $\mathbb{P}(S^e(S^\mu V)) \dashrightarrow \mathbb{P}(S^e(S^\lambda V))$, which we keep calling $\pi_{\lambda,\mu}$. The λ-*symmetrization* of a degree e homogeneous polynomial $f \in S^e(S^\mu V)$ is the image $\pi_{\lambda,\mu}(f)$ under the map already defined.

The case $\mu \vdash d$, $\lambda = (d) \vdash d$ is studied in detail by Oeding in [Oed]. Here we write $\pi_\mu := \pi_{(d),\mu}$ for brevity. We consider the projections

$$\mathrm{Chow}_\mu \mathbb{P}(V) := \pi_\mu(X_\mu) \subset \mathbb{P}(S^d V), \tag{5.3.9}$$

classically known as *multiple root loci* of $\mathbb{P}(S^d V)$. They are filled by (classes of) symmetric tensors of the form $l_1^{\mu_1} \cdots l_s^{\mu_s}$ for some linear forms $l_1, \ldots, l_s$. When $\mu = (d)$, the corresponding multiple root locus is the Veronese variety $X_{(d)}$. Oeding derived the following striking factorization formula for the μ-discriminant f_μ in terms of equations of dual multiple loci.

Theorem 5.3.5. *[Oed, Theorem 1.2] Let $\mu \vdash d \geq 2$, and let V be a complex vector space of dimension n with $n \geq 1$. Then*

$$X_\mu^\vee \cap \mathbb{P}(S^d V^*) = \bigcup_{\gamma \prec \mu} [\mathrm{Chow}_\gamma \mathbb{P}(V)]^\vee.$$

In particular,

$$\pi_\mu(f_\mu) = \prod_{\gamma \prec \mu} \Phi_{\gamma,n}^{M_{\gamma,\mu}},$$

where $\Phi_{\gamma,n}$ is the equation of $[\mathrm{Chow}_\gamma \mathbb{P}(V)]^\vee$ when it is a hypersurface in $\mathbb{P}(S^d V^*)$, and the multiplicity $M_{\gamma,\mu}$ is the number of partitions μ such that γ is a symmetrization of μ.

A more general result by Holweck and Oeding which we apply is the following.

Theorem 5.3.6. *[HO, Theorem 2.2] Let $X \subset \mathbb{P}(V)$ and $Y \subset \mathbb{P}(A)$ be algebraic varieties with $V = A \oplus B$. If for each smooth point $[y] \in Y$ there is a smooth point $[x] \in X$ such that $\pi_B(T_x X) \subset T_y Y$ (where in the inclusion X and Y are seen as affine cones), then*

$$Y^\vee \subset X^\vee \cap \mathbb{P}(A^*).$$

Moreover if $X^\vee$ and $Y^\vee$ are hypersurfaces defined respectively by polynomials f and g and, for every general point $[h] \in Y^\vee$, $H = \mathcal{V}(h)$, viewed as a hyperplane in $\mathbb{P}(V)$, is a point of multiplicity m of $X^\vee$, then g^m divides $\pi_B(f)$.

The following result an almost immediate consequence of Theorem 5.3.5 and is an instance of Theorem 5.3.6. It relates the λ-symmetrization of the μ-discriminant (the equation of $X_\mu^\vee$), where $\lambda \prec \mu$ are two partitions of d.

Proposition 5.3.7. *Let $\lambda \prec \mu$ be two partitions of d. Then*

$$X_\lambda^\vee \subset X_\mu^\vee \cap \mathbb{P}(S^\lambda V).$$

Moreover, f_λ is a factor of multiplicity one in $\pi_{W^{\lambda,\mu}}(f_\mu)$.

Proof. By Theorem 5.3.5, we have the inclusions $X_{(d)}^\vee \subset X_\mu^\vee \cap \mathbb{P}(S^d V)$ and $X_{(d)}^\vee \subset X_\lambda^\vee \cap \mathbb{P}(S^d V)$. We stress that, taking into account Definition 1.3.1, we are identifying $\mathbb{P}((S^\mu V/S^d V)^\perp)$ with $S^d V$ and by abuse of notation write $X_{(d)}^\vee \subset X_\mu^\vee \cap \mathbb{P}(S^d V)$. Again by Theorem 5.3.5, the discriminant $f_{(d)}$ is a factor of multiplicity one in both the polynomials $\pi_{(d),\mu}(f_\mu)$ and $\pi_{(d),\lambda}(f_\lambda)$. □

Remark 5.3.8. Fix a partition $\mu \vdash d$ and a vector space $S^\mu V$. As a completion of the remark in Definition 5.3.4, we stress that distinct symmetrizations $\lambda_1 \neq \lambda_2$ of μ yield isomorphic but distinct subspaces $S^{\lambda_1} V \neq S^{\lambda_2} V$ and, in turn, distinct (but isomorphic) varieties X_{λ_1} and X_{λ_2}, even if $\lambda_1 = \lambda_2$ as partitions of d. For example, when we write $X_{(2,1)} \subset X_{(1,1,1)}$ we do take into account which components of $\mu = (1,1,1)$ we are summing to get $\lambda = (2,1)$. Nevertheless, we choose to omit this assumption in our notation. In terms of projections $\pi_{\lambda,\mu}$, different symmetrizations $\lambda_1 \neq \lambda_2$ of μ yield different maps $\pi_{\lambda_1,\mu} \neq \pi_{\lambda_2,\mu}$.

The following key fact shows how the previous result shifts from dual Segre -Veronese varieties to their respective ED polynomials.

Proposition 5.3.9. *Let $\lambda \prec \mu$ be two partitions of d and let $u \in S^\lambda V$. Then the ED polynomial of $X_\lambda^\vee$ at u divides with multiplicity one the λ-symmetrization of the ED polynomial of $X_\mu^\vee$ at u.*

Proof. Let $u \in S^\lambda V$ and let $x \in X_\lambda \subset X_\mu$ be a λ-symmetric singular tensor for u. We show that x is also a μ-symmetric singular tensor for u. According to the decomposition $S^\mu V = S^\lambda V \oplus W^{\lambda,\mu}$, the tangent space of X_μ at x decomposes in a good way as $T_x X_\mu = T_x X_\lambda \oplus W$ for some subspace $W \subset W^{\lambda,\mu}$. In particular, any tangent vector of X_μ at x may be written in a unique way as $y = y_\lambda + w$ for some $y_\lambda \in T_x X_\lambda$ and $w \in W$. Then we have

$$q_{F,\mu}(u-x,y) = q_{F,\mu}(u-x,y_\lambda) + q_{F,\mu}(u-x,w) = 0 + 0 = 0$$

for all $y \in T_x X_\mu$. Thanks to Proposition 5.1.4, this fact means, at the level of ED polynomials, that there exists an integer $\beta \geq 0$ such that

$$\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2) = [\mathrm{EDpoly}_{X_\lambda^\vee,u}(\varepsilon^2)]^\beta \cdot h,$$

and $\mathrm{EDpoly}_{X_\lambda^\vee,u}(\varepsilon^2)$ is not a factor of h. By Lemma 5.2.3, the equations of $X_\mu^\vee$ and $X_\lambda^\vee$, namely f_μ and f_λ, appear with multiplicity 2 in the lowest terms of

$\mathrm{EDpoly}_{X^\vee_\mu,u}(\varepsilon^2)$ and $\mathrm{EDpoly}_{X^\vee_\lambda,u}(\varepsilon^2)$. Moreover, by Proposition 5.3.7, f_λ is a factor of multiplicity one in $\pi_{\lambda,\mu}(f_\mu)$. This implies that $\beta = 1$. □

Definition 5.3.10. Let $\mu = (\mu_1, \dots, \mu_s) \vdash d$. Consider a subset of indices $J \subset [s]$. We say that a partition $\lambda = (\lambda_1, \dots, \lambda_r) \prec \mu$ is *compatible* with J if for all $1 \le j \le r$ we may write

$$\lambda_j = \mu_{i_{j,1}} + \cdots + \mu_{i_{j,l_j}}$$

and the subset of indices $I_{\lambda,j} := \{i_{j,1}, \dots, i_{j,l_j}\}$ is either contained in J or in $[s] \setminus J$. Moreover, we define $J_\lambda := \{j \in [r] \mid I_{\lambda,j} \subset J\}$.

Example 5.3.11. Let $\mu = (\mu_1, \dots, \mu_s) \vdash d$. Consider a nonempty subset $J = \{p_1, \dots, p_n\} \subset [s]$ and its complement $[s] \setminus J = \{q_1, \dots, q_{s-n}\}$. Then any of the following partitions $\lambda \prec \mu$, which we use in Corollary 5.3.13 and Lemmas 5.3.15 and 5.3.17, is compatible with J:

1. $\lambda = (d) = (\mu_1 + \cdots + \mu_s)$ if $J = [s]$,
2. $\lambda = (\lambda_1, \lambda_2) = (\mu_{p_1} + \cdots + \mu_{p_n}, \mu_{q_1} + \cdots + \mu_{q_{s-n}})$,
3. $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (\mu_{p_1} + \cdots + \mu_{p_{n-1}}, \mu_{p_n}, \mu_{q_1} + \cdots + \mu_{q_{s-n}})$,
4. $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (\mu_{p_1} + \cdots + \mu_{p_n}, \mu_{q_1} + \cdots + \mu_{q_{s-n-1}}, \mu_{q_{s-n}})$.

Definition 5.3.10 is useful for introducing the following variation of Proposition 5.3.7.

Proposition 5.3.12. *Let $\mu = (\mu_1, \dots, \mu_s)$ be a partition of d. Consider a subset of indices $J \subset [s]$ and a partition $\lambda = (\lambda_1, \cdots, \lambda_r) \prec \mu$ compatible with J. If $X^\vee_{\mu,J}$ is a hypersurface, then $X^\vee_{\lambda,J_\lambda}$ is a hypersurface too, and*

$$X^\vee_{\lambda,J_\lambda} \subset X^\vee_{\mu,J} \cap \mathbb{P}(S^\lambda V). \tag{5.3.10}$$

Moreover, f_{λ,J_λ} is a factor of multiplicity one in $\pi_{\lambda,\mu}(f_{\mu,J})$.

Proof. In the first part, we derive the inclusion (5.3.10) by applying Theorem 5.3.6. Since the partition λ is compatible with J, we conclude immediately from Definition 5.3.10 that $X_{\lambda,J_\lambda} \subset X_{\mu,J} \cap \mathbb{P}(S^\lambda V)$. Observe that $X_{\mu,J}$ and X_{λ,J_λ} are smooth varieties. Pick any point $y = y_1^{\lambda_1} \otimes \cdots \otimes y_r^{\lambda_r} \in X_{\lambda,J_\lambda}$. First of all, we have $\pi^{-1}_{\lambda,\mu}(y) = \{x\}$, where $x = x_1^{\mu_1} \otimes \cdots \otimes x_s^{\mu_s} \in X_{\mu,J}$ is such that $x_i = y_j$ for all $i \in I_{\lambda,j}$ and for all $j \in [s]$. Secondly, we show that the projection $\pi_{\lambda,\mu}(T_x X_{\mu,J})$ of the tangent space of $X_{\mu,J}$ at x is contained in $T_y X_{\lambda,J_\lambda}$. By definition,

$$T_x X_{\mu,J} = \left\langle x, x_1^{\mu_1} \otimes \cdots v_i x_i^{\mu_i - 1} \otimes \cdots \otimes x_s^{\mu_s} \;\middle|\; v_i \in T_{x_i} Q \text{ if } i \in J,\ v_i \in V \text{ if } i \notin J \right\rangle.$$

If $i \in J$, then $x_1^{\mu_1} \otimes \cdots v_i x_i^{\mu_i - 1} \otimes \cdots \otimes x_s^{\mu_s} = x$ and we already showed that $\pi_{\lambda,\mu}(x) = y \in T_y X_{\lambda,J_\lambda}$. Now assume that $j \notin J$. Then

$$\pi_{\lambda,\mu}(x_1^{\mu_1} \otimes \cdots v_i x_i^{\mu_i - 1} \otimes \cdots \otimes x_s^{\mu_s}) = y_1^{\lambda_1} \otimes \cdots \otimes v_i y_j^{\lambda_j - 1} \otimes \cdots \otimes y_r^{\lambda_r} \in T_y X_{\lambda,J_\lambda},$$

where the index j is such that $i \in I_{\lambda,j}$. Hence $\pi_{\lambda,\mu}(T_x X_{\mu,J}) \subset T_y X_{\lambda,J_\lambda}$ as we wanted. Therefore the inclusion (5.3.10) follows by Theorem 5.3.6.

Now suppose by absurd that $X_{\lambda,J_\lambda}^\vee$ is not a hypersurface. Then necessarily $|J_\lambda| = r-1$ and $\lambda_k = 1$ for $k \notin J_\lambda$, by Lemma 5.2.2. By Definition 5.3.10, we have that $|J| = s-1$ and $\mu_j = 1$ for all $j \notin J$, that is, $X_{\mu,J}^\vee$ is not a hypersurface by Lemma 5.2.2, a contradiction.

Finally we assume that $X_{\mu,J}^\vee$ is a hypersurface and we verify that the equation f_{λ,J_λ} of $X_{\lambda,J_\lambda}^\vee$ appears with multiplicity one in $\pi_{\lambda,\mu}(f_{\mu,J})$. We mimick the argument used by Oeding in the second proof of [Oed, Lemma 5.1]. Let h be a general vector in $X_{\lambda,J_\lambda}^\vee$. In particular, h is a smooth point of $X_{\lambda,J_\lambda}^\vee$, so there is exactly one point $y \in X_{\lambda,J_\lambda}$ such that $h \in N_y X_{\lambda,J_\lambda}$. By the previous part of the proof, we know that $\pi_{\lambda,\mu}^{-1}(y) = \{x\}$ for some $x \in X_{\mu,J}$ and that $h \in N_x X_{\mu,J}$, where we are considering the inclusion $h \in S^\lambda V \subset S^\mu V$. If $X_{\lambda,J_\lambda}^\vee$ had multiplicity greater than one in $X_{\mu,J}^\vee \cap \mathbb{P}(S^\lambda V)$, there would be a point $\tilde{x} \in X_{\mu,J}$, distinct from x, such that $h \in N_{\tilde{x}} X_{\lambda,J_\lambda}$. Again the first part of the proof would imply that $\tilde{y} := \pi_{\lambda,\mu}(\tilde{x})$ is distinct from y and that $h \in N_{\tilde{y}} X_{\lambda,J_\lambda}$. But then $h \in N_y X_{\lambda,J_\lambda} \cap N_{\tilde{y}} X_{\lambda,J_\lambda}$, and this contradicts the fact that h is smooth on $X_{\lambda,J_\lambda}^\vee$. □

Propositions 5.3.9 and 5.3.12 yield the following useful corollary for the proof of Theorem 5.0.5.

Corollary 5.3.13. *Let $\mu = (\mu_1, \ldots, \mu_s)$ be a partition of d. Then*

$$\alpha_{\mu,J} = \alpha_{d,\sum_{j \in J} \mu_j}$$

for all $J \subset [s]$, where the integers $\alpha_{\mu,J}$ were defined in (5.3.4) and in (5.3.5), whereas the integers $\alpha_{d,j}$ were defined in (5.3.6).

Proof. Let $K \subset [d]$ such that $|K| = \sum_{j \in J} \mu_j$. By Proposition 5.3.12, $f_{\mu,J}$ is a factor of multiplicity one in $\pi_{\mu,1^d}(f_{d,K})$. Moreover, given any tensor $u \in S^\mu V$, by Proposition 5.3.9 the ED polynomial of $X_\mu^\vee$ at u divides the ED polynomial of $X_d^\vee$ at u with multiplicity one. Therefore, the exponents of $f_{d,K}$ and $f_{\mu,J}$, which are respectively $\alpha_{d,|K|} = \alpha_{d,\sum_{j \in J} \mu_j}$ and $\alpha_{\mu,J}$, must coincide. □

Thanks to Corollary (5.3.13) and the identities in (5.3.7), one might consider the linear system

$$\sum_{J \subset [s]} \alpha_{d,\sum_{j \in J} \mu_j} \theta_{\mu,J} + 2N = 0 \quad \forall \mu \vdash d \tag{5.3.11}$$

in the variables $\alpha_{d,1},\dots,\alpha_{d,d}$, and show that the system (5.3.11) admits the unique solution

$$(\alpha_{d,1},\dots,\alpha_{d,d})=(-1,0,1,\dots,d-2). \tag{5.3.12}$$

Note that the system (5.3.11) has d variables and as many equations as the number of partitions $\mu\vdash d$, which is considerably larger than d in general. Actually, it turns out that many equations are linearly dependent.

After initializing the system (5.3.11) on the mathematical software `Macaulay2`, we verified that it admits the unique solution (5.3.12) for small values of n (say, $n<100$). Modulo this issue, we are ready to conclude the proof of Theorem 5.0.5.

Proof of Theorem 5.0.5. Let $d\geq 1$ be an integer and let $\mu=(\mu_1,\dots,\mu_s)$ be a partition of d. We only need to show that the highest coefficient $a_N=a_N(u)$ and the lowest coefficient $a_0=a_0(u)$ of the ED polynomial of $X_\mu^\vee$ at $u\in S^\mu V$ are respectively

$$a_N=\prod_{j\,:\,\mu_j>1}f_{\mu,\{j\}}^{\mu_j-2}\cdot\prod_{|J|>1}f_{\mu,J}^{\sum_{k\in J}\mu_k-2},\quad a_0=f_\mu^2\cdot\prod_{j\,:\,\mu_j=1}f_{\mu,\{j\}}. \tag{5.3.13}$$

By Corollary 5.3.13, for every nonempty subset $J\subset[s]$, the exponent of $f_{\mu,J}$ is $\alpha_{\mu,J}=\alpha_{d,\sum_{k\in J}\mu_k}$. Moreover, by (5.3.12), we have that

$$\alpha_{d,\sum_{k\in J}\mu_k}=\sum_{k\in J}\mu_k-2,$$

thus completing the proof. □

In the last part of this section, we extract a system $\mathcal{S}_d$ of d equations from (5.3.11). In our computations, we observed (for small values of n) that $\mathcal{S}_d$ is of full rank when $n>2$, whereas is of rank $d-1$ for $n=2$. Actually, the case $n=2$ is completely described in Section 5.4, where we show in Proposition 5.4.4 that the system $\mathcal{S}_d$ admits essentially the unique solution (5.3.12).

The first equation to add in $\mathcal{S}_d$ corresponds to the trivial partition $\mu=(d)$, as explained in the following result.

Corollary 5.3.14. *For any partition $\mu=(\mu_1,\dots,\mu_s)\vdash d$, we have the relation*

$$\alpha_{\mu,[s]}=\alpha_{(d),\{1\}}=\alpha_{d,d}=d-2. \tag{5.3.14}$$

Proof. The proof is a direct application of Theorem 2.0.2. On one hand, the highest coefficient of $\mathrm{EDpoly}_{X_{(d)}^\vee,u}(\varepsilon^2)$ is

$$a_N(u)=\Delta_{\widetilde{Q}}(u)^{d-2},$$

where $\Delta_{\widetilde{Q}}(u) = f_{(d),\{1\}}(u)$ is the equation of the dual of the d-th Veronese embedding into $\mathbb{P}(S^d V)$ of $Q \subset V$. On the other hand, the lowest coefficient is

$$a_0(u) = \Delta_d(u)^2,$$

where $\Delta_d(u) = f_{(d)}$ is the discriminant of the form u. $\square$

Actually, the last corollary solves the first problem of determining the exponent $\alpha_{d,d}$ of $f_{\mu,[s]}$, for any partition $\mu = (\mu_1, \ldots, \mu_s) \vdash d$. The next two technical lemmas furnish a bunch of linear conditions involving the remaining exponents $\alpha_{d,j}$, to be added in the system $\mathcal{S}_d$.

Lemma 5.3.15. *Let $d \geq 2$ and consider the partition $\mu = (k, d-k) \vdash d$ for all $1 \leq k \leq \lfloor d/2 \rfloor$. Then for all $1 \leq k \leq \lfloor d/2 \rfloor$ we have the relation*

$$\theta_{\mu,\{1\}}\alpha_{d,k} + \theta_{\mu,\{2\}}\alpha_{d,d-k} + \theta_{\mu,[2]}\alpha_{d,d} = 2(\theta_\mu - N). \tag{5.3.15}$$

Proof. There are essentially three cases to discuss:

1. Assume $d = 2$. Then $\mu = (1,1)$ and we have

$$\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2) = \det(\varepsilon I - u)\det(\varepsilon I + u).$$

   Then $a_N(u) = 1$ and $f_\mu(u) = \det(u)$ is the determinant of the $n \times n$ matrix representing u. In particular, we get $\theta_{\mu,\{1\}} = \theta_{\mu,\{2\}} = 0$ and $\alpha_{2,2} = 0$, yielding the identity $2N = 2\theta_\mu$, which corresponds to (5.3.15) in this very special case.

2. Now assume $d > 2$ and $k = 1$, hence $\mu = (1, d-1)$. From Lemma 5.2.2 we have that $X_{\mu,\{2\}}^\vee$ is not a hypersurface, therefore $f_{\mu,\{2\}} = 1$ and $\theta_{\mu,\{2\}} = 0$. By Corollary 5.3.13 we have that $\alpha_{\mu,[2]} = \alpha_{d,d}$, whereas $\alpha_{\mu,\{1\}} = \alpha_{d,1}$. Therefore equations (5.3.4) and (5.3.5) become respectively

$$a_N(t) = f_{\mu,[2]}^{\alpha_{d,d}}, \quad a_0(t) = f_\mu^2 \cdot f_{\mu,\{1\}}^{-\alpha_{d,1}}.$$

   Equation (5.3.7) yields the identity

$$\theta_{\mu,\{1\}}\alpha_{d,1} + \theta_{\mu,[2]}\alpha_{d,d} = 2(\theta_\mu - N). \tag{5.3.16}$$

   Hence we get relation (5.3.15) for $k = 1$.

3. Finally we consider the case $d > 2$ and $2 \leq k \leq \lfloor d/2 \rfloor$. Equations (5.3.4) and (5.3.5) become respectively

$$a_N(t) = f_{\mu,[2]}^{\alpha_{\mu,[2]}} \cdot f_{\mu,\{1\}}^{\alpha_{\mu,\{1\}}} \cdot f_{\mu,\{2\}}^{\alpha_{\mu,\{2\}}}, \quad a_0(t) = f_\mu^2.$$

   Again by Corollary 5.3.13 we have that $\alpha_{\mu,[2]} = \alpha_{d,d}$, while $\alpha_{\mu,\{1\}} = \alpha_{d,k}$ and $\alpha_{\mu,\{2\}} = \alpha_{d,d-k}$. Then relation (5.3.15) follows by (5.3.7) after applying Theorem 5.4.2 and Corollary 5.4.1. $\square$

Corollary 5.3.16. *For all $n \geq 2$ and $d \geq 1$, we have that $\alpha_{d,1} = -1$. In particular, for all $\mu \vdash d$, the lowest coefficient of* $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ *is*

$$a_0(u) = f_\mu(u)^2 \cdot \prod_{j\,:\,\mu_j=1} f_{\mu,\{j\}}(u).$$

Therefore, the ε-offset of $X_\mu^\vee$, with respect to the squared distance function $\delta_{F,\mu}$, is a hypersurface of degree

$$\deg(\mathcal{O}_\varepsilon(X)) = 2\deg(f_\mu) + \sum_{j\,:\,\mu_j=1} \deg(f_{\mu,\{j\}}).$$

Proof. We know from Corollary 5.3.14 that $\alpha_{d,d} = d-2$ for all $n \geq 2$ and all $d \geq 1$. Then equation (5.3.16) is linear in the only variable $\alpha_{d,1}$. Similarly to Remark 2.3.6, one may check that $\alpha_{d,1} = -1$ is the solution of the above-mentioned equation. The rest of the statement follows from identity (5.3.5). $\square$

Lemma 5.3.17. *Let $d \geq 3$ and consider the partition $\mu = (k, d-k-1, 1) \vdash d$ for all $1 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$. Then for all $1 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$ we have the relation*

$$\begin{aligned}&\theta_{\mu,\{3\}}\alpha_{d,1} + \theta_{\mu,\{1\}}\alpha_{d,k} + \theta_{\mu,\{1,3\}}\alpha_{d,k+1} + \theta_{\mu,\{2\}}\alpha_{d,d-k-1}\\ &+ \theta_{\mu,\{2,3\}}\alpha_{d,d-k} + \theta_{\mu,\{1,2\}}\alpha_{d,d-1} + \theta_{\mu,[3]}\alpha_{d,d} = 2(\theta_\mu - N)\end{aligned} \tag{5.3.17}$$

Proof. We discuss three cases in this proof as well:

1. We start by considering the case $d = 3$, hence $\mu = (1,1,1)$. By Corollary 5.3.13 we have

$$\alpha_{\mu,\{j\}} = \alpha_{3,1}\ \forall 1 \leq j \leq 3, \quad \alpha_{\mu,\{i,j\}} = \alpha_{3,2}\ \forall 1 \leq i \neq j \leq 3, \quad \alpha_{\mu,[3]} = \alpha_{3,3}.$$

   The highest coefficient $a_N = a_N(u)$ and the lowest coefficient $a_0 = a_0(u)$ of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ become respectively

$$\begin{aligned}a_N &= f_{\mu,[3]}^{\alpha_{3,3}} \cdot \left(f_{\mu,\{1,2\}} f_{\mu,\{1,3\}} f_{\mu,\{2,3\}}\right)^{\alpha_{3,2}}\\ a_0 &= f_\mu^2 \cdot \left(f_{\mu,\{1\}} f_{\mu,\{2\}} f_{\mu,\{3\}}\right)^{-\alpha_{3,1}}.\end{aligned}$$

   In particular, we get the identity (5.3.17) in this special case. We refer the reader to Section 5.4 for a detailed treatise on this specific example.

2. Now we suppose that $d > 3$ and that $k = 1$, hence $\mu = (1, d-2, 1)$. By Corollary 5.3.13 we have that

$$\begin{aligned}&\alpha_{\mu,\{1\}} = \alpha_{\mu,\{3\}} = \alpha_{d,1}, \quad \alpha_{\mu,\{2\}} = \alpha_{d,d-2}\\ &\alpha_{\mu,\{1,2\}} = \alpha_{\mu,\{2,3\}} = \alpha_{d,d-1}, \quad \alpha_{\mu,\{1,3\}} = \alpha_{d,2}\end{aligned}$$

and $\alpha_{\mu,[3]} = \alpha_{d,d}$. The highest coefficient a_N and the lowest coefficient a_0 of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ become respectively

$$\begin{aligned} a_N &= f_{\mu,[3]}^{\alpha_{d,d}} \cdot \left(f_{\mu,\{1,2\}} f_{\mu,\{2,3\}}\right)^{\alpha_{d,d-1}} f_{\mu,\{1,3\}}^{\alpha_{d,2}} \cdot f_{\mu,\{2\}}^{\alpha_{d,d-2}} \\ a_0 &= f_\mu^2 \cdot \left(f_{\mu,\{1\}} f_{\mu,\{3\}}\right)^{-\alpha_{d,1}}. \end{aligned}$$

Putting all these facts together, we get equation (5.3.17) in this case.

3. Finally we suppose that $d > 3$ and that $2 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$. By Corollary 5.3.13 we derive the identities

$$\begin{aligned} &\alpha_{\mu,\{1\}} = \alpha_{d,k}, \quad \alpha_{\mu,\{2\}} = \alpha_{d,d-k-1}, \quad \alpha_{\mu,\{3\}} = \alpha_{d,1} \\ &\alpha_{\mu,\{1,2\}} = \alpha_{d,d-1}, \quad \alpha_{\mu,\{1,3\}} = \alpha_{d,k+1}, \quad \alpha_{\mu,\{2,3\}} = \alpha_{d,d-k} \end{aligned}$$

and $\alpha_{\mu,[3]} = \alpha_{d,d}$. The highest coefficient a_N and the lowest coefficient a_0 become respectively

$$a_N = f_{\mu,[3]}^{\alpha_{d,d}} \cdot f_{\mu,\{1,3\}}^{\alpha_{d,k+1}} \cdot f_{\mu,\{2,3\}}^{\alpha_{d,d-k}} \cdot f_{\mu,\{1\}}^{\alpha_{d,k}} \cdot f_{\mu,\{2\}}^{\alpha_{d,d-k-1}}, \quad a_0 = f_\mu^2 \cdot f_{\mu,\{3\}}^{-\alpha_{d,1}}.$$

Summing up, we obtain equation (5.3.17). □

The next proposition verifies the product formula (5.0.6) for all tensors of format $n^{\times d}$ with $d \in \{2,3,4\}$, possibly with partial symmetry.

Proposition 5.3.18 (Product formula for $d \in \{3,4\}$). *When $d \in \{2,3,4\}$, the product formula (5.0.6) is true for all $\mu \vdash d$ and all $n \geq 2$.*

Proof. The case $d = 2$ corresponds to the trivial case of $n \times n$ matrices, possibly symmetric if $\mu = (2)$. The first nontrivial case is $d = 3$. In this case, the unknown exponents are $\alpha_{3,1}$, $\alpha_{3,2}$ and $\alpha_{3,3}$. By Corollaries 5.3.14 and 5.3.16 we have $\alpha_{3,3} = 1$ and $\alpha_{3,1} = -1$, respectively. To conclude, we consider the relation (5.3.17) for $k = 1$, namely

$$-3\theta_{3,\{1\}} + 3\theta_{3,\{1,2\}}\alpha_{3,2} + \theta_{3,[3]} = 2\left[\theta_3 - \mathrm{EDdegree}(X_3)\right].$$

With an analogous check to Remark 2.3.6, one verifies that necessarily $\alpha_{3,2} = 0$.

Now suppose that $d = 4$. The unknown exponents are $\alpha_{4,1}$, $\alpha_{4,2}$, $\alpha_{4,3}$ and $\alpha_{4,4}$. Again Corollaries 5.3.14 and 5.3.16 yield $\alpha_{4,4} = 2$ and $\alpha_{4,1} = -1$, respectively. Consider the relation (5.3.15) for $k = 2$, namely

$$\theta_{(2,2),\{1\}}\alpha_{4,2} + \theta_{(2,2),[2]} = \theta_{(2,2)} - \mathrm{EDdegree}(X_{(2,2)}).$$

Solving for $\alpha_{4,2}$ we get that $\alpha_{4,2} = 0$. Finally, we consider the relation (5.3.17) for $k = 1$, which simplifies as

$$\theta_{(1,2,1),\{1,2\}}\alpha_{4,3} + \theta_{(1,2,1),[3]} = \theta_{(1,2,1)} + \theta_{(1,2,1),\{1\}} - \mathrm{EDdegree}(X_{(1,2,1)}).$$

Solving for $\alpha_{4,3}$, one verifies that $\alpha_{4,3} = 1$. □

Remark 5.3.19. Define $\mathcal{S}_d$ to be the linear system formed by equations (5.3.14), (5.3.15) and (5.3.17). In particular, $\mathcal{S}_d$ has $1+\lfloor\frac{d}{2}\rfloor+\lfloor\frac{d-1}{2}\rfloor=d$ equations in d unknowns $\alpha_{d,1},\dots,\alpha_{d,d}$. As we observed in Proposition 5.3.18, when $d\in\{2,3,4\}$ the linear system $\mathcal{S}_d$ is solved via substitution, since at every step we obtain a linear equation in only one variable. Things become more complicated for $d\geq 5$, where a more detailed study of the matrix of coefficients of $\mathcal{S}_d$ is required.

In order to distinguish between all different partitions, for the moment we rename all partitions in Lemmas 5.3.15 and 5.3.17 as

$$\mu^{(1)}(k)\coloneqq(k,d-k)\vdash d,\quad \mu^{(2)}(k)\coloneqq(k,d-k-1,1)\vdash d.$$

Observe that, when $n=1$, no equation in $\mathcal{S}_d$ involves the unknown $\alpha_{d,d-1}$, hence the rank of the matrix of coefficients of $\mathcal{S}_d$ is at most $d-1$. The geometrical reason is that, for any partition $\mu=(\mu_1,\dots,\mu_s)\vdash d$, with $\mu_1\geq\cdots\geq\mu_s$, the only subset $J\subset[s]$ such that $\alpha_{\mu,J}=\alpha_{d,d-1}$ is $J=[s-1]$, by Corollary 5.3.13. Indeed, by Lemma 5.2.2, the corresponding dual variety $X^\vee_{\mu,J}$ is not a hypersurface, thus $f_{\mu,J}=1$ and the exponent $\alpha_{d,d-1}$ remains undetermined. To be consistent with the higher dimensional results, we define $\alpha_{d,d-1}\coloneqq d-3$.

After substituting in (5.3.14), (5.3.15) and (5.3.17), we see that the vector $(\alpha_{d,1},\dots,\alpha_{d,d})=(-1,0,1,\dots,d-2)$ is a solution of $\mathcal{S}_d$.

It remains to show that the matrix of coefficients of $\mathcal{S}_d$ is of maximal rank d (or $d-1$ when $n=2$). First of all, we observe that all equations coming from (5.3.15) are pairwise linearly independent. The same holds for the set of equations coming from (5.3.15). Moreover, any equation coming from either (5.3.15) or (5.3.17) is linearly independent with (5.3.14). It may happen that the k-th equation in (5.3.17) is a linear combination of the k-th and $(k+1)$-th equations in (5.3.15). This happens only if the maximal minors of the submatrix

$$\mathcal{M}_{d,k}=\begin{pmatrix}\theta_{\mu^{(1)}(k),\{1\}} & 0 & 0 & \theta_{\mu^{(1)}(k),\{2\}}\\ 0 & \theta_{\mu^{(1)}(k+1),\{1\}} & \theta_{\mu^{(1)}(k+1),\{2\}} & 0\\ \theta_{\mu^{(2)}(k),\{1\}} & \theta_{\mu^{(2)}(k),\{1,3\}} & \theta_{\mu^{(2)}(k),\{2\}} & \theta_{\mu^{(2)}(k),\{2,3\}}\end{pmatrix}$$

obtained extracting the coefficients of α_k, α_{k+1}, α_{d-k-1} and α_{d-k} from the three mentioned equations, vanish simultaneously. We verified for small values of n that this is impossible for $1\leq k\leq\lfloor\frac{d-1}{2}\rfloor$. In particular, this could be checked easily for $n=2$, as showed in Proposition 5.4.4. The entries of the matrix $\mathcal{M}_{d,k}$ can be computed applying the formulas in Examples 5.3.2 and 5.3.3.

5.4 The case of binary tensors

Throughout this section, we set $n=2$, hence we concentrate on tensors of binary format. In the first part of this section, we derive degree formulas and we show in Proposition 5.4.4 that the linear system $\mathcal{S}_d$ defined in Remark 5.3.19 admits a unique solution. Afterwards, we show a sum of square property related to the equations $f_{\mu,J}$ of the dual affine cones $X_{\mu,J}$. We conclude the section studying in detail the case of a $2 \times 2 \times 2$ tensor.

In Proposition 5.3.1, we compute the degree $\theta_{\mu,J}$ of $X_{\mu,J}^{\vee}$ in full generality. However, that formula simplifies a lot in the binary setting. In the following, we define $e_j(p_1,\ldots,p_s) = \sum_{1\le k_1<\cdots<k_j\le s} p_{k_1}\cdots p_{k_j}$ to be the elementary symmetric polynomial of degree j.

Corollary 5.4.1. *For any subset $J \subset [s]$, let $\mu(J) \vdash d - \sum_{j\in J}\mu_j$ be the partition whose summands are all the μ_k such that $k \in [s]\setminus J$. In particular, $\mu(\emptyset) = \mu$. Then*

$$\theta_{\mu,J} = 2^{|J|}\sum_{i=0}^{s-|J|}(-2)^{s-|J|-i}(i+1)!e_i(\mu(J)). \tag{5.4.1}$$

Proof. In the notations of equation (5.3.3), we have respectively $m = s - |J|$, $\beta_j = 0$ if $j \in J$, otherwise $0 \le \beta_j \le 1$ if $j \notin J$. Then

$$\begin{aligned}
\theta_{\mu,J} &= 2^{|J|}\sum_{k=0}^{s-|J|}(-1)^k(s-|J|+1-k)!\sum_{|\beta|=k}\eta_\beta\left[\prod_{j\notin J}\binom{2}{\beta_j}\mu_j^{1-\beta_j}\right]\\
&= 2^{|J|}\sum_{k=0}^{s-|J|}(-1)^k(s-|J|+1-k)!2^k e_{s-|J|-k}(\mu(J))\\
&= 2^{|J|}\sum_{i=0}^{s-|J|}(-2)^{s-|J|-i}(i+1)!e_i(\mu(J)).
\end{aligned}$$

where in the last passage we used the change of indices $i = s - |J| - k$. $\square$

When $J = \emptyset$, we recover the degree of the so-called μ-discriminant of a binary tensor (see [GKZ, XIII, Theorem 2.4]).

Theorem 5.4.2. *Let $d \ge 1$ and suppose that $\mu = (\mu_1,\ldots,\mu_s) \vdash d$. Then*

$$\theta_\mu = \sum_{i=0}^{s}(-2)^{s-i}(i+1)!e_i(\mu). \tag{5.4.2}$$

Note that in the nonsymmetric case $\mu = 1^d$, we have $e_i(1^d) = \binom{d}{i}$ for all $0 \le i \le d$. Hence we recover the degree $\theta_d := \theta_{1^d}$ of the hyperdeterminant of a d-dimensional binary tensor:

$$\theta_d = \sum_{i=0}^{d}(-2)^{d-i}\binom{d}{i}(i+1)!. \tag{5.4.3}$$

Now we take a closer look at the relations obtained in Lemmas 5.3.15 and 5.3.17. In the binary case, we used the simplified formula in Corollary 5.4.1 to compute them more explicitly and show in Proposition 5.4.4 that the linear system $\mathcal{S}_d$ defined in Remark 5.3.19 is of rank $d-1$.

Proposition 5.4.3. *The equations in $\mathcal{S}_d$ coming from (5.3.15) simplify to*

$$(d-k-1)\alpha_{d,k}+(k-1)\alpha_{d,d-k}+\alpha_{d,d}=2(k(d-k)-d+1). \tag{5.4.4}$$

for all $1\leq k\leq \lfloor \frac{d}{2}\rfloor$. Instead, the equations coming from (5.3.17) become respectively

$$\begin{aligned}&(d-k-1)\alpha_{d,k}+2(d-k-2)\alpha_{d,k+1}+k\alpha_{d,d-k-1}\\ &+2(k-1)\alpha_{d,d-k}+2\alpha_{d,d}=2(3k(d-k-1)-2d+3)\end{aligned} \tag{5.4.5}$$

for all $1\leq k\leq \lfloor \frac{d-1}{2}\rfloor$.

Proof. We perform an explicit computation for the first set of equations. The proof is similar for the other set of equations. So consider the partition $\mu=(k,d-k)\vdash d$. We apply formula (5.4.1) in this special case:

$$\begin{aligned}\theta_\mu&=\sum_{i=0}^{2}(-2)^{2-i}(i+1)!e_i(\mu)=4-4d+6k(d-k),\\ \theta_{\mu,\{1\}}&=2\sum_{i=0}^{1}(-2)^{1-i}(i+1)!e_i(d-k)=4(d-k-1),\\ \theta_{\mu,\{2\}}&=2\sum_{i=0}^{1}(-2)^{1-i}(i+1)!e_i(k)=4(k-1),\\ \theta_{\mu,[2]}&=4\ .\end{aligned}$$

Moreover, by Corollary 5.1.2 we have that $N=\mathrm{EDdegree}(X_\mu)=2k(d-k)$. Substituting the preceding relations and simplifying we obtain the relations (5.4.4). $\square$

Proposition 5.4.4. *The linear system $\mathcal{S}_d$ defined by equations (5.3.14), (5.4.4) and (5.4.5) admits the unique solution $(\alpha_{d,1},\ldots,\alpha_{d,d})=(-1,0,1,\ldots,d-2)$, provided that $\alpha_{d,d-1}\coloneqq d-3$.*

Proof. After substituting in (5.3.14), (5.4.4) and (5.4.5), we see that the vector $(\alpha_{d,1},\ldots,\alpha_{d,d})=(-1,0,1,\ldots,d-2)$ is a solution of $\mathcal{S}_d$.

It remains to show that the matrix of coefficients of $\mathcal{S}_d$ is of maximal rank $d-1$. By Remark 5.3.19, we need to check that the maximal minors of the matrix

$\mathcal{M}_{d,k}$ do not vanish simultaneously for all $1 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$. In this case, the matrix $\mathcal{M}_{d,k}$ becomes

$$\mathcal{M}_{d,k} = \begin{pmatrix} d-k-1 & 0 & 0 & k-1 \\ 0 & d-k-2 & k & 0 \\ d-k-1 & 2(d-k-2) & k & 2(k-1) \end{pmatrix}$$

and the maximal minors $m_{d,k}^{(j_1,j_2,j_3)}$ obtained picking the columns j_1, j_2, j_3 of $\mathcal{M}_{d,k}$ are respectively

$$\begin{aligned} m_{d,k}^{(1,2,3)} &= -k\,(d-k-2)\,(d-k-1)\,, & m_{d,k}^{(1,3,4)} &= (k-1)\,(d-k-2)\,(d-k-1)\,, \\ m_{d,k}^{(1,2,4)} &= k\,(k-1)\,(d-k-1)\,, & m_{d,k}^{(2,3,4)} &= -k\,(k-1)\,(d-k-2)\,. \end{aligned}$$

From the above identities we see that the four maximal minors of $\mathcal{M}_{d,k}$ do not vanish simultaneously for all $1 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$. □

An immediate consequence of Theorem 5.0.5 is that, for any partition $\mu \vdash d$, we may write an identity involving the ED degree of X_μ and the degrees of the varieties $X_{\mu,J}^\vee$, as pointed out below.

Corollary 5.4.5. *Consider a partition $\mu = (\mu_1, \ldots, \mu_s) \vdash d$. Let $\mu(J)$ be the partition defined in Corollary 5.4.1 for all $J \subset [s]$. Recalling that the degrees of the μ-discriminant θ_μ and of the hyperdeterminant θ_j are defined in (5.4.2) and (5.4.3), respectively, then*

$$\begin{aligned} \mathrm{EDdegree}(X_\mu) &= s!\mu_1 \cdots \mu_s = \sum_{J \subset [s]} 2^{|J|-1} \left(2 - \sum_{k \in J} \mu_k\right) \theta_{\mu(J)}, \\ \mathrm{EDdegree}(X_d) &= d! = \sum_{j=0}^{d} \binom{d}{j} (2-j) 2^{j-1} \theta_{d-j}. \end{aligned}$$

In this section, we investigate also the non-negativity of the various factors $f_{\mu,J}$ appearing in the extreme coefficients of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$. Actually, they are (products of) SOS polynomials. This fact is useful for the considerations about tensors of format $2 \times 2 \times 2$ made in Proposition 5.4.9.

Proposition 5.4.6. *Let $J \subset [s]$, $J \neq \emptyset$. If $J = [s]$, then $f_{\mu,J}$ is the product of d SOS polynomials. If $J \neq [s]$, then $f_{\mu,J}$ is an SOS polynomial. In particular, $f_{\mu,J}$ is a nonnegative polynomial for all $J \neq \emptyset$.*

Proof. For $J = [s]$, the statement follows since $X^\vee_{\mu,J} \subset \mathbb{P}(V^{\otimes d})$ is the union of d pairwise conjugate hyperplanes. Now let $\mu = 1^d$ and $J = \{1\}$. More explicitly,

$$\begin{aligned} X^\vee_{d,\{1\}} &= \left[\mathrm{Seg}_d([(1,\sqrt{-1})] \times \mathbb{P}(V)^{\times(d-1)})\right]^\vee \\ &\cup \left[\mathrm{Seg}_d([(1,-\sqrt{-1})] \times \mathbb{P}(V)^{\times(d-1)})\right]^\vee \subset \mathbb{P}(V^{\otimes d}). \end{aligned} \tag{5.4.6}$$

Thus $X^\vee_{d,\{1\}}$ is isomorphic to two copies of $X^\vee_{d-1} \subset \mathbb{P}(V^{\otimes(d-1)})$. By Lemma 5.2.2, the varieties $X^\vee_{d,\{1\}}$ and $X^\vee_{d-1}$ are hypersurfaces when $d \geq 3$. If $\{a_{i_1\cdots i_d}\}$ and $\{b_{j_2\cdots j_d}\}$ are homogeneous coordinates for $\mathbb{P}(V^{\otimes d})$ and $\mathbb{P}(V^{\otimes(d-1)})$ respectively, the equations of the two components of $X^\vee_{d,\{1\}}$ in (5.4.6) are

$$\begin{aligned} f^+_{d-1}(\{a_{j_1\cdots j_d}\}) &\coloneqq f_{d-1}(\{b_{j_2\cdots j_d}\})_{|\{b_{j_2\cdots j_d} \,=\, a_{0i_2\cdots i_d}+\sqrt{-1}\,a_{1i_2\cdots i_d}\}}, \\ f^-_{d-1}(\{a_{j_1\cdots j_d}\}) &\coloneqq f_{d-1}(\{b_{j_2\cdots j_d}\})_{|\{b_{j_2\cdots j_d} \,=\, a_{0i_2\cdots i_d}-\sqrt{-1}\,a_{1i_2\cdots i_d}\}}. \end{aligned} \tag{5.4.7}$$

In particular, f^+_{d-1} and f^-_{d-1} are conjugate polynomials and their product is the equation $f_{d,\{1\}}$ of $X^\vee_{d,\{1\}}$. Therefore $f_{d,\{1\}}$ is the sum of two squared polynomials. In the same fashion, we show that $f_{d,\{1,2\}} = f^+_{d,\{2\}} \cdot f^-_{d,\{2\}}$, where the factors $f^+_{d,\{2\}}$ and $f^-_{d,\{2\}}$ are defined as in (5.4.7). Therefore, $f_{d,\{1,2\}}$ is again a sum of two squared polynomials. More in general, the iteration of this argument shows that, possibly after a permutation of the indices, the polynomial $f_{d,J}$ is a sum of two squared polynomials for any subset $J \subset [d]$.

Now consider a partition $\mu = (\mu_1, \ldots, \mu_s) \vdash d$ and a nonempty subset $J \subset [s]$. On one hand, By Proposition 5.3.12 there exists a subset $\tilde{J} \subset [d]$ with $|\tilde{J}| = \sum_{j\in J} \mu_j$ such that $f_{\mu,J}$ divides $\pi_{\mu,1^d}(f_{d,\tilde{J}})$ with multiplicity one. On the other hand, the first part of the proof implies that $f_{d,\tilde{J}} = h_1^2 + h_2^2$ for some homogeneous polynomials h_1 and h_2.

Summing up, there exist two homogeneous polynomials h_1' and h_2' such that $h_1' \pm \sqrt{-1}\,h_2'$ divides $\pi_{\mu,1^d}(h_1 \pm \sqrt{-1}\,h_2)$ with multiplicity one and

$$f_{\mu,J} = h_1'^2 + h_2'^2 = (h_1' + \sqrt{-1}\,h_2')(h_1' - \sqrt{-1}\,h_2'). \qquad \square$$

Problem 5.4.7. In particular, Proposition 5.4.6 tells us that, at least in the binary case, if $(X_\mu \cap Q_{F,\mu})^\vee$ is a hypersurface, then its equation is an SOS polynomial. We consider concrete examples in Remark 5.4.8 and in the last part of this section dealing with binary tensors of format $2 \times 2 \times 2$.

Looking at Remark 5.2.5, we observed that the variety $(X_\mu \cap Q_{F,\mu})^\vee$, in the case $\mu = 1^s$ is always a hypersurface (with several irreducible components), unless $s = 2$ and $n_1 = n_2$. In particular, it is a hypersurface when $s = 2$ and $n_1 < n_2$.

This is indeed the case of non-square matrices, where the determinant is not defined, namely $X_2^\vee$ is not a hypersurface. As pointed out in (5.0.2), the lowest coefficient of $\mathrm{EDpoly}_{X_2^\vee,u}(0)$ is equal to $\det(uu^T)$, which is in fact the equation of $X_{2,\{2\}}^\vee$ by Lemma 5.2.1. What is more, the polynomial $\det(uu^T)$ is an SOS polynomial by the classical Cauchy-Binet formula.

Beyond the study of varieties of rank-one tensors, we considered the variety $(X^\vee \cap Q)^\vee$ appearing in Corollary 4.4.6 in various examples (say, projective curves and surfaces) and verified (with the help of the `Macaulay2` package `"SOS"`) that, when it is a hypersurface, its equation admits an SOS decomposition. It would be interesting to solve the following problem suggested by Bernd Sturmfels: assuming that $(X^\vee \cap Q)^\vee$ is a hypersurface, is its equation a nonnegative polynomial? If so, is it an SOS polynomial?

Remark 5.4.8. Looking closely at the polynomials defined in (5.4.7), one may see that for all $d \geq 3$ and for all $j \in [d]$, the polynomial $f_{d,\{j\}}$ is written as

$$f_{d,\{j\}}(u) = \mathrm{Det}\left(u_j^{(1)} + \sqrt{-1}u_j^{(2)}\right) \cdot \mathrm{Det}\left(u_j^{(1)} - \sqrt{-1}u_j^{(2)}\right), \tag{5.4.8}$$

where $u_j^{(1)}$ and $u_j^{(2)}$ are the tensors in $V^{\otimes(d-1)}$ obtained considering in u the slices $\{u_{i_1\cdots i_d}\}$ with $i_j = 1$ and $i_j = 2$, respectively. The cases $d = 3$ and $d = 4$ are depicted in Figures 2 and 5.3, respectively.

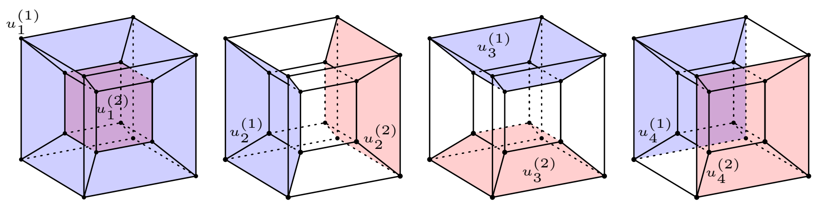

Figure 5.3: The slices $u_j^{(1)}$ and $u_j^{(2)}$ appearing in the computation of $f_{4,\{j\}}$.

As pointed out in the proof of Proposition 5.4.6, formula (5.4.8) may be generalized to any polynomial $f_{d,J} \neq 1$. For example, below we interpret the equation $f_{4,\{1,2\}}$ of $X_{4,\{1,2\}}^\vee$ in terms of the tensors $u_{\{1,2\}}^{(rs)} \in V \otimes V$, with $r, s \in [2]$, obtained extracting from u the slices $\{u_{rsij}\}$ highlighted in Figure 5.4.

$$\begin{aligned} f_{4,\{1,2\}}(u) = \det \Big\{ &\Big[\Big(u^{(11)}_{\{1,2\}} + \sqrt{-1}u^{(21)}_{\{1,2\}}\Big) + \sqrt{-1}\Big(u^{(12)}_{\{1,2\}} + \sqrt{-1}u^{(22)}_{\{1,2\}}\Big)\Big] \\ &\Big[\Big(u^{(11)}_{\{1,2\}} + \sqrt{-1}u^{(21)}_{\{1,2\}}\Big) - \sqrt{-1}\Big(u^{(12)}_{\{1,2\}} + \sqrt{-1}u^{(22)}_{\{1,2\}}\Big)\Big]\Big\} \\ \cdot \det \Big\{ &\Big[\Big(u^{(11)}_{\{1,2\}} - \sqrt{-1}u^{(21)}_{\{1,2\}}\Big) + \sqrt{-1}\Big(u^{(12)}_{\{1,2\}} - \sqrt{-1}u^{(22)}_{\{1,2\}}\Big)\Big] \\ &\Big[\Big(u^{(11)}_{\{1,2\}} - \sqrt{-1}u^{(21)}_{\{1,2\}}\Big) - \sqrt{-1}\Big(u^{(12)}_{\{1,2\}} - \sqrt{-1}u^{(22)}_{\{1,2\}}\Big)\Big]\Big\}. \end{aligned}$$

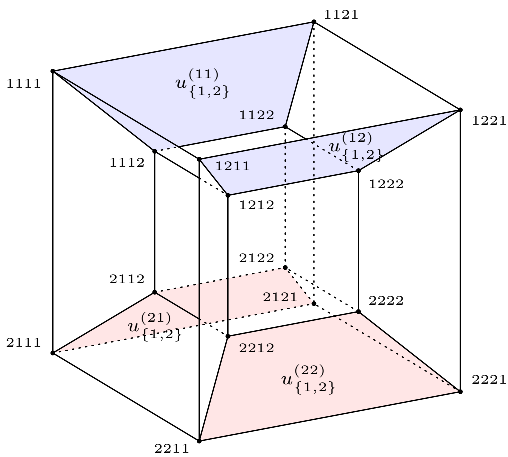

Figure 5.4: The slices $f^{(rs)}_{\{1,2\}}$ appearing in the expression of $f_{4,\{1,2\}}(u)$.

We conclude this chapter by studying in detail the ED polynomial of the variety $X_3^\vee$ at a given tensor of format $2 \times 2 \times 2$.

More in general, we recall that, thanks to Proposition 4.2.12, the coefficients of $\mathrm{EDpoly}_{X^\vee,u}(\varepsilon^2)$ are $\mathrm{SO}(V)^d$-invariants. Indeed we are interested in computing a minimal generating set for the invariant ring $S(V^{\otimes d})^{\mathrm{SO}(V)^d}$.

As in the previous sections, q is the standard Euclidean scalar product. We fix $x_{j,1}, x_{j,2}$ as coordinates for the j-th copy of V in $V^{\otimes d}$. Then, the associated quadratic form q is in coordinates $x_{j,1}^2 + x_{j,2}^2$ for all $j \in [d]$.

Now consider the change of coordinates

$$z_{j,1} = x_{j,1} + \sqrt{-1}\,x_{j,2}, \quad z_{j,2} = x_{j,1} - \sqrt{-1}\,x_{j,2}.$$

In these new coordinates, the expression for the quadratic form q on the j-th copy of V in $V^{\otimes d}$ becomes $z_{j,1}z_{j,2}$. Moreover, each binary tensor $u = (u_{i_1 \cdots i_d}) \in V^{\otimes d}$

may be written as

$$u = \sum_{(i_1,\dots,i_d)\in\{1,2\}^d} u_{i_1\cdots i_d} x_{1,i_1}\cdots x_{d,i_d} = \sum_{(i_1,\dots,i_d)\in\{1,2\}^d} t_{i_1\cdots i_d} z_{1,i_1}\cdots z_{d,i_d},$$

for some coefficients $t_{i_1\cdots i_d}$ depending on the old set of coordinates $\{u_{i_1\cdots i_d}\}$ via the following relations:

$$t_{i_1\cdots i_d} = \sum_{(j_1,\dots,j_d)\in\{1,2\}^d} \left[\sqrt{-1}^{\sum_{l=1}^d j_l}(-1)^{\sum_{l=1}^d (1-i_l)j_l}\right] u_{j_1\cdots j_d}.$$

One may verify by direct computation that, for all $(i_1,\dots,i_d)\in\{1,2\}^d$, the complex conjugate of $t_{i_1\cdots i_d}$ is $t_{k_1\cdots k_d}$, where $k_l = 1 - i_l$.

The new system of coordinates is more effective for computing invariants with respect to $\mathrm{SO}(V)^d$. Indeed, the torus $\mathrm{SO}(V)^d \cong (\mathbb{C}^*)^d = (\mathbb{C}\setminus\{0\})^d$ acts on $V^{\otimes d}$ by rescaling each coordinate $t_{i_1\cdots i_d}$ as shown below:

$$t_{i_1\cdots i_d} \mapsto \prod_{j=1}^d \xi_j^{(-1)^{i_j}} t_{i_1\cdots i_d},$$

for some $(\xi_1,\dots,\xi_d)\in(\mathbb{C}^*)^d$.

Using [Stu, Algorithm 1.4.5], we computed a minimal generating set of invariants of $S(V^{\otimes d})^{\mathrm{SO}(V)^d}$, a least for small values of d. Focusing on the case $d=3$, we get that

$$S(V^{\otimes d})^{\mathrm{SO}(V)^d} \cong \mathbb{C}[t_{i_1\cdots i_d}]^{\mathrm{SO}(V)^d} \cong \mathbb{C}[\theta_1,\theta_2,\theta_3,\theta_4,\varphi_1,\varphi_2],$$

where the θ_j's are four real invariants of degree two, whereas φ_1 and φ_2 are two non-real mutually conjugate invariants of degree four:

$$\begin{gathered}\theta_1 = t_{1,1,1}t_{2,2,2},\ \theta_2 = t_{1,1,2}t_{2,2,1},\ \theta_3 = t_{1,2,1}t_{2,1,2},\ \theta_4 = t_{1,2,2}t_{2,1,1},\\ \varphi_1 = t_{1,1,2}t_{1,2,1}t_{2,1,1}t_{2,2,2},\ \varphi_2 = t_{1,1,1}t_{1,2,2}t_{2,1,2}t_{2,2,1}.\end{gathered} \tag{5.4.9}$$

In addition, the only relation among them is $\theta_1\theta_2\theta_3\theta_4 - \varphi_1\varphi_2 = 0$. Since we are dealing with real binary tensors, the coefficients of the ED polynomial of $X_\mu^\vee$ at u are all real polynomials in the entries $\{u_{i_1\cdots i_d}\}$ of u. Indeed, they are elements of $\mathbb{R}[\theta_1,\theta_2,\theta_3,\theta_4,\varphi]$, where

$$\varphi := \frac{\varphi_1+\varphi_2}{2}. \tag{5.4.10}$$

In the old set of coordinates $\{u_{i_1\cdots i_d}\}$, these invariants become respectively

$$
\begin{aligned}
\theta_1 &= (u_{111} - u_{122} - u_{212} - u_{221})^2 + (u_{222} - u_{211} - u_{121} - u_{112})^2,\\
\theta_2 &= (u_{111} + u_{122} + u_{212} - u_{221})^2 + (u_{222} + u_{211} + u_{121} - u_{112})^2,\\
\theta_3 &= (u_{111} + u_{122} - u_{212} + u_{221})^2 + (u_{222} + u_{211} - u_{121} + u_{112})^2,\\
\theta_4 &= (u_{111} - u_{122} + u_{212} + u_{221})^2 + (u_{222} - u_{211} + u_{121} + u_{112})^2,\\
\varphi &= u_{111}^4 + 2u_{111}^2u_{112}^2 + u_{112}^4 + 2u_{111}^2u_{121}^2 - 2u_{112}^2u_{121}^2 + u_{121}^4\\
&\quad + 8u_{111}u_{112}u_{121}u_{122} - 2u_{111}^2u_{122}^2 + 2u_{112}^2u_{122}^2 + 2u_{121}^2u_{122}^2 + u_{122}^4\\
&\quad + 2u_{111}^2u_{211}^2 - 2u_{112}^2u_{211}^2 - 2u_{121}^2u_{211}^2 - 6u_{122}^2u_{211}^2 + u_{211}^4\\
&\quad + 8u_{111}u_{112}u_{211}u_{212} + 8u_{121}u_{122}u_{211}u_{212} - 2u_{111}^2u_{212}^2 + 2u_{112}^2u_{212}^2\\
&\quad - 6u_{121}^2u_{212}^2 - 2u_{122}^2u_{212}^2 + 2u_{211}^2u_{212}^2 + u_{212}^4 + 8u_{111}u_{121}u_{211}u_{221}\\
&\quad + 8u_{112}u_{122}u_{211}u_{221} + 8u_{112}u_{121}u_{212}u_{221} - 8u_{111}u_{122}u_{212}u_{221}\\
&\quad - 2u_{111}^2u_{221}^2 - 6u_{112}^2u_{221}^2 + 2u_{121}^2u_{221}^2 - 2u_{122}^2u_{221}^2 + 2u_{211}^2u_{221}^2\\
&\quad - 2u_{212}^2u_{221}^2 + u_{221}^4 - 8u_{112}u_{121}u_{211}u_{222} + 8u_{111}u_{122}u_{211}u_{222}\\
&\quad + 8u_{111}u_{121}u_{212}u_{222} + 8u_{112}u_{122}u_{212}u_{222} + 8u_{111}u_{112}u_{221}u_{222}\\
&\quad + 8u_{121}u_{122}u_{221}u_{222} + 8u_{211}u_{212}u_{221}u_{222} - 6u_{111}^2u_{222}^2 - 2u_{112}^2u_{222}^2\\
&\quad - 2u_{121}^2u_{222}^2 + 2u_{122}^2u_{222}^2 - 2u_{211}^2u_{222}^2 + 2u_{212}^2u_{222}^2 + 2u_{221}^2u_{222}^2 + u_{222}^4.
\end{aligned}
\tag{5.4.11}
$$

The geometrical properties of the extreme coefficients of the ED polynomial of $X_3^\vee \subset \mathbb{P}(V^{\otimes 3}) \cong \mathbb{P}^7$ at $u \in V^{\otimes 3}$ were previously described in Example 0.0.7. We recall that $\mathrm{EDdegree}(X_3) = 6$, hence $\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2)$ is written as

$$\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2) = a_6(u)\varepsilon^{12} + a_5(u)\varepsilon^{10} + \cdots + a_0(u).$$

We determined symbolically all the coefficients of $\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2)$ with respect to the generators $\theta_1,\ldots,\theta_4,\varphi$. In particular, $\deg(a_j) = 2(10-j)$ for all $0 \le j \le 6$. For example, the coefficient $a_5(u)$ is relevant since the ratio $a_5(u)/a_6(u)$ corresponds to the sum of the squares of the singular values of u, thanks to Proposition 5.1.4. We observed that the coefficients $a_j(u)$ may be written in a more concise way using the following symmetric polynomials of the four quadratic invariants $\theta_1,\ldots,\theta_4$:

$$
\begin{aligned}
e_1(\theta_1,\theta_2,\theta_3,\theta_4) &= \theta_1 + \theta_2 + \theta_3 + \theta_4,\\
e_2(\theta_1,\theta_2,\theta_3,\theta_4) &= \theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_4 + \theta_3\theta_4,\\
e_3(\theta_1,\theta_2,\theta_3,\theta_4) &= \theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_4 + \theta_2\theta_3\theta_4,\\
e_4(\theta_1,\theta_2,\theta_3,\theta_4) &= \theta_1\theta_2\theta_3\theta_4.
\end{aligned}
$$

The coefficients $a_j(u)$ are displayed below:

$$
\begin{aligned}
a_6(u) &= e_4,\\
a_5(u) &= \left(\frac{1}{2}\right)^3 (e_3\varphi - 3e_1e_4),\\
a_4(u) &= \left(\frac{1}{2}\right)^8 \left[4e_2\varphi^2 - 2(5e_1e_3 + 24e_4)\varphi + 15e_1^2e_4 - 12e_2e_4 + e_3^2\right],\\
a_3(u) &= \left(\frac{1}{2}\right)^{10} \left[2e_1\varphi^3 - 2(2e_1e_2 + 11e_3)\varphi^2 + \left(5e_1^2e_3 + 30e_1e_4 - 4e_2e_3\right)\varphi\right.\\
&\quad \left. -5e_1^3e_4 + 12e_1e_2e_4 - e_1e_3^2 - 18e_3e_4\right],\\
a_2(u) &= \left(\frac{1}{2}\right)^{16} \left[16\varphi^4 - 8\left(3e_1^2 + 20e_2\right)\varphi^3 + 8\left(3e_1^2e_2 + 18e_1e_3 - 4e_2^2 + 60e_4\right)\varphi^2\right.\\
&\quad - 4\left(5e_1^3e_3 + 18e_1^2e_4 - 12e_1e_2e_3 - 24e_2e_4 + 44e_3^2\right)\varphi\\
&\quad \left. +15e_1^4e_4 - 72e_1^2e_2e_4 + 6e_1^2e_3^2 + 144e_1e_3e_4 + 48e_2^2e_4 - 8e_2e_3^2 - 432e_4^2\right],\\
a_1(u) &= \left(\frac{1}{2}\right)^{19} \left[-160e_1\varphi^4 + 4\left(3e_1^3 + 4e_1e_2 + 88e_3\right)\varphi^3\right.\\
&\quad - 4\left(2e_1^3e_2 + 3e_1^2e_3 - 8e_1e_2^2 - 24e_1e_4 + 52e_2e_3\right)\varphi^2\\
&\quad + \left(5e_1^4e_3 - 12e_1^3e_4 - 24e_1^2e_2e_3 + 48e_1e_2e_4 + 80e_1e_3^2 + 16e_2^2e_3 - 288e_3e_4\right)\varphi\\
&\quad \left. -3e_1^5e_4 + 24e_1^3e_2e_4 - 2e_1^3e_3^2 - 36e_1^2e_3e_4 - 48e_1e_2^2e_4 + 8e_1e_2e_3^2 + 144e_2e_3e_4 - 32e_3^3\right],\\
a_0(u) &= \mathrm{Det}(u)^2 g_1(u),
\end{aligned}
$$

where the hyperdeterminant $\mathrm{Det}(u)$ and the polynomial $g_1(u)$ are expressed as

$$
\begin{aligned}
g_1(u) &= \left(\frac{1}{2}\right)^{12} \left[-8\varphi^3 + 4e_2\varphi^2 - 2(e_1e_3 - 4e_4)\varphi + e_1^2e_4 - 4e_2e_4 + e_3^2\right],\\
\mathrm{Det}(u) &= \left(\frac{1}{2}\right)^6 (-8\varphi - e_1^2 + 4e_2).
\end{aligned}
$$

A classical expression for the hyperdeterminant Det of a $2 \times 2 \times 2$ tensor was showed in (0.0.28).

Using the coefficients above, we computed symbolically the ε^2-discriminant of $\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2)$, denoted with $\Delta_{X_3^\vee}(u)$. We know from Corollary 4.2.9 that $\Delta_{X_3^\vee} = \Delta_{X_3}$. For the moment, let $f(u)$ and $b(u)$ be the equations of the ED discriminant $\Sigma_{X_3^\vee} = \Sigma_{X_3}$ and of the bisector hypersurface $B(X_3^\vee, X_3^\vee) = B(X_3, X_3)$, respectively. We know from Proposition 4.2.4 that $f(u)$ and $b(u)$ divide $\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2)$. Then we verified that

$$
\Delta_{X_3^\vee} = e_4 f^3 b^2, \tag{5.4.12}
$$

where, taking into account relation (5.4.10),

$$b(u) = (\varphi_1 - \varphi_2) \prod_{i<j} (\theta_i - \theta_j) \tag{5.4.13}$$

$$\begin{aligned} f(u) = {} & 8\varphi^9 - 3\left(9e_1^2 - 20e_2\right)\varphi^8 - 24\left(9e_1e_3 - 4e_2^2 - 12e_4\right)\varphi^7 \\ & - 4\left(54e_1^2e_4 - 54e_1e_2e_3 + 16e_2^3 - 252e_2e_4 + 159e_3^2\right)\varphi^6 \\ & + 24\left(9e_1e_3e_4 - 24e_2^2e_4 + 7e_2e_3^2 + 90e_4^2\right)\varphi^5 \\ & - 6\left(81e_1^2e_4^2 - 72e_1e_2e_3e_4 + 45e_1e_3^3 - 8e_2^2e_3^2 + 468e_2e_4^2 - 246e_3^2e_4\right)\varphi^4 \\ & + 24\left(81e_1e_3e_4^2 + 36e_2^2e_4^2 - 42e_2e_3^2e_4 + 7e_3^4 - 324e_4^3\right)\varphi^3 \\ & - 12\left(54e_1e_2e_3e_4^2 - 27e_1e_3^3e_4 + e_2e_3^4 - 324e_2e_4^3 + 135e_3^2e_4^2\right)\varphi^2 \\ & - 72e_4\left(27e_1e_3e_4^2 - 9e_2e_3^2e_4 + 2e_3^4 - 81e_4^3\right)\varphi \\ & + 729e_1^2e_4^4 - 54e_1e_3^3e_4^2 + e_3^6 - 2916e_2e_4^4 + 972e_3^2e_4^3. \end{aligned}$$

In particular, the ED discriminant of X_3 cut out by $f(u)$ is an irreducible hypersurface of degree 36 in $\mathbb{P}^7_{\mathbb{C}}$, whereas the polynomial $b(u)$, which defines set-theoretically the bisector hypersurface $B(X_3, X_3)$ (see Definition 4.1.5), has degree 16. Note also that the exponents appearing in (5.4.12) confirm the computations made in Example 4.2.6.

In the following, we assume that $u \in V^{\otimes 3}$ is μ-symmetric for $\mu \in \{(2,1),(3)\}$. Among the six critical binary tensors for u on X_3, $\mathrm{EDdegree}(X_\mu)$ of them are μ-symmetric. Below we describe the critical binary tensors that belong to $X_3 \setminus X_\mu$.

Proposition 5.4.9. (1) *Let $\mu = (2,1)$ and let $u \in S^\mu V$ be general. Then u admits four critical binary tensors on X_μ. The remaining two critical binary tensors are $x \otimes y \otimes z$ and $y \otimes x \otimes z$ for some $x, y, z \in V$. If u is real, the common singular value of the two critical points on $X \setminus X_\mu$ is real.*

(2) *Let $\mu = (3)$ and let $u \in S^\mu V$ be general. Then u admits three critical binary tensors on X_μ. The remaining three critical binary tensors are $x \otimes x \otimes y$, $x \otimes y \otimes x$ and $y \otimes x \otimes x$ for some $x, y \in V$. If u is real, the common singular value of the three critical points on $X \setminus X_\mu$ is real.*

Proof. By Proposition 5.3.9, $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ divides $\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2)$ with multiplicity one when $u \in S^\mu V$. Let us discuss part (1). Then u admits $\mathrm{EDdegree}(X_\mu) = 4$ (see (5.1.2)) critical binary tensors corresponding to four singular vector triples (x_j, x_j, y_j) for some $x_j, y_j \in V$, $j \in [4]$. Moreover, for any singular vector triple (x, y, z) for u with singular value σ and $x \neq y$, the permutation (y, x, z) is again a singular vector triple for u, and shares the same singular value σ. Hence, there is a linear polynomial $h(\varepsilon^2)$ such that

$$\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2) = \mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)\cdot h(\varepsilon^2)^2.$$

In conclusion, apart from the μ-symmetric singular vector tuples, there is room left only for one more nonsymmetric singular vector triple (x,y,z) and its permutation (y,x,z). Moreover, if $u\in S^\mu V^{\mathbb{R}}$, the root of the linear polynomial $h(\varepsilon^2)$ must be real.

Now let us look at part (2). Then u admits $\mathrm{EDdegree}(X_{(3)})=3$ (see (5.1.2)) critical binary tensors corresponding to three singular vector triples (x_j,x_j,x_j) for some $x_j\in V$, $j\in[3]$. With a similar argument of part (1), we observe that there is a linear polynomial $\tilde{h}(\varepsilon^2)$ such that

$$\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2) = \mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)\cdot h'(\varepsilon^2)^3.$$

We recall that the polynomial $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ was computed symbolically in Example 2.4.2. In conclusion, apart from the μ-symmetric singular vector tuples, there is room left only for one more nonsymmetric singular vector triple of the form (x,x,y) for some $x,y\in V$, together with its permutations (x,y,x) and (y,x,x). Moreover, if u has real entries, the root of $h'(\varepsilon^2)$ must be real. $\square$

Remark 5.4.10. Let us examine Proposition 5.4.9(1). In this case the invariants θ_2 and θ_3 introduced in (5.4.11) coincide. This implies that the highest coefficient $a_6=\theta_1\theta_2\theta_3\theta_4$ of $\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2)$ splits into two factors $\theta_1\theta_4$ and $\theta_2\theta_3=\theta_2^2$, which correspond to the highest coefficients of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ and $h(\varepsilon^2)^2$, respectively. About the lowest coefficient a_0, from (0.0.29) we see that in this case the polynomials $f_{3,\{1\}}$ and $f_{3,\{3\}}$ coincide. Indeed, the lowest coefficients of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ and $h(\varepsilon^2)^2$ are respectively $\mathrm{Det}^2\cdot f_{3,\{2\}}$ and $f_{3,\{1\}}\cdot f_{3,\{3\}}=f_{3,\{1\}}^2$. More precisely, $\mathrm{Det}=f_\mu$ and $f_{3,\{2\}}=f_{\mu,\{2\}}$. We computed symbolically the ED polynomial of $X_3^\vee$ at a μ-symmetric tensor u. In particular,

$$h(\varepsilon^2)=16\theta_2\varepsilon^2-\theta_1\theta_2-\theta_2\theta_4+2\varphi,$$

where φ was defined in (5.4.10). In addition, a consequence of Proposition 5.4.6 is that, up to sign multiplication, the highest and lowest coefficients of $h(\varepsilon^2)$ are SOS polynomials. In particular, the root of $h(\varepsilon^2)$ may be written as

$$\begin{aligned}16\varepsilon^2 &= \frac{\theta_1\theta_2+\theta_2\theta_4-2\varphi}{\theta_2}\\ &= \frac{(c_{01}c_{10}-c_{11}c_{20}-c_{00}c_{11}+c_{10}c_{21})^2+(c_{00}c_{21}-c_{01}c_{20})^2}{(c_{00}+c_{20})^2+(c_{01}+c_{21})^2},\end{aligned}$$

where we are using μ-symmetric variables $\{c_{ij}\}$ defined analogously to the beginning of Section 5.1.

Now take into account Proposition 5.4.9(2). Looking at their definition in (5.4.11), in this case the invariants θ_2, θ_3 and θ_4 coincide. Indeed the highest coefficient $a_6 = \theta_1\theta_2\theta_3\theta_4$ of $\mathrm{EDpoly}_{X_3^\vee,u}(\varepsilon^2)$ splits into two factors θ_1 and $\theta_2\theta_3\theta_4 = \theta_2^3$, which correspond to the highest coefficients of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ and $h'(\varepsilon^2)^3$, respectively. About the lowest coefficient a_0, from (0.0.29) we see that in this case the polynomials $f_{3,\{1\}}$, $f_{3,\{2\}}$ and $f_{3,\{3\}}$ coincide. Indeed, the lowest coefficients of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ and $h'(\varepsilon^2)^3$ are respectively Det^2 and $f_{3,\{1\}}\cdot f_{3,\{2\}}\cdot f_{3,\{3\}} = f_{3,\{1\}}^3$. More precisely, $\mathrm{Det} = f_\mu$. Moreover, in this case

$$h'(\varepsilon^2) = 16\theta_2\varepsilon^2 - \theta_1\theta_2 - \theta_2^2 + 2\varphi$$

and the root of $h'(\varepsilon^2)$ may be expressed as (using the coordinates $\{c_j\}$ of the symmetric tensor t)

$$16\varepsilon^2 = \frac{\theta_1\theta_2 + \theta_2^2 - 2\varphi}{\theta_2} = \frac{(c_1^2 - c_2^2 - c_0c_2 + c_1c_3)^2 + (c_0c_3 - c_1c_2)^2}{(c_0 + c_2)^2 + (c_1 + c_3)^2}.$$

Remark 5.4.11. More generally, one may verify that for any partition $\mu \vdash d$ and for a general symmetric tensor $u \in S^dV$ (not necessarily a binary tensor), the polynomial $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ is divided by $\mathrm{EDpoly}_{X_{(d)}^\vee,u}(\varepsilon^2)$ and by other factors. We observed that there is a precise relation between the factors of $\mathrm{EDpoly}_{X_\mu^\vee,u}(\varepsilon^2)$ and the dual multiple root loci $[\mathrm{Chow}_\lambda\mathbb{P}(V)]^\vee$ (see (5.3.9)) for all $\lambda \prec \mu$, that somehow shifts the work by Oeding [Oed] from symmetrizations of μ-discriminants to symmetrizations of their respective ED polynomials. This fact encourages another line of research on the ED polynomials of varieties of tensors.

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