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A Skew Invariant of Plane Cubic Curves Related to the Hesse Pencil

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Introduction

Let V be a vector space of dimension $n+1$, endowed with the natural action of the general linear group $GL(n+1)$. This action extends to an action of the same group on Sym^dV, the space of homogeneous polynomial of degree d in $n+1$ variables, which in turn can be viewed as an action on the coefficients a_i of such d-forms.
An invariant is a multilinear map $Sym^dV \times \cdots \times Sym^dV \longrightarrow \mathbb{C}$, which remains invariant under the action of $SL(n+1)$.
The theory of invariants has been extensively studied since the classical work of Hilbert, who proved the fundamental finiteness theorem for the ring of invariants (Theorem 4.9). Since then, the study of invariants has remained a central topic in algebra and geometry. When investigating a given invariant, a key question is to understand its vanishing locus, that is, the set of forms on which it vanishes, and what geometric or algebraic properties these forms posses.

Historically, much attention has been devoted to symmetric invariants, which reflect intrinsic properties of a single form. In this thesis we focus on an skew invariant $R \in \bigwedge^3(Sym^3\mathbb{C}^3)$ that relates three plane cubic curves:

$$R : Sym^3\mathbb{C}^3 \times Sym^3\mathbb{C}^3 \times Sym^3\mathbb{C}^3 \longrightarrow \mathbb{C}$$

We focused on the study of the locus defined by $R(f, g, -) \equiv 0$, while the question of determining the geometric conditions under which $R(f, g, h) = 0$ remains open. Our analysis begins by observing that $R(f, H(f), -) = 0$ for a generic smooth cubic f whose Hessian is denoted by $H(f)$. Then, we examine how this relation changes when f is a singular cubic.
We define an algebraic subvariety N of the Grassmannian of lines in $\mathbb{P}^9$, such that

$$N = \overline{\{L \in G(1,9) \mid R(f, g, -) = 0, \quad \forall \ f, g \in L\}}$$

The main result of this thesis is the proof that N coincides with the closure of the orbit of the Hesse pencil under the action of $SL(3)$. We denote this orbit by $O(\langle x^3 + y^3 + z^3, xyz\rangle)$, and refer to its closure as the Hesse Pencil Variety:

$$N = \overline{O(\langle x^3 + y^3 + z^3, xyz\rangle)}$$

We have shown that both varieties have dimension 8, and that their multidegree with respect to the Schubert cycle decomposition is the same, namely $(1, 3, 9, 12, 6)$. This guarantees that the two varieties coincide in dimension 8. As for the lower-dimensional orbits, we proceed by analyzing all the orbits contained in N, and verifying that they are also contained in the Hesse Pencil Variety.
Finally, we are also able to show that this variety is not smooth, as it contains two singular orbits, namely those of $\langle x^2y, x^2z\rangle$ and $\langle x^3, x^2y\rangle$.

We now provide a more detailed description of the contents of each chapter.
The first four chapters are introductory and serve to present the tools necessary for the subsequent analysis. In particular, the first chapter is devoted to the study and description of the irreducible representation of $GL(n+1)$, while the second chapter introduces fundamental concepts related to d-forms, that is, elements in Sym^dV. These topics form the essential background needed to approach the theory of invariants, which is the focus of Chapter 4. Indeed, the objects we aim to study arise

precisely from the action of $GL(n+1)$ on the space of d-forms. Chapter 3 is dedicated to Grassmann varieties, with particular emphasis on the decomposition of a variety into Schubert cycles. This decomposition, as anticipated, plays a key role in the main result of the thesis.
Chapter 5 serves as a "toy model" that helps to anticipate and better understand the construction later developed in the case of plane cubics. We begin by observing that binary quartics-elements in $Sym^4\mathbb{C}^2$- and plane cubics-elements of $Sym^3\mathbb{C}^3$- are the only two cases in which the Hessian map sends a d-forms to another d-forms of the same degree. For this reason, we first examine binary quartics, which involve lower-dimensional objects, before addressing the case of plane cubics. The analogy between the two settings lies in the fact that, in both, one can define a Hesse Pencil Variety, i.e., the orbit closure of the pencil generated by a smooth form and its Hessian. In both contexts, we study the dimension, the multidegree, the orbits contained in the variety, and its singular locus. In the binary quartic case, this variety is a smooth Fano threefold already studied in literature. In this thesis, we re-express it from a different prospective, providing an alternative description of this classical object.
Chapter 6 recalls some well-known results on plane cubics, with a particular focus on their relation with the Hessian. Of particular importance for the study in the following chapters is the $Hesse\ configuration$, namely arrangement of the nine inflection points of a smooth cubic. These nine points lie on twelve lines, with the property that any line through two of them contains a third. Equally relevant is the explicit description of the fibers of the Hessian map $H^{-1}(f)$, as f varies along the orbits of $SL(3)$.
The final two chapters are devoted to the original research. In Chapter 7, we introduce the invariant R, providing its explicit formula. We prove that for a generic cubic f, one has $R(f, H(f), -) = 0$, and we describe the space $\{g \in Sym^3\mathbb{C}^3 \ | R(f, g, -) = 0\}$ as f varies along the orbits of the action of $SL(3)$. In Chapter 8, we introduce the two varieties: the variety N, whose equations are explicitly known since they derive from the explicit expression of R, and the Hesse Pencil Variety S. The entire chapter is devoted to proving that $N = S$, following the strategy previously outlined. Since the equations of N are known, we are able to perform explicit computations using the software $Macaulay2$. On the other hand, we study S from a theoretical perspective, for instance to compute its dimension and multidegree. In particular, in order to compute the multidegree of S, we were naturally led to study the geometry of the Hesse configuration of nine points in $\mathbb{P}^2$. We proved that, given 4 general points in the plane, there exist exactly 6 distinct Hesse configurations containing them, and this turned out to be crucial for completing the proof.

1 Representation Theory

In this section, we will first provide general notions from representation theory, and then focus on the representations of the general linear group.

1.1 Fundamental Concepts

Definition 1.1 ($\mathbb{C}$-algebra)**.** *Let A be a complex vector space that also has a unital ring structure. If*

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \quad \forall a, b \in A \text{ and } \lambda \in \mathbb{C}$$

holds, then A is called a $\mathbb{C}$-algebra.

Given a finite group G it is possible to define a $\mathbb{C}$-algebra as

$$\mathbb{C}G := \left\{ \sum_{g \in G} \lambda_g e_g \; | \lambda_g \in \mathbb{C} \right\} \tag{1.1}$$

and this is called the *Group Algebra of G*. The elements $\{e_g | g \in G\}$ form a basis of $\mathbb{C}G$ as a vector space over $\mathbb{C}$. The addition and multiplication are defined as follows:

$$\sum_{g \in G} \lambda_g e_g + \sum_{h \in G} \mu_h e_h = \sum_{g \in G} (\lambda_g + \mu_g) e_g$$

$$\sum_{g \in G} \lambda_g e_g \cdot \sum_{h \in G} \mu_h e_h = \sum_{h,g \in G} (\lambda_g \mu_h) e_{gh}.$$

The neutral element is e_1. To simplify the notation, we will write g instead of e_g, and thus an element in $\mathbb{C}G$ is written as $\sum_{g \in G} \lambda_g g$.

Definition 1.2 (Representation)**.** *Let G be a group. A group homomorphism*

$$\rho : G \to GL(V) \simeq GL_{dim(V)}(\mathbb{C})$$

where $GL(V)$ denotes the general linear group of a complex vector space V, is called a representation of G.

Remark 1.3. *By linearity, ρ can be extended to a morphism $\overline{\rho} : \mathbb{C}G \longrightarrow Mat_{dim(V)}(\mathbb{C})$, that is*

$$\overline{\rho}\left(\sum_{g \in G} \lambda_g g \right) = \sum_{g \in G} \lambda_g \rho(g).$$

Since the construction can be inverted, we will refer to both simply as a representation of G.

Note that it follows immediately from the definition that this is equivalent to stating that V is a $G - module$, that is, there exists an action of G on V(we write $G \curvearrowright V$):

$$G \times V \longrightarrow V$$

with $g \cdot v := \rho(g)v$. In the following, we will not distinguish a representation from the $G - module$ associated with it.

Definition 1.4 (Character). *Let ρ be a representation of G. We define the character of G associated with ρ as:*

$$\chi_\rho : G \longrightarrow \mathbb{C}$$

where $\chi_\rho(g) := trace(\rho(G))$

If V is the G-module associated with ρ, we can also use the notation $\chi_V = \chi_\rho$. It follows from the properties of the trace of a matrix that $\chi_\rho(h^{-1}gh) = \chi_\rho(g)\ \forall \rho$ and from this, it follows that $\chi_\rho(g)$ can be expressed as a function of the eigenvalues of the matrix $\rho(g)$. Note that

$$\chi_\rho(1) = trace(Id) = dim(V). \tag{1.2}$$

An important non-trivial fact about characters is that the isomorphism class of a representation is uniquely determined by the character associated with it.

Remark 1.5. *[Formal Character] Suppose that $G = GL(n)$ is a linear group. Then, χ_ρ is a polynomial in the entries $g_{(i,j)}$ of the matrix $g \in GL(n)$. Moreover, due to the properties of characters, it is invariant under conjugation by $GL(n)$. It follows from Lemma 4.1.4 in [17] that this polynomial can be expressed as a symmetric polynomial in terms of the eigenvalues $t_1, .., t_n$ of a matrix $g \in G$. When written in this form, the character is referred to as the $formal\ \ character$ of ρ.*

Let (V, ρ) be a G-module. If W is a subspace of V invariant under the action of G, then we call $(W, \rho\restriction_W)$ a *submodule* of V. We say that (V, ρ) is *irreducible* if there does not exist any proper submodule of V. Another important non-trivial fact in representation theory is that every representation of a reductive[1] group G can be written as a direct sum of irreducible ones. Moreover, if $V = \oplus_i V_i$ is the decomposition in irreducible representations, choosing the basis of V as the union of the bases of V_i in an ordered manner, we have

$$\rho(g) = \begin{bmatrix} \rho_1(g) & & & & \\ & \rho_2(g) & & & \\ & & ... & & \\ & & & ... & \\ & & & & \rho_k(g) \end{bmatrix}$$

where (ρ_i, V_i) for $i = 1, .., k$ are the representations of V_i which respect the chosen basis, that is $\rho_i(g) \in GL_{dimV_i}\mathbb{C}$.

We denote by V^G the set of elements that remain invariant under the action of G, that is

$$V^G := \{v \in V | g \cdot v = v \quad \forall g \in G\}; \tag{1.3}$$

this is a submodule of V on which the action of G is trivial.

[1] In characteristic zero, saying that a group is reductive is equivalent to requiring that every rational representation is completely reducible. We will adopt this definition and restrict our attention to this setting. Notably, the groups $GL(n)$ and $SL(n)$ are reductive, and they will be the main focus in the discussion that follows.

Let V and W be two vector spaces, and let G be a group acting on both. Then, the action of G on $V \oplus W$ and $V \otimes W$ given by

$$g \cdot (v \oplus w) = (g \cdot v) \oplus (g \cdot w)$$

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$$

are well defined. Moreover, the character associated with these representations are the sum and the product of the two characters respectively, that is

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

$$\chi_{V \otimes W} = \chi_V \cdot \chi_W$$

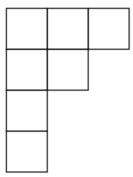
Let $\rho : G \longrightarrow GL(V)$ the action of G on V. We denote by V^* the dual vector space of V. The *dual representation* of V is define as the representation of G on V^* given by:

$$\rho^* : G \longrightarrow GL(V^*) \ \ such\,that \ \ \rho^*(g)(\phi)(v) = \phi(\rho(g^{-1})(v)) \quad for\,all \ \ g \in G, \ \phi \in V^* \ \ and \ \ v \in V$$

Moreover, if (ρ, V) is an irreducible representation, then its dual (ρ^*, V^*) is also irreducible.

1.2 $GL(V)$-Representations

Let Σ_d be the symmetric group, whose elements are the permutation of d points, its order is $d!$. To a partition $\lambda = (\lambda_1, .., \lambda_k)$ of d, with $\lambda_i \geq \lambda_{i+1}$, is associated a *Ferrers diagram* with λ_i boxes in the $i-th$ row. For example, the Ferrers diagram associated to $\lambda = (3, 2, 1, 1)$, that is a partition of 7, is



We define a *tableau* to be a numbering of the boxes by the integers $1, .., d$ and, by a slight abuse of notation, we shall continue to denote this object by λ. Given a tableau, define two subgroups of the symmetric group

$$P = P_\lambda = \{g \in \Sigma_d | g \ \ preserves \ \ each \ \ row\}$$

and

$$Q = Q_\lambda = \{g \in \Sigma_d | g \ \ preserves \ \ each \ \ column\}$$

Consider the group algebra $\mathbb{C}\Sigma_d$ as in definition (1.1), and define these two elements in it:

$$a_\lambda = \sum_{g \in P} e_g \ \text{ and } \ b_\lambda = \sum_{g \in Q} sgn(g) \cdot e_g$$

If V is any vector space, Σ_d acts on the $d-th$ tensor power $V^{\otimes d}$ by permuting factor, that is

$$(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}. \tag{1.4}$$

Therefore, Remark 1.3 gives a morphism $\mathbb{C}\Sigma_d \longrightarrow End(V^{\otimes d})$ and identifying a_λ and b_λ with their respective images, we have

$$Im(a_\lambda) = Sym^{\lambda_1}V \otimes \cdots \otimes Sym^{\lambda_k}V \subset V^{\otimes d}$$
$$Im(b_\lambda) = \wedge^{\mu_1}V \otimes \cdots \otimes \wedge^{\mu_k}V \subset V^{\otimes d}$$

where μ is the *conjugate partition* to λ, that is defined by interchanging rows and columns in the Ferrers diagram, i.e., reflecting the diagram in the 45° line. We set

$$c_\lambda := a_\lambda \cdot b_\lambda \in \mathbb{C}\Sigma_d; \tag{1.5}$$

this is called a *Young symmetrizer*. For example, when $\lambda = (d)$, $c_{(d)} = a_{(d)} = \sum_{g\in\Sigma_d} e_g$ and the image of $c_{(d)}$ on $V^{\otimes d}$ is $Sym^d V$. When $\lambda = (1,..,1)$, $c_{(1,..,1)} = b_{(1,..,1)} = \sum_{g\in\Sigma_d} sgn(g)e_g$, and the image of $c_{(1,..,1)}$ on $V^{\otimes d}$ is $\wedge^d V$.
We will see that the images of the symmetrizers c_λ in $V^{\otimes d}$ provide essentially all the finite-dimensional irreducible representations of $GL(V)$.

It is well known that for any finite-dimensional complex vector space V , we have the canonical decomposition

$$V \otimes V = Sym^2 V \oplus \wedge^2 V.$$

The group $GL(V)$ clearly acts on V and thus, it acts on $V \otimes V$ as seen in the previous paragraph. We will see that this is the decomposition of $V \otimes V$ into a direct sum of irreducible $GL(V)$-representations.
Instead, if we increase the number of factors and consider $V^{\otimes d}$, we have:

$$V^{\otimes d} = Sym^d V \oplus \wedge^d V \oplus \textit{other spaces}, \tag{1.6}$$

just as $Sym^d V$ and $\wedge^d V$ are images of symmetrizing operators from $V^{\otimes d}$ to itself, so are the other factors. The symmetric group Σ_d acts on $V^{\otimes d}$, say on the right, by permuting factors as in 1.4 and this action commutes with the left action of $GL(V)$.

Let λ be a partition of d, a *standard Young tableau of shape* λ (for short: $SYT\lambda$) is a filling T of the Ferrers diagram of λ with integers $1, 2, .., d$ (without repetitions) such that the rows and the columns are increasing. For instance, here are some $SYT\lambda$ corresponding to the partition $\lambda = (3, 2, 1, 1)$ of 7:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \\ 7 & & \end{matrix} \qquad \begin{matrix} 1 & 2 & 3 \\ 4 & 7 & \\ 5 & & \\ 6 & & \end{matrix} \qquad \cdots \qquad \begin{matrix} 1 & 5 & 7 \\ 2 & 6 & \\ 3 & & \\ 4 & & \end{matrix}$$

Now, we denote by c_T the Young symmetrizer associated with the standard Young tableau T, as we defined in 1.5.
Since the actions of Σ_d and $GL(V)$ on $V^{\otimes d}$ commute, we can give the following definition:

Definition 1.6. *Let λ be any partition of d and T a standard Young tableau of shape λ. We denote the image of c_T an $V^{\otimes d}$ by $\mathbb{S}_T V$ and we call it the Weyl module associated with T:*

$$\mathbb{S}_T V := Im(c_T : V^{\otimes d} \longrightarrow V^{\otimes d}) \tag{1.7}$$

which is a representation of $GL(V)$.

Observe that there is only one way to obtain a standard Young tableau (SYT) from the partitions $d = d$ e $d = 1 + ... + 1$. Thus, as we have previously observed, the partition $d = d$ corresponds to $\mathbb{S}_{(d)}V = Sym^dV$ while the partition $d = 1 + .. + 1$ corresponds to $\mathbb{S}_{(1,..,1)}V = \wedge^dV$.

Theorem 1.7. *The Weyl modules $\mathbb{S}_TV$, with T a $SYT\lambda$ and λ a partition of d, are precisely the irreducible $GL(V)-$modules that appears in the decomposition of $V^{\otimes d}$. The Young symmetrizers c_T define an isomorphism of $GL(V)-$module:*

$$V^{\otimes d} \cong \bigoplus_{\lambda \vdash d} \bigoplus_{T\ SYT\lambda} \mathbb{S}_TV$$

For instance, in the case $d = 2$, there are only two possible partitions: $2 = 2$ and $2 = 1 + 1$. For each of them, there is only one associated standard Young tableau. Thus, we have:

$$c_{\begin{smallmatrix}1&2\end{smallmatrix}} \oplus c_{\begin{smallmatrix}1\\2\end{smallmatrix}} : V^{\otimes 2} \longrightarrow \mathbb{S}_{\begin{smallmatrix}1&2\end{smallmatrix}}V \oplus \mathbb{S}_{\begin{smallmatrix}1\\2\end{smallmatrix}}V \cong Sym^2V \oplus \wedge^2V$$

For $d = 3$ there are three different partitions and four different SYT's. We have:

$$V^{\otimes 3} \cong \mathbb{S}_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}V \oplus \mathbb{S}_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}V \oplus \mathbb{S}_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}V \oplus \mathbb{S}_{\begin{smallmatrix}1&2&3\end{smallmatrix}}V$$

If T and T' are SYT of the same shape λ, then $\mathbb{S}_T$ and $\mathbb{S}_{T'}$ are isomorphic $GL(V)$-modules, and so we can write $\mathbb{S}_\lambda := \mathbb{S}_T \cong \mathbb{S}_{T'}$. The following theorem reformulates the previous one in terms of this information and adds important details about the characters associated with these Weyl modules.
We denote by $S_\lambda = S_\lambda(x_1, .., x_n)$ the *Schur polynomial* associated with the partition λ. A detailed analysis of these polynomials can be found in [8].

Theorem 1.8.

- *Let $n = dimV$. Then $\mathbb{S}_\lambda V$ is zero if $\lambda_{n+1} \neq 0$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, then*

$$dim\ \mathbb{S}_\lambda V = S_\lambda(1, .., 1) = \prod_{1\leq i\leq j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

- *Let m_λ be the number of SYT of shape λ. Then*

$$V^{\otimes d} \cong \bigoplus_{\lambda \vdash d} \mathbb{S}_\lambda V^{\oplus m_\lambda}.$$

- *For any $g \in GL(V)$, let $x_1, .., x_n$ be the eigenvalues of g:*

$$\chi_{\mathbb{S}_\lambda V}(g) = S_\lambda(x_1, .., x_n). \tag{1.8}$$

An important point we wish to emphasize about Schur polynomials is the following: as λ varies over the partitions of d into at most n parts, these polynomials S_λ form a basis for the symmetric polynomials of degree d in these n variables. If $\lambda = d$, then $S_\lambda(x_1, .., x_n)$ is the sum of all possible monomials of degree d in $x_1, .., x_n$. If $\lambda = 1 + .. + 1$, then $S_\lambda(x_1, .., x_n)$ is the $d-th$ elementary symmetric polynomial in $x_1, .., x_n$, that is, the sum of all possible monomials of degree d without repetitions. A $semi-standard\ \ Young\ \ tableau$ of shape λ (for short $SSYT\lambda$) is a filling U of the Ferrers diagram of λ with integers $1, .., n$ (repetition is allowed) such that the rows of U are weakly increasing and the columns are strictly increasing. This definition allows us to write a formula for Schur polynomials,that is

$$S_\lambda(x_1, .., x_n) = \sum_{U\ \ SSYT\lambda} \prod_{i=1}^{n} x_i^{\#i's\ in\ U} \tag{1.9}$$

Let's make an example. We consider $d = 3$, $n = 3$ and $\lambda = (2, 1)$. There are 8 possible $SSYT\lambda$ and they are:

$$\begin{matrix} 1 & 1 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 2 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 3 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 3 & \end{matrix}$$

$$\begin{matrix} 1 & 2 \\ 3 & \end{matrix} \qquad \begin{matrix} 1 & 3 \\ 3 & \end{matrix} \qquad \begin{matrix} 2 & 2 \\ 3 & \end{matrix} \qquad \begin{matrix} 2 & 3 \\ 3 & \end{matrix}$$

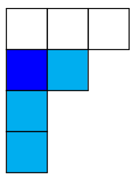
Thus, using the formula, we obtain

$$S_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

As for the number m_λ, it is given by the following formula:

$$m_\lambda = \frac{d!}{\prod(Hook\ \ lengths)}$$

In the denominator, we have the product of the lengths of all the hooks that appear in the representation of λ as a Ferrers diagram. To clarify, let us consider $\lambda = (3, 2, 1, 1)$, and choose a square in the Ferrers diagram (blue). The hook associated with this square consists of the set of all the squares below it and those to the left of it (cyan).



By counting all these squares, we obtain the length of the corresponding hook, which can then be recorded in the diagram. Finally we display the complete diagram with the lengths of all the hooks.

6	3	1
4	1	
2		
1		

In the denominator of the formula 1.2, all the numbers written in the diagram appear, so that we obtain

$$m_\lambda = \frac{7!}{6 \cdot 3 \cdot 4 \cdot 2} = 35$$

Theorem 1.9. *As $d \in \mathbb{N}$ and $\lambda \vdash d$ vary, the Weyl module $\mathbb{S}_\lambda V$ form the complete set of irreducible representations of $GL(V)$.*

One of the main problems in representation theory is to decompose a given representation into a direct sum of irreducible ones. In particular, if W is a $GL(V)$-module, then, as a consequence of the theory developed so far, it decomposes as

$$W \cong \bigoplus_\lambda c_\lambda \, \mathbb{S}_\lambda V$$

The coefficients c_λ are called *multiplicities* and finding them provides a solution to the problem stated above. Since we have stated that each representation is uniquely determined by its character, the problem of determining the multiplicities c_λ is a problem in the symmetric polynomials. In fact,

$$\chi_W = \sum_\lambda c_\lambda S_\lambda$$

where χ_W is the formal character of W (defined in Remark 1.5), that is a symmetric polynomial in the eigenvalues of a matrix. We have already recalled that Schur polynomials provide a basis for this space, so the coefficients c_λ are (theoretically) uniquely determined. Moreover, in [17] is presented the algorithm to compute them.

Remark 1.10. *Let us consider the irreducible representation $\mathbb{S}_\lambda V$ of $GL(n)$. As we know, its dual is also irreducible and, by Theorem 1.9, it is again associated with a Young diagram λ^*. In particular, if we embed the partition λ into a rectangular Young diagram with n rows and λ_1 columns, then the partition λ^* is the complement of λ within this rectangle.*
For example, if $n = 3$ and $\lambda = (5,1)$, then the dual representation corresponds to the partition $\lambda^* = (5,4)$, which is the complement of λ in the rectangle of shape $(5,5,5)$.

Let λ and μ be two partitions of d and m respectively, and consider the composition $\mathbb{S}_\mu(\mathbb{S}_\lambda V)$. The problem of decomposing this functor into irreducible representations is called *Plethysm problem.*

It can been shown and it is in [8] that

$$\mathbb{S}_\mu(\mathbb{S}_\lambda(V)) = \bigoplus_\nu M_{\lambda\mu\nu} \mathbb{S}_\nu V$$

where the sum is over all partitions $\nu \vdash dm$ and $M_{\lambda\mu\nu}$ are non-negative integers. In this regard, note that $\mathbb{S}_\lambda V \subset V^{\otimes d}$ and $\mathbb{S}_\mu(\mathbb{S}_\lambda) \subset (\mathbb{S}_\lambda V)^{\otimes m} \subset V^{\otimes dm}$.

$$\begin{array}{ccc}
(V^{\otimes d})^{\otimes m} \xrightarrow{(c_\lambda)^{\otimes m}} (\mathbb{S}_\lambda V)^{\otimes m} \subset (V^{\otimes d})^{\otimes m} & \xrightarrow{c_\mu} & \mathbb{S}_\mu(\mathbb{S}_\lambda V) \subset (\mathbb{S}_\lambda V)^{\otimes m} \\
 & \searrow^{\oplus M_{\lambda\mu\nu} c_\nu} & \updownarrow \\
 & & \bigoplus_\nu M_{\lambda\mu\nu} \mathbb{S}_\nu V
\end{array}$$

Proposition 1.11. *Let $\lambda \vdash d$ and $\mu \vdash m$ be two partitions. Suppose n divides dm and let $ng = dm$. Then*

$$\mathbb{S}_\mu(\mathbb{S}_\lambda V)^{GL(V)} \cong c_{(g,..,g)} \mathbb{S}_{(g,..,g)} V \cong c_{(g,..,g)} \mathbb{C}^*$$

where $c_(g,..,g) = M_{\lambda\mu(g,..,g)}$.
Moreover,

$$dim(\mathbb{S}_\mu(\mathbb{S}_\lambda V)^{GL(V)}) = \begin{cases} c_{(g,..,g)} & if \quad n \quad divides \quad dm \\ 0 & otherwise \end{cases} \tag{1.10}$$

Proof. Let $\overline{\nu}$ be the partition of dm given by $\overline{\nu} = \underbrace{(g, g, .., g)}_{n}$, and consider $\mathbb{S}_{\overline{\nu}}V$. It equals the one dimensional $GL(V)$-module given by

$$det^g : \; GL(V) \longrightarrow \mathbb{C}^*$$

This is a group morphism thanks to the proprieties of the determinant and thus it defines a one-dimensional representation. Moreover, its formal character is given by

$$\chi_{det^g}(x_1, .., x_n) = trace([det^g(diag(x_1, .., x_n))]) = det^g(diag(x_1, .., x_n)) = x_1^g x_2^g \cdots x_n^g$$

where $x_1, .., x_n$ are the eigenvalues of a matrix in $GL(V)$.
We also know from Theorem 1.8 that the character associated with $S_{\overline{\nu}}V$ is given by $S_{(g,g,..,g)}(x_1, .., x_n) = x_1^g \cdots x_n^g$. For this last equality, we can simply use the formula (1.9), noting that there is only one possible $SSYT$, which is obtained by placing the number $1, .., n$ in each column.
Hence, since these two modules have the same character, they must be isomorphic, that is, $\mathbb{S}_{\overline{\nu}}V \cong \mathbb{C}^*$. Furthermore, this means that

$$\mathbb{S}_\mu(\mathbb{S}_\lambda V) \cong \underbrace{\mathbb{C}^* \oplus \cdots \oplus \mathbb{C}^*}_{M_{\lambda\mu\overline{\nu}}} \oplus \cdots$$

as a $GL(V)$-module. For every $\mathbb{C}$ that appears in the decomposition of $\mathbb{S}_\mu(\mathbb{S}_\lambda V)$, there must be an element that generates it and, therefore, must be an invariant under the action of $GL(V)$.

To prove the converse, namely that all invariants are obtained this way, it will be sufficient to observe that no other module appearing in the decomposition can have dimension one. The dimension of a Weyl module is simply its formal character evaluated at $(1, .., 1)$, and thus, once again, we can use the formula in (1.9). Whenever we have a Ferrers diagram with a column shorter than n, there are always more than one way to fill it, and this implies that the dimension of the associated module cannot be 1. □

In the following chapters, we will be interested in $Sym^m(Sym^d V)$ and $\wedge^m(Sym^d V)$. Applying the previous discussion to these two cases, we will be able to determine, at least in principle, whether these spaces contain invariants or not.

2 On Degree-d Forms and Their Hessian

In this chapter, we briefly present some fundamental facts about homogeneous polynomials and their associated Hessians.

2.1 The Veronese Variety

Let V be a complex vector space of dimension n+1. The space $Sym^d(V)$ is the dth symmetric power of V and it can be identified with the space of all homogeneous polynomials of degree d in $n+1$ variables as follows:

Proposition 2.1. *We have $Sym^d(V) \cong \mathbb{C}[x_0, .., x_n]_d$, where the last one is the space of all homogeneous polynomials of degree d in $dim(V) = n+1$ variables.*

Proof. We define

$$\Psi : Sym^d(V) \longrightarrow \mathbb{C}[x_0, .., x_n]_d$$

$$\Psi(\Phi) = f_\Phi : V \to \mathbb{C}$$

where $f_\Phi(v) := \Phi(\underbrace{v, ..., v}_{d})$.

$\Phi : V^d \longrightarrow \mathbb{C}$ is a multilinear function invariant under permutation of the entries. Let $e_0, .., e_n$ be a basis of V and $x_0, .., x_n$ its dual basis, we can write

$$\Phi(v_1, .., v_d) = \sum_{i_1,..,i_d=0}^{n} \Phi(e_{i_1}, .., e_{i_d}) x_{i_1}(v_1) \cdots x_{i_d}(v_d)$$

From this, it is clear that f_Φ is a polynomial on the $x_i's$.
Vice versa if $f = \sum_{i_1,..,i_d=0}^{n} f_{i_1,..,i_d} x_{i_1} \cdot \cdot x_{i_d} \in \mathbb{C}[x_0, .., x_n]_d$ we define

$$\Phi_f(v_0, .., v_d) = \sum_{i_1,..,i_d=0}^{n} f_{i_1,..,i_d} x_{i_1}(v_0) \cdots x_{i_d}(v_d)$$

These two map are inverses, and the isomorphism is proved.

□

The map

$$\nu_d : \mathbb{P}V \longrightarrow \mathbb{P}(Sym^d(V)) \tag{2.1}$$

defined as $\nu_d(v) = v^d = v \otimes v \otimes .. \otimes v$ is called *the* $d-th$ $Veronese$ *map* and its image $\nu(\mathbb{P}V)$ is *the* $d-th$ $Veronese$ *variety*. It consists of all polynomials that are the dth power of a linear form.
Once a basis for V and the corresponding basis for $Sym^d V$ have been fixed, the Veronese map can be expressed in coordinates. Let us consider the case $n = 2$, $d = 2$:

$$\nu : \mathbb{P}^3 \longrightarrow \mathbb{P}^5$$

$$\nu([x_0, x_1, x_2]) = [x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2]$$

Keeping this in mind, the following result becomes extremely clear and simple.

Theorem 2.2. *A linear function on $Sym^d(V)$ is uniquely determined by its restriction to the Veronese variety.*

Proof. Let H be a linear function on $Sym^d(V)$. We evaluate the function H on a generic point of the Veronese variety. We obtain a homogeneous polynomial of degree d in n variables whose coefficients are the same of the linear function H. Since H is known on the Veronese variety, we can derive the coefficients. □

In equivalent way, Theorem 2.2 says that $\mathbb{P}Sym^dV$ is spanned by elements lying on the Veronese variety.
To make effective the previous Theorem, compare a general polynomial

$$f = \sum_{i_0+..+i_n=d} \frac{d!}{i_0! \cdot \cdot i_n!} a_{i_0,..,i_n} {x_0}^{i_0} \cdots {x_n}^{i_n}$$

with the d-th power of a linear form

$$(b_0x_0 + .. + b_nx_n)^d = \sum_{i_0+..+i_n=d} \frac{d!}{i_0! \cdot \cdot i_n!} {b_0}^{i_0} \cdots {b_n}^{i_n} {x_0}^{i_0} \cdots {x_n}^{i_n}$$

getting the correspondence

$$ {b_0}^{i_0} \cdots {b_n}^{i_n} \longrightarrow a_{i_0,..,i_n}. \tag{2.2}$$

2.2 The Hessian Map

We keep the notation from the previous paragraph and let $f \in Sym^dV$, that is, a homogeneous polynomial of degree d in $n+1$ variables, named x_i for $i \in \{0,..,n\}$. We define the *Hessian of f* as follows:

$$H(f) := det\left(\left(\frac{\partial^2 f}{\partial_{x_i}\partial_{x_j}}\right)_{i,j}\right) = 0 \qquad i,j \in \{0,1,..,n\}$$

It is a homogeneous polynomial of degree $(d-2)(n+1)$ in the variables x_i, that is an element of $Sym^{(d-2)(n+1)}V$.
Given a degree-d form, the variety

$$V(f) := \{p \in \mathbb{P}^n \mid f(p) = 0\} \subset \mathbb{P}^n$$

is well-defined and is the object of interest when we speak of such forms. The polynomial f uniquely determines $V(f)$, up to scalar multiplication of f itself. Moreover, we can identify f with its coefficients a_i for $i = 0,..,N$, where $N = \binom{n+d}{d} - 1$. Since scalar multiples define the same space, we will be interested in the projective space $\mathbb{P}^N = \mathbb{P}(Sym^dV)$. Thus, the Hessian map can be rewritten as:

$$H_{(d,n)} : \mathbb{P}^{\binom{n+d}{d}-1} \dashrightarrow \mathbb{P}^{\binom{n+\bar{d}}{\bar{d}}-1} \qquad with \;\; \bar{d} = (d-2)(n+1) \tag{2.3}$$

The points where this map is not defined correspond to polynomials f such that $H(f) \equiv 0$. These are called *hypersurfaces with vanishing Hessian*. This carries a natural scheme structure and is known as the $(d,n)-$*Gordan-Noether locus*, denoted by $GN_{d,n}$. We define a *cone* in $\mathbb{P}^n$ as something that does not depend on all the variables, up to projective transformation. Clearly, if the hypersurface $V(f) \subset \mathbb{P}^n$ is a cone, then f has vanishing Hessian. Hesse claimed that the converse was also true, regardless of the degree of f. Later, Gordan and Noether proved that Hesse's claim holds for $n \leq 3$, but it does not hold for $n \geq 4$. In fact, they provided a counterexample in the case $n = 4$.

Theorem 2.3. *If $n \leq 3$, then a hypersurface has vanishing hessian if and only if it is a cone.*

This topic is discussed in the article [9].
Other properties of the Hessian map are studied in [4], we limit ourselves to making a few remarks. Let us begin by noting that the only case in which this map contracts is the case $(3,1)$; in fact, we obtain $H_{(3,1)} : \mathbb{P}^3 \to \mathbb{P}^2$, and thus such a map cannot be generically finite, with generically finite meaning the generic fiber is finite. Moreover, in general, when $n = 1$, the locus where the Hessian map vanishes coincides with the rational normal curve of $\mathbb{P}^d$, which is the image of the $d-th$ Veronese map from $\mathbb{P}^1$ to $\mathbb{P}^d$:

$$[\lambda, \mu] \longmapsto [\lambda^d, \lambda^{d-1}\mu, ..., \mu^d]$$

corresponding to the binary forms of the type $(\lambda x + \mu y)^d$. We denote this curve by Γ. In particular, the following holds (Proposition 2.6 in [4]):

Proposition 2.4. *The Hessian map $H_{(d,1)}$ is generically finite onto its image, unless $d = 3$, in which case it has general fibers of dimension 1, i.e., the chords of Γ.*

Equality between the source and the target spaces of the Hessian map occurs only in the two cases $(4,1)$ and $(3,2)$, that corresponds to $\mathbb{P}(Sym^4\mathbb{C}^2)$ and $\mathbb{P}(Sym^3\mathbb{C}^3)$. Thus, in the case of binary quartics and plane cubics we have

$$H_{(4,1)} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4 \qquad\qquad H_{(3,2)} : \mathbb{P}^9 \dashrightarrow \mathbb{P}^9$$

and, as we shall see later, it is possible to consider the fixed points of these maps, which will themselves carry a scheme structure. From the previous proposition, these two maps are generically finite; in particular, $H_{(4,1)}$ has degree 2, and $H_{(3,2)}$ has degree 3.

3 Grassmannians

In this chapter, we explore Grassmann varieties, with a particular focus on Schubert calculus and the fundamental basis Theorem 3.9. This will be useful in the study of an invariant concerning plane cubics, which we will discuss in the final chapter. Before that, we will also introduce a "toy model", concerning binary quartics and the Grassmannian $G(\mathbb{P}^1, \mathbb{P}^4)$, in the chapter 5.

3.1 Definition

Let V be a vector space of dimension $n+1$. We define

$$G(d+1, V) := \{W \subset V \ linear\ subspace \mid dim(W) = d+1\},$$

there is a natural bijection with the space

$$G(d, \mathbb{P}(V)) = \{\mathbb{P}(W) \subset \mathbb{P}(V)\ projective\ subspace \mid dim(\mathbb{P}(W)) = d\}$$

We will denote this latter space by $G(\mathbb{P}^d, \mathbb{P}^n)$, or simply by $G(d,n)$.
Let $W \subset V$ be as in the definition, and let $\omega_0, ..., \omega_d$ and $\omega'_0, .., \omega'_d$ be two bases of W. Then, we have that $\omega_0 \wedge ... \wedge \omega_d = \lambda\ \omega'_0 \wedge ... \wedge \omega'_d$. Thus, each element of $G(d,n)$ determines, up to scalars, a unique element of $\bigwedge^{(d+1)} V$. This allow us to define a map, called the Plücker map,

$$\Phi : G(d,n) \longrightarrow \mathbb{P}\left(\bigwedge^{d+1} V\right) = \mathbb{P}^N$$

such that $\Phi(W) = [\omega_0 \wedge .. \wedge \omega_d]$ and $N = \binom{n+1}{d+1} - 1$.

Proposition 3.1. *The Plücker map is injective.*

Proof. Let W and Y be two subspaces of V of dimension $d+1$, and let $\omega_0, .., \omega_d$ and $y_0, .., y_d$ be two bases respectively of W and Y such that $\omega_0 \wedge ... \wedge \omega_d = \lambda\, y_0 \wedge ... \wedge y_d$. It follows that $\omega_0 \wedge ... \wedge \omega_d \wedge y_i = 0$ for all $i = 0, .., d$ which means that $y_i \in W$ for all i. Therefore, $Y \subset W$, and since they have the same dimension, it follows that $Y = W$. □

For any $(d+1) \times (n+1)$ matrix $[p_i(j)]$ with $i = 0, .., d$ and $j = 0, .., n$, and any sequence of $(d+1)$ integers $j_0, ..j_d$ with $0 \leq j_\beta \leq n$, let us denote by $p(j_0 \cdots j_d)$ the determinant of the $(d+1) \times (d+1)$-matrix $[p_i(j_\beta)]$ with $i = 0, .., d$ and $\beta = 0, .., d$. Of course, it holds that

$$p(j_0 \cdots j_d) = 0 \quad \textit{if any two of the } j_\beta \textit{ are equal,}$$

$$p(j_0 \cdots j_d) = -p(j_0 \cdots j_{\beta-1} j_{\beta+1} j_\beta j_{\beta+2} \cdots j_d) \quad for \quad \beta = 1, .., d-1$$

and

$$p(j_0 \cdots j_d) = sgn(\sigma)\ p(j_{\sigma(0)} \cdots j_{\sigma(d)}) \quad \forall\ \sigma \ \textit{permutation of} \ \{0, .., d\}$$

Thus, such a function p is uniquely determined by its values on sequences of the form $0 \leq p_0 < \cdots < p_d \leq n$, and there are exactly $N+1$ of them.
Note that, fixed a basis of V, $\{e_0, .., e_n\}$, and using coordinates with respect to the basis of $\bigwedge^{(d+1)} V$, $\{e_{i_0} \wedge ... \wedge e_{i_d}\ ,\ \ 0 \leq i_0 < ... < i_d \leq n\}$, $\Phi(W)$ is given by the minors of maximal order $d+1$ of the $(d+1) \times (n+1)$ matrix formed by the coordinates of the vectors of a basis of W. That is, if we define

$$M_W := \begin{bmatrix} - & w_0 & - \\ - & w_1 & - \\ & \cdot & \\ & \cdot & \\ - & w_d & - \end{bmatrix} \in Mat_{(d+1)\times(n+1)}\mathbb{C}$$

then the coordinate of $\Phi(W)$, corresponding to the basis element $e_{i_0} \wedge ... \wedge e_{i_d}$, is the determinant $p(i_0 \cdots i_d)$ of the matrix M_W. Therefore, considering all these determinants, they define a point $(.., p(i_0 \cdots i_n), ..)$ of $\mathbb{P}^N$. These are called *Plücker coordinates* of W.
Not every point of $\mathbb{P}^N$ arises from some d-plane in $\mathbb{P}^n$. In fact, the Plücker coordinates $p(j_0, .., j_d)$ of a d-plane must satisfy some quadratic relations. In particular, we state the following theorem, whose proof can be found in [11].

Theorem 3.2. *There is a natural bijective correspondence between the d-planes in $\mathbb{P}^n$ and the points of $\mathbb{P}^N$, whose coordinates satisfy the quadratic relations*

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0 \cdots j_{d-1} k_\lambda) p(k_0 \cdots \check{k}_\lambda \cdots k_{d+1}) = 0 \tag{3.1}$$

where $j_0 \cdots j_{d-1}$ and $k_0 \cdots k_{d+1}$ are any sequences of integers with $0 \leq j_\beta, k_\gamma \leq n$.

We can also find an open covering of $G(d,n)$ that endows it with the structure of a subvariety of $\mathbb{P}^N$. In particular, there exists a bijection between the points of $\mathbb{P}^N$ that satisfy (3.1) and for which a certain Plücker coordinate $p(k_0 \cdots k_d) \neq 0$ (for some fixed indices $0 \leq k_0 < .. < k_d \leq n$), and the set of $(d+1) \times (n+1)$ matrices whose submatrix $(d+1) \times (d+1)$ formed by the columns indexed by $k_0, .., k_d$ is the identity matrix. Clearly, such a matrix is mapped to the point in $\mathbb{P}^N$ given by the appropriate $(d+1) \times (d+1)$ minors, as explained above. Conversely, if $(\dots, p(j_0 \cdots j_d), \dots)$ is a point in $\mathbb{P}^N$ satisfying the stated properties, it corresponds to the matrix defined by $p_i(j) = \frac{p(k_0 \cdots k_{i-1} j k_{i+1} \cdots k_d)}{p(k_0 \cdots k_d)}$.
Thus, the variety of points of $\mathbb{P}^N$ whose coordinates satisfy the quadratic relations 3.1 is covered by $N+1$ copies of affine space of dimension $(d+1)(n-d)$ and the following holds:

Corollary 3.3. *$G(d,n)$ is an irreducible projective variety of dimension $(d+1)(n-d)$. Moreover, $G(d,n)$ is rational.*

Esempio 3.4 (Plücker relations of G(1,4))**.** *Let's now look at a practical example of how the Plücker relations can be derived from the open covering of the Grassmannian. We will analyze the case of projective lines in $\mathbb{P}^4$, that is, $G(1,4)$, which we will deal with later in the thesis, specifically when discussing binary quartics that give rise to the projective space $\mathbb{P}^4$.*
We know that $G(1,4)$ is embedded in a projective space of dimension 9, so we have 10 Plücker coordinates, which are:

$$p_{(i,j)} := p(i,j) \quad for \quad i = 0,..,3 \quad and \quad j = i+1,..,4$$

The local open set corresponding to $p_{(0,1)} \neq 0$ corresponds to matrices of the form

$$\begin{bmatrix} 1 & 0 & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & 1 & a_{2,3} & a_{2,4} & a_{2,5} \end{bmatrix}$$

and the Plücker coordinates of such a matrix are

$$p_{(0,1)} = 1 \qquad p_{(0,2)} = a_{2,3} \qquad p_{(0,3)} = a_{2,4} \qquad p_{(0,4)} = a_{2,5}$$

$$p_{(1,2)} = -a_{1,3} \qquad p_{(1,3)} = -a_{1,4} \qquad p_{(1,4)} = -a_{1,5}$$

$$p_{(2,3)} = a_{1,3}a_{2,4} - a_{2,3}a_{1,4} \qquad p_{(2,4)} = a_{1,3}a_{2,5} - a_{1,5}a_{2,3}$$

$$p_{(3,4)} = a_{1,4}a_{2,5} - a_{1,5}a_{2,4}$$

By substituting the previous equations into the last three and homogenizing with $p_{(0,1)}$, three Plücker relations are obtained:

$$p_{(0,1)}p_{(2,3)} = p_{(1,3)}p_{(0,2)} - p_{(1,2)}p_{(0,3)}, \qquad p_{(0,1)}p_{(2,4)} = p_{(1,4)}p_{(0,2)} - p_{(1,2)}p_{(0,4)},$$

$$p_{(0,1)}p_{(3,4)} = p_{(1,4)}p_{(0,3)} - p_{(1,3)}p_{(0,4)}$$

However, these are not sufficient to define the ideal of $G(1,4)$ in $\mathbb{P}^9$. We also need to consider the other local open sets and proceed in the same way. For example, by considering $p_{(0,2)} \neq 0$, we obtain three equations, one of which is new compared to the previous ones, namely

$$p_{(0,2)}p_{(3,4)} = p_{(0,3)}p_{(2,4)} - p_{(2,3)}p_{(0,4)}$$

By proceeding further, only one additional equation is found, which can be obtained, for instance, from $p_{(1,2)} \neq 0$ and it is

$$p_{(1,2)}p_{(3,4)} = p_{(2,4)}p_{(1,3)} - p_{(2,3)}p_{(1,4)}$$

There are thus 5 Plücker relations that define $G(1,4)$, and they are precisely the ones listed above.
Note that these equations correspond to the five pfaffians[2] 4×4 of the skew-symmetric matrix

$$\begin{bmatrix} 0 & p_{(0,1)} & p_{(0,2)} & p_{(0,3)} & p_{(0,4)} \\ -p_{(0,1)} & 0 & p_{(1,2)} & p_{(1,3)} & p_{(1,4)} \\ -p_{(0,2)} & -p_{(1,2)} & 0 & p_{(2,3)} & p_{(2,4)} \\ -p_{(0,3)} & -p_{(1,3)} & -p_{(2,3)} & 0 & p_{(3,4)} \\ -p_{(0,4)} & -p_{(1,4)} & -p_{(2,4)} & -p_{(3,4)} & 0 \end{bmatrix}$$

Thus, the Grassmannian $G(1,4)$ corresponds to the variety of 5×5 skew-symmetric matrices of rank ≤ 2.

3.2 Schubert Varieties

Let $A_0 \subset A_1 \subset \cdots \subset A_d$ be a strictly increasing sequence of $(d+1)$ linear spaces in $\mathbb{P}^n$. A $d-plane$ W in $\mathbb{P}^n$ is said to satisfy the *Schubert condition* defined by this sequence if $dim(A_i \cap W) \geq i$ for all i. We denote this set

$$\Omega(A_0, .., A_d) := \{W \in G(d,n) \mid dim(W \cap A_i) \geq i\} \subset G(d,n)$$

Esempio 3.5. *Let us consider the Grassmannian of lines in $\mathbb{P}^4$, namely $G(1,4)$. Here $d = 1$. Let A_0 be a fixed line in $\mathbb{P}^4$, and let A_1 be $\mathbb{P}^4$ itself. Then the subset $\Omega(A_0, a_1)$ of $G(1,4)$ represents the set of lines L such that:*

$$dim(A_0 \cap L) \geq 0 \quad and \quad dim(\mathbb{P}^4 \cap L) \geq 1$$

Since the second condition is automatically satisfied, $\Omega(A_0, A_1)$ represents the set of lines L intersecting A_0.

Proposition 3.6. *Let $0 \leq a_0 < \cdots < a_d \leq n$ be a sequence of integers and for $i = 0, .., d$ let A_i be the a_i-dimensional linear space in $\mathbb{P}^n$ whose points are of the form $(p(0), .., p(a_i), 0, .., 0)$. Then $\Omega(A_0, .., A_d)$ consists exactly of those points $(.., p(j_0, .., j_d), ..)$ in $G(1,d)$ satisfying $p(j_0, .., j_d) = 0$ whenever $j_i > a_i$ holds for some i.*

We will not provide a full proof of this statement, which is Proposition 3 in [11]. However, let us outline the main steps of the argument.
If the A_i are as in the proposition, and we consider a d-plane W satisfying the Schubert condition, then, by induction on i, we may pick a point $P_i = (p_0(i), .., p_n(i)) \in A_i \cap W$ such that $P_0, .., P_n$ are linearly independent and span W. The $(d+1) \times (n+1)$ matrix whose rows are the coordinates of the points P_i is of the form

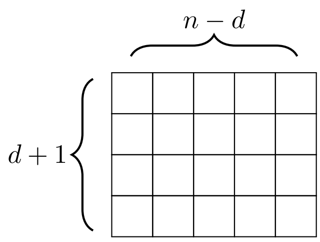
$$\begin{bmatrix} * & \cdots & * & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ * & \cdots & \cdots & * & 0 & \cdots & \cdots & \cdots & 0 \\ \\ \\ * & \cdots & \cdots & \cdots & \cdots & * & 0 & \cdots & 0 \end{bmatrix}$$

[2] Let A be a skew-symmetric matrix. Then $\det(A) = Pf(A)^2$, and $Pf(A)$ is called the Pfaffian of A. Moreover, skew-symmetric matrices have even rank.

and this is because $P_i \in A_i$.The number of zeros at the end of the $i-th$ row is exactly $n-a_i$. The minors of the matrix give the Plücker coordinates of $W \in G(d,n)$. In the proof of the proposition, this description of W is used to show that its Plücker coordinates satisfy the required property. The converse, however, requires a bit more care and is proved using what was observed after Theorem 3.2.
Let us now see how one can conventionally associate a tableau to such a set $\Omega(A_0,..,A_d)$, that is to a matrix as above. We began by observing that the first row must contain at least d zeros, the second at least one less, the third at least one less, and so on. We can therefore "exclude" these zeros from our analysis, since they arise solely from the strictly increasing condition for the sequence of spaces. In each row, there remain $n-a_i-(d-i)$ significant zeros, for $i=0,.,d$. By setting

$$\lambda_i = n-d-a_i+i \qquad for \;\; i=0,..,d \tag{3.2}$$

we obtain a Young tableau with d rows representing $\Omega(A_0,..,A_d)$. Note that $\lambda_i \geq \lambda_{i+1}$ for $i=0,..,d-1$ and that such a tableau is contained in a table of dimension $(d+1)\times(n-d)$, that is



Proposition 3.7. *Let $A_0 \subset \cdots \subset A_d$ and $B_0 \subset \cdots \subset B_d$ be two strictly increasing sequences of linear spaces in $\mathbb{P}^n$ and assume $dim(A_i)=dim(B_i)$ for $i=1,..,d$. Then there is an invertible linear transformation of $\mathbb{P}^N$ into itself which carries $G(d,n)$ into itself and $\Omega(B_0,..,B_d)$ into $\Omega(A_0,..,A_d)$.*

Proof. Since $dim(A_i)=dim(B_i)$ for all i, there obviously is an invertible $(n+1)\times(n+1)$ matrix $T=[t_{i,j}]$ such that the associated linear transformation of $\mathbb{P}^n$ into itself carries B_i into A_i for all i, $T(B_i)=A_i$. Clearly, T carries a d-plane W in $\mathbb{P}^n$ into another one $T(W)$, and if W satisfies the Schubert condition $dim(B_i\cap W)\geq i$ for all i, then $T(W)$ satisfies the Schubert condition $dim(A_i\cap T(W))\geq i$ for all i because $T(B_i)=A_i$.
Choose $(d+1)$ points $P_i=(p_i(0),..,p_i(n))$ for $i=0,..,d$ which span W. The images of these points under T generate $T(W)$; in particular, we denote

$$Q_i(j)=\sum_{\alpha=0}^{n} t_{j,\alpha}p_i(\alpha) \qquad for \;\; j=0,..,n$$

The Plücker coordinates obtained from the minors of the $(d+1)\times(n+1)$ matrix with entries given by the points Q_i are linear combinations of the Plücker coordinates of W, given by the points P_i.
We have thus obtained a linear transformation $\Lambda(t_{i,j})$ from $\mathbb{P}^N$ into itself, which maps the Grassmannian $G(d,n)$ into itself and $\Omega(B_0,..,B_d)$ into $\Omega(A_0,..,A_d)$.
Finally, since T is invertible, Λ is also invertible, and its inverse is given by $\Lambda(T^{-1})$. □

Thus, a set of the form $\Omega(B_0, .., B_d)$ is equivalent to one of the type described in Proposition 3.6. Moreover, as we have seen, to such sets one can associate a Young tableau, and conversely, a Young tableau contained in a $(d+1) \times (n-d)$ table corresponds to a well-defined sequence of integers a_i, which can be obtained from the formula (3.2). Therefore, we have obtained a correspondence between the Young tableau of this kind and the subsets of $G(d,n)$ of the form $\Omega(A_0, ..A_d)$, up to projective transformations.

Corollary 3.8. *Using the previous notation, $\Omega(B_0, .., B_d)$ consists of those points in $G(d,n)$ whose coordinates $q(j_0, ..j_d)$ satisfy certain linear equations. In particular, $\Omega(B_0, .., B_d)$ is an irreducible subvariety of $G(d,n)$ and, if we set $b_i = dim(B_i)$,*

$$dim(\Omega(B_0, .., B_d)) = \sum_{i=0}^{d} b_i - i$$

These subvariety $\Omega(B_0, .., B_d)$ of the Grassmannian $G(d,n)$ will be referred to as *Schubert varieties.*
To obtain the linear equations of a Schubert variety, it is enough to find the linear transformation that maps it to its "simple" form $\Omega(A_0, .., A_d)$, namely the one in Proposition 3.6. Since we know that such a variety is defined by $p(j_0, .., j_d) = 0$ whenever $j_i > a_i$, we can deduce the equations for the original variety.
Note that the number of boxes of the tableau associated to a Schubert variety satisfies:

$$\begin{aligned} |\lambda| &= \sum_{i=0}^{d}(n-d-b_i+i) = (d+1)(n-d) - \sum_{i=0}^{d} b_i + i \\ &= dim(G(d,n)) - dim(\Omega(B_0, .., B_d)) = codim(\Omega(B_0, .., B_d)) \end{aligned}$$

Finally, we recall here the formula for computing the degree of Schubert varieties in terms of the tableau λ associated with $\Omega(B_0, .., B_d)$. The following formula can be found in [7]:

$$degree(\Omega(B_0, .., B_d)) = \frac{|\lambda|!}{\prod(Hook\ \ lengths)} \tag{3.3}$$

where the denominator is the same of the one appearing in the formula of m_λ in section 1.2.

3.3 Cohomology Ring

In this section, we examine how Schubert varieties allows us to define a basis for the cohomology ring of the Grassmannian, which will be very useful for solving enumerative problems.
The cohomology group with the integers as coefficients $H^i(G(d,n), \mathbb{Z}) = H^i(G(d,n))$ is zero when i is greater than $2(d+1)(n-d) = 2\ dim(G(d,n)$ and the direct sum

$$H^*(G(d,n), \mathbb{Z}) = \bigoplus_i H^i(G(d,n))$$

is a graded ring under the *cup product*. Since $G(d,n)$ is a complex manifold, it is also an oriented real manifold, and therefore $H^{2(d+1)(n-d)}(G(d,n))$ is isomorphic to $\mathbb{Z}$.

$$H^{2(d+1)(n-d)}(G(d,n)) \xrightarrow{deg} \mathbb{Z}$$

If u is a cohomology class, then $deg(u)$ is called the *degree of u.*
To every subvariety X of a compact complex manifold M, it is possible to associate a cohomology class. In particular, if X is irreducible and has codimension p, then such a class lies in $H^{2p}(M, \mathbb{Z})$ (this can be found in [10]). We can apply this to the Grassmannian $G(d, n)$ and Schubert varieties in it.
Let $\Omega(A_0, .., A_d) \subset G(d, n)$ be a Schubert variety, it can be proved that its cohomology class depends only on the integers $a_i = dim(A_i)$ for $i = 0, .., d$. We can then denote these classes by $\Omega(a_0, .., a_d)$ and we call them *Schubert cycles.*

Theorem 3.9 (The basis theorem 1)**.** *Considered additively $H^*(G(d, n), \mathbb{Z})$ is a free abelian group and the Schubert cycles $\Omega(a_0, .., a_d)$ form a basis. In particular, the Schubert cycles $\Omega(a_0, .., a_d)$ with $[(d + 1)(n - d) - \sum_{i=0}^{d}(a_i - i)] = p$ form a basis of $H^{2p}(G(d, n))$ and each odd dimensional group is zero.*

Esempio 3.10. *Consider the Grassmannian of points in $\mathbb{P}^n$, we have $G(0, n) = \mathbb{P}^n$ and $\Omega(A_0) = A_0$. The basis theorem says that $H^{2p}(\mathbb{P}^n)$ is a free cyclic group generated by the class $\Omega(n - p)$. The other groups are zero.*

Remark 3.11. *Recalling the language of tableaux introduced in the previous section, we can easily identify the cycles that generate a cohomology group $H^{2p}(G(d, n))$. We know that $codim(\Omega(A_0, .., A_d)) = |\lambda|$, and thus it follows from the theorem that the relevant cycles are those corresponding to varieties with $|\lambda| = p$. Therefore, all tableaux contained in a table of dimension $(d+1) \times (n-d)$ and consisting of p boxes generates $H^{2p}(G(d, n))$.*

Theorem 3.12. *The basis $\{\cdots, \Omega(a_0, .., a_d), \cdots\}$ of the group $H^{2p}(G(d, n), \mathbb{Z}$ and the basis $\{\cdots, \Omega(n - a_d, .., n - a_0), \cdots\}$ of the group $H^{2[(d+1)(n-d)-p]}(G(d, n), \mathbb{Z})$ are dual under the pairing $v, w \to deg(v \cdot w)$ of Poincaré duality.*

In other words, the proposition says that an arbitrary element ν of $H^{2p}(G(d, n))$ can be written uniquely in the form

$$\nu = \sum \delta(n - a_d, .., n - a_0)\Omega(a_0, .., a_d) \tag{3.4}$$

with

$$\delta(n - a_d, .., n - a_0) = deg(\nu \cdot \Omega(n - a_d, .., n - a_0))$$

In particular, if we consider X an irreducible subvariety of $G(d, n)$ of dimension $n-p$, its cohomology class ν lies in $H^{2p}(G(d, n))$, and therefore can be written as a linear combination in (3.4). To determine the coefficient $\delta(n-a_d, .., n-a_0)$, we can choose a Schubert variety $\Omega(B_0, .., B_d)$ such that $dim(B_i) = n - a_{(d-i)}$, so that its cohomology class corresponds to $\Omega(n - a_0, .., n - a_d)$. This variety has dimension equal to the codimension of X, and thus, by choosing the B_i appropriately, we find that the intersection of X with $\Omega(B_0, .., B_d)$ consists of a finite number of points. The number of such points, counted with multiplicity, is precisely $\delta(n - a_d, .., n - a_0)$.

Definition 3.13. *(Multidegree) Let X be an irreducible subvariety of $G(d, n)$ of dimension $n - p$ and let ν be its cohomology class as in (3.4). We define $(..., \delta(n - a_d, .., n - a_0), ...)$ the multidegree of X.*

Let Y be an irreducible subvariety of $G(d, n)$ of dimension p and $(..., \epsilon(a_0, .., a_d), ...)$ its multidegree. If the intersection between X and Y is a finite set of points then this number $i(X \cap Y)$ is the degree of the product of

$$\sum \delta(n - a_d, .., n - a_0)\Omega(a_0, .., a_d) \quad \text{and} \quad \sum \epsilon(a_0, .., a_d)\Omega(n - a_0, .., n - a_d)$$

therefore, using the preposition we have

$$i(X \cap Y) = \sum \delta(n - a_d, .., n - a_0)\epsilon(a_0, .., a_d)$$

Esempio 3.14. *(Bézout's Theorem) Consider $G(0, n) = \mathbb{P}^n$ and let X be an irreducible subvariety of dimension $n - p$. Note that, in this case, the Schubert cycles correspond to the linear subspace of $\mathbb{P}^n$. In particular, $\Omega(n-p)$ is the cycle of a linear subspace of dimension $n-p$. From the previous theory, we have that $\nu = \delta(p)\Omega(n-p)$, where $\nu \in H^{2p}(\mathbb{P})$ is the cohomology class of X and $\delta(p)$ is the number of points with multiplicity in the intersection of X and a suitable chosen p-dimensional linear space, that is, the degree of X in the usual sense. Let Y be a p-dimensional subvariety of $\mathbb{P}^n$ with cohomology class $\epsilon(n - p)\Omega(p)$. As in the previous case, $\epsilon(p)$ denotes the degree of Y in the usual sense. If the intersection of X and Y consists of finitely many points, then from the previous formula we obtain $i(X \cap Y) = \delta(n - p)\epsilon(p)$, which is the well-known result of Bézout's theorem.*

4 Invariant Theory

In this section we will introduce the fundamentals of invariant theory for forms, examine its most important theorems, and demonstrate how to determine such objects.

4.1 Invariants and Covariants of Forms

Let V be a complex vector space of dimension $n+1$. The linear group of $(n+1)\times(n+1)$ matrices, $GL(V)$, clearly acts on V; consider the group action of $GL(V)$ on Sym^dV given by $c \cdot (v \otimes ... \otimes v) = (c \cdot v) \otimes ... \otimes (c \cdot v)\ \forall c \in GL(V), v \in V$. We would like to draw attention to this action for a moment. For simplicity, let us consider the case $dim(V) = 2$. The elements of Sym^dV can be regarded as homogeneous polynomials of degree d in two variables (see Proposition 2.1)

$$f(x, y) = \sum_{k=0}^{d} \binom{d}{k} a_k x^k y^{d-k}$$

Acting on x and y with an element of $GL(2)$ means to substitute $x = c_{1,1}\overline{x}+c_{1,2}\overline{y}$ and $y = c_{2,1}\overline{x} + c_{2,2}\overline{y}$ in the previous expression. The binary form $f(x, y)$ is transformed into another binary form $\overline{f}(\overline{x}, \overline{y})$. We call $\overline{a}_k$ the coefficients of this new form, they are a linear combinations of the a_i whose coefficients are polynomials in the $c_{i,j}$. The explicit representation of this coefficients is in [17] paragraph 3.6. Note that this action is defined by the equations:

$$x = (c_{i,j})\overline{x} \ \ and \ \ f(a, x) = f(\overline{a}, \overline{x}) \quad \forall(c_{i,j}) \in GL(2) \tag{4.1}$$

Consider the polynomial ring $\mathbb{C}[a_0, ..a_d, x, y]$ where the variables are those of V, namely (x,y) and the coefficients of the linear form a_i. The group $GL(2)$ acts on this space taking $x \to \overline{x}$, $y \to \overline{y}$ and $a_i \to \overline{a}_i$ as we described before. A polynomial $I \in \mathbb{C}[a_0, ..a_d, x, y]$ is said to be a *covariant* of *index g* if

$$I(\overline{a}_i, \overline{x}, \overline{y}) = (c_{1,1}c_{2,2} - c_{2,1}c_{1,2})^g \ I(a_i, x, y) \quad \forall(c_{i,j}) \in GL(2) \tag{4.2}$$

A covariant I is *homogeneous* if it is homogeneous both as a polynomial in the variables $a_0, .., a_d$ and in the variables x, y. In this case, the total degree of I in $a_0, .., a_d$ is called the the *degree* of the covariant I, and its total degree in x,y is called the *order* of I. A covariant of order 0, that is, a polynomial $I \in \mathbb{C}[a_0, .., a_d]$ that satisfies the (4.2) is said to be an *invariant*.
We can also consider a collection of binary forms

$$f_i(x, y) = \sum_{k=0}^{d_i} \binom{d_i}{k} a_{i,k} x^k y^{d_i - k} \quad for\ i = 1, 2, .., r$$

A polynomial $I \in \mathbb{C}[a_{1,0}, a_{1,1}, .., a_{1,d_1}, ..., a_{r,d_r}, x, y]$ is called a *joint covariant* of the forms $f_1, .., f_r$ if it is a relative invariant of the $GL(2)$-action. This means

$$I(\overline{a_{1,1}}, ., \overline{a_{1,d_1}}, .., \overline{a_{r,d_r}}, \overline{x}, \overline{y}) = (c_{1,1}c_{2,2} - c_{2,1}c_{1,2})^g I(a_{1,1}, ., a_{1,d_1}, .., a_{r,d_r}, x, y) \quad \forall\, (c_{i,j}) \in GL(2)$$

Where the $\overline{a_{k,i}}$ are obtained as in the case of a single form described above. We say that I is a *joint invariant* of $f_1, .., f_r$ if I does not depend on x and y at all.
When the dimension of V is bigger than 2 we can repeat all the same steps. A form of degree d is

$$f(x_0, .., x_n) = \sum_{i_0 + ... + i_n = d} \frac{d!}{i_0! ... i_n!} a_{i_0, .., i_n} {x_0}^{i_0} ... {x_n}^{i_n} \tag{4.3}$$

and the action of $GL(n+1)$ has a description analogous to that of the 2-dimensional case. In this case, however, we need to pay closer attention to the definition of covariant, which (as we will explain later in section 4.3) is nothing but a generalization of what we saw in the case of two variables.

Definition 4.1. *Let* $\underbrace{Sym^d V \otimes \cdots \otimes Sym^d V}_{m} = \bigoplus_{\lambda \vdash md} (\mathbb{S}_\lambda V)^{n_\lambda}$ *be the decomposition in irreducible representations of* $GL(V)$*. Each of the Weyl modules appearing in this decomposition is called a Covariant of degree* m *of* d*-forms.*

Assume $t = dim(\mathbb{S}_\lambda V)$ for some Weyl module in the decomposition, then $GL(V) \cdot \mathbb{S}_\lambda V = \mathbb{S}_\lambda V$, that is, every object in this space is mapped by the group action to another object in the same space. However, a polynomial expression in the variables a_i (the coefficients of the d-forms) and x_i (coordinates of V) does not always exist to describe these objects, unlike the case when $dim(V) = 2$. In the case $t = 1$, the submodule is generated by a single polynomial $I \in \mathbb{C}[a_{1,i_0,..,i_n}, .., a_{m,i_0,..,i_n}]$ invariant under the action of $GL(n+1)$. These are called $invariant\ of\ d-forms$.

We observe that the space $Sym^m(Sym^d V)$ is contained in $\underbrace{Sym^d V \otimes \cdots \otimes Sym^d V}_{m}$ and each of its submodules is a symmetric covariant of m forms. Note also that $Sym^m(Sym^d V)$ can be viewed as the space of homogeneous polynomials of degree m in the variables a_i, where the a_i represent the coefficients of a form in $Sym^d V$. Consequently, each submodules gives rise to an ideal of $\mathbb{C}[Sym^d V]$ whose generators has degree m.
The space $Sym^m(Sym^d V)^{GL(V)}$ is called the space of $symmetric\ invariants\ of\ degree\ m\ \ for\ forms\ \ of\ degree\ d$. The following sections will be dedicated to the study of

this space.

Similarly, the space $\bigwedge^m(Sym^dV)$ is contained in $\underbrace{Sym^dV \otimes \cdots \otimes Sym^dV}_{m}$. The Weyl submodules that appear in its decomposition correspond to skew covariants of m forms. When the submodule has dimension one, we obtain a single polynomial expression $I \in \mathbb{C}[a_{1,i_0,..,i_n},..,a_{m,i_0,..,i_n}]$ that is skew-symmetric which respect to the blocks of variables corresponding to the coefficients of the m forms. In this case, I is called a skew invariant. These objects can be interpreted as *joint invariant* of m forms of degree d. We will deal with one of these in section 7.

Invariants which do not depend on the determinant of the matrix $C \in GL(V)$, are called *absolute invariants*.
The special linear group $SL(V)$, consisting of $n \times n$ matrices with determinant equal to one, is a subgroup of $GL(V)$; therefore, the action on d-forms described above can be restricted to this subgroup. Moreover, the invariants with respect to the action of $GL(V)$ coincide with those under the action of $SL(V)$.

4.2 Invariance For the Torus

The group $SL(V)$ contains the torus $(\mathbb{C}^*)^n$ of diagonal matrices

$$T = \{D(t_1,..,t_n,\frac{1}{t_1\cdots t_n})|t_i \in \mathbb{C}^*\}.$$

T is a subgroup of $SL(V)$, so whenever there is an action of $SL(V)$ on a vector space, we can consider its restriction to T.
Let consider an element $f \in Sym^d(V)$ as in (4.3). The space $Sym^m(Sym^d(V))$ is spanned by monomials $f_{i_{0,1},..,i_{n,1}}...f_{i_{0,m},..,i_{n,m}}$.
The *weight* of the monomial $f_{i_{0,1},..,i_{n,1}}...f_{i_{0,m},..,i_{n,m}}$ is the vector

$$(\sum_{j=1}^m i_{0,j}, \sum_{j=1}^m i_{1,j}, ..., \sum_{j=1}^m i_{n,j}) \tag{4.4}$$

A monomial is called *isobaric* if its weight has all equal entries. Consider the double sum $\sum_{k=0}^n \sum_{j=1}^m i_{k,j} = \sum_{j=1}^m d = md$. So the weight of an isobaric monomial in $Sym^m(Sym^d(V))$ is $(\frac{md}{n+1},..,\frac{md}{n+1})$, in particular isobaric monomials exist if and only if $n+1$ divides md.

Proposition 4.2. *$I \in Sym^m(Sym^d(V))$ is invariant for the action of the torus of diagonal matrices $(\mathbb{C}^*)^n \subset SL(V)$ if and only if it is a sum of isobaric monomials.*

Note that it is enough to check if a monomial of given degree is isobaric for n places in the $(n+1)$-dimensional weight vector. Indeed, if we assume it to be true for the first n entries, we obtain:

$$md = \sum_{k=0}^n \sum_{j=1}^m i_{k,j} = \frac{n(md)}{n+1} + \sum_{j=1}^m i_{n,j} \implies \sum_{j=1}^m i_{n,j} = \frac{md}{n+1}$$

In particular, for binary form it is enough to check the condition just for one place.

Proof. Consider the polynomial $f = f_{i_0,..,i_n} {x_0}^{i_0} \cdots {x_n}^{i_n} \in Sym^d(V)$, to see the action of a diagonal matrix $D(a_0, .., a_n)$ on $f_{i_0,..,i_n}$ we have to substitute $x_i \to a_i x_i$ in f

$$f(\overline{x}_i) = f_{i_0,..,i_n} (a_0 x_0)^{i_0} \cdots (a_n x_n)^{i_n} = {a_0}^{i_0} \cdots {a_n}^{i_n} f_{i_0,..,i_n} {x_0}^{i_0} \cdots {x_n}^{i_n}$$

Therefore this action takes $f_{i_0,..,i_n}$ to ${a_0}^{i_0} \cdots {a_n}^{i_n} f_{i_0,..,i_n}$.
Consider the diagonal matrix with entries $(\frac{1}{t_1 \cdots t_n}, t_1, .., t_n)$ that acts on $f_{i_0,..,i_n}$ by multiplying by $(t_1 \cdots t_n)^{-i_0} {t_1}^{i_1} \cdots {t_n}^{i_n}$. Acting on the monomial

$$f_{i_{0,1},..,i_{n,1}} \cdots f_{i_{0,m},..,i_{n,m}}$$

we multiply it by $(t_1 \cdots t_n)^{-\sum_j i_{0,j}} {t_1}^{\sum_j i_{1,j}} \ldots {t_n}^{\sum_j i_{n,j}}$ and this is equal to 1 if and only if $\sum_j i_{k,j}$ does not depend on k. □

Corollary 4.3. *A necessary condition for the existence of a non-zero $I \in Sym^m(Sym^d(V))$ that is invariant for $SL(V)$ is that $n+1$ divides md.*

Proof. If I is invariant then it is also invariant for the torus of diagonal matrices. □

Note that this results had already been obtained by other means in the chapter on representation theory; in particular, it follows from Proposition 1.11.

4.3 The Symbolic Representation of Invariants

Remember that V is a complex vector space of dimension $n+1$, then the space $\wedge^{n+1} V$ has dimension 1 and it is isomorphic to $\mathbb{C}$. For every $v_1, .., v_{n+1} \in V$ the determinant $v_1 \wedge .. \wedge v_{n+1}$ is well defined and it is $SL(V)$-invariant. Let us consider a rectangular Young tableau of size $(n+1) \times g$ filled with numbers, where each number corresponds to a linear form on V. i corresponds to a linear form l_i. We construct a tableau function by taking the product of the determinant arising from each column. For example

1	1
2	3
3	4

represents $(l_1 \wedge l_2 \wedge l_3)(l_1 \wedge l_3 \wedge l_4)$. To define formally this notion, we set $[a] = 1, .., a$ for any natural number a and we associate to any tableau T a function $t : [n+1] \times [g] \longrightarrow [m]$, where g is the number of columns, m is the highest number appearing in the diagram and $t(i, j)$ is the entry at the place (i,j) of the tableau.

Definition 4.4. *For any Young tableau T of size $(n+1) \times g$ filled with numbers from 1 appearing h_1 times until m appearing h_m times, so that $h_1 + .. + h_m = g(n+1)$, we denote by G_T the multilinear function $Sym^{h_1} V \times .. \times Sym^{h_m} V \longrightarrow \mathbb{C}$ defined by*

$$G_T({l_1}^{h_1}, .., {l_m}^{h_m}) = \prod_{j=1}^{g} (l_{t(1,j)} \wedge ... \wedge l_{t(n+1,j)})$$

G_T is called a tableau function.

Note that G_T is well-defined by Theorem 2.2.

Proposition 4.5. *Every tableau function G_T is $SL(V)$-invariant.*

Proof. Immediate by the proprieties of the determinant. $\square$

This last result is equivalent to saying that G_T is a joint invariant.

Every $F \in Sym^m(Sym^d(V))$ corresponds to a multilinear function

$$F : \underbrace{Sym^d(V) \times ... \times Sym^d(V)}_{m} \longrightarrow \mathbb{C}$$

which is symmetric in the m entries.

Definition 4.6. *Let $md = (n+1)g$. Let T be a Young tableau of size $(n+1) \times g$ filled with numbers from 1 appearing d times until m appearing d times. Let G_T : $\underbrace{Sym^d(V) \times .. \times Sym^d(V)}_{m} \longrightarrow \mathbb{C}$ be the function introduces in Definition 4.4. We denote by F_T the polynomial obtained by symmetrizing G_T, that is $F_T(h) = G_T(\underbrace{h, ..., h}_{m})$ for any $h \in Sym^d(V)$. F_T is called a symmetrized tableau function.*

Proposition 4.7. *Any F_T as in Definition 4.6 is $SL(V)$-invariant.*

Proof. Let $g \in SL(V)$ and $h \in Sym^d(V)$. We have $F_T(g \cdot h) = G_T(g \cdot h, .., g \cdot h) = G_T(h, .., h) = F_T(h)$ where the second equality follows by Proposition 4.5. $\square$

Note that F_T is the polynomial associated with G_T due to the isomorphism in Proposition 2.1.
Now, let's explain how to obtain in practice the expression of such an invariant in terms of the coefficients of a general linear form in $Sym^d(V)$.
Let consider a Young tableau as in Definition 4.4 with $m = 3$. We can easily evaluate G_T on three $d-th$ powers of liner forms: $(a_0x_0 + ... + a_nx_n)^d$, $(b_0x_0 + ... + b_nx_n)^d$ and $(c_0x_0 + ... + c_nx_n)^d$. This give an expression for G_T as a sum of monomials in the a_i, b_i and c_i with total degree md and degree d separately in each of the a_i, b_i and c_i. Using the correspondence in 2.2, we obtain the general expression for $G_T((a_{i_1,..,i_n}), (b_{i_1,..,i_n})(c_{i_1,..,i_n}))$:

$$a_0^{i_0} \cdot \cdot a_n^{i_n} b_0^{j_0} \cdot \cdot b_n^{j_n} c_0^{k_0} \cdot \cdot c_n^{k_n} \longrightarrow a_{i_0,..,i_n} b_{j_0,..,j_n} c_{k_0,..,k_n} \tag{4.5}$$

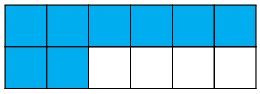
Here, $(a_{i_0,..,i_n})$,$(b_{j_0,..,j_n})$,$(c_{k_0,..,k_n})$ are three generic d-forms, and this is the explicit expression for a joint invariant.
To write the expression of $F_T((a_{i_0,..,i_n}))$ we use the same correspondence as in 4.5, applying the same coefficients to all the three generic d-forms in the target. Specifically:

$$a_0^{i_0} \cdot \cdot a_n^{i_n} b_0^{j_0} \cdot \cdot b_n^{j_n} c_0^{k_0} \cdot \cdot c_n^{k_n} \longrightarrow a_{i_0,..,i_n} a_{j_0,..,j_n} a_{k_0,..,k_n} \tag{4.6}$$

Remark 4.8. *The construction made for invariants can be generalized to suitable covariants. Recalling the definition of covariants, let us consider the space $Sym^m(Sym^dV)$, and let $\mathbb{S}_\lambda V$ be an irreducible representation of this space with dimension greater than 1. The associated Young diagram λ consists of $m \cdot d$ boxes, but it is not rectangular. For instance, consider the space $Sym^2(Sym^4V)$ where V has dimension 2. The decomposition of this space is*

$$Sym^2(Sym^4V) \cong \mathbb{S}_{(8)} \oplus \mathbb{S}_{(6,2)} \oplus \mathbb{S}_{(4,4)}$$

Let's consider the covariant $\mathbb{S}_{(6,2)}V$. We can complete the Young diagram $\lambda = (6,2)$ to make it rectangular and with columns of length dim(V).



In the boxes of the original partition, we place the numbers from 1 to 2, each repeated 4 times, as in the case of rectangular tableaux, while in the remaining boxes we place x.

1	1	1	1	2	2
2	2	x	x	x	x

To find the expression of the covariant, we proceed as in the case above, with the only difference that x represents a linear form with coefficients x, y (or x_i with $i = 0, .., n$ in the case dim$(V) = n+1$) and it is not modified using the correspondence in (2.2), unlike the coefficients of the other linear forms (4.6). From our example, we obtain:

$$I(a_i, x, y) =(-a_3^2 + a_2a_4)x^4 + (2a_2a_3 - 2a_1a_4)x^3y + (-3a_2^2 + 2a_1a_3 + a_0a_4)x^2y^2+ \\ + (2a_1a_2 - 2a_0a_3)xy^3 + (-a_1^2 + a_0a_2)y^4$$

This expression corresponds to the Hessian of a quartic form of the type $a_4x^4 + 4a_3x^3y + 6a_2x^2y^2 + 4a_1xy^3 + a_0y^4$, which therefore turns out to be a covariant. We will see that this is not a coincidence, but rather a general phenomenon for d-forms. Finally, we observe that this construction is only possible for covariants whose rectangular completion is missing boxes on the same row. This is why we mentioned that it is not always possible to express a covariant as a polynomial in the a_i (the coefficients of the forms) and the x_i (coordinates in V).

The entire constructions carried out for $Sym^m(Sym^dV)$ can be analogously repeated for invariants and covariants of $\bigwedge^m(Sym^dV)$. Clearly, in this case it makes sense only to obtain the explicit expression of G_T, that is, of the corresponding relative invariants. In fact, since we are dealing with antisymmetric functions, attempting to compute the expression of F_T would yield zero. Joint invariants of this type are called *skew invariants*. We will study an example of a skew invariant in Chapter 7 and use the construction just described to derive its explicit expression.

4.4 Finiteness of the Ring of Invariant

Let (W, ρ) be any rational representation of $G = SL(n+1)$. We denote by $\mathbb{C}[W]$ the space of homogeneous polynomials in $dim(W)$ variables. This space has the structure of a graded ring

$$\mathbb{C}[W] = \bigoplus_{m=0}^{\infty} Sym^mW$$

Theorem 4.9 (Hilbert's finiteness theorem)**.** *Using the previous notation, the invariant ring $\mathbb{C}[W]^G$ is finitely generated as a $\mathbb{C}-algebra$.*

The proof is in [16]. This result is very important; in the following, we will apply it with $W = Sym^dV$ and V a complex vector space of dimension $n+1$. In this case we have

$$\mathbb{C}[W] = \bigoplus_{m=0}^{\infty} Sym^m(Sym^d V) = \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda \vdash dm} c_\lambda \mathbb{S}_\lambda V$$

Moreover, from Preposition 1.11, it follows that

$$\mathbb{C}[W]^G = \bigoplus_{g=0}^{\infty} c_{(g,g,\dots,g)} \mathbb{S}_{(g,g,\dots,g)} V$$

where $g(n+1) = md$ and the coefficients $c_{(g,g,\dots,g)}$ may also vanish. The previous theorem tells us that this ring has finitely many generators, and the following result provides a method to find them.

Theorem 4.10. *The space of invariants of degree m, $Sym^m(Sym^d V)^G$, is generated by symmetrized tableau functions F_T, constructed as in Definition 4.6.*

Let $I_1, .., I_k$ be a finite set invariants that generate $\mathbb{C}[Sym^d V]^G$. A syzygy among them is an integral rational functions of $I_1, .., I_k$ that vanishes identically. It can be shown that the number of irreducible syzygies, that is, those which cannot be obtained as a linear combination of syzygies of lower degrees, is finite. Moreover, there relations are known, but we will not go into detail about this aspect.

5 Binary Quartics

The aim of this chapter is to introduce a *toy model* for the subsequent study of the Hesse variety associated with plane cubics. In particular, after introducing some basic notions concerning binary quartics, we will focus on the study of the subvariety of $G(1,4)$ given by the Zariski closure of the set of pencils generated by a quartic and its Hessian. We will also see that this variety is well known in the literature, and that this is simply an alternative way to define it.

5.1 Key Concepts

We denote by $[x, y]$ the homogeneous coordinates of $\mathbb{P}^1$. A quartic Q in this space is defined by an equation of the form $f = 0$ with

$$f = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4.$$

Thus, we can identify it with the coefficients of this polynomial and see it as a point in $\mathbb{P}^4$. A quartic is said to be *singular* if it has at least one multiple root; otherwise, it is called smooth.
The Hessian of a quartic is still a quartic; in fact it is defined by

$$H(f) = det\left(\left(\frac{\partial^2 f}{\partial_{x_i}\partial_{x_j}}\right)_{i,j}\right) \qquad i, j = 1, 2$$

where $x_1 = x$ and $x_2 = y$. $H(f)$ is a homogeneous polynomial of degree 4 in (x, y), whose coefficients are polynomial of degree 2 in the a_i. These coefficients can be easily computed, and by identifying the Hessian with its coefficients, it corresponds to the point of $\mathbb{P}^4$:

$$[-6a_1^2+6a_0a_2, -3a_1a_2+3a_0a_3, -3a_2^2+2a_1a_3+a_0a_4, -3a_2a_3+3a_1a_4, -6a_3^2+6a_2a_4]$$

It is clear that the Hessian of a quartic is not always well-defined. In fact, when all the previously expressions vanish, the Hessian cannot be determined. This also provides a way to obtain the ideal defining the variety Z of quartics for which the Hessian is not defined, which turns out to have dimension 1. Moreover, Z coincides with the variety of cones, that is, with the set of quartics defines by equations of the form (as discussed in Chapter 2, section 2.2):

$$f = (a_0x + a_1y)^4$$

This is called the *quartic rational normal curve*.
Let us now consider quartics that coincide with their own Hessian. Denote by Sq the variety obtained as the Zariski closure of this set. This produces a two-dimensional variety that contains the previously defined variety Z, that is, $Z \subset Sq$. Moreover, the equation of a quartic in this space is of the form:

$$f = (a_0x^2 + a_1xy + a_2y^2)^2$$

and thus, Sq is the variety of squares. It is defined by 7 equations of degree 3 in a_i given by the vanishing of the coefficients of $det\begin{pmatrix} f_x & f_y \\ H(f)_x & H(f)_y \end{pmatrix}$, which is a polynomial of degree 6 in (x, y). Finally, Sq is smooth.

5.2 Invariants of Binary Quartics

A binary quartic can be regarded as an element in Sym^4V, where V is a 2-dimensional vector space, identifying it with the polynomial that defines it. Sym^4V in fact, has dimension 5. An invariant of degree m for binary quartics will be given by an element in $Sym^m(Sym^4V)^{GL(2)}$, that is, by a homogeneous polynomial of degree m in the variables $a_0, .., a_4$ invariant under the action of $GL(2)$. From Proposition 4.9, we also know that the space of all invariants $\mathbb{C}[Sym^4V]^{GL(2)}$ is finitely generated as a ring, and in particular, we have:

Theorem 5.1. *The ring of invariants of binary quartics, that is* $\oplus_m Sym^m(Sym^4V)^{GL(2)}$, *is freely generated by two invariants* I *and* J *of degree* 2 *and* 3 *respectively.*

First of all, we can verify the existence of these two invariants according to Proposition 1.11, that is, by decomposing the spaces $Sym^2(Sym^4V)$ and $Sym^3(Sym^4V)$ into Weyl modules.

$$Sym^2(Sym^4V) \cong \mathbb{S}_{(8)} \oplus \mathbb{S}_{(6,2)} \oplus \mathbb{S}_{(4,4)}$$

$$Sym^3(Sym^4V) \cong \mathbb{S}_{(12)} \oplus \mathbb{S}_{(10,2)} \oplus \mathbb{S}_{(9,3)} \oplus \mathbb{S}_{(8,4)} \oplus \mathbb{S}_{(6,6)}$$

Thus, in the first decomposition, we find the invariant I, which corresponds to $\mathbb{S}_{(4,4)} \cong \mathbb{C}^*$, and in the second, the invariant J, which corresponds to $\mathbb{S}_{(6,6)} \cong \mathbb{C}^*$.

The invariant I corresponds to the tableau

1	1	1	1
2	2	2	2

and applying the procedure in 4.3, we obtain:

$$I = 3a_2^2 - 4a_1a_3 + a_0a_4$$

As for the invariant J, it corresponds to the tableau

1	1	1	1	2	2
2	2	3	3	3	3

and it is given by the formula

$$J = a_2^3 - 2a_1a_2a_3 + a_0a_3^2 + a_1^2a_4 - a_0a_2a_4 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}$$

The vanishing of the invariants I and/or J has geometric meaning: $I = J = 0$ if and only if the quartic has a triple or a quadruple root. Moreover, if $I = 0$ and $J \neq 0$ the roots form an *anharmonic quadruplet*, that is the cross-ratio of the four roots $(z_1, z_2, z_3, z_4) = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is equal to $-\omega$ or $-\omega^2$ where $\omega^3 = 1$. This requirement concerns the symmetric arrangement of the roots on the projective line $\mathbb{P}^1$. Whereas $J = 0$ if and only if the quartic can be written as the sum of two fourth powers, $f = (ax - by)^4 + (cx + dy)^4$, and the roots form an *harmonic quadruplet* (the cross-ration of the four roots is $\frac{1}{2}$, 2 or -1). See [6] for further details.
Both I^3 and J^2 have degree 6, and any combination of them is still an invariant of degree 6. The most important among these is the discriminant

$$\Delta = I^3 - 27J^2,$$

which vanishes if and only if the quartic is singular.
Finally, $\frac{I^3}{J^2}$ is an absolute invariant.

We now present the classification of orbits under the action of $GL(2)$ on the space of quartic forms. This can be found in [14].

Orbit representatives	**Hessian**	**Description**	$\mathbf{dim}(\overline{O(f)})$
$x^4 + 6\lambda x^2y^2 + y^4 \quad \lambda \neq \pm\frac{1}{3}$	$x^4 + y^4 + 6\frac{1-3\lambda^2}{6\lambda}x^2y^2$	*simple roots*	3
$x^4 + x^2y^2$	$6x^4 - x^2y^2$	*one double root*	3
x^2y^2	x^2y^2	*two double roots*	2
x^3y	x^4	*triple root*	2
x^4	0	*quadruple root*	1

Where $O(f)$ is the orbit of the quartic f, that is

$$O(f) = \{c \cdot f \mid c \in GL(2)\}$$

Remark 5.2. *(How to compute the dimension of an orbit) Let X be an algebraic variety and G an algebraic group such that G acts on X. Let $x \in X$ and $O(x)$ its orbit, consider*

$$G \xrightarrow{\phi} O(x)$$

given by $\phi(g) = g \cdot x$. The differential of this morphism at the identity $d\phi_e$ yields

$$T_eG \xrightarrow{d\phi_e} T_xO(x)$$

Denote by J_e the Jacobian matrix of ϕ evaluated at the identity element of G. It holds that:

$$rank(J_e) = dim_{\mathbb{A}}(T_xO(x)) = dim_{\mathbb{A}}(O(x))$$

where $dim_{\mathbb{A}}$ denotes the affine dimension.

When we view the Hessian of a quartic as a polynomial of degree 2 in (x, y) with coefficients that are degree 4 polynomials in the a_i, it follows from Proposition 4.4.2 in [17] that it is a covariant of degree 2 and order 4 (this property actually holds more generally for homogeneous forms, as stated in the proposition: the Hessian of a d-form is a covariant of index 2 and order (n+1)(d-2)). Therefore, if a quartic lies in a given orbit, then the Hessian of all quartics in the same orbit lie in the same orbit as the Hessian of f. This is equivalent to saying that the Hessian map is equivariant under the action of $GL(2)$.

5.3 Syzygetic Pencils and their Variety

Proposition 5.3. *Let f be a quartic in $\mathbb{P}^4$ that belongs to the image of H but not to Sq. Then it is the Hessian of two other quartics in $\mathbb{P}^4 - Sq$.*

This can be found in [4]. Note that this statement is equivalent to saying that the Hessian map is generically a [2 : 1] map.

From the orbits of $GL(2)$, we can see that a quartic in Sq with a non-vanishing Hessian is equivalent to x^2y^2. From this, it is easy to verify that the space of quartics whose Hessian is such an element has dimension 1. In particular, for x^2y^2, this space is given by $x^4+\lambda y^4$, to which we must add x^2y^2 itself. On the other hand, for elements of Sq with a vanishing Hessian, we can take x^4 as a representative. It is then found that the space of quartics whose Hessian is x^4 has dimension 1, and it is given by $x^4 + \lambda x^3y$.
We summarize these results in a table.

Orbit representatives	**Description** $H^{-1}(f)$
$x^4 + 6\lambda x^2y^2 + y^4 \quad \lambda \neq \pm\frac{1}{3}$	$x^4 + 6tx^2y^2 + y^4$ *with* $6\lambda t = 1 - 3t^2$
$x^4 + x^2y^2$	$x^4 - 2x^2y^2$
x^2y^2	$x^4 + \lambda y^4 \quad \lambda \in \mathbb{C}^*$ e x^2y^2
x^3y	*empty*
x^4	$x^4 + \lambda x^3y \quad \lambda \in \mathbb{C}$

Consider the Grassmannian of the lines in $\mathbb{P}^4$, $G(\mathbb{P}^1, \mathbb{P}^4)$. As we show in 3.4, it is embedded in a projective space of dimension 9. Taking the Plücker coordinates of this space $p_{(i,j)}$ when $i = 0, .., 3$ and $i < j \leq 4$, $G(\mathbb{P}^1, \mathbb{P}^4)$ is defined by five quadratic equations. Moreover, it has dimension 6 and degree 5. Consider $\mathbb{P}^4$ as the space of binary quartics and let $f \in \mathbb{P}^4$ be a quartic. Then, as long as the Hessian is well-defined and distinct from f, it generates a line $\langle f, H(f)\rangle$ in $\mathbb{P}^4$, that is a point in $G(\mathbb{P}^1, \mathbb{P}^4)$.

Proposition 5.4. *Let L be a line in $\mathbb{P}^4$ generated by a quartic and its Hessian, that is $L = \langle f, H(f)\rangle$, whit $H(f) \neq 0, f$. Then, the Hessian of every quartic on the line L still lies on L.*
Moreover, if a quartic is in $L - Sq$, then the quartics for which it is the Hessian must also be contained in L.

Proof. From the description of the $GL(2)$-orbits, it follows that f can be chosen in the form

$$x^4 + 6\lambda x^2y^2 + y^4 = 0 \qquad with \ \ \lambda \in \mathbb{C} \tag{5.1}$$

We can easily compute the Hessian of such quartic, we obtain:

$$x^4 + 6\frac{1 - 3\lambda^2}{6\lambda}x^2y^2 + y^4 = 0$$

which clearly still is of the same form.
This is equivalent to saying that every pencil of the form $\langle f, H(f)\rangle$, with f smooth is projectively equivalent to one of the form

$$x^4 + 6\lambda x^2y^2 + y^4 = 0 \qquad with \ \ \lambda \in \mathbb{C} \cup \{\infty\}$$

where for $\lambda = \infty$ the equation is $x^2y^2 = 0$. Moreover, from the previous computations, it also follows that the Hessian of a quartic in the pencil still belongs to the pencil.
For the second part, it is enough to observe that the correspondence $\lambda \to \frac{1-3\lambda^2}{6\lambda}$ defines a $[2:1]$ map from $\mathbb{C} \cup \{\infty\}$ to itself. That is, a generic quartic in the pencil is the Hessian of two other quartics in the same pencil.
There are two additional cases in which a pencil can be generated by a quartic and its Hessian, namely when f lies in the orbit of $x^4 + x^2y^2$ or x^3y. In the first case, everything works similarly to the case of smooth quartics; that is, given f of the form $x^4 + x^2y^2 + \lambda(6x^4 - x^2y^2)$, its Hessian has the same form with $\lambda \longrightarrow \frac{11\lambda+3}{18\lambda-4}$. Moreover, each quartic in this pencil is the Hessian of a unique other quartic, which also belongs to the same pencil. In the case of $x^3y + \lambda x^4$, each of the quartic in the pencil has x^4 as its Hessian, and it is not the Hessian of any quartic. □

From the canonical form of a pencil generated by a smooth quartic and its Hessian, we can observe that it contains three quartics that coincide with their own Hessian. One is given for $\lambda = \infty$, while the others are found by solving the equation $6\lambda^2 = 1 - 3\lambda^2$, that gives $\lambda \in \{+\frac{1}{3}, -\frac{1}{3}\}$.

Definition 5.5. *Consider the set of lines generated by a quartic and its Hessian. We denote by S the subvariety of $G(\mathbb{P}^1, \mathbb{P}^4)$ obtained as the Zariski closure of this space, that is*

$$S = \overline{\{\langle f, H(f)\rangle \in G(\mathbb{P}^1, \mathbb{P}^4) \ | f \ \ has \ \ 4 \ \ simple \ \ roots\}}$$

Let's now see how to calculate the dimension of S theoretically. Later, we will also discuss how to compute the equations of S computationally, and thus determine its dimension.
We define

$$P := \{(f, L) \in \mathbb{P}^4 \times S \mid f \in L\} \subset \mathbb{P}^4 \times G(\mathbb{P}^1, \mathbb{P}^4),$$

we denote by $p_2 : \mathbb{P}^4 \times G(1,4) \longrightarrow G(1,4)$ and $p_1 : \mathbb{P}^4 \times G(1,4) \longrightarrow \mathbb{P}^4$ the two projections. Let's consider the restriction of these two maps to P and examine their fibers.

$$\begin{array}{ccc} & P & \\ {\scriptstyle p_1|_P} \swarrow & & \searrow {\scriptstyle p_2|_P} \\ \mathbb{P}^4 & & S \end{array}$$

Let $g \in \mathbb{P}^4$ a quadric, $p_1|_P^{-1}(g) = \{(g, L) \in \{g\} \times S \mid g \in L\} \subset P$. So, for a generic quartic, that is, if g is not in C and its Hessian is defined, we have that there exists a unique $L \in S$ that satisfies this condition, and it is given by $L = \langle g, H(g) \rangle$. This follows from Proposition 5.4. We obtain

$$dim(P) = dim(\mathbb{P}^4) + dim(p_1|_P^{-1}(g)) = 4 + 0 = 4.$$

Let now $L \in S$, $p_2|_P^{-1}(L) = \{(f, L) \in \mathbb{P}^4 \times \{L\} \mid f \in L\} \subset P$. Clearly, this fiber consists of all the quadrics contained in L and therefore has dimension 1. We have

$$4 = dim(P) = dim(S) + dim(p_2|_p^{-1}(L)) = dim(S) + 1 \quad \Longrightarrow \quad dim(S) = 3.$$

To obtain the equations defining S in Plücker coordinates, we proceed as follows: we consider the 2×5 matrix where the first row contains the coefficients a_i of a generic quadric, and the second row contains the corresponding coefficients of the hessian, expressed in terms of the a_i.

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ -6a_1^2 + 6a_0a_2 & -3a_1a_2 + 3a_0a_3 & -3a_2^2 + 2a_1a_3 + a_0a_4 & -3a_2a_3 + 3a_1a_4 & -6a_3^2 + 6a_2a_4 \end{bmatrix}$$

To obtain the equation of the generic line for f and $H(f)$ in Plücker coordinates, we simply set the latter equal to the 2nd-order minors of the previous matrix, that is

$$p_{(i,j)} = det(A_i|A_j) \qquad for \;\; i = 0, .., 3 \;\; and \;\; i < j \leq 4$$

where A_i is the $i-th$ column of the matrix A. Thus, we obtain the Plücker coordinates as cubic expressions in the a_i. Finally, it will be sufficient to eliminate all the auxiliary variables a_i to obtain an ideal containing the desired relations among the $p_{(i,j)}$.
We carried out this computation using Macaulay2. It turns out that S is defined by 8 equations, three of which are linear, while the remaining five are quadratic. These last equations are precisely the quadratic equations that define $G(1,4)$. Thus, S is obtained as the intersection of three hyperplanes in $\mathbb{P}^9$ with the Grassmannian $G(1,4)$.
In particular,

$$(3p_{(2,3)} - p_{(1,4)},\; 2p_{(1,3)} - p_{(0,4)},\; 3p_{(1,2)} - p_{(0,3)},\; 2p_{(1,4)}^2 - 3p_{(0,4)}p_{(2,4)} + 2p_{(0,3)}p_{(3,4)},$$

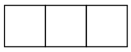
$$p_{(0,4)}p_{(1,4)} - 3p_{(0,3)}p_{(2,4)} + 3p_{(0,2)}p_{(3,4)},\; p_{(0,4)}^2 - 2p_{(0,3)}p_{(1,4)} + 2p_{(0,1)}p_{(3,4)},$$

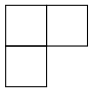
$$p_{(0,3)}p_{(0,4)} - 3p_{(0,2)}p_{(1,4)} + 3p_{(0,1)}p_{(2,4)},\ 2p_{(0,3)}^2 - 3p_{(0,2)}p_{(0,4)} + 2p_{(0,1)}p_{(1,4)})$$

are the equations that define S. Furthermore, we computed the dimension end the degree of S, obtaining:

$$dim(S) = 3 \qquad degree(S) = 5$$

This variety turns out to be smooth. To verify this, we computed the variety of singular points contained in S and checked that it is empty. This last variety is obtained by adding to the defining equations of S those given by the condition that the Jacobian matrix of S has rank less than $6 = codim(S)$.
In the following, we will see how to use Schubert calculus to compute the multidegree of S with respect to Schubert cycles. In $G(\mathbb{P}^1, \mathbb{P}^4)$ there are exactly nine Schubert cycles that generate $H^*(G(1,4), \mathbb{Z})$. Each of these cycles corresponds to a Young tableau that can be embedded in the representative 2×5 matrix. Additionally, we know that $codim(\Omega(\lambda)) = \#\ boxes\ in\ \lambda$. Since we are interested in evaluating the multidegree of S, which has dimension 3, we focus on $H^6(G(1,4), \mathbb{Z})$. From the basis theorem 3.9, it follows that it has two generator corresponding to the tableaux:





Indeed, these correspond to Schubert varieties of dimension 3. Using the notation in section 3.3, these two Schubert cycles are denoted respectively by $\Omega(0,4)$ and $\Omega(1,3)$. It follows from the theory that the degree of S can be written as follows:

$$deg(S) = \alpha\ deg(X_{\square\square\square}) + \beta\ deg(X_{\substack{\square\square\\ \square}}) \tag{5.2}$$

and the coefficients α and β can be computed as the degree of the intersection between S and the same Schubert variety. With the caution of choosing the Schubert varieties in such a way that their intersection with S consists of a finite number of points, the degree thus represents the number of points with multiplicity in the intersection. To clarify:

$$dim(S \cap X_{\square\square\square}) = 0 \implies \alpha = deg(S \cap X_{\square\square\square})$$
$$dim(S \cap X_{\substack{\square\square\\ \square}}) = 0 \implies \beta = deg(S \cap X_{\substack{\square\square\\ \square}})$$

For this last fact, we observe that, according to Proposition 3.12, the cycles $\Omega(0,4)$ and $\Omega(1,3)$ are self-dual.
The degrees of these two Schubert varieties can be compute using the Hook lengths formula in (3.3), we have

$$deg(X_{\square\square\square}) = \frac{3!}{3\cdot 2\cdot 1} = 1 \qquad deg(X_{\substack{\square\square\\ \square}}) = \frac{3!}{3\cdot 1\cdot 1} = 2$$

Proposition 5.6. *Using the previous notation, the multidegree of the variety S is given by*

$$\alpha = 1 \qquad \beta = 2$$

Proof. A Schubert variety corresponding to the cycle $\Omega(0,4)$ represents all the lines in $\mathbb{P}^4$ that pass through a fixed point. Intersecting with S, we must ask ourselves how many lines of the type $\langle f, H(f)\rangle \subset \mathbb{P}^4$ pass through a given point $g \in \mathbb{P}^4$. In the general case, there is only one such line, as follows from Proposition 5.4, and it is given by $\langle g, H(g)\rangle$. Thus, we obtain $\alpha = 1$.
A Schubert variety corresponding to the cycle $\Omega(1,3)$ represents all the lines in $\mathbb{P}^4$ that touch a given line r and are contained in a hyperplane $\mathbb{P}^3$ that also contains r. Similarity to the previous case, we need to count the number of lines of the form $\langle f, H(f)\rangle$ that satisfy these properties. Again, by Proposition 5.4, this number is equal to the number of quadrics on r whose Hessian belongs to the given hyperplane. The condition of belonging to r becomes $a_i = p_i + \lambda\, v_i$. The coefficients of the Hessian of such a quadric, denoted by $\overline{a}_i$, are polynomials of degree 2 in λ. A hyperplane of $\mathbb{P}^4$ is given by a linear equation of the form:

$$\sum_{i=0}^{4} \mu_i\ a_i = 0 \qquad\qquad \mu_i \in \mathbb{C}$$

where the a_i are the coordinates of $\mathbb{P}^4$. The condition that the Hessian belongs to this hyperplane can be rewritten as

$$\sum_{i=0}^{4} \mu_i\ \overline{a}_i = 0,$$

which turns out to be a quadratic equation in the parameter λ, and therefore there are two solutions. In this way, we obtain $\beta = 2$. $\square$

We performed these computations also using Macaulay2. The key observation we want to make is that, to find the Schubert varieties that satisfy the condition of having a finite intersection with S, one can follow a procedure similar to the one used in the proof of Proposition 3.7, where T is a random matrix, 5×5 in this case. We present the obtained results in the following table.

Schubert variety X_λ	$deg(X_\lambda)$	$codim(S + X_\lambda)$	$degree(S + X_\lambda)$
(0,4)	1	9	1
(1,3)	2	9	2

Note that the degrees of the two Schubert variety are consistent with the Hook-length formula (3.3) in section 3.2.
Thus, we obtain $\alpha = 1$ and $\beta = 2$ again. Rewriting the formula in 5.2, we have $dim(S) = 1 \cdot 1 + 2 \cdot 2 = 5$ which coincides with the direct computation of the degree of S.

5.4 The Orbit Closure of a Pencil

We recall that S denotes the closure of the family of pencils generated by a binary quartic and its Hessian. Since the Hessian map is $SL(2)$-equivariant, it follows that S is also $SL(2)$-invariant. Indeed, if $\langle f, H(f)\rangle \in S$, then for any $C \in SL(2)$, we have $C \cdot \langle f, H(f)\rangle = \langle C \cdot f, H(C \cdot f)\rangle \in S$. Thus, S contains orbits of pencils, and our goal in this section is to determine which ones and how many. In particular, we will show that the following theorem holds.

Theorem 5.7. *The variety S, defined in 5.5, coincides with the closure of the orbit of the pencil $< x^4 + y^4, x^2y^2 >$ under the action of $GL(2)$, that is*

$$S = \overline{O(\langle x^4 + y^4, x^2y^2 \rangle)}$$

Moreover, S consists of 3 orbits of pencils and is smooth.

Since we know (see preposition 5.4) that all the pencils of the form $\langle f, H(f) \rangle$ with f smooth are equivalent to $\langle x^4 + y^4, x^2y^2 \rangle$, the first part of the theorem follows immediately.
Let f be a quartic, we define

$$\mathbb{P}^3_f := \overline{\{\langle f, g \rangle \in G(1, 4) \mid g \in \mathbb{P}^4\}}$$

that is, the closure of the space of all pencils containing f. It has dimension 3.
For each orbit of quartics, we fix a representative and compute the dimension of $\mathbb{P}^3_f \cap S$. The results are summarized in the following table, along with a description of the corresponding space.

Representative f	**dim**$(\mathbb{P}^3_f \cap S)$	**Description**
$x^4 + y^4$	0	$\langle x^4 + y^4, x^2y^2 \rangle$
$x^4 + x^2y^2$	0	$\langle x^4, x^2y^2 \rangle$
x^2y^2	1	$\langle x^2y^2, ax^4 + by^4 \rangle \quad a, b \in \mathbb{C}$
x^3y	0	$\langle x^3y, x^4 \rangle$
x^4	1	$\langle x^4, ax^3y + bx^2y^2 \rangle \quad a, b \in \mathbb{C}$

We now observe that $\langle x^2y^2, a(x^4) + b(y^4) \rangle$ $a, b \in \mathbb{C}$ contains actually only finitely many orbits of pencils. Indeed, for $a, b \neq 0$, we have $H(ax^4 + by^4) = x^2y^2$. Thus, using the equivariance of the Hessian map, the following diagram commutes

$$\begin{array}{ccc} ax^4 + by^4 & \xrightarrow{\cdot C} & \overline{a}x^4 + \overline{b}y^4 \\ \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\ x^2y^2 & \xrightarrow{\cdot C} & x^2y^2 \end{array}$$

We have $\langle x^2y^2, ax^4 + by^4 \rangle \cdot C = \langle x^2y^2, \overline{a}x^4 + \overline{b}y^4 \rangle$ and the orbits of such pencils corresponds to the orbits of the quartic $ax^4 + by^4$ with $a, b \neq 0$. Clearly, all of them lie in the orbit of x^4+y^4. We must also consider the cases where $a = 0$ or $b = 0$. These two cases are equivalent up to switching x and y, so they yield a simple additional orbit. Therefore, we obtain exactly two orbits of pencils generated by

$$\langle x^2y^2, x^4 + y^4 \rangle \qquad \langle x^2y^2, x^4 \rangle$$

Similarly, also the pencils of the form $\langle x^4, ax^3y + bx^2y^2 \rangle$ turn out to form only finitely many orbits. However, more care is needed in this case. In particular, we have $H(ax^3y + bx^2y^2) = -3a^2x^4 - 4b(ax^3y + bx^2y^2) \in \langle x^4, ax^3y + bx^2y^2 \rangle$ for every

$a, b \neq 0$. The equivariance of the Hessian map gives the following diagram:

$$\begin{array}{ccc} f = ax^3y + bx^2y^2 & \xrightarrow{\quad \cdot C \quad} & \overline{a}x^3y + \overline{b}x^2y^2 \\ \Big\downarrow H & & \Big\downarrow H \\ H(f) \in \langle x^4, ax^3y + bx^2y^2 \rangle & \xrightarrow{\cdot C} & C \cdot H(f) \in \langle x^4, \overline{a}x^3y + \overline{b}x^2y^2 \rangle \end{array}$$

from which we deduce that $C \cdot \langle x^4, ax^3y + bx^2y^2 \rangle = C \cdot \langle f, H(f) \rangle = \langle \overline{a}x^3y + \overline{b}x^2y^2, C \cdot H(f) \rangle = \langle x^4, \overline{a}x^3y + \overline{b}x^2y^2 \rangle$. So, as in the previous case, the orbits correspond to those of $ax^3y + bx^2y^2$, to which we must add the ones for $a = 0$ or $b = 0$. We thus obtain three orbits, with the following representatives:

$$\langle x^4, x^3y + x^2y^2 \rangle \qquad \langle x^4, x^3y \rangle \qquad \langle x^4, x^2y^2 \rangle$$

Finally, observing that $x^3y + x^2y^2$ lies in the same orbit as $x^4 + x^2y^2$, and using once again the equivariance of the Hessian map, we get $\langle x^4, x^3y + x^2y^2 \rangle = \langle x^4, x^4 + x^2y^2 \rangle = \langle x^4, x^2y^2 \rangle$.

In Section 5.5, after determining the equations defining S, we also verified that it is smooth. As an additional check, we can compute the rank of the 8×10 Jacobian matrix for each orbit. At smooth points, the Jacobian should have rank equal to the codimension of S, which is 6; therefore, any point where the rank drops must be singular. In the following table, we summarize all the orbits contained in S along with their dimension and the rank of the Jacobian matrix.

Representative $\langle f, g \rangle$	$\mathbf{dim}(O(\langle f, g \rangle))$	$\mathbf{Rank}(J)$
$\langle x^4 + y^4, x^2y^2 \rangle$	3	6
$\langle x^4, x^2y^2 \rangle$	2	6
$\langle x^4, x^3y \rangle$	1	6

This concludes the proof of Theorem 5.7.

It is possible to see explicitly that both lower-dimensional orbits lie in the closure of the pencil $\langle x^4 + y^4, x^2y^2 \rangle$. Indeed, consider the two families of quartics $x^4 + x^2y^2 + ty^4$ and $x^3y + t(x^4 + y^4)$, depending on a parameter t. Each of these, together with its Hessian, generates a pencil in $O(\langle x^4 + y^4, x^2y^2 \rangle)$, which are, respectively,

$$\langle x^4 + x^2y^2 + ty^4,\ 12tx^2y^2 + 2ty^4 + 2x^4 - x^2y^2 \rangle, \quad \langle x^3y + tx^4 + ty^4, 16t^2x^2y^2 + 8txy^3 - x^4 \rangle$$

Taking the limit as $t \to 0$, we obtain, respectively, $\langle x^4, x^2y^2 \rangle$ and $\langle x^4, x^3y \rangle$, which must therefore lie in the closure of $O(\langle x^4 + y^4, x^2y^2 \rangle)$.

5.5 The Variety of Pencils Revisited

The variety S studied so far is actually a well-known variety in the literature. In fact, in the article [1], all the closed 3-fold $GL(2)$-orbits are classified. In particular we will see that our variety corresponds to the orbit of the sextic $x^5y - xy^5$.

To make the connection with that definition of S, let us recall that $G(1, 4)$ is embedded in $\mathbb{P}^9$. In particular, we can think of it as contained in $\bigwedge^2(Sym^4\mathbb{C}^2)$, which indeed has dimension 9. This space has the following decomposition in Weyl modules:

$$\bigwedge^2(Sym^4\mathbb{C}^2) \cong \mathbb{S}_{(7,1)}\mathbb{C}^2 \ \oplus \ \mathbb{S}_{(5,3)}\mathbb{C}^2$$

Since the representations of $SL(2)$ are isomorphic to their duals, we can also write

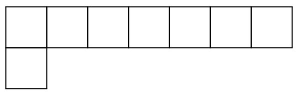
$$\bigwedge^2(Sym^4\mathbb{C}^2) \cong Sym^6\mathbb{C}^2 \ \oplus \ Sym^2\mathbb{C}^2$$

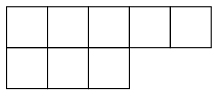
Thus, the orbit of the sextic $x^5y - xy^5$ is contained in $Sym^6\mathbb{C}^2 \subset \bigwedge^2(Sym^4\mathbb{C}^2)$ and, from the classification in article [1], we conclude that this orbit coincides with S.

Proposition 5.8. *Using the previous notation we have:*

$$G(1,4) \cap \mathbb{P}(Sym^6\mathbb{C}^2) = S \tag{5.3}$$

This will become clear once the correspondence between $\bigwedge^2(Sym^4\mathbb{C}^2)$ and the two submodules $Sym^6\mathbb{C}^2$ and $Sym^2\mathbb{C}^2$ is made explicit in Plücker coordinates. These two modules, of dimension 7 and 3 respectively, correspond to the tableaux





We can use the symbolic representation of covariants, as explained in Remark 4.8, to obtain explicit formulations of these covariants. In particular, we start from

1	1	1	1	2	2	2
2	x	x	x	x	x	x

1	1	1	1	2
2	2	2	x	x

and construct the two corresponding polynomials in the variables a_i, b_i of two quartics and x, y of $\mathbb{C}^2$. We report here the results obtained, along with their expression in Plücker coordinates.

$$\begin{aligned}
F_6 = {} & (-a_4b_3 + a_3b_4)x^6 + (3a_4b_2 - 3a_2b_4)x^5y + (-3a_4b_1 - 6a_3b_2 + 6a_2b_3 + 3a_1b_4)x^4y^2 + \\
& + (a_4b_0 + 8a_3b_1 - 8a_1b_3 - a_0b_4)x^3y^3 + (-3a_3b_0 - 6a_2b_1 + 6a_1b_2 + 3a_0b_3)x^2y^4 + \\
& + (3a_2b_0 - 3a_0b_2)xy^5 + (-a_1b_0 + a_0b_1)y^6 \\
= {} & p_{(3,4)}x^6 - 3p_{(2,4)}x^5y + (3p_{(1,4)} + 6p_{(2,3)})x^4y^2 - (p_{(0,4)} + 8p_{(1,3)})x^3y^3 + \\
& + (3p_{(0,3)} + 6p_{(1,2)})x^2y^4 - 3p_{(0,2)}xy^5 + p_{(0,1)}y^6
\end{aligned}$$

$$\begin{aligned}
F_2 = {} & (-a_4b_1 + 3a_3b_2 - 3a_2b_3 + a_1b_4)x^2 + (a_4b_0 - 2a_3b_1 + 2a_1b_3 - a_0b_4)xy + \\
& + (-a_3b_0 + 3a_2b_1 - 3a_1b_2 + a_0b_3)y^2 = \\
& (p_{(1,4)} - 3p_{(2,3)})x^2 - (p_{(0,4)} - 2p_{(1,3)})xy + (p_{(0,3)} - 3p_{(1,2)})y^2
\end{aligned}$$

These two expression show that $Sym^6\mathbb{C}^2$ can be obtained by intersecting $\bigwedge^2(Sym^4\mathbb{C}^2)$ with the three hyperplanes defined by the vanishing of the coefficients of F_2, namely:

$$p_{(1,4)} - 3p_{(2,3)} = 0 \qquad p_{(0,4)} - 2p_{(1,3)} = 0 \qquad p_{(0,3)} - 3p_{(1,2)} = 0$$

Observe that these hyperplanes are exactly those defining the variety S, which proves Proposition 5.8.
Furthermore, the expression of the sextic F_6 gives an explicit correspondence between quartic pencils and sextics. Let us consider, for example, the pencil $\langle x^4 + y^4, x^2y^2\rangle$. It must correspond to a sextic in the orbit of $x^5y - xy^5$. Consider the 2×5 matrix whose rows are the coefficients of $x^4 + y^4$ and x^2y^2

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The 2×2 minors of this matrix give the Plücker coordinates of the pencil generated by these two quartics, which are

$$\begin{array}{llll} p_{(0,1)} = 0 & p_{(0,2)} = 1 & p_{(0,3)} = 0 & p_{(0,4)} = 0 \\ p_{(1,2)} = 0 & p_{(1,3)} = 0 & p_{(1,4)} = 0 & \\ p_{(2,3)} = 0 & p_{(2,4)} = -1 & & \\ p_{(3,4)} = 0 & & & \end{array}$$

Using F_6, the corresponding sextic is $x^5y - xy^5$, as expected.
Similarly, we can proceed for the other lower-dimensional orbits in the closure.

Pencil	**Sextic**
$\langle x^4 + y^4, x^2y^2\rangle$	$x^5y - xy^5$
$\langle x^4, x^2y^2\rangle$	xy^5
$\langle x^4, x^3y\rangle$	x^6

What has been obtained for the lower-dimensional orbits also corresponds to what is stated in the article [1]. In the closure of the orbit of $x^5y - xy^5$, there are two other lower-dimensional orbits, those of x^5y and x^6, with dimensions 2 and 1, respectively.

Remark 5.9. *We have seen that the variety S is given by the intersection of the Grassmannian $G(1,4)$ with three hyperplanes. According to the article [12], it is a Fano 3-fold. A Fano variety X is characterized by having ample anticanonical bundle. In particular, for the Grassmannian $G(1,4)$, the canonical bundle satisfies $K_{G(1,4)} \cong O(-5)$. By intersecting with three hyperplanes, we obtain a 3-fold S with $K_S \cong O(-2)$. In this article S is also described as a variety of sums of squares.*

6 Plane Cubics

In this chapter, we will focus on the study of plane cubic curves, exploring their main invariants and concluding with an analysis of the Hessian map.

6.1 Key Concepts

Let's start by introducing these objects. A plane cubic C_f is defined by a homogeneous polynomial of degree three in the variables x, y, z, which represent the homogeneous coordinates in a projective space. Thus, as follow from Proposition 2.1, we can say that it is an object in Sym^3V, where V is a three-dimensional vector space. In particular, we have

$$C_f = \{[x,y,z] \in \mathbb{P}^2 \; | f(x,y,z) = 0\} = \{f = 0\} \quad with$$

$$f = a_0x^3+3a_1x^2y+3a_2x^2z+3a_3xy^2+6a_4xyz+3a_5xz^2+a_6y^3+3a_7y^2z+3a_8yz^2+a_9z^3.$$

We can thus identify C_f with f, and f with the ten homogeneous coordinates given by its coefficients a_i. Therefore, we can think of a cubic curve as a point in $\mathbb{P}^9$.
A point $p \in C_f$ is called *singular* if the gradient of f vanishes at that point. Otherwise, it is called *smooth*.
The group $SL(3)$ acts on cubic curves as explained at the beginning of chapter 3. The orbits of this action are all classified and can be found in [3], we will analyze them in detail later. In particular, there are infinitely many orbits, 8 of them contain singular cubics, while the others are of the form:

$$E_\lambda : \quad x^3 + y^3 + z^3 + 6\lambda xyz = 0 \qquad \lambda \in \mathbb{C} - \{-\frac{1}{2}, 0, 1\}$$

and contain smooth cubics.

The Hessian cubic of f, $H(a_i, x)$, defined in section 2.2, is an homogeneous polynomial of degree 3 in the variables x, y, z whose coefficients are polynomials of degree 3 in the a_i. It is still a plane cubic. A point $p \in C$ is called a *flex point* if it is a common point of C and its Hessian, that is, it satisfies $f(p) = 0$ and $H(f)(p) = 0$. It then follows from Bézout's theorem that, in general, a plane cubic has nine flex points. The tangent line to the cubic at these points intersects the cubic only at that point, with multiplicity 3.
It is called *triangle* a cubic whose Hessian coincides with the cubic itself. This is equivalent to saying that all the points of such a cubic are flex points and thus, it is the union of three lines.

Remark 6.1. *If we consider two curves of order n, they will intersect at n^2 points. However, only $\frac{n(n+3)}{2}$ points are needed to determine a curve of order n in the plane (Cramer's paradox). In general, with some exceptions, the n^2 intersection points of two curves of order n impose only $\frac{n(n+3)}{2} - 1$ conditions on curves of the same order that must pass through these points. Moreover, the curves of order n that pass through $\frac{n(n+3)}{2} - 1$ points of the plane also pass through other $n^2 - \left(\frac{n(n+3)}{2} - 1\right) = \frac{(n-1)(n-2)}{2}$ points.*
All of this can be applied to the case of cubics, and in particular, it holds that: Cubics passing through eight generic points of the plane also pass through a ninth. In particular, if $P_1, .., P_9$ are the nine intersection points of two cubics, the condition of passing through any eight of these points imply passing through the nine.
It can be found in [6], Book II, Chapter 15.

Theorem 6.2 (Mac-Laurin)**.** *The line joining two flex points of a smooth cubic contains a third flex point.*

This theorem can be found in [6], where a different proof is given from the one we outline here. Specifically, we observe that every smooth cubic is projectively equivalent to one of the form E_λ, and for these, the theorem can be verified explicitly.

In any case, we will explore this in more detail in the following sections. Furthermore, Mac-Laurin extends this statement to cubics with a node: the three inflection points of a cubic with a node are collinear.
The lines containing three flex points are called $Mac-Laurin\ \ lines$. In the smooth case, from the existence of nine flex points, it follows that there are exactly $\binom{9}{2}\frac{1}{3} = 12$ Mac-Laurin lines. Four of them pass through the same flex point.
Such a configuration of 9 points in $\mathbb{P}^2$ and 12 lines is called the $Hesse\ \ configuration$.

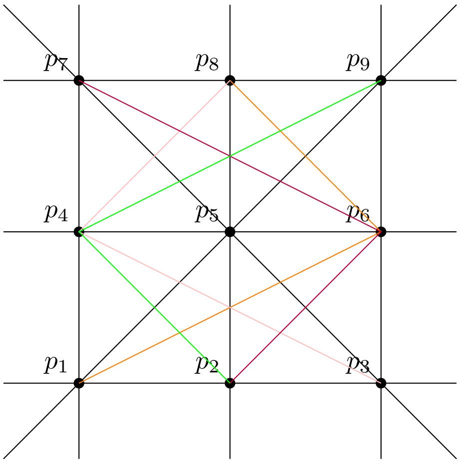


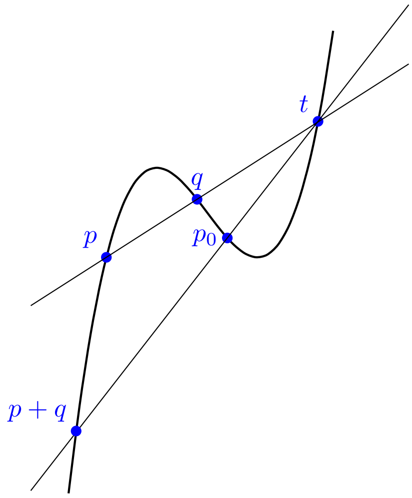
Figure 1: This figure represents nine points in Hesse configuration and the twelve lines that characterize them. The eight black lines are clearly visible, while the remaining four could not be explicitly drawn. Instead, four different colours were used to connect the three points on each of these lines.

The twelve Mac-Laurin lines form 4 triangles, each containing the nine inflection points. To construct a triangle, we choose one of the 12 lines, named a, through three flex points. As the second line, we take a line b that is not one of the $3 \cdot 3 + 1 = 10$ lines passing through one of the three flex points in a. Finally, there are exactly nine lines that connect one inflection point on a with one on b. Adding a and b, it remains only one line c that completes the trilateral.

The following theorem is also due to Mac-Laurin, we present here a slightly different proof from the one given in [6].

Theorem 6.3. *Let $p_i \in \mathbb{P}^2$ for $i = 1, .., 9$ be nine points in the Hesse configuration. If C_f is a smooth cubic passing through these nine points, then these points are flex points of f.*

Before proving this theorem, we first recall some aspects of the group structure of elliptic curves. We choose a point $p_0 \in C$, called the "origin", which will serve as the neutral element of the group. Let p and q be two other points on the curve. The sum $p + q$ is defined as follows: we draw the line through p and q, which intersect the cubic at a third point, denoted by t. Next, we draw the line through p_0 and t; this line intersects the cubic at a third point, which is precisely $p + q$. It can be proved that this defines a group structure on C.



Lemma 6.4. *Let p_0, the origin of the group, be a flex point of the cubic. The following proprieties hold:*

- *three points of C are collinear if and only if their sum equals p_0.*
- *p is a flex point of C if and only if $3p = p_0$.*

Proof. Let p, q and r be three points of C. First, suppose that the points are collinear. When we sum $p+q$, we obtain the third intersection point of the line through r and p_0 with the cubic. Now, adding r to this point means finding the intersection of the tangent to p_0 with C . However, since p_0 is a flex point, its tangent intersects the cubic only at p_0 with multiplicity 3. Thus, we obtain precisely p_0.
Conversely, supposed that $p+q+r = p_0$. This means that the points $p+q$ and r are collinear with p_0. It then follows from the definition of $p+q$ that r must be the third point of the line through p and q with C, which implies that p, q and r are collinear.
Now, let p be a flex point of C. The sum $p+p$ is given by the third intersection point of the line through p and p_0 with C. Summing again with p, we obtain p_0, which shows that $3p = p_0$.
Conversely, let $p \in C$ such that $3p = p_0$. This means that $p+p$ and p are collinear and this happens if and only if the third intersection point between the tangent line from p and the cubic C is again p, that is p is a flex point.

□

Proof of Theorem 6.3. Consider a group structure on C where the origin is a flex point. Since this point serves as the neutral element of the group, we will simply denote it by 0. Let p_i for $i = 1, .., 9$ be the nine points in the Hesse configuration. From the previous lemma, it follows that the relationship that must hold between these points can be rewritten as follows:

$$\begin{aligned} p_1 + p_2 + p_3 &= 0 \\ p_4 + p_5 + p_6 &= 0 \\ p_7 + p_8 + p_9 &= 0 \\ p_1 + p_4 + p_7 &= 0 \\ p_2 + p_5 + p_8 &= 0 \\ p_3 + p_6 + p_9 &= 0 \\ p_1 + p_5 + p_9 &= 0 \\ p_3 + p_5 + p_7 &= 0 \\ p_2 + p_6 + p_7 &= 0 \\ p_1 + p_6 + p_8 &= 0 \\ p_2 + p_4 + p_9 &= 0 \\ p_3 + p_4 + p_8 &= 0 \end{aligned}$$

Each equation represents one of the 12 lines, and each point belongs to 4 lines, meaning it lies on 4 of these equations.
There are no privileged points in this configuration, so it will be sufficient to show that one of these p_i is a flex point, that is, satisfies $3p_i = 0$. Let us choose, for example, p_1. Summing the first three equations that contain it, starting from the top, we obtain:

$$3p_1 + p_2 + p_3 + p_4 + p_7 + p_5 + p_9 = 0$$

Observing that $p_2 + p_4 + p_9 = 0$ and $p_3 + p_5 + p_7 = 0$ are two of the equations, we obtain the thesis: $3p_1 = 0$. □

6.2 Invariants of Plane Cubics

An invariant for plane cubics is an homogeneous polynomial $I \in \mathbb{C}[a_0, .., a_9]$ that is invariant under the action of $SL(3)$. As follows from the theorem 4.9, the space of all invariants must be finitely generated, and indeed we have the following proposition:

Theorem 6.5. *The ring of invariants of a plane cubic, that is $\bigoplus_m Sym^m(Sym^3\mathbb{C}^3)^{GL(3)}$, is freely generated by two invariants S and T of degree 4 and 6 respectively.*

We will see in a moment the definition and what these two invariants represent. First, let's just mention that this proposition can be found in [17] and it can be proved as an application of the constructive proof provided by Hilbert for the Finiteness Theorem 4.9.
From the proposition 1.11, we can also evaluate the presence of these invariants or, alternatively, the absence of lower-degree invariants by decomposing modules of the type $Sym^m(Sym^3\mathbb{C}^3)$ into irreducible representation of $GL(3)$. Here, we report the lowest degrees calculated with the help of the software Macaulay2.

$$Sym^2(Sym^3\mathbb{C}^3) = \mathbb{S}_6 \oplus \mathbb{S}_{(4,2)}$$

$$Sym^3(Sym^3\mathbb{C}^3) = \mathbb{S}_9 \oplus \mathbb{S}_{(7,2)} \oplus \mathbb{S}_{(6,3)} \oplus \mathbb{S}_{(5,2,2)} \oplus \mathbb{S}_{(4,4,1)}$$

$$Sym^4(Sym^3\mathbb{C}^3) = \mathbb{S}_{12} \oplus \mathbb{S}_{(10,2)} \oplus \mathbb{S}_{(9,3)} \oplus \mathbb{S}_{(8,4)} \oplus \mathbb{S}_{(8,2,2)} \oplus \mathbb{S}_{(7,4,1)} \oplus \mathbb{S}_{(7,3,2)} \oplus \mathbb{S}_{(6,6)} \oplus \mathbb{S}_{(6,4,2)} \oplus \mathbb{S}_{(4,4,4)}$$

Thus, we see that there are no invariants of degree lower than 4 and, at degree 4, we observe the factor $\mathbb{S}_{(4,4,4)}$ that corresponds to the invariant S mentioned in the Theorem. Continuing the calculation for greater degrees, it can be seen that there are no invariants of degree 5 and that there is one of degree 6 corresponding to the module $\mathbb{S}_{(6,6,6)}$.
The *Aronold invariant* S, it is of degree 4 and its symbolic representation using the tableau language is the following:

1	1	1	2
2	2	3	3
3	4	4	4

We denote by S the corresponding multilinear form G_T and its symmetrization F_T:

$$S(l_1^3, l_2^3, l_3^3, l_4^3) = (l_1 \wedge l_2 \wedge l_3)(l_1 \wedge l_2 \wedge l_4)(l_1 \wedge l_3 \wedge l_4)(l_2 \wedge l_3 \wedge l_4)$$

By calculating the expression of the invariant in terms of the coefficients a_i, as we explained at the end of paragraph 4.3, one obtains a sum of 25 monomials. The cubic forms for which $S = 0$ (equianharmonic cubic) can be expressed as the sum of three cubes.
The other invariant T, that appears in the proposition, has degree 6 and corresponds to the tableau:

1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2

This gives a sum of 103 monomials of degree 6. The cubic forms for which $T = 0$ (harmonic cubic) are hessian of their own hessian.
The explicit expressions of both these invariants, in terms of the coefficients of a plane cubic, can be found in [17].

There is a connection between these two invariants and the invariants I and J defined in the case of quartics in Section 5.2. Let C_f be a plane cubic and $P \in C_f$ a point, we define

$$\pi_P : C_f - \{P\} \longrightarrow \mathbb{P}^1$$

where $\mathbb{P}^1$ represents the line in $\mathbb{P}^2$ through P and $\pi_P(Z) = \langle P, Z \rangle$. The map is generically $2:1$, as a general line through P intersects the cubic curve C_f in two further points. Moreover, there are four branch points, namely the directions for which the line through P meets the cubic at a single point with multiplicity greater than one. These four points determine a binary quartic. Moreover, if we carry out the same construction with a different point $P' \in C_f$, the resulting quartic lies in the same SL(2)-orbit as the previous one. As P varies, all the quartics obtained in this way are SL(2)-equivalent. In particular, cubics with $S = 0$ determine quartics with $I = 0$ and are called *anharmonic* (among them we find the Fermat cubic). On the other hand, cubics with $T = 0$ give rise to quartics with $J = 0$ and are therefore called *harmonic*.

The two invariants S and T let us to introduce the *discriminant* of a plane cubic as

$$\Delta = T^2 - 64S^3 \tag{6.1}$$

which vanishes whenever the cubic is singular. Otherwise, the curve is an elliptic curve. The fact that the discriminant is an invariant implies that if a cubic is singular, then all cubics in its orbit are singular as well. We include here the table listing all the orbits of the $SL(3)$ action.

Table 1: Orbits with respect to the action of $SL(3)$

Cubic	**Hessian cubic**	**Description**
x^3		*triple line*
$xy(x+y)$		*three concurrent line*
x^2y		*double line + line*
$x(x^2+yz)$	$-3x^3+xyz$	*conic + sec.line*
$y(x^2+yz)$	y^3	*conic + tangent line*
$y^2z-x^3-x^2z$	$3xy^2-x^2z+y^2z$	*nodal*
y^2z-x^3	xy^2	*cusp*
$x^3+y^3+z^3+6txyz$ with $t^3 \neq 1$	same form with $t' = \frac{-1-2t^3}{6t^2}$	*smooth*
xyz	xyz	*triangle*

Moreover, cubics have an absolute invariant, given by $\frac{S^3}{T^2}$.

6.3 Hesse Pencil

The *syzygetic pencil* is defined as the pencil of smooth cubics generated by a cubic and its Hessian: $\lambda f + \mu H(f)$. Since a cubic and its Hessian have nine common points, which are the inflection points of f, all the cubics in this pencil must pass through these points, which we call the *base points of the pencil*. Being inflection points of f, they form a Hesse configuration and, therefore, by Theorem 6.3, they must be inflection points for all the cubics in the pencil. Thus, we have the following:

Proposition 6.6. *All the cubics in a syzygetic pencil have the nine base points as their inflection points.*

It follows directly from this result that the Hessian of any cubic in the pencil must still belong to the pencil. In this regard, note that the cubics passing through eight generic points of $\mathbb{P}^2$ determine a pencil, and thus, in this case, such a pencil must coincide with $\lambda f + \mu H(f)$. Hesse proved this fact in 1844, along with a more profound result which is in Theorem 6.7.
In general, the pencil of cubics given by

$$E_\lambda : \quad x^3 + y^3 + z^3 + 6\lambda xyz = 0 \qquad \lambda \in \mathbb{C} \cup \{\infty\} \tag{6.2}$$

where $E_\infty : xyz = 0$, is called the *Hesse pencil*.
This kind of pencil also appears in the context of binary quartic: indeed, it corresponds to the pencil of quartics generated by a smooth quartic and its Hessian $\langle x^4+y^4, x^2y^2\rangle$, as defined in (5.1).
E_λ is a *syzygetic pencil*: in fact, it is generated by the Fermat cubic $x^3 + y^3 + z^3$ and its Hessian xyz. By intersecting these two curves, we find the inflection points of the Fermat cubic, which are:

$$\begin{matrix} (0,1,-1) & (0,1,\epsilon) & (0,\epsilon,1) \\ (1,0,-1) & (1,0,\epsilon) & (\epsilon,0,1) \\ (1,-1,0) & (1,\epsilon,0) & (\epsilon,1,0) \end{matrix}$$

where $\epsilon = e^{2\pi i/3}$ is the primitive cubic root of unity.
To find the singular cubics that belongs to the pencil, we denote by (x_0, y_0, z_0) a singular point and solve the system:

$$\begin{cases} x_0^3 + y_0^3 + z_0^3 + 6\lambda x_0y_0z_0 = 0 \\ 3x_0^2 + 6\lambda y_0z_0 = 0 \\ 3y_0^2 + 6\lambda x_0z_0 = 0 \\ 3z_0^2 + 6\lambda x_0y_0 = 0 \end{cases}$$

Without loss of generality, we can take $z_0 = 1$ (otherwise, $z_0 = 0 \implies x_0 = 0, y_0 = 0$).

$$\begin{cases} 3x_0^3 + 6\lambda x_0y_0 = 0 \\ 3y_0^2 + 6\lambda x_0 = 0 \\ 2\lambda x_0y_0 = -1 \end{cases} \longrightarrow \begin{cases} x_0^3 = 1 \\ y_0 = \frac{-1}{2\lambda x_0} \\ 3y_0^2 + 6\lambda x_0 = 0 \end{cases}$$

Substituting $x_0 = \epsilon^k$ with $k = 0, 1, 2$ and $y = \frac{-1}{2\lambda\epsilon^k}$ into the last equation, we obtain $\lambda = \{-\frac{1}{2}, \frac{-1}{2}\epsilon, \frac{-1}{2}\epsilon^2\}$. Moreover, the cubic for $\lambda = \infty$, namely $xyz = 0$, is clearly singular. Thus, in conclusion, the Hesse pencil contains 4 singular cubics.
The Hessian of a cubic in the pencil is easily obtained as

$$\begin{vmatrix} 6x & 6\lambda z & 6\lambda y \\ 6\lambda z & 6y & 6\lambda x \\ 6\lambda y & 6\lambda x & 6z \end{vmatrix} = -6^3(\lambda^2x^3 + \lambda^2y^3 + \lambda^2z^3 - xyz(2\lambda^3 + 1)) = 0$$

By rearranging this expression, we obtain:

$$H(E_\lambda) : x^3 + y^3 + z^3 + 6\mu\, xyz = 0 \qquad where \qquad \mu = \frac{-1 - 2\lambda^3}{6\lambda^2}$$

This clearly shows that the Hessian of a cubic in the pencil still belongs to the pencil. Moreover, to find the inflection points of such a cubic, we must impose:

$$\begin{cases} x^3 + y^3 + z^3 + 6\lambda\, xyz = 0 \\ x^3 + y^3 + z^3 + 6\mu\, xyz = 0 \end{cases} \longrightarrow \begin{cases} x^3 + y^3 + z^3 = 0 \\ xyz = 0 \end{cases}$$

where the second system is obtained by subtracting the two previous equations. Thus, the solutions are the same as those of the Fermat cubic, and this proves that all the

cubic in the pencil have the same inflection points, which are the base points of the pencil.
For a fixed μ, the solutions in λ of the equation $6\lambda^2\mu = -1-2\lambda^3$ provide the equations of the three cubics of which the cubic E_μ is the Hessian. In the general case, these three are the only cubics with such a property. In fact, if there were a cubic not in the pencil E_λ whose Hessian coincides with E_μ, this would mean that E_μ lies in the intersection of two pencils of cubics. As seen above, it follows that the Hessian of E_μ should belong to both pencils, and thus they would coincide.
The cubic $xyz = 0$, which is obtained for $\lambda = \infty$, is a cubic whose Hessian coincides with itself. It is easy to see that there are three other cubics in the pencil with this propriety, which can be found by solving the equation $6\lambda^2\mu = -1 - 2\lambda^3$ where we set $\mu = \lambda$. We obtain $8\lambda^3 = -1$, that is $\lambda \in \{-\frac{1}{2}, -\frac{1}{2}\epsilon, -\frac{1}{2}\epsilon^2\}$. These 4 cubics are the triangles of the pencil and coincide with its 4 singular cubics.
The triangles are also the Hessian of two other cubics in the pencil, which we now aim to determine. To do so, we reconsider the equation $\mu = \frac{-1-2\lambda^3}{6\lambda^2}$ with $\mu \in \{\infty, -\frac{1}{2}, -\frac{1}{2}\epsilon, -\frac{1}{2}\epsilon^2\}$.

- For $\mu = \infty$ we have $\lambda^2 = 0$, that is, $\lambda = 0$ counted twice.
- For $\mu = -\frac{1}{2}$ we have $2\lambda^3 - 3\lambda^2 + 1 = (2\lambda+1)(\lambda^2 - 2\lambda+1) = (2\lambda+1)(\lambda-1)^2 = 0$. Thus, the other cubic is obtained from $\lambda = 1$, and it must be counted twice.
- For $\mu = -\frac{1}{2}\epsilon$ we have $2\lambda^3 - 3\lambda^2\epsilon + 1 = (2\lambda+\epsilon)(\lambda^2 - 2\epsilon\lambda+\epsilon^2) = (2\lambda+\epsilon)(\lambda-\epsilon)^2 = 0$. In this case as well, only one other cubic is found, for $\lambda = \epsilon$, and it is counted twice.
- For $\mu = -\frac{1}{2}\epsilon^2$ we have $2\lambda^3 - 3\lambda^2\epsilon^2 + 1 = (2\lambda+\epsilon^2)(\lambda - \epsilon^2)^2 = 0$. Thus we obtain, $\lambda = \epsilon^2$ counted twice.

In general, we can conclude that each triangle of the pencil is the Hessian of a unique irreducible cubic, counted with multiplicity 2.

Theorem 6.7 (Hesse's Theorem)**.** *The pencil determined by a smooth cubic and its Hessian contains the Hessians of all its curves. Moreover, each cubic is the Hessian of three others, which belong to the syzygetic pencil whose base points are its nine flex points.*

Proof of Theorem 6.7. Every smooth cubic is projectively equivalent to a cubic of the form E_λ (table 1). Therefore, it will be sufficient to prove the theorem for a cubic of this type. Since we have seen that such a cubic has as Hessian exactly E_μ with $\mu = \frac{1-2\lambda^3}{6\lambda^2}$, the generated pencil coincides with the Hesse pencil E_λ, and thus the theorem holds, as follows from the previous observations. □

6.4 The Hessian Map

We have previously defined the Hessian of a cubic f in (2.2). We will now show that the equation defining the Hessian cubic is a covariant of index 2. In fact, let $C = (c_{i,j})$ be an element of $GL(3)$, the action on f is defined by the equation $f(a_i, x) = f(\overline{a_i}, \overline{x})$ as shown in (4.1). Applying the differential operator $\frac{\partial^2}{\partial_{\overline{x}_i}\partial_{\overline{x}_j}}$ we obtain:

$$\frac{\partial^2}{\partial_{\overline{x}_i}\partial_{\overline{x}_j}} f(\overline{a}, \overline{x}) = \sum_{k,l=1}^{3} \frac{\partial^2 f(a,x)}{\partial_{x_i}\partial_{x_j}}\ c_{i,k}\ c_{j,l}$$

and the determinant of these expression gives

$$H(\overline{a},\overline{x}) = det\left(\frac{\partial^2 f(\overline{a},\overline{x})}{\partial_{\overline{x}_i}\partial_{\overline{x}_j}}\right) = det(C)^2 \cdot det\left(\frac{\partial^2 f(a,x)}{\partial_{x_i}\partial_{x_j}}\right) = det(C)^2 \cdot H(a,x). \quad (6.3)$$

If we now identify both f and $H(f)$ with their coefficients in $\mathbb{P}^9$, it is clear that the factor $det(C)^2$ no longer has any relevance, and thus we obtain a $GL(3)$-equivariant map

$$H : \mathbb{P}^9 \longrightarrow \mathbb{P}^9 \quad (6.4)$$

that take a cubic to its hessian cubic and satisfies $H(C \cdot f) = C \cdot H(f)$ where $C \in GL(3)$. To clarify this last statement, recall that, since $H(a_i, x)$ is a cubic form, acting on it with an element $C \in GL(3)$ means to replacing $x \to C\ \overline{x}$ and examining the new coefficients in $\overline{x}$. Thus, the expression obtained in 6.3 tells us that these coefficients are the same as those of $H(\overline{a_i}, \overline{x})$, that is the Hessian of $C \cdot f$ whose coefficients are the $\overline{a_i}$.
Let's write the explicit expression of the map H in terms of the coefficients a_i of f.
$H(f) =$

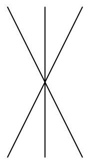
$$\begin{pmatrix} -a_2^2a_3 + 2a_1a_2a_4 - a_0a_4^2 - a_1^2a_5 + a_0a_3a_5 \\ 3a_1a_4^2 - 3a_1a_3a_5 - 3a_2^2a_6 + 3a_0a_5a_6 + 6a_1a_2a_7 - 6a_0a_4a_7 - 3a_1^2a_8 + 3a_0a_3a_8 \\ 3a_2a_4^2 - 3a_2a_3a_5 - 3a_2^2a_7 + 3a_0a_5a_7 + 6a_1a_2a_8 - 6a_0a_4a_8 - 3a_1^2a_9 + 3a_0a_3a_9 \\ 3a_3a_4^2 - 3a_3^2a_5 - 6a_2a_4a_6 + 3a_1a_5a_6 + 6a_2a_3a_7 - 3a_0a_7^2 - 3a_1a_3a_8 + 3a_0a_6a_8 \\ 12a_4^3 - 12a_3a_4a_5 - 6a_2a_5a_6 - 12a_2a_4a_7 + 18a_1a_5a_7 + 18a_2a_3a_8 - 12a_1a_4a_8 - 6a_0a_7a_8 - 6a_1a_3a_9 + 6a_0a_6a_9 \\ 3a_4^2a_5 - 3a_3a_5^2 - 3a_2a_5a_7 + 6a_1a_5a_8 - 3a_0a_8^2 + 3a_2a_3a_9 - 6a_1a_4a_9 + 3a_0a_7a_9 \\ -a_4^2a_6 + 2a_3a_4a_7 - a_1a_7^2 - a_3^2a_8 + a_1a_6a_8 \\ -6a_4a_5a_6 + 3a_4^2a_7 + 6a_3a_5a_7 - 3a_2a_7^2 + 3a_2a_6a_8 - 3a_1a_7a_8 - 3a_3^2a_9 + 3a_1a_6a_9 \\ -3a_5^2a_6 + 3a_4^2a_8 + 6a_3a_5a_8 - 3a_2a_7a_8 - 3a_1a_8^2 - 6a_3a_4a_9 + 3a_2a_6a_9 + 3a_1a_7a_9 \\ -a_5^2a_7 + 2a_4a_5a_8 - a_2a_8^2 - a_4^2a_9 + a_2a_7a_9 \end{pmatrix}$$

The Hessian map is not defined for cubic forms with a vanishing Hessian, that is, for all cubic forms (a_i) for which the ten cubic equations above vanish. These equations thus define the equations of the variety consisting of cubics with a vanishing Hessian, which we will denote by Z:

$$Z = \overline{\{f \in \mathbb{P}^9 \mid H(f) = 0\}} \subset \mathbb{P}^9$$

It has dimension 5, and we can also provide a characterization of such cubic forms. In fact, the following holds:

Proposition 6.8. *Let f be the polynomial defining a cubic in $\mathbb{P}^2$, and let $C_f = \{f = 0\} \subset \mathbb{P}^2$. Then, $H(f) = 0$ if and only if C_f is a cone, that is, if and only if C_f consists of three distinct lines passing through a common point.*



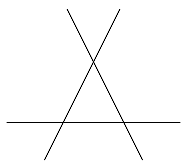
A proof of this fact can be found in [9].
Note that this is consistent with the classification of orbits under the action of $SL(3)$. In fact, the cubics with vanishing Hessian are precisely those given by triple line x^3, double line plus line x^2y and three concurrent lines $xy(x+y)$.

Let us now consider the cubics that are fixed points for this map, that is, those that coincide with their own Hessian. Let Tr denote the Zariski closure of this set:

$$Tr := \overline{\{f \in \mathbb{P}^9 \mid H(f) = f\}} \in \mathbb{P}^9 \tag{6.5}$$

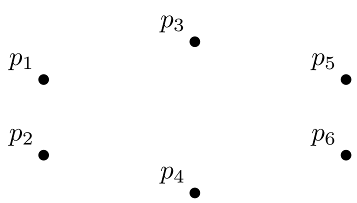
This variety has dimension 6 and contains Z, since the equations defining it are obtained by imposing the vanishing of the 2×2 minors of the matrix $\begin{bmatrix} f \\ H(f) \end{bmatrix}$. Again, we are able to provide a characterization of the cubic in Tr.

Proposition 6.9. *Let f be a polynomial defining a cubic in $\mathbb{P}^2$, and let $C_f = \{f = 0\} \subset \mathbb{P}^2$. Then, $f \in Tr$ if and only if C_f is the union of three lines, that is $f = (\alpha_0 x + \alpha_1 y + \alpha_2 z)(\beta_0 x + \beta_1 y + \beta_2 z)(\gamma_0 x + \gamma_1 y + \gamma_2 z)$.*



This is why Tr will be referred to as the variety of triangles. From the classification of the orbits under the action of $SL(3)$, we see that all the triangles lie in the same orbit, which is the one of xyz.

Proposition 6.10. *Given six generic points in $\mathbb{P}^2$, meaning that no three of them are collinear, there exist exactly 15 triangles passing through these points.*



Proof. Let us fix one side of the triangles as the line p_1p_2. We then need to construct two more lines passing through the remaining four points, and we have 3 possible choices for this. Since p_1 can initially be paired with any of the 5 points, we obtain a total of $3 \cdot 5 = 15$ triangles. □

We now study the preimages of H. As seen in Hesse's theorem, a generic cubic f is the Hessian of three other cubics that lie in the same syzygetic pencil as f. That is, the following holds:

Proposition 6.11. *The Hessian map for plane cubic is generically [3:1].*

However, this does not hold for the cubics in Tr. These certainly lie in the image of H, as they are fixed points of the map, but not only that. Since H is equivariant with respect to the action of $SL(3)$, we can reduce to studying the case of xyz, as a representative of $Tr - Z$. To find the cubics of which xyz is the Hessian, we need to determine for which coefficients a_i the 2×2 minors of the matrix vanish:

$$\begin{bmatrix} & & & & & H(f) & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $H(f)$ is a row vector containing the coefficients of the Hessian of a generic cubic. By saturating with Tr, one finds that the cubics whose Hessian is xyz are given by equations of the form

$$\lambda_1 x^3 + \lambda_2 y^3 + \lambda_3 z^3 = 0$$

which form a projective space of dimension 2 in $\mathbb{P}^9$, to which we must add xyz itself. If, on the other hand, we consider a cone, taking x^3 as a representative, we find that the space of cubics whose Hessian is the cone has dimension 4 and is defined by the following conditions:

$$\begin{cases} a_4^2 - a_3 * a_5 = 0 \\ a_6 = a_7 = a_8 = a_9 = 0 \end{cases}$$

We denote

$$\overline{H^{-1}(f)} := \overline{\{g \in \mathbb{P}^9 \mid H(g) = f\}} \subset \mathbb{P}^9 \tag{6.6}$$

Table 2: Space of the cubics with a fixed cubic as Hessian. Results for the $SL(3)$ orbits.

Cubic f	$dim(\overline{H^{-1}(f)})$	**Description of** $g \in H^{-1}(f)$
x^3	4	$a_0x^3 + 3a_1x^2y + 3a_2x^2z + 3a_3xy^2 + 6a_4xyz + 3a_5xz^2$ *with* $a_4^2 = a_3a_5$ *and* $a_2^2a_3 - 2a_1a_2a_4 + a_1^2a_5 \neq 0$
$xy(x+y)$	*empty*	
x^2y	3	$a_0x^3 + 3a_1x^2y + 3a_2x^2z + a_6y^3$ *with* $a_2, a_6 \neq 0$
$x(x^2+yz)$	0	$x^3 - 3xyz$
$y(x^2+yz)$	*empty*	
$y^2z - x^3 - x^2z$	0	$-2x^3 - 3x^2z + 3xy^2 + 3y^2z$
$y^2z - x^3$	*empty*	
$x^3 + y^3 + z^3 - 3txyz$ with $t^3 \neq 1$	0	$x^3 + y^3 + z^3 - 3\lambda xyz$ *with* $4 - \lambda^3 = 3\lambda^2 t$
xyz	2	$a_0x^3 + a_6y^3 + a_9z^3$ with $a_0, a_6, a_9 \neq 0$ *and* xyz

7 A Skew Invariant of Plane Cubics

In this chapter, we introduce a skew invariant of plane cubics. In particular, we deal with a joint invariant of three cubics. We will see that it is closely related to the Hessian of a cubic.

7.1 Definition

We identify the space of plane cubics with $Sym^3\mathbb{C}^3$ and consider the space $\bigwedge^3(Sym^3\mathbb{C}^3)$. We can decompose the latter into Weyl modules, obtaining:

$$\bigwedge^3(Sym^3\mathbb{C}^3) = \mathbb{S}_{(7,1,1)} \oplus \mathbb{S}_{(6,3)} \oplus \mathbb{S}_{(5,3,1)} \oplus \mathbb{S}_{(3,3,3)}$$

As follows from Proposition 1.11, $\mathbb{S}_{(3,3,3)}$ corresponds to a skew invariant of degree 3. Its explicit expression can be obtained thrugh symbolic representation.
From Theorem 2.2, we can define this invariant on the Veronese variety:

$$R : Sym^3(\mathbb{C}^3) \times Sym^3(\mathbb{C}^3) \times Sym^3(\mathbb{C}^3) \longrightarrow \mathbb{C}$$
$$R\ (l^3, m^3, n^3) := (l \wedge m \wedge n)^3 \tag{7.1}$$

where l, m, n are linear forms, and $l \wedge m \wedge n$ is the determinant.
In the language of tableau functions, as defined in 4.3, R corresponds to the $SL(3)$ invariant function associated with the 3×3 tableau:

1	1	1
2	2	2
3	3	3

From this definition, it is evident that R is an antisymmetric invariant.
The expression of R over three generic cubics can be obtained using the process described in paragraph 4.3. It is important to note that the only expression we need corresponds to G_T. In fact, if we attempt to derive an expression analogous to F_T as in Definition 4.6, the result is 0 due to the antisymmetric property of the invariant.
Below, we provide the Macaulay2 code to compute this expression:

```
KK=QQ
R1=KK[x_1..x_3,y_1..y_3,z_1..z_3,a_0..a_9,b_0..b_9,c_0..c_9]
inv=det(matrix{{x_1,x_2,x_3},{y_1,y_2,y_3},{z_1,z_2,z_3}})^3
syma=(x,h)->(contract(x,h)*transpose matrix{{a_0..a_9}})_(0,0)
symb=(x,h)->(contract(x,h)*transpose matrix{{b_0..b_9}})_(0,0)
symc=(x,h)->(contract(x,h)*transpose matrix{{c_0..c_9}})_(0,0)
invx=syma(symmetricPower(3,matrix{{x_1,x_2,x_3}}),inv)
invy=symb(symmetricPower(3,matrix{{y_1,y_2,y_3}}),invx)
invz=symc(symmetricPower(3,matrix{{z_1,z_2,z_3}}),invy)
factor invz
```

In this code x_i, y_i, z_i are the coefficients of the three linear forms and $(a_i), (b_i), (c_i)$ represent the three generic cubics. We apply the correspondence from 4.5 in the following way:

$$x_1, x_2, x_3 \longleftrightarrow a_i \quad y_1, y_2, y_3 \longleftrightarrow b_i \quad z_1, z_2, z_3 \longleftrightarrow c_i$$

We obtain the following expression for R, it is a sum of 54 monomials:

$R((a_i),(b_i),(c_i)) = (-1)(a_9b_6c_0 - 3a_8b_7c_0 + 3a_7b_8c_0 - a_6b_9c_0 - 3a_9b_3c_1 + 6a_8b_4c_1 - 3a_7b_5c_1 + 3a_5b_7c_1 - 6a_4b_8c_1 + 3a_3b_9c_1 + 3a_8b_3c_2 - 6a_7b_4c_2 + 3a_6b_5c_2 - 3a_5b_6c_2 + 6a_4b_7c_2 - 3a_3b_8c_2 + 3a_9b_1c_3 - 3a_8b_2c_3 - 6a_5b_4c_3 + 6a_4b_5c_3 + 3a_2b_8c_3 - 3a_1b_9c_3 - 6a_8b_1c_4 + 6a_7b_2c_4 + 6a_5b_3c_4 - 6a_3b_5c_4 - 6a_2b_7c_4 + 6a_1b_8c_4 + 3a_7b_1c_5 - 3a_6b_2c_5 - 6a_4b_3c_5 + 6a_3b_4c_5 + 3a_2b_6c_5 - 3a_1b_7c_5 - a_9b_0c_6 + 3a_5b_2c_6 - 3a_2b_5c_6 + a_0b_9c_6 + 3a_8b_0c_7 - 3a_5b_1c_7 - 6a_4b_2c_7 + 6a_2b_4c_7 + 3a_1b_5c_7 - 3a_0b_8c_7 - 3a_7b_0c_8 + 6a_4b_1c_8 + 3a_3b_2c_8 - 3a_2b_3c_8 - 6a_1b_4c_8 + 3a_0b_7c_8 + a_6b_0c_9 - 3a_3b_1c_9 + 3a_1b_3c_9 - a_0b_6c_9)$

Moreover, R can also be rewritten in terms of the exterior (wedge) product as follows:

$R((a_i),(b_i),(c_i)) = (-1)(a_9 \wedge a_6 \wedge a_0 - 3a_8 \wedge a_7 \wedge a_0 - 3a_9 \wedge a_3 \wedge a_1 + 6a_8 \wedge a_4 \wedge a_1 + 3a_7 \wedge a_5 \wedge a_1 + 3a_8 \wedge a_3 \wedge a_2 - 6a_7 \wedge a_4 \wedge a_2 - 3a_6 \wedge a_5 \wedge a_2 - 6a_5 \wedge a_4 \wedge a_3$

7.2 Relationship with the Hessian

As we know the Hessian of a plane cubic is also a plane cubic, we can identify it by its coefficients, which are polynomials in the a_i of degree 3. As a result, we obtain a vector with 10 entries, denoted by $\overline{a} = (\overline{a}_0,..,\overline{a}_9)$. Let $Q[a_0,..,a_9]_3$ be the space of homogeneous polynomial of degree 3 in the variables a_i, thus we have $\overline{a} \in (\mathbb{C}[a_0,..,a_9]_3)^{10}$ and a *Syzygy* among $\overline{a}_i$ is a 10-tuple of elements in $\mathbb{C}[a_0,..,a_9]$, $s = (s_0,..,s_9)$, such that $\overline{a}s^T = \sum_i \overline{a}_i s_i = 0$.
Macaulay2 provides a command to compute syzygies. Using the following code, we calculate the syzygies of $H(f)$:

```
f=x_1^3*a_0+3*x_1^2*x_2*a_1+3*x_1^2*x_3*a_2+3*x_1*x_2^2*a_3+6*x_1*x_2*x_3*a_4+
  +3*x_1*x_3^2*a_5+x_2^3*a_6+3*x_2^2*x_3*a_7+3*x_2*x_3^2*a_8+x_3^3*a_9
hf=det diff(x1,diff(transpose x1,f))
X=matrix{{ x_1^3,(1/3)* x_1^2*x_2,(1/3)* x_1^2*x_3,(1/3)* x_1*x_2^2,
        (1/6)* x_1*x_2*x_3,(1/3)* x_1*x_3^2, x_2^3,(1/3)* x_2^2*x_3,
        (1/3)* x_2*x_3^2, x_3^3}}
xhf=contract(X,hf)--coefficienti hessiana
kernel xhf
ghf=gens kernel(xhf)
```

At the end of this code, we obtain a 10×45 matrix with elements in K. Every column of this matrix represents a *syzygy* of $\overline{a}$. The first 10 columns have entries of degree 1 in the a_i while the other 35 has degree 3 in the a_i. We are now particularly interested in the 10×10 submatrix whose coefficients are linear in the a_i:

$$S = \begin{bmatrix} -a_6 & 0 & -a_7 & 0 & 0 & -a_8 & 0 & 0 & 0 & -a_9 \\ 3a_3 & -a_7 & 2a_4 & 0 & -a_8 & a_5 & 0 & 0 & -a_9 & 0 \\ 0 & a_6 & a_3 & 0 & a_7 & 2a_4 & 0 & 0 & a_8 & 3a_5 \\ -3a_1 & 2a_4 & -a_2 & -a_8 & a_5 & 0 & 0 & -a_9 & 0 & 0 \\ 0 & -2a_3 & -2a_1 & 2a_7 & 0 & -2a_2 & 0 & 2a_8 & 2a_5 & 0 \\ 0 & 0 & 0 & -a_6 & -a_3 & -a_1 & 0 & -a_7 & -2a_4 & -3a_2 \\ a_0 & -a_2 & 0 & a_5 & 0 & 0 & -a_9 & 0 & 0 & 0 \\ 0 & a_1 & a_0 & -2a_4 & -a_2 & 0 & 3a_8 & a_5 & 0 & 0 \\ 0 & 0 & 0 & a_3 & a_1 & a_0 & -3a_7 & -2a_4 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_6 & a_3 & a_1 & a_0 \end{bmatrix}$$

Let's return to discussing the invariant R and associate a matrix to it, which we will denote as $\overline{R}$. This matrix is defined such that $R((b_i),(a_i),(c_i)) = b\overline{R}c$, where $b = (b_0,..,b_9)$, $a = (a_0,..,a_9)$ and $c = (c_0,..,c_9)$ represent the coefficients of three generic cubic forms. It is evident that $\overline{R}$ is a matrix whose entries are linear in the a_i, and it is also straightforward to compute it:

$$\overline{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -a_9 & 3a_8 & -3a_7 & a_6 \\ 0 & 0 & 0 & 3a_9 & -6a_8 & 3a_7 & 0 & -3a_5 & 6a_4 & -3a_3 \\ 0 & 0 & 0 & -3a_8 & 6a_7 & -3a_6 & 3a_5 & -6a_4 & 3a_3 & 0 \\ 0 & -3a_9 & 3a_8 & 0 & 6a_5 & -6a_4 & 0 & 0 & -3a_2 & 3a_1 \\ 0 & 6a_8 & -6a_7 & -6a_5 & 0 & 6a_3 & 0 & 6a_2 & -6a_1 & 0 \\ 0 & -3a_7 & 3a_6 & 6a_4 & -6a_3 & 0 & -3a_2 & 3a_1 & 0 & 0 \\ a_9 & 0 & -a_5 & 0 & 0 & 3a_2 & 0 & 0 & 0 & -a_0 \\ -3a_8 & 3a_5 & 6a_4 & 0 & -6a_2 & -3a_1 & 0 & 0 & 3a_0 & 0 \\ 3a_7 & -6a_4 & -3a_3 & 3a_2 & 6a_1 & 0 & 0 & -3a_0 & 0 & 0 \\ -a_6 & 3a_3 & 0 & -3a_1 & 0 & 0 & a_0 & 0 & 0 & 0 \end{bmatrix}$$

This way of expressing the invariant R is useful for the following

Proposition 7.1. *Let R be the antisymmetric invariant of $Sym^3(\mathbb{C}^3)$ that we discussed earlier. Then, for any $f \in Sym^3(\mathbb{C}^3)$ for which the Hessian is define, we have $R(f, H(f), -) = 0$, where $H(f)$ is the Hessian of f.*

Proof. We begin this proof by observing the two matrices $\overline{R}$ and S . These matrices are identical up to a permutation of their columns, meaning there exists a permutation matrix P such that $\overline{R}P = S$. Constructing such a matrix P is straightforward:

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This observation implies that every column of $\overline{R}$ is also a syzygy for the coefficients of the Hessian of the cubic form whose coefficients are the a_i.
Therefore, we can express $-R(f, H(f), -) = R(H(f), f, -) = \overline{a}\overline{R}(-) = 0$ where $f = (a_i)_i$ and $H(f) = (\overline{a}_i)_i$ is the hessian of f. The final equality follows directly from the earlier observation. □

Corollary 7.2. *If f and g are two cubic forms belonging to the same pencil generated by a cubic form and its Hessian, then $R(f, g, -) = 0$.*

Proof. This result follows directly from the previous theorem and the linearity of the invariant R. □

We conclude this section by presenting the computation of the rank of the matrix $\overline{R}$ for all the possible orbits of the action of $SL(3)$ and making some observations.

Table 3: Rank of the matrix associated with the invariant R for a representative in each orbit of the action of $SL(3)$

Cubic f	**Hessian cubic** $H(f)$	**Rank**($\overline{R}$)
x^3		4
$xy(x+y)$		6
x^2y		6
$x(x^2+yz)$	$-3x^3+xyz$	8
$y(x^2+yz)$	y^3	8
$y^2z-x^3-x^2z$	$3xy^2-x^2z+y^2z$	8
y^2z-x^3	xy^2	8
$x^3+y^3+z^3-3txyz$ with $t^3 \neq 1$	same form with $t' = \frac{t^3-4}{6t^2}$	8
xyz	xyz	6

Note that the cubics for which the rank of $\overline{R}$ is less than 8 are all cubics for which the Hessian is either not defined or coincides with the cubic itself. We define

$$T := \{f = (a_i) \mid Rank(\overline{R}(a_i)) \leq 6\} \subset \mathbb{P}^9 \tag{7.2}$$

$$C(f) := \{g \in \mathbb{P}^9 | R(f, g, -) \equiv 0\} \subset \mathbb{P}^9 \tag{7.3}$$

For a fixed f, the Rank of $\overline{R}$ is the dimension of the image of the invariant R, whereas $C(f)$ defines the kernel. Thus, we have $T = \{f |\ dim(C(f)) \geq 3\}$. Moreover, from corollary 7.2, it follows that $C(f) = \langle f, H(f)\rangle$ when $Rank(\overline{R}(f)) = 8$, that is $f \notin T$.

7.3 The Vanishing Locus of the Invariant R

At this point, it becomes natural to ask for which f and g we have $R(f, g, -) = 0$, and, in the case this is not true, for which h we have $R(f, g, h) = 0$.

Let us begin by rewriting R in a similar but clearer formulation for what we need to do:

$$R : Sym^3(\mathbb{C}^3) \times Sym^3(\mathbb{C}^3) \longrightarrow (Sym^3(\mathbb{C}^3))^*$$
$$R(f,g) := R(f,g,-) : Sym^3(\mathbb{C}^3) \to \mathbb{C}$$

Let $f = (a_i)$, $g = (b_i)$ and $h = (c_i)$ the identification of the three cubic forms with their coefficients. Consider the expression of R we gave in 7.1 and reorder it according to the coefficients of h, that is:

$$\begin{aligned} R =&(-a_9b_6 + 3a_8b_7 - 3a_7b_8 + a_6b_9, 3a_9b_3 - 6a_8b_4 + 3a_7b_5 - 3a_5b_7 + 6a_4b_8 - 3a_3b_9,\\ &-3a_8b_3 + 6a_7b_4 - 3a_6b_5 + 3a_5b_6 - 6a_4b_7 + 3a_3b_8, -3a_9b_1 + 3a_8b_2 + 6a_5b_4 - 6a_4b_5 - 3a_2b_8 + 3a_1b_9,\\ &6a_8b_1 - 6a_7b_2 - 6a_5b_3 + 6a_3b_5 + 6a_2b_7 - 6a_1b_8, -3a_7b_1 + 3a_6b_2 + 6a_4b_3 - 6a_3b_4 - 3a_2b_6 + 3a_1b_7,\\ &a_9b_0 - 3a_5b_2 + 3a_2b_5 - a_0b_9, -3a_8b_0 + 3a_5b_1 + 6a_4b_2 - 6a_2b_4 - 3a_1b_5 + 3a_0b_8,\\ &3a_7b_0 - 6a_4b_1 - 3a_3b_2 + 3a_2b_3 + 6a_1b_4 - 3a_0b_7, -a_6b_0 + 3a_3b_1 - 3a_1b_3 + a_0b_6) \cdot (c_0, c_1, .., c_9) =\\ &= n(a_i, b_i) \cdot (c_0, .., c_9) \end{aligned} \tag{7.4}$$

This represents the scalar product between two vectors with ten entries. Note that $(c_0, .., c_9)$ is simply a point in $Sym^3(\mathbb{C})^3$, while $R(f, g, -)$ is a point in $(Sym^3(\mathbb{C}^3))^*$. Using the dual basis, the first 10-vector we introduced in the last expression for R, denoted as $n = n(f, g)$, is $R(f, g, -) \in (Sym^3(\mathbb{C}^3))^*$. Summarizing, $n \in (\mathbb{C}[a_i, b_i])^{10}$ such that $R(f, g, h) = n \cdot (c_0, .., c_9)$.

Let's begin by considering the cubic f of the form $x^3 + y^3 + z^3 - 3txyz$ with $t \in \mathbb{C}$. By rewriting the expression of n in this case, we obtain:

$n = (-b_6 + b_9, -3b_8t + 3b_3, 3b_7t - 3b_5, 3b_5t - 3b_1, 0, -3b_3t + 3b_2, b_0 - b_9, -3b_2t + 3b_8, 3b_1t - 3b_7, -b_0 + b_6)$

Then $R(f, g, -) = 0$ when this vector is identically 0. This leads to two distinct cases:

- If $t^3 \neq 1$ we have $R(f, g, -) = 0$ if and only if
$$\begin{cases} b_0 = b_6 = b_9 \\ b_1 = b_2 = b_3 = b_5 = b_7 = b_8 = 0 \end{cases}$$
This is equal to stating that $R(f, g, -) = 0$ if and only if g is a cubic form in the same pencil as f. In other words, $g = x^3 + y^3 + z^3 - 3sxyz$ for some $s \in \mathbb{C}$.
- If $t^3 = 1$, meaning f is a cubic that coincides with its Hessian, we have $R(f, g, -) = 0$ if and only if
$$\begin{cases} b_0 = b_6 = b_9 \\ b_3 = tb_8 = t^2b_2 \\ b_5 = tb_7 = t^2b_1 \end{cases}$$
That is, $g = x^3+y^3+z^3+6b_4xyz+3b_1(t^2xz^2+ty^2z+x^2y)+3b_2(t^2xy^2+tyz^2+x^2z)$.

Repeating the same argument for the other orbits under the action of $SL(3)$, excluding those with $Rang(\overline{R}) = 8$ for which we already know the result, we obtain the following results:

Table 4: Vanish locus of $R(f, g, -) = 0$ when f is a representative in each orbit of the action of $SL(3)$

Cubic f	$\{g \mid R(f, g, -) = 0\}$
x^3	$x^3 + 3b_1x^2y + 3b_2x^2z + 3b_3xy^2 + 6b_4xyz + 3b_5xz^2$
$xy(x + y)$	$x^3 + 3b_1(x^2y + xy^2) + 3b_2(x^2z + xyz + y^2z) + b_6y^3$
x^2y	$x^3 + 3b_1x^2y + 3b_2x^2z + b_6xyz$
$x(x^2 + yz)$	$x^3 + 6b_4xyz$
$y(x^2 + yz)$	$y(x^2 + yz) + b_6y^3$
$y^2z - x^3 - x^2z$	$y^2z - x^3 - x^2z + b_3(3xy^2 - x^2z + y^2z)$
$y^2z - x^3$	$y^2z - x^3 + 3b_3xy^2$
$x^3 + y^3 + z^3 - 3txyz$ with $t^3 \neq 1$	$x^3 + y^3 + z^3 + 6b_4xyz$
xyz	$x^3 + 6b_4xyz + b_6y^3 + b_9z^3$

Comparing these results with those reported in Table 2, we easily observe that, for triangles, the space where R vanishes is obtained by adding the triangle itself, as a linear combination, to the space of cubics for which the triangle is the Hessian. In fact, since R is antisymmetric, we have $R(f, f, -) = 0$, and from the linearity of R, it follows that the cubic itself must always be a generator of the space in which R vanishes. Regarding x^2y, the space shown in the table above exactly coincides with the space $H^{-1}(f)$, which already contains the cubic x^2y in its span. Finally, we observe that for the triple lines, that is x^3, the space $H^{-1}(f)$ is contained in the space where R vanishes because in the first space we have an additional condition on the coefficients.

8 The Hesse Pencil Variety

This chapter is devoted to the study of a variety $N \subset G(1, 9)$. The defining equations are obtained from the explicit expression of R; in particular, this variety describes the locus $R(f, g, -) \equiv 0$. At the end of the chapter, we will prove that N coincides with the Hesse Pencil Variety, that is, the closure of the orbit of the Hesse pencil under the action of $SL(3)$.

8.1 Connection with the Invariant

Let us consider the Grassmannian $Gr(\mathbb{P}^1, \mathbb{P}^9)$, that is the set of of projective lines in $\mathbb{P}^9$. $Gr(\mathbb{P}^1, \mathbb{P}^9)$ is a projective variety of dimension 16 (projective dimension) embedded in a projective space of dimension 44: $Gr(\mathbb{P}^1, \mathbb{P}^9) \hookrightarrow \mathbb{P}^{44}$. Observing the expression of R introduced in the previous section, we see that $n = n(a_i, b_i)$, defined in (7.4), is closely related to the Plücker coordinates of the line through points a_i and b_i. In fact, we have

$$p_{ij} = a_i b_j - a_j b_i \quad \forall\, i = 0, .., 9 \;\; and \;\; j > i$$

and the components of the vector n can be rewritten in terms of these coordinates.

$n(p_{ij}) = (3p_{7,8} - p_{6,9}, 3p_{5,7} - 6p_{4,8} + 3p_{3,9}, 3p_{5,6} - 6p_{4,7} + 3p_{3,8}, 6p_{4,5} + 3p_{2,8} - 3p_{1,9}, 6p_{3,5} + 6p_{2,7} - 6p_{1,8}, 3p_{2,5} - p_{0,9}, 6p_{3,4} + 3p_{2,6} - 3p_{1,7}, 6p_{2,4} + 3p_{1,5} - 3p_{0,8}, 3p_{2,3} + 6p_{1,4} - 3p_{0,7}, 3p_{1,3} - p_{0,6})$

We define a variety contained in $G(\mathbb{P}^1, \mathbb{P}^9)$, whose equations are given by the vanishing of the entries of the vector $n(p_{ij})$ and the equations of the Grassmannian. We denote this variety by N. With the help of Macaulay2, we can easily compute the dimension of the variety N, obtaining:

$$N = (n(p_{i,j}) = 0) \cap Gr(\mathbb{P}^1, \mathbb{P}^9) \;\subset Gr(\mathbb{P}^1, \mathbb{P}^9) \qquad dim(N) = 8. \tag{8.1}$$

Using the Betti command to determine the generators of the ideal, we get:

```
betti mingens N

          0   1
total: 1 210
    0: 1  10
    1: . 200
```

so, the ideal N has 10 generators of degree 1 and 200 generators of degree 2.
We can also evaluate the degree of N, which is 622 (the degree of G is 1430).

Let us now consider the space $\bigwedge^2 Sym^3\mathbb{C}^3$, which has dimension 45. We can think of $G(1, 9) \subset \mathbb{P}(\bigwedge^2(Sym^3\mathbb{C}^3))$. This space decomposes into Weyl modules as follows:

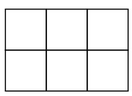
$$\bigwedge^2(Sym^3\mathbb{C}^3) = \mathbb{S}_{(5,1)}\mathbb{C}^3 \;\oplus\; \mathbb{S}_{(3,3)}\mathbb{C}^3$$

These two covariants have dimension 35 and 10, respectively.

Proposition 8.1. *Using the previous notations, we have*

$$G(1, 9) \cap \mathbb{S}_{(5,1)} = N$$

Proof. We proceed analogously to the case of quartics5.8. The covariant of dimension 10 corresponds to the tableau



Hence, as follows from Remark 4.8, we can obtain an explicit expression starting from the tableau

1	1	1
2	2	2
x	x	x

In this way we obtain a polynomial in $Sym^3\mathbb{C}^3$ whose coefficients can be rewritten in terms of Plücker coordinates:

$$\begin{aligned} F_{10} =&(3p_{(7,8)} - p_{(6,9)})x^3 + (3p_{(5,7)} - 6p_{(4,8)} + 3p_{(3,9)})x^2y + \\ &+ (3p_{(5,6)} - 6p_{(4,7)} + 3p_{(3,8)})x^2z + (6p_{(4,5)} + 3p_{(2,8)} - 3p_{(1,9)})xy^2 + \\ &+ (6p_{(3,5)} + 6p_{(2,7)} - 6p_{(1,8)})xyz + (3p_{(2,5)} - p_{(0,9)})xz^2 + (6p_{(3,4)} + 3p_{(2,6)} - 3p_{(1,7)})y^3 + \\ &+ (6p_{(2,4)} + 3p_{(1,5)} - 3p_{(0,8)})y^2z + (3p_{(2,3)} + 6p_{(1,4)} - 3p_{(0,7)})yz^2 + (3p_{(1,3)} - p_{(0,6)})z^3 \end{aligned}$$

The vanishing of the 10 coefficients of this polynomial corresponds to the equations defining the space $\mathbb{S}_{(5,1)}\mathbb{C}^3$, which is thus obtained by intersecting $\bigwedge^2(Sym^3\mathbb{C}^3)$ with this ten hyperplanes. To conclude the proof of the proposition, it is sufficient to observe that the ten coefficients of F_{10} coincide with the vector $n(p_{i,j})$. □

The variety N contains the lines of $\mathbb{P}^9$ for which $R(a_i, b_i, -) = 0$, where a_i and b_i are two points of the line itself. Remember that these points in $\mathbb{P}^9$ correspond to cubics, and according to corollary 7.2, a generic point of this variety corresponds to a syzygetic pencil, that is, a pencil of the form $\langle f, H(f)\rangle$.

Definition 8.2. *Consider in $G(1,9)$ the set of lines generated by a smooth cubic and its Hessian. We call Locus of Hesse pencils the subvariety of $G(1,9)$ defined as the Zariski closure of this set, and denote it by S.*

$$S = \overline{\{\langle f, H(f)\rangle \in G(1,9) \mid f \ \ smooth\}} \subset G(\mathbb{P}^1, \mathbb{P}^9)$$

As we observed above, it holds that $S \subset N$.

Let's now see how to obtain the dimension of S from a theoretical perspective. One could consider deriving computationally the equations that define the variety S, using a procedure similar to that applied in the case of binary quadrics and $G(1,4)$ in 5. The problem is that in this case we are dealing with a much larger number of variables, and in particular, we need to eliminate 10 variables from an ideal defined by 45 equations. The computation time increases so much that it can be considered computationally infeasible. We define

$$P := \overline{\{(f, L) \in \mathbb{P}^9 \times S \mid f \in L\}} \subset \mathbb{P}^9 \times G(1,9)$$

and we denote by $p_1 : \mathbb{P}^9 \times G(1,9) \longrightarrow \mathbb{P}^9$ and $p_2 : \mathbb{P}^9 \times G(1,9) \longrightarrow G(1,9)$ the projections onto the two spaces.

$$\begin{array}{ccc} & P & \\ {\scriptstyle p_1|_Z} \swarrow & & \searrow {\scriptstyle p_2|_Z} \\ \mathbb{P}^9 & & S \end{array}$$

Let's take $f \in \mathbb{P}^9$ and consider its fiber over P, that is $p_1|_P^{-1}(f) = \{(f, L) \in \{f\} \times S \mid f \in L\}$. In the general case, there is only a syzygetic pencil to which f belongs, and it is the one generated by f itself and its Hessian, $L = \langle f, H(f)\rangle$, as follows from Hesse's Theorem 6.7. Thus, the dimension of a generic fiber is 0 and we have

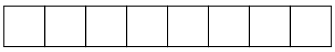
$$dim(P) = dim(\mathbb{P}^9) + 0 = 9.$$

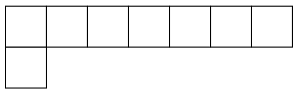
Now, let's consider $L \in S$. Its fiber is given by $p_2|_Z^{-1}(L) = \{(f, L) \in \mathbb{P}^9 \times \{L\} \mid f \in L\}$. This fiber is exactly composed of the cubics lying in L, and therefore has dimension 1. It follows that

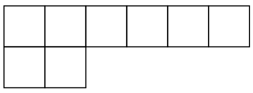
$$9 = dim(P) = dim(S) + 1 \qquad \Longrightarrow \qquad dim(S) = 8.$$

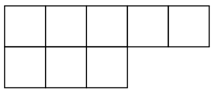
The goal of what we will do in the following of this chapter is to prove that the two varieties N and S are indeed the same. The strategy we intend to employ involves decomposing the classes of these two varieties into a linear combination of Schubert cycles, calculating their multidegrees. At the end of the process, the equality of these multidegrees will ensure the equality between S and N, except for subvarieties of dimension less than 8, as follows from Theorem 3.9. In the case of N, we will do this computationally, and since we already know the degree of N, the weighted sum of the multidegrees should result in 622. For S, since we do not have the equations that define it, we cannot compute its multidegree directly. However, since we know the theoretical description of S, we will derive its multidegree theoretically from this.

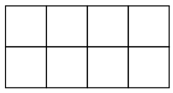
In $G(\mathbb{P}^1, \mathbb{P}^9)$ there are exactly five Schubert cycles with $dim = 8$. As it follows from Theorem 3.9, these cycles generates $H^{16}(G(1,9), \mathbb{Z})$. Therefore, since both S and N have dimension 8, their degree can be expressed as a linear combination of the degree of these cycles. Each of these cycles corresponds to a tableau consisting of 8 squares contained within a 2×10 matrix. In particular, these are:

$\lambda_1 =$ 

$\lambda_2 =$ 

$\lambda_3 =$ 

$\lambda_4 =$ 

$\lambda_5 =$ 

If λ_i represents one of these diagrams, we denote by X_i the corresponding Schubert variety. Using the notation in [11], these Schubert cycles correspond, in order, to $\Omega(0, 9)$, $\Omega(1, 8)$, $\Omega(2, 7)$, $\Omega(3, 6)$, $\Omega(4, 5)$. From the theory, we know that:

$$degree(N) = \sum_{i=1}^{5} \alpha_i \; degree(X_i)$$

$$degree(S) = \sum_{i=1}^{5} \beta_i \; degree(X_i)$$

Therefore, the two vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ represent the multidegrees of the two varieties N ans S respectively. Moreover, we also know that the coefficients α_i are found as the number of elements in the intersection between N and the Schubert variety dual to X_i. However, since all these varieties are self-dual (see Theorem 3.12), the intersection must be taken with X_i itself. We also recall that this number corresponds to the degree of the intersection of the two varieties, with the care of choosing the Schubert variety in such a way that has a zero-dimensional intersection with N. As we already observed in the case of the quartics, this can be achieved by following the approach used in the proof of Proposition 3.7, taking as T a random 10×10 matrix in this case. The same argument can be applied to S to determine β.

As I mentioned earlier, the multidegree of N was determined computationally. Below, we present the obtained results in a table.

Schubert variety X_λ	$deg(X_\lambda)$	$codim(N \cap X_\lambda)$	$degree(N \cap X_\lambda)$
(0,9)	1	44	1
(1,8)	7	44	3
(2,7)	20	44	9
(3,6)	28	44	12
(4,5)	14	44	6

We thus obtain that $\alpha = (1, 3, 9, 12, 6)$, and using the formula above for the degree of N, we get $degree(N) = 1 \cdot 1 + 3 \cdot 7 + 9 \cdot 20 + 12 \cdot 28 + 6 \cdot 14 = 622$, which indeed matches the degree of N previously found directly from the equations.

Proposition 8.3. *Using the notation above, the multedegree of the variety S is given by*

$$\beta = (1, 3, 9, 12, 6)$$

For this proof, we will use the theoretical description of the Schubert varieties and S to determine the number of elements in the intersection, that is, the degree. In any case, everything presented in the proof has been computationally verified whenever possible.

Let's start from $\Omega(0, 9)$. A Schubert variety corresponding to this cycle represents the lines in $\mathbb{P}^9$ passing through a fixed point $f \in \mathbb{P}^9$. When intersecting with S, we need to count the number of syzygetic pencils passing through that point, that is, for the fixed cubic f. In the general case, there is only one, which corresponds to $\langle f, H(f)\rangle$, as follows from Hesse's Theorem 6.7. Thus, we obtain $\beta_1 = 1$.

As for $\Omega(1, 8)$, a Schubert variety corresponding to this cycle represents the lines in $\mathbb{P}^9$ that meet a fixed line l and are contained in a hyperplane $\mathbb{P}^8$ that also contains l. Thus, as before, to find the intersection with S, we need to count the number of syzygetic lines with this property. This requirement can be reformulated as follows: how many cubics that belongs to l have their Hessian contained in the fixed hyperplane? The condition of belonging to l becomes $a_i = p_i + \lambda v_i$ for $i = 0, .., 9$. The coefficients of the Hessian of such a cubic, denoted by $\overline{a_i}$, are polynomials of degree 3 in λ. A hyperplane of $\mathbb{P}^9$ is given by a linear equation of the form:

$$\sum_{i=0}^{9} \mu_i \ a_i = 0 \qquad \mu_i \in \mathbb{C}$$

where the a_i are the coordinates of $\mathbb{P}^9$. The condition that the Hessian belongs to this hyperplane can be rewritten as

$$\sum_{i=0}^{9} \mu_i \ \overline{a_i} = 0,$$

which turns out to be a cubic equation in the parameter λ, and therefore there are three solution. In this way, we obtain $\beta_2 = 3$.

We proceed in an analogous way for the cycle $\Omega(2,7)$. A Schubvert variety associated with this cycle represents all the lines in $\mathbb{P}^9$ that meet a plane π and are contained in a subspace $\mathbb{P}^7$ of $\mathbb{P}^9$ that also contains the plane π. We need to count how many syzygetic lines satisfy this property. This number will coincide with the number of cubics in π whose Hessian lies in the fixed $\mathbb{P}^7$. The condition that a cubic belongs to π can be written as:

$$a_i = p_i + \lambda v_i + \mu w_i \qquad i = 0, .., 9$$

The coefficients of the Hessian of such a cubic $\overline{a_i}$ are polynomials of degree 3 in (λ, μ). Suppose that the space $\mathbb{P}^7$ is given by the intersection of the two hyperplanes:

$$\sum_{i=0}^{9} \delta_i \ a_i = 0 \qquad \sum_{i=0}^{9} \gamma_i \ a_i = 0 \qquad\qquad \beta_i, \gamma_i \in \mathbb{C}$$

Thus, the Hessian of a cubic in π belongs to these hyperplanes if and only if

$$\begin{cases} \sum_{i=0}^{9} \delta_i \ \overline{a}_i = 0 \\ \sum_{i=0}^{9} \gamma_i \ \overline{a}_i = 0 \end{cases}$$

That are two equations of degree 3 in the two variables (λ, μ). By Bézout's theorem, there must be 9 solutions, that is $\beta_3 = 9$.

It's time to study what happens for the cycle $\Omega(3,6)$. First, we observe that by using reasoning analogous to the previous cases, we obtain a greater number of solutions. The reason is that, up to this point, we have been able to disregard the variety of triangles T. In fact, the dimension of T is 6, so it has an empty intersection with a generic line or a generic plane in $\mathbb{P}^9$. A Schubert variety associated with the cycle $\Omega(3,6)$ represents the lines that intersect a $\mathbb{P}^3$ and are contained in a $\mathbb{P}^6$, which in turn contains the $\mathbb{P}^3$. We need to count the number of cubics that lies in the $\mathbb{P}^3$ and that has their Hessian in the $\mathbb{P}^6$. The cubics lying in $\mathbb{P}^3$ will be characterized by three variables. The condition that the Hessian belongs to a $\mathbb{P}^6$ will be given by three cubic equations in these three variables. Again, by Bézout's theorem, we should have 27 solutions. The fact is that these 27 solutions also count the triangles that belong to $\mathbb{P}^3$ and those, clearly, by coinciding with their own Hessian, satisfy the condition we used to count these cubics. However, a triangle does not generate a syzygetic pencil. Thus, the question now becomes: How many triangles lie in a generic $\mathbb{P}^3$? To answer this, we think of $\mathbb{P}^3$ as the space of cubic curves passing through six fixed points. From Proposition 6.10, we know that there are exactly 15 triangles passing through six generic points. In conclusion, the number of syzygetic pencils will be $27-15=12$, meaning $\beta_4 = 12$.

Remark 8.4. *It should be noted that a triangle does not generate a syzygetic pencil because it does not determine two points of $\mathbb{P}^9$ with its Hessian. However, the triangles belong to syzygetic pencils, specifically we know they belong to a 2-dimensional pencil space (corresponding to the preimage H^{-1} of the triangle itself). Therefore, in principle, there could be other pencils, beyond the ones we have counted, that lie in the $\mathbb{P}^6$ and pass through a triangle in $\mathbb{P}^3$. This does not happen because a 2-dimensional space in $\mathbb{P}^9$ has an empty intersection with a generic $\mathbb{P}^6$.*

The last case we need to address is that of the cycle $\Omega(4,5)$. A Schubert variety corresponding to this cycle represents the lines in $\mathbb{P}^9$ that intersect a fixed $\mathbb{P}^4$ nontrivially and are contained in a $\mathbb{P}^5$ which, in turn, contains the same $\mathbb{P}^4$. Since a general line in $\mathbb{P}^5$ always intersects a general $\mathbb{P}^4 \subset \mathbb{P}^5$, the condition can be simply reformulated as describing the lines in $\mathbb{P}^9$ that are contained in a general $\mathbb{P}^5$. In particular, we need to count the number of syzygetic pencils satisfying this property. To do this, we consider the $\mathbb{P}^5$ inside $\mathbb{P}^9$ as given by the space of cubics passing through four fixed points. Moreover, since any four points in general position in $\mathbb{P}^2$ can always be mapped, via a projectivity, to

$$(1,0,0) \qquad (0,1,0) \qquad (0,0,1) \qquad (1,1,1)$$

we may choose there four points to define $\mathbb{P}^5$. As we did in the previous steps, in order to count the pencils contained in this $\mathbb{P}^5$, we should ask how many cubics in this $\mathbb{P}^5$ have their Hessian also lying in the same $\mathbb{P}^5$. These are precisely the cubics that share with their Hessian the four fixed points, that is, the cubics that have four prescribed inflection points. Counting the smooth cubics with this property is equivalent to counting the number of Hesse configurations of nine points in $\mathbb{P}^2$ with four fixed points.

Lemma 8.5. *There are exactly six Hesse configurations passing through the four fixed points.*

Proof. We know that, in a Hesse configuration, any line passing through two points always contains a third points of the configuration. The four fixed points determine six lines, so two of these lines must necessarily intersect at a fifth point of the configuration. Indeed, if this were not the case, we would have 4 fixed points plus 6 additional ones from the lines, which would be too many to fit in the configuration. This gives us three ways to choose the fifth point, which correspond to:

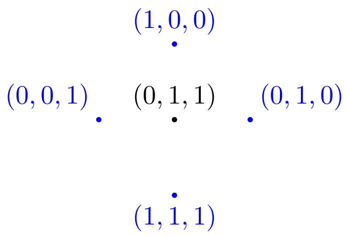


Figure 1: $\{y = z\} \cap \{x = 0\} = (0, 1, 1)$

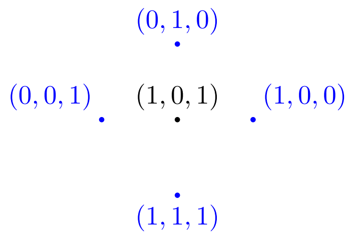


Figure 2: $\{x = z\} \cap \{y = 0\} = (1, 0, 1)$

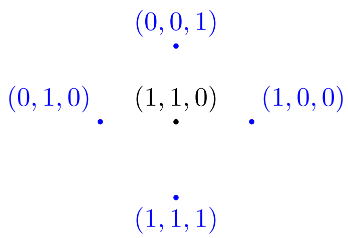
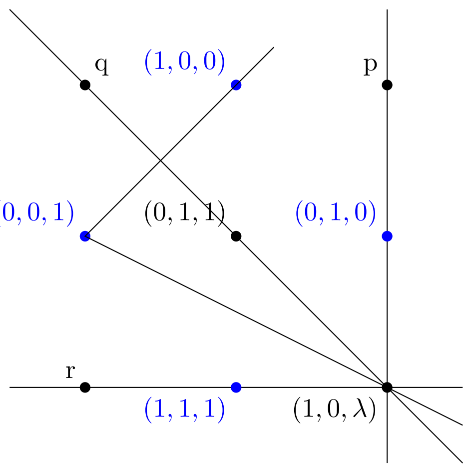


Figure 3: $\{x = y\} \cap \{z = 0\} = (1, 1, 0)$

We will show that for each of these cases, we obtain two Hesse configurations. Let us take the first case as an example; for the others, the procedure will be analogous. Consider the line through $(1, 0, 0)$ and $(0, 0, 1)$, that is, $y = 0$. On this line, there must be another point of the configuration, which we denotes as $(1, 0, \lambda)$. This point generates three other lines of the configuration with the points $(0, 1, 0)$, $(0, 1, 1)$ and $(1, 1, 1)$, which are respectively $z = \lambda x$, $z = y + \lambda x$ and $z = \lambda x + y(1 - \lambda)$. On each of these lines, we find another point of the configuration.



Let p denote the points of the configuration that lies on the line $z = \lambda x$. As we know, this point must lie on exactly three other lines of the configuration. However, we have four points available $(1, 0, 0)$, $(0, 1, 1)$, $(0, 0, 1)$, $(1, 1, 1)$ that, a priori, would determine four additional lines. Therefore, it follows that p must necessary be collinear with the line through two of these four points. Excluding the lines that already intersect the line $z = \lambda x$ at another point and those that already contain three points of the configuration, the only possibility is that p lies on the line through $(1, 1, 1)$ and $(0, 0, 1)$, that is $x = y$. We obtain

$$p \in \{z = \lambda x\} \cap \{x = y\} \quad \Longrightarrow \quad p = (1, 1, \lambda)$$

Let q denote the point on $z = y + \lambda x$. Using a similar reasoning, it follows that q must also lie on the line through $(1, 1, 1)$ and $(0, 1, 0)$, that is

$$q \in \{z = y + \lambda x\} \cap \{x = z\} \quad \Longrightarrow \quad q = (1, 1 - \lambda, 1)$$

Finally, let r denote the third point on $z = \lambda x + y(1 - \lambda)$. We obtain

$$r \in \{z = \lambda x + y(1 - \lambda)\} \cap \{z = 0\} \qquad \Longrightarrow \qquad r = (\lambda - 1, \lambda, 0)$$

The line through p and q must contain another point. The lines of the configuration for $(1, 0, 0)$ must be 4. From the previous calculation, we have already considered 3 of them, namely the lines for the points:

$$(1, 0, 0) \quad (0, 1, 1) \quad (1, 1, 1)$$

$$(1, 0, 0) \quad (0, 0, 1) \quad (1, 0, \lambda)$$

$$(1, 0, 0) \quad (0, 1, 0) \quad (\lambda - 1, \lambda, 0)$$

It follows that the line through p and the one through q must coincide, that is, the three points p, $(1, 0, 0)$ and q must be collinear.

$$0 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & \lambda \\ 1 & 1 - \lambda & 1 \end{vmatrix} = 1 + \lambda(\lambda - 1) = \lambda^2 - \lambda + 1$$

The solutions are $\lambda_1 = -\epsilon$ and $\lambda_2 = \epsilon + 1$ where ϵ is a primitive third root of unity. These two values give us the two possible configurations. The only thing left to verify is that r also lies on the line through q and $(0, 0, 1)$, and on the line through p and $(0, 1, 1)$.
For $\lambda = -\epsilon$

$$\begin{vmatrix} 1 & 1 + \epsilon & 1 \\ 0 & 0 & 1 \\ -\epsilon - 1 & -\epsilon & 0 \end{vmatrix} = \epsilon - (\epsilon + 1)^2 = -(\epsilon^2 + \epsilon + 1) = 0$$

$$\begin{vmatrix} 1 & 1 & -\epsilon \\ 0 & 1 & 1 \\ -\epsilon - 1 & -\epsilon & 0 \end{vmatrix} = -\epsilon(\epsilon + 1) - (-\epsilon + (\epsilon + 1)) = -\epsilon^2 - \epsilon - 1 = 0$$

For $\lambda = 1 + \epsilon$

$$\begin{vmatrix} 1 & -\epsilon & 1 \\ 0 & 0 & 1 \\ \epsilon & 1+\epsilon & 0 \end{vmatrix} = -(1 + \epsilon + \epsilon^2) = 0$$

$$\begin{vmatrix} 1 & 1 & 1+\epsilon \\ 0 & 1 & 1 \\ \epsilon & 1+\epsilon & 0 \end{vmatrix} = -\epsilon(1 + \epsilon) - (1 + \epsilon - \epsilon) = -\epsilon^2 - \epsilon - 1 = 0$$

We have thus found the two configurations corresponding to the first case. For the other two cases the procedure is the same; here we simply state the results, namely the six possible configurations.

First case:

$(1, \epsilon + 1, 1)$ $(1, 0, 0)$ $(1, 1, -\epsilon)$
$(0, 0, 1)$ $(0, 1, 1)$ $(0, 1, 0)$
$(1, \epsilon + 1, 0)$ $(1, 1, 1)$ $(1, 0, -\epsilon)$

$(1, -\epsilon, 1)$ $(1, 0, 0)$ $(1, 1, 1 + \epsilon)$
$(0, 0, 1)$ $(0, 1, 1)$ $(0, 1, 0)$
$(1, -\epsilon, 0)$ $(1, 1, 1)$ $(1, 0, \epsilon + 1)$

Second case:

$(\epsilon + 1, 1, 1)$ $(0, 1, 0)$ $(1, 1, -\epsilon)$
$(0, 0, 1)$ $(1, 0, 1)$ $(1, 0, 0)$
$(\epsilon + 1, 1, 0)$ $(1, 1, 1)$ $(0, 1, -\epsilon)$

$(-\epsilon, 1, 1)$ $(0, 1, 0)$ $(1, 1, 1 + \epsilon)$
$(0, 0, 1)$ $(1, 0, 1)$ $(1, 0, 0)$
$(-\epsilon, 1, 0)$ $(1, 1, 1)$ $(0, 1, \epsilon + 1)$

Third case:

$(-\epsilon, 1, 1)$ $(0, 0, 1)$ $(1, \epsilon + 1, 1)$
$(0, 1, 0)$ $(1, 1, 0)$ $(1, 0, 0)$
$(1, 0, \epsilon + 1)$ $(1, 1, 1)$ $(0, 1, -\epsilon)$

$(\epsilon + 1, 1, 1)$ $(0, 0, 1)$ $(1, -\epsilon, 1)$
$(0, 1, 0)$ $(1, 1, 0)$ $(1, 0, 0)$
$(1, 0, -\epsilon)$ $(1, 1, 1)$ $(0, 1, \epsilon + 1)$

□

Recalling where we started, each of these six configurations corresponds to a syzygetic pencil contained in the $\mathbb{P}^5$. We thus have 6 pencils in $S \cap X_{(5,5)}$, that is, $\beta_5 = 6$.

8.2 Pencil Orbits in the Variety

The action of $SL(3)$ on cubics induces an action of the same group on $G(1,9)$, that is, on pencils of cubics. Since R is an invariant under the action of $SL(3)$, and the variety N is defined in term of R, it follows that N must be invariant under this action, meaning that it is composed by $SL(3)$-orbits of pencils. The aim of this section is therefore to understand which and how many orbits are contained in N, and which of them form the singular locus of N. Moreover, this classification will allow us to complete the proof of the following theorem, which was initiated in the previous section.

Theorem 8.6. *In the previously introduced notation, we have*

$$N = S,$$

and in particular, this variety coincides with the closure of the orbit of the pencil $\langle x^3 + y^3 + z^3, xyz\rangle$ *under the action of* $SL(3)$, *and contains* 8 *additional orbits of pencils. Moreover, this variety is not smooth, and its singular locus coincides with the two orbits* $O(\langle x^3, x^2y\rangle)$ *and* $O(\langle x^2y, x^2z\rangle)$.

We began by defining, for each cubic $f \in \mathbb{P}^9$,

$$\mathbb{P}^8_f := \overline{\{\langle f, g\rangle \in G(1,9) \mid g \in \mathbb{P}^9\}} \tag{8.2}$$

which is the closure of the set of pencils through f. This space has dimension 8.
For each orbit of cubics, we fix a representative and compute the dimension of $\mathbb{P}^8_f \cap N$. The results are summarized in the following table, along with a description of the corresponding space. Let us note that this is just another way of viewing the information already presented in Table 8.2. Nevertheless, we repeat the results here for the reader's convenience.

Cubic f	$\mathbf{dim}(\mathbb{P}^8_f \cap N)$	**Description**
x^3	4	$\langle x^3, 3b_1x^2y + 3b_2x^2z + 3b_3xy^2 + 6b_4xyz + 3b_5xz^2\rangle$
$xy(x+y)$	2	$\langle xy(x+y), b_0x^3 + 3b_2(x^2z + xyz + y^2z) + b_6y^3\rangle$
x^2y	2	$\langle x^2y, b_0x^3 + 3b_2x^2z + b_6y^3\rangle$
$x(x^2+yz)$	0	$\langle x^3, xyz\rangle$
$x(y^2+xz)$	0	$\langle x(y^2+xz), x^3\rangle$
$y^2z - x^3 - x^2z$	0	$\langle y^2z - x^3 - x^2z, 3xy^2 - x^2z + y^2z\rangle$
$y^2z - x^3$	0	$\langle y^2z - x^3, xy^2\rangle$
$x^3 + y^3 + z^3 - 3txyz$ with $t^3 \neq 1$	0	$\langle x^3 + y^3 + z^3, xyz\rangle$
xyz	2	$\langle xyz, b_1x^3 + b_6y^3 + b_9z^3\rangle$

Note that the only cubic f for which a single pencil arise are those for which the Hessian is defined and distinct from the cubic f; in this case, the pencil is generated by the cubic itself and its Hessian. We can compute the dimension of the orbits of pencil using the Remark 5.2. Furthermore, to study the singular locus of N, we consider the 210×45 Jacobian matrix obtained from the equations defining N, and we evaluate its rank. Since N has dimension 8, if the rank is lower than 36, then the point is singular.

Pencil $\langle f, g\rangle$	$\mathbf{dim}(O(\langle f, g\rangle))$	**Rank** J
$\langle x^3 + y^3 + z^3, xyz\rangle$	8	36
$\langle y^2z - x^3 - x^2z, 3xy^2 - x^2z + y^2z\rangle$	7	36
$\langle y^2z - x^3, xy^2\rangle$	6	36
$\langle x^3, xyz\rangle$	6	36
$\langle x(y^2 + xz), x^3\rangle$	5	36

Table 5: Pencils of cubics contained in N that are of the form $\langle f, H(f)\rangle$

Remark 8.7. *Whenever a pencil in N contains a cubic f whose Hessian $H(f)$ is defined and different from f itself, it must necessarily coincide with $\langle f, H(f)\rangle$, and in particular, it must belong to one of the orbits just listed. This follows from the equivariance of the Hessian map with respect to the action of $SL(3)$.*

What we aim to prove in the following is that the pencils through x^4, $xy(x+y)$, x^2y or xyz contained in N form a finite number of orbits. Before proceeding, we state the following preliminary lemma.

Lemma 8.8. *Let $\langle f, g\rangle$ be a pencil of cubics and suppose that $H(g)$ is define, distinct from g and $H(g) \in \langle f, g\rangle$. Then, for any $C \in SL(3)$ such that $H(C \cdot g) \in \langle f, C \cdot g\rangle$, we have*

$$C \cdot \langle f, g\rangle = \langle f, C \cdot g\rangle$$

Proof. It is enough to recall that $H(C \cdot g) = C \cdot H(g)$, and thus we have:

$$C \cdot \langle f, g\rangle = C \cdot \langle g, H(g)\rangle = \langle C \cdot g, H(C \cdot g)\rangle = \langle f, C \cdot g\rangle$$

□

A We began our analysis with x^3. Our goal is to show that the pencils of the form

$$\langle x^3, 3b_1x^2y + 3b_2x^2z + 3b_3xy^2 + 6b_4xyz + 3b_5xz^2\rangle$$

form only finitely many $SL(3)$-orbits that we can control. We observe that all the cubics in the linear combination are of the form *line* + *conic* = $\{x = 0\}$ + *conic*. Let us distinguish several cases:

– **Conic(smooth) + Tangent line**$\{x = 0\}$
This condition is satisfied when the intersection of $\{x = 0\}$ and the conic

$\{3b_1xy + 3b_2xz + 3b_3y^2 + 6b_4yz + 3b_5z^2\}$ consists of a single point, and the conic does not degenerate into one or two lines. The first requirement leads to the condition $b_4^2 = b_3b_5$, while the second one is equivalent to requiring that the Hessian is non-zero and distinct from the cubic itself. Toghether these conditions are precisely those we found for cubics in $H^{-1}(x^3)$ (see Table 2). Moreover, all cubics of the form $conic(smooth) + tangent\ \ line$ lie in the same $SL(3)$-orbit. By Lemma 8.8, it follows that all the pencil of the form

$$\langle x^3, smooth\ \ conic\ \ tangent\ \ to\ \ \{x = 0\}\rangle$$

belong to the same orbit.

– **Conic(smooth) + Secant line**
From the classification of the $SL(3)$-orbits, we know that all cubics of this type lie in the same orbit. We claim that the Hessian of such a cubic lies in the pencil generated by it and by x^3. Then, by Lemma 8.8, this will imply that all pencils of the form

$$\langle x^3, smooth\ \ conic\ \ secant\ \ to\ \ \{x = 0\}\rangle$$

belong to the same orbit.
Let us consider the cubic $x(x^2 + yz)$, which is of this type. Its Hessian is $x(-3x^2 + xyz) \in \langle x^3, x(x^2 + yz)\rangle$ and has the same form. If now xq denotes another cubic of the same form, we know that there exist a matrix $C \in SL(3)$ such that $C \cdot x(x^2 + yz) = xq$. Moreover, C must send x to itself and the following diagram commutes:

$$\begin{array}{ccc} f = x(x^2+yz) & \xrightarrow{\quad \cdot C \quad} & xq \\ \Big\downarrow H & & \Big\downarrow H \\ H(f) \in \langle x^3, x(x^2+yz)\rangle & \xrightarrow{\cdot C} & C \cdot H(f) = H(xq) \end{array}$$

We obtain

$$H(xq) = C \cdot H(f) \in C \cdot \langle x^3, f\rangle = \langle C \cdot x^3, C \cdot f\rangle = \langle x^3, xq\rangle$$

Which prove our claim.

– **Triangles(three not collinear lines)**
First, observe that the pencil $\langle x^3, xyz\rangle$ contains the cubic $x^3 + xyz = x(x^2 + yz)$, and therefore it coincides with the pencil $\langle x^3, x(x^2 + yz)\rangle$, which is precisely the case previously analyzed.
If $x(x + \beta_1 y + \beta_2 z)(x + \gamma_1 y + \gamma_2 z)$ is another triangle of the same form, then $\beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$ and there exist $C \in SL(3)$ such that $C \cdot xyz = x(x + \beta_1 y + \beta_2 z)(x + \gamma_1 y + \gamma_2 z)$ and $C \cdot x = x$. Therefore, all pencils of the form

$$\langle x^3, triangle(three\ \ distinct\ \ lines)\rangle$$

lie in the same orbit. However, this orbit coincides with that of the pencils

$$\langle x^3, smooth\ \ conic\ \ secant\ \ to\ \ \{x = 0\}\rangle$$

– **Cone(three distinct line)**
All the cones consisting of three distinct lines lie in the same orbit. Moreover, if two such conics share the line $\{x = 0\}$, we can map one to the other using an element $C \in SL(3)$ that fixes x. Therefore, applying Lemma 8.8, we conclude that all pencils

$$\langle x^3, cone(three \ \ distinct \ \ line)\rangle$$

belong to the same orbit.

– **line** $\{x = 0\}$**+ double line**
First, observe that the pencil $\langle x^3, xy^2\rangle$ contains the cubic $x^3 + xy^2 = x(x^2+y^2)$ that is a cone with three distinct lines. Thus, this pencil coincides with $\langle x^3, x(x^2 + y^2)\rangle$ and it is in the previous orbit.
All cubics of the type xr^2 lie in the same orbit. In particular, since we must have $C \cdot xr^2 = xs^2$ with $r, s \neq x$, it follows that C must fix the line $\{x = 0\}$. Therefore, by applying again Lemma 8.8, we conclude that all pencils of the form

$$\langle x^3, \{x = 0\} + double \ \ line\rangle$$

lie in the same orbit. Moreover, based on the initial observation, this orbit coincides with

$$\langle x^3, cone(three \ \ distinct \ \ lines)\rangle$$

– **double line** $\{x = 0\}$**+ line**
All this kind of conics lie in the same $SL(3)$-orbit. Moreover, if $C \cdot x^2 r = x^2 s$ then C must fix x, and thus we have

$$\langle x^3, \{x = 0\}^2 + line\rangle$$

lie in the same orbit.

In summary, we have shown that all pencils containing x^3 and lying in N fall into four distinct orbits. We list in the following table a representative for each orbit, together with the dimension of the orbit and the rank of the Jacobian matrix 210×45 of N.

Description	**Representative** $\langle f, g\rangle$	$\mathbf{dim}(O(\langle f, g\rangle))$	**Rank** J
$\langle x^3, Trangle\rangle =$ $\langle x^3, Conic + secant \ \ line\rangle$	$\langle x^3, xyz\rangle$	6	36
$\langle x^3, Conic + tangent \ \ line\rangle$	$\langle x^3, x(y^2 + xz)\rangle$	5	36
$\langle x^3, x \cdot double \ \ line\rangle =$ $\langle x^3, Cone\rangle$	$\langle x^3, xy^2\rangle$	4	36
$\langle x^3, x^2 \cdot line\rangle$	$\langle x^3, x^2 y\rangle$	3	35

Table 6: Pencils of cubics contained in N that are of the form $\langle x^3, f\rangle$. Note that $\langle x^3, xyz\rangle$ and $\langle x^3, x(y^2 + xz)\rangle$ are two orbits of the form $\langle f, H(f)\rangle$, and they were already listed in Table 10.

B Let us now consider the case of x^2y. The pencils we need to analyze are of the form

$$\langle x^2y, b_0x^3 + 3b_2x^2z + b_6y^3\rangle$$

However, the space of cubics defined by such the cubics $b_0x^3 + +3b_2x^2z + b_6y^3$ with $b_2, b_6 \neq 0$ coincides with $H^{-1}(x^2y)$. All these pencils are of the form $\langle f, H(f)\rangle$, and therefore, by the Remark 8.7, they lie in the same orbit as $\langle y^2z - x^3, xy^2\rangle$, which was already listed in Table 10. For the remaining cases, with $b_2 = 0$ or $b_6 = 0$, it is straightforward to verify that there exists an element of $SL(3)$ preserving x^2y which maps $x^3 + 3b_2x^2z$ to x^2z and $x^3 + b_6y^3$ to $x^3 + y^3$. We find five orbits:

Representative $\langle f, g\rangle$	$\mathbf{dim}(O(\langle f, g\rangle))$	**Rank** J
$\langle x^2y, y^3 + x^2z\rangle,\ \langle x^2y, y^3 + x^2z\rangle$	6	36
$\langle x^2y, x^3 + y^3\rangle$	5	36
$\langle x^2y, x^2z\rangle,\ \langle x^2y, x^3 + x^2z\rangle$	4	35
$\langle x^2y, y^3\rangle$	4	36
$\langle x^2y, x^3\rangle$	3	35

Table 7: Pencils of cubics contained in N that are of the form $\langle x^2y, f\rangle$. It can be observed that the last two orbits in the table coincide with two that had already been listed. Thus, the only new orbits with respect to the previously classified ones are those of $\langle x^2y, x^3 + y^3\rangle$ and $\langle x^2y, x^2z\rangle$.

C Let us now consider the case of $xy(x + y)$. The pencils we need to analyze are of the form:

$$\langle xy(x + y), b_0x^3 + 3b_2(x^2z + xyz + y^2z) + b_6y^3\rangle$$

We denote by g the linear combination $b_0x^3 + 3b_2(x^2z + xyz + y^2z) + b_6y^3$. A straightforward computation of the Hessian shows that $H(g) \in \langle xy(x + y), g\rangle$ for every g whose Hessian is defined. Therefore, for such g, the orbit coincides with one of the form $\langle f, H(f)\rangle$ already listed in Table 10, namely the one corresponding to $\langle y^2z - x^3 - x^2z, 3xy^2 - x^2z + y^2z\rangle$. The remaining cases are pencil of the form $\langle xy(x+y), x^3 + b_6y^3\rangle$. For these we observe that the pencil contains a *triple line* only when $b_6 = 0$ or $b_6 = 1$; in all other cases, it instead contains a *double line + line*. We can thus reduce the first case to the orbit of $\langle x^3, xy^2\rangle$ and the second one to the orbit of $\langle x^2y, x^3 + y^3\rangle$, both of which have already been analyzed. We find three orbits of pencils.

Representative $\langle f, g\rangle$	$\mathbf{dim}(O(\langle f, g\rangle))$	**Rank** J
$\langle xy(x+y), z(x^2+xy+y^2)\rangle$, $\langle xy(x+y), x^3+z(x^2+xy+y^2)\rangle$, $\langle xy(x+y), x^3+y^3+z(x^2+xy+y^2)\rangle$	7	36
$\langle xy(x+y), x^3+y^3\rangle$, $\langle xy(x+y), x^3\rangle$	4	36
$\langle xy(x+y), x^3+b_6y^3\rangle$	5	36

Table 8: Pencils of cubics contained in N that are of the form $\langle xy(x+y), f\rangle$. These three orbits coincide with three that were already found previously.

D The last case we need to study is that of xyz. We consider pencils of the form

$$\langle xyz, b_0x^3 + b_6y^3 + b_9z^3\rangle$$

and observe that we can act with $SL(3)$ while fixing xyz. Therefore, there are three orbits.

Representative $\langle f, g\rangle$	$\mathbf{dim}(O(\langle f, g\rangle))$	**Rank** J
$\langle xyz, x^3+y^3+z^3\rangle$	8	36
$\langle xyz, x^3+y^3\rangle$	6	36
$\langle xyz, x^3\rangle$	6	36

Table 9: Pencils of cubics contained in N that are of the form $\langle xyz, f\rangle$. These three orbits are all of the form $\langle f, H(f)\rangle$, and therefore coincide with three of those in Table 10. In particular, the second one coincides with the orbit of $\langle y^2z - x^3, xy^2\rangle$.

To summarize what we have done so far: the analysis carried out up to this point has shown that N contains, in addition to the orbit of the pencil $\langle x^3+y^3+z^3, xyz\rangle$, which is, in particular, the only one of dimension 8, eight more orbits. Among these, four are of the form $\langle f, H(f)\rangle$, while the remaining four have dimensions $5, 4, 4$, and 3. Our goal is to show that all these orbits are actually contained in the closure of $O(\langle x^3+y^3+z^3, xyz\rangle)$. To do so, we will explicitly construct degenerations which, as $\epsilon \to 0$, tend to pencils lying in the smaller orbits.

The degenerations for the orbits of the form $\langle f, H(f)\rangle$ are obtained simply by taking families of smooth cubics tending to f, together with their corresponding Hessians. For the orbits of $\langle x^2y, x^3+y^3\rangle$ and $\langle x^3, xy^2\rangle$, we explicitly constructed two degenerations of the Hesse pencil. The degeneration for $\langle x^2y, x^2z\rangle$ is obtained from $\langle xy(x+y), z(x^2+xy+y^2)\rangle$ whose orbit is the same as the one generated by $\langle y^2z-x^3-x^2z, 3xy^2-x^2z+y^2z\rangle$ as follows from the analysis in point C. In particular, we take $\langle xy(x+\epsilon y), z(x^2+\epsilon xy+\epsilon^2y^2)\rangle \in O(\langle xy(x+y), z(x^2+xy+y^2)\rangle) = O(\langle y^2z - x^3 - x^2y, 3xy^2 - x^2z + y^2z\rangle)$. This shows that $\langle x^2y, x^2z\rangle$ belongs to the closure of the Hesse pencil, since we have already observed that the orbit $O(\langle y^2z - x^3 - x^2z, 3xy^2 - x^2z + y^2z\rangle$ belongs to it, and therefore any pencil in its closure does as well. Finally, the degeneration for $\langle x^3, x^2y\rangle$ from $\langle x^3, xy(z+\epsilon z)\rangle \in O(\langle x^3, xyz\rangle)$, which is suffices by an analogue reasoning.

Representative $\langle f,g\rangle$	**Degeneration**
$\langle y^2z - x^3 - x^2z, 3xy^2 - x^2z + y^2z\rangle$	$\langle y^2z - x^3 - x^2z + \epsilon z^3,\ 3xy^2 - x^2z + y^2z + \epsilon(-9xz^2 - 3z^3)\rangle$
$\langle y^2z - x^3, xy^2\rangle$	$\langle y^2z - x^3 - \epsilon z^3,\ xy^2 + 3\epsilon xz^2\rangle$
$\langle x^3, xyz\rangle$	$\langle x(x^2+yz) + \epsilon(y^3+z^3),\ -6x^3 + 2xyz + \epsilon(216xyz\epsilon - 6y^3 - 6z^3)\rangle$
$\langle x(y^2+xz), x^3\rangle$	$\langle x(y^2+xz) + \epsilon z^3,\ x^3 - \epsilon(-3y^2z + 3xz^2)\rangle$
$\langle x^2y, x^3+y^3\rangle$	$\langle x^3 + 2y^3 + (x+\epsilon z)^3, xy(x+\epsilon z)\rangle$
$\langle x^3, xy^2\rangle$	$\langle x^3 + y^3 + (\epsilon z - y)^3, xy(\epsilon z - y)\rangle$
$\langle x^2y, x^2z\rangle$	$\langle xy(x+\epsilon y), z(x^2 + \epsilon xy + \epsilon^2 y)\rangle$
$\langle x^3, x^2y\rangle$	$\langle x^3, xy(x+\epsilon z)\rangle$

Table 10: Classification of orbits in N and their degeneration families arising from $O(\langle x^3+y^3+z^3, xyz\rangle)$

Thus, the table shows that all the orbits in N lie in the closure of $O(\langle x^3 + y^3 + z^3, xyz\rangle)$, and we can conclude that $S = N$. Moreover, from the rank analysis of the Jacobian matrix, it follows that the singular locus of this variety consists of the two orbits $O(\langle x^2y, x^2z\rangle)$ and $O(\langle x^3, x^2y\rangle)$, of dimension 4 and 3, respectively. The proof of the Theorem 8.6 is now complete.

Remark 8.9. *The* 3*-dimensional orbit* $O(\langle x^3, x^2y\rangle)$ *is the only closed orbit of the action of* $SL(3)$ *on* $\mathbb{P}(\mathbb{S}_{(5,1)}\mathbb{C}^3)$*. This follows from the claim* 23.52 *in [8] since* $\mathbb{S}_{(5,1)}\mathbb{C}^3$ *is an irreducible* $SL(3)$*-module, indeed, if we consider the pencil* $\langle x^3, x^2y\rangle$ *and act with the Borel subgroup* $B \in SL(3)$*, that is, the group of upper triangular matrices, it is easy to see that* $\langle x^3, x^2y\rangle \cdot b = \langle x^3, x^2y\rangle$ *for all* $b \in B$*. Therefore, the orbit of this element must be closed and it is the unique one.*

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