Kronecker decomposition of pencils of quadrics and nonabelian apolarity

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Strict equivalence

Set $\mathrm{GL}_{k_1,\ldots,k_r} = \mathrm{GL}_{k_1}(\mathbb{C}) \times \ldots \times \mathrm{GL}_{k_r}(\mathbb{C})$.

Matrix pencil of size $m \times n$: $\mathcal{P} = \mu A + \lambda B$ where $A, B \in \mathfrak{M}_{m\times n}(\mathbb{C})$.

$$\mathfrak{M}_{m\times n}(\mathbb{C}[\mu,\lambda]_1) = \text{space of matrix pencils of size } m \times n$$

Two matrix pencils $\mathcal{P}$ and $\mathcal{P}'$ are **strictly equivalent** if they are in the same orbit with respect to the group action

$$\begin{array}{ccc} \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) & \longrightarrow & \mathrm{Aut}\left(\mathfrak{M}_{m\times n}(\mathbb{C}[\mu,\lambda]_1)\right) \\ (P,Q) & \mapsto & \left(\mu A + \lambda B \mapsto \mu(P \cdot A \cdot {}^tQ) + \lambda(P \cdot B \cdot {}^tQ) \right) \end{array}$$

Regular pencil: $m = n$ and $\det(\mu A + \lambda B) \neq 0$.
Singular pencil: $m \neq n$ or $\det(\mu A + \lambda B) = 0$.

Regular invariants

Set $g_k(\mu,\lambda) = \gcd(k \times k \text{ minors of } \mathcal{P})$ and $r = \max\{k \mid g_k(\mu,\lambda) \neq 0\}$.

Invariant polynomials: for $i = 1 : r$

$$d_i(\mu,\lambda) := \frac{g_i}{g_{i-1}} = \mu^{u_i} \prod_j e_{ij}(\mu,\lambda)^{w_{ij}} \overset{\overline{\mathbb{C}}=\mathbb{C}}{=} \mu^{u_i} \prod_j (a_{ij}\mu + \lambda)^{w_{ij}}$$

where $e_{ij}(1,\lambda)$ are irreducible. Note that $d_1|\dots|d_r$.

Elementary divisors: the factors μ^{u_i} and $e_{ij}(\mu,\lambda)^{w_{ij}}$.

They define pencils of size u_i and w_{ij} respectively of the form

$$H_{u_i} = \begin{bmatrix} \mu & \lambda & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda \\ & & & \mu \end{bmatrix}, \quad \mathfrak{J}_{w_{ij},a_{ij}} = \begin{bmatrix} \lambda + a_{ij}\mu & \mu & & \\ & \ddots & \ddots & \\ & & \ddots & \mu \\ & & & \lambda + a_{ij}\mu \end{bmatrix}$$

Singular invariants

Minimal indices for columns: the minima degrees $0 \leq \epsilon_1 \leq \ldots \leq \epsilon_p$ of the (linearly independent) solutions of the equation $(\mu A + \lambda B)x(\mu, \lambda) = 0$.

Minimal indices for rows: the minima degrees $0 \leq \eta_1 \leq \ldots \leq \eta_q$ of the (linearly independent) solutions of the equation $(\mu \cdot {}^t A + \lambda \cdot {}^t B)x(\mu, \lambda) = 0$.

Let g and h be such that $\epsilon_1 = \ldots = \epsilon_g = \eta_1 = \ldots = \eta_h = 0$.
For $i \geq g$, each ϵ_i defines the pencil of size $\epsilon_i \times (\epsilon_i + 1)$

$$R_{\epsilon_i} = \begin{bmatrix} \lambda & \mu & & \\ & \ddots & \ddots & \\ & & \lambda & \mu \end{bmatrix}$$

For $j \geq h$, each η_j defines the pencil ${}^t R_{\eta_j}$ of size $(\eta_j + 1) \times \eta_j$.

Kronecker-Weierstrass form

Theorem (Weierstrass, 1868 - Kronecker, 1890)

Every projective pencil $\mu A + \lambda B$ is strictly equivalent to a canonical block-direct-sum of the form

$$0_{h\times g} \boxplus \Big(\mathop{\boxplus}_{i=g+1}^{p} R_{\epsilon_i} \Big) \boxplus \Big(\mathop{\boxplus}_{j=h+1}^{q} {}^{t}R_{\eta_j} \Big) \boxplus \Big(\mathop{\boxplus}_{k=1}^{s} H_{u_k} \Big) \boxplus \Big(\mathop{\boxplus}_{l,z} \mathfrak{J}_{w_{lz},a_{lz}} \Big)$$

where ϵ_i and η_j are the minimal indices for columns and rows respectively, and μ^{u_s} and $(\lambda + a_{ij}\mu)^{w_{ij}}$ are the elementary divisors.

The Kronecker-Weierstrass form classifies the representatives in

$$\mathrm{GL}_m(\mathbb{C}) \backslash \mathfrak{M}_{m\times n}(\mathbb{C}[\mu,\lambda]_1) / \mathrm{GL}_n(\mathbb{C})$$

Symmetric pencils

We denote symmetric matrix pencils by $\operatorname{Sym}^2 \mathbb{C}^m[\mu, \lambda]_1$.
Two symmetric pencils are **congruent** if they are in the same orbit with respect to the group action

$$\begin{array}{ccc} \mathrm{GL}_m(\mathbb{C}) & \longrightarrow & \operatorname{Aut}\left(\operatorname{Sym}^2 \mathbb{C}^m[\mu, \lambda]_1\right) \\ T & \mapsto & \left(\mu A + \lambda B \mapsto \mu({}^tTAT) + \lambda({}^tTBT)\right) \end{array}$$

Proposition

Two symmetric pencils are strictly equivalent if and only if they are congruent.

Corollary

Two pencils of quadratic forms can be carried into one another by a non-singular transformation if and only if the corresponding symmetric pencils have same minimal indices and elementary divisors.

Segre symbol

The intersection of two quadrics $\mathcal{A} = {}^t XAX$ and $\mathcal{B} = {}^t XBX$ in $\mathbb{P}^m_{\mathbb{C}}$ is described by the symmetric pencil $\mathcal{P} = \mu A + \lambda B$. The **roots** of $\mathcal{P}$ are the roots $[y_i : -x_i] \in \mathbb{P}^1$ of the elementary divisors $(x_i\mu + y_i\lambda)^{e^i_j}$.
The **Segre symbol** of $\mathcal{P}$ is the ordered sequence of its invariants

$$\Sigma(\mathcal{P}) = \left[(e^1_1, \ldots, e^1_{r_1}) \ldots (e^k_1, \ldots, e^k_{r_k}); \epsilon_{g+1}, \ldots, \epsilon_p; g\right]$$

where k is the number of distinct roots and

$$r_1 \geq \ldots \geq r_k \quad , \quad e^i_1 \geq \ldots \geq e^i_{r_i} \quad , \quad \epsilon_{g+1} \leq \ldots \leq \epsilon_p$$

Example: $\Sigma\left(\begin{bmatrix} \lambda & & \\ & \mu & \\ & & 0 \end{bmatrix}\right) = [1\ 1;\ ;1]$, while $\Sigma\left(\begin{bmatrix} \mu & \lambda & \\ \lambda & & \\ & & 0 \end{bmatrix}\right) = [2;\ ;1]$.

The Segre symbol does not uniquely define the pencil even up to $\mathrm{GL}_{2,m}$-action (i.e. up to strict equivalence and to GL_2-action on $\mathbb{P}^1$).

Up to $GL_2 \curvearrowright \mathbb{P}^1$, we may assume the roots to be $[1 : -\frac{x_i}{y_i}]$, hence represent them by $z_i \in \mathbb{C}$ or better by a vector in $\mathbb{C}^{(k)}/_\sim$ where

$$\mathbb{C}^{(k)} = \left\{ z \in \mathbb{C}^k \mid z_i \neq z_j \ \forall i \neq j \right\}$$

$$z \sim w \iff \exists \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2 \ : \ \forall i = 1 : k, \ w_i = \frac{az_i + b}{cz_i + d}$$

The quotient $\mathbb{C}^{(k)}/_\sim$ parametrizes all the possible k-tuples of roots (up to $GL_2 \curvearrowright \mathbb{P}^1$): a class $[v]$ is called a **continuous modulus**.

Proposition

A pencil of quadrics $\mathcal{P}$ is uniquely determined (up to GL-action) by its Segre symbol and a continuous modulus $[v] \in \mathbb{C}^{(k)}/_\sim$.

Theorem

Let $\mathcal{P}$ and $\mathcal{Q}$ be two pencils of quadrics in $\mathbb{P}^m$ with roots $[\mu_i^{\mathcal{P}} : \lambda_i^{\mathcal{P}}]$ and $[\mu_i^{\mathcal{Q}} : \lambda_i^{\mathcal{Q}}]$ for $i = 1 : k$. Then $\mathcal{P}$ and $\mathcal{Q}$ are projectively equivalent in $\mathbb{P}^m$ if and only if they have the same Segre symbols.

Projective space of quadrics

Set $W = \{Q : \mathbb{C}^{m+1} \to \mathbb{C} \text{ quadric}\} \supset W_r = \{Q \in W \mid \text{Rk}(Q) = r\}$.
For $\mathcal{P} = \mu Q_1 + \lambda Q_2$ defined by linearly independent quadrics $Q_1, Q_2 \in W \setminus \{0\}$, set $L_{\mathcal{P}}$ its *projective line* in $\mathbb{P}W$ and $V(\mathcal{P}) \subset \mathbb{P}^m$.

Claim

The Kronecker class of a pencil of quadrics $\mathcal{P}$ is uniquely determined by the *position* of the line $L_{\mathcal{P}}$ with respect to the subvarieties $\overline{\mathbb{P}W_r}$ and by the *singular part* $\text{Sing}(V(\mathcal{P}))$ of the base locus $V(\mathcal{P})$.

Be careful!

Not only the schematically-singular parts, but also the ones of dimension greater than the expected one: e.g., in $\mathbb{P}^2$ of $\text{Sing}(V([2;;1]))$ is not only the double point (x^2, y) but also the line (x).

Position of $L_{\mathcal{P}}$

For $L_{\mathcal{P}} \subset \mathbb{P}W$ projective line of $\mathcal{P}$, set $m_0(L_{\mathcal{P}}) = \min\{r \mid L_{\mathcal{P}} \subset \overline{\mathbb{P}W_r}\}$.
Given $\{P_1, \dots, P_{q_L}\} = L_{\mathcal{P}} \cap \overline{\mathbb{P}W_{m_0(L)-1}}$, set $\forall i \leq q_L, \forall j \leq k_i(L_{\mathcal{P}})$
$k_i(L_{\mathcal{P}}) = \max\{k \mid P_i \in \overline{\mathbb{P}W_{m_0(L)-k}}\}$, $m_{ij}(L_{\mathcal{P}}) = \mathrm{mult}_{P_i}(L_{\mathcal{P}} \cap \overline{\mathbb{P}W_{m_0(L_{\mathcal{P}})-j}})$

The set of values m_0, q_L, k_i, m_{ij} determines the **position** of $L_{\mathcal{P}}$.

Proposition

If $\Sigma(\mathcal{P}) = \left[(e^1_1, \dots, e^1_{r_1}) \dots (e^k_1, \dots, e^k_{r_k}); \epsilon_{g+1}, \dots, \epsilon_p; g\right]$, then $L_{\mathcal{P}}$ has position:

(i) $m_0(L_{\mathcal{P}}) = m + 1 - p$;

(ii) $q(L_{\mathcal{P}}) = k$;

(iii) $k_i(L_{\mathcal{P}}) = r_i$ for all $i = 1 : k$;

(iv) $m_{ij}(L_{\mathcal{P}}) = \sum_{l=1}^{r_i-j+1} e^i_{r_i-l+1}$ for all $i = 1 : k$ and $j = 1 : r_i$.

Lemma

Given $\mathcal{P}, \mathcal{P}'$ two pencil of quadrics, their lines L, L' have similar position if and only if the pencils have Segre symbols with the same multiplicities (i.e $k = k'$ and $e^i_j = (e')^i_j$) and same number of minimal indices (i.e. same $p = p'$), other than same continuous moduli.

If $\mathcal{P}$ is regular, then it is uniquely determined by the position of $L_{\mathcal{P}}$. But if the pencil is singular, its position is enough <u>iff</u> $m = 2, 3$: this comes from combinatorial costraints on the sizes of Kronecker blocks.

$\Sigma(\mathcal{P})$	$L_{\mathcal{P}}$	$\det(\mathcal{P})$	$q(L_{\mathcal{P}})$	$L_{\mathcal{P}} \cap \overline{\mathbb{P}W_2}$	$L_{\mathcal{P}} \cap \overline{\mathbb{P}W_1}$
$[1\ 1\ 1]$	$\lambda x^2 + (\mu - \lambda)y^2 - \mu z^2$	$\lambda(\lambda+\mu)\mu$	3	$1+1+1$	$\emptyset$
$[2\ 1]$	$\mu x^2 - \mu z^2 + 2\lambda xy$	$\lambda^2\mu$	2	$2+1$	$\emptyset$
$[(1\ 1)\ 1]$	$\lambda x^2 - \lambda y^2 + \mu z^2$	$\lambda^2\mu$	2	$2+1$	1
$[3]$	$\lambda y^2 + 2\lambda xz + 2\mu xy$	λ^3	1	3	$\emptyset$
$[(2\ 1)]$	$\mu x^2 + 2\lambda xy + \lambda z^2$	λ^3	1	3	1
$[;1;]$	$\mu xz + \lambda xy$	0	0	$L_{\mathcal{P}}$	$\emptyset$
$[1\ 1;;1]$	$\mu y^2 + \lambda x^2$	0	2	$L_{\mathcal{P}}$	$1+1$
$[2;;1]$	$\mu x^2 + \lambda xy$	0	1	$L_{\mathcal{P}}$	2

Singular components in $V(\mathcal{P})$

Lemma

Set $\overline{k} = k - \#\{i \mid r_i = e^i_{r_i} = 1\}$. Then $\mathrm{Sing}(V(\mathcal{P}))$ has at least t components $\mathcal{S}_1, \ldots, \mathcal{S}_t$ (with reduced structure) where

$$t = \begin{cases} \overline{k} & \text{if } p = g = 0 \text{ (no minimal indices)} \\ \max\{\overline{k}, 1\} & \text{if } p = g > 0 \text{ (only zero minimal indices)} \\ \overline{k} + 1 & \text{if } p > g \text{ (there are non-zero minimal indices)} \end{cases}$$

Moreover, up to permutation of the $\mathcal{S}_i$'s, it holds:

(i) each $\mathcal{S}_i$ is either a linear subspace of dimension $d_i = r_i + p - 1$ (for $e^i_{r_i} > 1$) or a quadrics of dimension $d_i - 1$ and corank $d_i + 1 - \#\{j \mid e^i_j = 1\}$ (for $e^i_{r_i} = 1$).

(ii) If $p > g$ (i.e. there are non-zero minimal indices), then in addition $\mathcal{S}_t = \mathcal{S}_{\overline{k}+1}$ is either a projective bundle of type $P(\epsilon_{g+1} \ldots \epsilon_p)$ (for $g = 0$) or a join variety of type $J(\epsilon_{g+1} \ldots \epsilon_p; g - 1)$ (for $g > 0$).

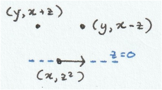
Theorem (Dimca, 1983)

Two pencils of quadrics $\mathcal{P}$ and $\mathcal{P}'$ are equivalent if and only if

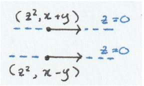
(i) the lines $L_{\mathcal{P}}, L_{\mathcal{P}'} \subset \mathbb{P}W$ have similar positions;

(ii) the irreducible components of $\mathrm{Sing}(V(\mathcal{P}))$ and $\mathrm{Sing}(V(\mathcal{P}'))$ are isomorphic.

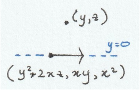
$\Sigma(\mathcal{P})$	$\overline{k}$	t	d_i	$\mathrm{Sing}(V(\mathcal{P}))$
[1 1 1]	0	0		$\emptyset$
[2 1]	1	1 (irred.)	0	one double point
[(1 1) 1]	1	1 (reducible)	0	two double points
[3]	1	1 (irred.)	0	one triple point
[(2 1)]	1	1 (irred.)	0	one (curv.) quadruple point
[; 1;]	0	1 (reducible)	1	a line and a disjoint point
[1 1; ; 1]	0	1 (irred.)	0	one (non-curv.) quadruple point
[2; ; 1]	1	1 (reducible)	1	a line with embedded double point



(a) [2 1]



(b) [(1 1) 1]



(c) [3]

2-slice tensors and $\mathrm{GL}_{2,m,n}$-action

2-slice tensor (of size $m \times n$): $T \in \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$.
Decomposable ones:

$$\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}) = \{[u \otimes v \otimes w] \mid u, v, w\} \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n)$$

$$\begin{array}{ccccc} \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n & \xrightarrow{\simeq} & \mathrm{Bil}(\mathbb{C}^m, \mathbb{C}^n; \mathbb{C}^2) & \xrightarrow{\simeq} & \mathfrak{M}_{m\times n}(\mathbb{C}) \times \mathfrak{M}_{m\times n}(\mathbb{C}) \\ T & \mapsto & (\phi_T : (v, w) \mapsto (a, b)) & \mapsto & (A, B) \end{array}$$

where A and B are such that ${}^t v \cdot A \cdot w = a$, ${}^t v \cdot B \cdot w = b$.

$$\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n \longleftrightarrow \mathfrak{M}_{m\times n}(\mathbb{C}[\mu, \lambda]_1)$$

Two 2-slice tensors are $\mathrm{GL}_{2,m,n}$**-equivalent** if they are in the same orbit with respect to the group action

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) & \longrightarrow & \mathrm{Aut}\left(\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n\right) \\ (M, P, Q) & \mapsto & \left(u \otimes v \otimes w \mapsto Mu \otimes Pv \otimes Qw \right) \end{array}$$

$\mathrm{GL}_{2,m,n}$-orbits

In general, there are infinitely many GL-orbits in $\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$

Proposition

The tensor space $\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ has finitely many GL-orbits if and only if $m \leq 3$ or $n \leq 3$.

$$\begin{array}{rccc} \gamma_T : & \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) & \longrightarrow & \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n \\ & G & \mapsto & G \cdot T \end{array}$$

$$d(\gamma_T)_I : \quad \mathfrak{gl}_2(\mathbb{C}) \times \mathfrak{gl}_m(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$$

We get:

$$\mathrm{Im}(\gamma_T) = \mathrm{orb}_{\mathrm{GL}}(T) \ , \ \ker(d(\gamma_T)_I) = \mathfrak{Lie}_I(\mathrm{stab}_{\mathrm{GL}}(T))$$

$$\dim(\mathrm{orb}_{\mathrm{GL}}(T)) = \mathrm{Rk}(d(\gamma_T)_I) = 4 + m^2 + n^2 - \dim\Big(\ker\left(d(\gamma_T)_I\right)\Big)$$

symRk_p in $\mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^{m+1})$

Symmetric 2-slice tensors: tensors in $\mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^{m+1})$.
Decomposable ones:

$$\mathrm{Seg}(\mathbb{P}^1 \times \nu_2(\mathbb{P}^m)) = \left\{[u \otimes l^2] \mid u \in \mathbb{C}^2,\ l \in \mathbb{C}^{m+1}\right\} \subset \mathbb{P}\left(\mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^{m+1})\right)$$

$$\mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^{m+1}) \longleftrightarrow \text{pencils of quadrics in } \mathbb{P}^m_{\mathbb{C}}$$

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_{m+1}(\mathbb{C}) & \longrightarrow & \mathrm{Aut}\left(\mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^{m+1})\right) \\ (M, P) & \mapsto & \left(u \otimes l^2 \mapsto Mu \otimes P \cdot l^2 \cdot {}^t P\right) \end{array}$$

The $\mathrm{GL}_{2,m+1}$-orbits are finitely many if and only if $m+1 \leq 3$.

Apolarity Theory

Waring decomposition problem

Express $f \in \mathrm{Sym}^d V$ as sum of powers of linear form $\sum_{i=0}^{r} l_i^d$.

Apolar ideal: $f^\perp = \{g \in \mathrm{Sym}^\bullet V^\vee \mid g \cdot f = 0\} \subset \mathbb{C}[\partial_0, \dots, \partial_m]$.

Lemma (Apolarity)

$\mathcal{Z}$ finite set of linear forms, $\mathcal{I}_\mathcal{Z} = \{g \in \mathrm{Sym}^\bullet V^\vee \mid g(l) = 0 \ \forall l \in \mathcal{Z}\}$.
Then

$$f = \sum_{l \in \mathcal{Z}} l^d \iff \mathcal{I}_\mathcal{Z} \subseteq f^\perp$$

Moral: We look for a decomposition of f in the base locus of 0-dimensional ideals in $f^\perp$.

$$f^\perp = \{g \in \operatorname{Sym}^\bullet V^\vee \mid g \cdot f = 0\} = \sum_k \left\{ \ker(\overbrace{C_{k,f} : \operatorname{Sym}^k V^\vee \to \operatorname{Sym}^{d-k} V}^{k\text{-th } \textbf{catalecticant} \text{ map}}) \right\}$$

Catalecticant algorithm

(1) Construct $C_{\lceil \frac{d}{2} \rceil, f} : \operatorname{Sym}^{\lceil \frac{d}{2} \rceil} V^\vee \to \operatorname{Sym}^{d - \lceil \frac{d}{2} \rceil} V$;

(2) Compute $\ker C_{\lceil \frac{d}{2} \rceil, f}$;

(3) Compute the Krull dimension $\dim_{\text{Krull}}(\ker C_{\lceil \frac{d}{2} \rceil, f})$:

  (a) if it is ≥ 1, the method fails!

  (b) else compute $\mathcal{Z} = \mathcal{Z}(\ker C_{\lceil \frac{d}{2} \rceil, f}) = \{[l_1], \ldots, [l_r]\}$;

(4) Solve the linear system $f = \sum_{i=1}^r c_i l_i^d$ where c_i are the indeterminates.

Since $\operatorname{Sym}^d(\mathbb{C}^{m+1})^\vee \simeq H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d))$,

$$C_{k,f} : H^0(\mathbb{P}^m, \mathcal{O}(k)) \to H^0(\mathbb{P}^m, \mathcal{O}(d-k)))^\vee$$

Nonabelian Apolarity

For $\mathcal{E}$ vector bundle over a variety X and $\mathcal{L} \in \mathrm{Pic}(X)$ such that $X \hookrightarrow \mathbb{P}(H^0(X,\mathcal{L})^\vee)$, the natural map

$$H^0(X,\mathcal{E}) \otimes H^0(X,\mathcal{E}^\vee \otimes \mathcal{L}) \to H^0(X,\mathcal{L})$$

leads to the linear map

$$H^0(X,\mathcal{E}) \otimes H^0(X,\mathcal{L})^\vee \to H^0(X,\mathcal{E}^\vee \otimes \mathcal{L})^\vee$$

by fixing $f \in H^0(X,\mathcal{L})^\vee$ we have

$$C_{\mathcal{E},f} : H^0(X,\mathcal{E}) \to H^0(X,\mathcal{E}^\vee \otimes \mathcal{L})^\vee$$

Let $f = \sum_{i=1}^r z_i$ minimal and $\mathcal{Z} = \{[z_1],\dots,[z_r]\} \subseteq \mathbb{P}(H^0(X,\mathcal{L})^\vee)$.

Lemma (Oeding-Ottaviani, 2013)

If $\mathrm{Rk}(C_{\mathcal{E},f}) = r \cdot \mathrm{Rk}(\mathcal{E})$, then $H^0(X,\mathcal{I}_\mathcal{Z} \otimes \mathcal{E}) = \ker(C_{\mathcal{E},f})$.

Eigenvectors of tensors

We can look for a decomposition of f in the base locus of $\ker(C_{\mathcal{E},f})$. But these are global sections. Anything better?

Get Q from the Euler SES $\quad 0 \to \mathcal{O}_{\mathbb{P}V}(-1) \to \mathcal{O}_{\mathbb{P}V} \otimes V \to Q \to 0.$
Set $\mathcal{E} = \bigwedge^a Q(e)$, $\mathcal{L} = \mathcal{O}(d)$ and $\rho : L_1 \to L_0$ presentation of $\mathcal{E}$:

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathrm{Sym}^e V, \bigwedge^a V) & & \mathrm{Hom}(\bigwedge^{m-a} V, \mathrm{Sym}^{d-e-1} V) \\
\simeq \big| & & \big| \simeq \\
H^0(\mathbb{P}^m, L_1) & \overset{P_{\mathcal{E},f}}{\dashrightarrow} & H^0(\mathbb{P}^m, L_0^\vee \otimes \mathcal{L})^\vee \\
\alpha \big\downarrow & \circlearrowleft & \big\uparrow \beta \\
H^0(\mathbb{P}^m, \bigwedge^a Q(e)) & \underset{C_{\mathcal{E},f}}{\longrightarrow} & H^0(\mathbb{P}^m, \bigwedge^a Q(e)^\vee \otimes \mathcal{L})^\vee
\end{array}
$$

Eigenvector of $M \in \mathrm{Hom}(\mathrm{Sym}^e V, \bigwedge^a V)$: $v \in V$ s.t. $M(v^e) \wedge v = 0$.

$\ker(C_{\mathcal{E},f})$ and $\ker(P_{\mathcal{E},f})$ have same common base locus, which corresponds to common eigenvectors for $\ker(P_{\mathcal{E},f})$.

Nonabelian Apolarity for pencils

Goal: Decompose a given $(B_1, B_2) \in \mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^{m+1})$.

Set

$$\mathcal{E} = \bigwedge^{a} Q(e) = Q(1) \simeq T\mathbb{P}^m \ , \ \mathcal{L} = \mathcal{O}(2) \ , \ \mathcal{E}^\vee \otimes \mathcal{L} = \Omega^1(2)$$

Then $(B_1, B_2) \in H^0(\mathbb{P}^m, \mathcal{O}(2))^\vee \oplus H^0(\mathbb{P}^m, \mathcal{O}(2))^\vee$ and $C_{\mathcal{E},f}$ is

$$C_{(B_1,B_2)} : H^0(\mathbb{P}^m, T\mathbb{P}^m) \to H^0(\mathbb{P}^m, \Omega^1(2))^\vee \oplus H^0(\mathbb{P}^m, \Omega^1(2))^\vee$$

- Up to isomorphism and up to scalars, $C_{(B_1,B_2)}$ is exactly

$$\begin{array}{rccc} C_{(B_1,B_2)} : & \mathfrak{sl}_{m+1}(\mathbb{C}) & \longrightarrow & \bigwedge^2 V \oplus \bigwedge^2 V \\ & A & \mapsto & \left(AB_1 - B_1({}^tA)\ ,\ AB_2 - B_2({}^tA)\right) \end{array}$$

- (B_1, B_2) is general, i.e. has Kronecker form of type $\mathrm{diag}(\lambda + a_i\mu)_i$ with $a_i \neq a_j \neq 0$;
- $\ker(C_{(B_1,B_2)})$ is invariant for GL_2-action.

Theorem

Let $(B_1, B_2) \in \mathbb{C}^2 \otimes \mathrm{Sym}^2 \mathbb{C}^{m+1}$ be a general symmetric pencil. Then:

(*i*) all matrices in $\ker(C_{(B_1,B_2)})$ have the same common eigenvectors $v_1, \ldots, v_{m+1}$ which are induced by the vectors $\tilde{v}_1, \ldots, \tilde{v}_{m+1}$ defining the Kronecker form

$$T_{(B_1,B_2)} \overset{\mathrm{GL}}{\sim} \sum_{i=1}^{m+1} \alpha_i \otimes \tilde{v}_i \otimes \tilde{v}_i$$

(*ii*) $\ker(C_{(B_1,B_2)})$ has dimension $m+1$ in $\mathfrak{gl}_{m+1}(\mathbb{C})$ and m in $\mathfrak{sl}_{m+1}(\mathbb{C})$;

(*iii*) for $C \in \ker(C_{(B_1,B_2)})$ general, in $\mathfrak{gl}_{m+1}(\mathbb{C})$ it holds $\ker(C_{(B_1,B_2)}) = \langle I, C, \ldots, C^m \rangle_{\mathbb{C}}$. In particular, in $\mathfrak{sl}_{m+1}(\mathbb{C})$ it holds $\ker(C_{(B_1,B_2)}) = \langle I, C, \ldots, C^m \rangle_{\mathbb{C}} \cap \mathfrak{sl}_{m+1}(\mathbb{C})$.

Key: The GL_{m+1}-action conjugates the kernels, that is
$\forall P \in \mathrm{GL}_{m+1}(\mathbb{C}),\ \ker(C_{(PB_1({}^tP), PB_2({}^tP))}) = P^{-1} \cdot \ker(C_{(B_1,B_2)}) \cdot P.$

Thanks for your attention!

For $\epsilon_1 = 0, \epsilon_2 = 1, \epsilon_3 = 2, \eta_1 = 0, \eta_2 = 0, \eta_3 = 2,\ \mu^3, (\lambda + \mu)^2$:

$$
\begin{bmatrix}
0 & & & & & & & & & & & & \\
0 & & & & & & & & & & & & \\
 & \lambda & \mu & & & & & & & & & & \\
 & & & \lambda & \mu & 0 & & & & & & & \\
 & & & 0 & \lambda & \mu & & & & & & & \\
 & & & & & & \lambda & 0 & & & & & \\
 & & & & & & \mu & \lambda & & & & & \\
 & & & & & & 0 & \mu & & & & & \\
 & & & & & & & & \mu & \lambda & 0 & & \\
 & & & & & & & & 0 & \mu & \lambda & & \\
 & & & & & & & & 0 & 0 & \mu & & \\
 & & & & & & & & & & & \lambda + \mu & \mu \\
 & & & & & & & & & & & 0 & \lambda + \mu
\end{bmatrix}
$$

For $\epsilon_1 = 0, \epsilon_2 = 0, \epsilon_3 = 2, \eta_1 = 0, \eta_2 = 1, \eta_3 = 2,\ \mu^2, (\lambda + \mu)^2, \lambda + \mu$:

$$
\begin{bmatrix}
0 & 0 & & & & & & & & & & & \\
 & & \lambda & \mu & 0 & & & & & & & & \\
 & & 0 & \lambda & \mu & & & & & & & & \\
 & & & & & \lambda & & & & & & & \\
 & & & & & \mu & & & & & & & \\
 & & & & & & \lambda & 0 & & & & & \\
 & & & & & & \mu & \lambda & & & & & \\
 & & & & & & 0 & \mu & & & & & \\
 & & & & & & & & \mu & \lambda & & & \\
 & & & & & & & & 0 & \mu & & & \\
 & & & & & & & & & & \lambda + \mu & & \\
 & & & & & & & & & & & \lambda + \mu & \mu \\
 & & & & & & & & & & & 0 & \lambda + \mu
\end{bmatrix}
$$

Rank in $\mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$

$\mathrm{Rk}(\mathfrak{J}_{w,a}) = w + (1 - \delta_{w1})$, $\mathrm{Rk}(R_\epsilon) = \epsilon + 1$

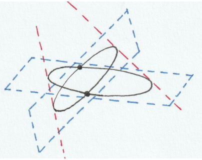
Theorem (Grigoriev-JàJà, 1979)

Let $T \in \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$ with minimal indices $\epsilon_1, \ldots, \epsilon_p, \eta_1, \ldots, \eta_q$ and regular part $\mathcal{K}$ of size N. Let $\delta(\mathcal{K})$ be the number of its non-squarefree invariant polynomials. Then

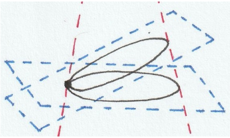
$$\mathrm{Rk}(T) = \sum_{i=1}^{p} (\epsilon_i + 1) + \sum_{j=1}^{q} (\eta_j + 1) + N + \delta(\mathcal{K})$$

The weight $\delta(\mathcal{P}_T)$ depends on the number of non-squarefree invariant polynomials and not on the number of non-squarefree elementary divisors.

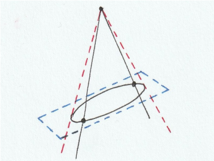
Some regular base loci in $\mathbb{P}^3_{\mathbb{C}}$



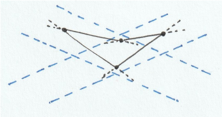
(a) [(1 1) 1 1]



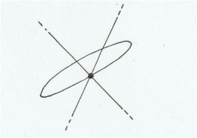
(b) [(2 1) 1]



(c) [(1 1) 2]



(d) [(1 1) (1 1)]



(e) [(3 1)]

Figura: Some base loci of pencils of quadrics in $\mathbb{P}^3$

Pencils of quadrics in $\mathbb{P}^2_{\mathbb{C}}$

Pencil	Segre sym.	$\mathcal{A}$	$\mathcal{B}$	$V(\mathcal{P})$
$\begin{bmatrix} \lambda & & \\ & \lambda+\mu & \\ & & \mu \end{bmatrix}$	[1 1 1]	$y^2 - z^2$	$x^2 - y^2$	four distinct points
$\begin{bmatrix} \mu & \lambda & \\ \lambda & & \\ & & \mu \end{bmatrix}$	[2 1]	$x^2 - z^2$	$2xy$	a double point and two other points
$\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{bmatrix}$	[(1 1) 1]	z^2	$x^2 - y^2$	two double points
$\begin{bmatrix} & \mu & \lambda \\ \mu & \lambda & \\ \lambda & & \end{bmatrix}$	[3]	$2xy$	$y^2 + 2xz$	a curvilinear triple point and another point
$\begin{bmatrix} \mu & \lambda & \\ \lambda & & \\ & & \lambda \end{bmatrix}$	[(2 1)]	x^2	$2xy + z^2$	a curvilinear quadruple point
$\begin{bmatrix} & \lambda & \mu \\ \lambda & & \\ \mu & & \end{bmatrix}$	[; 1;]	$2xz$	$2xy$	a line and a disjoint point
$\begin{bmatrix} \lambda & & \\ & \mu & \\ & & 0 \end{bmatrix}$	[1 1; ; 1]	y^2	x^2	a non-curvilinear quadruple point
$\begin{bmatrix} \mu & \lambda & \\ \lambda & & \\ & & 0 \end{bmatrix}$	[2; ; 1]	x^2	$2xy$	a line and an embedded double point

$\mathrm{GL}_{2,3,3}(\mathbb{C})$-orbits

$\mathcal{P}_T$	$\dim(\mathrm{orb}_{\mathrm{GL}}(T))$	Rk	$\underline{\mathrm{Rk}}$	T
$\begin{bmatrix} \lambda & & \\ & \mu & \\ & & \lambda+\mu \end{bmatrix}$	18	3	3	$a_2 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2 + (a_2 + a_1) \otimes b_3 \otimes c_3$
$\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{bmatrix}$	15	3	3	$a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_1 \otimes b_3 \otimes c_3$
$\begin{bmatrix} \lambda & \mu & \\ & \lambda & \\ & & \mu \end{bmatrix}$	17	4	3	$a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + (a_1 + a_2) \otimes b_3 \otimes c_3 + a_1 \otimes b_1 \otimes c_2$
$\begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}$	10	3	3	$a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_2 \otimes b_3 \otimes c_3$
$\begin{bmatrix} \lambda & \mu & \\ & \lambda & \\ & & \lambda \end{bmatrix}$	14	4	3	$a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_2 \otimes b_3 \otimes c_3 + a_1 \otimes b_1 \otimes c_2$
$\begin{bmatrix} \lambda & \mu & \\ & \lambda & \mu \\ & & \lambda \end{bmatrix}$	16	4	3	$a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_2 \otimes b_3 \otimes c_3 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_3$
$\begin{bmatrix} \lambda & \mu & \\ & \lambda & \mu \\ & & \end{bmatrix}$	14	3	3	$a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_3$
$\begin{bmatrix} \lambda & \mu & \\ & & \lambda \\ & & \mu \end{bmatrix}$	14	4	3	$a_2 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_2 \otimes b_2 \otimes c_3 + a_1 \otimes b_3 \otimes c_3$

Regular pencils in $\mathbb{C}^2 \otimes \mathrm{Sym}^2(\mathbb{C}^4)$

Segre symbol	dim	symRk_p	$\underline{\mathrm{Rk}}$	T
[1 1 1 1]	19	4	4	$\lambda\otimes x^2+(\lambda+\mu)\otimes y^2+(\lambda-\mu)\otimes z^2+\mu\otimes w^2$
[2 1 1]	19	5	4	$\lambda\otimes(x+y)^2+(\mu-\lambda)\otimes x^2-\lambda\otimes y^2+\mu\otimes z^2+(\lambda+\mu)\otimes w^2$
[(1 1) 1 1]	18	4	4	$\lambda\otimes x^2+\lambda\otimes y^2+\mu\otimes z^2+(\lambda+\mu)\otimes w^2$
[3 1]	18	5	4	
[(2 1) 1]	17	5	4	$\lambda\otimes(x+y)^2+(\mu-\lambda)\otimes x^2-\lambda\otimes y^2+\lambda\otimes z^2+\mu\otimes w^2$
[(1 1 1) 1]	15	4	4	$\lambda\otimes x^2+\lambda\otimes y^2+\lambda\otimes z^2+\mu\otimes w^2$
[2 2]	18	5	4	
[(1 1) 2]	17	5	4	$\lambda\otimes x^2+\lambda\otimes y^2+\mu\otimes(z+w)^2+(\lambda-\mu)\otimes z^2-\mu\otimes w^2$
[(1 1) (1 1)]	16	4	4	$\lambda\otimes x^2+\lambda\otimes y^2+\mu\otimes z^2+\mu\otimes w^2$
[4]	17	5	4	
[(3 1)]	17	5	4	
[(2 2)]	15	6	4	$\lambda\otimes(x+y)^2+(\mu-\lambda)\otimes x^2-\lambda\otimes y^2+$ $+\lambda\otimes(z+w)^2+(\mu-\lambda)\otimes z^2-\lambda\otimes w^2$
[(2 1 1)]	14	5	4	$\lambda\otimes(x+y)^2+(\mu-\lambda)\otimes x^2-\lambda\otimes y^2+\lambda\otimes z^2+\lambda\otimes w^2$

[2 2] has one only invariant polynomial (non-squarefree), hence $\delta = 1$;
[(2 2)] has two invariant polynomials (non-squarefree), hence $\delta = 2$.
This is why $\mathrm{symRk}_p([2\ 2]) = 5$ while $\mathrm{symRk}_p([(2\ 2)]) = 6$.