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Normal Bundle of Rational Curves and the Waring's Problem

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Contents

Introduction

In this thesis we address the problem to determine the splitting of the normal bundle of rational curves in $\mathbb{P}^m$ of fixed degree n.

This problem has been considered in the 80s in a series of papers by Ghione and Sacchiero (see [Ghione and Sacchiero, 1980] and [Sacchiero, 1982]) and in another series by Eisenbud and Van de Ven (see [Eisenbud and Van de Ven, 1981] and [Eisenbud and Van de Ven, 1982]) in the case of rational curves in $\mathbb{P}^3$.

In this case the normal bundle has rank 2 and its splitting is $\mathcal{O}_{\mathbb{P}^1}(2n-1-\rho) \oplus \mathcal{O}_{\mathbb{P}^1}(2n-1+\rho)$ with $0 \leq \rho \leq n-3$. The extreme case $\rho = n-3$ is achieved only if we allow singularities, while in the smooth case we have $0 \leq \rho \leq n-4$.

We can parametrize the rational curves in several ways. The most natural way is maybe the Hilbert scheme, as chosen by Eisenbud and Van de Ven. The scheme $H^{3,n}$ defined as the component of the Hilbert scheme $\mathrm{Hilb}_n \mathbb{P}^3$ of rational curves of degree n in $\mathbb{P}^3$ containing the smooth curves as an open subset, has dimension $4(n+1)-4 = 4n$. Indeed we may look at the parametrizations $\mathbb{P}^1 \to \mathbb{P}^3$ as given by 4 homogeneous polynomials of degree n in two variables, which make a rational variety $V^{3,n}$ of dimension $4(n+1)-1$.

Indeed $H^{3,n}$ is a quotient of (an open subset of) $V^{3,n}$ by the action of $SL(2)$, identifying different parametrizations when give the same curve.

In any case, one can define subschemes $N_3^n((2n-1-\rho),(2n-1+\rho)) = N_3^n(\rho)$ of $H^{3,n}$ or $V^{3,n}$ such that the normal bundle of curves in $N_3^n(\rho)$ splits exactly as $\mathcal{O}_{\mathbb{P}^1}(2n-1-\rho) \oplus \mathcal{O}_{\mathbb{P}^1}(2n-1+\rho)$. The main result achieved by Eisenbud and Van de Ven is that in the case of smooth rational space curves the corresponding subset $\mathcal{N}_3^n(\rho) \subset N_3^n(\rho)$ is irreducible of codimension $2\rho-1$ for $1 \leq \rho \leq n-4$. Instead the main result for rational space curves with only ordinary singularities achieved by Ghione and Sacchiero is that $N_3^n(\rho)$ is a quasi-projective, integral, Cohen-Macaulay variety of codimension $2\rho-1$ for $1 \leq \rho \leq n-3$.

Few facts on this flavour are known on $\mathbb{P}^m$, with $m \geq 4$. To our knowledge the only paper devoted to this topic remains that of Sacchiero (see [Sacchiero, 1980]). On $\mathbb{P}^m$ the normal bundle has rank $m-1$, and one cannot expect a simple well ordered filtration like in the case of $\mathbb{P}^3$, but likely there is a much more complicated filtration of the subschemes where the normal bundle has a fixed splitting.

In this thesis we choose a more projective point of view, by considering the rational normal curve $C_n \subset \mathbb{P}^n$ and we view our degree n curves as projections from a linear space $L = \mathbb{P}^{k-1}$. The projected curves lie in a projective space of dimension $(n-k)$. We point out that we are interested to the case with ordinary singularities as in the work of Ghione and Sacchiero (see [Ghione and Sacchiero, 1980]).

We define the scheme $H^{m,n}$ as the component of the Hilbert scheme $\mathrm{Hilb}_n \mathbb{P}^m$ of arithmetic genus zero curves of degree n in $\mathbb{P}^m$ containing the smooth curves as an open subset and we may look at the parametrizations $\mathbb{P}^1 \to \mathbb{P}^m$ as given by $m+1$ homogeneous polynomials of degree n in two variables, which make a rational variety $V^{m,n}$. Moreover we denote with $N^n_m(n_1, ..., n_{m-1})$ the subscheme of curves such that the splitting type of normal bundle is $(n_1, ..., n_{m-1})$. Note that $SL(m+1)$ acts on both $H^{m,n}$ and $V^{m,n}$, and that the subschemes $N^n_m(n_1, ..., n_{m-1})$ are invariant under this action. Furthermore the general isotropy subgroup of this action on $M^{m,n}$ is $SL(2)$. The action of $SL(m+1)$ is free on $V^{m,n}$ and an open subscheme of the quotient $V^{m,n}//SL(m+1)$. An open subscheme of the quotient of $V^{m,n}$ for $SL(m+1)$ has dimension $(n+1)(m+1)-1-((m+1)^2-1) = (m+1)(n-m)$. Indeed this quotient is isomorphic to an open subset of the Grassmannian $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ for $m = n-k$.

So in this thesis we choose to work directly on the Grassmannian $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$.

One advantage of working directly on the Grassmannian $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ is that the irreducible components and the codimension of the varieties $N^n_m(n_1, ..., n_{m-1})/SL(m+1)$ (parametrizing subspaces L such that the curve obtained by projecting from L has normal bundle isomorphic to $\bigoplus_{i=1}^{m-1} \mathcal{O}_{\mathbb{P}^1}(n_i)$) remain the same of $N^n_m(n_1, ..., n_{m-1})$. Then we can study directly the basic structures of these subvarieties in the Grassmannian. Since now the dimension of the Grassmannian is lower than the dimension of the Hilbert scheme, this allows easier computations, also with the help of a computer (we benefit through this thesis by the software Macaulay2).

But the main advantage of this approach is that we can relate the splitting of the normal bundle of the projected curve to geometric properties of the subspace L.

The main novelty of this thesis is the interplay between the Waring decomposition and some geometric properties of the subspace L.

As a sample of this interplay we quote the proposition 3.1.18 which says that the centre of projection $L \in Gr(\mathbb{P}^1, \mathbb{P}^5)$ lies on a $3-$secant plane to the rational normal curve C_5 if and only if the normal bundle of the projected rational curve $\pi_2(C_5)$ on $\mathbb{P}^3$ is $N_{\pi_2(C_5);\mathbb{P}^3} = \mathcal{O}(7) \oplus \mathcal{O}(11)$. This result gives the irreducibility of the corresponding scheme, in particular we quote the theorem 3.1.20 which says that $N_2^5(7, 11)$ is an irreducible variety of codimension 3 formed by the lines $L \cong \mathbb{P}^1$ that belong to a $3-$secant plane to the rational normal curve C_5.

Our main tool in this work to study the splitting of the normal bundle $N_{\pi_k(C_n);\mathbb{P}^{n-k}}$ of rational curves of degree n in $\mathbb{P}^{n-k}$ is the following exact cohomology sequence:

$$0 \longrightarrow H^0(N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) \longrightarrow H^0(\mathcal{O}^{n-1}_{\pi_k(C_n)}) \xrightarrow{N^L_{n,k}} H^0(\mathcal{O}_{\mathbb{P}^1}(2)^k) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1(N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) \longrightarrow 0\ ,$$

where we have indicated with π_k the projection map from the linear subspace $L \cong \mathbb{P}^{k-1}$ and $N^L_{n,k}$ the following $3k \times (n-1)$ matrix:

$$N^L_{n,k} = \begin{pmatrix} a^1_0 & \dots & a^1_{n-2} \\ -2a^1_1 & \dots & -2a^1_{n-1} \\ a^1_2 & \dots & a^1_n \\ \vdots & \ddots & \vdots \\ a^k_0 & \dots & a^k_{n-2} \\ -2a^k_1 & \dots & -2a^k_{n-1} \\ a^k_2 & \dots & a^k_n \end{pmatrix}.$$

By our preliminary assumption on singularities we obtain a lower bound on the rank of the above matrix:

$$\operatorname{rank} N^L_{n,k} = n - 1 - h^0(N^\vee_{\pi_k(C_n),\mathbb{P}^{n-k}}(n+2)) \geq k.$$

The knowledge of $\operatorname{rank}(N^L_{n,k})$ is sufficient to determine some splitting classes of the normal bundle, in particular we quote the proposition 3.2.8 which says that $N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-1-\operatorname{rank}(N^L_{n,k})} \oplus \mathcal{F}$, with $\mathcal{F}$ a vector bundle of rank $(\operatorname{rank}(N^L_{n,k}) - k)$ on $\mathbb{P}^1$ such that $\mathcal{F} \cong \bigoplus_i^{\operatorname{rank}(N^L_{n,k})-k} \mathcal{O}(l_i)$ with $l_i \geq n+3$ and $\deg(\mathcal{F}^\vee(n+2)) = -2k$.

We point out that in the above result about rational space curves of degree 5 we have $n = 5$, $k = 2$ and $\operatorname{rank} N^L_{5,2} = 3$.

We prove in the Main Theorem for Normal Bundle 3.4.16 that the following conditions are equivalent:

i) the centre of projection $L \cong \mathbb{P}^{k-1}$ is general in the (irreducible) variety of those $\mathbb{P}^{k-1}$ which belongs to a linear system Φ of affine dimension $n-1-\mathrm{rank}(N^L_{n,k})$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$;

ii) the curve of degree n projected from $L \cong \mathbb{P}^{k-1}$ has $N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-1-\mathrm{rank}(N^L_{n,k})} \oplus \mathcal{O}(n+2+B)^{A-2k+B\cdot A} \oplus \mathcal{O}(n+3+B)^{2k-B\cdot A}$, where we have indicated $A = \mathrm{rank}(N^L_{n,k}) - k$ and $B = \lfloor \frac{2k}{\mathrm{rank}(N^L_{n,k})-k} \rfloor$.

The third advantage is clearly that this approach allows to consider curves in spaces of arbitrary dimension, not only in $\mathbb{P}^3$ as seen above. This approach works well only in particular cases, and the general case remains very difficult.

In this thesis we work also with the splitting of the restricted tangent bundle $T\mathbb{P}^{n-k}|_{\pi_k(C_n)}$. The first results in this direction were given by Ramella in her doctoral thesis (see [Ramella, 1993]) where she proved that the splitting of the normal bundle and of the tangent bundle are surprisingly very few related. Let us mention that Verdier claimed (without proof) in [Verdier, 1983] a very general result, saying that the varieties of rational curves in any $\mathbb{P}^m$ of degree n with fixed splitting type of their restricted tangent bundle are irreducible and giving the codimension. The proof of this result was clarified later by Ramella (see [Ramella, 1990]).

As for the normal bundle we consider the following exact cohomology sequence:

$$0 \longrightarrow H^0((T\mathbb{P}^{n-k}|_{\pi_k(C_n)})^\vee(n+1))) \longrightarrow H^0(\mathcal{O}^n_{\pi_k(C_n)}) \xrightarrow{T^L_{n,k}} H^0(\mathcal{O}_{\mathbb{P}^1}(1)^k) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1((T\mathbb{P}^{n-k}|_{\pi_k(C_n)})^\vee(n+1))) \longrightarrow 0\,,$$

where we have indicated with $T^L_{n,k}$ the $2k \times n$ matrix:

$$T^L_{n,k} = \begin{pmatrix} a^1_0 & \dots & a^1_{n-1} \\ -a^1_1 & \dots & -a^1_n \\ \vdots & \ddots & \vdots \\ a^k_0 & \dots & a^k_{n-1} \\ -a^k_1 & \dots & -a^k_n \end{pmatrix},$$

and it must be:

$$\mathrm{rank}\, T^L_{n,k} = n - h^0((T\mathbb{P}^{n-k}|_{\pi_k(C_n)})^\vee(n+1))) \geq k+1.$$

We prove in the Main Theorem for Restricted Tangent Bundle 2.5.18 that the following conditions are equivalent:

i) the centre of projection L is general in the (irreducible) variety of those $\mathbb{P}^{k-1}$ which belongs to a linear system Φ of affine dimension $n - \mathrm{rank}(T^L_{n,k})$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$;

ii) the curve of degree n projected from $L \cong \mathbb{P}^{k-1}$ has $T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-\mathrm{rank}(T^L_{n,k})} \oplus \mathcal{O}(n+1+B)^{A-k+B\cdot A} \oplus \mathcal{O}(n+2+B)^{k-B\cdot A})$, where we have indicated $A = \mathrm{rank}(T^L_{n,k}) - k$ and $B = \lfloor \frac{k}{\mathrm{rank}(T^L_{n,k})-k} \rfloor$.

See corollary 2.5.19 for a more complete statement about the irreducibility and codimension of the varieties above.

The structure of the thesis is the following.

In chapter one we give preliminaries about general theory of vector bundle on projective spaces and about apolarity theory.

In chapter two we deal with the problem of rational curves with fixed splitting type of its Restricted Tangent Bundle.

In chapter three we deal with the problem of rational curves with fixed splitting type of its Normal Bundle.

So our basic objects of study in this thesis are the subvarieties $N^n_{n-k}(n_1, ..., n_{n-k-1})$ and $T^n_{n-k}(t_1, ..., t_{n-k})$ formed by the linear spaces $L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ such that the normal bundle (respectively the restricted tangent bundle) of the rational curves projected by L is isomorphic to $\oplus_i \mathcal{O}_{\mathbb{P}^1}(n_i)$ (respectively $\oplus_i \mathcal{O}_{\mathbb{P}^1}(t_i)$).

Chapter 1

Preliminaries

1.1 Vector Bundles on Projective Space

We will establish the notations and the most important facts about the cohomology of projective spaces with coefficients in an analytic cohorent sheaf (for example see [Okonek et al., 1980], [Le Potier, 1997]). Let V be an (n+1)-dimensional complex vector space, we denote by $\mathbb{P}^n = \mathbb{P}(V)$ the associated projective space of lines in V with a natural structure as compact complex manifold. Let X be a complex space with structure sheaf $\mathcal{O}_X$ and let F be a holomorphic vector bundle over X. Then we have a sheaf $\mathcal{O}_X(F)$ of germs of holomorphic sections in F. It is a locally free sheaf of rank equal to the rank of F. In what follow we shall not distinguish between a vector bundle F and the associated locally free sheaf $\mathcal{O}_X(F)$.

As usual we shall denote by $\mathcal{O}_{\mathbb{P}^n}(1)$ the hyperplane bundle over $\mathbb{P}^n$. The dual bundle $\mathcal{O}_{\mathbb{P}^n}(1)^\vee$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ will be denoted by $\mathcal{O}_{\mathbb{P}^n}(-1)$, it is the tautological line bundle over $\mathbb{P}^n$:

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \{(l, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \text{ such that } v \in [l]\}.$$

For any coherent analytic sheaf F over $\mathbb{P}^n$ we define $F(r) = F \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(r)$, where for $r \in \mathbb{Z}$ we denoted by:

$$\mathcal{O}_{\mathbb{P}^n}(r) = \begin{cases} \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes r} & \text{for } r \geq 0, \\ \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes |r|} & \text{for } r \leq 0 \end{cases}$$

The sections of the line bundle $\mathcal{O}_{\mathbb{P}^n}(r)$, when $r \geq 0$, can be identified with the homogeneous polynomial $P \in \mathbb{C}[z_0, ..., z_n]$ of degree r, or in equivalent way $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \cong$

$\operatorname{Sym}^r V = S^r V$ and:

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0 \text{ for } 0 < i < n \quad \forall r \in \mathbb{Z}.$$

The zero loci of sections of $\mathcal{O}_{\mathbb{P}^n}(r)$ are exactly the hypersurfaces of degree r. The zero loci of a general section of $\mathcal{O}_{\mathbb{P}^n}(r_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^n}(r_k)$ is called a complete intersection. It is important to underline that $\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(b)) \cong S^{b-a}V^\vee$, moreover a morphism $\oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i) \to \oplus_j \mathcal{O}_{\mathbb{P}^n}(b_j)$ is represented by a matrix whose entries are homogeneous polynomials.

We get an exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow Q \longrightarrow 0 \,, \tag{1.1}$$

regarding $\mathcal{O}_{\mathbb{P}^n}(-1)$ as subbundle of $\mathcal{O}_{\mathbb{P}^n}^{n+1}$, where Q is called the quotient bundle and has rank n. More invariant way we can rephrased above as:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \otimes V \longrightarrow Q \longrightarrow 0 \,. \tag{1.2}$$

It is well known (for example see [Okonek et al., 1980] pg.6, [Griffiths and Harris, 1994] pg.409, [Harris, 1995] pg.201, [Ottaviani and Vallès, 2001] pg.29) that :

$$Q \cong T\mathbb{P}^n(-1),$$

the twisted holomorphic tangent bundle.

The exact short sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \longrightarrow T\mathbb{P}^n \longrightarrow 0 \,, \tag{1.3}$$

is called the Euler sequence. Let $\Omega^p_{\mathbb{P}^n}$ be the sheaf of germs of holomorphic $p-$forms on $\mathbb{P}^n$, so:

$$\Omega^1_{\mathbb{P}^n} \cong (T\mathbb{P}^n)^\vee, \quad \Omega^p_{\mathbb{P}^n} \cong \bigwedge^p \Omega^1_{\mathbb{P}^n}.$$

Dualizing the Euler sequence and taking the $p-$th exterior power one gets the following exact sequence:

$$0 \longrightarrow \Omega^p_{\mathbb{P}^n}(p) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{p}} \longrightarrow \Omega^{p-1}_{\mathbb{P}^n}(p) \longrightarrow 0 \,, \tag{1.4}$$

For the canonical bundle $\omega_{\mathbb{P}^n} \cong \Omega^n_{\mathbb{P}^n} = \det \Omega^1_{\mathbb{P}^n}$, we have that $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Theorem 1.1.1 (Serre duality). *If X is an $n-$dimensional projective algebraic complex manifold with canonical line bundle ω_X, then we have for any holomorphic vector bundle E over X:*

$$H^q(X,E)^* \cong H^{n-q}(X,E^*\otimes\omega_X).$$

The above Serre duality implies:

$$h^q(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n}(r)) = h^{n-q}(\mathbb{P}^n,\Omega^{n-p}_{\mathbb{P}^n}(-r)).$$

Moreover we have the following theorem.

Theorem 1.1.2 (Bott formula).

$$h^q(\mathbb{P}^n,\Omega^p_{\mathbb{P}^n}(r)) = \begin{cases} \binom{r+n-p}{r}\binom{r-1}{p} & \textit{for } q=0 \quad 0\le p\le n, \quad r>p \\ 1 & \textit{for } r=0, \quad 0\le p=q\le n \\ \binom{-r+p}{-r}\binom{-r-1}{n-p} & \textit{for } q=n \quad 0\le p\le n, \quad r<p-n \\ 0 & \textit{otherwise} . \end{cases}$$

In particular for $p=0$ we have:

$$h^q(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(r)) = \begin{cases} \binom{r+n}{r} & \text{for } q=0 \quad r\ge 0 \\ \binom{-r-1}{-r-1-n} & \text{for } q=n \quad r\le -n-1 \\ 0 & \text{otherwise} . \end{cases} \tag{1.5}$$

Definition 1.1.3. A coherent analytic sheaf F over $\mathbb{P}^n$ is said to be generated by global sections if the canonical homomorphism of sheaves:

$$\varphi: H^0(\mathbb{P}^n,F)\otimes_{\mathbb{C}}\mathcal{O}_{\mathbb{P}^n}\to F, \qquad \varphi_x(s\otimes h)=hs_x, \qquad x\in\mathbb{P}^n,$$

is surjective.

Theorem 1.1.4 (A). *For every coherent analytic sheaf F over $\mathbb{P}^n$ there is a $r_0\in\mathbb{Z}$ so that for $r\ge r_0$ the sheaf $F(r)$ is generated by global sections.*

Theorem 1.1.5 (B). *For every coherent analytic sheaf F over $\mathbb{P}^n$ there is a $r_0\in\mathbb{Z}$ such that for $r\ge r_0$ and all $q>0$:*

$$H^q(\mathbb{P}^n,F(r))=0.$$

Definition 1.1.6. A vector bundle E over X is called spanned if there are global sections $s_1,...,s_k$ such that for all $x\in X$ the vectors $s_1(x),...,s_k(x)$ span the fiber $\pi^{-1}(x)$ where $\pi: E\to X$ is the surjective morphism associated to E.

Let E be a spanned vector bundle of rank r over X. We denote by $s_1, ..., s_{r-p+1}$ some $r-p+1$ generic sections of E. The subvariety:

$$\{x \in X : s_1(x), ..., s_{r-p+1}(x) \text{ are linearly dependents }\} \tag{1.6}$$

has codimension p and its homology class in $H_{2n-2p}(X, \mathbb{Z})$ does not depend on the sections.

Definition 1.1.7. The Chern classes $c_p(E) \in H^{2p}(X, \mathbb{Z})$ of a spanned vector bundle E are defined as the Poincaré dual of the class in (1.6).

Observation 1.1.8. *i) If $p = r$ in (1.6) we get the zero locus of a generic section of E.*

ii) If $p = 1$ in (1.6) we get that $c_1(E) = c_1(\det E)$.

iii) If $X = \mathbb{P}^1$ we get that $c_1(\mathcal{O}_{\mathbb{P}^1}(a)) = a$, we will indicate with $\deg E = c_1(E)$.

When E is not spanned there is a way to supply the definition of Chern classes. This is to tensor E with some ample line bundle L in order to get $E \otimes L$ spanned and then use the formula:

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) c_i(L)^{k-i},$$

in particular $c_1(E \otimes L) = c_1(E) + rc_1(L)$. The Chern polynomial is the formal expression:

$$c_E(t) := c_0(E) + c_1(E)t + c_2(E)t^2 + ...$$

In the case $X = \mathbb{P}^n$ we have $c_i(E) \in \mathbb{Z}$ and $c_E(t) \in \mathbb{Z}[t]/t^{n+1}$.

If we have the following exact sequence of vector bundle:

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0\,, \tag{1.7}$$

the Whitney formula is:

$$c_E(t) c_G(t) = c_F(t), \tag{1.8}$$

in particular we have:

$$c_1(F) = c_1(E) + c_1(G),$$

$$c_2(F) = c_2(E) + c_1(E)c_1(G) + c_2(G).$$

Theorem 1.1.9 (Gauss-Bonnet)**.** *For any compact complex variety of dimension n we have:*

$$\chi(X, \mathbb{Z}) = c_n(TX).$$

The Thom-Porteous formula allows to compute the homology class and even the class in the Chow ring of the degeneracy locus of a map between two vector bundles. Let $\phi : E \to F$ be a sheaf map between vector bundles of rank e and f. The $k-$degeneracy locus is $D_k(\phi) := \{x \in X |\, \mathrm{rank}(\phi_x) \leq k\}$. We have:

$$\mathrm{codim}\, D_k(\phi) \leq (e-k)(f-k),$$

in the generic case. Assume that $\mathrm{codim}\, D_k(\phi) = (e-k)(f-k)$, then the Thom-Porteous formula is:

$$[D_k(\phi)] = \det(c_{f-k+j-i}(F-E)_{1\leq i,j\leq e-k}),$$

where $c_i(E-F)$ is the $i-$th coefficient in the expansion of the quotient c_E/c_F ($c_i = 0$ if $i < 0$) and $[D_k(\phi)]$ is the fundamental class of $D_k(\phi)$.

Segre first and Grothendieck then prove that a rank r vector bundle on $\mathbb{P}^1$ splits in r line bundles:

Theorem 1.1.10 (Grothendieck-Segre, [Grothendieck, 1957],[Ghione and Ottaviani, 1992])**.** *Every holomorphic vector bundle E of rank k over $\mathbb{P}^1$ has the form:*

$$E \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_k),$$

with uniquely determined numbers $a_1, ..., a_k \in \mathbb{Z}$ with $\deg E = a_1 + ... + a_k$ and $a_1 \leq ... \leq a_k$.

Definition 1.1.11. We define the splitting type of vector bundle E the above twisting factors and we indicate it with $(a_1, ..., a_k)$.

Theorem 1.1.12. *The tangent bundle $T\mathbb{P}^n$ on $\mathbb{P}^n$ splits on any line as $\mathcal{O}_{\mathbb{P}^n}(1)^{n-1} \oplus \mathcal{O}_{\mathbb{P}^n}(2)$ and its generic section vanishes in $n+1$ points.*

1.2 Statement of the Problem and Known Results

The aim of our work is to study the varieties which parametrize the subschemes of the Hilbert scheme of algebraic rational curves with fixed splitting type of Normal Bundle and also which ones with fixed splitting type of Restricted Tangent Bundle. We are also interested on case when both are fixed.

1.2.1 Known Results on Normal Bundle of Rational Curves

In the case of rational space curves C of degree n the normal bundle has rank 2 and it has the splitting $\mathcal{O}_{\mathbb{P}^1}(2n-1-\rho)\oplus\mathcal{O}_{\mathbb{P}^1}(2n-1+\rho)$ with $0\le\rho\le n-3$. We can observe that the extreme case $\rho=n-3$ is achieved only if we allow singularities, while in the smooth case we have $0\le\rho\le n-4$.

The first studied case is the space rational quartic curve, it has only one possibility for the splitting type, which one balanced:

Theorem 1.2.1 ([Ghione and Sacchiero, 1980]). *Let C be a nonsingular rational quartic curve in $\mathbb{P}^3$. Then the normal bundle of C is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(7)\oplus\mathcal{O}_{\mathbb{P}^1}(7)$.*

Also the rational curve contained in a smooth quadric has normal bundle balanced:

Theorem 1.2.2 ([Eisenbud and Van de Ven, 1981]). *A smooth rational space curve C of degree $n\ge 3$ which is contained in a smooth quadric has normal bundle $N_{C;\mathbb{P}^3}=\mathcal{O}_{\mathbb{P}^1}(2n-1)\oplus\mathcal{O}_{\mathbb{P}^1}(2n-1)$.*

Instead the extreme case $\rho=n-3$ is achieved only if we allow singularities:

Theorem 1.2.3 ([Ghione and Sacchiero, 1980]). *C is contained in a quadratic cone and has a $(n-2)-$fold point in its vertex if and only if $\rho=n-3$.*

In order to address the problem we need to choose a parametrization of rational space curves. The most natural way to parametrize the rational space curves of degree n is maybe the Hilbert scheme $\mathrm{Hilb}_n\,\mathbb{P}^3$, as chosen by Eisenbud and Van de Ven.

Definition 1.2.4. The scheme $H^{3,n}$ defined as the component of the Hilbert scheme $\mathrm{Hilb}_n\,\mathbb{P}^3$ of rational space curves of degree n containing the smooth curves as an open subset, has dimension $4(n+1)-4=4n$. One can define subschemes $N_3^n((2n-1-\rho),(2n-1+\rho))=N_3^n(\rho)$ of $\mathrm{Hilb}_n\,\mathbb{P}^3$ such that the normal bundle of curves in $N_3^n(\rho)$ splits exactly as $\mathcal{O}_{\mathbb{P}^1}(2n-1-\rho)\oplus\mathcal{O}_{\mathbb{P}^1}(2n-1+\rho)$ and we indicate with $\mathcal{N}_3^n(\rho)$ the analogous one formed by smooth rational space curves.

Theorem 1.2.5 ([Eisenbud and Van de Ven, 1981]). *The sets $\mathcal{N}_3^n(\rho)$ form a stratification of $H^{3,n}$ by non empty, locally closed subsets for $0 \leq \rho \leq n-4$. $\mathcal{N}_3^n(\rho)$ is irreducible of codimension $2\rho - 1$ for $1 \leq \rho \leq n-4$.*

Ghione and Sacchiero solve the problem in the case of rational space curves with ordinary singularity. They consider the parametrization $\mathbb{P}^1 \to \mathbb{P}^3$ as given by 4 homogeneous polynomials of degree n in two variables, which make a rational variety $V^{3,n}$ of dimension $4(n+1) - 1 = 4n + 3$. We point out that $H^{3,n}$ is a quotient of an open subset of $V^{3,n}$ by the action of $SL(2)$, identifying different parametrizations when give the same curve.

$V^{3,n}$ is the Zariski open set of $\mathbb{P}^{4n+3}$ corresponding to the 4−tuples of polynomials which give curves with at most ordinary singularity. Instead we indicate with $\mathcal{V}^{3,n}$ the 4−tuples of polynomials which give an embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, i.e. $\mathcal{V}^{3,n}$ is formed by the smooth rational space curves of degree n.

Theorem 1.2.6 ([Ghione and Sacchiero, 1980]). *There exists an hypersurface $V_0^{3,n}$ of $V^{3,n} \subset \mathbb{P}^{4n+3}$ such that if $\psi \in V^{3,n} \setminus V_0^{3,n}$ then $N_{\psi(\mathbb{P}^1);\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^1}(2n-1)^2$.*

We consider the subsets of $V^{3,n}$:

$$V_\rho^{3,n} = \{\psi \in V^{3,n} : \rho(\psi) \geq \rho\} \quad 0 \leq \rho \leq n-3,$$

where $\rho(\psi)$ is the integer which determine the splitting of the normal bundle of $\psi(\mathbb{P}^1)$ i.e. $N_{\psi(\mathbb{P}^1);\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^1}(2n-1+\rho(\psi(\mathbb{P}^1)) \oplus \mathcal{O}_{\mathbb{P}^1}(2n-1-\rho(\psi(\mathbb{P}^1))$.

Theorem 1.2.7 ([Sacchiero, 1982]). *There exists a stratification of $V^{3,n}$ (resp. $\mathcal{V}^{3,n}$):*

$$\emptyset \neq V_{n-3}^{3,n} \subset V_{n-4}^{3,n} \subset \ldots \subset V_\rho^{3,n} \subset \ldots \subset V_1^{3,n} \subset V^{3,n};$$

$$(\textit{resp. } \emptyset \neq \mathcal{V}_{n-3}^{3,n} \subset \mathcal{V}_{n-4}^{3,n} \subset \ldots \subset \mathcal{V}_\rho^{3,n} \subset \ldots \subset \mathcal{V}_1^{3,n} \subset \mathcal{V}^{3,n}) \textit{ such that:}$$

1) $C \in V_\rho^{3,n}$ (resp. $C \in \mathcal{V}_\rho^{3,n}$), $1 \leq \rho \leq n-3$ (resp. $1 \leq \rho \leq n-4$) if and only if $N_{C;\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(2n-1-\overline{\rho}) \oplus \mathcal{O}_{\mathbb{P}^1}(2n-1+\overline{\rho})$ with $\overline{\rho} \geq \rho$.

2) $V_\rho^{3,n}$ (resp. $\mathcal{V}_\rho^{3,n}$), for $1 \leq \rho \leq n-3$ (resp. $1 \leq \rho \leq n-4$), is a quasi-projective, integral, Cohen-Macaulay variety of codimension $2\rho - 1$.

1.2.2 Normal Bundle of Rational Curves in $\mathbb{P}^{n-k}$

Only few facts are known for rational curves in $\mathbb{P}^{n-k}$. In this section we will explain the technique used by Sacchiero and his main result obtained in that way. At the end we present our approach use in this thesis.

Let $C \subset \mathbb{P}^{n-k}$, $n-k-1 \geq 1$, be a rational curve of degree n with only ordinary singularities. We can write $N_{C;\mathbb{P}^{n-k}} = \bigoplus_{i=1}^{n-k-1} \mathcal{O}(n+d_i)$. On the other hand we have $c_1(N_{C;\mathbb{P}^{n-k}}) = (n-k+1)n-2$, so $\sum_{i=1}^{n-k-1} d_i = 2n-2$. Let $t_0, t_1 \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$ be two section and $A = \bigoplus_{m\geq 0} A_m$. Let $\psi : \mathbb{P}^1 \to \mathbb{P}^{n-k}$ be the morphism defined by $\psi_i \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$. Let $C = \psi(\mathbb{P}^1)$, we suppose that it has only ordinary singularities, i.e. the map of differential fibre bundles $\Omega\psi : \psi^*\Omega_{\mathbb{P}^{n-k}} \to \Omega_{\mathbb{P}^1}$ is surjective.

As made for rational curves in $\mathbb{P}^3$ we may look at the parametrizations $\mathbb{P}^1 \to \mathbb{P}^m$ as given by $m+1$ homogeneous polynomials of degree n in two variables, which make a rational variety $V^{m,n}$.

Lemma 1.2.8 ([Sacchiero, 1980]). *Let $\omega = ((\partial\psi_i/\partial(t_0,t_1)))_{i=0,\ldots,n-k}$ be the Jacobian matrix. Then we have the following exact sequence:*

$$0 \longrightarrow N^{\vee}_{C;\mathbb{P}^{n-k}} \longrightarrow \mathcal{O}^{n-k+1}(-n) \xrightarrow{\omega} \mathcal{O}^2(-1) \longrightarrow 0.$$

From the above exact sequence we can give the following one:

$$0 \longrightarrow \bigoplus_{m\in\mathbb{Z}} \Gamma(C, N^{\vee}_{C;\mathbb{P}^{n-k}}(m)) \longrightarrow A^{n-k+1}(-n) \xrightarrow{\omega} A^2(-1),$$

which gives in degree $n+d$:

$$0 \longrightarrow \Gamma(C, N^{\vee}_{C;\mathbb{P}^{r+1}}(n+d)) \longrightarrow A_d^{n-k+1} \xrightarrow{\omega_{n+d}} A^2_{n+d-1}.$$

So $\ker\omega_{n+d} = \Gamma(C, N^{\vee}_{C;\mathbb{P}^{n-k}}(n+d))$ and if $d_h \leq d < d_{h+1}$ we have $\dim\ker\omega_{n+d} = h(d+1) - \sum_{i=1}^{h} d_i$.

Proposition 1.2.9. *(see [Sacchiero, 1980]) We have:*

i) $d_i \neq 1$ *for all* $i = 1, 2, ..., n-k-1$;

ii) $\dim\ker\omega_n = h$ *if and only if* $C \subset \mathbb{P}^{n-k-h}$.

Lemma 1.2.10 (see [Sacchiero, 1980]). *Let $2n-2 = \delta r + \rho$ with $0 \leq \rho < n-k-1$. Let*

$$V_d^{n-k,n} = \{C \in V^{n-k,n} : \mathrm{rank}(\omega_{n+d}(C)) \textit{ is not maximum } \}$$
$$W^{n-k,n} = V^{n-k,n}_{\delta-1} \cup V^{n-k,n}_{\delta}$$

be subset of $V^{n-k,n}$. Then:

i) $C \in V_{\delta-1}^{n-k,n}$ *if and only if* $d_1(C) \leq \delta - 1$*;*

ii) $C \in V_{\delta}^{n-k,n}$ *if and only if* $dim \ker \omega_{n+\delta}(C) > n-k-1-\rho$*;*

iii) $V_{\delta-1}^{n-k,n}$*,* $V_{\delta}^{n-k,n}$*,* $W^{n-k,n}$ *are proper closed subset of* $V^{n-k,n}$*;*

iv) $C \in V^{n-k,n} \setminus W^{n-k,n}$ *if and only if* $d_1(C) = d_2(C) = ... = d_{n-k-1-\rho}(C) = \delta$ *and* $d_{n-k-1-\rho+1}(C) = ... = d_{n-k-1}(C) = \delta + 1$.

Theorem 1.2.11 ([Sacchiero, 1980])**.** *If* C *is a generic rational curve of degree* n *in* $\mathbb{P}^{n-k}$ *and* $2n-2 = \delta(n-k-1) + \rho$ *with* $0 \leq \rho < n-k-1$*, then* $N_{C;\mathbb{P}^{n-k}} \simeq \mathcal{O}(n+\delta)^{n-k-\rho} \oplus \mathcal{O}(n+\delta+1)^{\rho}$*.*

In this thesis we consider the rational normal curve $C_n \subset \mathbb{P}^n$ and we obtain our degree n rational curves in $\mathbb{P}^{n-k}$ as projections from a linear space $L = \mathbb{P}^{k-1}$.

As in the work of Ghione and Sacchiero (see [Ghione and Sacchiero, 1980]) we are interested to the case with ordinary singularities.

Definition 1.2.12. For rational curves of degree n in $\mathbb{P}^m$ we define the scheme $H^{m,n}$ as the component of the Hilbert scheme $\text{Hilb}_n \mathbb{P}^m$ of rational curves of degree n in $\mathbb{P}^m$ containing the smooth curves as an open subset. Moreover we denote with $N_m^n(n_1, ..., n_{m-1})$ is the subscheme of curves with splitting type of normal bundle is $(n_1, ..., n_{m-1})$ and with $T_m^n(t_1, ..., t_m)$ is the subscheme of curves with splitting type of restricted tangent bundle is $(t_1, ..., t_m)$.

Note that $SL(m+1)$ acts on both $H^{m,n}$ and $V^{m,n}$, and that the subschemes $N_m^n(n_1, ..., n_{m-1})$ are invariant under this action. Furthermore the general isotropy subgroup of this action on $M^{m,n}$ is $SL(2)$. The action of $SL(m+1)$ is free on $V^{m,n}$ and an open subscheme of the quotient $V^{m,n}//SL(m+1)$. An open subscheme of the quotient of $V^{m,n}$ for $SL(m+1)$ has dimension $(n+1)(m+1)-1-((m+1)^2-1) = (m+1)(n-m)$. Indeed this quotient is isomorphic to an open subset of the Grassmannian $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ for $m = n-k$.

So in this thesis we choose to work directly on the Grassmannian $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$.

One advantage of working directly on the Grassmannian $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ is that the irreducible components and the codimension of the varieties $N_m^n(n_1, ..., n_{m-1})/SL(m+1)$ (parametrizing subspaces L such that the curve obtained by projecting from L has normal bundle isomorphic to $\bigoplus_{i=1}^{m-1} \mathcal{O}_{\mathbb{P}^1}(n_i)$) remain the same of $N_m^n(n_1, ..., n_{m-1})$ and similarly for $T_m^n(t_1, ..., t_m)$. Then we can study directly the basic structures of these subvarieties in the Grassmannian. Since now the dimension of the Grassmannian is lower than the dimension of the Hilbert scheme, this allows easier computations, also with the help of a computer (we benefit through this thesis by the software Macaulay2).

But the main advantage of this approach is that we can relate the splitting of the normal bundle of the projected curve to geometric properties of the subspace L.

1.2.3 Known Results on Restricted Tangent Bundle on Rational Curves

Some results for rational curves contained in a smooth quadric are proved by Eisenbud and Van de Ven about restricted tangent bundle:

Theorem 1.2.13 ([Eisenbud and Van de Ven, 1981]). *A smooth rational space-curve C of degree $n \geq 3$ is contained in a smooth quadric if and only if $T\mathbb{P}^3 \mid_C \simeq \mathcal{O}(2n-2) \oplus \mathcal{O}(n+1) \oplus \mathcal{O}(n+1)$.*

The main result was obtained by Verdier (see [Verdier, 1983]), but he does not write down a proof, which is shown later on by Ramella (see [Ramella, 1990]). If $a \in \mathbb{Z}$ we indicate $a^+ = \max\{0, a\}$, if $(a_1, ..., a_m)$ is a sequence of integers we will indicate $\delta(a_1, ..., a_m) = \sum_{i,j}(a_i - a_j - 1)^+$. We can observe that if $T\mathbb{P}^m(-1) \mid_C \cong \bigoplus_{i=1}^m \mathcal{O}(a_i)$, then $\dim Ext^1(T\mathbb{P}^m(-1) \mid_C, T\mathbb{P}^m(-1) \mid_C) = \delta(a_1, ..., a_m)$.

It seems natural to conjecture that the same applies to normal bundle of rational curves in $\mathbb{P}^m$. In fact, in this thesis there are only cases that verify this conjecture.

Theorem 1.2.14 ([Verdier, 1983]/[Ramella, 1990]). *Let $V^{m,d}$ be the family of morphisms $f : \mathbb{P}^1 \to \mathbb{P}^m$ of degree n. For every sequence of integers $(a_1, ..., a_m)$ such that $a_1 \geq ... \geq a_m$ and $\sum_{i=1}^m a_i = n$ we indicate with $V^{m,n}(a_1, ..., a_m)$ the family of morphisms $f \in V^{m,n}$ such that the splitting type of the fibre bundle $f^*(T\mathbb{P}^m(-1))$ is $(a_1, ..., a_m)$. Then:*

i) $V^{m,n}(a_1, ..., a_m) = \emptyset$ if $a_m < 0$;

ii) $V^{m,n}(a_1, ..., a_m)$ is a subvariety non empty of $V^{m,n}$ smooth and connected of codimension $\delta(a_1, ..., a_m)$.

We point out that Ramella prove the above theorem using a result on the Shatz stratification (see [Drezet and Le Potier, 1985] and [Bruguières, 1985]). In particular Ramella prove the following result:

Proposition 1.2.15 ([Ramella, 1990]). *The natural morphism $\varphi : V^{m,n} \times \mathbb{P}^1 \to \mathbb{P}^m$ and the vector bundle $T\mathbb{P}^m(-1)$ give a family of vector bundles of rank m and degree n on $\mathbb{P}^1$ parametrized by $V^{m,n}$.*

For all $f \in V^{m,n}$ the Kodaira-Spencer morphism:

$$ks(f) : T_f(V^{m,n}) \to Ext^1(f^*(T\mathbb{P}^m(-1)), f^*(T\mathbb{P}^m(-1))),$$

is surjective.

An other interesting thing is shown by Ramella is a necessary and sufficient condition for rational curve in $\mathbb{P}^m$ to stay on a $r-$plane:

Theorem 1.2.16 ([Ramella, 1990]). *Let f be an embedding of $\mathbb{P}^1$ in $\mathbb{P}^m$. The splitting type of $f^*(T\mathbb{P}^m(-1))$ is $(a_1, ..., a_m)$ with $a_i = 0$ for all $i \geq r+1$ if and only if the curve $f(\mathbb{P}^1)$ is on an $r-$plane.*

1.2.4 Relation Between Normal and Restricted Tangent Bundle

Only few results are known in the relation between both splitting type. We will indicate with $T_3^n(t_1, t_2, t_3)$ (respectively $N_3^n(n_1, n_2)$) the family of the smooth curves C in $H^{3,n}$ with the splitting type (t_1, t_2, t_3) of $T\mathbb{P}^3|_C$ (respectively with the splitting type (n_1, n_2) of $N_{C;\mathbb{P}^3}$).

Theorem 1.2.17 ([Ramella, 1993]). *i) The normal bundle of the general curve of $T_3^n(t_1, t_2, t_3)$ is of general type, i.e. in every $T_3^n(t_1, t_2, t_3)$ there is a general curve belonging to $N_3^n(2n-1, 2n-1)$.*

ii) If C belongs to $N_3^n(n_1, n_2)$ with $n_2 \leq \frac{n+6}{4} - n$, then the maximum degree of subfibre bundle of $T\mathbb{P}^3|_C$ is $4n + 2 - 2n_2$.

iii) If $t_3 + n_2 \geq 3n$, then $T_3^n(t_1, t_2, t_3)$ intersects $N_3^n(n_1, n_2)$ on a subscheme which has an irreducible component with right dimension, i.e. $4n - \delta(t_1, t_2, t_3) - \delta(n_1, n_2)$.

iv) If $t_3 \geq n_2$, then $T_3^n(t_1, t_2, t_3)$ does not intersect $N_3^n(n_1, n_2)$.

1.3 Apolarity and Waring's Problem

1.3.1 Catalecticant and Apolarity Setup

In the following section we want to fix the notation and remark principal results about connection with apolarity theory, catalecticant homomorphism and secant varieties of rational normal curve (see [Iarrobino and Kanev, 1999]).

Contraction Action and Catalecticant Morphism

Let V be a complex vector space $\dim_{\mathbb{C}} V = r+1$ with basis $x_0, ..., x_r$, we consider the ring of homogeneous polynomial $S = \bigoplus_{i \geq 0} \operatorname{Sym}^i V$, where $\operatorname{Sym}^i V$ is the $i-$th symmetric power of V. The dual basis of $V^\vee$ can be denoted by $\partial_0 = \frac{\partial}{\partial x_0}, ..., \partial_r = \frac{\partial}{\partial x_r}$ and $T = \bigoplus_{i \geq 0} S^i V^\vee$ is the dual ring of S, so that ∂_i is an operator acting on x_j, as well x_i is an operator acting on ∂_j:

$$\operatorname{Sym}^p V \otimes \operatorname{Sym}^q V^\vee \to \operatorname{Sym}^{p-q} V \qquad \text{for } p \geq q\ , \tag{1.9}$$

$$(f, \phi) \mapsto \phi \circ f$$

or

$$\operatorname{Sym}^p V \otimes \operatorname{Sym}^q V^\vee \to \operatorname{Sym}^{q-p} V^\vee \qquad \text{for } q \geq p\ , \tag{1.10}$$

$$(f, \phi) \mapsto f \circ \phi$$

this is called the contraction action. Both maps can be defined for any p, q with the convention that $\operatorname{Sym}^i V$ is zero for negative i.

Definition 1.3.1. If we fixed a form $f \in \operatorname{Sym}^n V$ we have the catalecticant homomorphism for all $1 \leq e \leq n-1$:

$$C_f(e, n-e) : \operatorname{Sym}^{n-e} V^\vee \to \operatorname{Sym}^e V \quad \phi \to \phi \circ f,$$

the relative matrix is called catalecticant matrix $\operatorname{Cat}_f(e, n-e; r+1)$ with basis for S of power monomials $x^{[E]} = \frac{1}{e_0! ... e_r!} x^{e_0} ... x^{e_r}$ with $E = (e_0, ..., e_r) \in \mathbb{N}^{r+1}$ and $e_0 + ... + e_r = e$.

Observation 1.3.2. *Clearly the transpose* $\operatorname{Cat}_f(e, n-e; r+1)^t$ *satisfies:*

$$\operatorname{Cat}_f(e, n-e; r+1)^t = \operatorname{Cat}_f(n-e, e; r+1).$$

Definition 1.3.3. We define the determinantal loci of catalecticant $U_s(e, n-e; r+1) \subset V_s(e, n-e; r+1)$ as:

$$U_s(e, n-e; r+1) = \{f \in \operatorname{Sym}^n V | \operatorname{rank} \operatorname{Cat}_f(e, n-e; r+1) = s\},$$

$$V_s(e, n-e; r+1) = \{f \in \operatorname{Sym}^n V | \operatorname{rank} \operatorname{Cat}_f(e, n-e; r+1) \leq s\}.$$

We denote by $\mathbb{V}_s(e, n-e; r+1)$ the scheme defined by the ideal $I_{s+1}(\operatorname{Cat}_f(e, n-e; r+1))$ generated by the $(s+1)\times(s+1)$ minors of $\operatorname{Cat}_f(e, n-e; r+1)$ and by $\mathbb{U}_s(e, n-e; r+1)$ its open subscheme whose closed points are those forms f for which $\operatorname{Cat}_f(e, n-e; r+1)$ has rank exactly s. We call the $V_s(e, n-e; r+1)$ catalecticant varieties and the $\mathbb{V}_s(e, n-e; r+1)$ catalecticant schemes.

Definition 1.3.4. If $f \in \operatorname{Sym}^n V$, we denoted by $S_e(f)$ the space $\operatorname{Sym}^{n-e} V^\vee \circ f = \{\phi \circ f | \phi \in \operatorname{Sym}^{n-e} V^\vee\}$, by definition it is the image of the catalecticant homomorphism $C_f(e, n-e)$. We let $S_f = \oplus_{0\leq i\leq n} S_i(f)$, which is an $T-$submodule of S. Let $s_i(f) = \dim_{\mathbb{C}} S_i(f)$ and denote by H_f the sequence:

$$H_f = (s_0(f), ..., s_i(f), ..., s_n(f)).$$

Observation 1.3.5. *The determinantal locus $V_s(e, n-e; r+1)$ equals the set of forms of degree n in $r+1$ variables, whose $(n-e)-$th partial derivatives span a subspace of dimension $\leq s$ in T_e. It's clear that:*

$$s_0 = 1, s_1 \leq r+1 \ \ \textit{and} \ \ s_{n-i}(f) = s_i(f).$$

Definition 1.3.6. To each element $f \in S_n$ one associates the ideal $I_f = Ann(f)$ in T consisting of polynomials ϕ such that $\phi \circ f = 0$ and we call ϕ and f apolar to each other, I_f is the apolar ideal of f.

One associates to f also the quotient algebra $A_f = T/I_f$. Macaulay called such an ideal a principal system, but we know them as Gorenstein ideals, since A_f is a Gorenstein Artin algebra. If $p = (a_0, ..., a_r) \in \mathbb{C}^{r+1}$, let L_p denote the linear form:

$$L_p = a_0 x_0 + ... + a_r x_r \in S_1.$$

Observation 1.3.7. *We have the following useful equality:*

$$\phi \circ L_p^{[N]} = \phi(p) L_p^{[N-E]},$$

for all $\phi \in T_e$, $e \leq n$ and any L_p. Abusing notation we will write $\phi(L_p)$ for $\phi(p)$.

Definition 1.3.8. Consider the forms $f \in S_n$ that can be written as a sum:

$$f = L_1^{[N]} + ... + L_s^{[N]},$$

for some choice of linear forms $L_1, ..., L_s \in S_1$. They form the image of the regular map:

$$\mu : \overbrace{S_1 \times ... \times S_1}^{s-times} \to S_n,$$

defined by $\mu(L_1, ..., L_s) = L_1^{[N]} + ... L_s^{[N]}$. Let us denote by $PS(s, n; r+1)$ this image. Its algebraic closure is an irreducible affine variety and it is invariant under multiplication by elements of $\mathbb{C}^*$.

Observation 1.3.9. *If we consider the Veronese variety $\nu_n(\mathbb{P}^r)$, $\mathbb{P}^r = \mathbb{P}(S_1)$ and if $s \leq \dim_{\mathbb{C}} T_n = \binom{n+r}{r}$ then for general enough forms $L_1, ..., L_s$ the projectivization of the span $< L^{[N]_1}, ..., L_s^{[N]} >$ is an $(s-1)-$plane that intersects the Veronese variety in the points $\nu_n(< L_i >) =< L_i^{[N]} >$, $i = 1, .., s$. Thus $\mathbb{P}\overline{PS(s, n; r+1)}$ is exactly the $s-$secant variety to the Veronese variety $\sigma_s(\nu_n(\mathbb{P}^r))$.*

Observation 1.3.10. *If $f \in PS(s, n; r+1)$ and $f = L_1^{[N]} + ... + L_s^{[N]}$, then for $e \leq n$ and every $\phi \in T_{n-e}$, we have:*

$$\phi \circ f = \phi(L_1)L_1^{[E]} + ... + \phi(L_s)L_s^{[E]}.$$

This shows that $S_e(f)$ the image of the catalecticant homomorphism $C_f(e, n-e)$ has dimension $\leq s$. Hence the $(s+1) \times (s+1)$ minors of the catalecticant matrices $\mathrm{Cat}_f(e, n-e; r+1)$ vanish on $PS(s, n; r+1)$.

Lemma 1.3.11 (Apolarity Lemma [Iarrobino and Kanev, 1999])**.** *Let $p_1, ..., p_s \in \mathbb{P}^r$, let $L_i = L_{p_i}$, let $P = \{[p_1], ..., [p_s]\} \subset \mathbb{P}^r$ and let $\mathcal{I}_P$ be the homogeneous ideal in T of polynomials vanishing on P. Then:*

i) For every $\phi \in R_e$:

$$\phi \circ (L_1^{[N]} + ... + L_s^{[N]}) = \phi(p_1)L_1^{[N-E]} + ... + \phi(p_s)L_s^{[N-E]}.$$

ii) With respect to the contraction paring $T_n \times S_n \to \mathbb{C}$ one has:

$$((\mathcal{I}_P)_n)^{\perp} =< L_1^{[N]}, ..., L_s^{[N]} > .$$

iii) The points $[p_1], ..., [p_s] \subset \mathbb{P}^r$ impose independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^n}(j)|$ if and only if $L_1^{[N]}, ..., L_s^{[N]}$ are linearly independent.

iv) Suppose $s \leq \dim_{\mathbb{C}} T_{n-e}$ *and the linear forms* $L_1^{[N]}, ..., L_s^{[N]}$ *have the property that the corresponding set* P *imposes independent conditions on the linear system* $|\mathcal{O}_{\mathbb{P}^r}(n-e)|$. *Let* $f = L_1^{[N]} + ... + L_s^{[N]}$. *Then we have for the apolar forms to* f *of degree* e *the equality:*

$$Ann(f)_e = (\mathcal{I}_P)_e.$$

Theorem 1.3.12 ([Landsberg and Teitler, 2010]). *Every* $f \in S_d = Sym^d\mathbb{C}^{r+1}$ *has an additive decomposition of length* s *no longer then* $\binom{r+d}{d} - r$.

Lemma 1.3.13. *If* $L_1, ..., L_s$ *are sufficiently general linear forms* $L_i = \sum_j \beta_{ij}x_j$ *in* S_1, *then* $F = L_1^{[N]} + ... + L_s^{[N]}$ *satisfies,* $H_f = H(s, n; r+1)$. *If* $s \leq min\{\dim_{\mathbb{C}} T_i, \dim_{\mathbb{C}} T_{n-i}\}$ *we have:*

$$T_i \circ f =< L_1^{[N-I]}, ..., L_s^{[N-I]} > .$$

Theorem 1.3.14 (O.Porras). *Let* $n \geq 2$, $1 \leq s \leq r-1$, *the following properties hold:*

i) the variety $V_s(1, n-1; r+1)$ *is normal, Cohen-Macaulay with rational singularities;*

ii) its ideal is generated by the $(s+1) \times (s+1)$ *minors of the catalecticant matrix* $\mathrm{Cat}_f(1, n-1; r+1)$;

iii) the singular locus of $V_s(1, n-1; r+1)$ *equals* $V_{s-1}(1, n-1; r+1)$.

Waring's Problem

The Waring's Problem can be formulated as follow:

Problem 1.3.15 (Waring's Problem). *What is the minimum integer* s *such that a general form of degree* n *in* $r+1$ *variables can be represented as sum of powers as:*

$$f = L_1^n + ... + L_s^n?$$

This was only recently solved by J. Alexander and A.Hirschowitz:

Theorem 1.3.16 ([Alexander and Hirschowitz, 1995]). *Let* $n \geq 3$. *Then we have* $\dim_{\mathbb{C}} PS(s, n; r+1) = \min((r+1)s, \dim_{\mathbb{C}} T_n)$, *except for the four triples* $(s, n, r+1) = (5, 4, 3), (9, 4, 4), (14, 4, 5)$ *and* $(7, 3, 5)$ *where the dimension is one less, equal to* $\dim_{\mathbb{C}} T_n - 1$.

Corollary 1.3.17 (Waring's Problem for General Forms)**.** *Suppose $n \geq 3$, then a sufficiently general homogeneous form $f(x_0, ..., x_r)$ of degree n can be represented as a sum of s powers of linear forms:*

$$f = L_1^n + ... + L_s^n,$$

where $s = \lceil \frac{1}{r+1}\binom{n+r}{r} \rceil$ except the cases $(n, r+1) = (4,3), (4,4), (4,5)$ where we needs $s = 6, 10, 15$ respectively and in the case $(n, r+1) = (3,5)$ where one needs $s = 8$.

1.3.2 Binary Forms

For a binary form $f \in \mathrm{Sym}^n V$ with $\dim_{\mathbb{C}} V = 2$:

$$f = a_0 x_0^n + ... + \binom{n}{d} a_d x_0^{n-d} x_1^d + ... + a_n x_1^n,$$

the relative $e-$th catalecticant matrix is :

$$\mathrm{Cat}_f(e, n-e) := \mathrm{Cat}_f(e, n-e; 2) = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-e-1} & a_{n-e} \\ a_1 & a_2 & \cdots & a_{n-e} & a_{n-e+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{e-1} & a_e & \cdots & a_{n-2} & a_{n-1} \\ a_e & a_{e+1} & \cdots & a_{n-1} & a_n \end{pmatrix}.$$

Observation 1.3.18. *Clearly we have $V_1(1, n-1; 2) = \nu_n(\mathbb{P}^1)$ which is the rational normal curve C_n in $\mathbb{P}^n$.*

Observation 1.3.19. *We observe that if $L \in S_1$, then:*

$$L = \alpha x_0 + \beta x_1 \leftrightarrow L^\perp = -\beta\partial_0 + \alpha\partial_1 \quad L^\perp \circ L = 0.$$

Let L_i be distinct for $i = 1, ..., e$. There are $c_i \in \mathbb{C}$ such that $f = \sum_{i=1}^e c_i L_i^n$ if and only if $(L_1^\perp \circ ... \circ L_e^\perp) f = 0$.

Definition 1.3.20. Let $f \in S_n$ be a binary form of degree n. Let $L_1, ..., L_m$ be a linear forms. A representation of f as a sum:

$$f = G_1 L_1^{n-g_1+1} + ... + G_s L_m^{n-g_m+1}, \tag{1.11}$$

where $G_i \in S_{g_i-1}$, is called a generalized additive decomposition (GAD) of f. A GAD is called normalized if no pair L_α, L_β is proportional to each other and none of the G_i is divisible by L_i. Its length is by definition $\sum_{i=1}^m g_i$.

If all $g_i = 1$ we obtain the classical additive decomposition:

$$f = c_1 L_1^n + \dots + c_s L_s^n, \tag{1.12}$$

with $c_i \in \mathbb{C}$.

The length of a binary form f is the minimum length of a GAD of f, we denote it by $l(f)$.

Lemma 1.3.21 ([Iarrobino and Kanev, 1999])**.** *Let $\phi = \prod_{i=1}^{m}(b_i\partial_0 - a_i\partial_1)^{g_i}$ be a prime decomposition of a nonzero form in T_s. Let $L_i = a_i x_0 + b_i x_1$. Then a form $f \in S_n$ with $n \geq s$ has a GAD as 1.11 if and only if ϕ is apolar to f. If all roots of ϕ are simple, then this is an additive decomposition.*

Lemma 1.3.22 ([Iarrobino and Kanev, 1999])**.** *Let $n = 2t$ or $n = 2t + 1$, let $f \in S_n$. Then $l(f) \leq t + 1$. If $I_f = Ann(f)$ is the ideal of forms apolar to f, then $l(f)$ equals the order d (i.e. the initial degree) of the graded ideal $I_f = (I_f)_d + (I_f)_{d+1} + \dots$.*

Lemma 1.3.23 (Jordan's Lemma)**.** *Suppose that the linear forms $L_i, i = 1, \dots, m$ are not proportional to each other and:*

$$0 = G_1 L_1^{n-g_1+1} + \dots + G_s L_m^{n-g_m+1},$$

with $\sum_{i=1}^{m} g_i \leq n + 1$. Then $G_i = 0$ for every i.

Proposition 1.3.24 (Uniqueness of GAD)**.** *Suppose $n = 2t$ or $n = 2t + 1$. Let:*

$$f = G_1 L_1^{n-g_1+1} + \dots + G_m L_m^{n-g_m+1},$$

be a normalized GAD of $f \in S_n$ of length $s = \sum_{i=1}^{m} g_i \leq t + 1$. Then f has no other GAD of length $\leq n+1-s$ and $l(f) = s$. In particular if $s \leq t$ or if $s = t+1, n = 2t+1$ (equivalently $2s \leq n + 1$), then the above is the unique normalized GAD of f having length $\leq t + 1$.

Definition 1.3.25. Let $f \in S_n$ and let $2l(f) \leq n + 1$. Then the unique normalized GAD of length $s = l(f)$ is called the canonical form of f.

Theorem 1.3.26 (Sylvester)**.**

i) For odd $n = 2t + 1$, the general $f \in S_n$ has a unique decomposition as a sum of $t + 1$ $n-th$ powers of linear forms.

ii) For even $n = 2d$, the general $f \in S_n$ has infinitely many decompositions as a sum of $t + 1$ $n-th$ powers of linear forms.

Theorem 1.3.27. *Let $n = 2t$ or $2t + 1$, let $f \in S_n$.*

i) Let $s = \operatorname{rank} \mathrm{Cat}_f(n - t, t; 2)$. Then $l(f) = s$. If $2s \leq n + 1$, then f has a unique generalized additive decomposition of length s. and no other GADs of length $\leq t + 1$.

ii) For every pair of integers s, e with $1 \leq s \leq e \leq n - e + 1$, if $l(f) = s$, then $l(f) = s = \operatorname{rank} \mathrm{Cat}_f(n - e, e; 2)$.

Theorem 1.3.28 ([Iarrobino and Kanev, 1999])**.** *Let f be a binary form of degree $n = 2t$ or $2t + 1$ and $I_f = Ann(f)$ be the ideal of forms apolar to f. Let $A_f = S/I_f$ be the associated Gorenstein Artin algebra. Let $s = \max\{\dim(A_f)_i\}$. Then:*

i. $s = l(f)$ and the Hilbert function of A_f satisfies:

$$H(A_f) = (1, 2, ..., s - 1, \begin{matrix} s-1 \\ s \end{matrix}, s, ..., \begin{matrix} n-s+1 \\ s \end{matrix}, s - 1, ..., 2, 1);$$

ii. Suppose $2s \leq n + 1$. Then $\dim I_s = 1$, $I_s =< \alpha >$ and for every integer v with $s \leq v \leq n - s + 1$ one has $I_v = S_{v-s} \circ \alpha$;

iii. The apolar ideal I_f is generated by two homogeneous polynomials $\alpha \in (I_f)_s$ and $\beta \in (I_f)_{n+2-s}$.
Equivalently the ring A_f is a complete intersection of generator degrees $s, n+2-s$.
The two polynomials above have no common zeros.

1.3.3 The Grassmannians of secant varieties of curves

We report also a result due to Chiantini and Ciliberto (see [Chiantini and Ciliberto, 2002]) that will be useful in our thesis. It is well known that curves C in $\mathbb{P}^n$ are not defective (see for example [Zak, 1993]) i.e the secant varieties $\sigma_r(C)$ all have the expected dimension $\min\{n, 2r + 1\}$. The start point is the following well known result (see for example [Harris, 1995]):

Proposition 1.3.29. *A smooth, non degenerate $n-$dimensional projective variety $X \subset \mathbb{P}^n$ is projected isomorphically from a point $p \in \mathbb{P}^n$ to $\mathbb{P}^{n-1}$ if and only if p does not belong to the secant variety $\sigma_1(X)$ of X.*

It is natural to consider projection of curves from some linear subspace. In this case if a linear span H of $r + 1$ points of X contains the center of projection, then it determines a $(r+1)-$secant space for the image X' of dimension less then r. In analogy with the theory of secant varieties $\sigma_r(X)$ one may ask about the expected dimension of these Grassmannians of secant varieties.

Definition 1.3.30. Let $C \subset \mathbb{P}^n$ be an irreducible non degenerate curve. We define the secant varieties as:

$$G_r(C) = \overline{\{H \subset \mathbb{P}^n : H \text{ is the span of } r+1 \text{ independent points of } C\}},$$

and

$$\sigma_r = \overline{\{p \in \mathbb{P}^n : p \in H \text{ for some } H \in G_r(C)\}},$$

Since C is irreducible, then $G_r(C)$ is irreducible of dimension $r+1$.

If one considers, in the incidence variety of $Gr(\mathbb{P}^r, \mathbb{P}^n) \times \mathbb{P}^n$ the subsets:

$$I(C) = \{(H,p) : p \in H, H \text{ is spanned by } r+1 \text{ independent points of } C\},$$

then $G_r(C), \sigma_r(C)$ correspond to the closures of the two natural projections $I(C) \to Gr(\mathbb{P}^r, \mathbb{P}^n)$ and $I(C) \to \mathbb{P}^n$. In particular $\dim I(C) = 2r+1$ and the result above says that the map $I(C) \to \mathbb{P}^n$ is generically finite when $2r+1 \leq n$, while otherwise it has general fibers of dimension $2r+1-n$.

Definition 1.3.31. We denote by $G_{s,r}(C)$ the following subset of $Gr(\mathbb{P}^s, \mathbb{P}^n)$:

$$G_{s,r}(C) = \overline{\{h \in Gr(\mathbb{P}^s, \mathbb{P}^n) : h \subset H \text{ for some } H \in G_r(C)\}}.$$

These objects are the Grassmannians of secant varieties of C. Observe that $G_{0,r}(C)$ coincides with $\sigma_r(C)$.

The elements of $G_{s,r}(C)$ are contained in the Grassmannian of $s-$planes of some $k-$plane $H \in G_r(C)$. Thus we always have:

$$\dim G_{s,r}(C) \leq \dim Gr(\mathbb{P}^s, \mathbb{P}^r) + G_r(C) = (s+1)(r-s) + k + 1.$$

Furthermore:

$$\dim G_{s,r}(C) \leq \dim Gr(\mathbb{P}^s, \mathbb{P}^n) = (s+1)(n-s).$$

Definition 1.3.32. We define the expected dimension of $G_{s,r}(C)$ as:

$$expdim(G_{s,r}(C) = \min\{(s+1)(n-s), (s+1)(r-s) + r + 1\}.$$

Chiantini and Ciliberto prove that the actual dimension of $G_{s,r}(C)$ is always equal to the expected one.

Theorem 1.3.33 ([Chiantini and Ciliberto, 2002])**.** *The dimension of $G_{s,r}(C)$ is equal to the expected dimension:*

$$\min\{(s+1)(n-s), (s+1)(r-s)+r+1\}.$$

Chapter 2

Restricted Tangent Bundle of Rational Curves

2.1 Restricted Tangent Bundle of Rational Curves in $\mathbb{P}^3$

Let $C_n \subset \mathbb{P}^n$ be the rational normal curve of degree n with $\nu_n : \mathbb{P}^1 \to \mathbb{P}^n$ and $\nu_n(\mathbb{P}^1) = C_n$, where ν_n is the Veronese map. Let $\pi_{n-3}(C_n)$ the rational curve obtained from C_n by projection from a $(n-4)$-dimensional linear subspace $L \subset \mathbb{P}^n$ on complementary $\mathbb{P}^3 \subset \mathbb{P}^n$, we will suppose that $\pi_{n-3}(C_n)$ has only ordinary singularities. In particular we want to exclude the case of $L \cap C_n \neq \emptyset$, otherwise the rational curve projected has degree one less and the case that L intersects a tangent line, otherwise we have a cusp, but these requests are not very restrictive.

Proposition 2.1.1 ([Harris, 1995],[Zak, 1993]). *Let $\sigma_2(C_n)$ be the Secant Variety of C_n.*

i) If $L \cap (\sigma_2(C_n)) = \emptyset$, then $\pi_{n-3}(C_n)$ is smooth.

ii) If L meets a secant line r, which is secant to C_n in two distinct points $p, q \in r$, then $\pi_{n-3}(p) = \pi_{n-3}(q)$ and $\pi_{n-3}(C_n)$ has a node.

iii) If L meets a tangent line, then $\pi_{n-3}(C_n)$ has a cusp.

The Euler exact sequence (see [Hartshorne, 1977], [Okonek et al., 1980]) on C_n and on $\pi_{n-3}(C_n)$ are respectively :

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\nu_n} \mathcal{O}_{C_n}(n)^{n+1} \xrightarrow{Syz(\nu_n)} \mathcal{O}_{C_n}(n+1)^n \longrightarrow 0, \tag{2.1}$$

(2.2)
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\pi_{n-3}\circ\nu_n} \mathcal{O}_{\pi_{n-3}(C_n)}(n)^4 \xrightarrow{Syz(\pi_{n-3}\circ\nu_n)} T\mathbb{P}^3|_{\pi_{n-3}(C_n)} \longrightarrow 0,$$

where

$$Syz(\nu_n) = \begin{pmatrix} t & -s & 0 & 0 & \dots & \dots & 0 \\ 0 & t & -s & 0 & \dots & \dots & 0 \\ 0 & 0 & t & -s & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & t & -s \end{pmatrix}$$

is a $n \times (n+1)$ matrix.

Let $p_1 = [a_0^1, ..., a_n^1], ..., p_{n-3} = [a_0^1, ..., a_n^1]$ be $n-3$ points which generate L, so we have the following exact sequence:

(2.3)
$$0 \longrightarrow \mathcal{O}_{C_n}(n)^{n-3} \xrightarrow{P} \mathcal{O}_{C_n}(n)^{n+1} \longrightarrow \mathcal{O}_{C_n}(n)^4 \longrightarrow 0,$$

where:

$$P = \left[\begin{array}{ccc} p_1^t & \dots & p_{n-3}^t \end{array} \right].$$

Therefore we have:

(2.4)
$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_{C_n}^{n-3}(n) \arrow[d, "P"] & & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\nu_n"] & \mathcal{O}_{C_n}(n)^{n+1} \arrow[r, "Syz(\nu_n)"] \arrow[d] & \mathcal{O}_{C_n}(n+1)^n \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\pi_{n-3}\circ\nu_n"] & \mathcal{O}_{\pi_{n-3}(C_n)}(n)^4 \arrow[r] \arrow[d] & T\mathbb{P}^3|_{\pi_{n-3}(C_n)} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}.$$

By the above diagram we can obtain:

(2.5)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}^{n-3}_{C_n}(n) \arrow[r, "\cong"] \arrow[d, "P"] & \mathcal{O}^{n-3}_{C_n}(n) \arrow[d, "(\mathcal{T}^L_{n,n-3})^t"] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\nu_n"] & \mathcal{O}_{C_n}(n)^{n+1} \arrow[r, "Syz(\nu_n)"] \arrow[d] & \mathcal{O}_{C_n}(n+1)^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\pi_{n-3}\circ\nu_n"] & \mathcal{O}_{\pi_{n-3}(C_n)}(n)^4 \arrow[r] \arrow[d] & T\mathbb{P}^3|_{\pi_{n-3}(C_n)} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & ,
\end{tikzcd}$$

where the map $(\mathcal{T}^L_{n,n-3})^t$ is:

$$(\mathcal{T}^L_{n,n-3})^t = Syz(\nu_n) \cdot \begin{pmatrix} a_0^1 & \dots & a_0^{n-3} \\ \vdots & \ddots & \vdots \\ a_n^1 & \dots & a_n^{n-3} \end{pmatrix} =$$

$$\begin{pmatrix} a_0^1 t - a_1^1 s & a_1^1 t - a_2^1 s & \dots & a_{n-1}^1 t - a_n^1 s \\ \vdots & \vdots & \ddots & \vdots \\ a_0^{n-3} t - a_1^{n-3} s & a_1^{n-3} t - a_2^{n-3} s & \dots & a_{n-1}^{n-3} t - a_n^{n-3} s \end{pmatrix}^t .$$

It is a $n \times (n-3)$ matrix. The last exact column of (2.5):

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n)^{n-3} \xrightarrow{(\mathcal{T}^L_{n,n-3})^t} \mathcal{O}_{\pi_{n-3}(C_n)}(n+1)^n \longrightarrow T\mathbb{P}^3|_{\pi_{n-3}(C_n)} \longrightarrow 0 \qquad (2.6)$$

gives us some information on the splitting type of $T\mathbb{P}^3|_{\pi_{n-3}(C_n)}$:

Lemma 2.1.2. *If $\pi_{n-3}(C_n)$ has only ordinary singularities, then the splitting type of $T\mathbb{P}^3|_{\pi_{n-3}(C_n)}$ must be (t_1, t_2, t_3) with $n+1 \leq t_1 \leq t_2 \leq t_3 \leq 2n-2$ and $t_1+t_2+t_3 = 4\cdot n$.*

Moreover the exact sequence 2.6 gives rise by duality and tensorizing by $\mathcal{O}_{\mathbb{P}^1}(n+1)$:

$$0 \longrightarrow (T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1) \longrightarrow \mathcal{O}^n_{\pi_{n-3}(C_n)} \xrightarrow{\mathcal{T}^L_{n,n-3}} \mathcal{O}_{\mathbb{P}^1}(1)^{n-3} \longrightarrow 0\,. \qquad (2.7)$$

If we pass to the exact cohomology sequence we get:

$$0 \longrightarrow H^0((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1))) \longrightarrow H^0(\mathcal{O}^n_{\pi_{n-3}(C_n)}) \xrightarrow{T^L_{n,n-3}} H^0(\mathcal{O}_{\mathbb{P}^1}(1)^{n-3}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1))) \longrightarrow 0\ ,$$

where we have indicated with $T^L_{n,n-3}$ the $2(n-3)\times n$ matrix:

$$T^L_{n,n-3} = \begin{pmatrix} a^1_0 & \dots & a^1_{n-1} \\ -a^1_1 & \dots & -a^1_n \\ \vdots & \ddots & \vdots \\ a^{n-3}_0 & \dots & a^{n-3}_{n-1} \\ -a^{n-3}_1 & \dots & -a^{n-3}_n \end{pmatrix}.$$

Observation 2.1.3. *The rank of $T^L_{n,n-3}$ does not depend from the points generating L.*

Observation 2.1.4. *Let F_{p_i} be the binary form of degree n which corresponds to the point $p_i \in \mathbb{P}^n$. If we indicate with $Cat_{F_{p_i}}(1, n-1; 2)$ the Hankel matrix $2\times n$ corresponding to F_{p_i}, then the matrix $T^L_{n,n-3}$ has rank:*

$$\operatorname{rank} T^L_{n,n-3} = \operatorname{rank} \begin{pmatrix} Cat_{F_{p_1}}(1, n-1; 2) \\ \vdots \\ Cat_{F_{p_{n-3}}}(1, n-1; 2) \end{pmatrix}.$$

Therefore we have that:

$$\operatorname{rank} T^L_{n,n-3} \geq 2,$$

otherwise each point belongs to C_n, which is impossible for our assumption on allowable singularities.

Observation 2.1.5. *We have that* $\deg((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1)) = -n+3$ *and*

$$h^0((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1)) = n - \operatorname{rank}(T^L_{n,n-3}) = \dim\ker(T^L_{n,n-3}).$$

Therefore we have:

$$2 \leq \operatorname{rank}(T^L_{n,n-3}) \leq \min\{n, 2(n-3)\},$$

so

$$n - \min\{n, 2(n-3)\} \leq h^0((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1)) \leq n-2,$$

but as $\operatorname{rank}((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1)) = 3$ *we have that*

$$(T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1) \textit{ splits in } \mathcal{O}(t'_1)\oplus\mathcal{O}(t'_2)\oplus\mathcal{O}(t'_3)$$

by Grothendieck-Segre's theorem (see [Grothendieck, 1957]) with

$$-n+3 \leq t'_1 \leq t'_2 \leq t'_3 \leq 0 \textit{ and } t'_1 + t'_2 + t'_3 = -n+3.$$

So we have that:

$$\operatorname{rank} T^L_{n,n-3} \geq n-2,$$

otherwise some t'_i should be ≥ 1, which is impossible as seen above.

Remark 2.1.6. It's clear from the above consideration that the value of $\operatorname{rank}(T^L_{n,n-3})$ corresponds to some splitting types of $T\mathbb{P}^3|_{\pi_{n-3}(C_n)}$.

We can distinguish two cases:

A) $\min\{n, 2(n-3)\} = 2(n-3)$, so $n = 3, 4, 5, 6$;

B) $\min\{n, 2(n-3)\} = n$, so $n \geq 6$.

These two cases will be the subject of our studies in the next two sections.

2.1.1 Case $3 \leq n \leq 6$

Rational Curves of degree 3 in $\mathbb{P}^3$

Proposition 2.1.7. *The restricted tangent bundle of rational normal curve C_3 of degree 3 is $(T\mathbb{P}^3|_{C_3}) = \mathcal{O}(4) \oplus \mathcal{O}(4) \oplus \mathcal{O}(4)$.*

Proof. Immediate from above. □

Rational Curves of degree 4 in $\mathbb{P}^3$

We observe that $\operatorname{rank} T^p_{4,1} = \operatorname{rank} Cat_{F_p}(1,3)$ where $p \in \mathbb{P}^4$ is the centre of projection π_1, therefore we have:

Observation 2.1.8. $\operatorname{rank}(T^p_{4,1}) = 2$ *if and only if $p \notin C_4$.*

Proposition 2.1.9. *$p \notin C_4$ if and only if $(T\mathbb{P}^3|_{\pi_1(C_4)})^\vee = \mathcal{O}(6) \oplus \mathcal{O}(5) \oplus \mathcal{O}(5)$.*

Proof. Immediate from above. □

Rational Curves of degree 5 in $\mathbb{P}^3$

In the case of $n = 5$ we have:

$$T^L_{5,2} = \begin{pmatrix} a^1_0 & \dots & a^1_4 \\ -a^1_1 & \dots & -a^1_5 \\ a^2_0 & \dots & a^2_4 \\ -a^2_1 & \dots & -a^2_5 \end{pmatrix}.$$

so $h^0((T\mathbb{P}^3|_{\pi_2(C_5)})^\vee(6)) \geq 1$.

We can analyze every possible value of rank $T^L_{5,2}$:

i) If $\mathrm{rank}(T^L_{5,2}) = 4$ we have that $(T\mathbb{P}^3|_{\pi_2(C_5)})^\vee(6) = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}$.

ii) If $\mathrm{rank}(T^L_{5,2}) = 3$ there are two possibilities or $(T\mathbb{P}^3|_{\pi_2(C_5)})^\vee(6) = \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}$ or $(T\mathbb{P}^3|_{\pi_2(C_5)})^\vee(6) = \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1)$, the last one is impossible by Lemma 2.1.2.

iii) Finally we have that $(T\mathbb{P}^3|_{\pi_2(C_5)})^\vee(6) = \mathcal{O}(-3) \oplus \mathcal{O} \oplus \mathcal{O}(1)$ if and only if $\mathrm{rank}(T_{5,2}) = 2$, but this is impossible by Lemma 2.1.2.

Proposition 2.1.10. *Let $\pi_2(C_5)$ be a space rational curve of degree 5 with only ordinary singularities:*

i) $\mathrm{rank}(T^L_{5,2}) = 4$ *if and only if* $T\mathbb{P}^3|_{\pi_2(C_5)} = \mathcal{O}(7) \oplus \mathcal{O}(7) \oplus \mathcal{O}(6)$, *this case is the general one;*

ii) $\mathrm{rank}(T^L_{5,2}) = 3$ *if and only if* $T\mathbb{P}^3|_{\pi_2(C_5)} = \mathcal{O}(8) \oplus \mathcal{O}(6) \oplus \mathcal{O}(6)$.

Lemma 2.1.11. *If the centre of projection $L \cong \mathbb{P}^1$ belongs to a pencil of 4−secant $\mathbb{P}^3$ to the rational normal curve C_5 in $\mathbb{P}^5$, then* $\mathrm{rank}\, T^L_{5,2} = 3$. *The coverse is true generically*

.

Proof. $\Rightarrow$ Let L be a line belongs to a pencil $\Phi = \{\pi_\lambda \cong \mathbb{P}^3 : \pi_\lambda = \lambda_0\pi_0 + \lambda_1\pi_1,\ \ \forall\lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1\}$ of 4−secant $\mathbb{P}^3$ to the rational normal curve C_5 in $\mathbb{P}^5$, where π_0, π_1 are two 4−secant $\mathbb{P}^3$. Let $q^i_1, ..., q^i_4 \in C_5$ be the points which generate π_i. Then there exist two points p_1, p_2 which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^1$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{4,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c^i_{1,\lambda} L^5_{1,\lambda} + ... + c^i_{4,\lambda} L^5_{4,\lambda},$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., 4$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^1$ there exists a differential form $\phi_\lambda \in T_4$ such that $\phi_\lambda \circ f_i = 0$. Therefore there exist two differential forms $\phi_1, \phi_2 \in T_4$ which for each $\lambda \in \mathbb{P}^1$ we have $\phi_\lambda = \lambda_0\phi_1 + \lambda_1\phi_2$ and in particular $\phi_1 \circ f_i = 0 = \phi_2 \circ f_i$, so $\mathrm{rank}\, T^L_{5,2} = 3$.

$\Leftarrow$ If rank $T^L_{5,2} = 3$, then there exist two binary form $\phi_1, \phi_2 \in T_4$ such that however we consider two generating points $p_1, p_2 \in \mathbb{P}^5$ of L it is $\phi_\alpha \circ f_i = (\alpha_0\phi_1 + \alpha_1\phi_2) \circ f_i = 0$ for all $\alpha = [\alpha_0, \alpha_1] \in \mathbb{P}^1$ and $i = 1, 2$, where $f_i \in S_5$ is the binary form corresponding to p_i. If we consider the primary decomposition of $\phi_\alpha = \prod_{l=1}^4 \phi^l_\alpha$ and we indicate with $(L_{l,\alpha})^\perp = \phi^l_\alpha$, then f_1, f_2 can be decomposed in ∞^1 different simultaneously ways, i.e.:

$$f_i = c^i_{1,\alpha} L^5_{1,\alpha} + \ldots + c^i_{4,\alpha} L^5_{4,\alpha},$$

for all $\alpha = [\alpha_0, \alpha_1] \in \mathbb{P}^1$ or in other words L belongs to a pencil $\Phi = \{\pi_\lambda \cong \mathbb{P}^3 : \pi_\lambda = \lambda_0\pi_0 + \lambda_1\pi_1, \quad \forall\lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1\}$ of 4−secant $\mathbb{P}^3$ to the rational normal curve C_5 in $\mathbb{P}^5$. Clearly this is true generically, it can happen that some ϕ_i which generate the linear system have multiple roots, so we have all possible degeneration of that linear system.

□

Lemma 2.1.12. *The variety of lines L that belong to a pencil is an irreducible variety of codimension 2 in* $Gr(\mathbb{P}^1, \mathbb{P}^5)$.

Proof. We can observe that a pencil of 4−secant $\mathbb{P}^3$ to the rational normal curve C_5 in $\mathbb{P}^5$ corresponds to a linear system of dimension two of quartic binary forms, therefore the set of these pencils corresponds to $Gr(\mathbb{P}^1, \mathbb{P}^4)$ which is irreducible and $\dim Gr(\mathbb{P}^1, \mathbb{P}^4) = 6$. Each projection line L belongs to one and only one of these pencils, so the variety of lines L that belong to a pencil is an irreducible variety of codimension 2 in $Gr(\mathbb{P}^1, \mathbb{P}^5)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

Observation 2.1.13. *By Theorem 1.2.14 (see [Verdier, 1983])* $T^5_3(6, 6, 8)$ *is an irreducible variety of* $\mathrm{codim}(T^5_3(6, 6, 8)) = 2$.

By Theorem 1.2.14[Verdier, 1983] and the above lemmas we have:

Theorem 2.1.14. *The centre of projection* $L \cong \mathbb{P}^1$ *belongs to a pencil of* 4−*secant* $\mathbb{P}^3$ *to the rational normal curve* C_5 *in* $\mathbb{P}^5$ *if and only if :*

$$T\mathbb{P}^3|_{\pi_2(C_5)} \cong \mathcal{O}(6)^2 \oplus \mathcal{O}(8).$$

Corollary 2.1.15. *The variety of lines L that, as centre of projection, give a rational curve of degree* 5 *in* $\mathbb{P}^3$ *which has* $T\mathbb{P}^3_{\pi_2(C_5),\mathbb{P}^3} \cong \mathcal{O}(6) \oplus \mathcal{O}(6) \oplus \mathcal{O}(8)$ *is an irreducible variety of codimension* 2 *in* $Gr(\mathbb{P}^1, \mathbb{P}^5)$ *formed by the lines L belonging to a pencil of* 4−*secant* $\mathbb{P}^3$ *to the rational normal curve* C_5 *in* $\mathbb{P}^5$.

Rational Curves of degree 6 in $\mathbb{P}^3$

Proposition 2.1.16. *Let $\pi_3(C_6)$ be a space rational curve of degree 6 with only ordinary singularities:*

i) $\operatorname{rank}(T^L_{6,3}) = 6$ *if and only if* $T\mathbb{P}^3|_{\pi_3(C_6)} = \mathcal{O}(8) \oplus \mathcal{O}(8) \oplus \mathcal{O}(8)$, *this case is the generic one;*

ii) $\operatorname{rank}(T^L_{6,3}) = 5$ *if and only if* $T\mathbb{P}^3|_{\pi_3(C_6)} = \mathcal{O}(9) \oplus \mathcal{O}(8) \oplus \mathcal{O}(7)$.

iii) $\operatorname{rank}(T^L_{6,3}) = 4$ *if and only if* $T\mathbb{P}^3|_{\pi_3(C_6)} = \mathcal{O}(10) \oplus \mathcal{O}(7) \oplus \mathcal{O}(7)$.

Case $\operatorname{rank} T^L_{6,3} = 5$

Lemma 2.1.17. $\operatorname{rank} T^L_{6,3} = 5$ *if and only if the forms f_i of degree n corresponding to the points p_i generating L can be represented by similar GAD, i.e. :*

$$f_i = G_{i_1} L_1^{6-g_1+1} + ... + G_{i_m} L_m^{6-g_m+1}.$$

Proof. $\Rightarrow$ If $\operatorname{rank} T^L_{6,3} = 5$, then there exists an element $\phi \in T_5$ such that for all forms f_i corresponding to the points p_i generating L we have $\phi \circ f_i = 0$. So we can consider the primary decomposition of $\phi = \prod_{i=1}^m (\phi_i)^{g_i}$, with $\phi_i \in T_1$ and $\sum_i g_i = 5$, so every f_i can be represented by the similar GAD, i.e. :

$$f_i = G_{i_1} L_1^{6-g_1+1} + ... + G_{i_m} L_m^{6-g_m+1},$$

where $(L_j)^\perp = \phi_j$ for all $j = 1, .., m$ and $G_{i_j} \in S_{g_j-1}$ for all $i = 1, 2, 3$ and $j = 1, ..., m$.

$\Leftarrow$ On the other hand if every f_i can be represented by the similar GAD, i.e. :

$$f_i = G_{i_1} L_1^{6-g_1+1} + ... + G_{i_m} L_m^{6-g_m+1},$$

then we can consider $\phi = \prod_{i=1}^m ((L_i)^\perp)^{g_i}$. By definition of GAD representation we have $\phi \circ f_i = 0$ for all $i = 1, 2, 3$, so $\phi \in \ker T^L_{6,3}$ and $\operatorname{rank} T^L_{6,3} = 5$.

□

Observation 2.1.18. *We can observe that if we fix $L_1, ..., L_5 \in S_1$ and for $1 \leq m \leq 5$ with $\sum_{i=1}^m g_i = 5$, then the spaces:*

$$\{G_1 L_1^{6-g_1+1} + ... + G_m L_m^{6-g_m+1} \text{ for all } G_1 \in S_{g_1-1}, ..., G_m \in S_{g_m-1}\}$$

are all possible degenerations of a 5−secant $\mathbb{P}^4$ to vary m and g_m.

Lemma 2.1.19. *The codimension in $Gr(\mathbb{P}^5, \mathbb{P}^6)$ of the variety of all $L \cong \mathbb{P}^2$ in $\mathbb{P}^6$ belonging to some* 5*-secant $\mathbb{P}^4$ to the rational normal curve in $\mathbb{P}^6$ is* 1*.*

Proof. We can consider the incidence variety $I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^2, \mathbb{P}^6), \pi \in S, L \subset S\}$ where S is the set of all 5-secant $\mathbb{P}^4$ to the rational normal curve in $\mathbb{P}^6$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^2, \mathbb{P}^6)$. We will indicated with ϕ_1 and ϕ_2 the natural projections:

$$\begin{tikzcd} & I_S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ Gr(\mathbb{P}^2, \mathbb{P}^6) & & S, \end{tikzcd}$$

so the codimension in $Gr(\mathbb{P}^2, \mathbb{P}^6)$ of $\phi_1(I_S)$ is equal to $\dim Gr(\mathbb{P}^2, \mathbb{P}^6) - \dim S - \dim \phi_2^{-1}(S) = 3(7-3) - 5 - 3(5-3) = 1$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

Observation 2.1.20. *By Theorem 1.2.14 (see [Verdier, 1983]) $T_3^6(7, 8, 9)$ is an irreducible variety of* $\operatorname{codim}(T_3^6(7, 8, 9)) = 1$.

By Theorem 1.2.14[Verdier, 1983] and the above lemmas we have:

Theorem 2.1.21. *The centre of projection $L \cong \mathbb{P}^2$ belongs to some $5-$secant $\mathbb{P}^4$ to the rational normal curve C_6 in $\mathbb{P}^6$ if and only if:*

$$T\mathbb{P}^3|_{\pi_3(C_6)} \cong \mathcal{O}(7) \oplus \mathcal{O}(8) \oplus \mathcal{O}(9).$$

Corollary 2.1.22. *The variety of linear spaces $L \cong \mathbb{P}^2$ that, as centre of projection, give a rational curve of degree* 6 *in $\mathbb{P}^3$ which has the restricted tangent bundle $T\mathbb{P}^3_{\pi_3(C_6), \mathbb{P}^3} \cong \mathcal{O}(7) \oplus \mathcal{O}(8) \oplus \mathcal{O}(9)$ is an irreducible variety of codimension* 1 *in $Gr(\mathbb{P}^2, \mathbb{P}^6)$ formed by the linear spaces L belong to some $4-$secant $\mathbb{P}^3$.*

Case $\operatorname{rank} T^L_{6,3} = 4$

Lemma 2.1.23. *If the centre of projection $L \cong \mathbb{P}^2$ belongs to some $4-$secant $\mathbb{P}^3$ to the rational normal curve C_6 in $\mathbb{P}^6$, then we have* $\operatorname{rank} T^L_{6,3} = 4$.

Proof. If $L \cong \mathbb{P}^2$ belongs to some $4-$secant $\mathbb{P}^3$, then there exist 3 points $p_1, p_2, p_3 \in L$ which generate L and the corresponding binary forms f_i have the apolar ideal respectively $Ann(f_i) = (\alpha, \beta_i)$ with $\deg(\alpha) = 4$, α has only simple roots and $\deg(\beta_i) = 4$ without common zeros with α. We have $\dim(\alpha)_{n-1} = 2$ and $\dim \bigcap_i (\alpha, \beta_i)_5 \geq 2$, so $\operatorname{rank} T^L_{6,3} = 6 - \dim \bigcap_i (\alpha, \beta_i)_5 \leq 4$. □

Lemma 2.1.24. *The codimension in $Gr(\mathbb{P}^2, \mathbb{P}^6)$ of the variety of all $L \cong \mathbb{P}^2$ in $\mathbb{P}^6$ belonging to some 4-secant $\mathbb{P}^3$ to the rational normal curve in $\mathbb{P}^6$ is 5.*

Proof. We can consider the incidence variety $I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^2, \mathbb{P}^6), \pi \in S, L \subset S\}$ where S is the set of all 4-secant $\mathbb{P}^3$ to the rational normal curve in $\mathbb{P}^6$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^2, \mathbb{P}^6)$. We will indicated with ϕ_1 and ϕ_2 the natural projections:

$$\begin{tikzcd} & I_S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ Gr(\mathbb{P}^2, \mathbb{P}^6) & & S, \end{tikzcd}$$

so the codimension in $Gr(\mathbb{P}^2, \mathbb{P}^6)$ of $\phi_1(I_S)$ is equal to $\dim Gr(\mathbb{P}^2, \mathbb{P}^6) - \dim S - \dim \phi_2^{-1}(S) = 3(7-3) - 4 - 3(4-3) = 5$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

Observation 2.1.25. *By Theorem 1.2.14 (see [Verdier, 1983]) $T_3^6((7)^2, 10)$ is an irreducible variety of* $\operatorname{codim}(T_3^6((7)^2, 10)) = 4$.

Theorem 2.1.26. *If the centre of projection $L \cong \mathbb{P}^2$ belongs to some 4−secant $\mathbb{P}^3$ to the rational normal curve C_6 in $\mathbb{P}^6$, then:*

$$T\mathbb{P}^3|_{\pi_3(C_6)} \cong \mathcal{O}(7) \oplus \mathcal{O}(7) \oplus \mathcal{O}(10).$$

Corollary 2.1.27. *The variety of linear spaces $L \cong \mathbb{P}^2$ that, as centre of projection, give a rational curve of degree 6 in $\mathbb{P}^3$ which has the restricted tangent bundle $T\mathbb{P}^3_{\pi_3(C_6), \mathbb{P}^3} \cong \mathcal{O}(7)^2 \oplus \mathcal{O}(10)$ has an irreducible subvariety of codimension 5 in $Gr(\mathbb{P}^2, \mathbb{P}^6)$ formed by the linear spaces L belong to some 4−secant $\mathbb{P}^3$.*

Lemma 2.1.28. *If the centre of projection $L \cong \mathbb{P}^2$ belongs to a pencil of 5−secant $\mathbb{P}^4$ to the rational normal curve C_6 in $\mathbb{P}^6$, then* $\operatorname{rank} T_{6,3}^L = 4$. *The converse is true generically.*

Proof. $\Rightarrow$ Let L be a $\mathbb{P}^2$ belongs to a pencil $\Phi = \{\pi_\lambda \cong \mathbb{P}^4 : \pi_\lambda = \lambda_0\pi_0 + \lambda_1\pi_1, \ \forall \lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1\}$ of 5−secant $\mathbb{P}^3$ to the rational normal curve C_6 in $\mathbb{P}^6$, where π_0, π_1 are 5−secant $\mathbb{P}^4$. Let $q_1^i, ..., q_5^i \in C_6$ be the points which generate π_i. Then there exist three points p_1, p_2, p_3 which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^1$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{5,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c^i_{1,\lambda} L^6_{1,\lambda} + ... + c^i_{5,\lambda} L^6_{5,\lambda},$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., 5$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^1$ there exists a differential forms $\phi_\lambda \in T_5$ such that $\phi_\lambda \circ f_i = 0$. Moreover there exist two differential forms $\phi_0, \phi_1 \in T_5$ which for each $\lambda \in \mathbb{P}^1$ we have $\phi_\lambda = \lambda_0\phi_0 + \lambda_1\phi_1$, in particular $\phi_j \circ f_i = 0$ for all $j = 0, 1$ and $i = 1, ..., 3$, so $\operatorname{rank} T^L_{6,3} = 4$.

$\Leftarrow$ If $\operatorname{rank} T^L_{6,3} = 4$, then there exist two binary form $\phi_1, \phi_2 \in T_5$ such that however we consider two generating points $p_1, p_2 \in \mathbb{P}^5$ of L it is $\phi_\alpha \circ f_i = (\alpha_0\phi_1 + \alpha_1\phi_2) \circ f_i = 0$ for all $\alpha = [\alpha_0, \alpha_1] \in \mathbb{P}^1$ and $i = 1, 2$, where $f_i \in S_6$ is the binary form corresponding to p_i. If we consider the primary decomposition of $\phi_\alpha = \prod_{l=1}^5 \phi^l_\alpha$ and we indicate with $(L_{l,\alpha})^\perp = \phi^l_\alpha$, then f_1, f_2 can be decomposed in ∞^1 different simultaneously ways, i.e.:

$$f_i = c^i_{1,\alpha} L^6_{1,\alpha} + ... + c^i_{5,\alpha} L^6_{5,\alpha},$$

for all $\alpha = [\alpha_0, \alpha_1] \in \mathbb{P}^1$ or in other words L belongs to a pencil $\Phi = \{\pi_\lambda \cong \mathbb{P}^3 : \pi_\lambda = \lambda_0\pi_0 + \lambda_1\pi_1, \quad \forall\lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1\}$ of $5-$secant $\mathbb{P}^4$ to the rational normal curve C_6 in $\mathbb{P}^6$. Clearly this is true generically, it can happen that some ϕ_i which generate the linear system have multiple roots, so we have all possible degenerations of that linear system.

$\square$

Lemma 2.1.29. *The variety of planes $L \cong \mathbb{P}^2$ that belong to a pencil Φ of $5-$secant $\mathbb{P}^4$ to the rational normal curve C_6 in $\mathbb{P}^6$ is an irreducible variety of codimension 4.*

Proof. We can observe that the pencil Φ of $5-$secant $\mathbb{P}^4$ to the rational normal curve C_6 in $\mathbb{P}^6$ correspond to the linear system of dimension 2 of binary forms of degree 5, therefore the set of these linear system corresponds to $Gr(\mathbb{P}^1, \mathbb{P}^5)$ which is irreducible and $\dim Gr(\mathbb{P}^1, \mathbb{P}^5) = 8$. Each centre of projection L belongs to a one and only one pencil therefore the variety of planes L that belong to a pencil Φ is an irreducible variety of codimension 4 in $Gr(\mathbb{P}^2, \mathbb{P}^6)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). $\square$

Theorem 2.1.30. *The centre of projection $L \cong \mathbb{P}^2$ belongs to a pencil Φ of $5-$secant $\mathbb{P}^4$ to the rational normal curve C_6 in $\mathbb{P}^6$ if and only if $T\mathbb{P}^3|_{\pi_3(C_6)} \cong \mathcal{O}(7)^2 \oplus \mathcal{O}(10)$.*

Corollary 2.1.31. *$T_3^6(7,7,10)$ is an irreducible variety of codimension 4 formed by the planes $L \cong \mathbb{P}^2$ that belong to a pencil Φ of $5-$secant $\mathbb{P}^4$ to the rational normal curve C_6 in $\mathbb{P}^6$.*

2.1.2 Case $n > 6$

Lemma 2.1.32. *We have only three different possibilities:*

a) $\mathrm{rank}(T^L_{n,n-3}) = n$ *if and only if*

$$T\mathbb{P}^3|_{\pi_{n-3}(C_n)} = \mathcal{O}(t_1) \oplus \mathcal{O}(t_2) \oplus \mathcal{O}(t_3),$$

where $n+2 \leq t_1 \leq t_2 \leq t_3 \leq 2n-4$ and $t_1+t_2+t_3 = 4n$.

b) $\mathrm{rank}(T^L_{n,n-3}) = n-1$ *if and only if*

$$T\mathbb{P}^3|_{\pi_{n-3}(C_n)} = \mathcal{O}(t_1) \oplus \mathcal{O}(t_2) \oplus \mathcal{O}(n+1),$$

where $n+2 \leq t_1 \leq t_2 \leq 2n-3$ and $t_1+t_2 = 3n-1$.

c) $\mathrm{rank}(T^L_{n,n-3}) = n-2$ *if and only if*

$$T\mathbb{P}^3|_{\pi_{n-3}(C_n)} = \mathcal{O}(2n-2) \oplus \mathcal{O}(n+1) \oplus \mathcal{O}(n+1).$$

No other possibilities can happen by conditions on t_i (see Lemma 2.1.2, Observation 2.1.5).

Unfortunately the *a*) and *b*) cases are formed by several possible splitting types, but we can study the rank of the following maps in order to discriminate the exact splitting:

$$0 \longrightarrow H^0((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+2))) \longrightarrow H^0(\mathcal{O}_{\pi_{n-3}(C_n)}(1)^n) \xrightarrow{T^L_{n,n-3}(1)} H^0(\mathcal{O}_{\mathbb{P}^1}(2)^{n-3}) \longrightarrow \cdots$$

$$\cdots \qquad \longrightarrow H^1((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+2))) \longrightarrow 0\ ,$$

where we have indicated with $T^L_{n,n-3}(1)$ the $3(n-3)\times 2n$ matrix:

$$T^L_{n,n-3}(1)=\begin{pmatrix}
a_0^1 & \dots & a_{n-1}^1 & 0 & \dots & 0\\
-a_1^1 & \dots & -a_n^1 & a_0^1 & \dots & a_{n-1}^1\\
0 & \dots & 0 & -a_1^1 & \dots & -a_n^1\\
\vdots & \ddots & \vdots & & & \\
a_0^{n-3} & \dots & a_{n-1}^{n-3} & 0 & \dots & 0\\
-a_1^{n-3} & \dots & -a_n^{n-3} & a_0^{n-3} & \dots & a_{n-1}^{n-3}\\
0 & \dots & 0 & -a_1^{n-3} & \dots & -a_n^{n-3}
\end{pmatrix},$$

and so on:

$$0 \longrightarrow H^0((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1+d))) \longrightarrow H^0(\mathcal{O}_{\pi_{n-3}(C_n)}(d)^n) \xrightarrow{T^L_{n,n-3}(d)} H^0(\mathcal{O}_{\mathbb{P}^1}(d+1)^{n-3}) \longrightarrow \cdots$$
$$\cdots \qquad \longrightarrow H^1((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1+d))) \longrightarrow 0\ .$$

Where we have indicate with:

$$T^L_{n,n-3}(d)=\begin{pmatrix} T^{p_1}_{n,1}(d)\\ \vdots \\ T^{p_{n-3}}_{n,1}(d)\end{pmatrix},$$

the matrix of dimensions $(d+2)(n-3)\times n(d+1)$ form by the following matrices of dimensions $(d+2)\times n(d+1)$:

$$(T^{p_i}_{n,n-3}(d))^t=\left.\begin{pmatrix}
(T^{p_i}_{n,1})^t & & \overbrace{\begin{matrix}0&0&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{matrix}}^{d} \\
\begin{matrix}0\\ \vdots \\ 0\end{matrix} & (T^{p_i}_{n,1})^t & \begin{matrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\0&\cdots&0\end{matrix}\\
\begin{matrix}0&\cdots&\cdots&\cdots\\ \vdots&\vdots&\vdots&\vdots\\ 0&\cdots&\cdots&\cdots\end{matrix} & & \begin{matrix}\cdots&\cdots&\cdots&0\\ \vdots&\vdots&\vdots&\vdots\\ \cdots&\cdots&\cdots&0\end{matrix}\\
\begin{matrix}0&\cdots&0&0\\ \vdots&\ddots&\vdots&\vdots\\0&\cdots&0&0\end{matrix} & & (T^{p_i}_{n,1})^t
\end{pmatrix}\right\}n(d+1)$$

Observation 2.1.33. *We can observe that:*

$$
\operatorname{rank} T^{p_i}_{n,1}(d) = \operatorname{rank}\left.\left(\begin{array}{cccccccc}
 & & & & \multicolumn{4}{c}{\overbrace{\hphantom{0\quad 0\quad \cdots\quad 0}}^{d}} \\
 & & & & 0 & 0 & \cdots & 0 \\
\multicolumn{4}{c}{Cat_{p_i}(n-1,1)} & \vdots & \vdots & \ddots & \vdots \\
 & & & & 0 & 0 & \cdots & 0 \\
0 & & & & & 0 & \cdots & 0 \\
\vdots & \multicolumn{3}{c}{Cat_{p_i}(n-1,1)} & & \vdots & \ddots & \vdots \\
0 & & & & & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & & & & \\
\vdots & \ddots & \vdots & \vdots & \multicolumn{4}{c}{Cat_{p_i}(n-1,1)} \\
0 & \cdots & 0 & 0 & & & &
\end{array}\right)\right\} n(d+1)
$$

2.1.3 Rational Curves of degree 7 in $\mathbb{P}^3$

Lemma 2.1.34. *a)* $\operatorname{rank}(T^L_{7,4}) = 7$ *if and only if* $T\mathbb{P}^3|_{\pi_4(C_7)} = \mathcal{O}(9) \oplus \mathcal{O}(9) \oplus \mathcal{O}(10)$;

b) $\operatorname{rank}(T^L_{7,4}) = 6$ *if and only if*

$$T\mathbb{P}^3|_{\pi_4(C_7)} = \mathcal{O}(8) \oplus \mathcal{O}(t_1) \oplus \mathcal{O}(t_2),$$

where $9 \leq t_1 \leq t_2 \leq 11$ *and* $t_1 + t_2 = 20$.

c) $\operatorname{rank}(T^L_{7,4}) = 5$ *if and only if*

$$T\mathbb{P}^3|_{\pi_4(C_7)} = \mathcal{O}(8) \oplus \mathcal{O}(8) \oplus \mathcal{O}(12).$$

No other possibilities can happen by conditions on t_i (see Lemma 2.1.2, Observation 2.1.5).

So case *b*) is formed by two possible splitting types :

b.1) $(8, 10, 10)$;

b.2) $(8, 9, 11)$.

To distinguish what it is we have to look at the following exact cohomology sequence and its relative map $T^L_{7,4}(1)$ (14×12 matrix):

$$0 \longrightarrow H^0((T\mathbb{P}^3|_{\pi_4(C_7)})^\vee(9))) \longrightarrow H^0(\mathcal{O}_{\pi_4(C_7)}(1)^7) \xrightarrow{T^L_{7,4}(1)} H^0(\mathcal{O}_{\mathbb{P}^1}(2)^4) \longrightarrow \cdots$$

$$\cdots \qquad \longrightarrow H^1((T\mathbb{P}^3|_{\pi_4(C_7)})^\vee(9))) \longrightarrow 0\,.$$

Proposition 2.1.35. *The splitting type of $T\mathbb{P}^3|_{\pi_4(C_7)}$ is:*

b.1) $(8, 10, 10)$ *if and only if* $\operatorname{rank} T^L_{7,4}(1) = 12$*;*

b.2) $(8, 9, 11)$ *if and only if* $\operatorname{rank} T^L_{7,4}(1) = 11$.

2.2 Varieties Parametrizing Subschemes of $\mathrm{Hilb}_n\,\mathbb{P}^3$

We define the varieties which parametrize the subscheme of the Hilbert scheme $T^n_3(t_1, t_2, t_3)$ as intersection of some of the following varieties:

$$V(T^L_{n,n-3}(d))^r := \{L \in Gr(\mathbb{P}^{n-4}, \mathbb{P}^n) : \operatorname{rank} T^L_{n,n-3}(d) \le r\}.$$

These are subvarieties of $Gr(\mathbb{P}^{n-4}, \mathbb{P}^n)$.

Since the rank condition is invariant under the action of $\mathrm{SL}(n-3, \mathbb{C})$ we are interested to study the determinantal varieties in $\mathrm{Hom}(H^0(\mathcal{O}^n(d)), H^0(\mathcal{O}^{n-3}(d+1))$:

$$D(T^L_{n,n-3}(d))^r = \{p_1, ..., p_{n-3} \in \mathbb{P}^n : \operatorname{rank} T^L_{n,n-3}(d) \le r\}.$$

Proposition 2.2.1. *We can compute the maximal codimension of the above varieties:*

$$\operatorname{codim} V(T^L_{n,n-3}(d))^r = \operatorname{codim} V(T^L_{n,n-3}(d))^r \le ((d+2)(n-3) - r)(n(d+1) - r).$$

2.3 Restricted Tangent Bundle of Rational Curves in Codim k

We can obtain as in the case of rational curves in $\mathbb{P}^3$ the following diagram for a projection π_k from $\mathbb{P}^n$ in a $\mathbb{P}^{n-k}$ determined by a linear space $L \cong \mathbb{P}^{k-1}$ generated by

k points $p_1 = (a_0^1, ..., a_n^1), ..., p_k = (a_0^k, ..., a_n^k) \in \mathbb{P}^n$:

(2.8)
$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}^k_{C_n}(n) \arrow[r, "\cong"] \arrow[d] & \mathcal{O}^k_{C_n}(n) \arrow[d, "(\mathcal{T}^L_{n,k})^t"] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\nu_n"] & \mathcal{O}_{C_n}(n)^{n+1} \arrow[r, "Syz(\nu_n)"] \arrow[d] & \mathcal{O}_{C_n}(n+1)^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\pi_k \circ \nu_n"] & \mathcal{O}_{\pi^{n-k+1}_{n-3}(C_n)}(n) \arrow[r] \arrow[d] & T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & .
\end{tikzcd}
$$

The last exact column of (2.8):

(2.9)
$$0 \longrightarrow \mathcal{O}_{C_n}(n)^k \longrightarrow \mathcal{O}_{C_n}(n+1)^n \longrightarrow T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \longrightarrow 0\,,$$

gives rise by duality and tensorizing by $\mathcal{O}(n+1)$:

(2.10)
$$0 \longrightarrow (T\mathbb{P}^{n-k}|_{\pi_k(C_n)})^\vee(n+1) \longrightarrow \mathcal{O}^n_{C_n} \xrightarrow{\mathcal{T}^L_{n,k}} \mathcal{O}_{C_n}(1)^k \longrightarrow 0\,.$$

We indicate with $T^L_{n,k}$ the $2k \times n$ matrix:

$$T^L_{n,k} = \begin{pmatrix} a_0^1 & \dots & a_{n-1}^1 \\ -a_1^1 & \dots & -a_n^1 \\ \vdots & \ddots & \vdots \\ a_0^k & \dots & a_{n-1}^k \\ -a_1^k & \dots & -a_n^k \end{pmatrix}.$$

So we have that $\deg((T\mathbb{P}^{n-k}|_{\pi_k(C_n)})^\vee(n+1)) = -k$ and

$$h^0((T\mathbb{P}^{n-k}|_{\pi_k(C_n)})^\vee(n+1)) = n - \mathrm{rank}(T^L_{n,k}) = \dim\ker(T^L_{n,k}).$$

As $\mathrm{rank}((T\mathbb{P}^{n-k}|_{\pi_k(C_n)})^\vee(n+1)) = n-k$ we have that $(T\mathbb{P}^{n-k}|_{\pi_k(C_n)})^\vee(n+1)$ splits in $\mathcal{O}(t'_1) \oplus ... \oplus \mathcal{O}(t'_{n-k})$ by Grothendieck-Segre's theorem (see [Grothendieck, 1957]) with $t'_1 \le ... \le t'_{n-k} \le 0$ and $t'_1 + .. + t'_{n-k} = -k$.

Therefore we have that:

$$\mathrm{rank}\, T^L_{n,k} \ge k+1,$$

otherwise some t'_i must be ≥ 1 which is impossible.

By the considerations above it's clear that:

Proposition 2.3.1. *$T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-\mathrm{rank}(T^L_{n,k})} \oplus \mathcal{F}$, with $\mathcal{F}$ a vector bundle of rank $\mathrm{rank}(T^L_{n,k}) - k$ on $\mathbb{P}^1$ and $\deg(\mathcal{F}^\vee(n+1)) = -k$ such that $\mathcal{F} \cong \bigoplus_{i=1}^{\mathrm{rank}(T^L_{n,k})-k} \mathcal{O}(l_i)$ with $l_i \geq n+2$.*

We define the varieties which parametrize the subscheme of the Hilbert scheme $T^n_{n-k}(t_1, ..., t_{n-k})$ as intersection of some of the following varieties:

$$V(T^L_{n,k}(d))^r := \{L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) : \mathrm{rank}\, T^L_{n,k}(d) \leq r\}.$$

These are subvarieties of $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$.

Since the rank condition is invariant under the action of $\mathrm{SL}(k, \mathbb{C})$ we are interested to study the determinantal varieties in $\mathrm{Hom}(H^0(\mathcal{O}^n(d)), H^0(\mathcal{O}^k(d+1))$:

$$D(T^L_{n,k}(d))^r = \{p_1, ..., p_k \in \mathbb{P}^n : \mathrm{rank}\, T^L_{n,k}(d) \leq r\}.$$

About the matrix $T^L_{n,k}$ we note that are two possible cases:

I) $k < \frac{n}{2}$;

II) $k \geq \frac{n}{2}$.

These two cases will be the subject of our studies in the next two sections.

2.4 Codimension k, for $1 \le k < \frac{n}{2}$

Observation 2.4.1. *Since we want to consider only the projections with only ordinary singularities, it is:*

$$2 \le \operatorname{rank}(T^L_{n,k}) \le 2k,$$

otherwise if $\operatorname{rank} T^L_{n,k} = 1$, *then* $p_1, ..., p_k \in C_n$. *Instead for conditions on splitting type of restricted tangent bundle:*

$$\operatorname{rank}(T^L_{n,k}) > k,$$

so we have:

$$k < \operatorname{rank}(T^L_{n,k}) \le 2k,$$

Proposition 2.4.2. $T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-\operatorname{rank}(T^L_{n,k})} \oplus \mathcal{F}$, *with* $\mathcal{F}$ *a vector bundle of rank* $\operatorname{rank}(T^L_{n,k}) - k$ *on* $\mathbb{P}^1$ *and* $\deg(\mathcal{F}^\vee(n+1)) = -k$ *such that* $\mathcal{F} \cong \bigoplus_{i=1}^{\operatorname{rank}(T^L_{n,k})-k} \mathcal{O}(l_i)$ *with* $l_i \ge n+2$.

In this case we have four possibilities:

1. $\mathcal{F} \cong \mathcal{O}(n+2)^k$ *if and only if* $\operatorname{rank}(T^L_{n,2}) = 2k$;
2. $\mathcal{F} \cong \mathcal{O}(n+2)^{k-2} \oplus \mathcal{O}(n+3)$ *if and only if* $\operatorname{rank}(T^L_{n,k}) = 2k-1$;
3. $\mathcal{F} \cong \mathcal{O}(n+2)^{k-2r} \oplus \mathcal{F}'$ *if and only if* $\operatorname{rank}(T^L_{n,k}) = 2k - r$ *with* $1 \le r \le k-2$, $\operatorname{rank}(\mathcal{F}') = r$ *and* $\deg(\mathcal{F}'^\vee(n+1)) = -k$;
4. $\mathcal{F} \cong \mathcal{O}(n+1+k)$ *if and only if* $\operatorname{rank}(T^L_{n,k}) = k+1$.

We can rephrased the above proposition as:

Proposition 2.4.3. *We have four possibilities:*

1. $\pi_k(C_n) \in T^n_{n-k}((n+1)^{n-2k}, (n+2)^k)$ *if and only if* $L \in V(T^L_{n,k})^{2k}$, *this case is the generic one;*
2. $\pi_k(C_n) \in T^n_{n-k}((n+1)^{n-2k+1}, (n+2)^{k-2}, (n+3))$ *if and only if* $L \in V(T^L_{n,k})^{2k-1}$;
3. $\pi_k(C_n) \in T^n_{n-k}((n+1)^{n-2k+r}, (n+2)^{k-2}, spt(\mathcal{F}'))$, *where* $spt(\mathcal{F}')$ *is the splitting type of* $\mathcal{F}'$ *with* $1 \le r \le k-2$, $\operatorname{rank}(\mathcal{F}') = r$ *and* $\deg(\mathcal{F}'^\vee(n+1)) = -k$, *if and only if* $L \in V(T^L_{n,k})^{2k-r}$ *with* $1 \le r \le k-2$;
4. $\pi_k(C_n) \in T^n_{n-k}((n+1)^{n-k-1}, (n+1+k))$ *if and only if* $L \in V(T^L_{n,k})^{k+1}$.

For the rest of this thesis we will write secant without further explanation meaning secant to the rational normal curve.

Observation 2.4.4. *We can observe that if* L *belong to a (n-1)-secant* $\mathbb{P}^{n-2}$ *generated by* $q_1, ..., q_{n-1} \in \mathbb{P}^n$*, then there exists an element* $\phi \in H^0(\mathcal{O}^n_{\pi_k(C_n)}) \cong S^{n-1}V^\vee = T_{n-1}$ *such that* $\phi \in \ker(T^L_{n,k}) = \bigcap_i \ker(Cat_{f_i}(1, n-1))$*, in fact we can take* $\phi = \prod_{i=1}^n (L^\perp_{q_i})^{n-1}$*, clearly this happens every time since* $\dim\ker T^L_{n,k} \geq 1$.

Unfortunately this condition is empty for $2k < n-1$*, in fact we can compute the codimension of the variety of every* $\mathbb{P}^{k-1}$ *which belong to some (n-1)-secant* $\mathbb{P}^{n-2}$ *constructing an incidence variety:*

$$I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n), \pi \in S, L \subset S\},$$

where S *is the variety of all (n-1)-secant* $\mathbb{P}^{n-2}$ *to* C_n*. In the usual way we can compute the codimension of the image of this incidence variety in* $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$*. We will indicated with* ϕ_1 *and* ϕ_2 *the natural projections:*

$$\begin{tikzcd}
 & I_S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\
Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) & & S,
\end{tikzcd}$$

so the codimension in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ *of* $\phi_1(I_S)$ *is equal to* $\dim Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) - \dim S - \dim \phi_2^{-1}(S) = k(n+1-k) - n + 1 - k(n-1-k)$*. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). We have that this variety has codimension* $2k - n + 1$*, but we are in the hypothesis* $2k < n-1$*, so* $2k - n + 1 < 0$.

For $2k = n-1$ *the condition gives* $\mathrm{codim} = 0$*, so it is verified for all L.*

Case $\mathrm{rank}\, T^L_{n,k} = 2k - 1$

Lemma 2.4.5. *Let* $2k < n \leq 3k - 1$*. If the centre of projection* $L \cong \mathbb{P}^{k-1}$ *belongs to a linear system of affine dimension* $n - 2k + 1$ *of* $(n-1)-$*secant* $\mathbb{P}^{n-2}$ *to the rational normal curve* C_n *in* $\mathbb{P}^n$*, then* $\mathrm{rank}\, T^L_{n,k} = 2k - 1$*. The converse is generically true.*

Proof. $\Rightarrow$ Let L be a $\mathbb{P}^{k-1}$ belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{n-2k}\pi_{n-2k},\ \forall\lambda = [\lambda_0, ..., \lambda_{n-2k}] \in \mathbb{P}^{n-2k}\}$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$, where $\pi_0, ..., \pi_{n-2k}$ are $(n-1)-$secant $\mathbb{P}^{n-2}$.

Let $q_1^i, ..., q_{n-1}^i \in C_n$ be the points which generate π_i. Then there exist k points $p_1, ..., p_k$ which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^{n-2k}$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{n-1,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c_{1,\lambda}^i L_{1,\lambda}^n + ... + c_{n-1,\lambda}^i L_{n-1,\lambda}^n,$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., n-1$. This means that L belongs to a $\mathbb{P}^{n-2(n-2k+1)} = \mathbb{P}^{4k-n-2}$, this is possible by the condition $n \leq 3k-1$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^{n-2k}$ there exists a differential forms $\phi_\lambda \in T_{n-1}$ such that $\phi_\lambda \circ f_i = 0$. Moreover there exist $n-2k+1$ differential forms $\phi_0, ..., \phi_{n-2k} \in T_{n-1}$ which for each $\lambda \in \mathbb{P}^{n-2k}$ we have $\phi_\lambda = \lambda_0\phi_0 + ... + \lambda_{n-2k}\phi_{n-2k}$, in particular $\phi_j \circ f_i = 0$ for all $j = 0, ..., n-2k$ and $i = 1, ..., k$, so $\operatorname{rank} T_{n,k}^L = 2k-1$.

$\Leftarrow$ If $\operatorname{rank} T_{n,k}^L = 2k-1$, then there exist $n-2k+1$ binary form $\phi_0, ..., \phi_{n-2k} \in T_{n-1}$ such that however we consider the generating points $p_1, ..., p_k \in \mathbb{P}^n$ of L, it is $\phi_\alpha \circ f_i = (\alpha_0\phi_0 + ... + \alpha_{n-2k}\phi_{n-2k}) \circ f_i = 0$ for all $\alpha = [\alpha_0, ..., \alpha_{n-2k}] \in \mathbb{P}^{n-2k}$ and $i = 1, ..., k$, where $f_i \in S_n$ is the binary form corresponding to p_i. In particular $\phi_j \circ f_i = 0$. We can consider the primary decomposition of $\phi_\alpha = \prod_{l=1}^{n-1} \phi_\alpha^l$ and we indicate with $(L_{l,\alpha})^\perp = \phi_\alpha^l$. Therefore $f_1, ..., f_k$ can be decompose in ∞^{n-2k} different simultaneously ways, i.e.:

$$f_i = c_{1,\alpha}^i L_{1,\alpha}^n + ... + c_{n-1,\alpha}^i L_{n-1,\alpha}^n,$$

for all $\alpha = [\alpha_0, ..., \alpha_{n-2k}] \in \mathbb{P}^{n-2k}$, so L belongs to a $\mathbb{P}^{4k-n-2}$. In other words L belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{n-2k}\pi_{n-2k}, \ \forall\lambda = [\lambda_0, ..., \lambda_{n-2k}] \in \mathbb{P}^{n-2k}\}$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$. Clearly it can happen that some ϕ_i have multiple roots, so we have all possible degenerations of the linear system Φ

$\square$

Observation 2.4.6. *By Theorem 1.2.14 (see [Verdier, 1983])* $T_{n-k}^n((n+1)^{n-2k+1}, (n+2)^{k-2}, n+3)$ *is an irreducible variety of* $\operatorname{codim}(T_{n-k}^n((n+1)^{n-2k+1}, (n+2)^{k-2}, n+3) = n-2k+1$.

Lemma 2.4.7. *Let $k-1 \leq 4k-n-2$ or equivalently $2k < n \leq 3k-1$. The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension $n-2k+1$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible variety of codimension $n-2k+1$.*

Proof. We can observe that the linear system Φ of affine dimension $n-2k+1$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ correspond to the linear system of dimension $n-2k+1$ of binary forms of degree $n-1$, therefore the set of these linear system corresponds to $Gr(\mathbb{P}^{n-2k}, \mathbb{P}^{n-1})$ which is irreducible and $\dim Gr(\mathbb{P}^{n-2k}, \mathbb{P}^{n-1}) = (n-2k+1)(2k-1)$. Each projection linear space L belongs to a $\mathbb{P}^{4k-n-2}$, so the dimension of the fibre is $k(3k-n-1)$, which is ≥ 0 with the condition $k-1 \leq 4k-n-2$. Therefore the variety of lines L that belong to a linear system Φ is an irreducible variety of codimension $n-2k+1$ in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

Theorem 2.4.8. *Let $2k < n \leq 3k-1$. The centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system Φ of affine dimension $n-2k+1$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ if and only if $T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-2k+1} \oplus \mathcal{O}(n+2)^{k-2} \oplus \mathcal{O}(n+3)$.*

Corollary 2.4.9. *$T^n_{n-k}((n+1)^{n-2k+1}, (n+2)^{k-2}, n+3)$ is an irreducible variety of* $\operatorname{codim}(T^n_{n-k}((n+1)^{n-2k+1}, (n+2)^{k-2}, n+3) = n-2k+1$ *formed by the linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension $n-2k+1$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$.*

Case $\operatorname{rank} T^L_{n,k} \leq 2k-1$

Lemma 2.4.10. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to some $(2k-r)-$secant $\mathbb{P}^{2k-r-1}$ to the rational normal curve C_n in $\mathbb{P}^n$, then we have* $\operatorname{rank} T^L_{n,k} \leq 2k-r$ *for $1 \leq r < k$.*

Proof. If $L \cong \mathbb{P}^{k-1}$ belongs to some $\mathbb{P}^{2k-r-1}$ $(2k-r)-$secant, then there exist k points $p_1, ..., p_k \in L$ which generate L and the corresponding binary forms f_i are generating by two forms $Ann(f_i) = (\alpha, \beta_i)$ with $\deg(\alpha) = 2k-r$, α has only simple roots and $\deg(\beta_i) = n-2k+r+2$ without common zeros with α. We have $\dim(\alpha)_{n-1} = n-2k+r$ and $\dim \bigcap_i (\alpha, \beta_i)_{n-1} \geq n-2k+r$, so $\operatorname{rank} T^L_{n,k} = n - \dim \bigcap_i (\alpha, \beta_i)_{n-1} \leq 2k-r$. □

Lemma 2.4.11. *The codimension in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ of the variety of all $L \cong \mathbb{P}^{k-1}$ in $\mathbb{P}^n$ belonging to some $(2k-r)$-secant $\mathbb{P}^{2k-r-1}$ to the rational normal curve in $\mathbb{P}^n$ is $k(n+r-1) - 2k^2 + r$.*

Proof. In fact we can consider the incidence variety $I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n), \pi \in S, L \subset S\}$ where S is the set of all $(2k-r)$-secant $\mathbb{P}^{2k-r-1}$ to the rational normal curve in $\mathbb{P}^n$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). $\square$

In particular for $r = k-1$ we have:

Theorem 2.4.12. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to some $\mathbb{P}^k$ $(k+1)-$secant to the rational normal curve C_n in $\mathbb{P}^n$, then we have:*

$$T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-k-1} \oplus \mathcal{O}(n+1+k).$$

Observation 2.4.13. *By Theorem 1.2.14 (see [Verdier, 1983]) $T^n_{n-k}((n+1)^{n-k-1}, n+1+k)$ is an irreducible variety of* $\operatorname{codim}(T^n_{n-k}((n+1)^{n-k-1}, n+1+k)) = (k-1)(n-k-1) \leq kn - k^2 - k - 1$.

Corollary 2.4.14. *The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that, as centre of projection, give a rational curve of degree n in $\mathbb{P}^{n-k}$ which has the restricted tangent bundle $T\mathbb{P}^{n-k}_{\pi_k(C_n),\mathbb{P}^{n-k}} \cong \mathcal{O}(n+1)^{n-k-1} \oplus \mathcal{O}(n+1+k)$ has an irreducible subvariety of codimension $kn - k^2 - k - 1$ in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ formed by the linear spaces L belong to some $(k+1)-$secant $\mathbb{P}^k$.*

We can prove a more general result:

Lemma 2.4.15. *Let $2k < n \leq 3k - 2r + 1$ and $2r < k+1$. If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system of dimension $n - 2k + r$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$, then* $\operatorname{rank} T^L_{n,k} = 2k - r$ *for $1 \leq r < k$. The converse is generically true.*

Proof. $\Rightarrow$ Let L be a $\mathbb{P}^{k-1}$ belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{n-2k+r-1}\pi_{n-2k+r-1}, \quad \forall\lambda = [\lambda_0, ..., \lambda_{n-2k+r-1}] \in \mathbb{P}^{n-2k+r-1}\}$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$, where $\pi_0, ..., \pi_{n-2k+r-1}$ are $(n-1)-$secant $\mathbb{P}^{n-2}$. Let $q^i_1, ..., q^i_{n-1} \in C_n$ be the points which generate π_i. Then there exist k points $p_1, ..., p_k$ which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^{n-2k+r-1}$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{n-1,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c^i_{1,\lambda} L^n_{1,\lambda} + ... + c^i_{n-1,\lambda} L^n_{n-1,\lambda},$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., n-1$. This means that L belongs to a $\mathbb{P}^{n-2(n-2k+r)} = \mathbb{P}^{4k-n-2r}$, this is possible by the condition $n \leq 3k - 2r + 1$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^{n-2k+r-1}$ there exists a differential form $\phi_\lambda \in T_{n-1}$ such that $\phi_\lambda \circ f_i = 0$. Moreover there exist $n - 2k + r$ differential forms $\phi_0, ..., \phi_{n-2k+r-1} \in T_{n-1}$ which for each $\lambda \in \mathbb{P}^{n-2k+r-1}$ we have $\phi_\lambda = \lambda_0\phi_0 + ... + \lambda_{n-2k+r-1}\phi_{n-2k+r-1}$, in particular $\phi_j \circ f_i = 0$ for all $j = 0, ..., n - 2k$ and $i = 1, ..., k$, so $\operatorname{rank} T^L_{n,k} = 2k - r$.

$\Leftarrow$ If $\operatorname{rank} T^L_{n,k} = 2k - r$, then there exist $n - 2k + r$ binary form $\phi_0, ..., \phi_{n-2k+r-1} \in T_{n-1}$ such that however we consider the generating points $p_1, ..., p_k \in \mathbb{P}^n$ of L, it is $\phi_\alpha \circ f_i = (\alpha_0\phi_0 + ... + \alpha_{n-2k+r-1}\phi_{n-2k+r-1}) \circ f_i = 0$ for all $\alpha = [\alpha_0, ..., \alpha_{n-2k+r-1}] \in \mathbb{P}^{n-2k+r-1}$ and $i = 1, ..., k$, where $f_i \in S_n$ is the binary form corresponding to p_i. In particular $\phi_j \circ f_i = 0$. We can consider the primary decomposition of $\phi_\alpha = \prod_{l=1}^{n-1} \phi^l_\alpha$ and we indicate with $(L_{l,\alpha})^\perp = \phi^l_\alpha$. Therefore $f_1, ..., f_k$ can be decomposed in $\infty^{n-2k+r-1}$ different simultaneously ways, i.e.:

$$f_i = c^i_{1,\alpha}L^n_{1,\alpha} + ... + c^i_{n-1,\alpha}L^n_{n-1,\alpha},$$

for all $\alpha = [\alpha_0, ..., \alpha_{n-2k+r-1}] \in \mathbb{P}^{n-2k+r-1}$, so L belongs to a $\mathbb{P}^{4k-n-2r}$. In other words L belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{n-2k+r-1}\pi_{n-2k+r-1}, \quad \forall\lambda = [\lambda_0, ..., \lambda_{n-2k+r-1}] \in \mathbb{P}^{n-2k+r-1}\}$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$. Clearly it can happen that some ϕ_i have any multiple roots, so we have all possible degenerations of the linear system Φ.

□

Lemma 2.4.16. *Let $2k < n \leq 3k - 2r + 1$ and $2r < k + 1$. The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension $n - 2k + r$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible variety of codimension $r(n - 2k + r)$.*

Proof. We can observe that the linear system Φ of affine dimension $n - 2k + r$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ corresponds to the linear system of dimension $n - 2k + r$ of binary forms of degree $n - 1$, therefore the set of these linear system corresponds to $Gr(\mathbb{P}^{n-2k+r-1}, \mathbb{P}^{n-1})$ which is irreducible and $\dim Gr(\mathbb{P}^{n-2k+r-1}, \mathbb{P}^{n-1}) = (n-2k+r)(2k-r)$. Each projection linear space L belongs to a $\mathbb{P}^{4k-n-2r}$, so the dimension of the fibre is $k(3k - n - 2r + 1)$, which is ≥ 0 with the condition $k - 1 \leq 4k - n - 2r$. Therefore the variety of linear spaces L that belong to a linear system Φ is an irreducible variety of codimension $r(n - 2k + r)$ in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. The above calculation is effective thanks to the result of Chiantini and

Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

By Theorem 1.2.14[Verdier, 1983] we have:

Theorem 2.4.17. *Let $2k < n \leq 3k - 2r + 1$ and $2r < k + 1$. The centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system Φ of affine dimension $n - 2k + r$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ if and only if $T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-2k+r} \oplus \mathcal{O}(n+2)^{k-2r} \oplus \mathcal{F}'$ with $\mathrm{rank}(\mathcal{F}') = r$ and $\deg(\mathcal{F}'^{\vee}(n+1)) = -k$.*

Corollary 2.4.18. *The union of all schemes $T^n_{n-k}((n+1)^{n-2k+r}, (n+2)^{k-2r}, spt(\mathcal{F}')$ with $\mathrm{rank}(\mathcal{F}') = r$ and*
$\deg(\mathcal{F}'^{\vee}(n+1)) = -k$ is an irreducible variety of codimension $r(n-2k+r)$ formed by the linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension $n - 2k + r$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$.

2.4.1 Restricted Tangent Bundle of Rational Curves in Codim 1

In the case of projections in codimension 1, i.e. from a point, we have:

$$0 \longrightarrow (T\mathbb{P}^{n-1}|_{\pi_1(C_n)})^{\vee}(n+1) \longrightarrow \mathcal{O}^n_{\pi_1(C_n)} \xrightarrow{T^L_{n,1}} \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0\,. \tag{2.11}$$

So we have that $\deg((T\mathbb{P}^{n-1}|_{\pi_1(C_n)})^{\vee}(n+1)) = -1$ and

$$h^0((T\mathbb{P}^{n-1}|_{\pi_1(C_n)})^{\vee}(n+1)) = n - \mathrm{rank}(T^L_{n,1}) = \ker(T^L_{n,1}).$$

We have only two possibilities:

1. $T\mathbb{P}^{n-1}|_{\pi_1(C_n)} \cong \mathcal{O}(n+1)^{n-2} \oplus \mathcal{O}(n+2)$ if and only if $\mathrm{rank}(T^L_{n,1}) = 2$;
2. $T\mathbb{P}^{n-1}|_{\pi_1(C_n)} \cong \mathcal{O}(n+1)^{n-3} \oplus \mathcal{O}(n) \oplus \mathcal{O}(n+3)$ if and only if $\mathrm{rank}(T^L_{n,1}) = 1$,

but the second one is impossible by (2.9).

So we have only two possibilities or the projection point is on the rational normal curve or it is not, but the first case does not happen by our preliminary hypothesis on projected curve.

Theorem 2.4.19. *The splitting type of restricted tangent bundle of a generic rational curve of degree $n-1$ in $\mathbb{P}^n$ is $((n+1)^{n-2}, n+2)$.*

2.4.2 Restricted Tangent Bundle of Rational Curves in Codim 2

In the case of projections in codimension 2, i.e. from a line, we have:

$$0 \longrightarrow (T\mathbb{P}^{n-2}|_{\pi_2(C_n)})^\vee(n+1) \longrightarrow \mathcal{O}^n_{\pi_2(C_n)} \xrightarrow{T^L_{n,2}} \mathcal{O}_{\mathbb{P}^1}(1)^2 \longrightarrow 0\,. \tag{2.12}$$

So we have that $\deg((T\mathbb{P}^{n-2}|_{\pi_2(C_n)})^\vee(n+1)) = -2$ and

$$h^0((T\mathbb{P}^{n-2}|_{\pi_2(C_n)})^\vee(n+1)) = n - \operatorname{rank}(T^L_{n,2}) = \ker(T^L_{n,2}).$$

In this case we have three possibility:

1. $T\mathbb{P}^{n-2}|_{\pi_2(C_n)} \cong \mathcal{O}(n+1)^{n-4} \oplus \mathcal{O}(n+2)^2$ if and only if $\operatorname{rank}(T^L_{n,2}) = 4$;
2. $T\mathbb{P}^{n-2}|_{\pi_2(C_n)} \cong \mathcal{O}(n+1)^{n-3} \oplus \mathcal{O}(n+3)$ if and only if $\operatorname{rank}(T^L_{n,2}) = 3$;
3. $T\mathbb{P}^{n-2}|_{\pi_2(C_n)} \cong \mathcal{O}(n+1)^{n-4} \oplus \mathcal{O}(n) \oplus \mathcal{O}(n+4)$ if and only if $\operatorname{rank}(T^L_{n,2}) = 2$.

But the last one is impossible by (2.9).

By Apolarity Lemma 1.3.11 we have an easy result:

Proposition 2.4.20. *If there exist two points $q_1, q_2 \in L$ each of them belongs to a different secant line, then:*

$$\ker(T^L_{n,2}) = (\mathcal{I}_{P_1} \cap \mathcal{I}_{P_2})_{n-1} = (\mathcal{I}_{P_1 \cup P_2})_{n-1},$$

where P_i is the set of points in $\mathbb{P}^1$ corresponds to the linear forms in the additive decomposition of f_i which corresponds to q_i.

Case $\operatorname{rank} T^L_{n,2} = 3$

Lemma 2.4.21. *The codimension in $Gr(\mathbb{P}^1, \mathbb{P}^n)$ of the variety of all lines in $\mathbb{P}^n$ belonging to some 3-secant $\mathbb{P}^2$ to the rational normal curve in $\mathbb{P}^n$ is $2n-7$.*

Proof. In fact we can consider the incidence variety $I_S = \{(L, \pi) : L \in \mathbb{G}r(1,n), \pi \in S, L \subset S\}$ where S is the set of all 3-secant $\mathbb{P}^2$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^1, \mathbb{P}^n)$. □

Theorem 2.4.22 (Case $\operatorname{rank} T^L_{n,2} = 3$)**.** *If the projection line L belongs to some $3-$secant $\mathbb{P}^2$, but it is not a secant line, then the splitting type of the restricted tangent bundle $T\mathbb{P}^{n-2}_{\pi_2(C_n),\mathbb{P}^{n-2}}$ is:*

$$((n+1)^{n-3}, n+3).$$

Proof. If L belongs to a $3-$secant $\mathbb{P}^2$, then there exist two points $q_1, q_2 \in L$ such that $Ann(f_1) = (\alpha, \beta_1)$ and $Ann(f_2) = (\alpha, \beta_2)$ with α has only simple roots and $\deg(\alpha) = 3$ and $\beta_1 \neq \beta_2$, $\deg(\beta_1) = \deg(\beta_2) = n-1$, with $\beta_i \in T/ < \alpha >$. So $\dim < \alpha >_{n-1}= n-3$, $\dim(< \alpha, \beta_i >_{n-1}= n-2$ and $n-2 \geq \dim(< \alpha, \beta_1 >_{n-1} \cap < \alpha, \beta_2 >_{n-1}) \geq n-3$.

Suppose $\dim(< \alpha, \beta_1 >_{n-1} \cap < \alpha, \beta_2 >_{n-1}) = n-2$, then $< \alpha, \beta_1 >_{n-1}=< \alpha, \beta_2 >_{n-1}$, but this is impossible because this means that $\beta_1 \in< \alpha, \beta_2 >_{n-1}$, so $\beta_1 \in Ann(f_2)$, which is impossible because otherwise $q_1 = q_2$, but it is must be $L =< q_1, q_2 >\cong \mathbb{P}^1$.

Therefore $\text{rank}(T^L_{n,2}) = n - \dim(< \alpha, \beta_1 >_{n-1} \cap < \alpha, \beta_2 >_{n-1}) = 3$.

□

Observation 2.4.23. *By Theorem 1.2.14[Verdier, 1983]* $T^n_{n-2}((n+1)^{n-3}, n+3)$ *is an irreducible variety of* $\text{codim}(T^n_{n-2}((n+1)^{n-3}, n+3)) = n-3 < 2n-7$.

Corollary 2.4.24. *The variety of lines that, as centre of projection, give a rational curve of degree n in $\mathbb{P}^{n-2}$ which has the restricted tangent bundle* $T\mathbb{P}^{n-2}_{\pi_2(C_n),\mathbb{P}^{n-2}} \cong \mathcal{O}(n+1)^{n-3} \oplus \mathcal{O}(n+3)$ *has an irreducible subvariety of codimension $2n-7$ in $Gr(\mathbb{P}^1, \mathbb{P}^n)$ formed by the lines belong to some $3-$secant $\mathbb{P}^2$, but it is not a secant line.*

Observation 2.4.25. *If* $\text{rank}(T_{n,2}) = 3$*, then or L belongs to a $3-$secant $\mathbb{P}^2$ or to $(r+1)-$secant $\mathbb{P}^r$ for $r < n$. In the second case there exist two points $p_1, p_2 \in L$ such that $Ann(f_1) = (\alpha, \beta_1)$ and $Ann(f_2) = (\alpha, \beta_2)$ with α has only simple roots and* $\deg(\alpha) = r+1$ *and* $\beta_1 \neq \beta_2$, $\deg(\beta_1) = \deg(\beta_2) = n-r+1$. *So* $\dim < \alpha >_{n-1}= n-r-1$ *and*

$$n-2 \geq \dim(< \alpha, \beta_1 >_{n-1} \cap < \alpha, \beta_2 >_{n-1}) \geq n-r-1.$$

Observation 2.4.26. *By Theorem 1.2.14 (see [Verdier, 1983])* $T^n_{n-2}((n+1)^{n-3}, n+3)$ *is an irreducible variety of* $\text{codim}(T^n_{n-2}((n+1)^{n-3}, n+3) = n-3$.

Theorem 2.4.27. *The centre of projection $L \cong \mathbb{P}^1$ belongs to a pencil of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ if ad only if* $T\mathbb{P}^{n-2}|_{\pi_2(C_n)} \cong \mathcal{O}(n+1)^{n-3} \oplus \mathcal{O}(n+3)$.

2.4.3 Restricted Tangent Bundle of Rational Curves in Codim 3

In the case of projections in codimension 3, i.e. from a line, we have:

$$0 \longrightarrow (T\mathbb{P}^{n-3}|_{\pi_3(C_n)})^\vee(n+1) \longrightarrow \mathcal{O}^n_{\pi_3(C_n)} \xrightarrow{\mathcal{T}^L_{n,3}} \mathcal{O}_{\mathbb{P}^1}(1)^3 \longrightarrow 0\,. \tag{2.13}$$

So we have that $\deg((T\mathbb{P}^{n-3}|_{\pi_3(C_n)})^\vee(n+1)) = -3$ and

$$h^0((T\mathbb{P}^{n-3}|_{\pi_3(C_n)})^\vee(n+1)) = n - \operatorname{rank}(T^L_{n,3}) = \ker(T^L_{n,3}).$$

In this case we have five possibilities:

1. $T\mathbb{P}^{n-3}|_{\pi_3(C_n)} \cong \mathcal{O}(n+1)^{n-6} \oplus \mathcal{O}(n+2)^3$ if and only if $\operatorname{rank}(T^L_{n,3}) = 6$;
2. $T\mathbb{P}^{n-3}|_{\pi_2(C_n)} \cong \oplus\mathcal{O}(n+1)^{n-5} \oplus \mathcal{O}(n+2) \oplus \mathcal{O}(n+3)$ if and only if $\operatorname{rank}(T^L_{n,3}) = 5$;
3. $T\mathbb{P}^{n-3}|_{\pi_3(C_n)} \cong \mathcal{O}(n+1)^{n-4} \oplus \mathcal{O}(n+4)$ if and only if $\operatorname{rank}(T^L_{n,3}) = 4$;
4. $T\mathbb{P}^{n-3}|_{\pi_3(C_n)} \cong \mathcal{O}(n) \oplus \mathcal{O}(n+1)^{n-5} \oplus \mathcal{O}(n+4)$ if and only if $\operatorname{rank}(T^L_{n,3}) = 3$;
5. $T\mathbb{P}^{n-3}|_{\pi_3(C_n)} \cong \mathcal{O}(n-1) \oplus \mathcal{O}(n+1)^{n-5} \oplus \mathcal{O}(n+4)$ if and only if $\operatorname{rank}(T^L_{n,3}) = 2$.

But the last two are impossible by (2.9).

Case $\operatorname{rank} T^L_{n,3} = 5$

Lemma 2.4.28. *The codimension in $Gr(\mathbb{P}^2, \mathbb{P}^n)$ of the variety of all lines in $\mathbb{P}^n$ belonging to some 5-secant $\mathbb{P}^4$ to the rational normal curve in $\mathbb{P}^n$ is $3n-17$.*

Proof. In fact we can consider the incidence variety $I_S = \{(L, \pi) : L \in \mathbb{G}r(2,n), \pi \in S, L \subset S\}$ where S is the set of all 5-secant $\mathbb{P}^4$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^2, \mathbb{P}^n)$. That calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

Theorem 2.4.29 (Case $\operatorname{rank} T^L_{n,3} = 5$)**.** *If the projection line L belongs to some $5-$secant $\mathbb{P}^4$, but it is not a secant line, then we have* $\operatorname{rank} T^L_{n,3} = 5$.

Proof. If L belongs to a $\mathbb{P}^4$ $5-$secant, then there exist three points $q_1, q_2, q_3 \in L$ such that $Ann(f_i) = (\alpha, \beta_i)$ with α has only simple roots and $\deg(\alpha) = 5$ and $\beta_1 \neq \beta_2 \neq \beta_3$, $\deg(\beta_1) = \deg(\beta_2) = \deg(\beta_3) = n-3$, with $\beta_i \in T/ < \alpha >$. So $\dim < \alpha >_{n-1} = n-5$, $\dim(< \alpha, \beta_i >_{n-1}) = n-2$ and $n-2 \geq \dim(< \alpha, \beta_1 >_{n-1} \cap < \alpha, \beta_2 >_{n-1}) \geq n-5$. □

Instead by the theorem 2.4.17 we have:

Theorem 2.4.30. *The centre of projection $L \cong \mathbb{P}^2$ belongs to a linear system Φ of affine dimension $n-5$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ if and only if $T\mathbb{P}^{n-3}|_{\pi_3(C_n)} \cong \mathcal{O}(n+1)^{n-5} \oplus \mathcal{O}(n+2) \oplus \mathcal{O}(n+3))$.*

Corollary 2.4.31. $T^n_{n-3}((n+1)^{n-5}, n+2, n+3)$ *is an irreducible variety of codimension* $n-5$ *formed by the linear spaces* $L \cong \mathbb{P}^2$ *that belong to a linear system* Φ *of affine dimension* $n-5$ *of* $(n-1)-$*secant* $\mathbb{P}^{n-2}$ *to the rational normal curve* C_n *in* $\mathbb{P}^n$.

Case $\operatorname{rank} T^L_{n,3} = 4$

Lemma 2.4.32. *The codimension in* $Gr(\mathbb{P}^2, \mathbb{P}^n)$ *of the variety of the linear spaces in* $\mathbb{P}^n$ *belonging to some* $4-$*secant* $\mathbb{P}^3$ *to the rational normal curve in* $\mathbb{P}^n$ *is* $3n-13$.

Proof. In fact we can consider the incidence variety $I = \{(L, \pi) : L \in Gr(\mathbb{P}^2, \mathbb{P}^n), \pi \in S, L \subset S\}$ where S is the set of all 4-secant $\mathbb{P}^3$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^2, \mathbb{P}^n)$. That calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). $\square$

Theorem 2.4.33 (Case $\operatorname{rank} T^L_{n,3} = 4$)**.** *If the projection line* L *belongs to some* $4-$*secant* $\mathbb{P}^3$*, but it is not a secant line, then the splitting type of the restricted tangent bundle* $T\mathbb{P}^{n-3}_{\pi_3(C_n), \mathbb{P}^{n-3}}$ *is:*

$$((n+1)^{n-4}, n+4).$$

Proof. If L belongs to a $4-$secant $\mathbb{P}^3$, then there exist three points $q_1, q_2, q_3 \in L$ such that $Ann(f_i) = (\alpha, \beta_i)$ with α has only simple roots and $\deg(\alpha) = 4$ and $\beta_1 \neq \beta_2 \neq \beta_3$, $\deg(\beta_1) = \deg(\beta_2) = \deg(\beta_3) = n-2$, with $\beta_i \in T/ < \alpha >$. So $\dim < \alpha >_{n-1} = n-4$, $\dim(< \alpha, \beta_i >_{n-1} = n-2$ and $n-2 \geq \dim(< \alpha, \beta_1 >_{n-1} \cap < \alpha, \beta_2 >_{n-1}) \geq n-4$. $\square$

Observation 2.4.34. *By Theorem 1.2.14 (see [Verdier, 1983])* $T^n_{n-3}((n+1)^{n-4}, n+4)$ *is an irreducible variety of* $\operatorname{codim}(T^n_{n-3}((n+1)^{n-4}, n+4)) = 2(n-4) < 3n-13$.

Theorem 2.4.35. *The variety of linear spaces* $L \cong \mathbb{P}^2$ *that, as centre of projection, give a rational curve of degree* n *in* $\mathbb{P}^{n-3}$ *which has the restricted tangent bundle* $T\mathbb{P}^{n-3}_{\pi_3(C_n), \mathbb{P}^{n-3}} \cong \mathcal{O}(n+1)^{n-4} \oplus \mathcal{O}(n+4)$ *has an irreducible subvariety of codimension* $3n-13$ *in* $Gr(\mathbb{P}^2, \mathbb{P}^n)$ *formed by the linear spaces* L *belong to some* $4-$*secant* $\mathbb{P}^3$.

By theorem 2.4.17 we have:

Theorem 2.4.36. *The centre of projection* $L \cong \mathbb{P}^2$ *belongs to a linear system* Φ *of affine dimension* $n-4$ *of* $(n-1)-$*secant* $\mathbb{P}^{n-2}$ *to the rational normal curve* C_n *in* $\mathbb{P}^n$ *if and only if* $T\mathbb{P}^{n-3}|_{\pi_3(C_n)} \cong \mathcal{O}(n+1)^{n-4} \oplus \mathcal{O}(n+4)$.

Corollary 2.4.37. *$T^n_{n-3}((n+1)^{n-4}, n+4)$ is an irreducible variety of codimension $2(n-4)$ formed by the linear spaces $L \cong \mathbb{P}^2$ that belong to a linear system Φ of affine dimension $n-4$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$.*

2.5 Codimension k, for $\frac{n}{2} \leq k \leq n-3$

Observation 2.5.1. *Since we want to consider only the projections with only ordinary singularities, it is:*

$$2 \leq \operatorname{rank}(T^L_{n,k}) \leq n,$$

otherwise if $\operatorname{rank} T^L_{n,k} = 1$*, then* $p_1, ..., p_k \in C_n$*. Moreover for conditions on splitting type of restricted tangent bundle:*

$$\operatorname{rank}(T^L_{n,k}) > k,$$

so we have:

$$k < \operatorname{rank}(T^L_{n,k}) \leq n,$$

Proposition 2.5.2. $T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-\operatorname{rank}(T^L_{n,k})} \oplus \mathcal{F}$*, with* $\mathcal{F}$ *a vector bundle of rank* $\operatorname{rank}(T^L_{n,k}) - k$ *on* $\mathbb{P}^1$ *and* $\deg(\mathcal{F}^\vee(n+1)) = -k$ *such that* $\mathcal{F} \cong \bigoplus_{i=1}^{\operatorname{rank}(T^L_{n,k})-k} \mathcal{O}(l_i)$ *with* $l_i \geq n+2$*.*

If $2n-2k \geq k$ *we have two possibilities:*

1. $\mathcal{F} \cong \mathcal{O}(n+2)^{2(n-k-r)-k} \oplus \mathcal{F}'$ *with* $\operatorname{rank}(\mathcal{F}') = n-r-2(n-k-r)$ *and* $\deg(\mathcal{F}'^\vee(n+1)) = -k+(2(n-k-r)-k)$ *if and only if* $\operatorname{rank}(T^L_{n,k}) = n-r$ *and* $2(n-k-r) \geq k$ *for* $0 \leq r \leq n-k-2$*;*
2. $\mathcal{F} \cong \mathcal{O}(n+1+k)$ *if and only if* $\operatorname{rank}(T^L_{n,k}) = k+1$*.*

However the last one is true also for $2n-2k < k$*.*

We can rephrased the above proposition as:

Proposition 2.5.3. *If* $2n-2k \geq k$ *we have two possibilities:*

1. $\pi_k(C_n) \in T^n_{n-k}((n+1)^r, (n+2)^{2(n-k-r)-k}, spt(\mathcal{F}'))$*, where* $spt(\mathcal{F}')$ *is the splitting type of* $\mathcal{F}'$ *with* $\operatorname{rank}(\mathcal{F}') = n-r-2(n-k-r)$ *and* $\deg(\mathcal{F}'^\vee(n+1)) = -k+(2(n-k-r)-k)$ *if and only if* $L \in V(T^L_{n,k})^{2k-r}$ *and* $2(n-k-r) \geq k$ *for* $0 \leq r \leq n-k-2$*;*
2. $\pi_k(C_n) \in T^n_{n-k}((n+1)^{n-k-1}, (n+1+k))$ *if and only if* $L \in V(T^L_{n,k})^{k+1}$*.*

However the last one is true also for $2n-2k < k$*.*

Lemma 2.5.4. $\operatorname{rank} T^L_{n,k} \leq n-1$ *if and only if the forms* f_i *of degree* n *corresponding to the points* p_i *generating* L *can be represented by the similar GAD, i.e. :*

$$f_i = G_{i_1} L_1^{n-g_1+1} + \ldots + G_{i_m} L_m^{n-g_m+1}.$$

Proof. $\Rightarrow$ If $\operatorname{rank} T^L_{n,k} \leq 2k-1$, then there exists at least an element $\phi \in T_{n-1}$ such that for all forms f_i corresponding to the points p_i generating L we have $\phi \circ f_i = 0$. So we can consider the primary decomposition of $\phi = \prod_{i=1}^m (\phi_i)^{g_i}$, with $\phi_i \in T_1$ and $\sum_i g_i = n-1$, so every f_i can be represented by the similar GAD, i.e. :

$$f_i = G_{i_1} L_1^{n-g_1+1} + \ldots + G_{i_m} L_m^{n-g_m+1},$$

where $(L_j)^\perp = \phi_j$ for all $j = 1,..,m$ and $G_{i_j} \in S_{g_j-1}$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, m$.

$\Leftarrow$ On the other hand if every f_i can be represented by the similar GAD, i.e. :

$$f_i = G_{i_1} L_1^{n-g_1+1} + \ldots + G_{i_m} L_m^{n-g_m+1},$$

then we can consider $\phi = \prod_{i=1}^m ((L_i)^\perp)^{g_i}$. By definition of GAD representation we have $\phi \circ f_i = 0$ for all $i = 1, \ldots, k$, so $\phi \in \ker T^L_{n,k}$ and $\operatorname{rank} T^L_{n,k} \leq 2k-1$.

□

Observation 2.5.5. *In particular we can observe that if* L *belong to a (n-1)-secant* $\mathbb{P}^{n-1}$ *generated by* $q_1, \ldots, q_n$*, then there exists an element* $\phi \in H^0(\mathcal{O}^n_{\pi_k(C_n)}) \cong S^{n-1}V^\vee = T_{n-1}$ *such that* $\phi \in \ker(T^L_{n,k}) = \bigcap_i \ker(Cat_{f_i}(1, n-1))$*, in fact we can take* $\phi = \prod_{i=1}^n (L^\perp_{q_i})^{n-1}$.

We can compute the codimension of the variety of every $\mathbb{P}^k$ *which belong to some n-secant* $\mathbb{P}^{n-1}$ *constructing an incidence variety:*

$$I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n), \pi \in S, L \subset S\},$$

where S *is the set of all n-secant* $\mathbb{P}^{n-1}$ *to* C_n*. In the usual way we can compute the codimension of the image of this incidence variety in* $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$*. We will indicated with* ϕ_1 *and* ϕ_2 *the natural projections:*

$$\begin{tikzcd} & I_S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) & & S, \end{tikzcd}$$

so the codimension in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ *of* $\phi_1(I_S)$ *is equal to* $\dim Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) - \dim S - \dim \phi_2^{-1}(S) = k(n+1-k) - n + 1 - k(n-1-k)$*. We have that this variety has* *codimension $2k - n + 1$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]).*

Lemma 2.5.6. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to some $(n-r)-$secant $\mathbb{P}^{n-r-1}$ to the rational normal curve C_n in $\mathbb{P}^n$, then we have $\operatorname{rank} T^L_{n,k} \leq n - r$ for $1 \leq r < n-k$.*

Proof. If $L \cong \mathbb{P}^{k-1}$ belongs to some $(n-r)-$secant $\mathbb{P}^{n-r-1}$, then there exist k points $p_1, ..., p_k \in L$ which generate L and the corresponding binary forms f_i are generating by two forms $Ann(f_i) = (\alpha, \beta_i)$ with $\deg(\alpha) = n-r$, α has only simple roots and $\deg(\beta_i) = r+2$ without common zeros with α. We have $\dim(\alpha)_{n-1} = r$ and $\dim \bigcap_i (\alpha, \beta_i)_{n-1} \geq r$, so $\operatorname{rank} T^L_{n,k} = n - \dim \bigcap_i (\alpha, \beta_i)_{n-1} \leq n - r$. □

Lemma 2.5.7. *The codimension in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ of the variety of $L \cong \mathbb{P}^{k-1}$ in $\mathbb{P}^n$ belonging to some $(n-r)$-secant $\mathbb{P}^{n-r-1}$ to the rational normal curve in $\mathbb{P}^n$ is $k - n + r + kr$.*

Proof. Infact we can consider the incidence variety $I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n), \pi \in S, L \subset S\}$ where S is the set of all $(n-r)$-secant $\mathbb{P}^{n-r-1}$ to the rational normal curve in $\mathbb{P}^n$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. That calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

In particular for $r = n - k - 1$ we have:

Theorem 2.5.8. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to some $(k+2)-$secant $\mathbb{P}^{k+1}$ to the rational normal curve C_n in $\mathbb{P}^n$, then we have:*

$$T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-k-1} \oplus \mathcal{O}(n+1+k).$$

Lemma 2.5.9. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a pencil Φ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$, then $\operatorname{rank} T^L_{n,k} = n-2$. The converse is generically true.*

Proof. $\Rightarrow$ Let L be a $\mathbb{P}^{k-1}$ belongs to a pencil $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0 \pi_0 + \lambda_1 \pi_1, \ \forall \lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1\}$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$, where π_0, π_1 are $(n-1)-$secant $\mathbb{P}^{n-2}$. Let $q^i_1, ..., q^i_{n-1} \in C_n$ be the points which generate π_i. Then there exists k points $p_1, ..., p_k$ which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^1$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{n-1,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c^i_{1,\lambda} L^n_{1,\lambda} + ... + c^i_{n-1,\lambda} L^n_{n-1,\lambda},$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., n-1$, so L belongs to a $\mathbb{P}^{n-4}$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^{n-2}$ there exists a differential form $\phi_\lambda \in T_{n-1}$ such that $\phi_\lambda \circ f_i = 0$. Moreover there exist two differential forms $\phi_0, \phi_1 \in T_{n-1}$ which for each $\lambda \in \mathbb{P}^1$ we have $\phi_\lambda = \lambda_0\phi_0 + \lambda_1\phi_1$, in particular $\phi_j \circ f_i = 0$, so $\operatorname{rank} T^L_{n,k} = n-2$.

$\Leftarrow$ If $\operatorname{rank} T^L_{n,k} = n-2$, then there exist two binary form $\phi_0, \phi_1 \in T_{n-1}$ such that however we consider the generating points $p_1, ..., p_k \in \mathbb{P}^n$ of L, it is $\phi_\alpha \circ f_i = (\alpha_0\phi_0 + \alpha_1\phi_1) \circ f_i = 0$ for all $\alpha = [\alpha_0, \alpha_1] \in \mathbb{P}^1$ and $i = 1, ..., k$, where $f_i \in S_n$ is the binary form corresponding to p_i. In particular $\phi_j \circ f_i = 0$, so if we consider the primary decomposition of $\phi_\alpha = \prod_{l=1}^{n-1} \phi^l_\alpha$ and we indicate with $(L_{l,\alpha})^\perp = \phi^l_\alpha$. Therefore $f_1, ..., f_k$ can be decompose in ∞^1 different simultaneously ways, i.e.:

$$f_i = c^i_{1,\alpha} L^n_{1,\alpha} + ... + c^i_{n-1,\alpha} L^n_{n-1,\alpha},$$

for all $\alpha = [\alpha_0, \alpha_1] \in \mathbb{P}^1$, so L belongs to a $\mathbb{P}^{n-4}$.

In other words L belongs to a pencil $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + \lambda_1\pi_1, \ \forall\lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^{n-2}\}$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$. Clearly it can happen that some ϕ_i have any multiple roots, so we have all possible degeneration of the linear system Φ.

$\square$

Lemma 2.5.10. *The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a pencil Φ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible variety of codimension $4k - 2n + 4$.*

Proof. We can observe that a pencil Φ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ corresponds to the linear system of dimension 2 of binary forms of degree $n-1$, therefore the set of these linear system corresponds to $Gr(\mathbb{P}^1, \mathbb{P}^{n-1})$ which is irreducible and $\dim Gr(\mathbb{P}^1, \mathbb{P}^{n-1}) = 2(n-2)$. Each projection linear space L belongs to a $\mathbb{P}^{n-4}$, so the dimension of the fibre is $k(n-3-k)$. Therefore the variety of linear spaces L that belong to a pencil Φ is an irreducible variety of codimension

$4k-2n+4$ in $Gr(\mathbb{P}^{k-1},\mathbb{P}^n)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). $\square$

Observation 2.5.11. *By Theorem 1.2.14 (see [Verdier, 1983]) $T^n_{n-k}((n+1)^2,(n+2)^{2n-3k-4},(n+3)^{2k-n+2})$ is an irreducile variety of* $\operatorname{codim}(T^n_{n-k}((n+1)^2,(n+2)^{2n-3k-4},(n+3)^{2k-n+2}) = 4k-2n+4$.

By Theorem 1.2.14[Verdier, 1983] we have:

Theorem 2.5.12. *The centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a pencil Φ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ if and only if $T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^2 \oplus \mathcal{F}$, with $\mathcal{F}$ a rank $n-2-k$ vector bundle on $\mathbb{P}^1$ and* $\deg(\mathcal{F}^\vee(n+1)) = -k$.

We can prove a more general result:

Lemma 2.5.13. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system of dimension r of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$, then* $\operatorname{rank} T^L_{n,k} = 2k-r$ *for $1 \le r < k$.*

Proof. Let L be a $\mathbb{P}^{k-1}$ belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{r-1}\pi_{r-1},\ \ \forall\lambda = [\lambda_0, ..., \lambda_{r-1}] \in \mathbb{P}^{r-1}\}$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$, where $\pi_0, ..., \pi_{r-1}$ are $(n-1)-$secant $\mathbb{P}^{n-2}$. Let $q^i_1, ..., q^i_{n-1} \in C_n$ be the points which generate π_i. Then there exist k points $p_1, ..., p_k$ which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^{r-1}$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{n-1,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c^i_{1,\lambda}L^n_{1,\lambda} + ... + c^i_{n-1,\lambda}L^n_{n-1,\lambda},$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., n-1$. This means that L belongs to a $\mathbb{P}^{n-2r}$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^{r-1}$ there exists a differential forms $\phi_\lambda \in T_{n-1}$ such that $\phi_\lambda \circ f_i = 0$. Moreover there exist r differential form $\phi_0, ..., \phi_{r-1} \in T_{n-1}$ which for each $\lambda \in \mathbb{P}^{r-1}$ we have $\phi_\lambda = \lambda_0\phi_0 + ... + \lambda_{r-1}\phi_{r-1}$, in particular $\phi_j \circ f_i = 0$ for all $j = 0, ..., r-1$ and $i = 1, ..., k$, so $\operatorname{rank} T^L_{n,k} = 2k-r$.

$\square$

Lemma 2.5.14. *Let $n > 2k+2r-1$. The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible variety of codimension $r(2k-n+1)$.*

Proof. We can observe that the linear system Φ of affine dimension r of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ correspond to the linear system of dimension r of binary forms of degree $n-1$, therefore the set of these linear system corresponds to $Gr(\mathbb{P}^{r-1}, \mathbb{P}^{n-1})$ which is irreducible and $\dim Gr(\mathbb{P}^{r-1}, \mathbb{P}^{n-1}) = r(n-r)$. Each projection linear space L belongs to a $\mathbb{P}^{n-2r}$, so the dimension of the fibre is $k(n-2r-k+1)$, which is ≥ 0 with the condition $n \geq 2k+2r-1$. Therefore the variety of lines L that belong to a linear system Φ is an irreducible variety of codimension $r(2k-n+1)$ in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

By Theorem 1.2.14[Verdier, 1983] we have:

Theorem 2.5.15. *Let $k < n-r+1$. The centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system Φ of affine dimension r of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ if and only if $T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^r \oplus \mathcal{F}$, with $\mathcal{F}$ a rank $n-r-k$ vector bundle on $\mathbb{P}^1$ and $\deg(\mathcal{F}^\vee(n+1)) = -k$ for $0 \leq r \leq n-k-2$.*

Corollary 2.5.16. *Let $k < n-r+1$. The union of all schemes $T^n_{n-k}((n+1)^r, spt(\mathcal{F}))$, with $spt(\mathcal{F})$ is the splitting type of $\mathcal{F}$ a rank $\operatorname{rank}(T^L_{n,k}) - k$ vector bundle on $\mathbb{P}^1$ and $\deg(\mathcal{F}^\vee(n+1)) = -k$ is an irreducible variety of codimension $r(2k-n+1)$ formed by the linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension r of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$.*

Main Theorem for Restricted Tangent Bundle

Proposition 2.5.17. *The generic splitting type of a vector bundle $\mathcal{F}$ of rank* $\mathrm{rank}(T^L_{n,k})-k$ *on* $\mathbb{P}^1$ *and* $\deg(\mathcal{F}^\vee(n+1)) = -k$ *such that* $\mathcal{F} \cong \bigoplus_{i=0}^{\mathrm{rank}(T^L_{n,k})-k} \mathcal{O}(l_i)$ *with* $l_i \geq n+2$ *is the following:*

$$((n+1+B)^{A-k+B\cdot A}, (n+2+B)^{k-B\cdot A}).$$

where we have indicated $A = \mathrm{rank}(T^L_{n,k}) - k$ *and* $B = \lfloor \frac{k}{\mathrm{rank}(T^L_{n,k})-k} \rfloor$.

By Theorem 2.4.17 and Theorem 2.5.15 we have covered all the possible cases and we can state the main result of this chapter, which is the following by the proposition above:

Theorem 2.5.18 (Main Theorem for Restricted Tangent Bundle)**.** *The following conditions are equivalent:*

i) the centre of projection L is general in the (irreducible) variety of those $\mathbb{P}^{k-1}$ which belongs to a linear system Φ of affine dimension $n-\mathrm{rank}(T^L_{n,k})$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$;

ii) the curve of degree n projected from $L = \mathbb{P}^{k-1}$ has $T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \cong \mathcal{O}(n+1)^{n-\mathrm{rank}(T^L_{n,k})} \oplus \mathcal{O}(n+1+B)^{A-k+B\cdot A} \oplus \mathcal{O}(n+2+B)^{k-B\cdot A})$, where we have indicated $A = \mathrm{rank}(T^L_{n,k}) - k$ and $B = \lfloor \frac{k}{\mathrm{rank}(T^L_{n,k})-k} \rfloor$.

Corollary 2.5.19. *Let* $k < 2\,\mathrm{rank}(T^L_{n,k}) - n + 1$. $T^n_{n-k}((n+1)^{n-\mathrm{rank}(T^L_{n,k})}, (n+1+B)^{A-k+B\cdot A}, (n+2+B)^{k-B\cdot A})$*, where we have indicated* $A = \mathrm{rank}(T^L_{n,k}) - k$ *and* $B = \lfloor \frac{k}{\mathrm{rank}(T^L_{n,k})-k} \rfloor$*, is an irreducible variety of codimension:*

$$(n - \mathrm{rank}\, T^L_{n,k})(2k - \mathrm{rank}\, T^L_{n,k}),$$

formed by the linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension $n - \mathrm{rank}(T^L_{n,k})$ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$.

2.5.1 Restricted Tangent Bundle of Rational Curves in Codim $n-3$

In the case of projections in codimension $n-3$, i.e. we get a space curve, we have:

$$0 \longrightarrow (T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1) \longrightarrow \mathcal{O}^n_{\pi_{n-3}(C_n)} \xrightarrow{\mathcal{T}^L_{n,n-3}} \mathcal{O}_{\mathbb{P}^1}(1)^{n-3} \longrightarrow 0\,. \tag{2.14}$$

So we have that $\deg((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1)) = -n+3$ and

$$h^0((T\mathbb{P}^3|_{\pi_{n-3}(C_n)})^\vee(n+1)) = n - \operatorname{rank}(T^L_{n,n-3}) = \ker(T^L_{n,n-3}).$$

In this case as shown in Lemma 2.1.32 we have three possibility:

a) $\operatorname{rank}(T^L_{n,n-3}) = n$ if and only if

$$T\mathbb{P}^3|_{\pi_{n-3}(C_n)} = \mathcal{O}(t_1) \oplus \mathcal{O}(t_2) \oplus \mathcal{O}(t_3),$$

where $n+2 \le t_1 \le t_2 \le t_3 \le 2n-4$ and $t_1+t_2+t_3 = 4n$.

b) $\operatorname{rank}(T^L_{n,n-3}) = n-1$ if and only if

$$T\mathbb{P}^3|_{\pi_{n-3}(C_n)} = \mathcal{O}(t_1) \oplus \mathcal{O}(t_2) \oplus \mathcal{O}(n+1),$$

where $n+2 \le t_1 \le t_2 \le 2n-3$ and $t_1+t_2 = 3n-1$.

c) $\operatorname{rank}(T^L_{n,n-3}) = n-2$ if and only if

$$T\mathbb{P}^3|_{\pi_{n-3}(C_n)} = \mathcal{O}(2n-2) \oplus \mathcal{O}(n+1) \oplus \mathcal{O}(n+1).$$

In particular for case c):

Theorem 2.5.20. *If the centre of projection $L \cong \mathbb{P}^{n-4}$ belongs to some $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, then we have:*

$$T\mathbb{P}^3|_{\pi_{n-3}(C_n)} \cong \mathcal{O}(n+1)^2 \oplus \mathcal{O}(2n-2).$$

Lemma 2.5.21. *The codimension in $Gr(\mathbb{P}^{n-4}, \mathbb{P}^n)$ of the variety of all $L \cong \mathbb{P}^{n-4}$ in $\mathbb{P}^n$ belonging to some $(n-r)$-secant $\mathbb{P}^{n-r-1}$ to the rational normal curve in $\mathbb{P}^n$ is $rn-2r-3$.*

Observation 2.5.22. *By Theorem 1.2.14 (see [Verdier, 1983]) $T^n_3((n+1)^2, 2n-2)$ is an irreducible variety of* $\operatorname{codim}(T^n_3((n+1)^2, 2n-2)) = 2(n-4) \le 2n-7$.

Corollary 2.5.23. *The variety of linear spaces $L \cong \mathbb{P}^{n-4}$ that, as centre of projection, give a rational curve of degree n in $\mathbb{P}^3$ which has the restricted tangent bundle $T\mathbb{P}^3_{\pi_{n-3}(C_n),\mathbb{P}^3} \cong \mathcal{O}(n+1)^2 \oplus \mathcal{O}(2n-2)$ has an irreducible subvariety of codimension $2n-7$ in $Gr(\mathbb{P}^{n-4}, \mathbb{P}^n)$ formed by the linear spaces L belong to some $(n-2)-$secant $\mathbb{P}^{n-3}$.*

Lemma 2.5.24. *The variety of linear spaces $L \cong \mathbb{P}^{n-4}$ that belong to a linear system Φ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible variety of codimension $2(n-5)$.*

Theorem 2.5.25. *The centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a pencil Φ of $(n-1)-$secant $\mathbb{P}^{n-2}$ to the rational normal curve C_n in $\mathbb{P}^n$ if and only if $T\mathbb{P}^3_{\pi_{n-3}(C_n),\mathbb{P}^3} \cong \mathcal{O}(n+1)^2 \oplus \mathcal{O}(2n-2)$.*

Chapter 3

Normal Bundle of Rational Curves

3.1 Normal Bundle of Rational Curves in $\mathbb{P}^3$

Let $C_n \subset \mathbb{P}^n$ be the rational normal curve of degree n with $\nu_n : \mathbb{P}^1 \to \mathbb{P}^n$ and $\nu_n(\mathbb{P}^1) = C_n$, where ν_n is the Veronese map. Let $\pi_{n-3}(C_n)$ the rational curve obtained from C_n by projection from a $(n-4)$-dimensional linear subspace $L \subset \mathbb{P}^n$ on complementary $\mathbb{P}^3 \subset \mathbb{P}^n$, we will suppose that $\pi_{n-3}(C_n)$ has only ordinary singularities. We denote by $f : \mathbb{P}^1 \to \mathbb{P}^3$ the composition morphism $f := \pi_{n-3} \circ \nu_n$. By Prop.1.1.9 and Def.3.4.5 [Sernesi, 2006] we have the following exact sequence:

$$0 \longrightarrow Tf(\mathbb{P}^1) \longrightarrow T\mathbb{P}^3 \mid_{f(\mathbb{P}^1)} \longrightarrow N_{\pi_{n-3}(C_n);\mathbb{P}^3} \longrightarrow T^1_{f(\mathbb{P}^1)} \longrightarrow 0\,, \tag{3.1}$$

where $T^1_{f(\mathbb{P}^1)}$ is the first cotangent sheaf of $\mathbb{P}^1$ and N_f is the normal sheaf of f (see [Sernesi, 2006]). If f is a closed immersion we have:

$$0 \longrightarrow T\mathbb{P}^1 \longrightarrow \mathrm{Hom}(f^*\Omega^1_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow N_f \longrightarrow 0\,. \tag{3.2}$$

Moreover if f is smooth we have that $N_f = 0$ and also $T^1_{f(\mathbb{P}^1)} = 0$.

Let $N'_{f(\mathbb{P}^1);\mathbb{P}^3} = \ker[N_{f(\mathbb{P}^1);\mathbb{P}^3} \to T^1_{f(\mathbb{P}^1)}]$ be the equisingular normal sheaf of $f(\mathbb{P}^1)$ in $\mathbb{P}^3$, we have the short exact sequence:

$$0 \longrightarrow Tf(\mathbb{P}^1) \longrightarrow T\mathbb{P}^3 \mid_{f(\mathbb{P}^1)} \longrightarrow N'_{f(\mathbb{P}^1);\mathbb{P}^3} \longrightarrow 0\,. \tag{3.3}$$

Let $J(\nu_n)$ be the Jacobian matrix of ν_n:

$$J(\nu_n) = \begin{pmatrix} ns^{n-1} & \dots & t^{n-1} & 0 \\ 0 & s^{n-1} & \dots & nt^{n-1} \end{pmatrix}^t.$$

The Euler sequences for $\mathbb{P}^1$ (see 2.1 with $n = 1$) and for $\mathbb{P}^3$ restricted to $\pi_{n-3}(C_n)$ (see 2.2) and the exact sequence of the Equisingular Normal Bundle of $\pi_{n-3}(C_n)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow T\mathbb{P}^3|_{\pi_{n-3}(C_n)} \longrightarrow N'_{\pi_{n-3}(C_n);\mathbb{P}^3} \longrightarrow 0\ , \tag{3.4}$$

give us the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{\pi_{n-3}(C_n)} \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^1}(1)^2 \arrow[d] & \mathcal{O}_{\pi_{n-3}(C_n)}(n)^4 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(2) \arrow[r] \arrow[d] & T\mathbb{P}^3|_{\pi_{n-3}(C_n)} \arrow[r] \arrow[d] & N'_{\pi_{n-3}(C_n);\mathbb{P}^3} \arrow[r] & 0 \\
 & 0 & 0 & & .
\end{tikzcd} \tag{3.5}$$

We can observe that the Jacobian matrix of $\pi_{n-3} \circ \nu_n$ gives a map:

$$\mathcal{O}_{\pi_{n-3}(C_n)}(1)^2 \xrightarrow{J(\pi_{n-3}\circ\nu_n)} \mathcal{O}_{\pi_{n-3}(C_n)}(n)^4\ . \tag{3.6}$$

So we can complete the previous diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{C_n} \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^1}(1)^2 \arrow[r, "J(\pi_{n-3}\circ\nu_n)"] \arrow[d] & \mathcal{O}_{\pi_{n-3}(C_n)}(n)^4 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(2) \arrow[r] \arrow[d] & T\mathbb{P}^3|_{\pi_{n-3}(C_n)} \arrow[r] \arrow[d] & N'_{\pi_{n-3}(C_n);\mathbb{P}^3} \arrow[r] & 0 \\
 & 0 & 0 & & .
\end{tikzcd} \tag{3.7}$$

By the above diagram (3.7) the following exact sequence holds:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)^2 \xrightarrow{J(\pi_{n-3}\circ\nu_n)} \mathcal{O}_{\pi_{n-3}(C_n)}(n)^4 \longrightarrow N'_{\pi_{n-3}(C_n);\mathbb{P}^3} \longrightarrow 0\ . \tag{3.8}$$

By tensoring (3.8) with $\mathcal{O}_{\mathbb{P}^1}(-n)$, we obtain this exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \xrightarrow{J(\pi_{n-3}\circ\nu_n)} \mathcal{O}^4_{\pi_{n-3}(C_n)} \longrightarrow N'_{\pi_{n-3}(C_n);\mathbb{P}^3}(-n) \longrightarrow 0 \,. \tag{3.9}$$

Observation 3.1.1. *We observe that* $degN'_{\pi_{n-3}(C_n);\mathbb{P}^3} = 4n-2$ *and it is a vector bundle of rank 2.*

Moreover by Grothendieck-Segre's theorem (see [Grothendieck, 1957]) $N_{\pi_{n-3}(C_n);\mathbb{P}^3}$ *splits in:*

$$\mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \text{ with } n_1, n_2 \in \mathbb{Z} \text{ such that } n_1 + n_2 = 4n-2,$$

where we abbreviate $\mathcal{O}_{\pi_{n-3}(C_n)}$ *to* $\mathcal{O}$ *and without loss of generality we can take* $n_1 \le n_2$.

Let us denote by:

$$Syz(J(\nu_n)) = \begin{pmatrix} t^2 & -2st & s^2 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & t^2 & -2st & s^2 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & t^2 & -2st & s^2 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & t^2 & -2st & s^2 \end{pmatrix}$$

the syzygy of the Jacobian matrix $J(\nu_n)$, where $Syz(J(\nu_n))$ is a matrix $(n-1) \times (n+1)$. So we have the following exact sequence for $J(\nu_n)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \xrightarrow{J(\nu_n)} \mathcal{O}^{n+1}_{C_n} \xrightarrow{Syz(J(\nu_n))} \mathcal{O}_{C_n}(2)^{n-1} \longrightarrow 0 \,, \tag{3.10}$$

which is similar to 3.9.

Therefore and for sequence 2.3 twisted by $\mathcal{O}(-n)$:

$$0 \longrightarrow \mathcal{O}^{n-3}_{C_n} \xrightarrow{P} \mathcal{O}^{n+1}_{C_n} \longrightarrow \mathcal{O}^4_{C_n} \longrightarrow 0, \tag{3.11}$$

we can obtain:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & \mathcal{O}^{n-3}_{C_n} \arrow[d, "P"] & & \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\nu_n)"] & \mathcal{O}^{n+1}_{C_n} \arrow[r, "Syz(J(\nu_n))"] \arrow[d] & \mathcal{O}_{C_n}(2)^{n-1} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\pi_{n-3}\circ\nu_n)"] & \mathcal{O}^4_{\pi_{n-3}(C_n)} \arrow[r] \arrow[d] & N'_{\pi_{n-3}(C_n);\mathbb{P}^3}(-n) \arrow[r] & 0 \\ & & 0 & & \end{tikzcd} \tag{3.12}$$

,

from which we can complete to the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{C_n}^{n-3} \arrow[r, "\cong"] \arrow[d, "P"] & \mathcal{O}_{C_n}^{n-3} \arrow[d, "(\mathcal{N}_{n,n-3}^L)^t"] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\nu_n)"] & \mathcal{O}_{C_n}^{n+1} \arrow[r, "Syz(J(\nu_n))"] \arrow[d] & \mathcal{O}_{C_n}(2)^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\pi_{n-3}\circ\nu_n)"] & \mathcal{O}_{\pi_{n-3}(C_n)}^4 \arrow[r] \arrow[d] & N'_{\pi_{n-3}(C_n);\mathbb{P}^3}(-n) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd} \tag{3.13}$$

where the map $(\mathcal{N}_{n,n-3}^L)^t$ is:

$$(\mathcal{N}_{n,n-3}^L)^t = Syz(J(\nu_n)) \cdot \begin{pmatrix} a_0^1 & \dots & a_0^{n-3} \\ \vdots & \ddots & \vdots \\ a_n^1 & \dots & a_n^{n-3} \end{pmatrix} =$$

$$\begin{pmatrix} a_0^1t^2 - 2a_1^1ts + a_2^1s^2 & a_1^1t^2 - 2a_2^1ts + a_3^1s^2 & \dots & a_{n-2}^1t^2 - 2a_{n-1}^1ts + a_n^1s^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_0^{n-3}t^2 - 2a_1^{n-3}ts + a_2^{n-3}s^2 & a_1^{n-3}t^2 - 2a_2^{n-3}ts + a_3^{n-3}s^2 & \dots & a_{n-2}^{n-3}t^2 - 2a_{n-1}^{n-3}ts + a_n^{n-3}s^2 \end{pmatrix}^t.$$

It is a $(n-1)\times(n-3)$ matrix. The last exact column of (3.13):

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{n-3} \xrightarrow{(\mathcal{N}_{n,n-3}^L)^t} \mathcal{O}_{\pi_{n-3}(C_n)}(2)^{n-1} \longrightarrow N'_{\pi_{n-3}(C_n);\mathbb{P}^3}(-n) \longrightarrow 0 \tag{3.14}$$

gives us a boundary information on the splitting type of $N'_{\pi_{n-3}(C_n),\mathbb{P}^3}$:

Lemma 3.1.2. *If $\pi_{n-3}(C_n)$ has only ordinary singularities, then the splitting type of $N'_{\pi_{n-3}(C_n),\mathbb{P}^3}$ must be (n_1, n_2) with $n+2 \le n_1 \le n_2 \le 2n-6$ and $n_1+n_2 = 4n-2$.*

Moreover the exact sequence 3.14 gives rise by duality and tensorizing by $\mathcal{O}_{\mathbb{P}^1}(2)$:

$$0 \longrightarrow N'^{\vee}_{\pi_{n-3}(C_n);\mathbb{P}^3}(n+2) \longrightarrow \mathcal{O}_{\pi_{n-3}(C_n)}^{n-1} \xrightarrow{\mathcal{N}_{n,n-3}^L} \mathcal{O}_{\mathbb{P}^1}(2)^{n-3} \longrightarrow 0\ . \tag{3.15}$$

Observation 3.1.3. *We can observe that if $\pi_{n-3}(C_n)$ has only ordinary singularities, then the map of sheaves of differentials is surjective, so we have:*

$$N'^{\vee}_{\pi_{n-3}(C_n);\mathbb{P}^3} \cong N^{\vee}_{\pi_{n-3}(C_n);\mathbb{P}^3}.$$

If we pass to the exact cohomology sequence we get:

$$0 \longrightarrow H^0(N^\vee_{\pi_{n-3}(C_n);\mathbb{P}^3}(n+2)) \longrightarrow H^0(\mathcal{O}^{n-1}_{\pi_{n-3}(C_n)}) \xrightarrow{N^L_{n,n-3}} H^0(\mathcal{O}_{\mathbb{P}^1}(2)^{n-3}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1(N^\vee_{\pi_{n-3}(C_n);\mathbb{P}^3}(n+2)) \longrightarrow 0\,,$$

where we have indicated with $N^L_{n,n-3}$ the $3(n-3)\times n-1$ matrix:

$$N^L_{n,n-3} = \begin{pmatrix} a^1_0 & \dots & a^1_{n-2} \\ -2a^1_1 & \dots & -2a^1_{n-1} \\ a^1_2 & \dots & a^1_n \\ \vdots & \ddots & \vdots \\ a^{n-3}_0 & \dots & a^{n-3}_{n-2} \\ -2a^{n-3}_1 & \dots & -2a^{n-3}_{n-1} \\ a^{n-3}_2 & \dots & a^{n-3}_n \end{pmatrix}.$$

Observation 3.1.4. *The rank of $N^L_{n,n-3}$ does not depend from the points generating L. Moreover let F_{p_i} be the binary form of degree n which corresponds to the point $p_i \in \mathbb{P}^n$. If we indicate with $Cat_{F_{p_i}}(2,n-2;2)$ the Hankel matrix $3\times n-1$ of F_{p_i} we have:*

$$\operatorname{rank} N^L_{n,n-3} = \operatorname{rank}\begin{pmatrix} Cat_{F_{p_1}}(2,n-2;2) \\ \vdots \\ Cat_{F_{p_{n-3}}}(2,n-2;2) \end{pmatrix}.$$

Observation 3.1.5. *We have that* $\deg(N^\vee_{\pi_{n-3}(C_n);\mathbb{P}^3}(n+2)) = -2n+6$ *and*

$$h^0(N^\vee_{\pi_{n-3}(C_n);\mathbb{P}^3}(n+2)) = n-1-\operatorname{rank}(N^L_{n,n-3}) = \dim\ker(N^L_{n,n-3}).$$

Therefore we have:

$$2 \le \operatorname{rank}(N^L_{n,n-3}) \le \min\{n-1, 3(n-3)\},$$

so

$$n-1-\min\{n, 2(n-3)\} \le h^0(N^\vee_{\pi_{n-3}(C_n);\mathbb{P}^3}(n+2)) \le n-3,$$

but as $\operatorname{rank}(N^\vee_{\pi_{n-3}(C_n);\mathbb{P}^3}(n+2)) = 2$ *we have that*

$$N^\vee_{\pi_{n-3}(C_n);\mathbb{P}^3}(n+2) \text{ splits in } \mathcal{O}(n'_1)\oplus\mathcal{O}(n'_2)$$

by Grothendieck-Segre's theorem (see [Grothendieck, 1957]) with

$$-2n+6 \le n'_1 \le n'_2 \le 0 \text{ and } n'_1+n'_2 = -2n+6.$$

So we have that:

$$\operatorname{rank}(N^L_{n,n-3}) \geq n-2,$$

otherwise one n'_i must be ≥ 1 which is impossible by above.

Remark 3.1.6. It's clear from the above consideration that the value of $\operatorname{rank}(N^L_{n,n-3})$ corresponds to some splitting types of $N'_{\pi_{n-3}(C_n);\mathbb{P}^3}$.

We can distinguish two cases:

A) $\min\{n-1, 3(n-3)\} = 3(n-3)$, so $n = 3, 4$;

B) $\min\{n-1, 3(n-3)\} = n-1$, so $n \geq 4$.

The cases above will be the subject of the next two sections.

3.1.1 Case $n = 3, 4$

Rational Curves of degree 3 in $\mathbb{P}^3$

Proposition 3.1.7. *The Normal Bundle of rational normal curve C_3 of degree 3 is $N_{\pi_{n-3}(C_n);\mathbb{P}^3} = \mathcal{O}(5) \oplus \mathcal{O}(5)$.*

Proof. Immediate from above. □

Rational Curves of degree 4 in $\mathbb{P}^3$

Let C_4 be a rational normal curve of degree 4 in $\mathbb{P}^4$ with usual Veronese embedding $\nu_4 : \mathbb{P}^1 \to \mathbb{P}^4$ and $\pi_1 : \mathbb{P}^4 \setminus \{p\} \to \mathbb{P}^3$ the projection from the point $p = (a_0, a_1, a_2, a_3, a_4) \in \mathbb{P}^4$. Thus we have:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{C_4} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{C_4} \arrow[d, "(\mathcal{N}^L_{4,1})^t"] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-3)^2 \arrow[r, "J(\nu_4)"] & \mathcal{O}^5_{C_4} \arrow[r, "Syz(J(\nu_4))"] \arrow[d] & \mathcal{O}_{C_4}(2)^3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-3)^2 \arrow[r, "J(\pi_1 \circ \nu_4)"] & \mathcal{O}^4_{\pi_1(C_4)} \arrow[r] \arrow[d] & N_{\pi_1(C_4);\mathbb{P}^3}(-4) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

So $N_{\pi_1(C_4);\mathbb{P}^3}(-4) = \mathcal{O}(2) \oplus \mathcal{O}(4)$ if and only if these three quadrics:

$$\begin{pmatrix} t^2 & -2st & s^2 & 0 & 0 \\ 0 & t^2 & -2st & s^2 & 0 \\ 0 & 0 & t^2 & -2st & s^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} =$$

$$\begin{pmatrix} a_0t^2 - 2a_1st + a_2s^2 \\ a_1t^2 - 2a_2st + a_3s^2 \\ a_2t^2 - 2a_3st + a_4s^2 \end{pmatrix}$$

are dependent. This is true if and only if

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ -2a_1 & -2a_2 & -2a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = \det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = 0$$

Note that we have actually proved that if p does not belong to the Secant Variety $\sigma_1(C_4)$ of C_4:

$$\sigma_1(C_4) = \{z = (z_0, z_1, ..., z_4) \in \mathbb{P}^4 : \operatorname{rank} \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix} = \operatorname{rank} Cat_z(2,4) < 3\},$$

then $N_{\pi_1(C_4);\mathbb{P}^3} = \mathcal{O}(7) \oplus \mathcal{O}(7)$.

Theorem 3.1.8. *Let $C_4 \subset \mathbb{P}^4$ the normal rational curve of degree 4 and $\pi_1 : \mathbb{P}^4 \setminus \{p\} \to \mathbb{P}^3$ be the projection from a point $p \in \mathbb{P}^4$. The rational curve $\pi_1(C_4) \subset \mathbb{P}^3$ has normal bundle balanced if and only if $p \notin \sigma_1(C_4)$. This is equivalent to be smooth for $\pi_1(C_4)$.*

3.1.2 Case $n > 4$

We have only three different possibilities:

Lemma 3.1.9. *a)* $\mathrm{rank}(N^L_{n,n-3}) = n-1$ *if and only if*

$$N_{\pi_{n-3}(C_n);\mathbb{P}^3} = \mathcal{O}(n_1) \oplus \mathcal{O}(n_2),$$

where $n+3 \leq n_1 \leq n_2 \leq 2n-8$ *and* $n_1 + n_2 = 4n-2$.

b) $\mathrm{rank}(N^L_{n,n-3}) = n-2$ *if and only if*

$$N_{\pi_{n-3}(C_n);\mathbb{P}^3} = \mathcal{O}(n+2) \oplus \mathcal{O}(3n-4).$$

No other possibilities can happen by conditions on n_i *(see Lemma 3.1.2, Observation 3.1.5).*

Unfortunately the $a)$ case are formed by several possible splitting types, but we can to study the rank of the following map in order to discriminate the exactly splitting type:

$$0 \longrightarrow H^0((N_{\pi_{n-3}(C_n);\mathbb{P}^3})^\vee(n+3))) \longrightarrow H^0(\mathcal{O}^{n-1}_{\pi_{n-3}(C_n)}(1)) \xrightarrow{N^L_{n,n-3}(1)} H^0(\mathcal{O}^{n-3}_{\pi_{n-3}(C_n)}(3)) \longrightarrow \cdots$$

$$\cdots \qquad \longrightarrow H^1((N_{\pi_{n-3}(C_n);\mathbb{P}^3})^\vee(n+3))) \longrightarrow 0\ ,$$

where we have indicated with $N^L_{n,n-3}(1)$ the $4(n-3) \times 2(n-1)$ matrix:

$$N^L_{n,n-3}(1) = \begin{pmatrix} a_0^1 & \dots & a_{n-2}^1 & 0 & \dots & 0 \\ -2a_1^1 & \dots & -2a_{n-1}^1 & a_0^1 & \dots & a_{n-2}^1 \\ a_1^2 & \dots & a_n^1 & -2a_1^1 & \dots & -2a_{n-1}^1 \\ 0 & \dots & 0 & a_2^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots & & & \\ a_0^{n-3} & \dots & a_{n-2}^{n-3} & 0 & \dots & 0 \\ -2a_1^{n-3} & \dots & -2a_{n-1}^{n-3} & a_0^{n-3} & \dots & a_{n-2}^{n-3} \\ a_2^{n-3} & \dots & a_n^{n-3} & -2a_1^{n-3} & \dots & -2a_{n-1}^{n-3} \\ 0 & \dots & 0 & a_2^{n-3} & \dots & a_n^{n-3} \end{pmatrix},$$

and so on:

$$0 \longrightarrow H^0((N_{\pi_{n-3}(C_n)};\mathbb{P}^3)^\vee(n+2+d))) \longrightarrow H^0(\mathcal{O}_{\pi_{n-3}(C_n)}(d)^{n-1}) \xrightarrow{N^L_{n,n-3}(d)} H^0(\mathcal{O}_{\mathbb{P}^1}(d+2)^{n-3}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1((N_{\pi_{n-3}(C_n)})^\vee(n+2+d))) \longrightarrow 0\ ,$$

with:

$$N^L_{n,n-3}(d) = \begin{pmatrix} N^{p_1}_{n,1}(d) \\ \vdots \\ N^{p_{n-3}}_{n,1}(d) \end{pmatrix},$$

where:

$$(N^{p_i}_{n,1}(d))^t = \left.\begin{pmatrix}
 & & & & & \overbrace{\begin{matrix}0 & 0 & \cdots & 0\end{matrix}}^{d} \\
 & (N^{p_i}_{n,1})^t & & & & \begin{matrix}\vdots & \vdots & \ddots & \vdots\end{matrix} \\
 & & & & & \begin{matrix}0 & 0 & \cdots & 0\end{matrix} \\
0 & & & & & \begin{matrix}0 & \cdots & 0\end{matrix} \\
\vdots & & (N^{p_i}_{n,1})^t & & & \begin{matrix}\vdots & \ddots & \vdots\end{matrix} \\
0 & & & & & \begin{matrix}0 & \cdots & 0\end{matrix} \\
\vdots & \cdots & \cdots & \cdots & & \cdots \quad \cdots \quad \cdots \quad \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots \quad \vdots \quad \vdots \\
\vdots & \cdots & \cdots & \cdots & & \cdots \quad \cdots \quad \cdots \quad \vdots \\
0 & \cdots & 0 & 0 & & \\
\vdots & \ddots & \vdots & \vdots & & (N^{p_i}_{n,1})^t \\
0 & \cdots & 0 & 0 & &
\end{pmatrix}\right\} (d+1)(n-1)$$

Observation 3.1.10. *We can observe that $N^L_{n,n-3}(d)$ is a $(d+3)(n-3)\times(d+1)(n-1)$ matrix, so:*

$$h^0(N^\vee_{\pi_{n-3}(C_n);\mathbb{P}^3}(d)) = (d+1)(n-1) - \operatorname{rank}(N^L_{n,n-3}(d)) = \dim\ker(N^L_{n,n-3}(d)),$$

and

$$\operatorname{rank}(N^L_{n,n-3}(d)) \le (d+1)(n-1) \textit{ if and only if } d \le n-4$$

Lemma 3.1.11. *We have that:*

$\ker(N^L_{n,n-3}(d)) \neq \emptyset$ *if and only if* $N_{\pi_{n-3}(C_n);\mathbb{P}^3} \cong \mathcal{O}(2n-1-\rho)\oplus\mathcal{O}(2n-1+\rho)$ *for* $n-3 \le \rho \le n-3+d$.

On the other hand we have:

$$0 \longrightarrow H^0((N_{\pi_{n-3}(C_n);\mathbb{P}^3})^\vee(2n-2))) \longrightarrow H^0(\mathcal{O}^{n-1}_{\pi_{n-3}(C_n)}(n-4)) \xrightarrow{N^L_{n,n-3}(n-4)} H^0(\mathcal{O}^{n-3}_{\pi_{n-3}(C_n)}(n-2)) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1((N_{\pi_{n-3}(C_n);\mathbb{P}^3})^\vee(2n-2))) \longrightarrow 0\ ,$$

where we have indicated with $N^L_{n,n-3}(1)$ the $4(n-3)\times 2(n-1)$ matrix.

Observation 3.1.12. *We can observe that* $\deg(N^{\vee}_{\pi_{n-3};\mathbb{P}^3}(2n-2)) = -2$*, so:*

$N^{\vee}_{\pi_{n-3};\mathbb{P}^3}(2n-2) \cong \mathcal{O}(-1+\dim\ker(N^L_{n,n-3}(n-4))) \oplus \mathcal{O}(-1+\dim\ker(N^L_{n,n-3}(n-4)))$

Lemma 3.1.13. *We have that:*

$\dim\ker(N^L_{n,n-3}(n-4)) \geq k$ *if and only if* $N_{\pi_{n-3}(C_n);\mathbb{P}^3} \cong \mathcal{O}(2n-1-k) \oplus \mathcal{O}(2n-1+k)$,

or equivalently:

$$\operatorname{rank}(N^L_{n,n-3}(n-4)) \leq r \text{ if and only if}$$

$$N_{\pi_{n-3}(C_n);\mathbb{P}^3} \cong \mathcal{O}(2n-1-\rho) \oplus \mathcal{O}(2n-1+\rho) \text{ for } n-3 \leq \rho \leq (n-3)(n-1)-r.$$

Observation 3.1.14. *We can remark that for the above consideration that* $\rho = \dim\ker(N^L_{n,n-3}(n-4))$.

We define the varieties which parametrize the subscheme of the Hilbert scheme $N_3^n(n_1, n_2)$ as intersection of some of the following varieties:

$$V(N^L_{n,n-3}(d))^r := \{L \in Gr(\mathbb{P}^{n-4}, \mathbb{P}^n) : \operatorname{rank} N^L_{n,n-3}(d) \leq r\}.$$

These are subvarieties of $Gr(\mathbb{P}^{n-4}, \mathbb{P}^n)$.

Since the rank condition is invariant under the action of $\mathrm{SL}(n-3, \mathbb{C})$ we are interested to study the determinantal varieties in $\operatorname{Hom}(H^0(\mathcal{O}^n(d)), H^0(\mathcal{O}^{n-3}(d+2))$:

$$D(N^L_{n,n-3}(d))^r = \{p_1, ..., p_{n-3} \in \mathbb{P}^n : \operatorname{rank} N^L_{n,n-3}(d) \leq r\}.$$

Proposition 3.1.15. *We can compute the maximal codimension of the above varieties:*

$\operatorname{codim} V(N^L_{n,n-3}(d))^r = \operatorname{codim} V(N^L_{n,n-3}(d))^r \leq ((d+3)(n-3)-r)((d+1)(n-1)-r)$.

By [Sacchiero, 1982] we have:

Theorem 3.1.16. *There exists a stratification of* $Gr(\mathbb{P}^{n-4}, \mathbb{P}^n)$*:*

$$\emptyset \neq V(N^L_{n,n-3})^{n-2} \subset V(N^L_{n,n-3}(1))^{2n-3} \subset ...$$

$$... \subset V(N^L_{n,n-3}(d))^{(d+1)(n-1)-1} \subset ... \subset V(N^L_{n,n-3}(n-4))^{(n-3)(n-1)-1} \subset Gr(\mathbb{P}^{k-1}, \mathbb{P}^n),$$

or equivalently:

$$\emptyset \neq V(N^L_{n,n-3}(n-4))^{(n-3)(n-2)} \subset V(N^L_{n,n-3}(n-4))^{(n-3)(n-1)-(n-4)} \subset ...$$

$$... \subset V(N^L_{n,n-3}(n-4))^{(n-3)(n-1)-\rho} \subset ... \subset V(N^L_{n,n-3}(n-4))^{(n-3)(n-1)-1} \subset Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$$

such that:

1) *$L \in V(N^L_{n,n-3})^{(n-3)(n-1)-\rho}$ (respectively $L \in V(N^L_{n,n-3}(d))^{(d+1)(n-1)-1}$ with $d = \rho - (n-3)$), for $1 \leq \rho \leq n-3$ if and only if $N_{\pi_{n-3}(C_n);\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(2n-1-\overline{\rho}) \oplus \mathcal{O}_{\mathbb{P}^1}(2n-1+\overline{\rho})$ with $\overline{\rho} \geq \rho$.*

2) *$V(N^L_{n,n-3})^{(n-3)(n-1)-\rho}$ (respectively $V(N^L_{n,n-3}(d))^{(d+1)(n-1)-1}$), for $1 \leq \rho \leq n-3$, is a quasi-projective, integral, Cohen-Macaulay variety.*

Invariant Approach

Let U be a vector complex space of dimension 2, we can write down the above diagram as following:

$$\begin{tikzcd}
 & & \mathbb{C}^{n-3}\otimes\mathcal{O}_{C_n} \arrow[d] & \mathbb{C}^{n-3}\otimes\mathcal{O}_{C_n} \arrow[d] & \\
0 \arrow[r] & U\otimes\mathcal{O}_{\mathbb{P}^1}(-n+1) \arrow[r, "J(\nu_n)"] & S^nU\otimes\mathcal{O}_{C_n} \arrow[r, "Syz(J(\nu_n))"] \arrow[d] & S^{n-2}U\otimes\mathcal{O}_{C_n}(2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U\otimes\mathcal{O}_{\mathbb{P}^1}(-n+1) \arrow[r, "J(\pi_{n-3}\circ\nu_n)"] & \frac{S^nU}{\mathbb{C}^{n-3}}\otimes\mathcal{O}_{\pi_{n-3}(C_n)} \arrow[r] & N_{\pi_{n-3}(C_n);\mathbb{P}^3}(-n) \arrow[r] & 0.
\end{tikzcd}$$

where the first row is SL(2)$-$invariant, but the second one is not.

By (3.15) we have the following diagram:

(3.16)

$$\begin{tikzcd}
0 \arrow[r] & H^0(N^{\vee}_{\pi_{n-3}(C_n);\mathbb{P}^3}(2n-2)) \arrow[r] \arrow[d] & H^0(\mathcal{O}^{n-1}_{C_n}(n-4)) \arrow[r] \arrow[d] & H^0(\mathcal{O}^{n-3}_{C_n}(n-2)) \arrow[r] & 0 \\
 & H^0(\mathcal{O}^4_{\pi_{n-2}(C_n)}(n-2)) \arrow[r] & H^0(\mathcal{O}^{n+1}_{C_n}(n-2)) \arrow[ur] & &
\end{tikzcd},$$

that is:

$$\begin{tikzcd}
0 \arrow[r] & S^mU \arrow[r] \arrow[d] & S^{n-4}U^*\otimes S^{n-2}U \arrow[r] \arrow[d] & S^{n-2}U^*\otimes S^{n-4}U \arrow[r] & 0 \\
 & S^{n-2}U^*\otimes S^3U \arrow[r] & S^{n-2}U^*\otimes S^nU \arrow[ur] & &
\end{tikzcd}. \tag{3.17}$$

We can decompose the above spaces in invariant subspaces and we get:

$$\begin{tikzcd}
0 \arrow[r] & S^mU \arrow[r] \arrow[d] & \sum_{i=0}^{n-1} S^{2n-6-2i}U \arrow[r] \arrow[d] & \sum_{i=0}^{n-1} S^{2n-6-2i}U \arrow[r] & 0 \\
 & \sum_{i=0}^{3} S^{n+1-2i}U \arrow[r] & \sum_{i=0}^{n-1} S^{2n-2-2i}U \arrow[ur] & &
\end{tikzcd}. \tag{3.18}$$

We observe that $\mathbb{P}(S^3U)$ is the projective space where we projected the normal rational curve.

Rational Curves of degree 5 in $\mathbb{P}^3$

Let C_5 be a rational normal curve of degree 5 in $\mathbb{P}^5$ with usual embedding $\nu_5:\mathbb{P}^1\to\mathbb{P}^5$ and $\pi_2:\mathbb{P}^5\backslash<p_1,p_2>\to\mathbb{P}^3$ the projection from the line L generated by two points $p_1=(a_0^1,...,a_5^1), p_2=(a_0^2,...,a_5^2)\in\mathbb{P}^5$. We have the following:

(3.19)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}^2_{C_5} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}^2_{C_5} \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-4)^2 \arrow[r, "J(\nu_5)"] & \mathcal{O}^6_{C_5} \arrow[r, "Syz(J(\nu_5))"] \arrow[d] & \mathcal{O}_{C_5}(2)^4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-4)^2 \arrow[r, "J(\pi_2 \circ \nu_5)"] & \mathcal{O}^4_{\pi_2(C_5)} \arrow[r] \arrow[d] & N'_{\pi_2(C_5);\mathbb{P}^3}(-5) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & ,
\end{tikzcd}$$

where $N'_{\pi_2(C_5);\mathbb{P}^3}$ is the equisingular normal sheaf.

Observation 3.1.17. *If $L \in Gr(\mathbb{P}^1, \mathbb{P}^5)$ does not lie on the Chow Hypersurface of the Secant Variety $S(C_5)$ then $\pi_{n-3}(C_5)$ is smooth.*

By dualizing the last exact column of (3.19) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(2)$, we get:

$$0 \longrightarrow N^{\vee}_{\pi_2(C_5);\mathbb{P}^3}(7) \longrightarrow \mathcal{O}^4_{C_5} \longrightarrow \mathcal{O}^2_{C_5}(2) \longrightarrow 0.$$

So that $deg(N^{\vee}_{\pi_2(C_5);\mathbb{P}^3}(7)) = -4$ and $N^{\vee}_{\pi_2(C_5);\mathbb{P}^3}(7) = \mathcal{O} \oplus \mathcal{O}(-4)$ if and only if $N_{\pi_2(C_5);\mathbb{P}^3}(-5) = \mathcal{O}(2) \oplus \mathcal{O}(6)$. If we denote the following matrix as:

$$N^L_{5,2} := \begin{pmatrix} a^1_0 & a^1_1 & a^1_2 & a^1_3 \\ -2a^1_1 & -2a^1_2 & -2a^1_3 & -2a^1_4 \\ a^1_2 & a^1_3 & a^1_4 & a^1_5 \\ a^2_0 & a^2_1 & a^2_2 & a^2_3 \\ -2a^2_1 & -2a^2_2 & -2a^2_3 & -2a^2_4 \\ a^2_2 & a^2_3 & a^2_4 & a^2_5 \end{pmatrix},$$

we have that $N_{\pi_2(C_5);\mathbb{P}^3}(-5) = \mathcal{O}(2) \oplus \mathcal{O}(6)$ if and only if $\mathrm{rank}(N^L_{5,2}) = 3$.

Proposition 3.1.18. *The centre of projection $L \in Gr(\mathbb{P}^1, \mathbb{P}^5)$ lies on a $3-$secant plane to the rational normal curve C_5 if and only if $N_{\pi_2(C_5);\mathbb{P}^3} = \mathcal{O}(7) \oplus \mathcal{O}(11)$.*

Proof. $\Leftarrow$ If L belongs to a $3-$secant plane, then there exist two points $p_1, p_2 \in L$ such that the corresponding forms $f_1, f_2 \in S_n$ have the same additive decomposition of length 3:

$$f_i = c^i_1(L_1)^5 + c^i_2(L_2)^5 + c^i_3(L_3)^5,$$

for $i = 1, 2$ and $c^i_j \in \mathbb{C}$. Therefore we can construct a differential form $\phi = \prod_{j=1}^{3} (L_j)^{\perp}$ which generate $\ker N^L_{5,2}$, so $\operatorname{rank} N^L_{5,2} = 3$.

$\Rightarrow$ If the rational curve of degree 5 projected from L has $N_{\pi_2(C_5);\mathbb{P}^3} = \mathcal{O}(7) \oplus \mathcal{O}(11)$, then $\operatorname{rank} N^L_{5,2} = 3$. Therefore there exists a differential form $\phi \in T_3$ apolar to both forms $f_1, f_2 \in S_5$ corresponding to the points $p_1, p_2 \in \mathbb{P}^5$ which generate L. It means that we have two possibilities or ϕ has only simple roots or it has some multiple roots. So we must analyse three cases:

i) ϕ has three simple roots;

ii) ϕ has a simple root and a double root;

iii) ϕ has one triple root.

In the first case the primary decomposition is $\phi = \prod_{j=1}^{3} \phi_j$, so we have that the forms f_1, f_2 has similar additive decompositions:

$$f_i = c^i_1 (L_1)^5 + c^i_2 (L_2)^5 + c^i_3 (L_3)^5,$$

for $i = 1, 2$ and $c^i_j \in \mathbb{C}$, where $L_j \in S_1$ and $(L_j)^{\perp} = \phi_j$ for $j = 1, 2, 3$ and L belongs to a 3−secant plane generated by $q_1, q_2, q_3 \in C_5$ corresponding to $(L_1)^5, (L_2)^5, (L_3)^5 \in S_5$.

In the second case the primary decomposition is $\phi = \phi_1 \phi_2^2$ with $\deg \phi_1 = \deg \phi_2 = 1$, we have:

$$f_i = c^i_1 (L_1)^5 + G_{i,2} (L_2)^4,$$

for $i = 1, 2$ and $c^i_1 \in \mathbb{C}$, where $L_j \in S_1$, $(L_j)^{\perp} = \phi_j$ for $j = 1, 2$ and $G_{i,2} \in S_1$. This means that L belongs to the plane generated by L_1^5 and the line parametrized by $\{G \cdot L_2^4 \in S_5 \text{ for all } G \in S_1\}$ i.e the plane generating by a point on C_5 and a tangent line to C_5 in a different point. But in this case L touches a tangent line, so the projected curve have a cusp which is excluded by our preliminary assumption on the singularities.

In the third case the primary decomposition is $\phi = \phi_1^3$ with $\deg \phi_1 = 1$, we have:

$$f_i = G_i L^3,$$

where $L \in S_1$, $L^{\perp} = \phi_1$ and $G_i \in S_2$. This means that L belongs to the plane parametrized by $\{G \cdot L^3 \in S_5 \text{ for all } G \in S_2\}$ i.e the tangent plane to C_5 in a point q corresponding to $L^5 \in S_5$. But in this case L touches a tangent line, so the projected curve have a cusp which is excluded by our preliminary assumption on the singularities.

□

Lemma 3.1.19. *The variety of lines $L \in Gr(\mathbb{P}^1, \mathbb{P}^5)$ lies on a $3-$secant plane to the rational normal curve C_5 is an irreducible variety of codimension 3.*

Proof. Infact we can consider the incidence variety $I_S = \{(L, \pi) : L \in \mathbb{G}r(\mathbb{P}^1, \mathbb{P}^5), \pi \in S, L \subset S\}$ where S is the set of all 3-secant planes to the rational normal curve in $\mathbb{P}^5$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^1, \mathbb{P}^5)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

Theorem 3.1.20. *$N_2^5(7, 11)$ is an irreducible variety of codimension 3 formed by the lines $L \cong \mathbb{P}^1$ that belong to a $3-$secant plane to the rational normal curve C_5.*

Observation 3.1.21. *If $L \in Gr(\mathbb{P}^1, \mathbb{P}^5)$ lies on a $3-$secant plane then it is contained in the Chow Hypersurface associated to the Secant Variety $\sigma_1(C_5)$.*

By dualizing the last exact column of (3.19) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(3)$, we get:

$$0 \longrightarrow N^{\vee}_{\pi_2(C_5);\mathbb{P}^3}(8) \longrightarrow \mathcal{O}^4_{C_5}(1) \longrightarrow \mathcal{O}^2_{C_5}(3) \longrightarrow 0,$$

and passing to the exact cohomology sequence, we have:

$$0 \longrightarrow H^0(N^{\vee}_{\pi_2(C_5);\mathbb{P}^3}(8)) \longrightarrow H^0(\mathcal{O}^4_{C_5}(1)) \longrightarrow H^0(\mathcal{O}^2_{C_5}(3)) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1(N^{\vee}_{\pi_2(C_5);\mathbb{P}^3}(8)) \longrightarrow 0.$$

So that $deg(N^{\vee}_{\pi_2(C_5);\mathbb{P}^3}(8)) = -2$ and $N^{\vee}_{\pi_2(C_5);\mathbb{P}^3}(8) = \mathcal{O} \oplus \mathcal{O}(-2)$ if and only if $N_{\pi_2(C_5);\mathbb{P}^3}(-5) = \mathcal{O}(3) \oplus \mathcal{O}(5)$. If we denote the following matrix as:

$$N^L_{5,2}(1) := \begin{pmatrix} a_0^1 & a_1^1 & a_2^1 & a_3^1 & 0 & 0 & 0 & 0 \\ -2a_1^1 & -2a_2^1 & -2a_3^1 & -2a_4^1 & a_0^1 & a_1^1 & a_2^1 & a_3^1 \\ a_2^1 & a_3^1 & a_4^1 & a_5^1 & -2a_1^1 & -2a_2^1 & -2a_3^1 & -2a_4^1 \\ 0 & 0 & 0 & 0 & a_2^1 & a_3^1 & a_4^1 & a_5^1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & 0 & 0 & 0 & 0 \\ -2a_1^2 & -2a_2^2 & -2a_3^2 & -2a_4^2 & a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_2^2 & a_3^2 & a_4^2 & a_5^2 & -2a_1^2 & -2a_2^2 & -2a_3^2 & -2a_4^2 \\ 0 & 0 & 0 & 0 & a_2^2 & a_3^2 & a_4^2 & a_5^2 \end{pmatrix},$$

we have that $N_{\pi_2(C_5);\mathbb{P}^3}(-5) = \mathcal{O}(3) \oplus \mathcal{O}(5)$ if and only if $\det(N^L_{5,2}(1)) = 0$.
By (3.17) we have:

$$\begin{tikzcd}
0 \arrow[r] & H^0(N^\vee_{\pi_2(C_5);\mathbb{P}^3}(8)) \arrow[r] \arrow[d] & H^0(\mathcal{O}^4_{C_5}(1)) \arrow[r] \arrow[d, "{}^tSyz(J(\nu_5))"'] & H^0(\mathcal{O}^2_{C_5}(3)) \arrow[r] & 0 \\
 & H^0(\mathcal{O}^4_{\pi_2(C_5)}(3)) \arrow[r] & H^0(\mathcal{O}^6_{C_5}(4)) \arrow[ur] & &
\end{tikzcd},$$

that is:

$$\begin{tikzcd}
0 \arrow[r] & S^mU \arrow[r] \arrow[d] & U^* \otimes S^3U \arrow[r, "M"] \arrow[d] & S^3U^* \otimes U \arrow[r] & 0 \\
 & S^3U^* \otimes S^3U \arrow[r] & S^3U^* \otimes S^5U \arrow[ur] & &
\end{tikzcd}. \tag{3.20}$$

We can decompose the above spaces in invariant subspaces and we get:

$$\begin{tikzcd}
0 \arrow[r] & S^mU \arrow[r] \arrow[d] & \sum_{i=0}^4 S^{4-2i}U \arrow[r] \arrow[d] & \sum_{i=0}^4 S^{4-2i}U \arrow[r] & 0 \\
 & \sum_{i=0}^3 S^{6-2i}U \arrow[r] & \sum_{i=0}^4 S^{8-2i}U \arrow[ur] & &
\end{tikzcd}, \tag{3.21}$$

where $\pi_2(C_5) \subset \mathbb{P}(S^3U)$.

If we denoted by

$$N^{p_i}_{5,1}(1) := \begin{pmatrix} a^i_0 & a^i_1 & a^i_2 & a^i_3 & 0 & 0 & 0 & 0 \\ -2a^i_1 & -2a^i_2 & -2a^i_3 & -2a^i_4 & a^i_0 & a^i_1 & a^i_2 & a^i_3 \\ a^i_2 & a^i_3 & a^i_4 & a^i_5 & -2a^i_1 & -2a^i_2 & -2a^i_3 & -2a^i_4 \\ 0 & 0 & 0 & 0 & a^i_2 & a^i_3 & a^i_4 & a^i_5 \end{pmatrix},$$

we can observe that :

$$N^L_{5,2}(1) := \begin{pmatrix} N^{p_1}_{5,1}(1) \\ N^{p_2}_{5,1}(1) \end{pmatrix},$$

so we can study the matrix $N^{p_i}_{5,1}(1)$.

Observation 3.1.22. *This is a map from $H^0(\mathcal{O}^4_{C_5}(1))$ to $H^0(\mathcal{O}_{C_5}(3))$, that is*

$$U^* \otimes S^3U \to S^3U^*,$$

equivalently

$$\mathbb{P}(U^* \otimes S^3U) \to \mathbb{P}(S^3U^*).$$

We observe that the Segre variety $\mathbb{P}(U^*\otimes S^3U) = \mathbb{P}^1\times\mathbb{P}^3$ *is bidual (see [Gel'fand et al., 1994]). We can consider the dual of the kernel of* $N^{p_i}_{5,1}(1)$*, that is the kernel of* $(N^{p_i}_{5,1}(1))^t$*. The* $\ker((N^{p_i}_{5,1}(1))^t)$ *is a* $\mathbb{P}^3$ *in* $\mathbb{P}^1\times\mathbb{P}^3$ *so the rank condition on* $(N^L_{5,2}(1))$ *is equivalent to ask that the corresponding two* $\mathbb{P}^3$ *span a* $\mathbb{P}^6$ *instead a* $\mathbb{P}^7$*.*

We want also show what happens in the case of Castelnuovo's Curve of degree 5 in $\mathbb{P}^3$.

Example 3.1.23 (Castelnuovo's Curve of degree 5 in $\mathbb{P}^3$). Let S be a ruled quadric surface in $\mathbb{P}^3$ and $C_C \subset S$ be the Castelnuovo's curve on S. It has degree 5 and arithmetic genus 2, so it has two singular point of type node.

Thus we have the following exact sequence:

$$0 \longrightarrow TC_c \longrightarrow T\mathbb{P}^3|_{C_C} \longrightarrow N_{C_C;\mathbb{P}^3} \longrightarrow \mathbb{C}\oplus\mathbb{C} \longrightarrow 0.$$

Let $f : \tilde{C_C} \to C_C \in \mathbb{P}^3$ be the normalization of C_c and $df : T\tilde{C_c} \to TC_C$. We have the following exact sequence (see [Ciliberto, 1987]):

$$0 \longrightarrow T\tilde{C_c} \longrightarrow f^*(T\mathbb{P}^3|_{C_C}) \longrightarrow N_f \longrightarrow 0,$$

where N_f is the normal sheaf to the map f (see [Ciliberto, 1987]). So we have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T\tilde{C_c} \arrow[r] \arrow[d] & f^*(T\mathbb{P}^3|_{C_C}) \arrow[r] \arrow[d] & N_f \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & TC_c \arrow[r] \arrow[d] & T\mathbb{P}^3|_{C_C} \arrow[r] \arrow[d] & N_{C_C;\mathbb{P}^3} \arrow[r] \arrow[d] & \mathbb{C}\oplus\mathbb{C} \arrow[r] & 0 \\
 & \mathbb{C}\oplus\mathbb{C} & \mathbb{C}^3\oplus\mathbb{C}^3 & \mathbb{C}^2\oplus\mathbb{C}^2 & & .
\end{tikzcd}$$

C_C is a type $(2,3)$ curve i.e. $C_C = 2F_1 + 3F_2$ where F_1, F_2 are the two fundamental divisor on S, so $C.C = 12$. Therefore we have that $\deg(N_{C_C}) = 12$ and $\deg(N_S|_{C_C}) = 10$. At the other hand we have the following exact sequence:

$$0 \longrightarrow N_{C_c}; S \longrightarrow N_{C_C;\mathbb{P}^3} \longrightarrow N_S|_{C_C} \longrightarrow 0,$$

so $N_{C_C;\mathbb{P}^3} = \mathcal{O}(10)\oplus\mathcal{O}(12)$.

Rational Normal Curve of degree 6 in $\mathbb{P}^3$

Let C_6 be a rational normal curve of degree 6 in $\mathbb{P}^6$ with usual morphism $\nu_6 : \mathbb{P}^1 \to \mathbb{P}^6$ and $\pi_3 : \mathbb{P}^6 \setminus < p_1, p_2, p_3 > \to \mathbb{P}^3$ the projection from the space generated by three points $p_i = (a_0^i, ..., a_6^i) \in \mathbb{P}^6$ for $i = 1, 2, 3$. Thus we have the following:

(3.22)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}^3_{C_6} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}^3_{C_6} \arrow[d, "N^L_{6,3}"] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-5)^2 \arrow[r, "J(\nu_6)"] & \mathcal{O}^7_{C_6} \arrow[r, "Syz(J(\nu_6))"] \arrow[d] & \mathcal{O}_{C_6}(2)^5 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-5)^2 \arrow[r, "J(\pi_3 \circ \nu_6)"] & \mathcal{O}^4_{\pi_3(C_6)} \arrow[r] \arrow[d] & N_{\pi_3(C_6);\mathbb{P}^3}(-6) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

By dualizing the last exact column of (3.22) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(2)$, we get:

$$0 \longrightarrow N^\vee_{\pi_3(C_6);\mathbb{P}^3}(8) \longrightarrow \mathcal{O}^5_{C_6} \longrightarrow \mathcal{O}^3_{C_6}(2) \longrightarrow 0.$$

So that $deg(N^\vee_{\pi_3(C_6);\mathbb{P}^3}(8)) = -6$ and $N^\vee_{\pi_3(C_6);\mathbb{P}^3}(9) = \mathcal{O} \oplus \mathcal{O}(-6)$ if and only if $N_{\pi_3(C_6);\mathbb{P}^3}(-6) = \mathcal{O}(2) \oplus \mathcal{O}(8)$. If we denote the following matrix as:

$$N^L_{5,2} := \begin{pmatrix} a_0^1 & a_1^1 & a_2^1 & a_3^1 & a_4^1 \\ -2a_1^1 & -2a_2^1 & -2a_3^1 & -2a_4^1 & -2a_5^1 \\ a_2^1 & a_3^1 & a_4^1 & a_5^1 & a_6^1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ -2a_1^2 & -2a_2^2 & -2a_3^2 & -2a_4^2 & -2a_5^2 \\ a_2^2 & a_3^2 & a_4^2 & a_5^2 & a_6^2 \\ a_0^3 & a_1^3 & a_3^2 & a_3^3 & a_4^3 \\ -2a_1^3 & -2a_2^3 & -2a_3^3 & -2a_4^3 & -2a_5^3 \\ a_2^3 & a_3^3 & a_4^3 & a_5^3 & a_6^3 \end{pmatrix},$$

we have that $N^\vee_{\pi_3(C_6);\mathbb{P}^3}(8) = \mathcal{O} \oplus \mathcal{O}(-6)$ if and only if $\text{rank}(N^L_{5,3}) = 5$.
By dualizing the last exact column of (3.22) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(3)$, we get:

$$0 \longrightarrow N^\vee_{\pi_3(C_6);\mathbb{P}^3}(9) \longrightarrow \mathcal{O}^5_{C_6}(1) \xrightarrow{N^L_{6,3}(1)} \mathcal{O}^3_{C_6}(3) \longrightarrow 0.$$

So that $deg(N^\vee_{\pi_3(C_6);\mathbb{P}^3}(9)) = -4$ and $N^\vee_{\pi_3(C_6);\mathbb{P}^3}(9) = \mathcal{O}\oplus\mathcal{O}(-4)$ if and only if $N_{\pi_3(C_6);\mathbb{P}^3}(-6) = \mathcal{O}(3) \oplus \mathcal{O}(7)$. If we denote the following matrix as:

$$N^L_{6,3}(1) := \begin{pmatrix} a_0^1 & a_1^1 & a_2^1 & a_3^1 & a_4^1 & 0 & 0 & 0 & 0 & 0 \\ -2a_1^1 & -2a_2^1 & -2a_3^1 & -2a_4^1 & -2a_5^1 & a_0^1 & a_1^1 & a_2^1 & a_3^1 & a_4^1 \\ a_2^1 & a_3^1 & a_4^1 & a_5^1 & a_6^1 & -2a_1^1 & -2a_2^1 & -2a_3^1 & -2a_4^1 & -2a_5^1 \\ 0 & 0 & 0 & 0 & 0 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & a_6^1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & 0 & 0 & 0 & 0 & 0 \\ -2a_1^2 & -2a_2^2 & -2a_3^2 & -2a_4^2 & -2a_5^2 & a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_2^2 & a_3^2 & a_4^2 & a_5^2 & a_6^2 & -2a_1^2 & -2a_2^2 & -2a_3^2 & -2a_4^2 & -2a_5^2 \\ 0 & 0 & 0 & 0 & 0 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & a_6^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 & 0 & 0 & 0 & 0 & 0 \\ -2a_1^3 & -2a_2^3 & -2a_3^3 & -2a_4^3 & -2a_5^3 & a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ a_2^3 & a_3^3 & a_4^3 & a_5^3 & a_6^3 & -2a_1^3 & -2a_2^3 & -2a_3^3 & -2a_4^3 & -2a_5^3 \\ 0 & 0 & 0 & 0 & 0 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & a_6^3 \end{pmatrix},$$

we have that $N^\vee_{\pi_3(C_6);\mathbb{P}^3}(9) = \mathcal{O} \oplus \mathcal{O}(-4)$ if and only if $\mathrm{rank}(N^L_{6,3})(1) = 9$.

Instead by tensorizing with $\mathcal{O}_{\mathbb{P}^1}(4)$, we get:

$$0 \longrightarrow N^\vee_{\pi_3(C_6);\mathbb{P}^3}(10) \longrightarrow \mathcal{O}^5_{C_6}(2) \xrightarrow{N^L_{6,3}(2)} \mathcal{O}^3_{C_6}(4) \longrightarrow 0.$$

So that $deg(N^\vee_{\pi_3(C_6);\mathbb{P}^3}(10)) = -2$ and $N^\vee_{\pi_3(C_6);\mathbb{P}^3}(10) = \mathcal{O} \oplus \mathcal{O}(-2)$ if and only if $N_{\pi_3(C_6);\mathbb{P}^3}(-6) = \mathcal{O}(4) \oplus \mathcal{O}(6)$.

We have that $N^\vee_{\pi_3(C_6);\mathbb{P}^3}(10) = \mathcal{O} \oplus \mathcal{O}(-2)$ if and only if $\det(N^L_{6,3}(2)) = 0$.

3.2 Normal Bundle of Rational Curves in codimension k

Let $C_n \subset \mathbb{P}^n$ be the rational normal curve of degree n and $\nu_n : \mathbb{P}^1 \to \mathbb{P}^n$ be the Veronese map and we will indicate with $C_n = \nu_n(\mathbb{P}^1)$ the image. Let $J(\nu_n)$ be the Jacobian matrix of ν_n:

$$J(\nu_n) = \begin{pmatrix} ns^{n-1} & \dots & t^{n-1} & 0 \\ 0 & s^{n-1} & \dots & nt^{n-1} \end{pmatrix}$$

and let $\pi_k(C_n)$ the rational curve obtained from C_n by projection from a $(k-1)$-dimensional linear subspace $L \subset \mathbb{P}^n$ on $\mathbb{P}^{n-k} \subset \mathbb{P}^n$. Obviously we restrict our attention at the case $k < n-2$.

In the following we will indicate with $p_i = (a_0^i, ..., a_n^i) \in \mathbb{P}^n$, for $i = 0, \dots, k-1$, the k points which generate the $(k-1)-$dimensional linear subspace $L \subset \mathbb{P}^n$.

So the Euler's exact sequence on $\mathbb{P}^{n-k}$ restricted to $\pi_k(C_n)$ (see [Hartshorne, 1977], [Okonek et al., 1980]):

$$0 \to \mathcal{O}_{C_n} \to \mathcal{O}_{\pi_k(C_n)}(n)^{n-k+1} \to T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \to 0,$$

and the usual one for Normal Bundle on $\pi_k(C_n)$ give rise to the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "="] \arrow[d] & \mathcal{O}_{C_n} \arrow[d] & & \\
 & \mathcal{O}_{\mathbb{P}^1}(1)^2 \arrow[r, "J(\pi_k \circ \nu_n)"] \arrow[d] & \mathcal{O}_{\pi_k(C_n)}(n)^{n-k+1} \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(2) \arrow[r] \arrow[d] & T\mathbb{P}^{n-k}|_{\pi_k(C_n)} \arrow[r] \arrow[d] & N_{\pi_k(C_n);\mathbb{P}^{n-k}} \arrow[r] & 0 \\
 & 0 & 0 & & .
\end{tikzcd}$$

Therefore the following exact sequence holds:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)^2 \xrightarrow{J(\pi_k \circ \nu_n)} \mathcal{O}_{\pi_k(C_n)}(n)^{n-k+1} \longrightarrow N_{\pi_k(C_n);\mathbb{P}^{n-k}} \longrightarrow 0\,. \tag{3.23}$$

By tensoring (3.23) with $\mathcal{O}_{\mathbb{P}^1}(-n)$, we obtain this exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \xrightarrow{J(\pi_k \circ \nu_n)} \mathcal{O}_{\pi_k(C_n)}^{n-k+1} \longrightarrow N_{\pi_k(C_n);\mathbb{P}^{n-k}}(-n) \longrightarrow 0\,. \tag{3.24}$$

Observation 3.2.1. *We observe that* $degN'_{\pi_{n-3}(C_n);\mathbb{P}^3}(-n) = 2n-2$*, equivalently* $degN'_{\pi_{n-3}(C_n);\mathbb{P}^3} = n^2-(k-1)n-2$*. It is a vector bundle of* rank $n-k-1$.

Moreover by Grothendieck-Segre's theorem (see [Grothendieck, 1957]) $N_{\pi_{n-3}(C_n);\mathbb{P}^3}$ *splits in:*

$$\bigoplus_{i=1}^{n-k-1} \mathcal{O}(n_i) \text{ with } n_i \in \mathbb{Z} \text{ such that } \sum_{i=1}^{n-k-1} n_i = n^2-(k-1)n-2,$$

where we abbreviate $\mathcal{O}_{\pi_{n-3}(C_n)}$ *to* $\mathcal{O}$ *and without loss of generality we can take* $n_1 \leq \cdots \leq n_{n-k-1}$.

Let us denote by:

$$Syz(J(\nu_n)) = \begin{pmatrix} t^2 & -2st & s^2 & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & t^2 & -2st & s^2 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & t^2 & -2st & s^2 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & t^2 & -2st & s^2 \end{pmatrix},$$

the syzygy of the Jacobian matrix $J(\nu_n)$, where $Syz(J(\nu_n))$ is a matrix $n-1 \times n+1$. So we have the following sequence for $J(\nu_n)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \xrightarrow{J(\nu_n)} \mathcal{O}_{C_n}^{n+1} \xrightarrow{Syz(J(\nu_n))} \mathcal{O}_{C_n}(2)^{n-1} \longrightarrow 0\,, \tag{3.25}$$

which is similar to 3.24.

Moreover we have an exact sequence similar to 2.3 twisted by $\mathcal{O}(-n)$:

$$0 \longrightarrow \mathcal{O}_{C_n}^k \xrightarrow{P} \mathcal{O}_{C_n}^{n+1} \longrightarrow \mathcal{O}_{C_n}^{n-k+1} \longrightarrow 0, \tag{3.26}$$

where:

$$P = \begin{bmatrix} p_1 & \cdots & p_k \end{bmatrix},$$

therefore we obtain:

(3.27)
$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}^k_{C_n} \arrow[d, "P"] & & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\nu_n)"] & \mathcal{O}^{n+1}_{C_n} \arrow[r, "Syz(J(\nu_n))"] \arrow[d] & \mathcal{O}_{C_n}(2)^{n-1} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\pi_k \circ \nu_n)"] & \mathcal{O}^{n-k+1}_{\pi_k(C_n)} \arrow[r] \arrow[d] & N'_{\pi_k(C_n);\mathbb{P}^{n-k}}(-n) \arrow[r] & 0 \\
 & & 0 & , &
\end{tikzcd}$$

which we can complete to the following diagram:

(3.28)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}^k_{C_n} \arrow[r, "\cong"] \arrow[d, "P"] & \mathcal{O}^k_{C_n} \arrow[d, "(\mathcal{N}^L_{n,k})^t"] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\nu_n)"] & \mathcal{O}^{n+1}_{C_n} \arrow[r, "Syz(J(\nu_n))"] \arrow[d] & \mathcal{O}_{C_n}(2)^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\pi_{n-3} \circ \nu_n)"] & \mathcal{O}^{n-k+1}_{\pi_k(C_n)} \arrow[r] \arrow[d] & N'_{\pi_k(C_n);\mathbb{P}^{n-k}}(-n) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

where the map $(\mathcal{N}^L_{n,k})^t$ is:

$$(\mathcal{N}^L_{n,k})^t = Syz(J(\nu_n)) \cdot \begin{pmatrix} a^1_0 & \dots & a^k_0 \\ \vdots & \ddots & \vdots \\ a^1_n & \dots & a^k_n \end{pmatrix} =$$

$$\begin{pmatrix} a^1_0 t^2 - 2a^1_1 ts + a^1_2 s^2 & a^1_1 t^2 - 2a^1_2 ts + a^1_3 s^2 & \dots & a^1_{n-2} t^2 - 2a^1_{n-1} ts + a^1_n s^2 \\ \vdots & \vdots & \ddots & \vdots \\ a^k_0 t^2 - 2a^k_1 ts + a^k_2 s^2 & a^k_1 t^2 - 2a^k_2 ts + a^k_3 s^2 & \dots & a^k_{n-2} t^2 - 2a^k_{n-1} ts + a^k_n s^2 \end{pmatrix}^t.$$

It is a $(n-1)\times k$ matrix. The last exact column of (3.28):

$$0 \longrightarrow \mathcal{O}^k_{\mathbb{P}^1} \xrightarrow{(\mathcal{N}^L_{n,k})^t} \mathcal{O}_{\pi_{n-k}(C_n)}(2)^{n-1} \longrightarrow {N'}_{\pi_k(C_n);\mathbb{P}^{n-k}}(-n) \longrightarrow 0 \tag{3.29}$$

gives us some information about the splitting type of ${N'}_{\pi_k(C_n),\mathbb{P}^{n-k}}$:

Lemma 3.2.2. *If $\pi_k(C_n)$ has only ordinary singularities, then the splitting type of ${N'}_{\pi_k(C_n),\mathbb{P}^{n-k}}$ must be $(n_1,...,n_{n-k-1})$ with $n+2\leq n_1\leq ...\leq n_{n-k-1}\leq n+2+2k$.*

Moreover the exact sequence 3.29 gives rise by duality and tensorizing by $\mathcal{O}_{\mathbb{P}^1}(2)$:

$$0 \longrightarrow {N'}^{\vee}_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2) \longrightarrow \mathcal{O}^{n-1}_{\pi_k(C_n)} \xrightarrow{\mathcal{N}^L_{n,k}} \mathcal{O}_{\mathbb{P}^1}(2)^k \longrightarrow 0\,. \tag{3.30}$$

Observation 3.2.3. *We can observe that if $\pi_k(C_n)$ has only ordinary singularities, then the map of differential is surjective, so:*

$$ {N'}^{\vee}_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong N^{\vee}_{\pi_k(C_n);\mathbb{P}^{n-k}}.$$

Therefore we will not distinguish between ${N'}_{\pi_k(C_n);\mathbb{P}^{n-k}}$ and $N_{\pi_k(C_n);\mathbb{P}^{n-k}}$, where it is clear what we must consider.

If we pass to the exact cohomology sequence we get:

$$0 \longrightarrow H^0(N^{\vee}_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) \longrightarrow H^0(\mathcal{O}^{n-1}_{\pi_k(C_n)}) \xrightarrow{N^L_{n,k}} H^0(\mathcal{O}_{\mathbb{P}^1}(2)^k) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^1(N^{\vee}_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) \longrightarrow 0\,,$$

where we have indicated with $N^L_{n,k}$ the $3k\times(n-1)$ matrix:

$$N^L_{n,k} = \begin{pmatrix} a^1_0 & \dots & a^1_{n-2} \\ -2a^1_1 & \dots & -2a^1_{n-1} \\ a^1_2 & \dots & a^1_n \\ \vdots & \ddots & \vdots \\ a^k_0 & \dots & a^k_{n-2} \\ -2a^k_1 & \dots & -2a^k_{n-1} \\ a^k_2 & \dots & a^k_n \end{pmatrix}.$$

Observation 3.2.4. *The rank of $N^L_{n,k}$ does not depend from the points generating L. Let F_{p_i} be the binary form of degree n which corresponds to the point $p_i\in\mathbb{P}^n$. If we* *indicate with* $Cat_{F_{p_i}}(2, n-2; 2)$ *or* $Cat_{p_i}(2, n-2; 2)$ *the Hankel matrix* $3 \times (n-1)$ *of* F_{p_i} *we have:*

$$\operatorname{rank} N^L_{n,k} = \operatorname{rank} \begin{pmatrix} Cat_{F_{p_1}}(2, n-2; 2) \\ \vdots \\ Cat_{F_{p_k}}(2, n-2; 2) \end{pmatrix}.$$

Observation 3.2.5. *We have that* $\deg(N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) = -2k$ *and*

$$h^0(N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) = n - 1 - \operatorname{rank}(N^L_{n,k}) = \dim \ker(N^L_{n,k}).$$

Therefore we have:

$$2 \le \operatorname{rank}(N^L_{n,k}) \le \min\{n-1, 3k\},$$

so

$$0 \le h^0(N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) \le n - k - 1,$$

but as $\operatorname{rank}(N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) = n - k - 1$ *we have that*

$$N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2) \text{ splits in } \bigoplus_{i=1}^{n-k-1} \mathcal{O}(n'_i)$$

by Grothendieck-Segre's theorem (see [Grothendieck, 1957]) with

$$-2k \le n'_1 \le \dots \le n'_{n-k-1} \le 0 \text{ and } n'_1 + \dots + n'_{n-k-1} = -2k.$$

Remark 3.2.6. It's clear from the above consideration that the value of $\operatorname{rank}(N^L_{n,k})$ corresponds to some splitting types of $N'_{\pi_k(C_n);\mathbb{P}^{n-k}}$.

Observation 3.2.7. *We have that* $\operatorname{rank}(N^L_{n,k}) = \min\{n-1, 3k\} \Leftrightarrow h^0(N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) = 0 \Leftrightarrow N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2) = \mathcal{O}(n'_0) \oplus \dots \oplus \mathcal{O}(n'_{n-k-2})$, *with all* $n'_i \ne 0$. *Instead* $\operatorname{rank}(N^L_{n,k}) = r < \min\{n-1, 3k\} \Leftrightarrow h^0(N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2)) = n - 1 - r \Leftrightarrow N^\vee_{\pi_k(C_n);\mathbb{P}^{n-k}}(n+2) = \mathcal{O}^{n-1-r} \oplus \mathcal{F}^\vee(n+2)$ *with* $\operatorname{rank}(\mathcal{F}) = r - k$ *until* $k \le r$, *and* $\deg(\mathcal{F}^\vee(n+2)) = -2k$ *or equivalently* $\deg(\mathcal{F}) = (r-k)(n+2) + 2k$. *If* $r < k$, *then* $n - 1 - r > n - 1 - k$, *so the situation is more complicated, we will see something about that after. On the other hand we have:*

$$\operatorname{rank}(N^L_{n,k}) \ge k + 1,$$

otherwise some n'_i *must be* ≥ 1, *so we have that some* n_i *must be* $\le n + 1$, *but this is impossible by Lemma 3.2.2.*

Proposition 3.2.8. *$N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-1-\mathrm{rank}(N^L_{n,k})} \oplus \mathcal{F}$, with $\mathcal{F}$ a vector bundle of rank $\mathrm{rank}(N^L_{n,k})-k$ on $\mathbb{P}^1$ and $\deg(\mathcal{F}^\vee(n+2)) = -2k$ such that $\mathcal{F} \cong \bigoplus_i^{\mathrm{rank}(N^L_{n,k})-k} \mathcal{O}(l_i)$ with $l_i \geq n+3$.*

We define the varieties which parametrize the subscheme of the Hilbert scheme $N^n_{n-k}(n_1, ..., n_{n-k-1})$ as intersection of some of the following varieties:

$$V(N^L_{n,k}(d))^r := \{L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) : \mathrm{rank}\, N^L_{n,k}(d) \leq r\}.$$

These are subvarieties of $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$.

Since the rank condition is invariant under the action of $\mathrm{SL}(k, \mathbb{C})$ we are interested to study the determinantal varieties in $\mathrm{Hom}(H^0(\mathcal{O}^n(d)), H^0(\mathcal{O}^k(d+2))$:

$$D(N^L_{n,k}(d))^r = \{p_1, ..., p_k \in \mathbb{P}^n : \mathrm{rank}\, N^L_{n,k}(d) \leq r\}.$$

About the matrix $N^L_{n,k}$ we note that are two possible cases:

i) $3k \geq n-1$, so $\frac{n-1}{3} \leq k < n-3$.

ii) $3k < n-1$, so $k < \frac{n-1}{3}$.

These two cases will be the subject of our studies in the next two sections.

3.3 Codimension k, for $k < \frac{n-1}{3}$

Observation 3.3.1. $k+1 \leq \mathrm{rank}\, N^L_{n,k} = n-1-h^0(N^\vee_{\pi_k(C_n),\mathbb{P}^{n-k}}(n+2)) \leq 3k$

Proposition 3.3.2. *$N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-1-\mathrm{rank}(N^L_{n,k})} \oplus \mathcal{F}$, with $\mathcal{F}$ a vector bundle of rank $\mathrm{rank}(N^L_{n,k})-k$ on $\mathbb{P}^1$ and $\deg(\mathcal{F}^\vee(n+2)) = -2k$ such that $\mathcal{F} \cong \bigoplus_i^{\mathrm{rank}(N^L_{n,k})-k} \mathcal{O}(l_i)$ with $l_i \geq n+3$.*

In this case we have three possibilities:

1. *$\mathcal{F} \cong \mathcal{O}(n+3)^{2k}$ if and only if $\mathrm{rank}(N^L_{n,2}) = 3k$;*

2. *$\mathcal{F} \cong \mathcal{O}(n+3)^{2k-2} \oplus \mathcal{O}(n+4)$ if and only if $\mathrm{rank}(N^L_{n,k}) = 3k-1$;*

3. *$\mathcal{F} \cong \mathcal{O}(n+3)^{2k-2r} \oplus \mathcal{F}'$ with $\mathrm{rank}(\mathcal{F}') = r$ and $\deg(\mathcal{F}'^\vee(n+2)) = -2k$ if and only if $\mathrm{rank}(N^L_{n,k}) = 3k-r$ with $1 < r \leq 2k-1$;*

4. *$\mathcal{F} \cong \mathcal{O}(n+2+2k)$ if and only if $\mathrm{rank}(N^L_{n,k}) = k+1$.*

Observation 3.3.3. *If $L \cong \mathbb{P}^{k-1}$, as centre of projection, belongs to a $(k+1)-$secant $\mathbb{P}^k$, then there exist k points $q_1, .., q_k \in L$ such that $Ann(f_1) = (\alpha, \beta_1), ..., Ann(f_k) = (\alpha, \beta_r)$ with α has only simple roots and $\deg(\alpha) = k+1$ and $\beta_1 \neq ... \neq \beta_k$, $\deg(\beta_1) = ... = \deg(\beta_k) = n-k+1$. So $\dim < \alpha >_{n-2} = n-k-2$ and $\dim < \beta_i >_{n-2} = k$, therefore $n-k-1 \leq \operatorname{rank}(N^L_{n,k}) = n-1-\dim(< \alpha, \beta_1 >_{n-2} \cap ... \cap < \alpha, \beta_k >_{n-2}) \leq k+1$.*

Observation 3.3.4. *We can observe that if L belongs to a $(n-2)-$secant $\mathbb{P}^{n-3}$ generated by $q_1, ..., q_{n-1}$, then there exists an element $\phi \in H^0(\mathcal{O}^n_{\pi_k(C_n)}) \cong S^{n-1}V^\vee = T_{n-1}$ such that $\phi \in \ker(N^L_{n,k}) = \bigcap_i \ker(Cat_{f_i}(2, n-2))$, in fact we can take $\phi = \prod_{i=1}^n L^\perp_{q_i}$, clearly this happen every time since $\dim \ker N^L_{n,k} \geq 1$.*

Unfortunately this condition is empty for $3k < n-2$, in fact we can compute the codimension of the variety of every $\mathbb{P}^{k-1}$ which belong to some (n-2)-secant $\mathbb{P}^{n-3}$ constructing an incidence variety:

$$I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n), \pi \in S, L \subset S\},$$

where S is the set of all (n-2)-secant $\mathbb{P}^{n-3}$ to C_n. In the usual way we can compute the codimension of the image of this incidence variety in $\mathbb{G}r(k-1, n)$. We will indicated with ϕ_1 and ϕ_2 the natural projections:

$$\begin{tikzcd} & I_S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) & & S, \end{tikzcd}$$

so the codimension in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ of $\phi_1(I_S)$ is equal to $\dim Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) - \dim S - \dim \phi_2^{-1}(S) = k(n+1-k) - n + 2 - k(n-2-k)$. That calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). We have that this variety has codimension $3k - n + 1$, but we are in the hypothesis $3k < n-1$, so $3k - n + 1 < 0$.

For $3k = n-1$ the condition gives $\operatorname{codim} = 0$, so it is verified for all L.

Case $\operatorname{rank} N^L_{n,k} = 3k-1$

Lemma 3.3.5. *Let $3k - 1 < n < 4k+1$. If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system of dimension $n - 3k$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, then $\operatorname{rank} N^L_{n,k} = 3k-1$. The converse is generically true.*

Proof. $\Rightarrow$ Let L be a $\mathbb{P}^{k-1}$ belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-3} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{n-3k-1}\pi_{n-3k-1}, \ \forall \lambda = [\lambda_0, ..., \lambda_{n-3k-1}] \in \mathbb{P}^{n-3k-1}\}$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, where $\pi_0, ..., \pi_{n-3k-1}$ are $(n-2)-$secant $\mathbb{P}^{n-3}$. Let $q_1^i, ..., q_{n-2}^i \in C_n$ be the points which generate π_i. Then there exists k points $p_1, ..., p_k$ which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^{n-3k-1}$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{n-2,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c_{1,\lambda}^i L_{1,\lambda}^n + ... + c_{n-2,\lambda}^i L_{n-2,\lambda}^n,$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., n-2$, so L belongs to a $\mathbb{P}^{n-3(n-3k)} = \mathbb{P}^{9k-2n}$, this is possible thanks to the condition $n < 4k+1$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^{n-3k-1}$ there exists a differential form $\phi_\lambda \in T_{n-2}$ such that $\phi_\lambda \circ f_i = 0$. Moreover there exist $n-3k$ differential form $\phi_0, ..., \phi_{n-3k-1} \in T_{n-2}$ which for each $\lambda \in \mathbb{P}^{n-3k-1}$ we have $\phi_\lambda = \lambda_0\phi_0 + ... + \lambda_{n-3k-1}\phi_{n-3k-1}$, in particular $\phi_j \circ f_i = 0$ for all $j = 0, ..., n-3k-1$ and $i = 1, ..., k$, so rank $N_{n,k}^L = 3k-1$.

$\Leftarrow$ If rank $N_{n,k}^L = 3k-1$, then there exist $n-3k$ binary form $\phi_0, ..., \phi_{n-3k-1} \in T_{n-2}$ such that however we consider the generating points $p_1, ..., p_k \in \mathbb{P}^n$ of L, it is $\phi_\alpha \circ f_i = (\alpha_0\phi_0 + ... + \alpha_{n-3k-1}\phi_{n-3k-1}) \circ f_i = 0$ for all $\alpha = [\alpha_0, ..., \alpha_{n-3k-1}] \in \mathbb{P}^{n-3k-1}$ and $i = 1, ..., k$, where $f_i \in S_n$ is the binary form corresponding to p_i. In particular $\phi_j \circ f_i = 0$, so if we consider the primary decomposition of $\phi_\alpha = \prod_{l=1}^{n-2} \phi_\alpha^l$ and we indicate with $(L_{l,\alpha})^\perp = \phi_\alpha^l$. Therefore $f_1, ..., f_k$ can be decomposed in ∞^{n-3k-1} different simultaneously ways, i.e.:

$$f_i = c_{1,\alpha}^i L_{1,\alpha}^n + ... + c_{n-2,\alpha}^i L_{n-2,\alpha}^n,$$

for all $\alpha = [\alpha_0, ..., \alpha_{n-3k-1}] \in \mathbb{P}^{n-3k-1}$ or in other words L belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{n-3k-1}\pi_{n-3k-1}, \ \forall\lambda = [\lambda_0, ..., \lambda_{n-3k-1}] \in \mathbb{P}^{n-3k-1}\}$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$. So L belongs to a $\mathbb{P}^{n-3(n-3k)} = \mathbb{P}^{9k-2n}$, this is possible thanks to the condition $n < 4k+1$. Clearly it can happen that some ϕ_i have any multiple roots, so we have all possible degenerations of the linear system Φ.

$\square$

Theorem 3.3.6. *Let $3k-1 < n < 4k+1$. If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system Φ of affine dimension $n-3k$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, then $N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-3k} \oplus \mathcal{O}(n+3)^{2k-2} \oplus \mathcal{O}(n+4)$.*

Corollary 3.3.7. *Let $k-1 \leq 4k-n-2$ or equivalently $n \leq 4k-1$. The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension $n-3k$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible component of codimension $n-3k$ of the subscheme $N^n_{n-k}((n+2)^{n-3k}, (n+3)^{2k-2}, (n+4))$.*

Proof. We can observe that the linear systems Φ of dimension $n-3k$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$ correspond to the linear systems of dimension $n-3k$ of binary forms of degree $n-2$, therefore the set of these linear system corresponds to $Gr(\mathbb{P}^{n-3k-1}, \mathbb{P}^{n-2})$ which is irreducible and $\dim Gr(\mathbb{P}^{n-3k-1}, \mathbb{P}^{n-2}) = (n-3k)(3k-1)$. Each projection linear space L belongs to a $\mathbb{P}^{n-3(n-3k)} = \mathbb{P}^{9k-2n}$, so the dimension of the fiber is $k(8k-2n+1)$ with the condition $k-1 \leq 9k-n$. Therefore the variety of linear spaces L that belong to a linear system Φ is an irreducible variety of codimension $n-3k$ in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. □

Case $\operatorname{rank} N^L_{n,k} \leq 3k-1$

Lemma 3.3.8. *Let $n \leq \frac{8k-3r+3}{2}$ with $1 \leq r \leq 2k-1$. The centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system Φ of affine dimension $n-3k+r-1$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, then* $\operatorname{rank} N^L_{n,k} = 3k-r$. *The converse is generically true.*

Proof. $\Rightarrow$ Let L be a $\mathbb{P}^{k-1}$ belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-3} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{n-3k+r-2}\pi_{n-3k+r-2}, \quad \forall\lambda = [\lambda_0, ..., \lambda_{n-3k+r-2}] \in \mathbb{P}^{n-3k+r-2}\}$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, where $\pi_0, ..., \pi_{n-3k+r-2}$ are $(n-2)-$secant $\mathbb{P}^{n-3}$. Let $q^i_1, ..., q^i_{n-2} \in C_n$ be the points which generate π_i. Then there exist k points $p_1, ..., p_k$ which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^{n-3k+r-2}$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{n-2,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c^i_{1,\lambda}L^n_{1,\lambda} + ... + c^i_{n-2,\lambda}L^n_{n-2,\lambda},$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., n-2$, so L belongs to a $\mathbb{P}^{n-3(n-3k+r-1)} = \mathbb{P}^{9k-3r-2n+3}$, this is possible thanks to the condition $n \leq \frac{8k-3r+3}{2}$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^{n-3k+r-2}$ there exists a differential form $\phi_\lambda \in T_{n-2}$ such that $\phi_\lambda \circ f_i = 0$. Moreover there exist $n-3k+r-1$ differential form $\phi_0, ..., \phi_{n-3k+r-2} \in T_{n-2}$ which for each $\lambda \in \mathbb{P}^{n-3k+r-2}$ we have $\phi_\lambda = \lambda_0\phi_0 + ... + \lambda_{n-3k+r-2}\phi_{n-3k+r-2}$, in particular $\phi_j \circ f_i = 0$ for all $j = 0, ..., n-3k+r-1$ and $i = 1, ..., k$, so $\operatorname{rank} N^L_{n,k} = 3k-r$.

$\Leftarrow$ If rank $N^L_{n,k} = 3k-r$, then there exist $n-3k+r-1$ binary form $\phi_0, ..., \phi_{n-3k+r-2} \in T_{n-2}$ such that however we consider the generating points $p_1, ..., p_k \in \mathbb{P}^n$ of L, it is $\phi_\alpha \circ f_i = (\alpha_0\phi_0 + ... + \alpha_{n-3k+r-2}\phi_{n-3k+r-2}) \circ f_i = 0$ for all $\alpha = [\alpha_0, ..., \alpha_{n-3k+r-2}] \in \mathbb{P}^{n-3k+r-2}$ and $i = 1, ..., k$, where $f_i \in S_n$ is the binary form corresponding to p_i. In particular $\phi_j \circ f_i = 0$, so if we consider the primary decomposition of $\phi_\alpha = \prod_{l=1}^{n-2} \phi^l_\alpha$ and we indicate with $(L_{l,\alpha})^\perp = \phi^l_\alpha$. Therefore $f_1, ..., f_k$ can be decomposed in $\infty^{n-3k+r-2}$ different simultaneously ways, i.e.:

$$f_i = c^i_{1,\alpha} L^n_{1,\alpha} + ... + c^i_{n-2,\alpha} L^n_{n-2,\alpha},$$

for all $\alpha = [\alpha_0, ..., \alpha_{n-3k+r-2}] \in \mathbb{P}^{n-3k+r-2}$ or in other words L belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{n-3k+r-2}\pi_{n-3k+r-2}, \quad \forall\lambda = [\lambda_0, ..., \lambda_{n-3k+r-2}] \in \mathbb{P}^{n-3k+r-2}\}$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$. So L belongs to a $\mathbb{P}^{n-3(n-3k+r-1)} = \mathbb{P}^{9k-2n-3r+3}$, this is possible thanks to the condition $n \leq \frac{8k-3r+3}{2}$. Clearly it can happen that the binary differential forms ϕ_i have any multiple roots, so we have all possible degenerations of the linear system Φ.

□

Theorem 3.3.9. *Let $n \leq \frac{8k-3r+3}{2}$ with $1 < r \leq 2k-2$. If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to to a linear system Φ of affine dimension $n-3k+r-1$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, then $N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-1-3k+r} \oplus \mathcal{O}(n+3)^{2k-2r} \oplus \mathcal{F}'$ with* $\operatorname{rank}(\mathcal{F}') = r$ *and* $\deg(\mathcal{F}'^\vee(n+2)) = -2k$.

Corollary 3.3.10. *Let $n \leq \frac{8k-3r+3}{2}$ with $1 < r \leq 2k-2$. The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension $n-3k+r-1$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible component of codimension $r(n-3k-1+r)$ of $N^n_{n-k}((n+2)^{n-1-3k+r}, (n+3)^{2k-2r}, spt(\mathcal{F}'))$ with $spt(\mathcal{F}')$ is the splitting type,* $\operatorname{rank}(\mathcal{F}') = r$ *and* $\deg(\mathcal{F}'^\vee(n+2)) = -2k$.

Proof. We can observe that the linear system Φ of affine dimension $n-3k+r-1$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$ corresponds to the linear system of dimension $n-3k+r-1$ of binary forms of degree $n-2$, therefore the set of these linear system corresponds to $Gr(\mathbb{P}^{n-3k+r-2}, \mathbb{P}^{n-2})$ which is irreducible and $\dim Gr(\mathbb{P}^{n-3k+r-2}, \mathbb{P}^{n-2}) = (n-3k+r-1)(3k-r)$. Each projection linear space L belongs to a $\mathbb{P}^{n-3(n-3k+r-1)} = \mathbb{P}^{9k-3r-2n+3}$, so the dimension of the fibre is $k(8k-3r-2n+4)$ with the condition $k-1 \leq 9k-3r-2n+3$. Therefore the variety of linear spaces L that belong to a linear system Φ is an irreducible variety of codimension

$r(n-3k-1+r)$ in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

Theorem 3.3.11. *Let $n \leq 2k+3$. If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to to a linear system Φ of affine dimension $n-k-2$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, then $N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-k-2} \oplus \mathcal{O}(n+2+2k)$.*

Corollary 3.3.12. *The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension $n-k-2$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible component of codimension $(2k-1)(n-k-2)$ of the subscheme $N^n_{n-k}((n+2)^{n-k-2}, (n+2+2k))$.*

3.3.1 Normal Bundle of Rational Curves in Codimension 1

Theorem 3.3.13. *Let $C_n \subset \mathbb{P}^n$ the normal rational curve of degree n and $\pi_1 : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^n$ be the projection from a point $p \in \mathbb{P}^n$. The rational curve $\pi_1(C_n) \subset \mathbb{P}^{n-1}$ has normal bundle $N_{\pi_1(C_n);\mathbb{P}^{n-1}} = \mathcal{O}(n+2)^{n-4} \oplus \mathcal{O}(n+3)^2$ if and only if $p \notin \sigma_2(C_n)$. This is equivalent to be smooth for $\pi_1(C_n)$.*

Proof. Let C_n be a rational normal curve of degree n in $\mathbb{P}^n$ with usual embedding $\nu_n : \mathbb{P}^1 \to \mathbb{P}^n$ and $\pi_1 : \mathbb{P}^n \setminus < p > \to \mathbb{P}^{n-1}$ the projection from the point $p = (a_0, ..., a_n) \in \mathbb{P}^n$. We have the following:

(3.31)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{C_n} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{C_n} \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\nu_n)"] & \mathcal{O}_{C_n}^{n+1} \arrow[r, "Syz(J(\nu_n))"] \arrow[d] & \mathcal{O}_{C_n}(2)^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\pi_1\circ\nu_n)"] & \mathcal{O}^n_{\pi_1(C_n)} \arrow[r] \arrow[d] & N_{\pi_1(C_n);\mathbb{P}^{n-1}}(-n) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since the map $\mathcal{O}_{C_n}(2)^{n-1} \to N_{\pi_1(C_n);\mathbb{P}^{n-1}}(-n)$ is surjective, it follows that $2 \leq l_0 \leq ... \leq l_{n-3}$ and $l_0 + ... + l_{n-3} = 2n-2$, where $N_{\pi_1(C_n);\mathbb{P}^{n-1}}(-n) = \mathcal{O}(l_0) \oplus ... \oplus \mathcal{O}(l_{n-3})$.

But we can write $l_i = l_i^1 + 2$, so $l_0^1 + ... + l_{n-3}^1 = 2$. Therefore we have $l_0 = ... = l_{n-5} = 2$ and $l_{n-4} = 2, 3$ $l_{n-3} = 4, 3$ respectively, so it is either

$N_{\pi_1(C_n);\mathbb{P}^{n-1}}(-n) = \mathcal{O}(2)^{n-3} \oplus \mathcal{O}(4)$ or

$N_{\pi_1(C_n);\mathbb{P}^4}(-n) = \mathcal{O}(2)^{n-4} \oplus \mathcal{O}(3)^2$,

not any other case occurred if the projection has only ordinary singularities .

By dualizing the last exact column of (3.31) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(2)$, we get:

$$0 \longrightarrow N^{\vee}_{\pi_1(C_n);\mathbb{P}^{n-1}}(n+2) \longrightarrow \mathcal{O}_{C_n}^{n-1} \longrightarrow \mathcal{O}_{C_n}(2) \longrightarrow 0,$$

and we have $deg(N^{\vee}_{\pi_1(C_n);\mathbb{P}^{n-1}}(n+2)) = -2$. Now either $N_{\pi_1(C_n);\mathbb{P}^{n-1}}(-n) = \mathcal{O}(2)^{n-4} \oplus \mathcal{O}(3)^2$ if and only if $N^{\vee}_{\pi_1(C_5);\mathbb{P}^4}(n+2) = \mathcal{O}^{n-2} \oplus \mathcal{O}(-1)^2$ or $N_{\pi_1(C_n);\mathbb{P}^{n-1}}(-n) = \mathcal{O}(2)^{n-3} \oplus \mathcal{O}(4)$ if and only if $N^{\vee}_{\pi_1(C_n);\mathbb{P}^{n-1}}(n+2) = \mathcal{O}^{n-3} \oplus \mathcal{O}(-2)$.

If we denote the following matrix as:

$$N^p_{n,1} := \begin{pmatrix} a_0 & \dots & a_{n-2} \\ -2a_1 & \dots & -2a_{n-1} \\ a_2 & \dots & a_n \end{pmatrix},$$

the smoothness condition corresponds to rank$(N^p_{n,1}) = 3$, so the second case above is impossible. $\square$

Example 3.3.14 (Rational Curve of degree 5 in $\mathbb{P}^4$). Let C_5 be a rational normal curve of degree 5 in $\mathbb{P}^5$ with usual embedding $\nu_5 : \mathbb{P}^1 \to \mathbb{P}^5$ and $\pi_1 : \mathbb{P}^5 \setminus < p > \to \mathbb{P}^4$ the projection from the point $p = (a_0, a_1, a_2, a_3, a_4, a_5) \in \mathbb{P}^5$. We have the following:

(3.32)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{C_5} \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{C_5} \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-4)^2 \arrow[r, "J(\nu_5)"] & \mathcal{O}_{C_5}^6 \arrow[r, "Syz(J(\nu_5))"] \arrow[d] & \mathcal{O}_{C_5}(2)^4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-4)^2 \arrow[r, "J(\pi_1 \circ \nu_5)"] & \mathcal{O}_{\pi_1(C_5)}^5 \arrow[r] \arrow[d] & N_{\pi_1(C_5);\mathbb{P}^4}(-5) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since the map $\mathcal{O}_{C_5}(2)^4 \to N_{\pi_1(C_5);\mathbb{P}^4}(-5)$ is surjective, it follows that $2 \leq x \leq y \leq z$. Therefore it is either $N_{\pi_1(C_5);\mathbb{P}^4}(-5) = \mathcal{O}(2)^2 \oplus \mathcal{O}(4)$ or $N_{\pi_1(C_5);\mathbb{P}^4}(-5) = \mathcal{O}(2) \oplus \mathcal{O}(3)^2$, not any other case occurred if the projection is smooth.

By dualizing the last exact column of (3.32) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(2)$, we get:

$$0 \longrightarrow N^{\vee}_{\pi_1(C_5);\mathbb{P}^4}(7) \longrightarrow \mathcal{O}^4_{C_5} \longrightarrow \mathcal{O}_{C_5}(2) \longrightarrow 0,$$

and we have $deg(N^{\vee}_{\pi_1(C_5);\mathbb{P}^4}(7)) = -2$. Now either $N_{\pi_1(C_5);\mathbb{P}^4}(-5) = \mathcal{O}(2) \oplus \mathcal{O}(3)^2$ if and only if $N^{\vee}_{\pi_1(C_5);\mathbb{P}^4}(7) = \mathcal{O} \oplus \mathcal{O}(-1)^2$ or $N_{\pi_1(C_5);\mathbb{P}^4}(-5) = \mathcal{O}(2)^2 \oplus \mathcal{O}(4)$ if and only if $N^{\vee}_{\pi_1(C_5);\mathbb{P}^4}(7) = \mathcal{O}^2 \oplus \mathcal{O}(-2)$.

If we denote the following matrix as:

$$N^p_{5,1} := \begin{pmatrix} a_0 & \dots & a_3 \\ -2a_1 & \dots & -2a_4 \\ a_2 & \dots & a_5 \end{pmatrix},$$

the smoothness condition correspond to $\text{rank}(N^p_{5,1}) = 3$, so the second case above is impossible.

Theorem 3.3.15. *Let $C_5 \subset \mathbb{P}^5$ the normal rational curve of degree 5 and $\pi_1 : \mathbb{P}^5 \setminus \{p\} \to \mathbb{P}^4$ be the projection from a point $p \in \mathbb{P}^5$. The rational curve $\pi_1(C_5) \subset \mathbb{P}^4$ has normal bundle $N_{\pi_1(C_5);\mathbb{P}^4} = \mathcal{O}(7) \oplus \mathcal{O}(8)^2$ if and only if $p \notin \sigma_2(C_5)$. This is equivalent to be smooth for $\pi_1(C_5)$.*

Codimension 1, non-degenerate case, $n \geq 5$

Splitting Type of $T\mathbb{P}^{n-1}\|_{\pi_1(C_n)}$	Rank of $T^L_{n,1}$	Geometric Meaning	Codimension	Degree
$(\overbrace{n+1,...,n+1}^{n-3}, n+1, n+2)$	2			
$(\overbrace{n+1,...,n+1}^{n-3}, n, n+3)$	1	Impossible		
Splitting of $N_{\pi_1(C_n),\mathbb{P}^{n-1}}$	**Rank of $N^L_{n,1}$**	**Geometric Meaning**	**Codimension**	**Degree**
$(\overbrace{n+2,...,n+2}^{n-4}, n+3, n+3)$	3	$p \notin \sigma_2(C_n)$		
$(\overbrace{n+2,...,n+2}^{n-4}, n+2, n+4)$	2	$p \in \sigma_2(C_n)$	n-3	$\frac{n(n-1)}{2}$

3.3.2 Normal Bundle of Rational Curves in Codimension 2

Let C_n be a rational normal curve of degree n in $\mathbb{P}^n$ with Veronese embedding $\nu_n : \mathbb{P}^2 \to \mathbb{P}^n$ and $\pi_2 : \mathbb{P}^n \setminus < p_1, p_2 > \to \mathbb{P}^{n-2}$ be the projection from the line generated by $p_1 = (a^1_0, ..., a^1_n), p_2 = (a^2_0, ..., a^2_n) \in \mathbb{P}^n$. We have the following:

(3.33)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{C_n}^2 \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{C_n}^2 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\nu_n)"] & \mathcal{O}_{C_n}^{n+1} \arrow[r, "Syz(J(\nu_n))"] \arrow[d] & \mathcal{O}_{C_n}(2)^{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-n+1)^2 \arrow[r, "J(\pi_1\circ\nu_n)"] & \mathcal{O}_{\pi_2(C_n)}^{n-1} \arrow[r] \arrow[d] & N_{\pi_2(C_n);\mathbb{P}^{n-2}}(-n) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since the map $\mathcal{O}_{C_n}(2)^{n-1} \to N_{\pi_2(C_n);\mathbb{P}^{n-2}}(-n)$ is surjective, it follows that $2 \leq n'_0 \leq ... \leq n'_{n-4}$ and $n'_0+...+n'_{n-4} = 2n-2$, where $N_{\pi_2(C_n);\mathbb{P}^{n-2}}(-n) = \mathcal{O}(n'_0)\oplus...\oplus\mathcal{O}(n'_{n-4})$. But we can write $n'_i = n''_i + 2$, so $n''_0 + ... + n''_{n-4} = 4$.

Therefore we have $n'_0 = ... = n'_{n-8} = 2$, if we can write:

$$N_{\pi_2(C_n);\mathbb{P}^{n-2}} = \mathcal{O}(n+2)^{n-7} \oplus \mathcal{F};$$

where $\mathcal{F}$ is a rank 4 vector bundle on $\mathbb{P}^1$. Hence we must study the splitting of $\mathcal{F}$, if we indicate with $\mathcal{F} = \mathcal{O}(f_0) \oplus ... \oplus \mathcal{O}(f_3)$, where $f_0 + ... + f_3 = 4n + 14$. Therefore it is one of the following cases:

1. $\mathcal{F} = \mathcal{O}(n+5)^4 =: \mathcal{F}_1$;

2. $\mathcal{F} = \mathcal{O}(n+4) \oplus \mathcal{O}(n+5)^2 \oplus \mathcal{O}(n+6) =: \mathcal{F}_2$;

3. $\mathcal{F} = \mathcal{O}(n+4)^2 \oplus \mathcal{O}(n+6)^2 =: \mathcal{F}_3$;

4. $\mathcal{F} = \mathcal{O}(n+4)^2 \oplus \mathcal{O}(n+5) \oplus \mathcal{O}(n+7) =: \mathcal{F}_4$;

5. $\mathcal{F} = \mathcal{O}(n+4)^3 \oplus \mathcal{O}(n+8) =: \mathcal{F}_5$;

not any other case occurred if the projection has only ordinary singularities.

By dualizing the last exact column of (3.33) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(2)$, we get:

$$0 \longrightarrow N^{\vee}_{\pi_2(C_n);\mathbb{P}^{n-2}}(n+2) \longrightarrow \mathcal{O}_{C_n}^{n-1} \xrightarrow{\mathcal{N}^L_{n,2}} \mathcal{O}_{C_n}^2(2) \longrightarrow 0,$$

and we have $deg(N^{\vee}_{\pi_2(C_n);\mathbb{P}^{n-2}}(n+2)) = -2$. But $N^{\vee}_{\pi_2(C_n);\mathbb{P}^{n-2}}(n+2) = \mathcal{O}^{n-7}\oplus\mathcal{F}^{\vee}(n+2)$, so $h^0(N^{\vee}_{\pi_2(C_n);\mathbb{P}^{n-2}}(n+2)) = n - 7 + h^0(\mathcal{F}^{\vee}(n+2))$. Therefore we have that $0 \leq h^0(\mathcal{F}^{\vee}(n+2)) \leq 3$, where we have:

a) $h^0(\mathcal{F}^\vee(n+2)) = 0 \Leftrightarrow \mathcal{F} = \mathcal{F}_1$;

b) $h^0(\mathcal{F}^\vee(n+2)) = 1 \Leftrightarrow \mathcal{F} = \mathcal{F}_2$;

c) $h^0(\mathcal{F}^\vee(n+2)) = 2 \Leftrightarrow \mathcal{F} = \mathcal{F}_3$ *or* $\mathcal{F}_4$;

d) $h^0(\mathcal{F}^\vee(n+2)) = 3 \Leftrightarrow \mathcal{F} = \mathcal{F}_5$.

If we denote the following matrix as:

$$N_{n,2}^{p_i} := \begin{pmatrix} a_0^i & \dots & a_{n-2}^i \\ -2a_1^i & \dots & -2a_{n-1}^i \\ a_2^i & \dots & a_n^i \end{pmatrix},$$

we have the following matrix:

$$N_{n,2}^L = \begin{pmatrix} N_n^{p_1} \\ N_n^{p_2} \end{pmatrix} = \begin{pmatrix} a_0^1 & \dots & a_{n-2}^1 \\ -2a_1^1 & \dots & -2a_{n-1}^1 \\ a_2^1 & \dots & a_n^1 \\ a_0^2 & \dots & a_{n-2}^2 \\ -2a_1^2 & \dots & -2a_{n-1}^2 \\ a_2^2 & \dots & a_n^2 \end{pmatrix}.$$

So we have the following cases:

A) $\operatorname{rank}(N_{n,2}) = 6 \Leftrightarrow h^0(\mathcal{F}^\vee(2)) = 0 \Leftrightarrow \mathcal{F} = \mathcal{F}_1$;

B) $\operatorname{rank}(N_{n,2}) = 5 \Leftrightarrow h^0(\mathcal{F}^\vee(2)) = 1 \Leftrightarrow \mathcal{F} = \mathcal{F}_2$;

C) $\operatorname{rank}(N_{n,2}) = 4 \Leftrightarrow h^0(\mathcal{F}^\vee(2)) = 2 \Leftrightarrow \mathcal{F} = \mathcal{F}_3$ *or* $\mathcal{F}_4$;

D) $\operatorname{rank}(N_{n,2}) = 3 \Leftrightarrow h^0(\mathcal{F}^\vee(2)) = 3 \Leftrightarrow \mathcal{F} = \mathcal{F}_5$.

If we consider the following exact sequence:

$$0 \longrightarrow N^\vee_{\pi_2(C_n);\mathbb{P}^{n-2}}(n+3) \longrightarrow \mathcal{O}_{C_n}^{n-1}(1) \xrightarrow{\mathcal{N}_{n,2}^L} \mathcal{O}_{C_n}^2(3) \longrightarrow 0,$$

and we have that:

C1) $\operatorname{rank}(N_{n,2}^L(1)) = 3 \Leftrightarrow h^0(\mathcal{F}^\vee(3)) = 8 \Leftrightarrow \mathcal{F} = \mathcal{F}_3$;

C2) $\operatorname{rank}(N_{n,2}^L(1)) \leq 4 \Leftrightarrow h^0(\mathcal{F}^\vee(3)) = 7 \Leftrightarrow \mathcal{F} = \mathcal{F}_4$.

Example 3.3.16 (Rational Curve of degree 7 in $\mathbb{P}^5$). In the table at the end of this section we have summarize the codimensions of the determinantal varieties which parametrize the splitting type of normal bundle for rational curves in $\mathbb{P}^5$ of degree 7 obtained by projection from the rational normal curve C_7 of degree 7. We will denote by $N^L_{7,2}$ the following matrix:

$$N^L_{7,2} := \begin{pmatrix} a^1_0 & \dots & a^1_5 \\ -2a^1_1 & \dots & -2a^1_6 \\ a^1_2 & \dots & a^1_7 \\ a^2_0 & \dots & a^2_5 \\ -2a^2_1 & \dots & -2a^2_6 \\ a^2_2 & \dots & a^2_7 \end{pmatrix},$$

which is the map of the following exact sequence:

$$0 \longrightarrow H^0(N^\vee_{\pi_2(C_7);\mathbb{P}^5}(9)) \longrightarrow H^0(\mathcal{O}^6_{C_7}) \xrightarrow{N^L_{7,2}} H^0(\mathcal{O}^2_{C_7}(2)) \longrightarrow H^1(N^\vee_{\pi_2(C_7);\mathbb{P}^5}(9)) \longrightarrow 0,$$

so that $h^0(N^\vee_{\pi_2(C_7);\mathbb{P}^5}(9)) = 6 - \operatorname{rank} N^L_{7,2}$.

We denote by $N^L_{7,2}(1)$ the following matrix:

$$N^L_{7,2}(1) := \begin{pmatrix} a^1_0 & \dots & a^1_5 & 0 & \dots & 0 \\ -2a^1_1 & \dots & -2a^1_6 & a^1_0 & \dots & a^1_5 \\ a^1_2 & \dots & a^1_7 & -2a^1_1 & \dots & -2a^1_6 \\ 0 & \dots & 0 & a^1_2 & \dots & a^1_7 \\ a^2_0 & \dots & a^2_5 & 0 & \dots & 0 \\ -2a^2_1 & \dots & -2a^2_6 & a^2_0 & \dots & a^2_5 \\ a^2_2 & \dots & a^2_7 & -2a^2_1 & \dots & -2a^2_6 \\ 0 & \dots & 0 & a^2_2 & \dots & a^2_7 \end{pmatrix},$$

which is the map of the following exact sequence:

$$0 \longrightarrow H^0(N^\vee_{\pi_2(C_7);\mathbb{P}^5}(10)) \longrightarrow H^0(\mathcal{O}^6_{C_7}(1)) \xrightarrow{N^L_{7,2}(1)} H^0(\mathcal{O}^2_{C_7}(3)) \longrightarrow H^1(N^\vee_{\pi_2(C_7);\mathbb{P}^5}(10)) \longrightarrow 0,$$

so that $h^0(N^\vee_{\pi_2(C_7);\mathbb{P}^5}(10)) = 12 - \operatorname{rank} N^L_{7,2}(1)$.

We will indicate with ExpCodim the codimension of the general determinantal variety of the same type, with ConjCodim the following one:

$$\sum_{i,j} \max\{n_i - n_j - 1, 0\},$$

where (n_1, n_2, n_3, n_4) is the splitting type of $N_{\pi_2(C_7),\mathbb{P}^5}$ and with Total Codim we will indicate the intersection of the two determinantal varieties from $N^L_{7,2}$ and $N^L_{7,2}(1)$.

Proposition 3.3.17. *The splitting type of the normal bundle $N_{\pi_2(C_7),\mathbb{P}^4}$ is:*

i) $(10,10,10,10)$ *if and only if* $\mathrm{rank}(N^L_{7,2}) = 6$*;*

ii) $(11,10,10,9)$ *if and only if* $\mathrm{rank}(N^L_{7,2}) = 5$*;*

iii) $(11,11,9,9)$ *if and only if* $\mathrm{rank}(N^L_{7,2}) = 4$ *and* $\mathrm{rank}(N^L_{7,2}(1)) = 8$*;*

iv) $(12,10,9,9)$ *if and only if* $\mathrm{rank}(N^L_{7,2}) = 4$ *and* $\mathrm{rank}(N^L_{7,2}(1)) = 7$*;*

v) $(13,9,9,9)$ *if and only if* $\mathrm{rank}(N^L_{7,2}) = 3$*;*

Splitting of $N_{\pi_2(C_7),\mathbb{P}^5}$	Rank of $N^L_{7,2}$	Codim with M2	Rank of $N^L_{7,2}(1)$	Codim with M2	Total Codim with M2	ExpCodim	ConjCodim
$(10,10,10,10)$	6		8				
$(11,10,10,9)$	5	1	8		1	1	1
$(11,11,9,9)$	4	4	8		4	4	4
$(12,10,9,9)$	4	4	7	4	5	5	5
$(13,9,9,9)$	3	7	6		6	9	9

Remark 3.3.18. If we consider the generic determinantal variety $M_k^{m\times n}$ made of the $m\times n$ matrix of rank k it's well known that:

$$\mathrm{codim}(M_k^{m\times n}) = (m-k)(n-k),$$

$$\deg(M_k^{m\times n}) = \prod_{i=0}^{n-k-1} \frac{\binom{m+i}{k}}{\binom{k+i}{k}}$$

Splitting of $T\mathbb{P}^{n-2}\|_{\pi_2(C_n)}$	Rank of $T^L_{n,2}$		Codim in $\mathrm{Hom}(H^0(\mathcal{O}^n), H^0(\mathcal{O}^2(1)))$	ExpCodim	Degree
$((n+1)^{n-4}, (n+2)^2)$	4				
$((n+1)^{n-4}, n+1, n+3)$	3		n-3	n-3	$\binom{n}{3}$
$((n+1)^{n-4}, n, n+4)$	2	Impossible			

Splitting of $N_{\pi_2(C_n),\mathbb{P}^{n-2}}$	Rank of $N^L_{n,2}$	Rank of $N^L_{n,2}(1)$	Codim in $\mathrm{Hom}(H^0(\mathcal{O}^{n-1}), H^0(\mathcal{O}^2(2)))$	ExpCodim	ConjCodim	Deg
$((n+2)^{n-7}, (n+3)^4)$	6					
$((n+2)^{n-7}, n+2, (n+3)^2, n+4)$	5		n-6	n-6	n-6	$\binom{n}{5}$
$((n+2)^{n-7}, (n+2)^2, (n+4)^2)$	4	8	2(n-5)	2(n-5)	2(n-5)	
$((n+2)^{n-7}, (n+2)^2, n+3, n+5)$	4	7	2(n-5)+1	2n-9	2(n-5)+1	
$((n+2)^{n-7}, (n+2)^2, n+6)$	3		3(n-4)	3(n-4)	3(n-4)	

Proposition 3.3.19. *If there exist two points $q_1, q_2 \in L$ each of them belongs to a different $3-$secant $\mathbb{P}^2$, then:*

$$\ker(N^L_{n,2}) = (\mathcal{I}_{P_1} \cap \mathcal{I}_{P_2})_{n-2} = (\mathcal{I}_{P_1\cup P_2})_{n-2},$$

where P_i is the set of points in $\mathbb{P}^1$ corresponds to the linear forms in the additive decomposition of f_i which corresponds to q_i.

Case rank $N^L_{n,2} = 3$

Lemma 3.3.20. *The splitting type of the normal bundle* $N_{\pi_2(C_n),\mathbb{P}^{n-2}}$ *is:*

$$((n+2)^{n-4}, n+6)$$

if and only if $\operatorname{rank}(N^L_{n,2}) = 3$.

Theorem 3.3.21 (Case Rank 3)**.** *If the projection line L belongs to some* $3-$*secant* $\mathbb{P}^2$*, but it is not a secant line, then the splitting type of the normal bundle* $N_{\pi_2(C_n),\mathbb{P}^{n-2}}$ *is:*

$$((n+2)^{n-4}, n+6).$$

Proof. If L belongs to a $3-$secant $\mathbb{P}^2$, then there exist two points $q_1, q_2 \in L$ such that $Ann(f_1) = (\alpha, \beta_1)$ and $Ann(f_2) = (\alpha, \beta_2)$ with α has only simple roots and $\deg(\alpha) = 3$ and $\beta_1 \neq \beta_2$, $\deg(\beta_1) = \deg(\beta_2) = n-1$. So $\dim <\alpha>_{n-2} = n-4$ and $<\alpha, \beta_1>_{n-2} = <\alpha, \beta_2>_{n-2} = <\alpha>_{n-2}$. Therefore $\operatorname{rank}(N^L_{n,2}) = n-1-\dim(<\alpha, \beta_1>_{n-2} \cap <\alpha, \beta_2>_{n-2}) = 3$.

□

Corollary 3.3.22. *The variety of lines that, as centre of projection, gives a rational curve of degree n in $\mathbb{P}^{n-2}$ which has the splitting type of the normal bundle* $N_{\pi_2(C_n),\mathbb{P}^{n-2}}$*:*

$$((n+2)^{n-4}, n+6)$$

has an irreducible subvariety of codimension $2n-7$ in $Gr(\mathbb{P}^1, \mathbb{P}^n)$ formed by the lines belonging to some $3-$*secant* $\mathbb{P}^2$*, but they are not a secant line.*

Case rank $N^L_{n,2} = 4$

Lemma 3.3.23. *The codimension in $Gr(\mathbb{P}^1, \mathbb{P}^n)$ of the variety of all lines in $\mathbb{P}^n$ belonging to some 4-secant $\mathbb{P}^3$ to the rational normal curve in $\mathbb{P}^n$ is $2n-10$*

Proof. In fact we can consider the incidence variety $I = \{(L, \pi) : L \in Gr(\mathbb{P}^1, \mathbb{P}^n), \pi \in S, L \subset S\}$ where S is the set of all 4-secant $\mathbb{P}^3$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^1, \mathbb{P}^n)$. That calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). □

Lemma 3.3.24. *If the splitting type of the normal bundle $N_{\pi_2(C_n),\mathbb{P}^{n-2}}$ is:*

$$((n+2)^{n-5},(n+4)^2),$$

then $\mathrm{rank}(N^L_{n,2}) = 4$. *Moreover the codimension of the variety which parametrize the above splitting is* $2(n-5)$ *in* $Gr(\mathbb{P}^1,\mathbb{P}^n)$.

Theorem 3.3.25 (Case Rank 4)**.** *If the projection line L belongs to some $4-$secant $\mathbb{P}^3$, but it does not belong to some $3-$secant $\mathbb{P}^2$, then the splitting type of the normal bundle $N_{\pi_2(C_n),\mathbb{P}^{n-2}}$ is:*

$$((n+2)^{n-5},(n+4)^2).$$

Proof. If L belongs to a $4-$secant $\mathbb{P}^3$, then there exist two points $q_1, q_2 \in L$ such that $Ann(f_1) = (\alpha, \beta_1)$ and $Ann(f_2) = (\alpha, \beta_2)$ with α has only simple roots and $\deg(\alpha) = 4$ and $\beta_1 \neq \beta_2$, $\deg(\beta_1) = \deg(\beta_2) = n-2$. So $\dim <\alpha>_{n-2} = n-5$ and $\dim(<\beta_1>_{n-2} \cap <\beta_2>_{n-2}) = 0$ otherwise $\beta_1 = \beta_2$ and $q_1 = q_2$, but this is impossible. Therefore $\mathrm{rank}(N^L_{n,2}) = n-1-\dim(<\alpha,\beta_1>_{n-2} \cap <\alpha,\beta_2>_{n-2}) = 4$. □

Corollary 3.3.26. *The variety of lines that, as centre of projection, give a rational curve of degree n in $\mathbb{P}^{n-2}$ which has the splitting of the normal bundle $N_{\pi_2(C_n),\mathbb{P}^{n-2}}$:*

$$((n+2)^{n-5},(n+4)^2)$$

has an irreducible component formed by the lines belonging to some $4-$secant $\mathbb{P}^3$, but they don't belong to some $3-$secant $\mathbb{P}^2$.

Case $\mathrm{rank}\, N^L_{n,2} = 5$

Lemma 3.3.27. *The splitting type of the normal bundle $N_{\pi_2(C_n),\mathbb{P}^{n-2}}$ is:*

$$((n+2)^{n-6},(n+3)^2,n+4)$$

if and only if $\mathrm{rank}(N^L_{n,2}) = 5$.

Observation 3.3.28. *If L belongs to a $5-$secant $\mathbb{P}^4$, then there exist two points $q_1, q_2 \in L$ such that $Ann(f_1) = (\alpha,\beta_1)$ and $Ann(f_2) = (\alpha,\beta_2)$ with α has only simple roots and* $\deg(\alpha) = 5$ *and* $\beta_1 \neq \beta_2$, $\deg(\beta_1) = \deg(\beta_2) = n-3$. *So* $\dim <\alpha>_{n-2} = n-6$, *so* $\mathrm{rank}(N^L_{n,2}) = n-1-\dim(<\alpha,\beta_1>_{n-2} \cap <\alpha,\beta_2>_{n-2}) \leq 5$.

Observation 3.3.29. *If* $\mathrm{rank}(N^L_{n,2}) = 5$, *then or $L \subset \mathbb{P}^4$ which is $5-$secant to C_n or $L \subset \mathbb{P}^k$ which is $(k+1)-$secant. In the second case there exist two points $q_1, q_2 \in L$ such that $Ann(f_1) = (\alpha,\beta_1)$ and $Ann(f_2) = (\alpha,\beta_2)$ with α has only simple roots and* $\deg(\alpha) = k+1$ *and* $\beta_1 \neq \beta_2$, $\deg(\beta_1) = \deg(\beta_2) = n-k+1$.

3.3.3 Normal Bundle of Rational Curves in Codimension 3

Case rank $N^L_{n,3} = 6, 5$

Observation 3.3.30. *If $L \cong \mathbb{P}^2$, as centre of projection, belongs to a $6-$secant $\mathbb{P}^5$, then there exist three points $q_1, q_2, q_3 \in L$ such that $Ann(f_1) = (\alpha, \beta_1), Ann(f_2) = (\alpha, \beta_2)$ and $Ann(f_3) = (\alpha, \beta_3)$ with α has only simple roots and $\deg(\alpha) = 6$ and $\beta_1 \neq \beta_2 \neq \beta_3$, $\deg(\beta_1) = \deg(\beta_2) = \deg(\beta_3) = n-4$. So $\dim < \alpha >_{n-2} = n-7$, so $\operatorname{rank}(N^L_{n,2}) = n-1-\dim(< \alpha, \beta_1 >_{n-2} \cap < \alpha, \beta_2 >_{n-2}) \leq 6$.*

Observation 3.3.31. *If $L \cong \mathbb{P}^2$, as centre of projection, belongs to a $5-$secant $\mathbb{P}^4$, then there exist three points $q_1, q_2, q_3 \in L$ such that $Ann(f_1) = (\alpha, \beta_1), Ann(f_2) = (\alpha, \beta_2)$ and $Ann(f_3) = (\alpha, \beta_3)$ with α has only simple roots and $\deg(\alpha) = 5$ and $\beta_1 \neq \beta_2 \neq \beta_3$, $\deg(\beta_1) = \deg(\beta_2) = \deg(\beta_3) = n-3$. So $\dim < \alpha >_{n-2} = n-6$, so $\operatorname{rank}(N^L_{n,2}) = n-1-\dim(< \alpha, \beta_1 >_{n-2} \cap < \alpha, \beta_2 >_{n-2}) \leq 5$.*

Case rank $N^L_{n,3} = 4$

Proposition 3.3.32. *If the $L \cong \mathbb{P}^2$ as a centre of projection plane belongs to a $4-$secant $\mathbb{P}^3$, but it is not a $3-$secant $\mathbb{P}^2$, then the splitting type of the normal bundle $N_{\pi_3(C_n),\mathbb{P}^{n-3}}$ is:*

$$((n+2)^{n-5}, n+8).$$

This is an irreducible component of the varieties of projection planes which give the splitting type above of codimension $3n-13$.

Proof. If $L \cong \mathbb{P}^2$, as centre of projection, belongs to a $4-$secant $\mathbb{P}^3$, then there exist three points $q_1, q_2, q_3 \in L$ such that $Ann(f_1) = (\alpha, \beta_1)$, $Ann(f_2) = (\alpha, \beta_2)$ and $Ann(f_3) = (\alpha, \beta_3)$ with α has only simple roots and $\deg(\alpha) = 4$ and $\beta_1 \neq \beta_2 \neq \beta_3$, $\deg(\beta_1) = \deg(\beta_2) = \deg(\beta_3) = n-2$. So $\dim < \alpha >_{n-2} = n-5$, so $\operatorname{rank}(N^L_{n,3}) = n-1-\dim(< \alpha, \beta_1 >_{n-2} \cap < \alpha, \beta_2 >_{n-2} \cap < \alpha, \beta_3 >_{n-2}) = 4$, otherwise $\beta_1 = \beta_2 = \beta_3$, but this is impossible. □

Example 3.3.33 (Rational Curve of degree 7 in $\mathbb{P}^4$). In the table at the end of this section we have summarize the codimensions of the determinantal varieties those parametrize the splitting type of normal bundle for rational curves in $\mathbb{P}^4$ of degree 7 obtained by projection from the rational normal curve C_7 in $\mathbb{P}^7$.

We will denote by $N^L_{7,3}$ the following matrix:

$$N^L_{7,3} := \begin{pmatrix} a^1_0 & \dots & a^1_5 \\ -2a^1_1 & \dots & -2a^1_6 \\ a^1_2 & \dots & a^1_7 \\ a^2_0 & \dots & a^2_5 \\ -2a^2_1 & \dots & -2a^2_6 \\ a^2_2 & \dots & a^2_7 \\ a^3_0 & \dots & a^3_5 \\ -2a^3_1 & \dots & -2a^3_6 \\ a^3_2 & \dots & a^3_7 \end{pmatrix},$$

which is the map of the following exact sequence:

$$0 \longrightarrow H^0(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(9)) \longrightarrow H^0(\mathcal{O}^6_{C_7}) \xrightarrow{N^L_{7,3}} H^0(\mathcal{O}^3_{C_7}(2)) \longrightarrow H^1(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(9)) \longrightarrow 0,$$

so that $h^0(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(9)) = 6 - \operatorname{rank} N^L_{7,3}$.

Again we will denote by $N^L_{7,3}(1)$ the following matrix:

$$N^L_{7,3}(1) := \begin{pmatrix} a^1_0 & \dots & a^1_5 & 0 & \dots & 0 \\ -2a^1_1 & \dots & -2a^1_6 & a^1_0 & \dots & a^1_5 \\ a^1_2 & \dots & a^1_7 & -2a^1_1 & \dots & -2a^1_6 \\ 0 & \dots & 0 & a^1_2 & \dots & a^1_7 \\ a^2_0 & \dots & a^2_5 & 0 & \dots & 0 \\ -2a^2_1 & \dots & -2a^2_6 & a^2_0 & \dots & a^2_5 \\ a^2_2 & \dots & a^2_7 & -2a^2_1 & \dots & -2a^2_6 \\ 0 & \dots & 0 & a^2_2 & \dots & a^2_7 \\ a^3_0 & \dots & a^3_5 & 0 & \dots & 0 \\ -2a^3_1 & \dots & -2a^3_6 & a^3_0 & \dots & a^3_5 \\ a^3_2 & \dots & a^3_7 & -2a^3_1 & \dots & -2a^3_6 \\ 0 & \dots & 0 & a^3_2 & \dots & a^3_7 \end{pmatrix},$$

which is the map of the following exact sequence:

$$0 \longrightarrow H^0(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(10)) \longrightarrow H^0(\mathcal{O}^6_{C_7}(1)) \xrightarrow{N_{7,3}(1)} H^0(\mathcal{O}^3_{C_7}(3)) \longrightarrow H^1(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(10)) \longrightarrow 0,$$

so that $h^0(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(10)) = 12 - \operatorname{rank} N^L_{7,3}(1)$.

Moreover we will denote by $N^L_{7,3}(2)$ the following matrix:

$$N^L_{7,3}(2) := \begin{pmatrix} a^1_0 & \dots & a^1_5 & 0 & \dots & 0 & 0 & \dots & 0 \\ -2a^1_1 & \dots & -2a^1_6 & a^1_0 & \dots & a^1_5 & 0 & \dots & 0 \\ a^1_2 & \dots & a^1_7 & -2a^1_1 & \dots & -2a^1_6 & a^1_0 & \dots & a^1_5 \\ 0 & \dots & 0 & a^1_2 & \dots & a^1_7 & -2a^1_1 & \dots & -2a^1_6 \\ 0 & \dots & 0 & 0 & \dots & 0 & a^1_2 & \dots & a^1_7 \\ a^2_0 & \dots & a^2_5 & 0 & \dots & 0 & 0 & \dots & 0 \\ -2a^2_1 & \dots & -2a^2_6 & a^2_0 & \dots & a^2_5 & 0 & \dots & 0 \\ a^2_2 & \dots & a^2_7 & -2a^2_1 & \dots & -2a^2_6 & a^2_0 & \dots & a^2_5 \\ 0 & \dots & 0 & a^2_2 & \dots & a^2_7 & -2a^2_1 & \dots & -2a^2_6 \\ 0 & \dots & 0 & 0 & \dots & 0 & a^2_2 & \dots & a^2_7 \\ a^3_0 & \dots & a^3_5 & 0 & \dots & 0 & 0 & \dots & 0 \\ -2a^3_1 & \dots & -2a^3_6 & a^3_0 & \dots & a^3_5 & 0 & \dots & 0 \\ a^3_2 & \dots & a^3_7 & -2a^3_1 & \dots & -2a^3_6 & a^3_0 & \dots & a^3_5 \\ 0 & \dots & 0 & a^3_2 & \dots & a^3_7 & -2a^3_1 & \dots & -2a^3_6 \\ 0 & \dots & 0 & 0 & \dots & 0 & a^3_2 & \dots & a^3_7 \end{pmatrix},$$

which is the map of the following exact sequence:

$$0 \longrightarrow H^0(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(10)) \longrightarrow H^0(\mathcal{O}^6_{C_7}(2)) \xrightarrow{N^L_{7,3}(2)} H^0(\mathcal{O}^3_{C_7}(4)) \longrightarrow H^1(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(10)) \longrightarrow 0,$$

so that $h^0(N^\vee_{\pi_3(C_7);\mathbb{P}^4}(10)) = 18 - \operatorname{rank} N^L_{7,3}(2)$.

Proposition 3.3.34. *The splitting type of the normal bundle $N_{\pi_3(C_7),\mathbb{P}^4}$ is:*

i) $(11, 11, 11)$ *if and only if* $\operatorname{rank}(N^L_{7,3}) = 6$*;*

ii) $(10, 11, 12)$ *if and only if* $\operatorname{rank}(N^L_{7,3}) = 6$ *and* $\operatorname{rank}(N^L_{7,3}(1)) = 11$*;*

iii) $(10, 10, 13)$ *if and only if* $\operatorname{rank}(N^L_{7,3}) = 6$, $\operatorname{rank}(N^L_{7,3}(1)) = 10$ *and* $\operatorname{rank}(N^L_{7,3}(2)) = 14$*;*

iv) $(9, 12, 12)$ *if and only if* $\operatorname{rank}(N^L_{7,3}) = 5$, $\operatorname{rank}(N^L_{7,3}(1)) = 10$ *and* $\operatorname{rank}(N^L_{7,3}(2)) = 15$*;*

v) $(9, 11, 13)$ *if and only if* $\operatorname{rank}(N^L_{7,3}) = 5$, $\operatorname{rank}(N^L_{7,3}(1)) = 10$ *and* $\operatorname{rank}(N^L_{7,3}(2)) = 14$*;*

vi) $(9, 10, 14)$ *if and only if* $\operatorname{rank}(N^L_{7,3}) = 5$, $\operatorname{rank}(N^L_{7,3}(1)) = 9$ *and* $\operatorname{rank}(N^L_{7,3}(2)) = 13$*;*

vii) $(9,9,15)$ *if and only if* $\operatorname{rank}(N^L_{7,3}) = 4$, $\operatorname{rank}(N^L_{7,3}(1)) = 8$ *and* $\operatorname{rank}(N^L_{7,3}(2)) = 12$.

Splitting of $N_{\pi_3(C_7),\mathbb{P}^4}$	Rank of $N^L_{7,3}$	Codim with M2	Rank of $N^L_{7,3}(1)$	Rank of $N^L_{7,3}(2)$	ExpCodim	ConjCodim
$(11,11,11)$	6		12	15		
$(10,11,12)$	6		11	15	1	1
$(10,10,13)$	6		10	14	4	4
$(9,12,12)$	5	4	10	15	4	4
$(9,11,13)$	5	4	10	14	4	5
$(9,10,14)$	5	4	9	13	10	7
$(9,9,15)$	4	8	8	12	18	10
$(9,8,16)$	3	Impossible				

Codimension 3, non-degenerate case, $n \geq 10$

Splitting of $T\mathbb{P}^{n-3}\|_{\pi_3(C_n)}$	Rank of $T^L_{n,3}$	Codim in $\mathrm{Hom}(H^0(\mathcal{O}^n), H^0(\mathcal{O}^3(1)))$
$((n+1)^{n-6}, (n+2)^3)$	6	
$((n+1)^{n-6}, n+1, n+2, n+3)$	5	n-5
$((n+1)^{n-6}, (n+1)^2, n+4)$	4	2(n-4)

Splitting of $N_{\pi_3(C_n),\mathbb{P}^{n-3}}$	Rank of $N^L_{n,3}$	Rank of $N^L_{n,3}(1)$	Rank of $N^L_{n,3}(2)$	Codim in	ExpCodim	ConjCodim
$((n+2)^{n-10}, (n+3)^6)$	9	12	15			
$((n+2)^{n-10}, n+2, (n+3)^4, n+4)$	8	12	15	n-9	n-9	n-9
$((n+2)^{n-10}, (n+2)^2, (n+3)^3, n+5)$	7	11	15	2(n-8)+3	2(n-8)+3	2(n-8)+3
$((n+2)^{n-10}, (n+2)^2, (n+3)^2, (n+4)^2)$	7	12	15	2(n-8)	2(n-8)	2(n-8)
$((n+2)^{n-10}, (n+2)^3, (n+4)^3)$	6	12	15	3(n-7)	3(n-7)	3(n-7)
$((n+2)^{n-10}, (n+2)^3, n+3, n+4, n+5)$	6	11	15			3(n-7)+1
$((n+2)^{n-10}, (n+2)^3, (n+3)^2, n+6)$	6	10	14	3(n-7)+4	3(n-7)+4	3(n-7)+4
$((n+2)^{n-10}, (n+2)^4, (n+5)^2)$	5	10	15	4(n-6)	4(n-6)	4(n-6)
$((n+2)^{n-10}, (n+2)^4, n+4, n+6)$	5	10	14			4(n-6)+1
$((n+2)^{n-10}, (n+2)^4, n+3, n+7)$	5	9	13	4(n-6)+3	4(n-6)+3	4(n-6)+3
$((n+2)^{n-10}, (n+2)^5, n+8)$	4	8	12			

3.4 Codimension k, for $\frac{n-1}{3} \leq k \leq n-3$

Observation 3.4.1. $k \leq \operatorname{rank} N^L_{n,k} = n-1-h^0(N^\vee_{\pi_k(C_n),\mathbb{P}^{n-k}}(n+2)) \leq n-1$

Proposition 3.4.2. *$N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-1-\operatorname{rank}(N^L_{n,k})} \oplus \mathcal{F}$, with $\mathcal{F}$ a vector bundle of rank $\operatorname{rank}(N^L_{n,k})-k$ on $\mathbb{P}^1$ and $\deg(\mathcal{F}^\vee(n+2)) = -2k$ such that $\mathcal{F} \cong \bigoplus_{i=0}^{\operatorname{rank}(N^L_{n,k})-k} \mathcal{O}(l_i)$ with $l_i \geq n+3$.*

If $2(n-k) \geq 2k$ we have two possibilities:

1. *$N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{r-1} \oplus \mathcal{O}(n+3)^{2(n-k-r)-2k} \oplus \mathcal{F}'$ with $\operatorname{rank}(\mathcal{F}') = n-r-2(n-k-r)+k$ and $\deg(\mathcal{F}'^\vee(n+2)) = -2k+2(2(n-k-r)-2k)$ if and only if $\operatorname{rank}(N^L_{n,k}) = n-r$ and $2(n-k-r) \geq 2k$ for $1 \leq r \leq n-k-2$;*

2. *$N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-k-2} \oplus \mathcal{O}(n+2+2k)$ if and only if $\operatorname{rank}(N^L_{n,k}) = k+1$.*

However the last one is true also for $2n-2k < 2k$.

We can rephrased the above proposition as:

Proposition 3.4.3. *If $2(n-k) \geq 2k$ we have two possibilities:*

1. *$\pi_k(C_n) \in N^n_{n-k}((n+2)^{r-1}, (n+3)^{2(n-k-r)-k}, spt(\mathcal{F}'))$, where $spt(\mathcal{F}')$ is the splitting type of $\mathcal{F}'$ with $\operatorname{rank}(\mathcal{F}') = n-r-2(n-k-r)$ and $\deg(\mathcal{F}'^\vee(n+2)) = -2k+2(2(n-k-r)-2k)$ if and only if $L \in V(N^L_{n,k})^{n-r}$ and $2(n-k-r) \geq 2k$ for $1 \leq r \leq n-k-2$;*

2. *$\pi_k(C_n) \in N^n_{n-k}((n+2)^{n-k-1}, (n+2+k))$ if and only if $L \in V(N^L_{n,k})^{k+1}$.*

However the last one is true also for $2n-2k < 2k$.

Lemma 3.4.4. *$\operatorname{rank} N^L_{n,k} \leq n-2$ if and only if the forms f_i of degree n corresponding to the points p_i generating L can be represented by the similar GAD, i.e. :*

$$f_i = G_{i_1} L_1^{n-g_1+1} + ... + G_{i_m} L_m^{n-g_m+1}.$$

Proof. $\Rightarrow$ If $\operatorname{rank} N^L_{n,k} \leq n-2$, then there exists at least an element $\phi \in T_{n-2}$ such that for all forms f_i corresponding to the points p_i generating L we have $\phi \circ f_i = 0$. So we can consider the primary decomposition of $\phi = \prod_{i=1}^m (\phi_i)^{g_i}$, with $\phi_i \in T_1$ and $\sum_i g_i = n-2$, so every f_i can be represented by the similar GAD, i.e. :

$$f_i = G_{i_1} L_1^{n-g_1+1} + ... + G_{i_m} L_m^{n-g_m+1},$$

where $(L_j)^\perp = \phi_j$ for all $j = 1,..,m$ and $G_{i_j} \in S_{g_j-1}$ for all $i = 1,...,k$ and $j = 1,...,m$.

$\Leftarrow$ On the other hand if every f_i can be represented by the similar GAD, i.e. :

$$f_i = G_{i_1} L_1^{n-g_1+1} + ... + G_{i_m} L_m^{n-g_m+1},$$

then we can consider $\phi = \prod_{i=1}^{m}((L_i)^{\perp})^{g_i}$. By definition of GAD representation we have $\phi \circ f_i = 0$ for all $i = 1, ..., k$, so $\phi \in \ker N_{n,k}^L$ and $\operatorname{rank} N_{n,k}^L \leq n-2$.

□

Observation 3.4.5. *In particular we can observe that if L belong to a (n-2)-secant $\mathbb{P}^{n-3}$ generated by $q_1, ..., q_{n-2}$, then there exists an element $\phi \in H^0(\mathcal{O}_{\pi_k(C_n)}^{n-1}) \cong S^{n-2}V^{\vee} = T_{n-2}$ such that $\phi \in \ker(N_{n,k}^L) = \bigcap_i \ker(Cat_{f_i}(2, n-2))$, in fact we can take $\phi = \prod_{i=1}^{n-2} L_{q_i}^{\perp}$.*

We can compute the codimension of the variety of every $\mathbb{P}^{k-1}$ which belongs to some $(n-2)-$secant $\mathbb{P}^{n-3}$ constructing an incidence variety:

$$I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n), \pi \in S, L \subset S\},$$

where S is the set of all $(n-2)$-secant $\mathbb{P}^{n-3}$ to C_n. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. We will indicated with ϕ_1 and ϕ_2 the natural projections:

$$\begin{tikzcd} & I_S \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) & & S, \end{tikzcd}$$

so the codimension in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ of $\phi_1(I_S)$ is equal to $\dim Gr(\mathbb{P}^{k-1}, \mathbb{P}^n) - \dim S - \dim \phi_2^{-1}(S) = k(n+1-k) - n + 2 - k(n-2-k)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). We have that this variety has codimension $3k - n + 2$ which is the codimension expected as determinantal variety.

In general we can prove:

Lemma 3.4.6. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to some $(n-1-r)-$secant $\mathbb{P}^{n-r-2}$ to the rational normal curve C_n in $\mathbb{P}^n$, then we have $\operatorname{rank} N_{n,k}^L \leq n-r$ for $1 \leq r < n-k-1$.*

Proof. If $L \cong \mathbb{P}^{k-1}$ belongs to some $(n-r-1)-$secant $\mathbb{P}^{n-r-2}$, then there exist k points $p_1, ..., p_k \in L$ which generate L and the corresponding binary forms f_i are generating by two forms $Ann(f_i) = (\alpha, \beta_i)$ with $\deg(\alpha) = n - r - 1$, α has only simple roots and $\deg(\beta_i) = r + 3$ without common zeros with α. We have $\dim(\alpha)_{n-2} = r$ and $\dim \bigcap_i (\alpha, \beta_i)_{n-2} \geq r$, so $\operatorname{rank} N^L_{n,k} = n - 1 - \dim \bigcap_i (\alpha, \beta_i)_{n-2} \leq n - r - 1$. $\square$

Lemma 3.4.7. *The codimension in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$ of the variety of all $L \cong \mathbb{P}^{k-1}$ in $\mathbb{P}^n$ belonging to some $(n-r-1)$-secant $\mathbb{P}^{n-r-2}$ to the rational normal curve in $\mathbb{P}^n$ is $2k + kr - n + r + 1$.*

Proof. Infact we can consider the incidence variety $I_S = \{(L, \pi) : L \in Gr(\mathbb{P}^{k-1}, \mathbb{P}^n), \pi \in S, L \subset S\}$ where S is the set of all $(n-r-1)$-secant $\mathbb{P}^{n-r-2}$ to the rational normal curve in $\mathbb{P}^n$. In the usual way we can compute the codimension of the image of this incidence variety in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). $\square$

In particular for $r = n - k - 1$ we have:

Theorem 3.4.8. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to some $(k+2)-$secant $\mathbb{P}^{k+1}$ to the rational normal curve C_n in $\mathbb{P}^n$, then we have:*

$$N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-k-2} \oplus \mathcal{O}(n+1+2k).$$

We can prove a more strong result:

Lemma 3.4.9. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system Φ of affine dimension r of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, then* $\operatorname{rank} N^L_{n,k} = n - r - 1$ *for $1 \leq r < n - k$. The converse is generically true.*

Proof. $\Rightarrow$ Let L be a $\mathbb{P}^{k-1}$ belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-3} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{r-1}\pi_{r-1}, \quad \forall\lambda = [\lambda_0, ..., \lambda_{r-1}] \in \mathbb{P}^{r-1}\}$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, where $\pi_0, ..., \pi_{r-1}$ are $(n-2)-$secant $\mathbb{P}^{n-3}$. Let $q^i_1, ..., q^i_{n-1} \in C_n$ be the points which generate π_i. Then there exists k points $p_1, ..., p_k$ which generate L, such that each p_i belongs to π_λ for all $\lambda \in \mathbb{P}^{r-1}$ and $\pi_\lambda =< q_{1,\lambda}, ..., q_{n-2,\lambda} >$. By Observation 1.3.9 the binary forms f_i corresponding to p_i can be decomposed as:

$$f_i = c^i_{1,\lambda} L^n_{1,\lambda} + ... + c^i_{n-2,\lambda} L^n_{n-2,\lambda},$$

where $L_{j,\lambda}$ is the linear binary form corresponding in the usual way to $q_{j,\lambda}$ for $j = 1, ..., n-1$. This means that L belongs to a $\mathbb{P}^{n-3r}$. So by Lemma 1.3.11 for each $\lambda \in \mathbb{P}^{r-1}$ there exists a differential forms $\phi_\lambda \in T_{n-2}$ such that $\phi_\lambda \circ f_i = 0$. Moreover there exist r differential form $\phi_0, ..., \phi_{r-1} \in T_{n-2}$ which for each $\lambda \in \mathbb{P}^{r-1}$ we have $\phi_\lambda = \lambda_0\phi_0 + ... + \lambda_{r-1}\phi_{r-1}$, in particular $\phi_j \circ f_i = 0$ for all $j = 0, ..., r-1$ and $i = 1, ..., k$, so $\operatorname{rank} N^L_{n,k} = n - r - 1$.

$\Leftarrow$ If $\operatorname{rank} N^L_{n,k} = n - 1 - r$, then there exist r binary form $\phi_0, ..., \phi_{r-1} \in T_{n-2}$ such that however we consider the generating points $p_1, ..., p_k \in \mathbb{P}^n$ of L, it is $\phi_\alpha \circ f_i = (\alpha_0\phi_0 + ... + \alpha_{r-1}\phi_{r-1}) \circ f_i = 0$ for all $\alpha = [\alpha_0, ..., \alpha_{r-1}] \in \mathbb{P}^{r-1}$ and $i = 1, ..., k$, where $f_i \in S_n$ is the binary form corresponding to p_i. In particular $\phi_j \circ f_i = 0$, so if we consider the primary decomposition of $\phi_\alpha = \prod_{l=1}^{n-2} \phi^l_\alpha$ and we indicate with $(L_{l,\alpha})^\perp = \phi^l_\alpha$. Therefore $f_1, ..., f_k$ can be decomposed in ∞^{r-1} different simultaneously ways, i.e.:

$$f_i = c^i_{1,\alpha}L^n_{1,\alpha} + ... + c^i_{n-2,\alpha}L^n_{n-2,\alpha},$$

for all $\alpha = [\alpha_0, ..., \alpha_{r-1}] \in \mathbb{P}^{r-1}$ or in other words L belongs to a linear system $\Phi = \{\pi_\lambda \cong \mathbb{P}^{n-2} : \pi_\lambda = \lambda_0\pi_0 + ... + \lambda_{r-1}\pi_{r-1}, \quad \forall\lambda = [\lambda_0, ..., \lambda_{r-1}] \in \mathbb{P}^{r-1}\}$ of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$. So L belongs to a $\mathbb{P}^{n-3r}$, this is possible thanks to the condition $n \leq k + 3r - 1$. Clearly it can happen that the binary differential forms ϕ_i have any multiple roots, so we have all possible degenerations of the linear system Φ.

$\square$

Theorem 3.4.10. *If the centre of projection $L \cong \mathbb{P}^{k-1}$ belongs to a linear system Φ of affine dimension r of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$, then $N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^r \oplus \mathcal{F}$, with $\mathcal{F}$ a vector bundle of rank $\operatorname{rank}(N^L_{n,k}) - k$ on $\mathbb{P}^1$ and $\deg(\mathcal{F}^\vee(n+2)) = -2k$ such that $\mathcal{F} \cong \bigoplus_{i=0}^{\operatorname{rank}(N^L_{n,k})-k} \mathcal{O}(l_i)$ with $l_i \geq n+3$.*

Corollary 3.4.11. *Let $n \geq k + 3r - 1$.The variety of linear spaces $L \cong \mathbb{P}^{k-1}$ that belong to a linear system Φ of affine dimension r of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$ is an irreducible component of codimension $r(3k + r - n + 1)$ of the union of all subschemes $N^n_{n-k}((n+2)^r, spt(\mathcal{F}))$, with $\mathcal{F}$ a vector bundle of rank $\operatorname{rank}(N^L_{n,k}) - k$ on $\mathbb{P}^1$ and $\deg(\mathcal{F}^\vee(n+2)) = -2k$ such that $\mathcal{F} \cong \bigoplus_{i=0}^{\operatorname{rank}(N^L_{n,k})-k} \mathcal{O}(l_i)$ with $l_i \geq n+3$.*

Proof. We can observe that the linear system Φ of affine dimension r of $(n-2)-$secant $\mathbb{P}^{n-3}$ to the rational normal curve C_n in $\mathbb{P}^n$ corresponds to the linear system of dimension r of binary forms of degree $n-1$, therefore the set of these linear system corresponds to $Gr(\mathbb{P}^{r-1}, \mathbb{P}^{n-2})$ which is irreducible and $\dim Gr(\mathbb{P}^{r-1}, \mathbb{P}^{n-2}) = r(n-1-r)$. Each projection linear space L belongs to a $\mathbb{P}^{n-3r}$, so the dimension of the fibre is $k(n-3r+1-k)$, which is ≥ 0 with the condition $n \geq k+3r-1$. Therefore the variety of lines L that belong to a linear system Φ is an irreducible variety of codimension $r(3k+r-n+1)$ in $Gr(\mathbb{P}^{k-1}, \mathbb{P}^n)$. The above calculation is effective thanks to the result of Chiantini and Ciliberto on the non-defectivity of the Grassmannians of secant varieties of curves (see [Chiantini and Ciliberto, 2002]). $\square$

Example 3.4.12 (Rational Curve of degree 6 in $\mathbb{P}^4$). Let C_6 be a rational normal curve of degree 6 in $\mathbb{P}^6$ with Veronese embedding $\nu_6 : \mathbb{P}^1 \to \mathbb{P}^6$ and $\pi_2 : \mathbb{P}^6 \setminus < p_1, p_2 > \to \mathbb{P}^4$ the projection from the point $p_1 = (a_0^1, ..., a_6^1), p_2 = (a_0^2, ..., a_6^2) \in \mathbb{P}^6$. We have the following:

(3.34)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{C_6}^2 \arrow[r, "\cong"] \arrow[d] & \mathcal{O}_{C_6}^2 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-5)^2 \arrow[r, "J(\nu_6)"] & \mathcal{O}_{C_6}^7 \arrow[r, "Syz(J(\nu_6))"] \arrow[d] & \mathcal{O}_{C_6}(2)^5 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-5)^2 \arrow[r, "J(\pi_2 \circ \nu_6)"] & \mathcal{O}_{\pi_2(C_6)}^5 \arrow[r] \arrow[d] & N_{\pi_2(C_6);\mathbb{P}^4}(-6) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Since the map $\mathcal{O}_{C_n}(2)^5 \to N_{\pi_2(C_6);\mathbb{P}^4}(-6)$ is surjective, it follows that $n+2 \leq n_1 \leq n_2 \leq n_3$. Therefore it can happen one of these four cases:

i) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(8) \oplus \mathcal{O}(12)$;

ii) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(9) \oplus \mathcal{O}(11)$;

iii) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(10) \oplus \mathcal{O}(10)$;

vi) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(9) \oplus \mathcal{O}(9) \oplus \mathcal{O}(10)$.

Not any other case occurred if the projection is with only ordinary singularities.

By dualizing the last exact column of (3.34) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(2)$, we get:

$$0 \longrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8) \longrightarrow \mathcal{O}^5_{C_6} \xrightarrow{\mathcal{N}^L_{6,2}} \mathcal{O}^2_{C_6}(2) \longrightarrow 0,$$

and we have $deg(N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8)) = -4$. Now we have:

i) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(8) \oplus \mathcal{O}(12) \Longleftrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8) = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-4)$;

ii) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(9) \oplus \mathcal{O}(11) \Longleftrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8) = \mathcal{O}(-3) \oplus \mathcal{O}(-1) \oplus \mathcal{O}$;

iii) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(10) \oplus \mathcal{O}(10) \Longleftrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8) = \mathcal{O}(-2) \oplus \mathcal{O}(-2) \oplus \mathcal{O}$;

iv) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(9) \oplus \mathcal{O}(9) \oplus \mathcal{O}(10) \Longleftrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8) = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$.

We will denote by $N^L_{6,2}$ the following matrix:

$$N^L_{6,2} = \begin{pmatrix} a^1_0 & \dots & a^1_4 \\ -2a^1_1 & \dots & -2a^1_5 \\ a^1_2 & \dots & a^1_6 \\ a^2_0 & \dots & a^2_4 \\ -2a^2_1 & \dots & -2a^2_5 \\ a^2_2 & \dots & a^2_6 \end{pmatrix},$$

which is the map of the following exact sequence:

$$0 \longrightarrow H^0(N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8)) \longrightarrow H^0(\mathcal{O}^5_{C_6}) \xrightarrow{N^L_{6,2}} H^0(\mathcal{O}^2_{C_6}(2)) \longrightarrow H^1(N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8)) \longrightarrow 0,$$

so that $h^0(N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(8)) = 5 - \operatorname{rank} N^L_{6,2}$.

By dualizing the last exact column of (3.34) and tensorizing with $\mathcal{O}_{\mathbb{P}^1}(3)$, we get:

$$0 \longrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(9) \longrightarrow \mathcal{O}^5_{C_6}(1) \xrightarrow{\mathcal{N}^L_{6,2}} \mathcal{O}^2_{C_6}(3) \longrightarrow 0,$$

and we have $deg(N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(9)) = -1$. Now we have :

i) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(8) \oplus \mathcal{O}(14) \Longleftrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(9) = \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-3)$;

ii) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(9) \oplus \mathcal{O}(13) \Longleftrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(9) = \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1)$;

iii) $N_{\pi_2(C_6);\mathbb{P}^4} = \mathcal{O}(8) \oplus \mathcal{O}(10) \oplus \mathcal{O}(10) \Longleftrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(9) = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1)$;

iv) $N_{\pi_2(C_6);\mathbb{P}^4}(-6) = \mathcal{O}(9) \oplus \mathcal{O}(9) \oplus \mathcal{O}(10) \Longleftrightarrow N^{\vee}_{\pi_2(C_6);\mathbb{P}^4}(9) = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}$.

Again we will denote by $N^L_{6,2}(1)$ the following matrix:

$$N^L_{6,2}(1) := \begin{pmatrix} a^1_0 & a^1_1 & a^1_2 & a^1_3 & a^1_4 & 0 & 0 & 0 & 0 & 0 \\ -2a^1_1 & -2a^1_2 & -2a^1_3 & -2a^1_4 & -2a^1_5 & a^1_0 & a^1_1 & a^1_2 & a^1_3 & a^1_4 \\ a^1_2 & a^1_3 & a^1_4 & a^1_5 & a^1_6 & -2a^1_1 & -2a^1_2 & -2a^1_3 & -2a^1_4 & -2a^1_5 \\ 0 & 0 & 0 & 0 & 0 & a^1_2 & a^1_3 & a^1_4 & a^1_5 & a^1_6 \\ a^2_0 & a^2_1 & a^2_2 & a^2_3 & a^2_4 & 0 & 0 & 0 & 0 & 0 \\ -2a^2_1 & -2a^2_2 & -2a^2_3 & -2a^2_4 & -2a^2_5 & a^2_0 & a^2_1 & a^2_2 & a^2_3 & a^2_4 \\ a^2_2 & a^2_3 & a^2_4 & a^2_5 & a^2_6 & -2a^2_1 & -2a^2_2 & -2a^2_3 & -2a^2_4 & -2a^2_5 \\ 0 & 0 & 0 & 0 & 0 & a^2_2 & a^2_3 & a^2_4 & a^2_5 & a^2_6 \end{pmatrix},$$

which is the map of the following exact sequence:

$$0 \longrightarrow H^0(N^\vee_{\pi_2(C_6);\mathbb{P}^4}(9)) \longrightarrow H^0(\mathcal{O}^5_{C_6}(1)) \xrightarrow{N^L_{6,2}(1)} H^0(\mathcal{O}^2_{C_6}(3)) \longrightarrow H^1(N^\vee_{\pi_2(C_6);\mathbb{P}^4}(9)) \longrightarrow 0,$$

so that $h^0(N^\vee_{\pi_2(C_6);\mathbb{P}^4}(9)) = 10 - \operatorname{rank} N^L_{6,2}(1)$.

Proposition 3.4.13. *The splitting type of the normal bundle $N_{\pi_2(C_6),\mathbb{P}^4}$ is:*

i) $(9, 9, 10)$ *if and only if* $\operatorname{rank}(N^L_{6,2}) = 5$*;*

ii) $(8, 10, 10)$ *if and only if* $\operatorname{rank}(N^L_{6,2}) = 4$ *and* $\operatorname{rank}(N^L_{6,2}(1)) = 8$*;*

iii) $(8, 9, 11)$ *if and only if* $\operatorname{rank}(N^L_{6,2}) = 4$ *and* $\operatorname{rank}(N^L_{6,2}(1)) = 7$*;*

iv) $(8, 8, 12)$ *if and only if* $\operatorname{rank}(N^L_{6,2}) = 3$.

Theorem 3.4.14. *If the centre of projection $L \cong \mathbb{P}^1$ belongs to a pencil of $4-$secant $\mathbb{P}^3$ to the rational normal curve C_6 in $\mathbb{P}^6$, then $N_{\pi_2(C_6);\mathbb{P}^4} \cong \mathcal{O}(9)^2 \oplus \mathcal{O}(10)$.*

We will indicate with ExpCodim the codimension of the general determinantal variety of the same type and with ConjCodim the following one:

$$\sum_{i,j} \max\{n_i - n_j - 1, 0\},$$

where (n_1, n_2, n_3) is the splitting type of $N_{\pi_2(C_6),\mathbb{P}^3}$. In the fifth column of the following matrix we have indicated the codimension calculated by Macaulay2 symbolic calculation software.

Splitting of $N_{\pi_2(C_6),\mathbb{P}^3}$	Rank of $N^L_{6,2}$	Rank of $N^L_{6,2}(1)$	Codim with M2	ExpCodim	ConjCodim	Degree
$(9,9,10)$	5					
$(8,10,10)$	4	8	2	2	2	$\binom{6}{4}$
$(8,9,11)$	4	7	3	3	3	$\binom{6}{4}\cdot\binom{10}{7}$
$(8,8,12)$	3		6	6	6	1

Main Theorem for Normal Bundle

Proposition 3.4.15. *The generic splitting type of a vector bundle $\mathcal{F}$ of rank* $\mathrm{rank}(N^L_{n,k})-k$ *on* $\mathbb{P}^1$ *and* $\deg(\mathcal{F}^\vee(n+2)) = -2k$ *such that* $\mathcal{F} \cong \bigoplus_{i=0}^{\mathrm{rank}(N^L_{n,k})-k} \mathcal{O}(l_i)$ *with* $l_i \geq n+3$ *is the following:*

$$((n+2+B)^{A-2k+B\cdot A}, (n+3+B)^{2k-B\cdot A}).$$

where we have indicated $A = \mathrm{rank}(N^L_{n,k}) - k$ *and* $B = \lfloor \frac{2k}{\mathrm{rank}(N^L_{n,k})-k} \rfloor$.

By Theorem 3.3.9 and Theorem 3.4.10 we have covered all the possible cases and we can state the main result of this chapter, which is the following by the above proposition:

Theorem 3.4.16 (Main Theorem for Normal Bundle)**.** *The following conditions are equivalent:*

i) the centre of projection $L \cong \mathbb{P}^{k-1}$ *is general in the (irreducible) variety of those* $\mathbb{P}^{k-1}$ *which belongs to a linear system* Φ *of affine dimension* $n-1-\mathrm{rank}(N^L_{n,k})$ *of* $(n-2)-$*secant* $\mathbb{P}^{n-3}$ *to the rational normal curve* C_n *in* $\mathbb{P}^n$*;*

ii) the curve of degree n *projected from* $L = \mathbb{P}^{k-1}$ *has* $N_{\pi_k(C_n);\mathbb{P}^{n-k}} \cong \mathcal{O}(n+2)^{n-1-\mathrm{rank}(N^L_{n,k})} \oplus \mathcal{O}(n+2+B)^{A-2k+B\cdot A} \oplus \mathcal{O}(n+3+B)^{2k-B\cdot A}$*, where we have indicated* $A = \mathrm{rank}(N^L_{n,k}) - k$ *and* $B = \lfloor \frac{2k}{\mathrm{rank}(N^L_{n,k})-k} \rfloor$.

Corollary 3.4.17. *Let* $n \geq k + 3(n-1-\mathrm{rank}(N^L_{n,k})) - 1$.*The variety of linear spaces* $L \cong \mathbb{P}^{k-1}$ *that belong to a linear system* Φ *of affine dimension* $n-1-\mathrm{rank}(N^L_{n,k})$ *of* $(n-2)-$*secant* $\mathbb{P}^{n-3}$ *to the rational normal curve* C_n *in* $\mathbb{P}^n$ *is an irreducible component of codimension:*

$$(n-1-\mathrm{rank}\, N^L_{n,k})(3k - \mathrm{rank}\, N^L_{n,k}),$$

of $N^n_{n-k}((n+2)^{n-1-\mathrm{rank}(N^L_{n,k})}, (n+2+B)^{A-2k+B\cdot A}, (n+3+B)^{2k-B\cdot A})$*, where we have indicated* $A = \mathrm{rank}(N^L_{n,k}) - k$ *and* $B = \lfloor \frac{2k}{\mathrm{rank}(N^L_{n,k})-k} \rfloor$.

Theorem 3.4.18. *The varieties of rational curves* C *of degree* n *in* $\mathbb{P}^{n-k}$ *whose normal bundle* $N_{C,\mathbb{P}^{n-k}}$ *has the summand* $\mathcal{O}_C(n+2)^\alpha$ *for* $0 \leq \alpha \leq n-k-2$ *are irreducible.*

3.5 Further Questions

At the end of this thesis we want to introduce some interesting open problems:

i) What happens geometrically in the unknown cases?

ii) What is the geometric meaning of $N^L_{n,k}(d)$ and $T^L_{n,k}(d)$?

iii) Is it true that $\operatorname{codim} N^n_{n-k}(n_1, ..., n_{n-k-1}) = \delta(n_1, ..., n_{n-k-1})$ in all cases?

iv) What happens if both splittings are fixed?

v) What happens in the Elliptic Case?

The first two questions are clearly related and they are very important to complete the classification of the subschemes of Hilbert Scheme of rational curves with fixed degree and splitting type of normal bundle, unfortunately we have only few clue obtained with software Macaulay2.

We stress that in our thesis we have found only cases for which the codimension of variety $N^n_{n-k}(n_1, ..., n_{n-k-1})$ is the same of the conjectured codimension on the third question. We don't have any counterexample yet, so it seems very hopeful.

The fourth question was investigated by Ramella (see [Ramella, 1993]), but only in the case of rational space curves, no much it is known for general case.

For the last we note that the situation of the splitting of vector bundle on elliptic curves is more complicated, an analogous result as Grothendieck-Segre theorem is the Atiyah teorem (see [Atiyah, 1957]). The splitting type of normal bundle to elliptic curve was investigated by several mathematicians (see [Ellingsrud and Laksov, 1981], [Hulek, 1983],[Hulek and Sacchiero, 1983]), but mostly for space elliptic curve. Moreover it is not known if the varieties where the splitting of the normal bundle is fixed are irreducible.

We point out that the approach in the work of Ellingsrud and Laksov and in the work of Hulek is to project the elliptic normal curve in some subspace, so it seems very hopeful to look at the problem in similar way as made for rational curves.

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