SO-invariant bilinear forms on tensor spaces.

Giorgio Ottaviani

February 9, 2023

Abstract

We classify SO-invariant symmetric bilinear forms on tensor spaces, focusing on special properties of Bombieri-Weyl form.

1 Scalar products on polynomials which are SO-invariant

Let V be a $(n+1)$-dimensional real vector space on $\mathbb{K}$ (as usual $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Fix a nondegenerate $q \in \mathrm{Sym}^2 V$, we have a corresponding action of $SO(V, q) = SO(V)$ on V. It is convenient to choose from the beginning a coordinate system such that q has the standard Euclidean expression

$$q = \sum_{i=0}^{n} x_i^2.$$

In section on harmonic decomposition we saw how to extend the SO-action from V to $\mathrm{Sym}^d V$ by the rule $(gf)(x) = f(g^{-1}x)$ for $f \in \mathrm{Sym}^d V$, $x \in V^\vee$, $g \in SO(V)$. Equivalently , the action on powers is $g \cdot l^d = (g \cdot l)^d$, $\forall l \in V$.

It is important to recall that $Sym^d V$ is a irreducible $SL(V)$-module but a reducible $SO(V)$-module, according to the harmonic decomposition $\mathrm{Sym}^d V = \oplus_{i \geq 0} H_{d-2i}$ (*** quote precedent section). This feature allows to construct several different SO-invariant scalar products on $\mathrm{Sym}^d V$

The quadratic form q gives a symmetric bilinear nondegenerate form $(v \cdot w)$, which in the above coordinate system corresponds to $(v \cdot w) = v^t w$. We denote $|v|^2 = q(v, v) = v^t v$.

We denote $(\alpha_0, \ldots, \alpha_n)! = \prod(\alpha_i!)$ and for any partition α of d we denote the multinomial coefficient $\binom{d}{\alpha} = \frac{d!}{\alpha!}$ so that we have the expansion

$$(\sum_{i=0}^{n} l_i x_i)^d = \sum_{\alpha} \binom{d}{\alpha} l^\alpha x^\alpha \tag{1.1}$$

Theorem 1.1. *1. For any $i = 0, \ldots, \lfloor d/2 \rfloor$ there is a unique SO-invariant bilinear form on $\mathrm{Sym}^d V$ such that on decomposable elements*

$$\langle v^d, w^d \rangle_i = (v \cdot w)^{d-2i} |v|^{2i} |w|^{2i}$$

2. *The expression of the bilinear form in 1. is the following. To any polynomial p we associate the differential operator $\widetilde{p} = p(\partial_0 \dots \partial_n)$ and $\langle p, f\rangle_0 = \frac{1}{d!}\widetilde{p}(f)$, $\langle p, f\rangle_i = \frac{(d-2i)!}{d!^2}\widetilde{(\Delta^i p)}(\Delta^i f)$. In particular* $\mathrm{diff}(x^\alpha, x^\beta) = \frac{\partial x^\beta}{\partial x^\alpha} = \begin{cases} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$

3. *Every SO-invariant bilinear form is a linear combination of the $\langle , \rangle_i$ in 1..*

Remark 1.2. *Note the products in 1. are not distinguished by the norm on decomposable tensors, indeed when $v = w$ we get that $\langle v^d, v^d\rangle_i = (v \cdot v)^d$ does not depend on i. Of course the norms on the whole space depend on i.*

Proof. Define for $i = 0$ the product on the monomial basis by the formula in 2. Then note that the scalar product $\langle\binom{d}{\alpha}x^\alpha, \binom{d}{\alpha}x^\alpha\rangle_0 = \binom{d}{\alpha}$ so that by linearity

$$\langle\sum_\alpha \binom{d}{\alpha} f_\alpha x^\alpha, \sum_\alpha \binom{d}{\beta} g_\beta x^\beta\rangle_0 = \sum_\alpha \binom{d}{\alpha} f_\alpha g_\alpha$$

and in the case $f_\alpha = v^\alpha$, $g_\alpha = w^\alpha$, corresponding to Newton expansion (1.1), the result coincides with $(v \cdot w)^d$.

Now note that

$$\Delta^i(v^d) = \frac{d!}{(d-2i)!} v^{d-2i}|v|^{2i}.$$

Then

$$(\widehat{\Delta^i(v^d)})\Delta^i(w^d)) = \frac{d!^2}{((d-2i)!)^2}|v|^{2i}|w|^{2i}\widehat{v^{d-2i}}(w^{d-2i}) = \frac{d!^2}{(d-2i)!}|v|^{2i}|w|^{2i}(v \cdot w)^{d-2i}$$

So the formula in 2. satisfies 1. also for $i > 0$.

The SO-invariance is immediate from 1. Indeed if $g \in SO(V)$ then

$$\langle g(v^d), g(w^d)\rangle_i = (gv \cdot gw)^{d-2i}|gv|^{2i}|gw|^{2i} = (v \cdot w)^{d-2i}|v|^{2i}|w|^{2i} = \langle v^d, w^d\rangle_i$$

3. For every irreducible SO-module H, the SO-representation $\mathrm{Sym}^2 H$ contains a unique trivial summand, hence $\mathrm{Sym}^2(\mathrm{Sym}^d V)$, which can be seen as the space of symmetric invariant maps $H_d \oplus H_{d-2} \oplus \dots \to H_d \oplus H_{d-2} \oplus \dots$ which contains at most $\lfloor (d+2)/2 \rfloor$ trivial summands by Schur Lemma. Hence the linear combinations of $\langle , \rangle_i$ fill all the SO-invariant bilinear symmetric forms. $\square$

The case $i = 0$ is called the Bombieri-Weyl product and it is defined on decomposable elements by

$$\langle v^d, w^d\rangle = (v \cdot w)^d$$

Its expression is $\frac{1}{d!}\mathrm{diff}(f, g)$.

The product $\langle , \rangle_i$ is nondegenerate on the summand $\oplus_{j \geq i} H_{d-2j}$. The general linear combination of $\langle , \rangle_i$ is nondegenerate on the whole $\mathrm{Sym}^d V$. Note any linear combination of $\langle , \rangle_i$ coincides with Bombieri-Weyl product $\langle , \rangle_0$ on the harmonic part H^d.

Remark 1.3. *Theorem 1.1 gives a proof of the First Fundamental Theorem of Invariant theory for this special case. Namely, if q is a $SO(V,q)$-invariant bilinear symmetric form*

$$q\colon V \times V \to \mathbb{K},$$

then $q(v^d, w^d)$ is a polynomial in $q(v,w), q(v,v), q(w,w)$.

Remark 1.4. *The M2 command for Bombieri-Weyl product is*

```
diff(f,g)
```

The M2 commands for the i-th product of Theorem 1.1 are

```
q=sum_{i=0}^n x_i
diff(diff(q^i,f),diff(q^i,g))
```

A orthogonal basis for Bombieri-Weyl product is given by scaled monomials, namely $\frac{1}{\sqrt{\alpha!}}x^\alpha$.

Theorem 1.5. *Different summands of the harmonic decomposition are orthogonal for any SO-invariant scalar product.*

Proof. We prove the claim first for Bombieri-Weyl product. We may pick $q^a v^{d-2a}$ and $q^b w^{d-2b}$ with v, w isotropic, and we may assume by symmetry $a > b$. Then

$$\langle q^a v^{d-2a}, q^b w^{d-2b}\rangle = \partial_v^{d-2a}\Delta^a(q^b w^{d-2b}) = \partial_v^{d-2a}\Delta^{a-1}(q^{b-1}w^{d-2b}) = \dots$$

$$\dots = \partial_v^{d-2a}\Delta^{a-b}w^{d-2b} = 0.$$

For the scalar product $\langle,\rangle_i$ we have $\Delta^i q^a v^{d-2a} = q^{a-i}v^{d-2a}$, $\Delta^i q^b w^{d-2b} = q^{b-i}w^{d-2b}$ and we reduce to previous case. □

Remark 1.6. *A consequence of Theorem 1.5 is that, given the harmonic decomposition*

$$f = \sum_{i=0} \lfloor d/2 \rfloor q^i f_i$$

*of **, for any SO-invariant norm it follows from Pythagorean Theorem*

$$|f|^2 = \sum_{i=0} \lfloor d/2 \rfloor |q^i f_i|^2$$

1.1 The polyhedral structure in the real case

The only invariant for a bilinear complex form is its rank. Over reals we have a finer invariant which is the signature. We are interested especially in positive definite bilinear forms. Remind that semipositive bilinear form make a closed convex cone, with interior part given by positive definite ones. The cone obtained cutting the semipositive cone with a linear affine subspace is called a *spectrahedron*.

Theorem 1.7. *The spectrahedron of semipositive $SO(V)$-invariant bilinear forms on* $\mathrm{Sym}^d V$ *is a polyhedral cone having $\lfloor \frac{d+1}{2} \rfloor$ vertices, corresponding to the Bombieri-Weyl forms on each summand of harmonic decomposition, extended to zero.*

Proof. Let $\mathrm{Sym}^d V = \oplus_{i=0}^{\lfloor d/2 \rfloor} q^i H_{d-2i}$ be the harmonic decomposition and for every $f = \sum_{i=0}^{\lfloor d/2 \rfloor} q^i f_i$ and $i = 0, \dots \lfloor d/2 \rfloor$ we may define $\langle f, g \rangle_{BW,i} = \langle f_i, g_i \rangle_{BW}$, which is a SO-invariant bilinear form of rank =dim H_{d-2i}. The convex envelope of these forms is contained in the spectrahedron of semipositive $SO(V)$-invariant bilinear forms. Now for any SO-invariant bilinear form we can consider its restriction to the orthogonal (by Theorem 1.5) summands of orthogonal decomposition. Since these restrictions are nonnegative scalar multiples of Bombieri-Weyl scalar product on each summand (by Schur Lemma on the complexification, hence also over $\mathbb{R}$), the result follows. $\square$

1.2 The complex Hermitian case

Lat V be a complex vector space equipped with a Hermitian scalar product q, which is sesquilinear, Hermitian and positive definite. The corresponding unitary group $U(V)$ acts on $\mathrm{Sym}^d V$

Proposition 1.8. *The $U(V)$-module* $\mathrm{Sym}^d V$ *is irreducible. It is irreducible also as a $SU(V)$-module.*

Proof. Both the statements follow by the unitary trick and the analogous results for $GL(V)$ and, respectively, $SL(V)$. In alternative, we show a geometric argument. It is enough to prove the second statement. Assume $dimV = n+1$, then $SU(V)$ acts transitively on the sphere $S^{2n+1} \subset V$. We claim that the d-Veronese span $v_d(S^{2n-1})$ is nondegenerate. Assume there is a real hyperplane $H \subset V \otimes \mathbb{R}$ containing $v_d(S^{2n-1})$. Then, posing $z_j = x_j + iy_j$ we have a homogeneous polynomial $f(x_0, \dots, x_n, y_0, \dots, y_n)$ which vanishes for any $(x_0, \dots, x_n, y_0, \dots, y_n)$ such that $\sum_{i=0}^n (x_i^2 + y_i^2) = 1$ and by the homogeneity it vanishes for any $(x_0, \dots, x_n, y_0, \dots, y_n) \neq 0$, which implies $f = 0$. $\square$

Theorem 1.9.

1. *There is a unique $SU(V)$-invariant Hermitian product on* $\mathrm{Sym}^d V$*, up to positive scalar multiples. It is also $U(V)$-invariant and on decomposable elements*
$$\langle v^d, w^d \rangle = (v \cdot \overline{w})^d$$
2. *The expression of the scalar product is $\frac{1}{d!}\mathrm{diff}(f, \overline{g})$ where $\overline{\sum_\alpha g_\alpha x^\alpha} := \sum_\alpha \overline{g_\alpha} x^\alpha$,* $\mathrm{diff}(x^\alpha, x^\beta) = \frac{\partial x^\beta}{\partial x^\alpha} = \begin{cases} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$

Proof. There is a technical issue here. The form is sesquilinear and it is an element of $\mathrm{Sym}^d V \otimes \mathrm{Sym}^d \overline{V} = \mathrm{Sym}^d V \otimes \mathrm{Sym}^d V^\vee$. So, even id $\mathrm{Sym}^d V$ is not self-duel for $n \geq 2$, we get a map $\mathrm{Sym}^d V \to \mathrm{Sym}^d V$ which by Schur Lemma is a multiple of the identity (the representation $\mathrm{Sym}^d V$ has complex type). A $SU(V)$-invariant Hermitian product gives a $SU(V)$-equivariant map $A\colon \mathrm{Sym}^d V \to \mathrm{Sym}^d V$. By Proposition

$\square$

Remark 1.10. *If f g are harmonic of degree $d-2k$, then $\langle q^k f, q^k g\rangle$ is proportional to $\langle f, g\rangle$ for Bombieri-Weyl product. This follows since on harmonic summands there is only one SO-invariant bilinear form, up to scalar multiples. Proportionality may fail for not harmonic polynomials.*

Note also, if v, w have norm 1, we have $|v-w|^2 = 2 - q(v,w) - q(w,v) = 2 - \mathrm{R}eq(v,w)$ so that the "round distance" defined as angle can be computed by $\mathrm{R}eq(v,w)$.

For $k=2$ it is $q(A,B) = tr(AB^H)$ and $|A|^2 = tr(AA^H)$, note that AA^H is Hermitian matrix. But the distance is computed through $\mathrm{R}e\ tr\ AB^H$.

1.3 Exercises

1. The modules $\wedge^2 V$ and $\mathrm{Sym}^2 V$ are orthogonal for the scalar product $\mathrm{t}r(AB^t)$ on $V \otimes V$.

2. The operator $L = q\Delta\colon \mathrm{Sym}^d V \to \mathrm{Sym}^d V$ of exercise ** regarding harmonic decomposition is self-adjoint with respect to Bombieri-Weyl. This implies immediately the orthogonality of the summands. Is L self adjoint with respect to any orthogonally invariant scalar product ?

1.4 Complements for Bombieri-Weyl product on symmetric tensors

The following is one of the most useful properties of Bombieri-Weyl product, which characterizes it

Proposition 1.11. *(Bombieri-Weyl computes evaluation) For Bombieri-Weyl product*

$$\langle l^d, f\rangle = cf(l) \text{ for a nonzero scalar } c$$

Proof. The property is true if $f = m^d$, so the result follows by expanding f as sum of powers (Waring expansion). $\square$

Remark 1.12. *In the same way, the product $\langle \Delta^i l^d, \Delta^i f$ is equal to $c(\Delta^i f)(l)$ for a nonzero scalar c.*

Proposition 1.13. *Let $\langle , \rangle$ be the Bombieri-Weyl product.*

1. *Let l, m_i be linear forms. Then $\langle l^d, \prod m_i\rangle = \prod \langle l, m_i\rangle$*

2. *Let l_i, m_j be linear forms. Then$\langle \prod l_i, \prod m_j\rangle = \frac{1}{d!}\sum_\sigma \prod \langle i, m_\sigma(j)\rangle$, i.e., up to scalar multiples $\langle \prod l_i, \prod m_j\rangle$ is the permanent of the Gram matrix $\langle l_i, m_j\rangle$*

Note indeed the permanent is invariant with respect to permutation of rows and/or columns.

Proof. The proof of first item follows from the interpretation of Bombieri-Weyl product as differentiation, indeed by $SL(2)$-action we may assume $l = x_0$ and it is obvious that $\partial_0^d(\prod_{i=1}^d m_i) = d! \prod_{i=1}^d (\partial_0 m_i)$. The second item follows the same pattern by iteration of Leibniz rule. Another interpretation of the first item is with evaluation of second factor at l. $\square$

Remark 1.14. *Let $P \in \mathrm{Sym}^d V, Q \in \mathrm{Sym}^e V$, then, according to the Bombieri-Weyl norm, it holds the Bombieri inequality[Ze]*

$$\sqrt{\frac{d!e!}{(d+e)!}}|P||Q| \leq |P \cdot Q| \leq |P||Q|$$

In the reference [Ze] there is a short proof, in the non homogeneous setting, and the norm used is equivalent to our Bombieri-Weyl polynomials.

1.5 The binary case

Among all scalar products $SO(2)$-invariant there is one which is $SL(2)$-invariant when d is even.

Proposition 1.15. *On $\mathrm{Sym}^d\mathbb{C}^2$ for d even the $SL(2)$-quadratical form is*

$$\langle f, g\rangle_{SL(2)} = \sum_{i=0}^{d/2}(-1)^i\binom{d/2}{i}\langle f, g\rangle_i$$

This coincides with the $SL(2)$-invariant quadric for the binary form of even degree d $\sum_{i=0}^{d} a_i\binom{d}{i}x^iy^{d-i}$ which is

$$\sum_{i=0}^{d/2-1}(-1)^i a_i a_{d-i}\binom{d}{i} + \frac{1}{2}a_{d/2}^2\binom{d}{d/2} = \frac{1}{2}\sum_{i=0}^{d}(-1)^i a_i a_{d-i}\binom{d}{i} \tag{1.2}$$

(note that the analogous expression in odd degree vanishes). Note that over the real this quadratic form has signature $(d/2, d/2+1)$, far from being positive (or negative) definite. Indeed the equation (1.2) polarizes to

$$\langle\sum_{i=0}^{d} a_i\binom{d}{i}x^iy^{d-i}, \sum_{i=0}^{d} b_i\binom{d}{i}x^iy^{d-i}\rangle = \frac{1}{2}\sum_{i=0}^{d}(-1)^i a_i b_{d-i}\binom{d}{i},$$

which specializes to

$$\langle(v_0x + v_1y)^d, (w_0x + w_1y)^d\rangle = \sum_{i=0}^{d}(v_0w_1)^i(v_1w_0)^{(d-i)}\binom{d}{i}(-1)^i = (v_0w_1 - v_1w_0)^d$$

which is called a transvectant ([Ot13, §4.3]).

This last expression is equal to

$$\left(|v|^2|w|^2 - (v\cdot w)^2\right)^{d/2} = \sum_{i=0}^{d/2}|v|^{2i}|w|^{2i}(v\cdot w)^{d-2i}\binom{d/2}{i}(-1)^i$$

1.6 Scalar products as integrals

In complex case we have, denoting by $|\ |$ a Hermitian form on $\mathbb{C}^{n+1}$ and $f, g \in Sym^d\mathbb{C}^{n+1}$

$$\langle f, g\rangle_{\mathbb{C}} := (\frac{i}{2\pi})^{n+1} \int_{\mathbb{C}^{n+1}} f(z)\overline{g(z)}e^{-|z|^2} dz_0\wedge d\overline{z_0}\wedge\ldots\wedge dz_n\wedge d\overline{z_n} = C_n \int_{\mathbb{P}^n(\mathbb{C})} f\overline{g}d\sigma \quad (1.3)$$

where $d\sigma$ is the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$ induced by the Hermitian form and C_n is a nonzero scalar. The scalar $(\frac{i}{2\pi})^{n+1}$ is chosen in order to have the normalization $\langle 1, 1\rangle_{\mathbb{C}} = 1$. According to this product, monomials are orthogonal but not orthonormal.

This product is different from Bombieri-Weyl in the sense that the result is always real, but for real polynomials it has the same value !

The path to prove the integral expression is the following.

Lemma 1.16. *According to (1.3) we have*

$$\langle z^\alpha, z^\beta\rangle = \begin{cases} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

Hence on the monomial basis the result is proportional to Bombieri-Weyl and it follows that Bombieri-Weyl scalar product on real polynomials has the integral representation (1.3)

Proof. Replace $z_j = x_j + iy_j$ and $\frac{i}{2}dz_j \wedge d\overline{z_j}$ get

$$\langle z^\alpha, z^\beta\rangle = \prod_{j=0}^{n} \frac{1}{\pi} \int_{\mathbb{C}} (x_j + iy_j)^{\alpha_j}(x_j - iy_j)^{\beta_j} e^{-xJ^2-y_j^2} dx_j dy_j =$$

$$\prod_{j=0}^{n} \frac{1}{\pi} \int_0^{+\infty} \int_0^{2\pi} \rho^{\alpha_j+\beta_j+1}e^{-\rho^2} \left(\cos((\alpha_j - \beta_j)\theta) + i\sin((\alpha_j - \beta_j)\theta)\right) d\theta d\rho.$$

If $\alpha \neq \beta$ there exists $\alpha_j \neq \beta_j$ so that

$$\int_0^{2\pi} \cos((\alpha_j - \beta_j)\theta)d\theta = \int_0^{2\pi} \sin((\alpha_j - \beta_j)\theta)d\theta = 0$$

and we get $\langle x^\alpha, x^\beta\rangle = 0$.

If $\alpha = \beta$ we get

$$\langle z^\alpha, z^\alpha\rangle = \prod_{j=0}^{n} \frac{1}{\pi} \int_0^{+\infty} \int_0^{2\pi} \rho^{2\alpha_j+1}e^{-\rho^2} d\theta d\rho = \prod_{j=0}^{n} 2\int_0^{+\infty} \rho^{2\alpha_j+1}e^{-\rho^2} d\rho =$$

$$\prod_{j=0}^{n} \int_0^{+\infty} t^{\alpha_j}e^{-t}dt = \prod_{j=0}^{n} \Gamma(\alpha_j + 1) = \alpha!$$

□

When restricted to real polynomials, the scalar product in (1.3) lies in the interior of the polyhedral cone defined in Theorem 1.7.

This result makes natural to consider another SO-invariant scalar product for real polynomials, it is

$$\langle f, g\rangle_{FSR} := \int_{\mathbb{R}^{n+1}} f(x)g(x)e^{-|x|^2}dx \tag{1.4}$$

According to this product, monomials are no more orthogonal. For example , if x^α and x^β have both all exponents being even, then from (1.4 it follows

$$\langle x^\alpha, x^\beta\rangle_{FSR} > 0$$

Anyway it follows from Theorem 1.5 that the summands of harmonic decomposition are still orthogonal.

Due to homogeneity, the integral in (1.4) coincides, up to a scalar multiple, to the same integral restricted to the sphere. This is meaningful in the case $n = 1$, where we get the restriction to the unit circle.

An interpretation of this integral is that it gives a probability distribution on the space of real polynomials. Indeed we can measure volumes of any measurable subset of $\mathrm{Sym}^d\mathbb{R}^{n+1}$, and in particular of cones. On a real or complex Hilbert space W it is defined the volume form

$$(2\pi)^{-\frac{\dim_{\mathbb{R}} W}{2}} e^{-\langle x,x\rangle} d\mu$$

Changing the scalar product amounts to fix that the probability distribution of a polynomial $f = \sum f_\alpha x^\alpha$ is prescribed by a Gaussian normal distribution, for each coefficient a_α, with prescribed variance. Different scalar product amount simply to different variances. Bombieri-Weyl product amounts to the variance $\sqrt{\binom{d}{\alpha}}$. When $W = \mathrm{Sym}^d V$ any orthogonally invariant scalar product gives an interesting volume form, accordingly we can measure volumes of subsets of $\mathrm{Sym}^d V$. One interesting application is to compute the expected value of real zeroes of a homogeneous polynomial of degree d in two variables.

This changes accordingly to the scalar product chosen in the polyhedral cone of Theorem 1.7.

The degenerate cases on the vertices give a definite answer, since in two variables polynomials chosen from H_d have always d real zeroes (see remark ** in Harmonic Decomposition section).

The answer can be obtained by integral geometry, as computed by Edelman and Kostlan[EK95], putting in a general framework a result first obtained by Shub and Smale. Their elegant proof shows that

Theorem 1.17. *The expected value for number of real zeroes according to Bombieri-Weyl distribution is* $\sqrt{d}$.

In [LL] there is the analogous (and easier) computation for real Fubini-Study distribution coming from (1.4).

Theorem 1.18. *The expected value for number of real zeroes according to real Fubini-Study distribution is* $\sqrt{\frac{d(d+2)}{3}}$.

Exercise Prove that (according to Hilbert, see [Rez, prop. 8.12]), there is a scalar $C_{n,d}$ such that, for any real form m,

$$\int_{S^n} m^d = \begin{cases} C_{n,d}|m|^d & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd} \end{cases}$$

1.7 Exercises on orthogonally invariant bilinear forms

Under the action of $SO(n+1)$, symmetric matrices split as

$$\langle I_{n+1}\rangle \oplus \langle \text{traceless}\rangle$$

We have a unique decomposition

$$A = \left(\frac{\operatorname{tr}(A)}{n+1} I_{n+1}\right) + \left(A - \frac{\operatorname{tr}(A)}{n+1} I_{n+1}\right)$$

The Bombieri-Weyl metric is $\langle A, A\rangle = \operatorname{tr}(A^2)$ and satisfies

$$|A|^2 = \left|\frac{\operatorname{tr}(A)}{n+1} I_{n+1}\right|^2 + \left|A - \frac{\operatorname{tr}(A)}{n+1} I_{n+1}\right|^2$$

It is convenient to write as scalar product

$$\langle A, B\rangle_\alpha = (2-\alpha)\operatorname{tr}(AB) + (\alpha - 1)\frac{2}{n+1}\operatorname{tr}(A)\operatorname{tr}(B) \tag{1.5}$$

for $0 \le \alpha \le 2$, the Bombieri-Weyl case corresponds to $\alpha = 1$.

Exercise** For which α we get real Fubini-Study product as in (1.5) ? *Answer: for $n = 1$ we have $\alpha = \frac{4}{3}$. Note that by multilinearity real Fubini-Study product for $n = 2$ is*

$$\langle A, B\rangle_{RFS,2} = \frac{1}{8}\operatorname{tr}(A)\operatorname{tr}(B) + \frac{1}{4}\operatorname{tr}(AB)$$

Exercise Prove that, for binary cubics, the real Fubini-Study product gives a scalar multiple of

$$(v \cdot w)\left(3|v|^2|w|^2 + 2(v\cdot w)^2\right)$$

The expression corresponding for binary cubics to $(v \cdot w)|v|^2|w|^2$ is

$$\langle \sum_{i=0}^{3} a_i \binom{3}{i} x^i y^{3-i}, \sum_{j=0}^{3} b_j \binom{3}{i} x^j y^{3-j}\rangle = (a_0 + a_2)(b_0 + b_2) + (a_1 + a_3)(b_1 + b_3)$$

which reads as

$$\langle f, g\rangle_{chess} = \frac{1}{6^2}\langle \Delta f, \Delta g\rangle$$

and correspond to the chessboard matrix (degenerate of rank 2)

$$\begin{pmatrix} a_0 & \dots & a_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_3 \end{pmatrix}$$

In conclusion the matrix for real Fubini-Study for binary cubics is

$$3 \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The discriminants are $(xf_y - yf_x) + t(x^2 + y^2)(y(a_0 + a_2) - x(a_1 + a_3))$ when the metric changes. Denote $Df = yf_x - xf_y$. Note that up to scalar multiples

$$y(a_0 + a_2) - x(a_1 + a_3) = \Delta(Df) = D(\Delta(f))$$

Remark 1.19. *The two operators Δ and $P(f) = (x^2+y^2)f$ behave like twin operators in quantum mechanics (position and momentum, ∂_x and multiplication by x) in the sense that*

$$[\Delta, P] = c \cdot \mathrm{Id}$$

for a scalar c, of course these operators live only allowing infinite dimension, that is on the whole polynomial ring $S = \oplus_{d \geq 0} S_d$.

For binary quintics, the expression corresponding to $(v \cdot w)|v|^4|w|^4$ is
$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 2 & 0 & 4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 & 0 & 2 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{pmatrix}$$
which has rank 2, reads as

$$\langle f, g \rangle_{new} = \frac{1}{(5!)^2} \langle \Delta^2 f, \Delta^2 g \rangle$$

and gives as discriminant
$(x^2 + y^2)^2 (y(a_0 + 2a_2 + a_4) - x(a_1 + 2a_3 + a_5)) = \Delta\Delta(Df) = \Delta(D(\Delta f)) = D\Delta\Delta f.$

The expression corresponding to $(v \cdot w)^3|v|^2|w|^2$ is $\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 \\ 1 & 0 & 4 & 0 & 3 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$

which has rank 4, reads as

$$\langle f,g\rangle_{new}=\frac{1}{20^2}\langle \Delta f,\Delta g\rangle$$

and gives as discriminant

$$(x^2+y^2)\left(x^3(-a_1-a_3)+x^2y(a_0-a_2-2a_4)+xy^2(2a_1+a_3-a_5)+y^3(a_2+a_3)\right)$$

Note that the cubic polynomial in the discriminant $\left(x^3(-a_1-a_3)+x^2y(a_0-a_2-2a_4)+xy^2(2a_1+a_3-a_5)\right.$ is equal up to scalar multiples to $\Delta(Df)=D(\Delta(f))$.

2 Scalar products on partially symmetric tensors

Let V_i equipped with a bilinear symmetric form q_i, the group $SO(V_i,q_i)=\{f\in GL(V_i)|f^*q_i=q_i\}=\{f\in GL(V_i)|q_i(v,w)=q_i(f(v),f(w)\quad\forall v,w\in V_i\}=\{f\in GL(V_i)|q_i(v,v)=q_i(f(v),f(v)\quad\forall v\in V_i\}$ acts on V_i. Then there is a form q on $V=V_1\otimes\ldots\otimes V_k$ (that induces a natural $SO(V,q)$ action), defined on decomposable elements as

$$q(v_1\otimes\ldots\otimes v_k,w_1\otimes\ldots\otimes w_k)=\prod_{i=1}^k q_i(v_i,w_i),\tag{2.1}$$

which is called the Bombieri-Weyl form. Note that the formula (2.1) can be extended by linearity to all the space, this means

$$q(\sum_{i=1}^p\alpha_i v_{1,i}\otimes\ldots\otimes v_{k,i},\sum_{j=1}^q\beta_j w_{1,j}\otimes\ldots\otimes w_{k,j})=\sum_{i=1}^p\sum_{j=1}^q\alpha_i\beta_j q(v_{1,i}\otimes\ldots\otimes v_{k,i},w_{1,j}\otimes\ldots\otimes w_{k,j})$$

Since the decomposition as sum of decomposable elements is not unique, such extension requires some care. A simple way to check the existence is to write the tensors in orthonormal coordinates, then Bombieri-Weyl product corresponds to differentiation of the second entry by the first one. In other words, if $\{x_{i,0},\ldots,x_{i,m_i}\}$ are orthonormal coordinates on V_i we have

$\langle f(x_{1,0},\ldots,x_{1,m_1},\ldots,x_{k,0},\ldots,x_{k,m_k}),g(x_{1,0},\ldots,x_{1,m_1},\ldots,x_{k,0},\ldots,x_{k,m_k})\rangle=$
$f(\partial_{1,0},\ldots,\partial_{1,m_1},\ldots,\partial_{k,0},\ldots,\partial_{k,m_k}),g(x_{1,0},\ldots,x_{1,m_1},\ldots,x_{k,0},\ldots,x_{k,m_k})$.

Moreover on decomposable elements we get on the i-th factor

$(\sum_{j=0}^{m_i}\alpha_j\partial_{x_{i,j}})^{d_i}(\sum_{j=0}^{m_i}\beta_j x_{i,j})^{d_i}=d_i!\left(\sum_{j=0}^{m_i}\alpha_j\beta_j\right)^{d_i}=d_i!q_i(\sum_{j=0}^{m_i}\alpha_j x_{i,j},\sum_{j=0}^{m_i}\beta_j x_{i,j})^{d_i}$

in agreement with (2.1) . This formula shows also that the construction does not depend on the choice of orthonormal coordinates.

2.1 Classification of SO-invariant quadratic forms

Let $V=\mathrm{Sym}^{d_1}V_1\otimes\ldots\otimes\mathrm{Sym}^{d_k}V_k$. We denote by $\Delta_j\colon\mathrm{Sym}^qV_j\to\mathrm{Sym}^{q-2}V_j$ ($\forall q$) the Laplacian operator associated to q_j.

Theorem 2.1. *For every $i_1=0,\ldots\lfloor d_1/2\rfloor$, ..., $i_k=0,\ldots\lfloor d_k/2\rfloor$, there is a unique $\prod_{j=1}^k SO(V_j,q_j)$-invariant bilinear symmetric form*

$$q_{i_1,\ldots,i_k}\colon V\times V\to\mathbb{K}$$

such that on decomposable elements

$$q_{i_1,\ldots,i_k}(v_1^{d_1}\ldots v_k^{d_k}, w_1^{d_1}\ldots w_k^{d_k}) = \prod_{j=1}^{k} q_j(v_j,w_j)^{d_j-2i_j} q_j(v_j,v_j)^{i_j} q_j(w_j,w_j)^{i_j} \tag{2.2}$$

$\forall v_j, w_j \in V_j$.

On $\mathbb{K} = \mathbb{C}$, *the space of* $\prod_{j=1}^{k} SO(V_j,q_j)$*-invariant bilinear symmetric forms is the space generated by* $q_{i_1,\ldots,i_k}$ *which form a basis of this space whih has dimension* $\prod_{j=1}^{k}\lfloor\frac{d+2}{2}\rfloor$.

The rank of $q_{0,\ldots,0}$ *is maximal (it is a nondegenerate form), the form* $q_{0,\ldots,0}$ *is called the Bombieri-Weyl form (it is sometimes called also Kostlan form of Frobenius form). In particular the general* $\prod_{j=1}^{k} SO(V_j,q_j)$*-invariant bilinear symmetric form is nondegenerate.*

The kernel of $q_{i_1,\ldots,i_k}$ *consists of* $\oplus_{j=1}^{k}\left[\left(\ker\Delta_j^{i_j}\right)\bigoplus\oplus_{p\neq j}V_p\right]$. *On* $\mathbb{K}=\mathbb{R}$, *if each* q_i *is positive definite, the set of* $\prod_{j=1}^{k} SO(V_j,q_j)$*-invariant scalar products consist of a polyhedral cone generated by the rays* $q_{i_1,\ldots,i_k}$, *in other words any* $\prod_{j=1}^{k} SO(V_j,q_j)$*-invariant scalar product has the form*

$$\sum_{i_1,\ldots i_k} c_{i_1,\ldots,i_k} q_{i_1,\ldots,i_k}$$

with the requirement being positive definite.

Proof of Theorem 2.1 Uniqueness follows from the fact that any tensor is sum of rank one tensors. In order to prove independence we can reduce to $k=1$, where the statement on the kernels impies independence. Existence follows from

$$\Delta^{i_1}\otimes\ldots\otimes\Delta^{i_k}\colon \mathrm{Sym}^{d_1}V_1\otimes\ldots\otimes\mathrm{Sym}^{d_k}V_k \to \mathrm{Sym}^{d_1-2i_1}V_1\otimes\ldots\otimes\mathrm{Sym}^{d_k-2i_k}V_k$$

$$q_{i_1,\ldots,i_k}(f,g) = \langle\Delta^{i_1}\otimes\ldots\otimes\Delta^{i_k}f, \Delta^{i_1}\otimes\ldots\otimes\Delta^{i_k}g\rangle$$

where $\langle,\rangle$ is the differentiation (scaled by $\frac{1}{d!}$) as in the special case (2.1). The fact that these invariant forms generate the space of invariant forms follows from the harmonic decomposition (see ***) , $\mathrm{Sym}^{d_i}V_i = \oplus H_{d_i-2j}$, indeed it follows that $\mathrm{Sym}^2\left(\mathrm{Sym}^{d_i}V_i\right)$ has an invariant part of dimension $\lfloor d_i/2\rfloor+1$, the global invariant part is the tensor product of the invariant parts of each factor.

□

Remark 2.2. *Again, Theorem 2.1 gives a proof os First Fundamental Theorem of Invariant theory for this special case. Namely, if* q *is a* $\prod_{j=1}^{k} SO(V_j,q_j)$*-invariant bilinear symmetric form*

$$q\colon V\times V\to\mathbb{K},$$

then $q(v_1^{d_1}\ldots v_k^{d_k}, w_1^{d_1}\ldots w_k^{d_k})$ *is a polynomial in* $q_j(v_j,w_j), q_j(v_j,v_j), q_j(w_j,w_j)$.

Corollary 2.3. *The spectrahedron of semipositive* $\prod_{j=1}^{k} SO(V_j,q_j)$*-invariant bilinear forms on* $\mathrm{Sym}^d V$ *is a polyhedral cone having* $\prod_{j=1}^{k}\lfloor\frac{d_j+1}{2}\rfloor$ *vertices.*

Moreover if $q_i\colon V_i \times V_i$ are Hermitian products, let $V = V_1 \otimes \ldots \otimes V_k$, then there is $q\colon V \times V$ Hermitian products such that again on decomposable elements $q(v_1 \otimes \ldots \otimes v_k, w_1 \otimes \ldots \otimes w_k) = \prod_{i=1}^k q_i(v_i, w_i)$. Note Hermitian distance on $\mathbb{C}^{n+1}$ is Euclidean distance through the isomorphism $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$.

Note

$$|v_1 \otimes \ldots \otimes v_k|^2 = \prod_{i=1}^k |v_i|^2$$

Exercise On the real side, if $q_i \in SO(p_i, q_i)$ then $V_1 \otimes V_2$ has a $SO(p_1p_2+q_1q_2, p_1q_2+q_1p_2)$

References

[EK95] A. Edelman, E. Kostlan, *How many zeros of a random polynomial are real ?*, Bull. AMS **32** (1995), 1–37.

[LL] A. Lerario, E. Lundman, *Statistics on the Hilbert XVI problem*, Int. Math. Res. Not. IMRN (2015).

[Ot13] *Five Lectures on projective Invariants*, Rendiconti del Seminario Matematico Univ. Politec. Torino, vol. 71, 1 (2013), 119-194.

[Rez] B. Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc. 96 (1992), no. 463.

[Ze] D. Zeilberger, *Chu's 1303 identity implies 1990 Bombieri's 1990 norm-inequality*, Amer. Math. Monthly 101(1994), 894–896.