manuscripta math. 85, 97 - 107 (1994)

manuscripta mathematica

The Tangent Space at a Special Symplectic Instanton Bundle on $\mathbb{P}_{2n+1}$

Giorgio Ottaviani and **Günther Trautmann**

Introduction

Mathematical instanton bundles on $\mathbb{P}_3$ have their analogues in rank-$2n$ instanton bundles on odd dimensional projective spaces $\mathbb{P}_{2n+1}$. The families of special instanton bundles on these spaces, which generalize the special 'tHooft bundles on $\mathbb{P}_3$, were constructed and described in [OS] and [ST]. More general instanton bundles have recently been constructed in [AO2]. Let $MI_{2n+1}(k)$ denote the moduli space of all instanton bundles on $\mathbb{P}_{2n+1}$ with second Chern class $c_2 = k$. In order to obtain a first impression of this space it is important to know its tangent dimension $h^1End(\mathcal{E})$ at a stable bundle $\mathcal{E}$ and the dimension $h^2End(\mathcal{E})$ of the space of obstructions to smoothness.

In this paper we prove that for a special symplectic bundle $\mathcal{E} \in MI_{2n+1}(k)$

$$h^2End(\mathcal{E}) = (k-2)^2\binom{2n-1}{2}.$$

Such bundles are stable by [AO1]. So for $n \geq 2$ the situation is quite different to that of $\mathbb{P}_3$, where this number becomes zero, which was shown in [HN]. Since $H^iEnd(\mathcal{E}) = 0$ for $i \geq 3$, our result and the Hirzebruch–Riemann–Roch formula, see Remark 2.4,

$$h^1End(\mathcal{E}) - h^2End(\mathcal{E}) = -k^2\binom{2n-1}{2} + k(8n^2) + 1 - 4n^2$$

give

$$h^1End(\mathcal{E}) = 4(3n-1)k + (2n-5)(2n-1).$$

Therefore the dimension of $MI_{2n+1}(k)$ grows linearly in k, whereas the difference $h^1End(\mathcal{E}) - h^2End(\mathcal{E})$ becomes negative for $n \geq 2$ and grows quadratically in k. A more important consequence, however, is that in general we have to expect that $MI_{2n+1}(k)$ is singular at special symplectic bundles: The spaces $MI_5(3)$ and $MI_5(4)$ are singular at those points, which follows from theorem 4.1 and [AO2].

In order to derive our result we fix a 2–dimensional vector space U and consider the natural action of $SL(2)$ on $\mathbb{P}_{2n+1} = \mathbb{P}(U \otimes S^n U)$ as in [ST]. The special instanton bundles are related to the $SL(2)$–homomorphisms β, see 1.4, and are $SL(2)$–invariant. We prove that there is an isomorphism of $SL(2)$–representations

$$H^2(End\,\mathcal{E}) \cong S^{k-3}(U) \otimes S^{k-3}(U) \otimes S^2(U \otimes S^{n-2}U).$$

Acknowledgement This work was supported by the **Deutsche Forschungsgemeinschaft**. The first author wishes to thank the Fachbereich Mathematik of the University of Kaiserslautern, where this work was begun, for its hospitality.

Notation

- Throughout the paper K denotes an algebraically closed ground field of characteristic 0.
- U denotes a 2–dimensional K–vector space, $S_n = S^n U$ its nth symmetric power and $V_n = U \otimes S_n$.
- There is the natural exact squence of $GL(U)$–equivariant maps for any k, $n \geq 1$

$$0 \to \Lambda^2 U \otimes S_{k-1} \otimes S_{n-1} \xrightarrow{\beta} S_k \otimes S_n \xrightarrow{\mu} S_{k+n} \to 0$$

  where μ is the multiplication map and β is defined by $(s \wedge t) \otimes f \otimes g \mapsto sf \otimes tg - tf \otimes sg$. This sequence splits and leads to the Clebsch–Gordan decomposition of $S_k \otimes S_n$ by induction. When we tensorize the sequence with U we obtain the exact sequence

$$0 \to \Lambda^2 U \otimes S_{k-1} \otimes V_{n-1} \xrightarrow{\beta} S_k \otimes V_n \xrightarrow{\mu} V_{k+n} \to 0.$$

- $\mathbb{P} = \mathbb{P}_{2n+1} = \mathbb{P}V_n$ is the projective space of one dimensional subspaces of V_n.
- The terms vector bundle and locally free sheaf are used synonymously.
- $\mathcal{O}(d)$ denotes the invertible sheaf of degree d on $\mathbb{P}$, Ω^p the locally free sheaf of differential p–forms on $\mathbb{P}$, such that $\Omega^p(p) = \Lambda^p \mathcal{Q}^\vee$ where $\mathcal{Q} = \mathcal{T}(-1)$ is the canonical quotient bundle on $\mathbb{P}$.
- We use the abbreviations $\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(d)$ for any sheaf $\mathcal{F}$ of $\mathcal{O}$–modules on $\mathbb{P}$, $H^i\mathcal{F} = H^i(\mathcal{F}) = H^i(\mathbb{P}, \mathcal{F})$, $h^i\mathcal{F} = \dim H^i\mathcal{F}$. If E is a finite dimensional K–vector space, $E \otimes \mathcal{O}$ denotes the sheaf of sections of the trivial bundle $\mathbb{P} \times E$, and $E \otimes \mathcal{F} = (E \otimes \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{F}$. We also write $m\mathcal{F} = K^m \otimes \mathcal{F}$.
- We use the Euler sequence $0 \to \Omega^1(1) \to V_n^\vee \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$ and the derived sequences in its Koszul complex $0 \to \Omega^p(p) \to \Lambda^p V_n^\vee \otimes \mathcal{O} \to \Omega^{p-1}(p) \to 0$ without extra mentioning.
- $Ext^i(\mathcal{F}, \mathcal{G}) = Ext^i_{\mathcal{O}}(\mathbb{P}, \mathcal{F}, \mathcal{G})$ for any two $\mathcal{O}$–modules $\mathcal{F}$ and $\mathcal{G}$.

1 Instanton bundles

1.1 An instanton bundle on $\mathbb{P} = \mathbb{P}_{2n+1}$ with instanton number k or a k–instanton is an algebraic vector bundle $\mathcal{E}$ on $\mathbb{P}$ satisfying:

(i) $\mathcal{E}$ has rank $2n$ and Chern polynomial $c(\mathcal{E}) = (1 - h^2)^{-k} = 1 + kh^2 + \ldots$.

(ii) $\mathcal{E}$ has natural cohomology in the range $-2n - 1 \leq d \leq 0$, that is for any d in that range $h^i\mathcal{E}(d) \neq 0$ for at most one i.

A k–instanton bundle $\mathcal{E}$ is called **symplectic** if there is an isomorphism $\mathcal{E} \xrightarrow{\varphi} \mathcal{E}^\vee$ satisfying $\varphi^\vee = -\varphi$. In this case the spaces A and B below are Serre–duals of each other, since $H^{2n}(\mathcal{E}(-2n-1))^\vee \cong H^1\mathcal{E}^\vee(-1) \cong H^1\mathcal{E}(-1)$.

Remark: In the original definition in [OS] the additional conditions

(iii) $\mathcal{E}$ is simple, that is $Hom(\mathcal{E}, \mathcal{E}) = K$,

(iv) the restriction of $\mathcal{E}$ to a general line is trivial

are imposed. It was shown in [AO1] that (iii) is already a consequence of (i) and (ii). Condition (iv) seems to be independent but we do not need it in this paper. By [ST] special instantons satisfy (iv).

1.2 Let now A, B, C be vector spaces of dimensions $k, k, 2n(k-1)$ respectively. A pair of linear maps

$$A \xrightarrow{a} B \otimes \Lambda^2 V_n^\vee, \quad B \otimes V_n^\vee \xrightarrow{b} C$$

corresponds to a pair of sheaf homomorphisms

$$A \otimes \mathcal{O}(-1) \xrightarrow{\tilde{a}} B \otimes \Omega^1(1), \quad B \otimes \Omega^1(1) \xrightarrow{\tilde{b}} C \otimes \mathcal{O}.$$

Here $\tilde{a}$ is the composition of the induced homomorphisms $A \otimes \mathcal{O}(-1) \to B \otimes \Lambda^2 V_n^\vee \otimes \mathcal{O}(-1) \twoheadrightarrow B \otimes \Omega^1(1)$ and $\tilde{b}$ is the composition of the induced homomorphismus $B \otimes \Omega^1(1) \rightarrowtail B \otimes V_n^\vee \otimes \mathcal{O} \to C \otimes \mathcal{O}$. Conversely, a and b are determined by $\tilde{a}$ and $\tilde{b}$ respectively as $H^0(\tilde{a}(1))$ and $H^0(\tilde{b}^\vee)^\vee$. Moreover, the sequence

$$A \otimes \mathcal{O}(-1) \xrightarrow{\tilde{a}} B \otimes \Omega^1(1) \xrightarrow{\tilde{b}} C \otimes \mathcal{O} \tag{1}$$

is a complex if and only if the induced sequence

$$A \longrightarrow B \otimes \Lambda^2 V_n^\vee \longrightarrow C \otimes V_n^\vee$$

is a complex. We say that (1) is a **monad** if it is a complex and if in addition $\tilde{a}$ is a subbundle and $\tilde{b}$ is surjective.

Proposition 1.3 *The cohomology sheaf $\mathcal{E} = Ker\,\tilde{b}/Im\,\tilde{a}$ of a monad (1) is a k–instanton and conversely any k–instanton is the cohomology of a monad (1). Moreover, the spaces A, B, C of such a monad can be identified with $H^{2n}\mathcal{E}(-2n-1)$, $H^1\mathcal{E}(-1)$, $H^1\mathcal{E}$ respectively.*

Sketch of a proof: if a monad (1) is given it is easy to derive the properties of the definition. Conversely using Beilinson's spectral sequence, Riemann–Roch and in particular (ii), one obtains a monad with the identification of the vector spaces as in [OS]. The map b is then nothing but the natural map $H^1\mathcal{E}(-1) \otimes V_n^\vee \to H^1\mathcal{E}$ and the map a is given as the composition of the cup product

$$H^{2n}\mathcal{E}(-2n-1) \otimes \Lambda^2 V_n \to H^{2n}\mathcal{E} \otimes \Omega^{2n-1}(-1)$$

and the natural isomorphisms

$$H^{2n}\mathcal{E} \otimes \Omega^{2n-1}(-1) \cong H^{2n-1}\mathcal{E} \otimes \Omega^{2n-2}(-1) \cong \ldots \cong H^1\mathcal{E}(-1)$$

arising from the Koszul sequences and condition (ii), see [V] in case of $\mathbb{P}_3$.

1.4 Existence and special instanton bundles: Using the special structure $V_n = U \otimes S^n U$ and the Clebsch–Gordan type exact sequence

$$0 \longrightarrow \Lambda^2 U \otimes S_{k-2} \otimes V_{n-1} \xrightarrow{\beta} S_{k-1} \otimes V_n \xrightarrow{\mu} V_{k+n-1} \longrightarrow 0,$$

see notation, we define the special homomorphism

$$S_{k-1}^\vee \otimes \Omega^1(1) \xrightarrow{\tilde{b}} \Lambda^2 U^\vee \otimes S_{k-2}^\vee \otimes V_{n-1}^\vee \otimes \mathcal{O}$$

by $b = \beta^\vee$. We denote $\mathcal{N} = Ker\ (\tilde{b})$. It was shown in [ST] that $\tilde{b}$ is surjective and that

$$H^0\mathcal{N}(1) \subset S_{k-1}^\vee \otimes H^0\Omega^1(2)$$

can be identified with a canonical injective $GL(U)$-homomorphism

$$S_{2n+k-1}^\vee \otimes \Lambda^2 U^\vee \xrightarrow{\kappa} S_{k-1}^\vee \otimes \Lambda^2 V_n^\vee,$$

dual to the map

$$S_{k-1} \otimes \Lambda^2 V_n \to S_{2n+k-1} \otimes \Lambda^2 U$$

which is defined by $f \otimes (s \otimes g) \wedge (t \otimes h) \mapsto (fgh) \otimes (s \wedge t)$.

In order to construct instanton bundles we have to find k-dimensional subspaces

$$A \subset S_{2n+k-1}^\vee \otimes \Lambda^2 U^\vee \subset S_{k-1}^\vee \otimes \Lambda^2 V_n^\vee$$

such that the induced homomorphism $\tilde{a}$ is a subbundle. By [ST], Lemma 3.7.1, this is the case exactly when $\mathbb{P}A \subset \mathbb{P}(S_{2n+k-1}^\vee)$ does not meet the secant variety $Sec_n(C_{2n+k-1})$ of $(n-1)$-dimensional secant planes of the canonical rational curve C_{2n+k-1} of $\mathbb{P}S_{2n+k-1}^\vee$, given by $u \mapsto u^{2n+k-1}$. By dimension reasons such subspaces exist, [ST], 3.7, and hence instanton bundles exist.

A k-instanton bundle $\mathcal{E}$ is called **special** if the map b of its monad is isomorphic to the $GL(U)$-homomorphism $\beta^\vee$, that is if there are isomorphisms φ and ψ and $g \in GL(V_n)$ with the commutative diagram

$$\begin{array}{ccc} H^1\mathcal{E}(-1)\otimes V_n^\vee & \xrightarrow{b} & H^1\mathcal{E} \\ \varphi\otimes g^\vee \downarrow \wr & & \psi \downarrow \wr \\ S_{k-1}^\vee \otimes V_n^\vee & \xrightarrow{\beta^\vee} & \Lambda^2 U^\vee \otimes S_{k-2}^\vee \otimes V_{n-1}^\vee . \end{array}$$

Whereas in [ST] the family of all special k–instanton bundles was described, examples of different types of general instanton bundles were found in [AO2].

Remark 1.5 If $\mathcal{E}$ is special and symplectic then, in addition to the special $GL(U)$–homomorphism $b = \beta^\vee$ of its monad, the map a is given by an element $\alpha \in S_{2n+2k-2}^\vee$ as $a = \kappa \circ \tilde{\alpha}$ where $S_{k-1} \xrightarrow{\tilde{\alpha}} S_{2n+k-1}^\vee$ is defined by $\tilde{\alpha}(f)(g) = \alpha(fg)$ and $S_{2n+k-1}^\vee \xrightarrow{\kappa} S_{k-1}^\vee \otimes \Lambda^2 V_n^\vee$ is as above, [ST], 4.3 and 5.8. In particular a is a $GL(U)$–homomorphism, too, and can be represented by a persymmetric matrix.

Remark 1.6 It is shown in [AO1] that special symplectic instanton bundles are stable in the sense of Mumford–Takemoto.

2 Representing $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$

Proposition 2.1 *Let $\mathcal{E}$ be a symplectic k–instanton and let $\mathcal{N}$ be the kernel of the monad (1). Then $Ext^2(\mathcal{E},\mathcal{E}) \cong H^2(\mathcal{N}\otimes\mathcal{N})$.*

Proof: The monad (1) gives rise to the exact sequences

$$0 \longrightarrow \mathcal{N} \longrightarrow B\otimes\Omega^1(1) \xrightarrow{\tilde{b}} C\otimes\mathcal{O} \longrightarrow 0 \tag{2}$$

and

$$0 \longrightarrow A\otimes\mathcal{O}(-1) \longrightarrow \mathcal{N} \longrightarrow \mathcal{E} \longrightarrow 0. \tag{3}$$

After tensoring we have the exact sequences

$$0 \longrightarrow A\otimes\mathcal{N}(-1) \longrightarrow \mathcal{N}\otimes\mathcal{N} \longrightarrow \mathcal{E}\otimes\mathcal{N} \longrightarrow 0 \tag{4}$$

and

$$0 \longrightarrow A\otimes\mathcal{E}(-1) \longrightarrow \mathcal{N}\otimes\mathcal{E} \longrightarrow \mathcal{E}\otimes\mathcal{E} \longrightarrow 0. \tag{5}$$

Since $\mathcal{E}\cong\mathcal{E}^\vee$ we obtain $Ext^2(\mathcal{E},\mathcal{E}) \cong H^2(\mathcal{E}\otimes\mathcal{E})$. Sequence (2) implies $h^2\mathcal{N}(-1) = h^3\mathcal{N}(-1) = 0$ and from this and (3) also $h^2\mathcal{E}(-1) = h^3\mathcal{E}(-1) = 0$. Now sequences (4) and (5) yield isomorphisms $H^2(\mathcal{E}\otimes\mathcal{E}) \cong H^2(\mathcal{N}\otimes\mathcal{E}) \cong H^2(\mathcal{N}\otimes\mathcal{N})$. □

2.2 In order to represent $H^2(\mathcal{N}\otimes\mathcal{N})$ we note that the sequence (2) is part of the exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{N} & \longrightarrow & B\otimes\Omega^1(1) & \xrightarrow{\tilde{b}} & C\otimes\mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H\otimes\mathcal{O} & \longrightarrow & B\otimes V^\vee\otimes\mathcal{O} & \xrightarrow{b} & C\otimes\mathcal{O} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & B\otimes\mathcal{O}(1) & = & B\otimes\mathcal{O}(1) & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{6}$$

where H is the kernel of the operator b, which is surjective because $\tilde{b}$ is surjective. The left-hand column of (6) gives us after tensoring by $\Omega^1(1)$

$$B\otimes H^0\Omega^1(2)\overset{\delta}{\cong}H^1(\mathcal{N}\otimes\Omega^1(1)) \text{ and } H^2(\mathcal{N}\otimes\Omega^1(1))=0. \tag{7}$$

Since $\tilde{b}$ is the Beilinson representation of $\mathcal{N}$, we have the commutative diagram

$$\begin{array}{ccc} H^1\mathcal{N}(-1)\otimes H^0\mathcal{O}(1) & \xrightarrow{cup} & H^1\mathcal{N} \\ \|\wr & & \|\wr \\ B\otimes V^\vee & \xrightarrow{b} & C. \end{array} \tag{8}$$

Moreover, δ in (7) coincides also with cup:

$$\begin{array}{cc} B\otimes H^0\Omega^1(2) & \overset{\delta}{\approx} H^1(\mathcal{N}\otimes\Omega^1(1)) \\ \|\wr & \nearrow\text{cup} \\ H^1\mathcal{N}(-1)\otimes H^0\Omega^1(2) & \end{array} \tag{9}$$

Tensoring the top row of (6) with $\mathcal{N}$ and using (7) we obtain the following diagram with exact row:

$$\begin{array}{ccccccc} 0\to H^1(\mathcal{N}\otimes\mathcal{N})\to & B\otimes H^1(\mathcal{N}\otimes\Omega^1(1)) & \to & C\otimes H^1(\mathcal{N}) & \to H^2(\mathcal{N}\otimes\mathcal{N})\to 0 \\ & \|\wr & & \|\wr & \\ & B\otimes B\otimes\Lambda^2V_n^\vee & \overset{\Phi}{\dashrightarrow} & C\otimes C. & \end{array} \tag{10}$$

It follows that

$$H^2(\mathcal{N}\otimes\mathcal{N})=Coker(\Phi)=Ker(\Phi^\vee)^\vee. \tag{11}$$

Lemma 2.3 *The induced operator Φ is the composition $B\otimes B\otimes\Lambda^2V_n^\vee\xrightarrow{id\otimes\sigma}B\otimes B\otimes V_n^\vee\otimes V_n^\vee\xrightarrow{b\otimes b}C\otimes C$, where σ denotes the canonical desymmetrization.*

Proof: The computation of Φ is achieved by the diagram

$$
\begin{array}{ccc}
B\otimes B\otimes \Lambda^2 V_n^\vee & \xrightarrow{\quad id_{B\otimes B}\otimes \sigma \quad} & B\otimes B\otimes V_n^\vee \otimes V_n^\vee \\
\downarrow \approx & \mathrm{I} & \downarrow \approx \\
B\otimes H^1\mathcal{N}(-1)\otimes H^0\Omega^1(2) & \xrightarrow{\quad id\otimes H^0(\iota(1)) \quad} & B\otimes H^1\mathcal{N}(-1)\otimes V_n^\vee\otimes H^0\mathcal{O}(1) \\
\downarrow id_B\otimes cup & \mathrm{II} & \downarrow id_{B\otimes V_n^\vee}\otimes cup \\
B\otimes H^1(\mathcal{N}\otimes\Omega^1(1)) & \xrightarrow{\quad id_B\otimes H^1(id_{\mathcal{N}}\otimes\iota) \quad} & B\otimes V_n^\vee\otimes H^1\mathcal{N} \\
\downarrow H^1(id_{\mathcal{N}}\otimes\tilde{b}) & \mathrm{III} \quad \swarrow b\otimes id_{H^1\mathcal{N}} & \downarrow \approx \\
C\otimes H^1\mathcal{N} & \mathrm{IV} & B\otimes V_n^\vee\otimes C \\
\downarrow \approx & \swarrow b\otimes id_C & \\
C\otimes C & &
\end{array}
$$

In this diagram ι denotes the canonical inclusion $\Omega^1(1) \hookrightarrow V_n^\vee \otimes \mathcal{O}$, and up to $\Lambda^2 V_n^\vee \cong H^0\Omega^1(2)$ and $V_n^\vee \cong H^0\mathcal{O}(1)$ the map σ can be identified with $H^0(\iota(1))$. Therefore, the square I is commutative. Square II is a canonically induced diagram of cup-operations and commutative using $B \cong H^1\mathcal{N}(-1)$. The triangle III is induced by the commutative triangle

$$
\begin{array}{ccc}
B\otimes\mathcal{N}\otimes\Omega^1(1) & \xrightarrow{id\otimes\iota} & B\otimes V^\vee\otimes\mathcal{N} \\
\downarrow \tilde{b}\otimes id & \swarrow_{b\otimes id} & \\
C\otimes\mathcal{N} & &
\end{array}
$$

and hence commutative, and the commutativity of IV results just from the identification $H^1\mathcal{N} \cong C$. Now by definition the composition of the left-hand column is Φ and the composition of the right-hand column is $id_B \otimes id_{V_n^\vee} \otimes b$ since b is defined by (8).

It follows that $\Phi = (b\otimes id_C)\circ(id_B\otimes id_{V_n^\vee}\otimes b)\circ(id_{B\otimes B}\otimes\sigma) = (b\otimes b)\circ(id\otimes\sigma)$.

Remark 2.4 If $\mathcal{E}$ is a k-instanton bundle it is easily checked that $h^i\mathcal{E}(d) = h^i\mathcal{E}^\vee(d) = 0$ for $i \geq 2$ and $d \geq -1$. Using $\mathcal{E}^\vee \otimes \mathcal{N}$ again it follows that $Ext^i(\mathcal{E},\mathcal{E}) = H^i(\mathcal{E}^\vee\otimes\mathcal{E}) = H^i(\mathcal{E}^\vee\otimes\mathcal{N}) = 0$ for $i \geq 3$. This and the Riemann-Roch formula, which can also ad hoc be derived from the monad representation, give

$$h^1(\mathcal{E}^\vee\otimes\mathcal{E}) - h^2(\mathcal{E}^\vee\otimes\mathcal{E}) = -k^2\binom{2n-1}{2} + 8kn^2 - 4n^2 + 1.$$

3 Determination of $\mathbf{Ext^2}(\mathcal{E},\mathcal{E})$

We are now able to determine $Ext^2(\mathcal{E},\mathcal{E})$ as a $GL(2)$-representation space in case of a special instanton bundle. In that case b is the dual of the operator $\beta : \Lambda^2 U \otimes S_{k-2}\otimes V_{n-1} \to S_{k-1}\otimes V_n$, see notation or 1.4. Then $\Phi^\vee$ is the composition of $\beta\otimes\beta$

and the multiplication map $V_n \otimes V_n \to \Lambda^2 V_n$. In order to simplify we choose a fixed basis $s, t \in U$ and the isomorphism $\Lambda^2 U \cong k$ given by $s \wedge t$. Then

$$S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1} \xrightarrow{\Phi^\vee} S_{k-1} \otimes S_{k-1} \otimes \Lambda^2 V_n$$

is explicitly given by

$$\begin{aligned} \Phi^\vee(g \otimes g' \otimes v \otimes v') &= sg \otimes sg' \otimes (tv \wedge tv') - sg \otimes tg' \otimes (tv \wedge sv') \\ &- tg \otimes sg' \otimes (sv \wedge tv') + tg \otimes tg' \otimes (sv \wedge sv'). \end{aligned}$$

In order to determine the kernel of $\Phi^\vee$ we consider the $GL(U)$-homomorphism

$$S_{k-3} \otimes S_{k-3} \otimes V_{n-2} \otimes V_{n-2} \xrightarrow{\epsilon'} S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1}$$

defined similarly by

$$\begin{aligned} \epsilon'(f \otimes f' \otimes u \otimes u') &= sf \otimes sf' \otimes tu \otimes tu' - sf \otimes tf' \otimes su \otimes tu' \\ &- tf \otimes sf' \otimes tu \otimes su' + tf \otimes tf' \otimes su \otimes su'. \end{aligned}$$

Up to the order of factors the map ϵ' is the tensor product $\beta' \otimes \beta'$ where $\beta' : S_{k-3} \otimes V_{n-2} \to S_{k-2} \otimes V_{n-1}$ is defined as β. Hence, ϵ' is injective. Finally, we define ϵ as the composition

$$S_{k-3} \otimes S_{k-3} \otimes S^2 V_{n-2} \overset{id \otimes \iota}{\rightarrowtail} S_{k-3} \otimes S_{k-3} \otimes V_{n-2} \otimes V_{n-2} \xrightarrow{\epsilon'} S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1}$$

where ι is the canonical desymmetrization. Then also ϵ is injective.

Proposition 3.1 *$(S_{k-3} \otimes S_{k-3} \otimes S^2 V_{n-2}, \epsilon)$ is the kernel of $\Phi^\vee$.*

Proof: A straightforward computation shows that $Im(\epsilon) \subset Ker(\Phi^\vee)$. In order to show equality we reduce $Ker(\Phi^\vee)$ modulo $Im(\epsilon)$ using canonical bases of the vector spaces. A more elegant proof using Clebsch–Gordan decompositions seems much harder to achieve. Let us denote the bases as follows:

basis of S_{k-3} :	$e_\alpha = s^{k-3-\alpha} t^\alpha$	$0 \le \alpha \le k-3$
basis of S_{k-2} :	$f_\alpha = s^{k-2-\alpha} t^\alpha$	$0 \le \alpha \le k-2$
basis of S_{k-1} :	$g_\alpha = s^{k-1-\alpha} t^\alpha$	$0 \le \alpha \le k-1$
basis of V_{n-2} :	$u_\mu = s \otimes s^{n-2-\mu} t^\mu$ $\bar{u}_\mu = t \otimes s^{n-2-\mu} t^\mu$	$0 \le \mu \le n-2$
basis of V_{n-1} :	$x_\mu = s \otimes s^{n-1-\mu} t^\mu$ $\bar{x}_\mu = t \otimes s^{n-1-\mu} t^\mu$	$0 \le \mu \le n-1$
basis of V_n :	$y_\mu = s \otimes s^{n-\mu} t^\mu$ $\bar{y}_\mu = t \otimes s^{n-\mu} t^\mu$.	$0 \le \mu \le n$

For the basis $f_\alpha \otimes f_\beta \otimes x_\mu \otimes x_\nu$, $f_\alpha \otimes f_\beta \otimes x_\mu \otimes \bar{x}_\nu$, $f_\alpha \otimes f_\beta \otimes \bar{x}_\mu \otimes x_\nu$, $f_\alpha \otimes f_\beta \otimes \bar{x}_\mu \otimes \bar{x}_\nu$ we use the index tuplets $(\alpha, \beta, \mu, \nu)$, $(\alpha, \beta, \mu, \bar{\nu})$, $(\alpha, \beta, \bar{\mu}, \nu)$, $(\alpha, \beta, \bar{\mu}, \bar{\nu})$ respectively. The set of these indices will be ordered **lexicographically** with the additional assumption that always $\mu < \bar{\nu}$. Then, for example, $(\alpha, \beta, \mu, \bar{\nu}) < (\alpha, \beta, \bar{\lambda}, \delta)$.

Accordingly, the coefficients of an element $\xi \in S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1}$ will be denoted by $c(\alpha, \beta, \mu, \nu)$, $c(\alpha, \beta, \mu, \bar{\nu})$, $c(\alpha, \beta, \bar{\mu}, \nu)$, $c(\alpha, \beta, \bar{\mu}, \bar{\nu})$.

By the formula for $\Phi^\vee$ we obtain the

Lemma 3.2 *Let* $\xi \in S_{k-2} \otimes S_{k-2} \otimes V_{n-1} \otimes V_{n-1}$.

(i) The coefficient of $\Phi^\vee(\xi)$ *at the basis element* $g_\alpha \otimes g_\beta \otimes y_\mu \wedge \bar{y}_\nu$ *in* $S_{k-1} \otimes S_{k-1} \otimes \Lambda^2 V_n$ *is*

$$\begin{array}{l} c(\alpha,\beta,\mu-1,\overline{\nu-1}) - c(\alpha,\beta,\overline{\nu-1},\mu-1) \\ -c(\alpha,\beta-1,\mu-1,\bar{\nu}) + c(\alpha,\beta-1,\overline{\nu-1},\mu) \\ -c(\alpha-1,\beta,\mu,\overline{\nu-1}) + c(\alpha-1,\beta,\bar{\nu},\mu) \\ +c(\alpha-1,\beta-1,\mu,\bar{\nu}) - c(\alpha-1,\beta-1,\bar{\nu},\bar{\mu}). \end{array}$$

Here we agree that each of these coefficients is 0 if one of $\alpha, \alpha-1, \beta, \beta-1 \notin [0, k-2]$ *or if one of* $\mu, \mu-1, \nu, \nu-1 \notin [0, n-1]$.

(ii) Analogous statements hold for the coefficient of $\Phi^\vee(\xi)$ *at* $g_\alpha \otimes g_\beta \otimes y_\mu \wedge y_\nu$ *for* $\mu < \nu$ *(without bars) and at* $g_\alpha \otimes g_\beta \otimes \bar{y}_\mu \wedge \bar{y}_\nu$ *for* $\mu < \nu$ *(with two bars).*

Lemma 3.3 *Let the notation be as above. If* $\Phi^\vee(\xi) = 0$ *then:*

(i) If $c(\alpha,\beta,\mu,\nu)$ *is the first non-zero coefficient of* ξ *(in the lexicographical order), then* $0 < \mu \leq \nu$.

(ii) If $c(\alpha,\beta,\mu,\bar{\nu})$ *is the first non-zero coefficient of* ξ, *then* $\mu \neq 0$, $\nu \neq 0$.

(iii) $c(\alpha,\beta,\bar{\mu},\nu)$ *is never a first non-zero coefficient of* ξ.

(iv) If $c(\alpha,\beta,\bar{\mu},\bar{\nu})$ *is the first non-zero coefficient of* ξ, *then* $0 < \mu \leq \nu$.

Proof: (i) Let $c(\alpha,\beta,\mu,\nu)$ be the first coefficient of ξ. Then, by Lemma 3.2 the coefficient of $0 = \Phi^\vee(\xi)$ at $g_\alpha \otimes g_\beta \otimes y_{\mu+1} \wedge y_{\nu+1}$ is

$$\begin{array}{rcl} 0 & = & c(\alpha,\beta,\mu,\nu) - c(\alpha,\beta,\nu,\mu) - c(\alpha,\beta-1,\mu,\nu+1) + c(\alpha,\beta-1,\nu,\mu+1) \\ & - & c(\alpha-1,\beta,\mu+1,\nu) + c(\alpha-1,\beta,\nu+1,\mu) - \ldots \end{array}$$

Since $c(\alpha,\beta,\mu,\nu)$ is the first coefficient, only the first two in this formula could be non-zero because the others have smaller index in the lexicographical order. Hence

$$c(\alpha,\beta,\mu,\nu) = c(\alpha,\beta,\nu,\mu).$$

If $\mu > \nu$ then $c(\alpha,\beta,\nu,\mu)$ would be earlier and non-zero. Hence, $\mu \leq \nu$. Assume now that $\mu = 0$. The coefficient of $\Phi^\vee(\xi)$ of $g_\alpha \otimes g_{\beta+1} \otimes y_0 \wedge y_{\nu+1}$ is

$$\begin{array}{rcl} 0 & = & c(\alpha,\beta+1,-1,\nu) - c(\alpha,\beta+1,\nu,-1) \\ & - & c(\alpha,\beta,-1,\nu+1) + c(\alpha,\beta,\nu,0) \mp \ldots \end{array}$$

In this sum all but $c(\alpha,\beta,\nu,0)$ are automatically zero because $(\alpha-1,\beta,\ldots) \leq (\alpha,\beta,0,\nu)$ and -1 occurs. Hence, $c(\alpha,\beta,0,\nu) = c(\alpha,\beta,\nu,0) = 0$, contradiction.

The statements (ii), (iii), (iv) are proved analogously. □

Now we continue the proof of Proposition 3.1. We reduce an element $\xi \in Ker(\Phi^\vee)$ to $0 \, mod \, Im(\epsilon)$ using Lemma 3.3.

a) Assume that the first non–zero coefficient of ξ is

$$c(\alpha,\beta,\mu,\nu).$$

Then by Lemma 3.3 $0<\mu\leq\nu$. Then the element

$$\xi'=\xi-c(\alpha,\beta,\mu,\nu)\epsilon(e_\alpha\otimes e_\beta\otimes u_{\mu-1}\cdot u_{\nu-1})$$

belongs to $Ker(\Phi^\vee)$. We have

$$\begin{aligned}&\epsilon(e_\alpha\otimes e_\beta\otimes u_{\mu-1}\cdot u_{\nu-1})\\&=f_\alpha\otimes f_\beta\otimes(x_\mu\otimes x_\nu+x_\nu\otimes x_\mu)\\&-f_\alpha\otimes f_{\beta+1}\otimes(x_{\mu-1}\otimes x_\nu+x_{\nu-1}\otimes x_\mu)\\&-f_{\alpha+1}\otimes f_\beta\otimes(x_\mu\otimes x_{\nu-1}+x_\nu\otimes x_{\mu-1})\\&+f_{\alpha+1}\otimes f_{\beta+1}\otimes(x_{\mu-1}\otimes x_{\nu-1}+x_{\nu-1}\otimes x_{\mu-1})\end{aligned}$$

and therefore ξ' is a sum of monomials of index $>(\alpha,\beta,\mu,\nu)$. Hence, we can assume that $\xi \, mod\, Im(\epsilon)$ has no coefficient with index (α,β,μ,ν).

b) By Lemma 3.3 we can assume that the first non–zero coefficient of ξ has index $(\alpha,\beta,\mu,\bar\nu)$ or $(\alpha,\beta,\bar\mu,\bar\nu)$. In the first case we know by Lemma 3.3 that $0<\mu,\nu$. When we consider again

$$\xi'=\xi-c(\alpha,\beta,\mu,\bar\nu)\epsilon(e_\alpha\otimes e_\beta\otimes u_{\mu-1}\cdot\bar u_{\nu-1})$$

we have $\Phi^\vee(\xi')=0$ and ξ' is a sum of monomials of index $>(\alpha,\beta,\mu,\bar\nu)$. Hence, we may assume that $\xi \, mod\, Im(\epsilon)$ has $c(\alpha,\beta,\bar\mu,\bar\nu)$ as first non–zero coefficient. Again by Lemma 3.3 $0<\mu,\nu$ and

$$\xi'=\xi-c(\alpha,\beta,\bar\mu,\bar\nu)\epsilon(e_\alpha\otimes e_\beta\otimes \bar u_{\mu-1}\cdot\bar u_{\nu-1})$$

is a sum of monomials of index $>(\alpha,\beta,\bar\mu,\bar\nu)$.

This finally shows that $\xi=0 \, mod\, Im(\epsilon)$.

This completes the proof of Proposition 3.1.

4 Conclusions

By Proposition 2.1, Proposition 3.1, (11) and Lemma 2.3 we have determined the space $Ext^2(\mathcal{E},\mathcal{E})$. Together with Remark 2.4 we obtain

Theorem 4.1 *For any special symplectic k–instanton bundle $\mathcal{E}$ on $\mathbb{P}_{2n+1}$*

(1) $Ext^2(\mathcal{E},\mathcal{E})\cong S^\vee_{k-3}\otimes S^\vee_{k-3}\otimes S^2V^\vee_{n-2}$

(2) $\dim Ext^2(\mathcal{E},\mathcal{E})=(k-2)^2\binom{2n-1}{2}$

(3) $\dim Ext^1(\mathcal{E},\mathcal{E})=4k(3n-1)+(2n-5)(2n-1)$.

Let $MI_{2n+1}(k)$ denote the open part of the Maruyama scheme of semi-stable coherent sheaves on $\mathbb{P}_{2n+1}$ with Chern polynomial $(1-h^2)^{-k}$ consisting of instanton bundles. By [AO1] any special symplectic instanton bundle $\mathcal{E}$ is stable. Therefore, $Ext^1(\mathcal{E},\mathcal{E})$ can be identified with the tangent space of $MI_{2n+1}(k)$ at $\mathcal{E}$. In [AO2] deformations $\mathcal{E}'$ of special symplectic instanton bundles in $MI_{2n+1}(k)$ have been found for $n=2$ and $k=3,4$ which satisfy $Ext^2(\mathcal{E}',\mathcal{E}')=0$. This shows that in these cases there are components $MI'_{2n+1}(k)$ of $MI_{2n+1}(k)$ of the expected dimension $4(3n-1)k+(2n-5)(2n-1)$ containing the set of special instanton bundles. In particular, see [AO2]:

for $k=3,4$ the moduli space $MI_5(k)$ is singular at least in special symplectic bundles.

However, in case $2n+1=3$ we obtain the vanishing result of [HN]:

any special k-instanton bundle $\mathcal{E}$ on $\mathbb{P}_3$ satisfies $Ext^2(\mathcal{E},\mathcal{E})=0$ and is a smooth point of $MI_3(k)$,

since any rank-2 instanton bundle is symplectic.

References

[AO1] Ancona, V.; Ottaviani, G.: On the stability of special symplectic instanton bundles on $\mathbb{P}^{2n+1}$. To appear in Trans. Am. Math. Soc.

[AO2] Ancona, V.; Ottaviani, G.: On moduli of instanton bundles on $\mathbb{P}^{2n+1}$. To appear in Pacific J. Math.

[HN] Hirschowitz, A.; Narasimhan, M.S.: Fibrés de 'tHooft speciaux et applications. Proc. Nice Conf. 1981, 143–163, Birkhäuser 1982

[M] Maruyama, M.: Moduli of stable sheaves I, II. J. Math. Kyoto Univ. **17**, 91 – 126 (1977) and **18**, 557 – 614 (1978)

[OS] Okonek, C.; Spindler, H.: Mathematical instanton bundles on $\mathbb{P}^{2n+1}$. Journal reine angew. Math. **364**, 35 – 50 (1986)

[ST] Spindler, H.; Trautmann, G.: Special instanton bundles on $\mathbb{P}^{2n+1}$, their geometry and their moduli. Math. Ann. **286**, 559 – 592 (1990)

[V] Verdier, J.L.: Instantons, Les équations de Yang–Mills, Séminaire E.N.S., 1977/78, exp. VII, Astérisque 71 – 72 (1980)

Giorgio Ottaviani
Dipartimento di Matematica Applicata
Via S. Marta 3
50139 Firenze
Italy
ottaviani@ingfi1.ing.unifi.it

Günther Trautmann
Fachbereich Mathematik
Erwin–Schrödinger Straße
67663 Kaiserslautern
Germany
trm@mathematik.uni-kl.de

(Received March 18, 1994)