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REAL AND COMPLEX 'T HOOFT INSTANTON BUNDLES OVER $\mathbb{P}^{2n+1}(\mathbb{C})$

Conferenza tenuta il 10 giugno 1996

We give an explicit description of the Kodaira-Spencer deformation theory for symplectic instanton bundles. The 't Hooft bundles over $\mathbb{P}^3(\mathbb{C})$ can be constructed from $(k+1)$ disjoint lines. We introduce the 't Hooft bundles E over $\mathbb{P}^{2n+1}(\mathbb{C})$ and we study some of their properties, relating them to Yang-Mills $Sp(n)$-connections over $\mathbb{P}^n(\mathbb{H})$. In particular we prove that generically $h^0(E(1)) = n$ for $c_2(E) = k \geq 3$.

't Hooft bundles are invariant by small deformations for $n \geq 2$, $k \geq 9$.

Introduction

A (mathematical) instanton bundle over $\mathbb{P}^{2n+1}(\mathbb{C})$ with $c_2 = k$ is a stable bundle of rank $2n$ satisfying the following condition:

$$E \text{ is the cohomology bundle of a monad} \qquad \text{(M)}$$
$$L \otimes \mathcal{O}(-1) \xrightarrow{B^t} M \otimes \mathcal{O} \xrightarrow{A} N \otimes \mathcal{O}(1)$$

where L, M, N are complex vector spaces respectively of dimension $k, 2n+2k, k$.

This means that B^t is injective (as bundle map), A is surjective, $A \circ B^t = 0$ and $E \simeq \ker A/\mathrm{Im} B^t$.

The main interest about instanton bundles comes from the following construction, due to Atiyah, Drinfeld, Hitchin and Manin ([ADHM]) in the case $n = 1$ and to Salamon in the case $n \geq 2$ ([Sal], see also [Ni]).

Let us regard $\mathbb{P}^{2n+1}(\mathbb{C})$ as the twistor space of $\mathbb{P}^n(\mathbb{H})$ and let

$$\mathbb{P}^{2n+1}(\mathbb{C}) \xrightarrow{\pi} \mathbb{P}^n(\mathbb{H})$$

be the twistor fibration. The fibers of π are embedded lines $\mathbb{P}^1(\mathbb{C})$ and are called *real lines*.

Let E' be a $2n$-bundle over $\mathbb{P}^n(\mathbb{H})$.
Let A be a $\mathrm{Sp}(n)$-connection over E' which is an absolute minimum of the Yang-Mills functional

$$YM(A) = \int_{\mathbb{P}^n(\mathbb{H})} \|F_A\|^2$$

where F_A is the curvature of A.

Then π^*A induces an holomorphic structure over $\pi * E'$ (some details for $n \geq 2$ are given in section 4) and gauge equivalent connections induce the same holomorphic structure.

The main result of [ADHM] is the following: if $n = 1$ $\pi^*E' = E$ is a symplectic instanton bundle with the additional condition to have a fixed trivialization on the real lines.

Conversely Salamon proves that if E is a symplectic instanton bundle over $\mathbb{P}^{2n+1}(\mathbb{C})$ which has a trivialization on the real lines then E is induced by a unique (up to gauge) $\mathrm{Sp}(n)$-connection over E' which gives an absolute minimum of the Yang-Mills functional.

In the first section of the paper we study the deformation of the instanton bundles in terms of the monad (M). This topic can be regarded as a coordinate-free version of the description given in [AABOP]. We outline also an extension of the algorithms of [AABOP] to the symplectic case.

In the section 2 we remark that the postulated dimensions for the moduli spaces of instanton (according to the Kodaira-Spencer theory) coincide with those obtained by using elementary rank considerations on the Kronecker modules (see [Tyu] §1 for $n = 1$).

In the section 3, which is the main part of this paper, we introduce the (generalized) 't Hooft bundles.

For $n = 1$ the 't Hooft bundles were considered in [H-N], they come through the Serre correspondance from the union of $k+1$ disjoint lines. For $n = 2$ some cases are treated in [M-S].

It is surprising that 't Hooft bundles are invariant by small deformatins for $n \geq 2$, $k \geq 9$ (this is not true for $n = 1$) so that they solve completely the problem to find the dimension of one irreducible component of the moduli space of symplectic instanton bundles.

We prove that for a generic 't Hooft bundle we have $h^0(E(1)) = n$ for $k \geq 3$.

At the end of the section 3 we compare this approach with some recent results about special instanton bundles.

In section 4 we give an explicit description of the Atiyah-Ward correspondance for $n \geq 2$, according to Salamon, in order to have an explicit description of the real 't Hooft bundles. We sketch some results whose proofs will appear in [AO3].

In the appendix we list two Macaulay scripts.

I have begun to work about instanton bundles in collaboration with V. Ancona, who has to be considered a coauthor of all the original aspects of this exposition.

The author benefited also of many insights during talks with A. Tikhomirov and G. Trautmann.

I wish to thank the Seminario Matematico e Fisico di Milano for the opportunity to talk about this subject in Milano.

It is a pleasure to thank warmly the colleagues of the Department of Mathematics of Milano A. Alzati, M. Bertolini, A. Lanteri, M. Palleschi, C. Tibiletti and C. Turrini for a lot of fruitful discussions through the last years and moreover for their help, encouragement and friendship just from the beginning of my mathematical activity.

1. Deformations of instanton bundles

Instanton bundles have been defined in the introduction as stable cohomology bundles of the monad (M).

It seems likely that the stability condition is contained in (M), this is true on $\mathbb{P}^3$ and $\mathbb{P}^5$ ([AO2]). Instanton bundles with $c_2 = k$ define a moduli space $MI_{\mathbb{P}^{2n+1}}(k)$ which is an open subset of the corresponding Maruyama scheme.

The symplectic instanton bundles with $c_2 = k$ carry a natural structure of closed subscheme of $MI_{\mathbb{P}^{2n+1}}(k)$ that we denote by

$$MIS_{\mathbb{P}^{2n+1}}(k)\,.$$

Let E be an instanton bundle with $c_2 = k$ over $\mathbb{P}^{2n+1}(\mathbb{C})$.

LEMMA 1.1 –

$$L \simeq H^{2n}(E(-2n-1))$$

$$M \simeq H^1(E \otimes \Omega^1) \simeq H^{2n}(E^\vee \otimes \Omega^{2n})^\vee$$

$$N \simeq H^1(E(-1))$$

Proof – Straightforward from (M).

The dual bundle $E^\vee$ is induced by the dual monad

$$N^\vee \otimes \mathcal{O}(-1) \xrightarrow{A^t} M^\vee \otimes \mathcal{O} \xrightarrow{B} L^\vee \otimes \mathcal{O}(+1)$$

If E is symplectic then the symplectic isomorphism $E \longrightarrow E^\vee$ is induced by a morphism of monads and we have $N^\vee \simeq L$, $M \simeq M^\vee$. In fact, according to the Serre duality

$$N^\vee \simeq H^1(E(-1))^\vee \simeq H^1(E^\vee(-1))^\vee \simeq H^{2n}(E(-2n-1)) \simeq L$$

Moreover, there is a symplectic isomorphism $J : M \longrightarrow M^\vee$ such that $B^t = JA^t$ and E is induced by the monad

$$N^\vee \otimes \mathcal{O}(-1) \xrightarrow{J \circ A^t} M^\vee \otimes \mathcal{O} \xrightarrow{A} N \otimes \mathcal{O}(1) \qquad (1.0)$$

where

$$A \in H^0(M \otimes N \otimes \mathcal{O}(1)) = \mathrm{Hom}\,(M^\vee \otimes \mathcal{O}, N \otimes \mathcal{O}(1)) \simeq$$

$$\simeq \mathrm{Hom}\,(N^\vee \otimes \mathcal{O}(-1), M \otimes \mathcal{O})$$

We have a natural map

$$\overset{2}{\Lambda} A : H^0(M \otimes N \otimes \mathcal{O}(1)) \longrightarrow H^0(\overset{2}{\Lambda} N \otimes \mathcal{O}(2))$$

$$A' \mapsto A'JA^t + AJA'^t$$

which is intrinsecally given by composing $A' \in \mathrm{Hom}\,(N^\vee \otimes \mathcal{O}(-1), M \otimes \mathcal{O})$ with AJ and projecting over the summand $\overset{2}{\Lambda} N$ of $N \otimes N$.

PROPOSITION 1.2 –

$$H^2(S^2E) \simeq \frac{\overset{2}{\Lambda} N \otimes H^0(\mathcal{O}(2))}{\mathrm{Im}(\overset{2}{\Lambda} A)} = \mathrm{Coker}\,(\overset{2}{\Lambda} A)$$

Let K be the kernel bundle occurring in the sequences

$$0 \longrightarrow K \longrightarrow M^\vee \otimes \mathcal{O} \xrightarrow{A} N \otimes \mathcal{O}(1) \longrightarrow 0 \tag{1.1}$$

$$0 \longrightarrow N^\vee \otimes \mathcal{O}(-1) \longrightarrow K \longrightarrow E \longrightarrow 0 \tag{1.2}$$

From (1.2) we get

$$0 \longrightarrow \overset{2}{\Lambda} N^\vee \otimes \mathcal{O}(-2) \longrightarrow N^\vee \otimes K(-1) \longrightarrow S^2K \longrightarrow S^2E \longrightarrow 0$$

Since $H^i(K(-1)) = 0$ for $i \geq 2$ it follows

$$H^2(S^2K) \simeq H^2(S^2E)$$

From (1.1) it follows

$$o \longrightarrow S^2K \longrightarrow S^2M \otimes \mathcal{O} \longrightarrow M \otimes N \otimes \mathcal{O}(1) \longrightarrow \overset{2}{\Lambda} N \otimes \mathcal{O}(2) \longrightarrow 0$$

The last nonzero morphism computed at the level of global sections is $\overset{2}{\Lambda} A$. Hence the result follows.

REMARK 1.3 – The space $\ker(\overset{2}{\Lambda} A)$ contains the subspace $\mathrm{End}\,(N) \oplus \boldsymbol{Sp}\,(M)$ according to the inclusion

$$\mathrm{End}\,(N) \oplus \boldsymbol{Sp}\,(M) \longrightarrow \mathrm{Hom}\,(M^\vee \otimes \mathcal{O}, N \otimes \mathcal{O}(1))$$

$$(\alpha, \gamma) \mapsto \alpha A + A\gamma$$

In fact

$$\overset{2}{\Lambda} A(\alpha A + A\gamma) = (\alpha A + A\gamma)JA^t + AJ(A^t\alpha^t + \gamma^t A^t) =$$

$$= A(\gamma J + J\gamma^t)A^t = 0$$

PROPOSITION 1.4 –

$$H^1(S^2E) \simeq \ker(\overset{2}{\Lambda} A)\big/\mathrm{End}\,(N) \oplus \boldsymbol{Sp}\,(M)$$

Proof – As in the proof of Prop. 1.2 we get from the same sequences:

$$0 \longrightarrow \mathrm{End}\,(N) \longrightarrow H^1(S^2K) \longrightarrow H^1(S^2E) \longrightarrow 0$$

$$0 \longrightarrow S^2M \longrightarrow \ker(\overset{2}{\Lambda} A) \longrightarrow H^1(S^2K) \longrightarrow 0$$

Since $S^2M \simeq \boldsymbol{Sp}\,(M)$ through the map

$$P \mapsto PJ$$

the result follows

REMARK 1.5 – $\ker(\overset{2}{\Lambda} A)$ is the tangent space in A to the subvariety $Q \subset H^0(M \otimes N \otimes \mathcal{O}(1))$ defined by $Q = \{A' \mid A'JA'^t = 0\}$.

The moduli space $MIS_{\mathbb{P}^{2n+1}}(k)$ can be defined as in [OS] as the $GL(N) \times \boldsymbol{Sp}\,(M)$ quotient of an open subspace of Q.

The action of $\mathrm{End}\,(N) \oplus \boldsymbol{Sp}\,(M)$ over $\ker(\overset{2}{\Lambda} A)$ given by $(\alpha, \gamma, A') \mapsto A' + \alpha A + A\gamma$ is the derivative in (id, A) of the $GL(N) \times \boldsymbol{Sp}\,(M)$ action over Q. Thus the prop. 1.4 identifies $H^1(S^2E)$ as the tangent space at $MIS_{\mathbb{P}^{2n+1}}(k)$ in E, according to Kodaira-Spencer theory.

The propositions 1.2 and 1.4 can be carried out in the same way for any instanton bundle, not necessarily symplectic.

We get the following propositions 1.6 and 1.7, that can be regarded as a coordinate-free version of the theorems 2.1.2 and 2.1.6 of [AABOP].

Let $A^t \otimes B$ be the map

$$H^0(N \otimes M^\vee \otimes \mathcal{O}(1)) \oplus H^0(M \otimes L^\vee \otimes \mathcal{O}(1)) \longrightarrow H^0(N \otimes L^\vee \otimes \mathcal{O}(2)),$$

$$(X, Y) \mapsto XB^t + AY^t.$$

We have also a morphism

$$\gamma : \operatorname{End}(L) \oplus \operatorname{End}(N) \oplus \operatorname{End}(M)$$
$$\longrightarrow H^0(N \otimes M^\vee \otimes \mathcal{O}(1)) \oplus H^0(M \otimes L^\vee \otimes \mathcal{O}(1)),$$
$$(\alpha, \beta, \gamma) \mapsto (\beta A - A\gamma^t, \alpha B + B\gamma)$$

with one-dimensional kernel.

It is straightforward to check that $\operatorname{Im}(\gamma) \subset \ker(A^t \otimes B)$.

PROPOSITION 1.6 –

$$H^2(\operatorname{End} E) \simeq \operatorname{Coker}(A^t \otimes B)$$

PROPOSITION 1.7 –

$$H^1(\operatorname{End} E) \simeq \ker(A^t \otimes B) \big/ \gamma(\operatorname{End} L \oplus \operatorname{End} N \oplus \operatorname{End} M).$$

The geometrical interpretation of remark 1.5 holds also in this case.

Deformations of bundles with structural group G

Let E be a vector bundle of rank r with structural group $G \subset GL(r)$. Let $\operatorname{Ad} : G \longrightarrow GL(g)$ be the adjoint representation and let $\operatorname{Ad} E$ be the corresponding adjoint bundle.

By Kodaira-Spencer theory among the small deformations of E preserving the structural group there is a versal deformation. Let X be its base space; then the germ of X at [E] is the zero locus of the Kuranishi map

$$\Phi_E : H^1(\operatorname{Ad} E) \longrightarrow H^2(\operatorname{Ad} E)$$

In particular $H^1(\operatorname{Ad} E)$ identifies with the Zariski tangent space to X at [E], and X is smooth at E if and only if $\Phi_E = 0$.

When E is stable the germ of the deformation space can be identified with the germ of the corresponding Maruyama scheme.

In particular if $G = \boldsymbol{Sp}(r)$ then E is called a symplectic bundle and we have $\operatorname{Ad} E = S^2 E$.

In equivalent way, a symplectic bundle E can be defined as a bundle E with an isomorphism $\phi : E \longrightarrow E^\vee$ such that $\phi = -\phi^t$.

The Kuranishi map

It follows by Artin deformation theory that the Kuranishi map of instanton bundles lifts through the diagram

$$\begin{array}{ccc} \ker(A^t \otimes B) & \xrightarrow{\Phi} & H^0(L^V \otimes N \otimes \mathcal{O}(2)) \\ \downarrow & & \downarrow \\ H^1(\mathrm{End}\ E) & \longrightarrow & H^2(\mathrm{End}\ E) \end{array}$$

where $\Phi(X,Y) = XY^t$.

In the symplectic case, the symplectic Kuranishi map lifts through the diagram

$$\begin{array}{ccc} \ker(\overset{2}{\Lambda} A) & \xrightarrow{\Phi} & H^0(\overset{2}{\Lambda} N \otimes \mathcal{O}(2)) \\ \downarrow & & \downarrow \\ H^1(S^2E) & \longrightarrow & H^2(S^2E) \end{array}$$

where $\Phi(X) = XJX^t$.

The computations of $h^1(S^2E)$, $h^2(S^2E)$ according to the prop. 1.6 and 1.7 can be easily implemented. We join a short Macaulay script in the appendix.

The interested reader can adapt the other scripts described in [An] (e.g. for the computation of the Kuranishi map) to the symplectic case.

Some of these scripts are available upon request to the author.

2. Elementary estimates for the dimension of the moduli spaces

The following proposition is a straightforward computation, by using (1.1) and (1.2)

PROPOSITION 2.1 – *Let E be an instanton bundle with $c_2 = k$ over $\mathbb{P}^{2n+1}$.*

(i) $h^1(End\,E) - h^2(End\,E) = -k^2\binom{2n-1}{2} + k(8n^2) + 1 - 4n^2$,

(ii) $h^1(End\,E(-1)) - h^2(End\,E(-1)) = -k^2(2n-2) + 4nk$.

If E is symplectic:

(iii) $h^1(S^2E) - h^2(S^2E) = -\frac{k^2}{?}\binom{2n-1}{2} + k\left(\frac{10n^2+5n+1}{2}\right) +$ $(-2n^2 - n)$,

(iv) $h^1(S^2E) - h^2(S^2E(-1)) = -k^2(n-1) + k(3n+1)$. (-1)

Let us observe that for $n = 1$ i) and iii) collapse to the well known $8k-3$.

(i) and (iii) give a lower bound for the dimension of the moduli spaces. (ii) and (iv) give similar bounds for the subscheme of the moduli space consisting of bundles which restrict to a fixed bundle on a fixed hyperplane.

For $n \geq 2$ the expressions from (i) to (iv) become negative for $k \gg 0$. This simple remark already shows that the moduli are obstructed for $n \geq 2$.

We recall that instanton bundles can be defined by a second monad (see [OS]), which is dual to (M) in the sense of Beilinson theorem (see [AO1]), precisely

$$L \otimes \mathcal{O}(-1) \xrightarrow{m} N \otimes \Omega^1(1) \xrightarrow{n} P \otimes \mathcal{O} \qquad (2.1)$$

where $P \simeq H^1(E)$ so that $\dim P = 2n(k-1)$. The role of the two morphisms m, n is not symmetric as in the first monad (M).

In particular m itself determines the bundle in the following way. Let $\mathbb{P}^{2n+1} = \mathbb{P}(V)$.

We have

$$m \in L^\vee \otimes N \otimes \overset{2}{\Lambda} V \subset \mathrm{Hom}\,(L \otimes V^\vee, N \otimes V) \qquad (2.2)$$

(see [Tyu] §1) and m satisfies the following condition

$$\mathrm{corank}\, m \geq 2n(k-1)\,. \qquad (2.3)$$

Conversely every m in (2.2) which satisfies (2.3) fits in a monad (2.1) and determines the instanton bundle (with a slight abuse of notation).

m is a Kronecker module in the language of [OS]. Two instanton bundles defined by the Kronecker modules m, m' are isomorphic if and only if there exists $(P, Q) \in GL(H) \times GL(H)/\mathbb{C}^*$ such that $m' = P^t m Q$.

THEOREM 2.2 – ([OS]) *An instanton bundle E defined by the Kronecker module m is symplectic if and only if the following two equivalent conditions hold:*

(i) $L^\vee \simeq N$ *and* $m \in \overset{2}{\Lambda}(L \otimes V)$

(ii) $L^\vee \simeq N$ *and* $m \in S^2 L \otimes \overset{2}{\Lambda} V$

The equivalence between (i) and (ii) follows from the canonical decomposition

$$\overset{2}{\Lambda}(L \otimes V) \simeq (\overset{2}{\Lambda} L \otimes S^2 V) \otimes (S^2 L \otimes \overset{2}{\Lambda} V)$$

COROLLARY 2.3 – ([OS]) *Two symplectic instanton bundles defined by the Kronecker modules m, m' are isomorphic if and only if there exist $P \in GL(H)$ such that*

$$m' = P^t m P$$

The action of $GL(H) \times GL(H)/\mathbb{C}^*$ (respectively of $GL(H)$ in the symplectic case) is free (up to finite subgroups) so that the moduli spaces of instantons can be studied by intersecting

(i) the variety of corank $2n(k-1)$ matrices in $\mathrm{Hom}\,(L \otimes V^\vee, N \otimes V)$ with $L^\vee \otimes N \otimes \overset{2}{\Lambda} V$

(ii) in the symplectic case the variety of corank $2n(k-1)$ matrices in $\overset{2}{\Lambda}(L \otimes V)$ with $S^2 L \otimes \overset{2}{\Lambda} V$.

Looking at the dimensions, we get the following simple but interesting computations:

$$\dim MI_{\mathbb{P}^{2n+1}}(k) \geq \dim L^\vee \otimes N \otimes \overset{2}{\Lambda} V +$$

$$-\operatorname{codim}\{\text{matrices of corank } 2n(k-1)\}+$$

$$-\dim GL(H) \times GL(H)\big/\mathbb{C}^* =$$

$$= \tbinom{2n+2}{2} k^2 - (2n(k-1))^2 - (2k^2-1) =$$

$$= -k^2 \tbinom{2n-1}{2} + k(8n^2) + 1 - 4n^2$$

which is exactly (i) of prop. 2.1.

$$\dim MIS_{\mathbb{P}^{2n+1}}(k) \geq \dim S^2 L \otimes \overset{2}{\Lambda} V +$$

$$-\operatorname{codim}\{\text{skew-symmetric matrices of corank } 2n(k-1)\}$$

$$-\dim GL(H) =$$

$$= \tbinom{2n+2}{2}\tbinom{k+1}{2} - \tbinom{2n(k-1)}{2} - k^2 =$$

$$= -\frac{k^2}{2}\binom{2n-1}{2} + k\left(\frac{10n^2+5n+1}{2}\right) + (-2n^2-n)$$

which is exactly (iii) of prop. 2.1.

3. The 't Hooft bundles

Instanton bundles from $k+1$ skew lines in $\mathbb{P}^3$

Let us begin with an explicit description of the monads for instanton bundles E over $\mathbb{P}^3$ such that $E(1)$ has a section vanishing on $k+1$ disjoint lines.

By using the well known Serre correspondence ([OSS]) it is possible to construct E from the variety Z given by the union of the $(k+1)$ disjoint lines and from an element of

$$\mathbb{P}(\operatorname{Ext}^1(Y_Z \otimes \det(N_{Z,\mathbb{P}^3}), \mathcal{O})) \simeq \mathbb{P}(\mathbb{C}^{k+1})$$

Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ be the standard skewsymmetric form. Let $A : \mathcal{O} \longrightarrow \mathcal{O}(1)^{2k+2}$ defined by

$$\begin{bmatrix} a_1 & & & \alpha_1 a & b_1 & & & \alpha_1 b \\ & \ddots & & \vdots & & \ddots & & \vdots \\ & & a_k & \alpha_k a & & & b_k & \alpha_k b \end{bmatrix} \tag{3.1}$$

with $\alpha_i \in \mathbb{C}$ generic and $a_1, \ldots, a_k, b_1, \ldots, b_k, a, b$ generic linear forms.

PROPOSITION 3.1 – *Let A as in (3.1). Then*

i) $AJA^t = 0$,

ii) $rk\ A = k$ *for every point in* $\mathbb{P}^3$.

Proof – i) is straightforward.

ii) is easy and will be proved in a more general setting in the prop.-def. 3.3.

PROPOSITION 3.2 – (description of 't Hooft bundles over $\mathbb{P}^3$)

Let A as in (3.1). Then A defines an instanton bundle E such that E(1) has a section vanishing on the $(k+1)$ *lines* $\{a = b = 0\}$ *and* $\{a_i = b_i = 0\}$ *for* $i = 1, \ldots, k$, *so that E is a 't Hooft bundle. Conversely the generic 't Hooft bundle can be described by a monad with A as in (3.1).*

Proof – A defines an instanton E from the prop. 3.1. The matrix

$$C^t = \left[\begin{array}{cccc|cccc} -b_1 & & & & a_1 & & & \\ & \ddots & & & & \ddots & & \\ & & -b_k & & & & a_k & \\ & & & -b & & & & a \end{array}\right]$$

satisfies

$$A^t \circ C = 0$$

so that it defines $k+1$ sections of the kernel bundle (twisted by $\mathcal{O}(1)$), that is one section of $E(1)$.

This section vanishes where the rank of C is not maximum, and this proves the first statement.

The second statement follows from the Serre correspondence.

REMARK – If the union Z of the $(k+1)$ lines is contained in a quadric q we get a special 't Hooft bundle E with $h^0(E(1)) = 2$.

A second independent section of $E(1)$ can be found in the following way: there exist linear forms s_i, t_i $i = 1, \dots, k+1$ such that

$$q = s_i a_i + t_i b_i = s_{k+1} a + t_{k+1} b$$

Then stack the following row to C^t:

$$(\alpha, s_1, \alpha_2 s_2, \dots, \alpha_k s_k, -s_{k+1}, \alpha_1 t_1, \dots, \alpha_k t_k, -t_{k+1})$$

and we still have for the new C':

$$A^t \circ C' = 0$$

't Hooft bundles over $\mathbb{P}^{2n+1}$

Denote by z_i, w_j $(i, j = 1, \dots, k)$ $2k$ generic linear forms over $\mathbb{P}^{2n+1}$. Let ξ_i, η_j $(i, j = 1, \dots, n)$ be $2n$ generic linear forms (that can be choosen as a part of a system of homogeneous coordinates over $\mathbb{P}^{2n+1}$).

Let (a_{ij}) be a generic $k \times n$ matrix with complex coefficients. Denote by $D(f_i)$ the diagonal matrix with p-th entry equal to f_p.

Let $A : \mathcal{O}^k \longrightarrow \mathcal{O}(1)^{2n+2k}$ defined by:

$$[D(z_i) \mid a \cdot D(\xi_i) \mid D(w_i) \mid a \cdot D(\eta_i)] \tag{3.2}$$

For $n = 1$ (3.2) reduces to (3.1).

PROPOSITION-DEFINITION 3.3 – *Let A as in (3.2). We have*

(i) $AJA^t = 0$,

(ii) $rk A = k$ *for every point in* $\mathbb{P}^{2n+1}$.

We call the symplectic instanton bundle defined by A (see (1.0)) a 't Hooft bundle.

Proof – (i) is straightforward.
In order to prove (ii) for a generic choice of A we can make the following choices: divide a system of homog. coord. into two subsets

$$\{x_0, \ldots, x_n\} \quad \{x_{n+1}, \ldots, x_{2n+1}\}$$

Pick up z_i, ξ_i as linear combination of $\{x_0, \ldots, x_n\}$. Pick up w_i, η_i as linear combination of $\{x_{n+1}, \ldots, x_{2n+1}\}$. It is then sufficient to check that the matrix $[D(z_i) \mid a \cdot D(\xi_i)]$ has rank k for every $(x_0, \ldots, x_n) \in \mathbb{P}^n$.

Choose (a_{ij}) with *all* nonzero minors of any order and choose z_i, ξ_i in such a way that for every subset of $n+1$ elements of them the intersection of the corresponding $n+1$ hyperplanes is empty. This means that the corresponding hyperplanes have normal crossing.

Let $\bar{x} \in \mathbb{P}^n$. Let $z_{i_1}, \ldots, z_{i_j}$ (with $j \leq n$) be the forms such that $z_{i_p}(\bar{x}) = 0$ for $p = 1, \ldots, j$.

Consider $\xi_1, \ldots, \xi_{n+1-j}$. By the assumption there exists $q_1 \in \{1, \ldots, n+1-j\}$ such that $\xi_{q_1}(\bar{x}) \neq 0$.

Now consider $\xi_1, \ldots, \hat{\xi}_{q_1}, \ldots, \xi_{n+1-j}, \xi_{n+2-j}$.

By the assumption there exists ξ_{q_2} among the above such that $\xi_{q_2}(\bar{x}) \neq 0$.

In this way we get

$$z_1 \ldots \hat{z}_{i_1} \ldots \hat{z}_{i_j} \ldots z_k \xi_{q_1} \ldots \xi_{q_j}$$

that are k forms all nonvanishing in $\bar{x}$.

It follows that the minor of $[D(z_i) \mid a \cdot D(\xi_i)]$ corresponding to these forms does not vanish in $\bar{x}$.

PROPOSITION 3.3 – *Let E be a 't Hooft bundle. $H^0(E(1))$ contains a n-dimensional subspace W_n such that the degeneracy locus of the evaluation map*

$$\mathcal{O} \otimes W_n \longrightarrow E(1)$$

is given by the union Z of the $(k+n)$ linear subspaces of codimension 2

$$\{z_i = w_i = 0\} \qquad i = 1, \ldots, k,$$

$$\{\xi_j = \eta_j = 0\} \qquad j = 1, \ldots, n.$$

Proof – Define

$$C^t = \left[\begin{array}{cccccc|cccccc} -w_1 & & & & & & -z_1 & & & & & \\ & \ddots & & & & & & \ddots & & & & \\ & & w_k & & & & & & -z_k & & & \\ & & & \eta_1 & & & & & & -\xi_1 & & \\ & & & & \ddots & & & & & & \ddots & \\ & & & & & \eta_n & & & & & & -\xi_n \end{array}\right] \tag{3.3}$$

and check that $A^t \circ C = 0$.

Then argue as in the proof of prop. 3.2.

Of course we have

$$C^t J C = 0 \tag{3.4}$$

REMARK – The codimension of Z in the prop. 3.3 is the expected one *only* for $n = 1$. Moreover the theorem 3.7 will show that for $k \geq 3$ $H^0(E(1)) = W_n$. For $k = 1,2$ W_n is a proper subspace of $H^0(E(1))$.

For every $j \times n$ complex matrix F define a subspace $\mathbb{P}_F^{2n+1-2j}$ by the equations

$$\begin{cases} F \cdot \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = 0 \\ F \cdot \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = 0 \end{cases} \tag{3.5}$$

PROPOSITION 3.4 – *Let E be a 't Hooft bundle.*

For every $\mathbb{P}_F^{2n+1-2j}$ as in (3.5) we have $E|_{\mathbb{P}_F^{2n+1-2j}} \simeq E' \oplus O^{2j}$ where E' is a 't Hooft bundle over $\mathbb{P}^{2n+1-2j}$.

Proof – Straightforward

REMARK – The family of $\mathbb{P}_F^{2n+1-2j}$ considered in the prop. 3.4 is parametrized by the Grassmannian $\mathrm{Gr}(\mathbb{P}^j, \mathbb{P}^{n-1})$.

COROLLARY 3.5 – *Every 't Hooft bundle is trivial on the generic line.*

THEOREM 3.6 – *Every 't Hooft bundle is stable in the sense of Mumford.*

Proof – It follows word by word by the theorem 3.7 of [AO2], using the prop. 3.4 at the place of the theor. 3.1 of [AO2].

THEOREM 3.7 – *Let E be a generic 't Hooft bundle over $\mathbb{P}^{2n+1}$ with $c_2 = k \geq 3$. Then $h^0(E(1)) = n$.*

Proof – For $n = 1$ the result is well known. For simplicity we give all the details in the case $n = 2$, the same pattern works for $n \geq 3$.

Let $(z_0, z_1, z_2, w_0, w_1, w_2)$ be a system of homogeneous coordinates over $\mathbb{P}^5$.

We consider a 't Hooft bundle (as in the proof of the prop. 3.3) corresponding to the following simple form:

$$A = z_0(d_0 \mid 0 \mid 0 \mid 0) + z_1(d_1 \mid (a^1, 0) \mid 0 \mid 0) + z_2(d_2 \mid (0, a^2) \mid 0 \mid 0) +$$

$$+ w_0(0 \mid 0 \mid \delta_0 \mid 0) + w_1(0 \mid 0 \mid \delta_1 \mid (a^1, 0)) + w_2(0 \mid 0 \mid \delta_2 \mid (0, a^2))$$

where d_i, δ_j are $k \times k$ diagonal matrices and a^i are $k \times 1$ column vectors.

Let b be an unknown $(4 + 2k) \times 1$column vector representing an element of $H^0(E(1))$ (as a column of C in (3.3)) such that $AJb = 0$.

Let us denote

$$b^t = (b_0^1, b_0^2, b_0^3, b_0^4)z_0 + (b_1^1, b_1^2, b_1^3, b_1^4)z_1 + (b_2^1, b_2^2, b_2^3, b_2^4)z_2 +$$

$$+ (c_0^1, c_0^2, c_0^3, c_0^4)w_0 + (c_1^1, c_1^2, c_1^3, c_1^4)w_1 + (c_2^1, c_2^2, c_2^3, c_2^4)w_2$$

where, as above, the 4 blocks have size respectively $k, 2, k, 2$.

The condition $AJb = 0$ divides into 21 blocks, each one consisting of k equations. Each block correspond to a quadratic monomial over $\mathbb{P}^5$.

We consider first the 6 blocks corresponding to the 6 monomials $z_i z_j$ $i, j = 0, \ldots, 2$.

This system of 6k equations in the $3k+6$ scalar unknowns $b_0^3, b_0^4, b_1^3, b_1^4, b_2^3, b_2^4$ can be explicitly solved and the only solution is the zero one.

In a more elegant way, consider that the kernel bundle

$$0 \longrightarrow K \longrightarrow \mathcal{O}^{4+2k} \xrightarrow{A} \mathcal{O}(1)^k \longrightarrow 0$$

restricts on the $\mathbb{P}^2$ where $w_j = 0$ to:

$$K|_{\mathbb{P}^2} \simeq F \oplus \mathcal{O}^{k+2}$$

where F is a 2-bundle over $\mathbb{P}^2$ such that

$$0 \longrightarrow F \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{k+2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^k \longrightarrow 0$$

The claim is equivalent to the statement that $H^0(F(1)) = 0$. This vanishing follows easily by the isomorphism

$$F(1) \simeq F^\vee(-k+1)$$

For $n \geq 3$ this argument can be repeated by using the theorem 2.2 of [AO2].

In the same way we consider the 6 blocks of equations corresponding to the 6 monomials $w_i w_j$ $i, j = 0, \ldots, 2$.

It is then sufficient to show that the system of $9k$ equations corresponding to the monomials $z_i w_j$ has $k+2$ solutions in the $6k+12$ scalar unknowns

$$b_0^1, b_0^2, b_1^1, b_1^2, b_2^1, b_2^2, c_0^3, c_0^4, c_1^3, c_1^4, c_2^3, c_2^4$$

let us denote

$$b_0^2 = (\beta_{01}, \beta_{02}) \qquad c_0^4 = (\gamma_{01}, \gamma_{02})$$

$$b_2^1 = (\beta_{11}, \beta_{12}) \qquad c_1^4 = (\gamma_{11}, \gamma_{12})$$

$$b_2^2 = (\beta_{21}, \beta_{22}) \qquad c_2^4 = (\gamma_{21}, \gamma_{22})$$

so that the system of these $9k$ equations (9 matrix equations) is:

$$d_0c_0^3 - \delta_0 b_0^1 = 0$$

$$d_0c_1^3 - \delta_1 b_0^1 - a^1\beta_{01} = 0$$

$$d_0c_2^3 - \delta_2 b_0^1 - a^2\beta_{02} = 0$$

$$d_1c_0^3 - \delta_0 b_1^1 - a^1\gamma_{01} = 0$$

$$d_1c_1^3 - \delta_1 b_1^1 - a^1(\gamma_{11} - \beta_{11}) = 0$$

$$d_1c_2^3 - \delta_2 b_1^1 - a^1\gamma_{21} - a^2\beta_{12} = 0$$

$$d_2c_0^3 - \delta_0 b_2^1 + a^2\gamma_{02} = 0$$

$$d_2c_1^3 - \delta_1 b_2^1 + a^2\gamma_{12} - a^1\beta_{21} = 0$$

$$d_2c_2^3 - \delta_2 b_2^1 + a^2(\gamma_{22} - \beta_{22}) = 0$$

We perform a Gaussian elimination. More precisely we can

eliminate	b_0^1	by	2 and 3	(adding	multiples of	1)
”	c_0^3	”	3	”	”	2
”	b_1^1	”	5 and 6	”	”	4
”	c_0^3	”	6	”	”	5
”	b_2^1	”	8 and 9	”	”	7
”	c_0^3	”	9	”	”	8
”	c_0^3	”	5 and 8	”	”	2
”	c_2^3	”	6 and 9	”	”	3

We get the system

$$d_0c_0^3 - \delta_0 b_0^1 = 0$$

$$d_0c_1^3 - a^1\beta_{01} - \delta_1\delta_0^{-1}d_0c_0^3 = 0$$

$$d_0c_2^3 - a^2\beta_{02} - \delta_2\delta_1^{-1}d_0c_1^3 + \delta_2\delta_1^{-1}a^1\beta_{01} = 0$$

$$-\delta_0 b_1^1 + d_1c_0^3 + a^1\gamma_{01} = 0$$

$$a^1(\gamma_{11} - \beta_{11}) - \delta_1\delta_0^{-1}a^1\gamma_{01} + d_1 d_0^{-1}a^1\beta_{01} = 0$$

$$a^1\gamma_{21} - a^2\beta_{12} - \delta_2\delta_1^{-1}a^1(\gamma_{11} - \beta_{11}) + d_1 d_0^{-1}a^2\beta_{02}$$
$$-d_1 d_0^{-1}\delta_2\delta_1^{-1}a^1\beta_{01} = 0$$

$$d_2 c_0^3 - \delta_0 b_2^1 + a^2\gamma_{02} = 0$$

$$a^2\gamma_{12} - a^1\beta_{21} - \delta_1\delta_0^{-1}a^2\gamma_{02} + d_2 d_0^{-1}a^1\beta_{01} = 0$$

$$a^2(\gamma_{22} - \beta_{22}) - \delta_2\delta_1^{-1}a^2\gamma_{12} + \delta_2\delta_1^{-1}a^1\beta_{21} + d_2 d_0^{-1}a^2\beta_{02}$$
$$-d_2 d_0^{-1}\delta_2\delta_1^{-1}a^1\beta_{01} = 0$$

Now the matrix equations n. 1, 2, 3, 4, 7 are a system of $5k$ equations of rank $5k$. The other $4k$ equations in the 10 remaining unknowns $\beta_{01}, \beta_{02}, \beta_{11} - \gamma_{11}, \beta_{12}, \beta_{21}, \beta_{22} - \gamma_{22}, \gamma_{01}, \gamma_{02}, \gamma_{12}, \gamma_{21}$ have rank 10 (here we need $k \geq 3$).

So the rank of the system is $5k + 10$ as we wanted.

This ends the proof of the theorem 3.7.

REMARK – From the theorem 3.7 it follows that generic 't Hooft bundles have *not* natural cohomology

THEOREM 3.8 – *'t Hooft bundles depend on* $5kn + 4n^2$ *parameters for* $k \geq 3$.

Proof – We sketch two different arguments.

By the prop. 3.3 and the theorem 3.7 the degeneracy locus of the evaluation map

$$\mathcal{O} \otimes H^0(E(1)) \longrightarrow E(1)$$

is the union Z of $k + n$ linear subspaces of codimension 2. The form of the matrix A in (3.2) shows that, for a fixed Z, there are kn parameters more. So the number of parameters is

$$(k + n) \cdot \dim \mathrm{Gr}\,(\mathbb{P}^{2n-1}, \mathbb{P}^{2n+1}) + kn = 5kn + 4n^2$$

A second argument is the following: we can arrange the first row of matrix a in (3.2) to be $(1,\dots,1)$. Then A depends on

$$(2k+2n)\ (2n+2)+(k-1)n$$

parameters.

Now $GL(k)\times Sp(n+k)$ acts over A by

$$(g,s),A\mapsto gAs$$

It can be shown that the isotropy subgroups of this action is given by diagonal matrices g and symplectic matrices s consisting of 4 diagonal blocks.

Subtracting to the above expression the dimension $4k+3n$ of the isotropy subgroups we get $5kn+4n^2$.

REMARK – Moreover a stronger statement is true, that is $h^1\ (S^2E)=5kn+4n^2$ for a generic 't Hooft bundle E. This result will appear in [AO3] and will be sketched in the section 4. From the explicit description of the elements of $H^1(S^2E)$ it follows that every small deformation of a generic 't Hooft bundle is again a 't Hooft bundle, confirming the theorem 3.7.

In [S-T] it was defined the class of special symplectic instanton bundles.

They are invariant for a particular action of $SL(2)$ over $\mathbb{P}^{2n+1}$. By using this $SL(2)$-action it was computed in [O-T] for a special symplectic instanton bundle E:

$$h^1(\operatorname{End}E)=4(3n-1)k+(2n-5)(2n-1)$$

Moreover the following is true [D]:

$$h^1(S^2E)=k(10n-2)+(4n^2-10n+3)$$

These results give another proof of the fact that $MI_{\mathbb{P}^{2n+1}}(k)$ (resp. $MIS_{\mathbb{P}^{2n+1}}(k)$) is singular at a special symplectic instanton bundle for $n\geq 2$, $k\gg 0$ [M-O].

One reason for a singularity to occur is now clear: two irreducible components meet at the points corresponding to special symplectic instanton bundles. A surprising fact is that the component of symplectic 't Hooft bundles has dimension bigger than one of the components of $U(n)$-bundles. The details will appear in [AO3].

4. The twistor fibration and the reality condition

Let A, B matrices over $\mathbb{H} = \mathbb{C} + j\mathbb{C}$. We emphasize that in general $(AB)^t$ is different from $(B^t A^t)$. Anyway the equality $\overline{\alpha \cdot \beta} = \bar{\beta} \cdot \bar{\alpha} \quad \forall \alpha, \beta \in \mathbb{H}$ implies that

$$\overline{A \cdot B}^t = \bar{B}^t \cdot \bar{A}^t .$$

Moreover $\alpha j = j\bar{\alpha} \quad \forall \alpha \in \mathbb{H}$.

We can define the twistor fibration

$$\mathbb{P}^{2n+1}(\mathbb{C}) \xrightarrow{\pi} \mathbb{P}^n(\mathbb{H})$$

$$(z_0, \ldots, z_{2n+1}) \mapsto (q_0, \ldots, q_n)$$

by the formulas $q_i = z_i + j z_{n+1+i}$. Salamon in [Sal] defines a $Sp(n)$-connection on a complex $2n$-bundle over $\mathbb{P}^n(\mathbb{H})$ starting by $n+1$ quaternionic $k \times n$ matrices $A_0, \ldots, A_n$ satisfying the following two conditions:

$$\text{i)} \ \sum_{i=0}^{n} A_i q_i \text{ is invertible } \forall (q_0, \ldots, q_n) \in \mathbb{P}^n(\mathbb{H})$$

$$\text{ii)} \ A_i \bar{A}_j^t \text{ is symmetric } \forall i, j \tag{4.1}$$

In fact $\sum_{i=0}^{n} A_i q_i$ define a bundle map

$$f : H^{\oplus k} \longrightarrow \mathbb{H}^{\oplus n+k}$$

where H is the tautological line bundle over $\mathbb{P}^n(\mathbb{H})$. Let $A_i = \alpha_i + j\beta_i$. Over $\mathbb{H}^{\oplus n+k}$ there is a hermitian metric. We have the splitting

$\mathbb{H}^{n+k} = E' \otimes f(H^{\oplus k})$ and we denote by $p : \mathbb{H}^{n+k} \longrightarrow E'$ the orthogonal projection. Then $p \circ d$ is a $\boldsymbol{Sp}(n)$-connection over E'. Moreover $\pi^* E' = E$ where E is a symplectic instanton bundle defined by the matrix

$$A = \sum_{i=o}^{n} z_i(\alpha_i \mid \bar{\beta}_i) + z_{n+1+i}(-\beta_i \mid \bar{\alpha}_i) \tag{4.2}$$

The link between (4.1) and the monad condition can be explicitly seen in our case by the following lemmas:

LEMMA 4.1 – *Let $A_i = \alpha_i + j\beta_i$ for $i = 0, \ldots, n$ be the $k \times n$ quaternionic matrices. Define A as in (4.2).*

Then the following properties are equivalent

i) $AJA^t = 0$

ii) $A_i \bar{A}_j^t$ *is symmetric* $\forall\, i, j$

iii) $(\sum q_i A_i)\ (\overline{\sum q_i A_i})^t$ *is real* $\forall\, q \in \mathbb{P}^n(\mathbb{H})$

Proof – Straightforward.

LEMMA 4.2 – *With the notations as in the lemma 4.1, the following properties are equivalent:*

i) A_i *satisfy the two conditions (4.1)*

ii) $(\sum q_i A_i)(\overline{\sum q_i Ai})^t \in GL(k, \mathbb{R}) \quad \forall\, q \in \mathbb{H}^{\oplus n+1} \backslash \{0\}$

iii) A *define a bundle in the monad (1.0).*

Proof – Straightforward.

Let $\sigma : H^0(M \otimes N \otimes \mathcal{O}(1)) \longrightarrow H^0(M \otimes N \otimes \mathcal{O}(1))$

$$\sum_{i=0}^{n} z_i(\alpha_i \mid \beta_i) + z_{n+1+i}(\gamma_i \mid \delta_i) \mapsto \sum_{i=0}^{n} z_i(\bar{\delta}_i \mid -\bar{\gamma}_i) + z_{n+1+i}(-\bar{\beta}_i \mid \bar{\alpha}_i)$$

PROPOSITION 4.3 –

i) $\sigma^2 = id$

ii) $\sigma(G \cdot A) = \bar{G}\sigma(A) \quad \forall G \in GL(k)$

iii) $\sigma(A \cdot L) = \sigma(A)c(L) \quad \forall L \in \mathrm{Sp}(n+k)$ *where*

$$c : \mathrm{Sp}(n+k) \longrightarrow \mathrm{Sp}(n+k)$$

is the involution such that $dc(l) = -\bar{l}^t$

iv) $\sigma(Q) \subset Q$ *(see the remark 1.5)*

v) A *has rank* k $\forall z \in \mathbb{P}^{2n+1}(\mathbb{C}) \Longleftrightarrow \sigma(A)$ *has rank* k $\forall z \in \mathbb{P}^{2n+1}(\mathbb{C})$

vi) $A = \sigma(A)$ *if and only if* A *has the form in (4.2)*

vii) *If* $A = \sigma(A)$ *then* A *defines an instanton trivial on the real lines.*

Proof – The properties from (i) to (vi) are straightforward. In order to prove (vii) consider that

$$P_1 = (z_0, \ldots, z_n, \ z_{n+1}, \ldots, z_{2n+1})$$

$$P_2 = (-\bar{z}_{n+1}, \ldots, -\bar{z}_{2n+1}, \ \bar{z}_0, \ldots, \bar{z}_n)$$

are always two distinct points on the π-fiber of $(z_0 + jz_{n+1}, \ldots, z_n + jz_{2n+1})$. In fact $(z_i + jz_{n+1+i}) \cdot j = -\bar{z}_{n+1+i} + j\bar{z}_i$.

Hence from $A = \sigma(A)$ it follows

$$A(P_1)J = \overline{A(P_2)}$$

and $A(P_1)JA(P_2)^t = \overline{A(P_2)}A(P_2)^t$ which is hermitian positive definite.

By the lemma 1.8 of [OS] this implies the result.

The prop. 4.3 shows that σ is an involution on $MIS_{\mathbb{P}^{2n+1}}(k)$ and the fixed points of this involution correspond to real instanton bundles which come from the Salamon construction.

Description of real 't Hooft bundles

The 't Hooft bundles which satisfy the condition $\sigma(A) = A$ (and then come from $\boldsymbol{Sp}(n)$ Yang-Mills connections over $\mathbb{P}^n(\mathbb{H})$ can be described in the following way (in a convenient system of coordinates):

$$A = \Big[\sum_{i=0}^{n} z_i\, D(b_i) + z_{n+1+i} \cdot D(c_i) \mid a \cdot D(z_i) \mid -\sum_{i=0}^{n} z_i\, D(\bar{c}_i) + \\ + z_{n+1+i} \cdot D(\bar{b}_i) \mid a \cdot D(z_{n+1+i})\Big] \tag{4.3}$$

where $c_i, b_j \in \mathbb{C},\ a_{ij} \in \mathbb{R}$.

We call the instanton bundles defined by A as in (4.3) the real 't Hooft bundles.

In quaternionic form we have $\sum A_i q_i$ with

$$A_i = \Big[D(b_i) - j\, D(c_i) \mid (0, \ldots, a^i, \ldots 0)\Big]$$

where a^i is a $k \times 1$ real column vector.

The proofs of the following theorems will appear in [AO3].

THEOREM 4.4 – *For a generic 't Hooft bundle E we have*

$$h^1(S^2E) = 5kn + 4n^2$$

Moreover $\ker(\overset{2}{\Lambda} A)$ *(see prop. 1.4) is generated by the following elements:*

α) $A \cdot L \quad \forall L \in \mathrm{Sp}(n+k)$

β) $G \cdot A \quad \forall G \in GL(K)$

γ) $M \cdot C^t \cdot J$ *where C is in (3.3) and M is a $k \times (k+n)$ matrix*

δ) $\sum_{j=0}^{2n+1} z_j F_j$ *where* $F_j = \Big[D(e_i) \mid a \cdot D(f_i) \mid D(g_i) \mid a \cdot D(h_i)\Big]$ $e_i, f_i, g_i, h_i,\ h_p \in \mathbb{C}$, *$a$ is a $(k \times n)$ matrix.*

REMARK – The elements in α) and in β) have an obvious meaning. The elements inγ) correspond to the image of the map

$$H^0(E(1)) \otimes H^1(E(-1) \longrightarrow H^1(S^2E).$$

The elements in δ) correspond to the image of the map

$$H^1(S^2E(-1)) \otimes V \longrightarrow H^1(S^2E).$$

THEOREM 4.5 – *For a generic real 't Hooft bundle the real kernel of* $\overset{2}{\Lambda} A$ *is generated by*

α) as in 4.4 with $L = -\bar{L}^t$

β) as in 4.4 with G real

γ) as in 4.4 with M real

δ) as in 4.4 with $\sigma(F_j) = F_j$ *(with obvious notations)*

The real dimension of this kernel is $5kn + 4n^2$, *equal to the number of real parameters for real 't Hooft bundles.*

The following result is simple but important.

THEOREM 4.6 – *The Kuranishi map vanishes for 't Hooft bundles.*

Proof – Here we take the theorem 4.4 for granted.
We have a case by case analysis:
We call Φ the bilinear form associated to the Kuranishi map.

$\alpha - \alpha$) $\Phi(AL_1, AL_2) = AL_1J(L_2^tA^t) + AL_2J(L_1^tA^t) =$
$= -AL_1L_2JA^t - AJL_2^tL_1^tA^t \in \mathrm{Im}(\overset{2}{\Lambda} A)$

$\alpha - \beta$) $\Phi(AL, GA) = -AJL^tA^tG^t - GALJA^t \in \mathrm{Im}(\overset{2}{\Lambda} A)$

$\beta - \beta$) $\Phi(G_1A, G_2A) = 0$

$\alpha - \gamma$) $\Phi(AL, MC^tJ) = -AJL^tJ^tCM^t - MC^tJLJA^t \in \mathrm{Im}(\overset{2}{\Lambda} A)$

$\beta-\gamma)$ $\Phi(GA, MC^tJ)=0$

$\gamma-\gamma)$ $\Phi(M_1C^tJ, M_2C^tJ)=0$ (by using (3.4))

$\alpha-\delta)$ $\Phi(AL, F_j) = -AJL^tF_j^t - F_jLJA^t \in \mathrm{Im}(\overset{2}{\Lambda} A)$

$\beta-\delta)$ $\Phi(GA, F_j) = GAJF_j^t + F_jJA^tG^t = -GF_jJA^t - AJF_j^tG^t \in \mathrm{Im}(\overset{2}{\Lambda} A)$ (because AJF_j^t is symmetric)

$\gamma-\delta)$ $\Phi(MC^tJ, F_j) = -MC^tF_j^t + F_jCM^t = (C^tF_j^t = C'JA^t$ for some $C') = -MC'JA^t - AJC'^tM^t \in \mathrm{Im}(\overset{2}{\Lambda} A)$

$\delta-\delta)$ $\Phi(F_j, F_k) = 0$

Appendix

We list two Macaulay scripts ([BS]).
The first one computes the dimension of $\ker(\overset{2}{\Lambda} A)$ and prints the dimensions $h^1(S^2E), h^2(S^2E)$ for a given symplectic instanton.
The second one produces a generic 't Hooft bundle.

```
;SCRIPT h1h2s2
;USAGE: h1h2s2 a
incr-set prlevel 1
if #0=1 start
incr-set prlevel -1
;h1h2s2 A
;Calcola le dimensioni di H1(S^ 2(E)) 3 H2(S^ 2(E)) per E dato
;dalla matrice A e le scrive su video.
incr-set prlevel 1
jump end

start:
nrows #1 nr@
poly one@ 1
nvars R nv@
int nn@ (nv@-2)/2
```

```
tensor #1 #1 tcc@
submat tcc@ tcc1@
1..nr@* (2*nn@+2*nr@)
submat tcc@ tcc2@
nr@*(2*nn@+2*nr@)+1..2*nr@*(2*nn@+2*nr@)
subtract tcc1@ tcc2@ tcc@
set autocalc 1
set autodegree 2
syz tcc@ -1 s1tcc@
ncols s1tcc@ ncs1tcc@
set autodegree 1
syz tcc@ -1 s0tcc@
ncols s0tcc@ ncs0tcc@
int syz1@ (nv@-1) *ncs0tcc@+ncs1tcc@
int h2@ syz1@-nr@*(2*nn@+2*nr@) * (2*nn@+2) +nr@* (nr@-1) *
(2*nn@+3) * (nn@+1)/2
int h1@ syz1@-nr@*nr@-(nn@+nr@) * (2*nn@+2*nr@+1)
shout type h1@
shout type h2@
set autocalc -1
end:
incr-set prlevel -1


;SCRIPT RANDSIM
;USAGE: RANDSIM k n a

incr-set prlevel 1
if #0=3 start
incr-set prlevel -1
;randsim k n a
;definisce un istantone simplettico generico
;con c2=k su P^ (2n+1) e lo chiama a
;tale che a.j.a^ t=0
incr-set prlevel 1
```

```
jump end
start:
ring S@
17
1
a@

random #1 #2*2+2 r1@
random #1 #2*2+2 r2@
random #1 #2 r3@
<ring 2*#2+2 x[0]-x[2*#2+1] R@
fetch r1@ r1@
fetch r2@ r2@
fetch r3@ r3@
<getvars rv@
transpose rv@ rvt@
mult r1@ rvt@ a1@
mult r2@ rvt@ a2@
submat a1@ f1@
1

submat a2@ f2@
1

if #1=1 saltaloop1
int i@ 2
loop1:
submat a1@ b1@
i@

submat a2@ b2@
i@

dsum f1@ b1@ f1@
dsum f2@ b2@ f2@
int i@ i@+1
if i@<#1+1 loop1
```

```
saltaloop1:
submat rvt@ g1@
1

submat rvt@ g2@
#2+2

if #2=1 saltaloop2
int i@ 2
loop2:
submat rvt@ b1@
i@

submat rvt@ b2@
#2+1+i@

dsum g1@ b1@ g1@
dsum g2@ b2@ g2@
int i@ i@+1
int i@<#2+1 loop 2
saltaloop2:
mult r3@ g1@ g1@
mult r3@ g2@ g2@
concat f1@ g1@ f2@ g2@
copy f1@ #3
end:
incr-set prlevel -1
```

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Pervenuta in Redazione il 25 luglio 1997