J. reine angew. Math. **409** (1990), 35—40

Journal für die reine und angewandte Mathematik

A linear bound on the t-normality of codimension two subvarieties of $\mathbb{P}^n$

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§ 1. Introduction

Let X be a nondegenerate (i.e. not contained in a hyperplane), codimension two, smooth subvariety of $\mathbb{P}^n(\mathbb{C})$, $n \geqq 6$. The well known Hartshorne conjecture says that X is a complete intersection. To prove the conjecture is equivalent to prove the statement that X is t-normal $\forall t \geqq 1$; i.e. the natural maps $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(X, \mathcal{O}_X(t))$ are surjective $\forall t \geqq 1$, ([E-G], th. 2. 4).

If $t=1$ the theorem of Zak on linear normality (see [Z]), says that X is 1-normal for $n \geqq 5$ (and this is the best possible value). If $t \geqq 2$ the best global result we know about t-normality is Ran's inequality: X is t-normal if $n \geqq 3t^2 + 2t + 2$ (see [R]); it has been improved only for little values of t (see for instance: [E], [P-P-S], [A-O; 1], [A-O; 2]).

Now the recent work of Ein (see [E]) about this subject, and the techniques we have developed in [A-O; 2], are sufficient to prove the following:

Theorem 1. 1. *Let X be a non degenerate, codimension two, smooth subvariety of $\mathbb{P}^n(\mathbb{C})$, $n \geqq 6$, then*:

i) *X is t-normal if* $n \geqq 6t - 2$ $(t \geqq 2)$.

ii) $H^r(X, \mathcal{O}_X(t)) = 0$ *if* $r \geqq 1$ *and* $n \geqq 6t + r$ $(t \geqq 1)$.

Remark 1. 2. For $t=2$ i) is proved in [E], in [A-O; 1] and (essentially) in [P-P-S]; for $t=3$ and $t=4$ it is also proved in [A-O; 2] by different methods. For $t=1$ and $t=2$ ii) is proved also in [E].

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§ 2. Notations and preliminaries

$\mathbb{P}^n$: n-dimensional projective space on $\mathbb{C}$.

Variety (subvariety): by this term we mean a *smooth* projective variety (subvariety) on complex numbers.

$\mathcal{O}_W$: structural sheaf of the variety W.

I_W: $\mathcal{O}_{\mathbb{P}^n}$-ideal sheaf of the variety W.

$N_{U|W}$: normal bundle of the subvariety U in W.

$I_{U|W}$: $\mathcal{O}_W$-ideal sheaf of the subvariety U in W.

$F(s)=F\otimes\mathcal{O}_{\mathbb{P}^n}(s)$ where F is a coherent sheaf and s is an integer.

E^*: dual bundle of the vector bundle E.

$E_{/U}$: vector bundle E restricted to the variety U.

K.v.t.: Kodaira vanishing theorem.

Proposition 2. 1 (see [H-S], th. 5. 1). *Let X be a 2-codimensional projective variety in $\mathbb{P}^n$, $n\geqq 6$; suppose that X is not a complete intersection, then*:

$$\det(N_{X|\mathbb{P}^n})=\mathcal{O}_X(a) \quad \textit{with} \quad a\geqq 2n+3.$$

Now we recall some results from [A-O; 1] and [E]. From now on $X=X_1$ will be a 2-codimensional subvariety of $\mathbb{P}^n$. There is a chain of varieties:

$$\mathbb{P}^n=X_0\supset X_1\supset X_2\supset X_3\supset \dots \supset X_k\supset \dots$$

such that $\dim(X_k)=n-2k$ $\forall k\geqq 0$ and such that X_k is the smooth zero-locus of a suitable section of $N_{X_{k-1}|X_{k-2}}(-1)$ $\forall k\geqq 2$. In fact we can proceed as follows: we fix X_1 in $\mathbb{P}^n$ and we choose a generic point $P\notin X_1$; we define X_2 as the locus of points of X_1 such that the $(n-2)$-linear tangent space at them, passes through P.

We define X_3 as the locus of points of X_2 such that the $(n-4)$-linear tangent space at them, passes through P; and so on.

Remark that it is always possible to choose a point P such that every X_k is smooth, of dimension $n-2k$, because if we consider the projection Φ of X_1 from P into a generic hyperplane of $\mathbb{P}^n$, we have that the X_k $(k\geqq 2)$ are Thom-Boardman varieties associated to the map Φ between the complex manifolds X_1 and $\mathbb{P}^{n-1}$, (see [Bo], p. 22, 24, 32; and [A-O; 2], § 3); and in this case, by a result of Mather (see [M; 1], [M; 2], th. 6 and p. 229, 244) there exists an open Zarisky set U in $\mathbb{P}^n$ such that if $P\notin U$ the X_k have the expected properties.

In this way it is easy to see that X_k is the smooth zero-locus of a suitable section of $N_{X_{k-1}|X_{k-2}}(-1)$ $\forall k \geqq 2$ (see [A-O; 2], § 3).

Now we put:

$$\mathcal{O}_k = \mathcal{O}_{X_k}, \ \forall k \geqq 0; \quad \mathcal{I}_k = I_{X_k|X_{k-1}}, \ N_k = N_{X_k|\mathbb{P}^n}, \ \mathcal{N}_k = N_{X_k|X_{k-1}}, \ \forall k \geqq 1.$$

We suppose that X_1 is not a complete intersection, so that we can use proposition 2.1; by this assumption, by obvious facts, by the properties of $\mathcal{N}_k(-1)$ and the definition of X_k, we can say that the following statements are true:

(2.2) There exist:

$$0 \to N^*_{k-1/X_k} \to N^*_k \to \mathcal{N}^*_k \to 0 \quad \forall k \geqq 2,$$

$$0 \to \det[\mathcal{N}^*_k(1)] \to \mathcal{N}^*_k(1) \to \mathcal{I}_{k+1} \to 0 \quad \forall k \geqq 1,$$

$$0 \to \mathcal{I}_k \to \mathcal{O}_{k-1} \to \mathcal{O}_k \to 0 \quad \forall k \geqq 1.$$

(2.3) $\mathcal{N}_k \approx \mathcal{N}_{k-1}(-1)_{/X_k} \quad \forall k \geqq 2.$

(2.4) By Barth theorem, (see [Ba]), $H^r(X_k, \mathcal{O}_k) = 0$ if $n \geqq 4k + r$ and $\mathrm{Pic}(X_k) \approx \mathbb{Z}$ if $n \geqq 4k + 2$, hence $\det(\mathcal{N}_k) = \mathcal{O}_k(a - 2k + 2)$ and $\det[\mathcal{N}_k(b)](c)$ is ample if

$$2n + 4 + 2b + c \geqq 2k, \quad k \geqq 1.$$

Now we apply to X_k, $k \geqq 1$, the results of Ein (see [E], th. 2.4; see also [P-P-S]); we have the following

Proposition 2.5. *Suppose $n \geqq 6k$, $k \geqq 1$, then:*

a) $H^r(X_k, N^*_k(1)) = 0$ *if* $n \geqq 6k + r - 1$,

b) $H^r(X_k, \mathcal{O}_k(1)) = 0$ *if* $n \geqq 6k + r$,

c) X_k *is t-normal if* $H^1(X_k, N^*_k(j)) = 0 \quad \forall j = 1, 2, \ldots, t$,

d) $H^{r-1}(X_k, \mathcal{O}_k(t)) = 0$ *if* $n \geqq 4k + r \quad r \geqq 2$ *and*

$H^r(X_k, N^*_k(j)) = 0 \quad \forall j = 1, 2, \ldots, t.$

Remark 2.6. By using Zak's classification of Severi varieties (see [L-VV]), it is easy to see that X_k, $k \geqq 1$, is 1-normal when $n \geqq \max\{6k - 2, 5\}$ and that propositions 2.5.c) and 2.5.d) are true even when $n \geqq 6k - 2$ (see [E], th. 2.4).

§ 3. Proof of theorem 1. 1

We get the proof of 1. 1 by considering the case $k=1$ in the following:

Theorem 3. 1. *Let* X_k *be a variety as in* § 2, $k \geqq 1$, *then*:

a) $H^q(X_k, N_k^*(t))=0$ *if* $t \geqq 1$, $q \geqq 2$, $n \geqq 6(t+k-1)+q-1$ *and*

$H^q(X_k, \mathcal{O}_k(t))=0$ *if* $t \geqq 1$, $q \geqq 1$, $n \geqq 6(t+k-1)+q$.

b) X_k *is t-normal* *if* $t \geqq 1$, $n \geqq \max\{6(t+k-1)-2, 5\}$.

Proof. We may suppose that X_k, $k \geqq 1$, is not a complete intersection. We remark that, in our assumptions, X_k is always a positive dimension variety.

a) We fix n and q and we proceed by descending induction on k. We choose $k_0 = \max\{k \mid n \geqq 6k+q-1\} = \left[\dfrac{n-q+1}{6}\right]$; for $k=k_0$.

3. 1.a) is true because in this case $t=1$ and we can use propositions 2. 5.a) and 2. 5.b). Now we suppose 3. 1.a) true for $k+1$ and we prove it for $k<k_0$. We use ordinary induction on t: for $t=1$ 3. 1.a) is true by 2. 5.a) and 2. 5.b); now we suppose 3. 1.a) true for $t-1$ and we prove it for $t>1$.

In our assumptions, by 2. 5.d), it suffices to show that $H^q(X_k, N_k^*(t))=0$ for $q \geqq 2$.

By using the first exact sequence of (2. 2) as many times as we need, we have only to show that $H^q(X_k, \mathcal{N}_{k-s/X_k}^*(t))=0$ $\forall s=0, 1, \dots, k-1$. By (2. 3) it is equivalent to show that

$$H^q(X_k, \mathcal{N}_k^*(t-s))=0 \quad \forall s=0, 1, \dots, k-1 \text{ (recall that } N_1=\mathcal{N}_1).$$

Now we use the second exact sequence of (2. 2) and we have:

$$0 \to \det[\mathcal{N}_k^*(1)]\,(t-s-1) \to \mathcal{N}_k^*(t-s) \to \mathcal{I}_{k+1}(t-s-1) \to 0.$$

$H^q(X_k, \det[\mathcal{N}_k^*(1)]\,(t-s-1))=0$ by using (2. 4) and K.v.t.

Now we use:

$$0 \to \mathcal{I}_{k+1}(t-s-1) \to \mathcal{O}_k(t-s-1) \to \mathcal{O}_{k+1}(t-s-1) \to 0.$$

If $t-s-1 \leqq -1$ we use K.v.t.; if $t-s-1=0$ we use Barth theorem; if $t-s-1 \geqq 1$ we use induction.

b) We fix n and we use descending induction on k again: we choose $k_0 = \max\{k \mid n \geqq 6k-2\} = \left[\frac{n+2}{6}\right]$; for $k=k_0$ 3. 1.b) is true because in this case $t=1$ and we can use remark 2. 6. Now we suppose 3. 1.b) true for $k+1$ and we prove it for $k<k_0$. We use ordinary induction on t: for $t=1$ 3. 1.b) is true by 2. 6; now we suppose 3. 1.b) true for $t-1$ and we prove it for $t>1$.

In our assumptions, by 2. 5.c), it suffices to show that $H^1(X_k, N_k^*(t))=0$. We proceed as in the previous case until we reduce to show that

$$H^1(X_k, \mathscr{I}_{k+1}(t-s-1))=0 \qquad \forall s=0,1,\dots,k-1.$$

Now we use the following diagram:

$$\begin{array}{ccccccc}
\longrightarrow & H^0(X_k, \mathcal{O}_k(t-s-1)) & \longrightarrow & H^0(X_{k+1}, \mathcal{O}_{k+1}(t-s-1)) & \longrightarrow & H^1(X_k, \mathscr{I}_{k+1}(t-s-1)) \\
 & \uparrow & \nearrow & & & \downarrow \\
 & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t-s-1)) & & & & H^1(X_k, \mathcal{O}_k(t-s-1)). \\
 & & & & & \downarrow
\end{array}$$

If $t-s-1 \leqq -1$ we get our thesis by K.v.t. If $t-s-1=0$ we have only to show that $H^q(X_k, \mathcal{O}_k)=0$ and we use Barth theorem.

If $t-s-1 \geqq 1$ by 3. 1.a) we have that $H^1(X_k, \mathcal{O}_k(t-s-1))=0$; by induction X_{k+1} is $(t-s-1)$-normal, so $H^1(X_k, \mathscr{I}_{k+1}(t-s-1))=0$ by looking at the previous diagram.

Added in proof. A slight modification of the argument given here allows to show that $H^q(I_X(t))=0$ for $n \geqq q+4t+3$ and $1 \leqq q \leqq n-2$.

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Eingegangen 31. März 1989