TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 1, Pages 49–55
S 0002-9947(02)03126-4
Article electronically published on September 6, 2002

NONDEGENERATE MULTIDIMENSIONAL MATRICES AND INSTANTON BUNDLES

LAURA COSTA AND GIORGIO OTTAVIANI

Abstract. In this paper we prove that the moduli space of rank $2n$ symplectic instanton bundles on $\mathbb{P}^{2n+1}$, defined from the well-known monad condition, is affine. This result was not known even in the case $n = 1$, where by Atiyah, Drinfeld, Hitchin, and Manin in 1978 the real instanton bundles correspond to self-dual Yang Mills $Sp(1)$-connections over the 4-dimensional sphere. The result is proved as a consequence of the existence of an invariant of the multidimensional matrices representing the instanton bundles.

1. Introduction

A symplectic instanton bundle on $\mathbb{P}^{2n+1}_{\mathbb{C}}$ is a bundle of rank $2n$ defined as the cohomology bundle of a well-known monad (see Definition 2.2).

In [ADHM78] it was shown that instanton bundles on $\mathbb{P}^3$ satisfying a reality condition correspond to self-dual Yang Mills $Sp(1)$-connections over the 4-dimensional sphere $S^4 = \mathbb{P}^1_{\mathbb{H}}$. This correspondence was generalized by Salamon ([Sal84]) who showed that instanton bundles on $\mathbb{P}^{2n+1}$ which are trivial on the fiber of the twistor map $\mathbb{P}^{2n+1} \to \mathbb{P}^n_{\mathbb{H}}$ correspond to $Sp(n)$-connections which minimize a certain Yang Mills functional over $\mathbb{P}^n_{\mathbb{H}}$. We denote by $MI_{\mathbb{P}^{2n+1}}(k)$ the moduli space of symplectic instanton bundles on $\mathbb{P}^{2n+1}$ with $c_2 = k$ (see Definition 2.4) and we denote by $I_{\mathbb{P}^{2n+1}}(k)$ the moduli space of k-instanton bundles on $\mathbb{P}^{2n+1}$ (see Definition 4.1).

Up to now, very little is known concerning the geometry of the moduli spaces $I_{\mathbb{P}^{2n+1}}(k)$ and a few results have been proved regarding $MI_{\mathbb{P}^{2n+1}}(k)$. For instance, up to the authors' knowledge, the only results concerning $MI_{\mathbb{P}^{2n+1}}(k)$ deal with small values of n and k. Indeed, it is known ([ADHM78]) that $MI_{\mathbb{P}^{2n+1}}(k)$ has a component of dimension $8k - 3$ for $n = 1$, that it is smooth for $n = 1$ and $k \leq 5$ ([KO99]) but, it is conjectured that it is singular and reducible for $n \geq 2$ and $k \geq 4$ (see [AO00]).

The goal of this paper is to show that all the moduli spaces $MI_{\mathbb{P}^{2n+1}}(k)$, for any $n \geq 1$ and any $k \geq 1$, share the following surprising property:

Theorem 1.1. *$MI_{\mathbb{P}^{2n+1}}(k)$ is affine.*

In addition, we will see that the same holds for all moduli spaces parametrizing k-instanton bundles on $\mathbb{P}^{2n+1}$. Indeed, we will prove

Theorem 1.2. *$I_{\mathbb{P}^{2n+1}}(k)$ is affine.*

Received by the editors October 23, 2001.
2000 *Mathematics Subject Classification.* Primary 14D21, 14J60; Secondary 15A72.
The first author was partially supported by DGICYT BFM2001-3584.
The second author was partially supported by Italian MURST.

As a by-product of Theorems 1.1 and 1.2, we will contribute to the study of a problem posed in the 80's (see for instance [HH86]) that, in the context of instanton bundles on $\mathbb{P}^{2n+1}$, reads as follows:

Problem. Determine the maximal dimension of complete subvarieties lying on $MI_{\mathbb{P}^{2n+1}}(k)$ (resp. $I_{\mathbb{P}^{2n+1}}(k)$).

More precisely, in this case, we will completely solve the problem and in Corollaries 3.5 and 4.6 we will see that $MI_{\mathbb{P}^{2n+1}}(k)$ (resp. $I_{\mathbb{P}^{2n+1}}(k)$) does not contain any complete subvariety of positive dimension.

The technique we use to prove our main results is to exhibit $MI_{\mathbb{P}^{2n+1}}(k)$ (resp. $I_{\mathbb{P}^{2n+1}}(k)$) as the GIT-quotient of an affine variety $\mathcal{Q}^0$ (resp. $\mathcal{P}^0$) and then use standard results in invariant theory. The fact that $\mathcal{Q}^0$ (resp. $\mathcal{P}^0$) is affine is a consequence of the existence of an invariant of multidimensional matrices representing the instanton bundles, which generalizes the hyperdeterminant (see [GKZ94] and [AO99]).

The first named author would like to thank the Dipartimento di Matematica, U. Dini for their hospitality and support at the time of the preparation of this paper.

2. Notation and preliminaries

We will start by fixing some notation and recalling some facts about k-instanton bundles on $\mathbb{P}^{2n+1} = \mathbb{P}(V)$, where V is a complex vector space of dimension $2n+2$. (See, for instance, [OS86] and [AO94].)

Notation 2.1. $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^{2n+1}}(d)$ denotes the invertible sheaf of degree d on $\mathbb{P}^{2n+1}$ and for any coherent sheaf E on $\mathbb{P}^{2n+1}$ we denote $E(d) = E \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(d)$.

Definition 2.2. A symplectic instanton bundle E over $\mathbb{P}^{2n+1} = \mathbb{P}(V)$ is a bundle of rank $2n$ which appears as a cohomology bundle of a monad,

$$I^* \otimes \mathcal{O}(-1) \xrightarrow{A} W \otimes \mathcal{O} \xrightarrow{A^t} I \otimes \mathcal{O}(1), \tag{1}$$

where (W, J) is a symplectic complex vector space of dimension $2n+2k$ and I is a complex vector space of dimension k.

We do not assume in the definition that E is stable, so we have to recall some results.

The monad condition means that A is injective (as a bundle morphism), A^t is surjective and $\mathrm{im}A \subset \ker A^t$ so that $E \simeq \ker A^t/\mathrm{im}A$. The fact that the map $W \otimes \mathcal{O} \xrightarrow{A^t} I \otimes \mathcal{O}(1)$ is surjective, is equivalent to the fact that the matrix $A \in Hom(V^* \otimes I^*, W)$ representing E is nondegenerate according to [GKZ94] (see Definition 2.3 for the precise definition).

$Hom(V^* \otimes I^*, W)$ contains the subvariety $\mathcal{Q}$ given by matrices A for which the sequence (1) is a complex, that is, such that $A^t J A = 0$. $GL(I) \times Sp(W)$ acts on $\mathcal{Q}$ by $(g, s) \cdot A = sAg$.

Definition 2.3. A matrix $A \in Hom(V^* \otimes I^*, W)$ is called degenerate if the multilinear system $A(v \otimes i) = 0$ has a solution such that $0 \neq v \in V^*$ and $0 \neq i \in I^*$.

By [GKZ94], Theorem 14.3.1, this is equivalent to the standard definition of degeneracy given in chapter 14.1 of [GKZ94]. It is easy to check that degenerate matrices fill an irreducible subvariety N of $Hom(V^* \otimes I^*, W)$ of codimension k (see [WZ96]). Hence, only in the case $k = 1$ is it well-defined as a hyperdeterminant according to [GKZ94]. In the next section we will define an $SL(I) \times Sp(W)$-invariant on $Hom(V^* \otimes I^*, W)$, called D, which generalizes the hyperdeterminant and is suitable for our purposes.

It was shown in [AO94] that all instanton bundles are simple, so that they carry a unique symplectic form. Moreover, for $n = 1, 2$ it was proved in [AO94] that all instanton bundles are stable, and it is expected that the same result is true for $n \geq 3$.

Recall that given $X = Spec(A)$, an affine scheme, and a reductive group G acting on X, then a theorem of Hilbert and Nagata shows that the ring of invariants A^G is finitely generated and $X/G := Spec(A^G)$ is what is called the affine algebro-geometric quotient of X by G. In addition, X/G is a good quotient and it is a geometric quotient if and only if all orbits are closed. In this setting, every orbit contains a unique closed orbit in its closure and a point in X is called stable if its orbit is closed and has the maximal dimension (see [PV89]).

In [BH78] it was essentially proved that there is a natural one-to-one correspondence between

i) isomorphism classes of symplectic instanton bundles, and

ii) orbits of $GL(I) \times Sp(W)$ on the open subvariety $\mathcal{Q}^0$ of $\mathcal{Q}$ given by nondegenerate matrices.

In fact, using the quoted results of [AO94], one can see that [BH78], Section 4 and the Theorem on page 19, adapt literally to our situation.

Moreover, in Theorem 3.3 we will see that $\mathcal{Q}^0$ is affine. Hence, if we denote by G the quotient of $GL(I) \times Sp(W)$ by $\pm(id, id)$, Barth and Hulek proved in [BH78] that G acts freely on $\mathcal{Q}^0$ and, in particular, all orbits are closed (in fact, any orbit contains in the closure orbits of smaller dimension). Therefore, all points of $\mathcal{Q}^0$ are stable for the action of $GL(I) \times Sp(W)$ and $\mathcal{Q}^0 \to \mathcal{Q}^0/G$ is a geometric quotient.

Definition 2.4. The GIT-quotient $\mathcal{Q}^0/GL(I) \times Sp(W)$ is denoted by $MI_{\mathbb{P}^{2n+1}}(k)$ and is called the moduli space of symplectic k-instanton bundles on $\mathbb{P}^{2n+1}$. It is a geometric quotient.

The above discussion shows that $MI_{\mathbb{P}^{2n+1}}(k)$ coincides for $n = 1, 2$ with the open subset $\mathcal{MI}_{\mathbb{P}^{2n+1}}(k)$ of the Maruyama scheme of symplectic stable bundles on $\mathbb{P}^{2n+1}$ of rank $2n$ and Chern polynomial $\frac{1}{(1-t^2)^k}$ which are instanton bundles (this is an open condition because by Beilinson's theorem, it is equivalent to certain vanishing in cohomology; see [OS86]). In particular, our notation for $MI_{\mathbb{P}^3}(k)$ is consistent with the usual one. For $n \geq 3$ it is expected that the same result is true, but at present we can only say that $\mathcal{MI}_{\mathbb{P}^{2n+1}}(k)$ is an open subset of $MI_{\mathbb{P}^{2n+1}}(k)$.

3. The invariant D and the proof of the main result

First, we remark that the vector spaces $W \otimes S^n I$ and $V \otimes S^{n+1} I$ have the same dimension $(2n+2k)\binom{k+n-1}{n} = (2n+2)\binom{k+n}{n+1}$. We can construct from

$$W \xrightarrow{A^t} V \otimes I$$

the morphisms

$$A^t \otimes id_{S^n I} : W \otimes S^n I \to V \otimes I \otimes S^n I,$$
$$id_V \otimes \pi : V \otimes I \otimes S^n I \to V \otimes S^{n+1} I,$$

where π is the natural projection, and we consider the composition

$$\Delta_A = (id_V \otimes \pi) \cdot (A^t \otimes id_{S^n I}) \colon W \otimes S^n I \to V \otimes S^{n+1} I. \tag{2}$$

Definition 3.1. Let $A \in Hom(V^* \otimes I^*, W)$. We define $D(A)$ to be the usual determinant of the morphism Δ_A in (2) induced by A.

Notice that

$$D\colon Hom(V^* \otimes I^*, W) \to (\det W)^{\alpha} \otimes (\det V)^{\beta}$$

where $\alpha = -\binom{k+n-1}{n}$ and $\beta = \binom{k+n}{n+1}$ is a $GL(V) \times GL(I) \times Sp(W)$-equivariant map and $D(A) = 0$ defines a homogeneous hypersurface of degree $(2n+2k)\binom{k+n-1}{n} = (2n+2)\binom{k+n}{n+1}$. After a basis has been fixed in each of the vector spaces V, I and W, the map D can be seen as an $SL(V) \times SL(I) \times Sp(W)$-invariant.

In fact, this definition generalizes the hyperdeterminant of boundary format as introduced in Theorem 14.3.3 of [GKZ94].

Lemma 3.2. *If A is degenerate, then $D(A) = 0$.*

Proof. There are $0 \neq v \in V^*$ and $0 \neq i \in I^*$ such that $A(v \otimes i) = 0$. Hence, $v \otimes S^{n+1} i \in V^* \otimes S^{n+1} I^*$ goes to zero under the dual of (2). □

If A is nondegenerate, we get $D(A) \neq 0$ only in the case $k = 1$ and, in general, it can happen that $D(A) = 0$, because the codimension of N is k. Our main technical result is the following.

Theorem 3.3. *If A defines an instanton (that is, A belongs to $\mathcal{Q}^0$), then $D(A) \neq 0$.*

Proof. From (1) we get the exact sequence

$$0 \to K \to W \otimes \mathcal{O} \to I \otimes \mathcal{O}(1) \to 0. \tag{3}$$

The $(n+1)$-th wedge power twisted by $\mathcal{O}(-n)$ gives the exact sequence

$$0 \to \wedge^{n+1} K(-n) \to \wedge^{n+1} W(-n) \to \dots$$
$$\dots \to \wedge^2 W \otimes S^{n-1} I(-1) \to W \otimes S^n I \to S^{n+1} I(1) \to 0$$

where the H^0 of the last morphism corresponds to Δ_A in (2). Taking cohomology, it is enough to prove

$$H^n(\wedge^{n+1} K(-n)) = 0. \tag{4}$$

The $(n+1)$-th wedge power twisted by $\mathcal{O}(-n)$ of the sequence

$$0 \to I^* \otimes \mathcal{O}(-1) \to K \to E \to 0$$

gives the sequence

$$0 \to S^{n+1} I^* \otimes K(-2n-1) \to \dots \to \wedge^{n-1} K \otimes S^2 I^*(-n-2) \to \dots$$
$$\dots \to \wedge^n K \otimes I^*(-n-1) \to \wedge^{n+1} K(-n) \to \wedge^{n+1} E(-n) \to 0.$$

In order to prove (4), taking cohomology, we need $H^{n+i}(\wedge^{n-i} K(-n-i-1)) = 0$ for $i = 0, \dots, n$ and $H^n(\wedge^{n+1} E(-n)) = 0$. The first group of vanishing is easily obtained by taking suitable wedge powers of (3). The crucial point used to get the last vanishing is the isomorphism $\wedge^{n+1} E \simeq \wedge^{n-1} E$; it is true because E is a rank $2n$ vector bundle with $c_1 = 0$. From the sequence

$$0 \to S^{n-1} I^*(-2n-1) \to S^{n-2} I^* \otimes K(-2n) \to \dots$$
$$\dots \to \wedge^{n-1} K(-n) \to \wedge^{n-1} E(-n) \to 0,$$

in order to prove $H^n(\wedge^{n-1} E(-n)) = 0$, we only need to see that

$$H^{n+i}(\wedge^{n-1-i} K(-n-i)) = 0 \quad \text{for } i = 0, \dots, n,$$

which follows by using the exact sequence (3) exactly as above. □

Now, we can state and prove the main result of this section.

Theorem 3.4. *$MI_{\mathbb{P}^{2n+1}}(k)$ is affine.*

Proof. By Theorem 3.3, we get that $\mathcal{Q} \setminus N = \mathcal{Q}^0 = \mathcal{Q} \setminus \{D = 0\}$ is affine. It follows that $MI_{\mathbb{P}^{2n+1}}(k)$ is affine too, because it is the quotient of an affine variety by a reductive group; see, e.g., [PV89], section 4.4. □

As a consequence we deduce

Corollary 3.5. *$MI_{\mathbb{P}^{2n+1}}(k)$ does not contain any complete subvariety of positive dimension.*

Proof. This follows from the fact that a quasi-affine complete variety is a finite set. □

Remark 3.6. The invariant D is meaningful even in the case $n = 0$. In this case it corresponds to the usual determinant of the map $\mathbb{C}^{2k} \to \mathbb{C}^2 \otimes \mathbb{C}^k$. For example, for $n = 0$ and $k = 2$ the degenerate $2 \times 2 \times 4$ matrices fill a variety of codimension 2 and degree 12 ([BS]) in $\mathbb{P}^{15}$ whose ideal is generated by one quartic (which is our invariant D), 10 sextics and one octic. We remark that the case $2 \times 2 \times 3$ is of boundary format. The case $2 \times 2 \times 5$ is interesting. Here degenerate matrices fill a variety of codimension 3 and degree 20, and its ideal is generated (at least) by 5 quartics, 50 sextics and 12 octics. The 5 quartics define a variety of codimension 2 and degree 10. Hence, in this case no analog of the invariant D can exist.

4. Instanton bundles with structure group $GL(2n)$

Definition 4.1. A k-instanton bundle E on $\mathbb{P}^{2n+1}$ is the cohomology bundle of a monad

$$K \otimes \mathcal{O}(-1) \xrightarrow{A} W \otimes \mathcal{O} \xrightarrow{B} I \otimes \mathcal{O}(1) \tag{5}$$

where W is a complex vector space of dimension $2n + 2k$ and I, K are complex vector spaces of dimension k.

Notice that E is not necessarily symplectic and that this notion is a true generalization of the one above only for $n \geq 2$, because all rank 2 bundles on $\mathbb{P}^3$ with $c_1 = 0$ are symplectic.

Let $(A, B) \in Hom(K \otimes V^*, W) \times Hom(W, I \otimes V)$ defining E. The monad condition is now equivalent to the fact that the matrices A and B are both nondegenerate and $B \cdot A = 0$.

$Hom(K \otimes V^*, W) \times Hom(W, I \otimes V)$ contains the subvariety $\mathcal{P}$ given by pairs of matrices (A, B) for which the sequence (5) is a complex, that is, such that $B \cdot A = 0$. $GL(I) \times GL(K) \times GL(W)$ acts on $\mathcal{P}$ by $(a, b, c) \cdot (A, B) = (cAb, aBc^{-1})$.

Arguing, as in the previous section, we can see that there is a natural one-to-one correspondence between

i) isomorphism classes of instanton bundles, and

ii) orbits of $GL(I) \times GL(K) \times GL(W)$ on the open subvariety $\mathcal{P}^0$ of $\mathcal{P}$ given by pairs of nondegenerate matrices.

Moreover, as in the second section and using Theorem 4.4, if we denote by H the quotient of $GL(I) \times GL(K) \times GL(W)$ by $(\lambda \cdot id, \lambda^{-1} \cdot id, \lambda \cdot id)$, then H acts freely on $\mathcal{P}^0$. In particular, all points of $\mathcal{P}^0$ are stable for the action of $GL(I) \times GL(K) \times GL(W)$.

Definition 4.2. The GIT-quotient $\mathcal{P}^0/GL(I) \times GL(K) \times GL(W)$ is denoted by $I_{\mathbb{P}^{2n+1}}(k)$ and is called the moduli space of k-instanton bundles on $\mathbb{P}^{2n+1}$. It is a geometric quotient.

$I_{\mathbb{P}^{2n+1}}(k)$ coincides for $n = 1, 2$ with the open subset $\mathcal{I}_{\mathbb{P}^{2n+1}}(k)$ of the Maruyama scheme of stable bundles on $\mathbb{P}^{2n+1}$ of rank $2n$ and Chern polynomial $\frac{1}{(1-t^2)^k}$, which are instanton bundles. For $n \geq 3$ we can say that $\mathcal{I}_{\mathbb{P}^{2n+1}}(k)$ is an open subset of $I_{\mathbb{P}^{2n+1}}(k)$. We remark that $MI_{\mathbb{P}^3}(k) = I_{\mathbb{P}^3}(k)$. $\mathcal{I}_{\mathbb{P}^{2n+1}}(k)$ is known to be singular for $n \geq 2$ and $k \geq 3$ (see [MO97]) and reducible for $n \geq 4$ (see [AO00]).

Definition 4.3. Let $(A, B) \in Hom(K \otimes V^*, W) \times Hom(W, I \otimes V)$. We define

$$\tilde{D}(A, B) := \det S(A) \cdot \det R(B)$$

where det denotes the usual determinant and $S(A)$, $R(B)$ are the morphisms

$$S(A) : S^{n+1}K \otimes V^* \to S^n K \otimes W,$$
$$R(B) : S^n I \otimes W \to S^{n+1} I \otimes V,$$

induced by A and B respectively, as in Definition 3.1.

Theorem 4.4. *If (A, B) defines an instanton (that is, (A, B) belongs to $\mathcal{P}^0$), then $\tilde{D}(A, B) \neq 0$.*

Proof. First, we will see that $\det S(A) \neq 0$. From (5) we get the exact sequence

$$0 \to K \otimes \mathcal{O}(-1) \to W \otimes \mathcal{O} \to Q \to 0. \tag{6}$$

The $(n + 1)$-th wedge power twisted by $\mathcal{O}(-n - 2)$ gives the exact sequence

$$0 \to S^{n+1}K \otimes \mathcal{O}(-2n - 3) \to S^n K \otimes W \otimes \mathcal{O}(-2n - 2) \to \dots$$
$$\dots \to \wedge^{n+1}W \otimes \mathcal{O}(-n - 2) \to \wedge^{n+1}Q(-n - 2) \to 0$$

where the H^{2n+1} of the first morphism corresponds to $S(A)$. Hence, taking cohomology, it is enough to prove

$$H^n(\wedge^{n+1}Q(-n - 2)) = 0.$$

This is shown by considering the $(n + 1)$-wedge sequence of the exact sequence

$$0 \to E \to Q \to I \otimes \mathcal{O}(1) \to 0$$

and arguing as in the proof of Theorem 3.3.

In order to prove $\det R(B) \neq 0$, we proceed exactly as in Theorem 3.3 and we leave the details to the reader. □

Theorem 4.5. *$I_{\mathbb{P}^{2n+1}}(k)$ is affine.*

Proof. First, notice that given $(A, B) \in \mathcal{P}$, if A or B is degenerate, then $\det S(A) \cdot \det R(B) = 0$. Hence, by Theorem 4.4 we get that $\mathcal{P}^0 = \mathcal{P} \setminus \{\tilde{D} = 0\}$ is affine. Therefore, by [PV89] section 4.4, $I_{\mathbb{P}^{2n+1}}(k)$ is affine also. □

As a by-product of Theorem 4.5, we deduce

Corollary 4.6. *$I_{\mathbb{P}^{2n+1}}(k)$ does not contain any complete subvariety of positive dimension.*

References

[AO94] V. Ancona and G. Ottaviani. Stability of special instanton bundles on $\mathbb{P}^{2n+1}$. *Trans. Amer. Math. Soc.*, 341, (1994), 677–693. MR **94d:**14017

[AO99] V. Ancona and G. Ottaviani. Unstable hyperplanes for Steiner bundles and multidimensional matrices. *Advances in Geometry*, 1, (2001), 165–192.

[AO00] V. Ancona and G. Ottaviani. On the irreducible components of the moduli space of instanton bundles on $\mathbb{P}^5$. *Geometry Seminars 1998-1999* (S. Coen, ed.), 95-100, Bologna 2000. MR **2001h:**14055

[ADHM78] M. F. Atiyah, V. G. Drinfeld, N. J. Hitchin, and Yu. I. Manin. Construction of instantons, *Phys Lett.*, A65, (1978), 185 –187. MR **82g:**81049

[BH78] W. Barth and K. Hulek. Monads and moduli of vector bundles. *Manuscripta Math.*, 25, (1978), 323 –347. MR **80f:**14005

[BS] D. Bayer and M. Stillman. Macaulay, a computer algebra system for algebraic geometry (http://www.math.columbia.edu/ bayer/Macaulay.html).

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston, 1994. MR **95e:**14045

[HH86] A. Hirschowitz and K. Hulek. Complete families of stable vector bundles over $\mathbb{P}^2$. Complex analysis and algebraic geometry, Proc. Conf. Göttingen 1985, Lect. Notes Math., 1194, 1986, 19-40. MR **87j:**14019

[KO99] P. I. Katsylo and G. Ottaviani. Regularity of the moduli space of instanton bundles $MI_{\mathbb{P}^3}(5)$. *AG/ 9911184*, 1999.

[OS86] C. Okonek and H. Spindler. Mathematical instanton bundles on $\mathbb{P}^{2n+1}$. *J. Reine Angew. Math.*, 364, (1986), 35–50. MR **87e:**14016

[MO97] R. M. Miró-Roig and X. Orus-Lacort. On the smoothness of the moduli space of mathematical instanton bundles. *Comp. Math.*, 105, (1997), 109-119. MR **97m:**14011

[PV89] V. L. Popov and E. B. Vinberg. Invariant theory. Algebraic geometry. IV: Linear algebraic groups, invariant theory, Encycl. Math. Sci. 55, (1994), 123-278; translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 55, (1989), 137-309.

[Sal84] S. Salamon. Quaternionic structures and twistor spaces. Global Riemannian Geometry (eds. N. J. Willmore and N. J. Hitchin), Ellis Horwood, London 1984.

[WZ96] J. Weyman and A. Zelevinsky. Singularities of hyperdeterminants. *Ann. Inst. Fourier, Grenoble*, 46, (1996), 3, 591–644. MR **97m:**14050

Departament Algebra i Geometria, Universitat de Barcelona, Gran Via, 585, 08007 Barcelona, Spain

E-mail address: `costa@mat.ub.es`

Dipartimento di Matematica "U. Dini", Università di Firenze, viale Morgagni 67/A, I 50134 Firenze, Italy

E-mail address: `ottavian@math.unifi.it`