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Boundedness for Nongeneral-Type 3-Folds in $\mathbb{P}_5$

Robert Braun, Giorgio Ottaviani, Michael Schneider, and Frank Olaf Schreyer

One of the tantalizing problems in projective geometry is Hartshorne's conjecture: smooth subvarieties $X \subset \mathbb{P}_n(\mathbb{C})$ with $\dim X > \frac{2}{3}n$ are complete intersections. Due to Serre's correspondence the most interesting case is $\operatorname{codim} X = 2$. In fact, in this case even 4-folds in $\mathbb{P}_6$ should be complete intersections. For $n \leq 5$ the remaining cases of "low codimension" are surfaces in $\mathbb{P}_4$ and 3-folds in $\mathbb{P}_5$. For surfaces in $\mathbb{P}_4$, Ellingsrud and Peskine [8] have established the following beautiful boundedness result.

THEOREM 1 (*Ellingsrud and Peskine*). *There are only finitely many families of smooth surfaces in $\mathbb{P}_4$ that are not of general type.*

This result supports (at least psychologically) the many recent efforts to classify nongeneral-type surfaces in $\mathbb{P}_4$ of low degree [1, 3, 14, 15]. The main purpose of this chapter is to establish a similar result for 3-folds in $\mathbb{P}_5$.

THEOREM 2. *There are only finitely many families of smooth 3-folds in $\mathbb{P}_5$ that are not of general type.*

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This result had been conjectured in Ref. 2 and partly proved in Ref. 6. Putting these two theorems and Ref. 11 together, we obtain the next result.

THEOREM 3. *Let $n \geq 4$. Then there are only finitely many families of smooth 2-codimensional submanifolds $X \subset \mathbb{P}_n$ that are not of general type.*

The main technical result is the following: Let σ be a positive integer. Then there exists a polynomial P_σ of degree 8 with positive leading term such that

$$-\chi(\mathcal{O}_X) \geq P_\sigma(\sqrt{d})$$

for all smooth 3-folds $X \subset \mathbb{P}_5$ of degree d contained in a (reduced, irreducible) hypersurface of degree σ.

The proof relies completely on the ideas and results of the analogous result for surfaces in $\mathbb{P}_4$ by Ellingsrud and Peskine. In contrast to the surface case this is not enough to conclude finiteness. The reason is the lack of a classification for 3-folds. We overcome this difficulty by using the generalized Hodge index theorem and the semipositivity of $N_{X/\mathbb{P}_5}(-1)$, N being the normal bundle. The bounds obtained are very far from what one expects to be best possible. For instance, for surfaces $S \subset \mathbb{P}_4$ lying on quintic hypersurfaces the optimistic estimate would be deg $S \leq 15$, provided $p_g(S) \leq 1$. In a final section we give some evidence toward this by deriving estimates for surfaces that are hyperplane sections of 3-folds in $\mathbb{P}_5$. These estimates have a topological nature and have emerged in one dimension less in the paper of Ellingsrud and Peskine. More precisely we prove the following result.

PROPOSITION 1. *Let E be a vector bundle of rank r on a projective manifold X admitting a morphism $\varphi: \mathcal{O}^{\oplus(r+1)} \to E$ such that $\Sigma = \{p \in X: \text{rk}\,\varphi(p) < r\}$ is generically a local complete intersection variety of codimension 2. Then*

1. $c_1(E) \geq 0$.
2. $c_1^2(E) \geq c_2(E) \geq 0$.
3. $c_1(E)c_2(E) \geq c_3(E) \geq 0$.

Here ≥ 0 means effective.

1. Notations and Preliminaries

We use the following notation:

X	smooth 3-fold in $\mathbb{P}_5$ of degree d
H	class of a hyperplane section of X

K	class of the canonical bundle of X
S	generic hyperplane section of X
C	generic hyperplane section of S
g	genus of C

We also use the following formulas (e.g., Chang [7]):

$$H^3 = d, \tag{1.1}$$

$$H^2K = 2g - 2 - 2d, \tag{1.2}$$

$$HK^2 = \tfrac{1}{2}d(d+1) - 9(g-1) + 6\chi(\mathcal{O}_S), \tag{1.3}$$

$$K^3 = -5d^2 + d(2g+25) + 24(g-1) - 36\chi(\mathcal{O}_S) - 24\chi(\mathcal{O}_X). \tag{1.4}$$

THEOREM 1.1 (*Riemann–Roch for a Vector Bundle E of Rank r on X*).

$$\chi(E) = \frac{1}{6}[c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)] + \frac{1}{4}c_1[c_1(E)^2 - 2c_2(E)]$$
$$+ \frac{1}{12}[c_1^2 + c_2]c_1(E) + \frac{r}{24}c_1c_2,$$

where $c_i = c_i(T_X)$.

In particular,

$$\chi(\mathcal{O}_X) = \tfrac{1}{24}c_1c_2.$$

Furthermore,

$$c_1 = -K,$$

and

$$c_2 = (15-d)H^2 + 6HK + K^2,$$

which follows from the exact sequence

$$0 \to T_X \to T_{\mathbb{P}_5}|X \to N_{X/\mathbb{P}_5} \to 0,$$

and

$$c_2(N_{X/\mathbb{P}_5}) = dH^2.$$

THEOREM 1.2 (*Roth* [16]). *If C is contained in a hypersurface of degree σ and if $\sigma^2 < d$, then S is contained in a hypersurface of degree σ. This is also true if C is replaced by S and S is replaced by X.*

THEOREM 1.3 (*Gruson and Peskine* [9]). *If C is not contained in a hypersurface of degree $\sigma - 1$, then*

$$g - 1 \le \frac{d}{2\sigma}\{d + \sigma(\sigma - 4)\}.$$

THEOREM 1.4 (*Castelnuovo bound* (*Harris* [10]). *Let $V \subset \mathbb{P}_n$ be an irreducible nondegenerate variety of dimension k and degree d. Put*

$$M = \left[\frac{d-1}{n-k}\right] \quad \text{and} \quad \varepsilon = d - 1 - M(n-k),$$

where $[x]$ is the greatest integer less than or equal to x. Then

$$p_g(V) = h^0(\tilde{V}, \Omega^k) \le \binom{M}{k+1}(n-k) + \binom{M}{k}\varepsilon,$$

where $\tilde{V}$ is a resolution of V (i.e., $\tilde{V}$ is a smooth variety mapping holomorphically and birationally to V).

THEOREM 1.5 (*Generalized Hodge Index Theorem* [5]). *Let L and A be line bundles on X such that A is ample. Then*

$$(L^2 \cdot A)A^3 \le (A^2 \cdot L)^2.$$

In Ref. 5 this is proved only for L nef. But the inequality does not change if we replace L by $L + kA$. Now just take k large enough to make $L + kA$ nef (or ample) and apply Ref. 5.

By the Barth–Lefschetz theorem [4] we always have

$$H^1(X, \mathcal{O}_X) = 0, \tag{1.5}$$

and, therefore,

$$H^1(S, \mathcal{O}_S) = 0. \tag{1.6}$$

2. First Estimates

In this section we prove an inequality between d, g, $\chi(\mathcal{O}_S)$, and $\chi(\mathcal{O}_X)$ that is deduced from the semipositivity of $N(-1)$, N being the normal bundle of X in $\mathbb{P}_5$. To bound the number of families, we need only bound the degree. This is the content of the following proposition.

PROPOSITION 2.1. *For any integer d_0 there are only finitely many irreducible components of the Hilbert scheme of 3-folds in $\mathbb{P}_5$ that contain 3-folds with $d \le d_0$.*

PROOF. The Hilbert polynomial of a 3-fold X in $\mathbb{P}_5$ is (see Theorem 1.1)

$$\begin{aligned}\chi(\mathcal{O}_X(t)) &= \tfrac{1}{6}t^3H^3 - \tfrac{1}{4}Kt^2H^2 + \tfrac{1}{12}[(15-d)H^2 + 6HK + 2K^2]tH + \chi(\mathcal{O}_X)\\ &= \tfrac{1}{6}t^3d - \tfrac{1}{4}t^2H^2K + \tfrac{1}{12}t[(15-d)d + 6H^2K + 2K^2H] + \chi(\mathcal{O}_X).\end{aligned}$$

Assume $d \le d_0$. By Theorem 1.4 there are only finitely many possible values for g, $p_g(S)$, and $p_g(X)$ and, hence, for $\chi(\mathcal{O}_S)$ and $\chi(\mathcal{O}_X)$ since $h^2(\mathcal{O}_X) \le p_g(S)$ and $h^1(\mathcal{O}_S) = h^1(\mathcal{O}_X) = 0$. By (1.1)–(1.4) there are only finitely many possibilities for the above polynomial. □

PROPOSITION 2.2. *Let X be a 3-fold in $\mathbb{P}_5$. Then*

1. $12\chi(\mathcal{O}_S) \ge d^2 - 7d - 18(g-1)$.
2. $24\chi(\mathcal{O}_S) \ge d^2 - 3d + (d-15)(g-1) + 12\chi(\mathcal{O}_X)$.

PROOF. Since $N_{X/\mathbb{P}_5}(-1)$ is globally generated, the Segre classes (defined as the inverse Chern classes of the dual bundle) satisfy

$$s_2(N(-1)) \cdot H \ge 0 \quad \text{and} \quad s_3(N(-1)) \ge 0.$$

Now

$$\begin{aligned}s_2 &= c_1^2 - c_2,\\ s_3 &= c_1(c_1^2 - 2c_2),\\ c_1(N(-1)) &= 4H + K,\\ c_1^2(N(-1)) &= 16H^2 + 8HK + K^2,\\ c_2(N(-1)) &= c_2(N) + c_1(N)\cdot(-H) + (-H)^2 = (d-5)H^2 - HK.\end{aligned}$$

Hence, by (1.1)–(1.4),

$$\begin{aligned}0 &\le s_2(N(-1))\cdot H = [(21-d)H^2 + 9HK + K^2]\cdot H\\ &= (21-d)d + 9(2g-2-2d) + \tfrac{1}{2}d(d+1) - 9(g-1) + 6\chi(\mathcal{O}_S)\\ &= -\tfrac{1}{2}d^2 + \tfrac{7}{2}d + 9(g-1) + 6\chi(\mathcal{O}_S),\\ 0 &\le s_3(N(-1)) = (4H+K)\cdot[2(13-d)H^2 + 10HK + K^2]\\ &= 8(13-d)H^3 + 40H^2K + 4HK^2 + 2(13-d)H^2K + 10HK^2 + K^3\\ &= 8(13-d)d + 2(33-d)H^2K + 14HK^2 + K^3\\ &= 104d - 8d^2 + (66-2d)(2g-2-2d)\\ &\quad + 14[\tfrac{1}{2}d(d+1) - 9(g-1) + 6\chi(\mathcal{O}_S)]\\ &\quad - 5d^2 + d(2g+25) + 24(g-1) - 36\chi(\mathcal{O}_S) - 24\chi(\mathcal{O}_X)\\ &= 104d - 8d^2 + 132(g-1) - 132d\\ &\quad - 4dg + 4d(d+1) + 7d(d+1) - 126(g-1)\\ &\quad + 84\chi(\mathcal{O}_S) - 5d^2 + 2dg + 25d + 24(g-1) - 36\chi(\mathcal{O}_S) - 24\chi(\mathcal{O}_X)\\ &= 8d - 2d^2 + 30(g-1) - 2dg + 48\chi(\mathcal{O}_S) - 24\chi(\mathcal{O}_X)\\ &= 6d - 2d^2 + (30-2d)(g-1) + 48\chi(\mathcal{O}_S) - 24\chi(\mathcal{O}_X).\end{aligned}$$ □

To obtain finiteness results for 3-folds X in $\mathbb{P}_5$, we cannot directly apply the finiteness results of Ellingsrud and Peskine since S is mostly of general type even if X is not. In the rest of this section we derive, however, some special finiteness results that can be obtained by using the technical result of Ref. 9, Proposition 3.

PROPOSITION 2.3. *Let X be a 3-fold in $\mathbb{P}_5$. If $(c_1^2 - c_2)\cdot H \le 0$ and if $d \ge 148$, then X is contained in a hypersurface of degree 6.*

PROOF. By (1.1)–(1.4) and Theorem 1.1,

$$\begin{aligned}(c_1^2 - c_2)\cdot H &= [(d-15)H^2 - 6HK]\cdot H\\ &= (d-15)d - 6(2g-2-2d) = d^2 - 3d - 12(g-1).\end{aligned}$$

If X is not contained in a hypersurface of degree 6 and if $d > 36$, then by Theorems 1.2 and 1.3,

$$g - 1 \le \frac{d}{14}(d+21) = \frac{1}{14}d^2 + \frac{3}{2}d.$$

Hence,

$$(c_1^2 - c_2)\cdot H \ge d^2 - 3d - \tfrac{6}{7}d^2 - 18d = \tfrac{1}{7}d^2 - 21d > 0 \qquad \text{for } d \ge 148.$$ □

COROLLARY 2.1. *Assume $c_2 \cdot H \ge 0$ and $(c_1^2 - ac_2)\cdot H \le 0$ for some $a < 1$. Then $\deg X$ is bounded.*

PROOF. The assumptions imply $(c_1^2 - c_2)\cdot H < 0$. By Proposition 2.3, X is contained in a hypersurface of degree 6 if $d \ge 148$, and so are S and C. By Proposition 3 of Ref. 8,

$$\chi(\mathcal{O}_S) \ge cd^3 + \text{l.t. in } \sqrt{d},$$

where c is a positive constant. Hence, by (1.1)–(1.4) and Theorem 1.1,

$$\begin{aligned}0 &\ge (c_1^2 - ac_2)\cdot H\\ &= [(-a)K^2 - a((15-d)H^2 + 6HK)]\cdot H\\ &= (1-a)[\tfrac{1}{2}d(d+1) - 9(g-1) + 6\chi(\mathcal{O}_S)]\\ &\quad - a[(15-d)d + 6(2g-2-2d)]\\ &\ge 6c(1-a)d^3 + \text{l.t. in } \sqrt{d} \qquad \text{(see Theorem 1.3).}\end{aligned}$$

Since $6c(1-a) > 0$, d is bounded. The corresponding statement for surfaces (i.e., $c_1^2 - ac_2 \le 0$ implies d is bounded) implies the finiteness result by applying surface classification. □

3. 3-Folds on Hypersurfaces of Fixed Degree

In this section we prove the main technical result (Proposition 3.1), which is the analog of Proposition 3 in Ref. 8, and whose proof follows the one there.

PROPOSITION 3.1. *Let $X \subset \mathbb{P}_5$ be a smooth 3-fold contained in a hypersurface V of degree σ with σ minimal. There is a constant d_0 depending only on σ, and there is a polynomial P_σ of degree 8 in $\sqrt{d}$ with positive leading coefficient such that, for $d \ge d_0$,*

$$P_\sigma(\sqrt{d}) \le -\chi(\mathcal{O}_X) = p_g(X) - h^2(\mathcal{O}_X) + 1.$$

The proof follows along the line in Ref. 8. First define $\mu = c_2(N_X(-\sigma)) \cdot H$ and assume $\sigma^2 < d$. Then by Lemma 1 of Ref. 8,

$$0 \leq \mu \leq (\sigma - 1)^2 d, \tag{3.1}$$

$$2\sigma(g - 1) = d^2 + d\sigma(\sigma - 4) - \mu. \tag{3.2}$$

Consequently,

$$g - 1 \geq \frac{d^2}{2\sigma} + \frac{d}{2}\left[(\sigma - 4) - \frac{(\sigma - 1)^2}{\sigma}\right]. \tag{3.3}$$

LEMMA 3.1. *For $t \geq \sigma$,*

$$\begin{aligned}\chi(\mathcal{I}_{X|V}(t)) &= \tfrac{1}{24}\sigma t^4 + \tfrac{1}{6}[\tfrac{1}{2}(6 - \sigma)\sigma - d]t^3 \\ &+ \tfrac{1}{4}\left[\tfrac{1}{6}(51 - 18\sigma + 2\sigma^2)\sigma + \frac{1}{\sigma}(d^2 + \sigma d(\sigma - 6) - \mu)\right]t^2 \\ &+ \tfrac{1}{24}\left[(90 - 51\sigma + 12\sigma^2 - \sigma^3)\sigma - 8d \right.\\ &\qquad \left. + \frac{6}{\sigma}[d(d + \sigma(\sigma - 4)) - \mu]\right]t \\ &+ \tfrac{1}{120}(274 - 225\sigma + 85\sigma^2 - 15\sigma^3 + \sigma^4)\sigma - t\chi(\mathcal{O}_S) - \chi(\mathcal{O}_X) \\ &=: Q(t) - \chi(\mathcal{O}_X).\end{aligned}$$

LEMMA 3.2. *Let $t_1 = \min\{t : t\sigma - d \geq \sigma$ and $(t\sigma - d)^2 - \mu - (t\sigma - d)\sigma(\sigma - 4) > 0\}$. Then*

1. $d/\sigma < t_1 \leq d/\sigma + \sqrt{d} + \sigma$.
2. $\chi(\mathcal{I}_{X|V}(t_1)) \geq A\sqrt{d^7}$ *+ l.t. in $\sqrt{d}$, where A is a constant depending only on σ.*

PROOF OF PROPOSITION 3.1. We may assume $t_1 \geq \sigma$. Recall from Ref. 8 that

$$\chi(\mathcal{O}_S) \geq \frac{1}{6\sigma^2}d^3 + \text{l.t. in } \sqrt{d}.$$

By Lemma 3.1,

$$-\chi(\mathcal{O}_X) = \chi(\mathcal{I}_{X|V}(t)) - Q(t).$$

Lemma 3.2(1) yields

$$\begin{aligned}-Q(t_1) &\geq -\frac{1}{24\sigma^3}d^4 + \frac{1}{6\sigma^3}d^4 - \frac{1}{4\sigma^3}d^4 + \frac{1}{6\sigma^3}d^4 + \text{l.t. in } \sqrt{d} \\ &= \frac{1}{24\sigma^3}d^4 + \text{l.t. in } \sqrt{d}.\end{aligned}$$

Using Lemma 3.2(2), we obtain

$$-\chi(\mathcal{O}_X) \geq \frac{1}{24\sigma^3}d^4 + \text{l.t. in } \sqrt{d}.$$

□

PROOF OF LEMMA 3.1. From the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}_5}(t - \sigma) \to \mathcal{O}_{\mathbb{P}_5}(t) \to \mathcal{O}_V(t) \to 0,$$
$$0 \to \mathcal{I}_{X|V}(t) \to \mathcal{O}_V(t) \to \mathcal{O}_X(t) \to 0,$$

we have

$$\chi(\mathcal{I}_{X|V}(t)) = \chi(\mathcal{O}_{\mathbb{P}_5}(t)) - \chi(\mathcal{O}_{\mathbb{P}_5}(t - \sigma)) - \chi(\mathcal{O}_X(t)).$$

Step 1. $t \geq 0$.

$$\begin{aligned}\chi(\mathcal{O}_{\mathbb{P}_5}(t)) &= \binom{t+5}{5} = \frac{1}{5!}(t + 5)(t + 4)(t + 3)(t + 2)(t + 1) \\ &= \frac{1}{5!}(t + 5)(t^2 + 7t + 12)(t^2 + 3t + 2) \\ &= \frac{1}{5!}(t + 5)(t^4 + 10t^3 + 35t^2 + 50t + 24) \\ &= \frac{1}{5!}(t^5 + 15t^4 + 85t^3 + 225t^2 + 274t + 120).\end{aligned}$$

Step 2. $t \geq \sigma$.

$$\chi(\mathcal{O}_{\mathbb{P}_5}(t-\sigma)) = \binom{t+5-\sigma}{5}$$

$$= \frac{1}{5!}(t+5-\sigma)(t+4-\sigma)(t+3-\sigma)(t+2-\sigma)(t+1-\sigma)$$

$$= \frac{1}{5!}(t+5-\sigma)[t^2+(7-2\sigma)t+(12-7\sigma+\sigma^2)]$$
$$\times [t^2+(3-2\sigma)t+(2-3\sigma+\sigma^2)]$$

$$= \frac{1}{5!}(t+5-\sigma)[t^4+(10-4\sigma)t^3+(35-30\sigma+6\sigma^2)t^2$$
$$+(50-70\sigma+30\sigma^2-4\sigma^3)t$$
$$+(24-50\sigma+35\sigma^2-10\sigma^3+\sigma^4)]$$

$$= \frac{1}{5!}[t^5+(15-5\sigma)t^4+(85-60\sigma+10\sigma^2)t^3$$
$$+(225-255\sigma+90\sigma^2-10\sigma^3)t^2$$
$$+(274-450\sigma+255\sigma^2-85\sigma^3+5\sigma^4)t$$
$$+(120-274\sigma+225\sigma^2-85\sigma^3+15\sigma^4-\sigma^5)].$$

Step 3. By Theorem 1.1 (also see proof of Proposition 2.1), (1.1)–(1.4), and (3.2),

$$\chi(\mathcal{O}_X(t)) = \tfrac{1}{6}t^3d - \tfrac{1}{4}t^2H^2K$$
$$+ \tfrac{1}{12}t[(15-d)d+6H^2K+2HK^2] + \chi(\mathcal{O}_X)$$
$$= \tfrac{1}{6}t^3d - \tfrac{1}{4}t^2(2g-2-2d)$$
$$+ \tfrac{1}{12}t[(15-d)d+6(2g-2-2d)$$
$$+ d(d+1) - 18(g-1) + 12\chi(\mathcal{O}_S)] + \chi(\mathcal{O}_X)$$
$$= \tfrac{1}{6}t^3d - \tfrac{1}{4}t^2\left[\frac{d^2}{\sigma} + d(\sigma-6) - \frac{\mu}{\sigma}\right]$$
$$+ \tfrac{1}{12}t\left[4d - 3\left\{\frac{d}{\sigma}(d+\sigma(\sigma-4)) - \frac{\mu}{\sigma}\right\}\right] + t\chi(\mathcal{O}_S) + \chi(\mathcal{O}_X).$$

Putting all this together, we obtain the assertion. □

To prove Lemma 3.2(2) we use the following two lemmas.

LEMMA 3.3. *Let $X \subset \mathbb{P}_n$ be a smooth variety of dimension $n-2$ ($n \geq 4$), and let $C := X \cap \mathbb{P}_3$ be a generic curve section of X. Assume that X is contained in a hypersurface V of degree σ with σ minimal ($d > \sigma^2$). Let $V_C := V \cap \mathbb{P}_3$ and $\tilde{V}_C$ be the normalization of V_C. Then for t_1 as in Lemma 3.2, there is a constant A_1 depending only on σ such that, for $k > t_1$,*

$$\sum_{\nu=t_1}^{k} h^1(\mathcal{J}_{C|\tilde{V}_C}(\nu)) \leq A_1\sqrt{d^3} + l.t.\ in\ \sqrt{d}, \tag{3.4}$$

$$\sum_{\nu=0}^{t_1-1} h^1(\mathcal{J}_{C|\tilde{V}_C}(\nu)) \leq \frac{1}{2\sigma}\sqrt{d^5} + l.t.\ in\ \sqrt{d}. \tag{3.5}$$

PROOF. The choice of t_1 implies [8, Lemma 5] $H^0(\mathcal{J}_{C|\tilde{V}_C}(t_1)) \neq 0$.

Proof of (3.4). Let $L := \mathcal{J}_{C|\tilde{V}_C}(t_1)$ and $\delta_1 := t_1\sigma - d$ (i.e., δ_1 is the degree of L restricted to a generic section of $\mathcal{O}_{\tilde{V}_C}(1)$). Recall that $\delta_1 < \sigma\sqrt{d} + \sigma^2$ by Lemma 3.2(1).

[8, Lemme C]:

$$\omega_{\tilde{V}_C} \text{ is } \tau\text{-regular if } \tau \geq \tfrac{1}{2}(\sigma^3 - 2\sigma^2 + 4\sigma - 9).$$

[8, Lemme D]:

If r is an integer such that $\omega_{\tilde{V}_C}$ is $(r-2)$-regular and $r \geq 2\sigma - 2$, then

1. L is $(r\delta_1)$-regular.
2. $\mathcal{O}_{\tilde{V}_C}$ is r-regular.

(We have $h^0(L) \neq 0$ since $h^0(\mathcal{J}_{C|\tilde{V}_C}(t_1)) \neq 0$.) Fix r as above and let Γ be a generic plane section of C, V_Γ the corresponding section of V_C, and $\tilde{V}_\Gamma$ the normalization of V_Γ.

Observation. There is a constant B depending only on σ such that

$$\sum_{n=0}^{r} h^1(\mathcal{O}_{\tilde{V}_C}(n)) \leq B.$$

Proof of Observation. From

$$0 \to \mathcal{O}_{\tilde{V}_C}(n-1) \to \mathcal{O}_{\tilde{V}_C}(n) \to \mathcal{O}_{\tilde{V}_\Gamma}(n) \to 0,$$

we deduce

$$h^1(\mathcal{O}_{\tilde{V}_C}(n)) \le \sum_{i=0}^{n} h^1(\mathcal{O}_{\tilde{V}_\Gamma}(i)).$$

Then

$$0 \to \mathcal{O}_{V_\Gamma}(i) \to \mathcal{O}_{\tilde{V}_\Gamma}(i) \to Q \to 0$$

implies

$$h^1(\mathcal{O}_{\tilde{V}_\Gamma}(i)) \le h^1(\mathcal{O}_{V_\Gamma}(i)).$$

Hence,

$$\sum_{n=0}^{r} h^1(\mathcal{O}_{\tilde{V}_C}(n)) \le \sum_{n=0}^{r}\sum_{i=0}^{n} h^1(\mathcal{O}_{V_\Gamma}(i)) \le \sum_{i=0}^{r} (i+1)h^1(\mathcal{O}_{V_\Gamma}(r-i)) =: B.$$

Since V_Γ is a plane curve of degree σ, the choice of r implies that B does depend only on σ.

Now let $\bar{L}$ be defined by

$$0 \to \mathcal{O}_{\tilde{V}_C} \to L \to \bar{L} \to 0.$$

The regularity of L implies

$$h^1(\bar{L}(r\delta_1)) = 0.$$

Consider

$$0 \to \bar{L}(r\delta_1 - 1) \to \bar{L}(r\delta_1) \to \bar{L}(r\delta_1)\,|\,_{\tilde{V}_\Gamma} \to 0.$$

Since $\deg \bar{L} = \delta_1$ and $\dim(\mathrm{Supp}(\bar{L})) = 1$, we have, for all s,

$$h^0(\bar{L}(s)\,|\,_{\tilde{V}_\Gamma}) = \delta_1, \qquad h^1(\bar{L}(s)\,|\,_{\tilde{V}}) = 0.$$

This implies

$$h^1(\bar{L}(r\delta_1 - 1)) \le \delta_1,$$

and by an easy induction for all $i > 0$,

$$h^1(\bar{L}(r\delta_1 - i)) \le i\delta_1. \tag{3.6}$$

Consequently, for $k > t_1$,

$$\begin{aligned}\sum_{\nu=t_1}^{k} h^1(\mathcal{I}_{C\,|\,\tilde{V}_C}(\nu)) &= \sum_{n=0}^{k-t_1} h^1(L(n)) \\ &\le \sum_{n=0}^{k-t_1} [h^1(\mathcal{O}_{\tilde{V}_C}(n)) + h^1(\bar{L}(n))] \\ &\le \sum_{n=0}^{r} h^1(\mathcal{O}_{\tilde{V}_C}(n)) + \sum_{i=1}^{r\delta_1} h^1(\bar{L}(r\delta_1 - i)) \\ &\le B + \sum_{i=1}^{r\delta_1} (i\delta_1) = \tfrac{1}{2}\delta_1(r\delta_1)(r\delta_1 + 1) + B \\ &\le \tfrac{1}{2}r^2\sigma^3\sqrt{d^3} + \text{l.t. in } \sqrt{d}.\end{aligned}$$

This is the assertion of (3.4).

Proof of (3.5). Again let $L := \mathcal{I}_{C\,|\,\tilde{V}_C}(t_1)$ and consider, for $0 < n \le t_1$,

$$0 \to \mathcal{O}_{\tilde{V}_C}(-n) \to L(-n) \to \bar{L}(-n) \to 0.$$

Since $\tilde{V}_C$ is normal we have, for $0 < n \le t_1$,

$$h^1(\mathcal{O}_{\tilde{V}_C}(-n)) = 0.$$

This implies, by (3.6),

$$h^1(L(-n)) \le h^1(\bar{L}(-n)) = h^1(\bar{L}(r\delta_1 - (r\delta_1 + n))) \le (r\delta_1 + n)\delta_1$$

(where r and δ_1 are as in the proof of (3.4)). Therefore,

$$\begin{aligned}\sum_{\nu=0}^{t_1-1} h^1(\mathcal{I}_{C\,|\,\tilde{V}_C}(\nu)) &= \sum_{n=1}^{t_1} h^1(L(-n)) \\ &\le \sum_{n=1}^{t_1} (r\delta_1 + n)\delta_1 = rt_1\delta_1^2 + \tfrac{1}{2}\delta_1 t_1(t_1 + 1) \\ &\le \left(r\frac{d}{\sigma}\sigma^2 d + \text{l.t. in } \sqrt{d}\right) + \left(\frac{d^2\sigma}{2\sigma^2}\sqrt{d} + \text{l.t. in } \sqrt{d}\right) \\ &= \frac{1}{2\sigma}\sqrt{d^5} + \text{l.t. in } \sqrt{d}.\end{aligned}$$

This is the assertion of (3.5). □

LEMMA 3.4. *Keeping the notations of Lemma 3.3, we have*

$$h^0(\mathscr{I}_{X|V}(t_1)) \le B_0\sqrt{d^{2\dim(V)-3}} + l.t.\ in\ \sqrt{d}. \tag{3.7}$$

For $i = 1, n-3, n-2, n-1$,

$$h^i(\mathscr{I}_{X|V}(t_1)) \le B_i\sqrt{d^{2\dim(V)-1}} + l.t.\ in\ \sqrt{d}, \tag{3.8}$$

where the B_i's are positive constants depending only on σ.

PROOF.

1. *Claim.* Let $Y \subset \mathbb{P}_N$ be a smooth $(N-2)$-dimensional variety $(N \ge 3)$ contained in a hypersurface V_Y of degree σ. Then, for $i = 0, 1$ and all $k \le 0$,

$$h^i(\mathscr{I}_{Y|V_Y}(k)) = 0.$$

Proof. This is an easy consequence of the long exact cohomology sequences of the following two exact sequences:

$$0 \to \mathscr{I}_{Y|\mathbb{P}_N}(k) \to \mathcal{O}_{\mathbb{P}_N}(k) \to \mathcal{O}_Y(k) \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}_N}(k-\sigma) \to \mathscr{I}_{Y|\mathbb{P}_N}(k) \to \mathscr{I}_{Y|V_Y}(k) \to 0.$$

Now let X be as in the assertion and consider generic hyperplane sections:

$$\begin{array}{ccccccc} & \mathbb{P}_n & \supset & \mathbb{P}_{n-1} & \supset \cdots \supset & \mathbb{P}_3 & \\ & \cup & & \cup & & \cup & \\ V = & V_0 & \supset & V_1 & \supset \cdots \supset & V_{n-3} & \leftarrow \tilde{V}_{n-3} \\ & \cup & & \cup & & \cup & \\ X = & X_0 & \supset & X_1 & \supset \cdots \supset & X_{n-3} & \end{array}$$

where $\tilde{V}_{n-3}$ is the normalization of V_{n-3}.

2. *Claim.* Let Q be defined by

$$0 \to \mathscr{I}_{X_{n-3}|V_{n-3}} \to \mathscr{I}_{X_{n-3}|\tilde{V}_{n-3}} \to Q \to 0.$$

Then, for all $k \ge 0$,

$$h^0(Q(k)) \le D(k+1),$$

where D is a positive constant depending only on σ.

Proof. Notice first that $h^0(Q(k)) = 0$ for all $k < 0$. Let Γ be a generic plane section of X_{n-3} and consider

$$0 \to Q(k-1) \to Q(k) \to Q_\Gamma(k) \to 0.$$

Since the support of Q_Γ is precisely the singular points of a generic plane section of V_{n-3}, we obtain, for all $r \in \mathbb{Z}$,

$$h^0(Q_\Gamma(r)) = D,$$

where D depends only on σ. Hence,

$$h^0(Q(k)) \le \sum_{r=0}^{k} h^0(Q_\Gamma(r)) \le D(k+1).$$

Recall Lemme B from Ref. 8: there exists a constant A depending only on σ such that

$$\sum_{k=1}^{t_1} h^0(\mathscr{I}_{X_{n-3}|\tilde{V}_{n-3}}(k)) \le A\sqrt{d^3} + \text{l.t. in } \sqrt{d}.$$

Look at the exact sequences

$$0 \to \mathscr{I}_{X_j|V_j}(k-1) \to \mathscr{I}_{X_j|V_j}(k) \to \mathscr{I}_{X_{j+1}|V_{j+1}}(k) \to 0,$$
$$0 \to \mathscr{I}_{X_{n-3}|V_{n-3}}(k) \to \mathscr{I}_{X_{n-3}|\tilde{V}_{n-3}}(k) \to Q(k) \to 0.$$

From the long exact cohomology sequences and the above preparations we obtain

$$\begin{aligned} h^0(\mathscr{I}_{X|V}(t_1)) &\le \sum_{k=1}^{t_1} h^0(\mathscr{I}_{X_1|V_1}(k)) \le \cdots \le \sum_{1}^{t_1} \cdots \sum_{1}^{t_1} h^0(\mathscr{I}_{X_{n-3}|V_{n-3}}(k)) \\ &\le \sum_{1}^{t_1} \cdots \sum_{1}^{t_1} h^0(\mathscr{I}_{X_{n-3}|\tilde{V}_{n-3}}(k)) \\ &\le t_1^{n-4}(A\sqrt{d^3} + \text{l.t. in } \sqrt{d}) \le B_0\sqrt{d^{2n-5}} + \text{l.t. in } \sqrt{d}. \end{aligned}$$

Analogously, using Lemma 3.3, we have

$$\begin{aligned}
h^1(\mathcal{I}_{X|V}(t_1)) &\le \sum_1^{t_1} \cdots \sum_1^{t_1} h^1(\mathcal{I}_{X_{n-3}|V_{n-3}}(k))\\
&\le \sum_1^{t_1} \cdots \sum_1^{t_1} [h^1(\mathcal{I}_{X_{n-3}|P_{n-3}}(k)) + h^0(Q(k))]\\
&\le [t_1^{n-4}(F\sqrt{d^5} + \text{l.t. in } \sqrt{d})] + \sum_1^{t_1} \cdots \sum_1^{t_1} D(k+1)\\
&\le (F_0\sqrt{d^{2n-3}} + \text{l.t. in } \sqrt{d}) + (D_0 t_1^{n-2} + \text{l.t. in } t_1)\\
&\le B_1\sqrt{d^{2n-3}} + \text{l.t. in } \sqrt{d}.
\end{aligned}$$

3. *Claim.* Let $Y \subset \mathbb{P}_N$ be a smooth $(N-2)$-dimensional variety of degree d ($N \ge 4$); assume that Y is contained in a hypersurface of degree σ. Then, for $i = N-3$, $N-2$, $N-1$ and all $k \ge d$,

$$h^i(\mathcal{I}_{Y|V_Y}(k)) = 0.$$

Proof. If $N = 4$, $\mathcal{I}_{Y|\mathbb{P}_4}$ is $(d-1)$-regular by Ref. 12. Hence, the claim follows from the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}_4}(k-\sigma) \to \mathcal{I}_{Y|\mathbb{P}_4}(k) \to \mathcal{I}_{Y|V_Y}(k) \to 0.$$

Now let $N > 4$ and consider the exact sequences

$$0 \to \mathcal{I}_{Y|V_Y}(k) \to \mathcal{I}_{Y|V_Y}(k+1) \to \mathcal{I}_{Y\cap\mathbb{P}_{N-1}|V_Y\cap\mathbb{P}_{N-1}}(k+1) \to 0.$$

The long exact cohomology sequences yield

$$h^i(\mathcal{I}_{Y|V_Y}(k)) \le \sum_{r>k} h^{i-1}(\mathcal{I}_{Y\cap\mathbb{P}_{N-1}|V_Y\cap\mathbb{P}_{N-1}}(r)) = 0$$

via the induction hypothesis if $k \ge d$ and $i - 1 = N-4$, $N-3$, $N-2$.

Now we can prove the remaining cases $i = n-3$, $n-2$, $n-1$ of (3.8) (again using Lemma 3.3):

$$\begin{aligned}
h^i(\mathcal{I}_{X|V}(t_1)) &\le \sum_{k>t_1} h^{i-1}(\mathcal{I}_{X_1|V_1}(k)) \le \sum_{k=1}^{d} h^{i-1}(\mathcal{I}_{X_1|V_1}(k))\\
&\le \cdots \le \sum_1^d \cdots \sum_1^d h^1(\mathcal{I}_{X_{i-1}|V_{i-1}}(k)) \qquad ((i-1) \text{ sums})\\
&\le \cdots \le \sum_1^d \cdots \sum_1^d \sum_{k=1}^{d} h^1(\mathcal{I}_{X_i|V_i}(k))\\
&\le \cdots \le \sum_1^d \cdots \sum_1^d \sum_1^d \cdots \sum_1^d h^1(\mathcal{I}_{X_{n-3}|V_{n-3}}(k)) \qquad ((n-3) \text{ sums})\\
&\le \cdots \le \sum_1^d \cdots \sum_1^d \sum_1^d \cdots \sum_1^d [h^1(\mathcal{I}_{X_{n-3}|P_{n-3}}(k)) + h^0(Q(k))]\\
&\le d^{n-4}\left[(F\sqrt{d^5} + \text{l.t. in } \sqrt{d}) + \sum_{k=1}^{d} D(k+1)\right]\\
&\le B_i\sqrt{d^{2n-3}} + \text{l.t. in } \sqrt{d}.
\end{aligned}$$

Thus, the proof of Lemma 3.4 is complete. □

Proof of Lemma 3.2.

1. See Ref. 8.
2. This follows from Lemma 3.4 and the obvious inequality

$$\chi(\mathcal{I}_{X|V}(t_1)) \ge -h^1(\mathcal{I}_{X|V}(t_1)) - h^3(\mathcal{I}_{X|V}(t_1)).$$ □

Corollary 3.1. *Let σ be a positive integer. There exist only finitely many families of smooth 3-folds in $\mathbb{P}_5$ that are not of general type and are contained in a hypersurface of degree σ.*

Proof. Let $X \subset \mathbb{P}_5$ be a smooth 3-fold that is not of general type and contained in a hypersurface of degree σ. Since X is not of general type, we have $H^0(X, \omega_X(-1)) = 0$, and hence

$$p_g(X) \le p_g(S). \tag{3.9}$$

But from Theorem 1.4 we know that

$$p_g(S) \le \frac{d^3}{24} + \text{l.t. in } d. \tag{3.10}$$

On the other hand,

$$p_g(X) = 1 + h^2(\mathcal{O}_X) - \chi(\mathcal{O}_X) \ge -\chi(\mathcal{O}_X).$$

By the proof of Proposition 3.1 we therefore obtain

$$p_g(X) \ge \frac{d^4}{24\sigma^3} + \text{l.t. in } \sqrt{d}. \tag{3.11}$$

From (3.9)–(3.11) it follows that d is bounded, and an application of Proposition 2.1 concludes the proof. □

Corollary 3.2. *There exist only finitely many families of smooth 3-folds in $\mathbb{P}_5$ that are not of general type and satisfy*

$$(c_1^2 - c_2) \cdot H \le 0.$$

Proof. It suffices to combine Proposition 2.3 and Corollary 3.1. □

4. Boundedness

In this section we prove our main finiteness result. We prove an inequality for 3-folds in $\mathbb{P}_5$ that comes from the generalized Hodge index theorem.

Proposition 4.1. *Let $X \subset \mathbb{P}_5$ be a smooth 3-fold. Then*

$$6\chi(\mathcal{O}_S) \le \frac{4}{d}(g-1)^2 + (g-1) - \frac{d^2}{2} + \frac{7}{2}d. \tag{4.1}$$

Proof. We apply the generalized Hodge index theorem (1.5) to obtain

$$(K \cdot H^2)^2 \ge d(K^2 \cdot H). \tag{4.2}$$

By (1.1)–(1.4) we have

$$K \cdot H^2 = 2g - 2 - 2d$$

and

$$K^2 \cdot H = \frac{d^2}{2} + \frac{d}{2} - 9(g-1) + 6\chi(\mathcal{O}_S).$$

Inserting these expressions into (4.2) yields the desired inequality. □

We need another easy tool.

Proposition 9.2. *Let $X \subset \mathbb{P}_5$ be a smooth 3-fold that is not of general type. Then*

$$-\chi(\mathcal{O}_X) \le \chi(\mathcal{O}_S).$$

Proof.

$$-\chi(\mathcal{O}_X) = p_g(X) - 1 - h^2(\mathcal{O}_X) \le p_g(X).$$

Consider the exact sequence

$$0 \to \omega_X(-1) \to \omega_X \to \omega_S(-1) \to 0.$$

Since X is not of general type, we have $H^0(X, \omega_X(-1)) = 0$, and therefore

$$p_g(X) \le h^0(S, \omega_S(-1)) \le p_g(S).$$

Thus, by (1.5) and (1.6), we get

$$-\chi(\mathcal{O}_X) \le p_g(X) \le p_g(S) \le 1 + p_g(S) = \chi(\mathcal{O}_S).$$ □

Now we can prove our finiteness result.

Theorem 4.1. *There are only finitely many irreducible components of the Hilbert scheme of smooth 3-folds in $\mathbb{P}_5$ that are not of general type.*

Proof. Let X be a smooth 3-fold in $\mathbb{P}_5$ that is not of general type. It is enough to show that $d = \deg X$ is bounded. Recall the inequality

$$24\chi(\mathcal{O}_S) \ge d^2 - 3d + (d-15)(g-1) + 12\chi(\mathcal{O}_X)$$

from Proposition 2.2. Using Proposition 4.2, we therefore obtain

$$24\chi(\mathcal{O}_S) \geq d^2 - 3d + (d-15)(g-1) - 12\chi(\mathcal{O}_S);$$

i.e.,

$$36\chi(\mathcal{O}_S) \geq d^2 - 3d + (d-15)(g-1). \tag{4.3}$$

Inequality (4.1) yields

$$36\chi(\mathcal{O}_S) \leq \frac{24}{d}(g-1)^2 + 6(g-1) - 3d^2 + 21d. \tag{4.4}$$

Combining (4.3) and (4.4) leads to

$$0 \leq (g-1)\left[\frac{24}{d}(g-1) - (d-15) + 6\right] - 4d^2 + 24d. \tag{4.5}$$

Assuming first that X is not contained in a hypersurface of degree 12 (assume $d > 12^2 = 144$), we have by Theorems 1.2 and 1.3 the estimate

$$g - 1 \leq \frac{d^2}{26} + \frac{9d}{2}.$$

Inserting this into (4.5) gives

$$0 \leq (g-1)\left[-\frac{d}{13} + 129\right] - 4d^2 + 24d.$$

The right side of this inequality is negative for $d \geq 1677$. Hence, we conclude that $d \leq 1676$ in this case.

If X is contained in a hypersurface of degree 12, it is enough to apply Corollary 3.1. □

This result, together with Refs. 8 and 11, yields a solution to the finiteness conjecture in codimension 2.

THEOREM 4.2. *Let $n \geq 4$. There exist only finitely many families of smooth 2-codimensional submanifolds of $\mathbb{P}_n$ that are not of general type.*

PROOF. The case $n = 4$ is treated in Ref. 8, and $n = 5$ is the content of Theorem 4.1. For $n \geq 6$ it was shown in Ref. 11 that, for $X \subset \mathbb{P}_n$, smooth of codimension 2, either X is a complete intersection or $\omega_X = \mathcal{O}_X(e)$, with $e \geq n + 2$. Hence, X is of general type or a complete intersection. If X is a complete intersection of two hypersufaces of degree a and b, which is not of general type, we therefore have $\omega_X = \mathcal{O}_X(a + b - n - 1)$ and $a + b \leq n$. This gives a bound for the degree of X:

$$d = ab \leq \frac{n^2}{4}.$$

□

5. Inequalities of Topological Type

In this section we point out that the inequality

$$\chi(\mathcal{O}_S) \geq c \cdot d^3 + \text{l.t. in } \sqrt{d}$$

of Ellingsrud and Peskine for smooth surfaces $S \subset \mathbb{P}_4$, contained in a hypersurface of fixed degree σ, can be improved for a large class of surfaces that extend to smooth 3-folds in $\mathbb{P}_5$ by a Chern class inequality. The fact that Castelnuovo-type inequalities between the degree d and the sectional genus g of a smooth surface $S \subset \mathbb{P}_4$ can be derived by a Chern class inequality was discovered by Ellingsrud and Peskine [8, Lemme 1].

PROPOSITION 5.1. *Let $X \subset \mathbb{P}_5$ be a smooth hypersurface contained in a hypersurface V of minimal degree σ. Then V defines a nontrivial section s of $N^{\vee}_{X/\mathbb{P}_5}(\sigma)$. Assume that $\Sigma = \{s = 0\}$ has no divisorial component. Then*

$$\deg \Sigma = d^2 - 4\sigma d + \sigma^2 d - 2\sigma(g-1), \tag{5.1}$$

$$p_a(\Sigma) = (\sigma - 3)\deg \Sigma + 1. \tag{5.2}$$

PROOF OF (5.1).

$$\begin{aligned}\deg \Sigma &= c_2(N^{\vee}_X(\sigma)) \cdot H \\ &= [c_2(N^{\vee}) + c_1(N^{\vee}) \cdot \sigma H + \sigma^2 H^2] \cdot H \\ &= [dH^2 - (6H + K)\sigma H + \sigma^2 H^2] \cdot H \\ &= (d - 6\sigma + \sigma^2)d - \sigma(2g - 2 - 2d).\end{aligned}$$

PROOF OF (5.2). The exact sequence

$$0 \to \mathcal{J}_{\Sigma|X} \to \mathcal{O}_X \to \mathcal{O}_\Sigma \to 0$$

yields

$$p_a(\Sigma) = 1 - \chi(\mathcal{O}_\Sigma) = \chi(\mathcal{I}_{\Sigma|X}) - \chi(\mathcal{O}_X) + 1.$$

The Koszul complex of s reads

$$0 \to \det(N(-\sigma)) \to N(-\sigma) \to \mathcal{I}_{\Sigma|X} \to 0.$$

Hence, by Theorem 1.1,

$$\begin{aligned}\chi(\mathcal{I}_{\Sigma|X}) &= \chi(N(-\sigma)) - \chi(\det(N(-\sigma)))\\ &= -\tfrac{1}{2}c_1(N(-\sigma))\cdot c_2(N(-\sigma)) + \tfrac{1}{2}K\cdot c_2(N(-\sigma)) + \chi(\mathcal{O}_X)\\ &= -\tfrac{1}{2}[(6-2\sigma)H + K]\cdot c_2(N(-\sigma)) + \tfrac{1}{2}K\cdot c_2(N(-\sigma)) + \chi(\mathcal{O}_X)\\ &= (\sigma-3)H\cdot c_2(N^\vee(-\sigma)) + \chi(\mathcal{O}_X)\\ &= (\sigma-3)\deg\Sigma + \chi(\mathcal{O}_X).\end{aligned}$$

This implies the assertion. □

Lemma 5.1. *Let G be a rank-n vector bundle on a projective manifold X. Suppose there is a morphism $\varphi\colon \bigoplus_1^{n+1}\mathcal{O} \to G$ such that $\Sigma := \{p \in X : \operatorname{rk}\varphi(p) < n\}$ is generically a local complete intersection subvariety of (expected) codimension 2. Then*

1. $c_1(G) \geq 0$.
2. $c_1^2(G) \geq c_2(G) \geq 0$.
3. $c_1(G)\cdot c_2(G) \geq c_3(G) \geq 0$.

Here $c \geq 0$ means that c is represented by an effective cycle.

Proof. By definition the ideal sheaf of Σ is generated by the maximal minors of $\varphi\colon \bigoplus_1^{n+1}\mathcal{O} \to G$; i.e.,

$$\mathcal{I}_\Sigma = \operatorname{Im}\left(\Lambda^n\varphi\colon \Lambda^n\left(\bigoplus_1^{n+1}\mathcal{O}\right)\otimes\Lambda^n G^\vee \to \mathcal{O}\right).$$

On the other hand, the dependency locus

$$C_i := \{s_1 \wedge \cdots \wedge s_{n+1-i} = 0\}$$

of any $n+1-i$ sections $s_1, \ldots, s_{n+1-i} \in H^0(X, G)$ represents the ith Chern class $c_i(G)$ of G provided that C_i has the expected codimension i. So the submatrices of $\varphi\colon \bigoplus_1^{n+1}\mathcal{O} \to G$ carry information about the Chern classes of G. Since Σ has codimension 2, we may assume that the sections $s_1, \ldots, s_{n+1}$ defining φ are such that the ith minor of φ,

$$D_i := \{s_1 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge s_{n+1} = 0\},$$

is an effective divisor representing $c_1(G)$, and any two of these divisors intersect in a subvariety of codimension 2. It follws that $c_1(G)^2 = D_n \cdot D_{n+1}$ is represented by an effective cycle, and, since $C_2 = \{s_1 \wedge \cdots \wedge s_{n-1} = 0\}$ is a subvariety of $D_n \cap D_{n+1}$, also $c_2(G)$ and $c_1(G)^2 - c_2(G)$ are effective. This proves (2). Actually (cf. Ref. 13),

$$c_1(G)^2 - c_2(G) = \Sigma.$$

For (3) we need that Σ is generically a local complete intersection. By this assumption we may assume that C_2 and Σ have no common component and that $C_2 \cap D_{n-1}$ has codimension 3. Thus, $c_1(G)\cdot c_2(G)$ is effective, and, since $C_3 = \{s_1 \wedge \cdots \wedge s_{n-2} = 0\}$ is a subscheme of $C_2 \cap D_{n-1}$, $c_3(G)$ and $c_1(G)\cdot c_2(G) - c_3(G)$ are also effective. □

Proposition 5.2. *With the hypothesis of Proposition 5.1, we have $0 \leq s_3 \leq s_1 s_2$, where $s_i := s_i(\mathcal{I}_{\Sigma|X}(\sigma-1))$ are the Segre classes of $\mathcal{I}_{\Sigma|X}(\sigma-1)$.*

Proof. Note first that $\mathcal{I}_{\Sigma|X}(\sigma-1)$ is globally generated by the partial derivatives of the equation defining V. This yields an exact sequence

$$0 \to F \to \bigoplus_{i=1}^{6}\mathcal{O}_X \to \mathcal{I}_{\Sigma|X}(\sigma-1) \to 0,$$

where F is locally free as syzygy module of $\mathcal{I}_{\Sigma|X}(\sigma-1)$. Furthermore, by definition of the Segre classes,

$$s_i = c_i(F^\vee).$$

By dualizing the above sequence, we obtain a map

$$\bigoplus_{i=1}^{6}\mathcal{O}_X \to F^\vee,$$

which is generically surjective and drops rank precisely in Σ. From Lemma 5.1 we obtain the assertion. □

Corollary 5.1. *With the hypothesis of Proposition* 5.1 *we have*

$$g - 1 \le \frac{d^2}{2\sigma} + \frac{d}{2}(\sigma - 4), \tag{5.3}$$

$$g - 1 \ge \frac{d^2}{2\sigma} - \frac{d}{2\sigma}(2\sigma + 1), \tag{5.4}$$

$$\chi(\mathcal{O}_S) \ge \frac{1}{6\sigma}\left[(g-1)(2d - 9\sigma) + d^2\left(\frac{\sigma}{2} + 1\right) + d\left(1 - \frac{7}{2}\sigma\right)\right], \tag{5.5}$$

$$\chi(\mathcal{O}_S) \le \frac{1}{6\sigma}\left[(g-1)(2d - 11\sigma + 2\sigma^2) - d^2\left(\frac{\sigma}{2} - 2\right) + d\left(2\sigma^2 - \frac{9}{2}\sigma\right)\right]. \tag{5.6}$$

If $d \ge \frac{9}{2}\sigma$, *then*

$$\chi(\mathcal{O}_S) \ge \frac{1}{6\sigma}\left[\frac{1}{\sigma}d^3 + d^2\left(\frac{\sigma}{2} - \frac{1}{\sigma} - \frac{11}{2}\right) + d\left(\frac{11}{2}\sigma + \frac{11}{2}\right)\right]. \tag{5.7}$$

If $d \ge \frac{11}{2}\sigma - \sigma^2$, *then*

$$\chi(\mathcal{O}_S) \le \frac{1}{6\sigma}\left[\frac{1}{\sigma}d^3 + d^2\left(\frac{3}{2}\sigma - \frac{15}{2}\right) + d\left(\sigma^3 - \frac{15}{2}\sigma^2 + \frac{35}{2}\sigma\right)\right]. \tag{5.8}$$

Proof. Inequality (5.3) follows from (5.1) since $\deg \Sigma \ge 0$. Consider the above Koszul complex of the section s twisted by $\mathcal{O}_X(\sigma - 1)$:

$$0 \to \mathcal{O}_X((5-\sigma)H + K) \to N_{X|\mathbb{P}_5}(-1) \to \mathcal{I}_{\Sigma|X}(\sigma - 1) \to 0.$$

Let $c_i := c_i(\mathcal{I}_{\Sigma|X}(\sigma - 1))$. Then

$$s_1 = c_1, \qquad s_2 = c_1^2 - c_2, \qquad s_3 = c_1^3 - 2c_1c_2 + c_3.$$

From the above sequence,

$$\begin{aligned}
c_1 &= (\sigma - 1)H,\\
c_2 &= c_2(N(-1)) - c_1(\mathcal{O}_X((5-\sigma)H + K)) \cdot c_1\\
&= dH^2 - (6H + K)H + H^2 - ((5-\sigma)H + K)(\sigma - 1)H\\
&= (d + \sigma^2 - 6\sigma)H^2 - \sigma HK,\\
c_1c_2 &= (\sigma - 1)(d + \sigma^2 - 6\sigma)d - \sigma(\sigma - 1)(2g - 2 - 2d)\\
&= (\sigma - 1)(\sigma^2 - 4\sigma)d + (\sigma - 1)d^2 - 2\sigma(\sigma - 1)(g - 1),\\
c_3 &= -c_1(\mathcal{O}_X((5-\sigma)H + K)) \cdot c_2\\
&= -((5-\sigma)H + K) \cdot ((d + \sigma^2 - 6\sigma)H^2 - \sigma HK)\\
&= (\sigma - 5)(d + \sigma^2 - 6\sigma)H^3\\
&\quad + [\sigma(5-\sigma) - (d + \sigma^2 - 6\sigma)]H^2K + \sigma HK^2\\
&= (\sigma - 5)(d + \sigma^2 - 6\sigma)d + [\sigma(5-\sigma) - (d + \sigma^2 - 6\sigma)](2g - 2 - 2d)\\
&\quad + \sigma[\tfrac{1}{2}d(d+1) - 9(g-1) + 6\chi(\mathcal{O}_S)]\\
&= d\left[(\sigma - 5)(\sigma^2 - 6\sigma) - 2\sigma(5-\sigma) + 2(\sigma^2 - 6\sigma) + \frac{\sigma}{2}\right]\\
&\quad + d^2\left[\sigma - 5 + 2 + \frac{\sigma}{2}\right]\\
&\quad + (g-1)[2\sigma(5-\sigma) - 2(d + \sigma^2 - 6\sigma) - 9\sigma] + 6\sigma\chi(\mathcal{O}_S)\\
&= d\left(\sigma^3 - 7\sigma^2 + 8\sigma + \frac{\sigma}{2}\right) + d^2\left[\sigma - 3 + \frac{\sigma}{2}\right]\\
&\quad + (g-1)(-2d - 4\sigma^2 + 13\sigma) + 6\sigma\chi(\mathcal{O}_S).
\end{aligned}$$

Hence,

$$\begin{aligned}
s_1s_2 &= (\sigma - 1)^3 d - [(\sigma - 1)(\sigma^2 - 4\sigma)d + (\sigma - 1)d^2 - 2\sigma(\sigma - 1)(g - 1)]\\
&= (\sigma - 1)(2\sigma + 1)d - (\sigma - 1)d^2 + 2\sigma(\sigma - 1)(g - 1).
\end{aligned}$$

Since $s_1s_2 \geq 0$ by Proposition 5.2, we obtain (5.4). Furthermore,

$$\begin{aligned} s_3 &= s_1s_2 - c_1c_2 + c_3 \\ &= (\sigma-1)(2\sigma+1)d - (\sigma-1)d^2 + 2\sigma(\sigma-1)(g-1) \\ &\quad - [(\sigma-1)(\sigma^2-4\sigma)d + (\sigma-1)d^2 - 2\sigma(\sigma-1)(g-1)] \\ &\quad + \left(\sigma^3 - 7\sigma^2 + 8\sigma + \frac{\sigma}{2}\right)d + \left(\sigma - 3 + \frac{\sigma}{2}\right)d^2 \\ &\quad + (-2d - 4\sigma^2 + 13\sigma)(g-1) + 6\sigma\chi)\mathcal{O}_S) \\ &= \left(\frac{7}{2}\sigma - 1\right)d - \left(\frac{\sigma}{2}+1\right)d^2 + (9\sigma - 2d)(g-1) + 6\sigma\chi(\mathcal{O}_S). \end{aligned}$$

Now $s_3 \geq 0$ gives (5.5).

To obtain (5.6) we look at $s_1s_2 - s_3 \geq 0$:

$$\begin{aligned} s_1s_2 - s_3 &= (\sigma-1)(2\sigma+1)d - (\sigma-1)d^2 + 2\sigma(\sigma-1)(g-1) \\ &\quad - \left[\left(\frac{7}{2}\sigma - 1\right)d - \left(\frac{\sigma}{2}+1\right)d^2 + (9\sigma-2d)(g-1) + 6\sigma\chi(\mathcal{O}_S)\right] \\ &= \left(2\sigma^2 - \frac{9}{2}\sigma\right)d - \left(\frac{\sigma}{2} - 2\right)d^2 \\ &\quad + (2\sigma^2 - 11\sigma + 2d)(g-1) - 6\sigma\chi(\mathcal{O}_S). \end{aligned}$$

Inequality (5.7) is simply (5.4) and (5.5), and (5.8) is (5.3) and (5.6)! □

REMARK. An optimistic point of view would be to prove the inequalities in Corollary 5.1 for surfaces in $\mathbb{P}_4$. This would lead to very sharp estimates. For instance, for $\sigma = 5$ and $p_g(S) \leq 1$, one would get $d \leq 14$. "Unfortunately" there is a surface of degree 15 in $\mathbb{P}_4$ with $p_g(S) = 1$ lying on a quintic [2].

Still we believe in the existence of good estimates for surfaces in $\mathbb{P}_4$ as in Corollary 5.1 that might possibly be proved by "topological" means by passing to a suitable 3-fold.

Suggested Readings

M. Beltrametti, M. Schneider, and A. J. Sommese, Threefolds of degree 9 and 10 in $\mathbb{P}^5$, *Math. Ann.* **288**, 613–644 (1990).

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14

The Curvature of the Petersson–Weil Metric on the Moduli Space of Kähler–Einstein Manifolds

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The Petersson–Weil metric is a main tool for investigating the geometry of moduli spaces. When A. Weil considered the classical Teichmüller space from the viewpoint of deformation theory, he suggested, in 1958, investigating the Petersson inner product on the space of holomorphic quadratic differentials. He conjectured that it induced a Kähler metric on the Teichmüller space. After proving this property, Ahlfors showed, in 1961, that the holomorphic sectional and Ricci curvatures were negative. Royden's conjecture of a precise upper bound for the holomorphic sectional curvature was proven by Wolpert and Tromba in 1986 along with the negativity of the sectional curvature.

The Petersson–Weil metric is strongly related to the variation of the hyperbolic metrics on the fibers of a holomorphic family. For compact manifolds of higher dimension the considerations have to be based on the existence of Kähler–Einstein metrics according to Yau, for negative and zero Ricci curvature k, and Siu [13], Tian [14], Tian and Yau [15], and Nadel

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Complex Analysis and Geometry, edited by Vincenzo Ancona and Alessandro Silva. Plenum Press, New York, 1993.