# HARMONIC DECOMPOSITION OF POLYNOMIALS

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Abstract. The space of homogeneous polynomials of degree $d$ splits according to the natural $SO$-action into irreducible summands that contain polynomial of the form $q^k g$ where $g$ is harmonic. This is called the harmonic decomposition.

## 1. Harmonic decomposition and harmonic part

Let $V$ be a $(n+1)$-dimensional real vector space on $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Fix a nondegenerate $q \in \mathrm{S}ym^2V$, we have the group $SO(V,q) = SO(V)$ of endomorphisms preserving $q$ and the corresponding action of $SO(V)$ on $V$. It is convenient to choose from the beginning a coordinate system such that $q$ has the standard Euclidean expression

$$q = \sum_{i=0}^{n} x_i^2.$$

Let $\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator, which corresponds to $q$ in the dual coordinates $\partial_0, \ldots, \partial_n$. As usual a polynoial $f$ is called *harmonic* if $\Delta f = 0$.

**Proposition 1.1.** *The group $SO(V,q)$ has exactly two orbits on $\mathbb{P}_{\mathbb{C}}(V)$, namely the quadric $Q_{n-1} = \{q = 0\}$ and its complement $\mathbb{P}_{\mathbb{C}}(V) \setminus Q_{n-1}$. The real group $SO(V_{\mathbb{R}}$ acts transitively on $\mathbb{P}_{\mathbb{R}}(V)$.*

*Proof.* An orthogonal transformation on $\mathbb{C}$ takes any non isotropic $v$ to $e_1$ and any isotropic $v$ to $e_1 + \sqrt{-1}e_2$. An orthogonal transformation on $\mathbb{R}$ takes any non zero $v$ to $e_1$. □

The natural way to define the action of $SO(V)$ to polynomials , is to consider polynomials as functions. In this way, if $f(x)$ is the function associated to the polynomial $f$, then for any $g \in SO(V)$ we define the function $gf$ by $(gf)(x) = f(g^{-1}x)$ which can be written as $(gf)(x) = f(g^t x)$. In the case of powers of linear forms, if $f(x) = (\sum v_i x_i)^d = (v^t x)^d$ then $f(g^t x) = (v^t(g^t x))^d = ((gv)^t x)^d$. In conclusion, we may identify $v$ with the linear form $v^t x$ and we get that $g(v^d) = (gv)^d$. This is coherent with the inclusion $\mathrm{S}ym^d V \subset V \otimes \ldots \otimes V$ and the action extended from $V$ to the tensor product as $g(v_1 \otimes \ldots \otimes v_d) = gv_1 \otimes \ldots \otimes gv_d$ on decomposable elements. For polynomials that are powers of linear forms the action is very simple, namely $g \cdot l^d = (g \cdot l)^d$, $\forall l \in V$.

**Remark 1.2.** *By the unitary trick* [22, 2.7]*, there is an equivalence of categories between holomorphic representations of $SO(n+1,\mathbb{C})$ and continuous representations of its real form $SO(n+1,\mathbb{R})$. In particular it is equivalent to prove harmonic* *decomposition over $\mathbb{C}$ and over $\mathbb{R}$. The unitary trick shows also an equivalence of categories between holomorphic representations of $SL(n+1,\mathbb{C})$ and continuous representations of its real form $SU(n+1)$.*

The Laplace operator $\Delta$ is invariant and the space of harmonic polynomials is an $SO$-module, which means that $f$ is harmonic if and only if $g\cdot f$ is harmonic $\forall g \in SO(V)$ . The following proposition gives a geometric point of view on this basic fact. It also enlights the fact that to understand real polynomials it is useful looking at the larger space of complex ones.

**Proposition 1.3.** *(i) Let $d\geq 2$. Given a linear form $l$, the power $l^d$ is harmonic if and only if $l$ is isotropic.*

*(ii) The space $H_d = \ker\Delta$ of harmonic polynomials of degree $d$ is spanned by powers $v^d$ with $v$ isotropic.*

*(iii) Any space $H_d$ is irreducible as $SO$-module.*

*Proof.* Compute $\Delta(l^d) = d(d-1)q(l,l)l^{d-2}$, which proves (i). Regarding (ii), assume the span of $v^d$ with $v$ isotropic is contained in a hyperplane of $H_d$ with equation $g$. Then $g$ corresponds to a harmonic polynomial which vanishes over the isotropic quadric $q$, then it is divisible by $q$ and it follows $g=0$ by Lemma 1.5, which proves (ii). An alternative argument is that the span of powers $v^d$ with $v$ isotropic is the span of $v_d(Q_{n-1})$ where $Q_{n-1}$ is the quadric of isotropic vectors and $H^0(v_d(Q_{n-1}),\mathcal{O}(1))$ corresponds to $H^0(Q_{n-1},\mathcal{O}(d))$ which can be computed by taking cohomology of the exact sequence

$$0\longrightarrow \mathcal{O}_{\mathbb{P}^n}(d-2)\longrightarrow \mathcal{O}_{\mathbb{P}^n}(d)\longrightarrow \mathcal{O}_{Q_{n-1}}(d)\longrightarrow 0$$

and turns out to have dimension $h^0(\mathcal{O}_{\mathbb{P}^n}(d)) - h^0(\mathcal{O}_{\mathbb{P}^n}(d-2))$, which is the same dimension of the space of harmonic polynomials of degree $d$. (iii) follows because $SO$ acts transitively on $v^d$ with $v$ isotropic. □

**Lemma 1.4.** *Let $f\in\mathbb{C}[x_0,\ldots,x_n]$ be a homogeneous polynomial of degree $d$.*

(1) *i)*
$$\Delta(qf) - q\Delta f = [\Delta, q]f = (4d+2n+2)f$$
(2) *ii)*
$$\Delta(q^kf) - q^k\Delta f = (2k(n+2d+2k-1))q^{k-1}f$$
(3) *iii)*
$$\Delta^k(qf) - q\Delta^k f = (2k(n+2d-2k+3))\Delta^{k-1}f$$

*Proof.* In order to prove (i), we may assume $f=l^d$. Compute

$\partial_x(ql^d) = 2xl^d + dql^{d-1}l_x$

$\partial_{xx}(ql^d) = 2l^d + 4dxl_xl^{d-1} + d(d-1)ql^{d-2}l_x^2$

$q\partial_{xx}l^d = d(d-1)ql^{d-2}l_x^2$

so that summing over all the variables

$\Delta(ql^d) = 2(n+1)l^d + 4dl^d + q\Delta(l^d)$, which proves (i).

(ii) is proved by induction on $k$, the case $k=1$ being item i). Indeed

$\Delta(q^kf) = \Delta q^{k-1}(qf) = q^{k-1}\Delta(qf)+q^{k-2}(4(k-1)(d+2)+2(k-1)n+2(k-1)(2k-3))(qf) =$

$q^{k-1}q\Delta(f) + q^{k-1}\left(4d+2n+2+4(k-1)(d+2)+2(k-1)n+2(k-1)(2k-3)\right)f$

(iii) is proved by induction on $k$, the case $k=1$ being item i). Indeed

$\Delta^k(qf) = \Delta(\Delta^{k-1}(qf) = \Delta(q\Delta^{k-1}(f) + (4(k-1)d + 2(k-1)n - 2(k-1)(2k-5))\,\Delta^{k-2}(f) =$
$= q\Delta^k(f) + (4(d-2k+2) + 2n + 2 + 4(k-1)d + 2(k-1)n - 2(k-1)(2k-5))\,\Delta^{k-1}(f)$

□

**Lemma 1.5.** $\Delta(qf) = 0$ *if and only if* $f = 0$

*Proof.* Assume $f \neq 0$ and let $f = q^k f'$ with maximal $k$. Lemma 1.4 implies $0 = \Delta q^{k+1} f' = q^{k+1}\Delta f' + cq^k f' = q^k(q\Delta f' + cf')$ for a nonzero scalar $c$. It follows that $q$ divides $f'$, which is a contradiction. □

**Proposition 1.6** (Harmonic Part)**.** *For any $f$ there are unique $f_0$ harmonic and $f_1$ such that*

$$f = f_0 + qf_1$$

*The polynomial $f_0$ is called the harmonic part of $f$.*

*If $q$ is real then $(\overline{f})_0 = \overline{f_0}$.*

*If $f$ and $q$ are real then also $f_0$ is real.*

*Proof.* The two subspaces $H_d, q\mathrm{S}ym^{d-2}V$ of $\mathrm{S}ym^dV$ have empty intersection by Lemma 1.5, hence by dimensional reasons there is a direct sum

$$\mathrm{S}ym^dV = H_d \oplus q\mathrm{S}ym^{d-2}V$$

We will see in *** that the two summands are orthogonal, so that the two summands $f_0 \in H_d$ and $qf_1 \in q\mathrm{S}ym^{d-2}V$ in the statement are unique and they correspond to two orthogonal projections. We explain now a naive way to get them. Consider the square system $\Delta f = \Delta(qf_1)$, in the unknown $f_1$. The associated homogeneous system is $\Delta(qf_1) = 0$ which has only the zero solution by Lemma 1.5. Hence there is a solution $f_1$ such that $\Delta f = \Delta(qf_1)$ and $f_0 = f - qf_1$ is harmonic. The uniqueness is now obvious. The statement on the conjugation follows from the uniqueness and also from the explicit construction. □

**Theorem 1.7** (Harmonic Decomposition)**.** *For any $f$ there are unique $f_i$ harmonic of degree $d-2i$ such that*

$$f = \sum_{i=0}^{\lfloor d/2 \rfloor} q^i f_i$$

*If $q$ is real then $(\overline{f})_i = \overline{f_i}$.*

*If $f$ and $q$ are real then also $f_i$ are real. The decomposition corresponds to the splitting of $SO(V)$-modules*

$$\mathrm{S}ym^dV = \oplus_{i\geq 0} H_{d-2i}.$$

*Proof.* By iterating the Proposition 1.10. □

**Corollary 1.8.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

- *When $d$ is odd the only homogeneous polynomial of degree $d$ in $\mathrm{S}ym^d\mathbb{K}^{n+1}$ which is $SO(n+1)$-invariant is zero.*

- *When $d$ is even the only homogeneous polynomial of degree $d$ in $\mathrm{Sym}^d\mathbb{K}^{n+1}$ which are $SO(n+1)$-invariant are scalar multiples of $q^{d/2}$.*

**Remark 1.9.** *Harmonic decomposition is a multi-variable generalization of Fourier expansion. Indeed for $n=1$ each space $H_d$ is two dimensional, spanned by $(x_0+ix_1)^d, (x_0-ix_1)^d$ or, in real polar coordinates $x_0=\rho\cos\theta, x_1=\rho\sin\theta$, by $\rho^d\cos(d\theta), \rho^d\sin(d\theta)$. Restricting to the unit circle we get the standard Fourier expansion.*

**Remark 1.10.** *The proof of Proposition gives an algorithm to compute explicitly the harmonic decomposition of $f$.*

*The first step is to compute $g_1$ such that $\Delta(f-qg_1)=0$, then pose $f_0=f-qg_1$ is the harmonic part and the step can be iterated with $g_1$ at the place of $f$. So compute $g_2$ such that $\Delta(g_1-qg_2)=0$, pose $f_1=g_1-qg_2$. Compute $g_3$ such that $\Delta(g_2-qg_3)=0$, pose $f_2=g_2-qg_3$, and so on. At the end*

$$f=\sum_{i=0}^{\lfloor d/2\rfloor} q^i f_i \qquad \deg f_i = d-2i$$

*Luckily, there is a more efficient way to compute the harmonic decomposition, due to the fact that $f_i$ is linear combination of $\Delta^i f, q\Delta^{i+1}f,\ldots,q^{d/2-i}\Delta^{d/2}f$ when $d$ is even and it is a linear combination of $\Delta^i f, q\Delta^{i+1}f,\ldots,q^{(d-1)/2-i}\Delta^{(d-1)/2}f$ when $d$ is odd. We will see this fact in the section on scalar products.*

## 1.1. **Some special cases.**

**Lemma 1.11.** *$\Delta(qf)$ is harmonic if and only if $f$ is harmonic.*

*Proof.* If $f$ is harmonic then $\Delta^2(qf)=\Delta(q\Delta(f)+cf)=0$. Conversely, let $\Delta^2(qf)=0$. Assume $\Delta(f)\neq 0$ and let $\Delta(f)=q^k f'$ with $f'$ not divisible by $q$, $\deg f'=d-2-2k\geq 0$. Then $0=\Delta^2(qf)=\Delta(q\Delta(f))+\Delta((4d+2n+2)f)=\Delta(q^{k+1}f')+(4d+2n+2)q^k f' = q^{k+1}\Delta f' + (4(k+1)(d-2-2k)+2(k+1)n+2(k+1)(2k+1)+4d+2n+2)q^k f'$ which implies that $q$ divides $f'$ which is a contradiction. □

**Lemma 1.12.** *Let $l$ be a linear form.*

$$\Delta(fl)=l\Delta f+2\partial_l(f)$$

*Proof.*

$$\partial_{xx}(fl)=(\partial_{xx}f)l+2\partial_x f\partial_x l$$

□

**Proposition 1.13.** *Let $f$ harmonic and $l$ be a linear form. Then we have the harmonic decomposition $fl=f_0+qf_1$ with $f_1$ harmonic.*

*Proof.* Let $f_1$ such that $\Delta q f_1=\Delta(fl)=2\partial_l(f)$. Since $\partial_l(f)$ is harmonic we get from Lemma 1.11 that $f_1$ is harmonic. □

For $d=2$ we have the harmonic decomposition

$$f=(f-q\frac{\Delta f}{2n+2})+q\frac{\Delta f}{2n+2}$$

Note if $\deg l = 1$ $\Delta(ql) = (6+2n)l$. Hence for $d = 3$ the harmonic decomposition is

$$f = \left(f - q\frac{1}{6+2n}\Delta f\right) + q\frac{1}{6+2n}\Delta f$$

## 2. Exercises

(1) Prove that $\Delta^i(q^k) = cq^{k-i}$ for a scalar factor $c \in \mathbb{R}$.
(2) Prove that $[\Delta^i, q^k]f = \Delta^i(q^k f) - q^k\Delta^i f = ***$.
(3) Prove that $\ker \Delta^i = \oplus_{j=0}^{i-1} H_{d-2j}$
(4) Prove that $m_q \colon \mathrm{S}ym^d V \to \mathrm{S}ym^{d+2} V$ maps $H_{d-2i}$ to $H_{(d+2)-2(i+1)}$ ***
(5) $\Delta$ maps $H^{d-2i}$ to $H_{(d-2)-2(i+1)}$ ***
(6) Consider the operator $L = q\Delta \colon \mathrm{S}ym^d V \to \mathrm{S}ym^d V$, prove that the summands of harmonic decomposition are eigenspaces of $L$, with some integer eigenvalues. The kernal of $L$ is the space $H_d$.

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