Some constructions of projective varieties

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Algebraic geometry studies the solutions of polynomial systems, which are called varieties. The first goal is to attach to a variety some numerical invariants, like the dimension and the degree, which describe some of its features.

The Hilbert polynomial of $X \subset \mathbf{P}^n$ encodes the more important numerical invariants of X. It is worth to remark that today many computer algebra systems allow to compute the Hilbert polynomial but, due to the big dimension of the employed memory, the range of the problems which are solvable with the help of a computer is still quite limited. The Hilbert polynomial can be easily computed from a resolution of the ideal sheaf of the variety.

The varieties which arise from the applications have often some additional structure or some symmetry. We will see the examples of scrolls, blow-up and conic bundles.

One cultural aspect we want to underline is that the study of varieties mixes two different roots of the mathematical thinking. One is the symbolic manipulation of numbers and polynomials, which goes back (in the Mediterranean world) to Arabs. The other is the geometric insight into drawings, which goes back to Greeks.

Most of the material of this note comes from [Ot].

1 The twisted cubic and complete intersections

Projective varieties of codimension one are given by the zero locus of one homogeneous polynomial. Two independent polynomial define in general a codimension two variety, but the converse is not true. The basic and classical example about this phenomenon is the following.

Example 1.1 The twisted cubic *The twisted cubic is the curve $C \subset \mathbf{P}^3$ given by $f: \mathbf{P}^1 \to \mathbf{P}^3$*

$$f(s,t) = (s^3, s^2t, st^2, t^3)$$

Hence if $(x_0, x_1, x_2, x_3) \in C$ $\exists$ $(s,t) \neq (0,0)$ such that

$$\begin{cases} x_0 &= s^3 \\ x_1 &= s^2t \\ x_2 &= st^2 \\ x_3 &= t^3 \end{cases}$$

Cutting C with a plane we find an equation of degree 3 in s, t, then deg $C = 3$. Let $q_1 := -(x_0x_3 - x_1x_2)$, $q_2 := x_0x_2 - x_1^2$. It is easy to check that $C \subset \{q_1 = q_2 = 0\}$ but the right side contains also the line $L := \{x_0 = x_1 = 0\}$, exactly we have $\{x_0x_3 - x_1x_2 = x_0x_2 - x_1^2 = 0\} = C \cup L$. L is residual, or better C and L are linked (the french term liaison is often used). This is in agreement with Bezout theorem: the (transverse) intersection of two quadrics has degree $2 \cdot 2 = 4$. We have to add the third equation $q_0 := x_1x_3 - x_2^2 = 0$ in order to get C, namely C is defined by the condition

$$rank \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix} \leq 1 \tag{1}$$

Exercise 1.2 *(i) Prove that the three quadrics q_0, q_1, q_2 which are the minors of (1) cut transversely, that is for every point of C the intersection among the three tangent planes coincides with the tangent line of C.*

(ii) Prove that the two first quadrics cut transversely for every point of C with only one exception (which point?)

(iii) Prove that L is tangent to C.

Definition 1.3 *A smooth variety $X \subset \mathbf{P}^n$ of codimension c is called a complete intersection if it is the transverse intersection of c hypersurfaces $F_1, \ldots, F_c$. If deg $F_i = d_i$ this is equivalent to say that the morphism $\oplus\mathcal{O}(-d_i) \stackrel{F_1,\ldots,F_c}{\longrightarrow} \mathcal{I}_X$ is surjective.*

Exercise 1.4 *Prove that, as a set, the twisted cubic C is given by the equations*

$$\begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & 0 \end{vmatrix} = 0$$

Prove that the intersection is not transverse. In the language of schemes, the previous two equations define a double structure on C.

Exercise 1.5 *Prove that a curve in $\mathbf{P}^3$ of prime degree and not contained in a plane is not a complete intersection.*

Theorem 1.6 (Canonical bundle of $\mathbf{P}^n$)

$$K_{\mathbf{P}^n} = \mathcal{O}(-n-1)$$

Theorem 1.7 *Let C be a curve of genus g. Then*

$$deg K_C = 2g - 2$$

Theorem 1.8 (Adjunction formula for hypersurfaces of $\mathbf{P}^n$) *Let $X \subset \mathbf{P}^n$ be a hypersurface of degree d. Then*

$$K_X = \mathcal{O}(-n-1+d)$$

Corollary 1.9 *Let $X \subset \mathbf{P}^n$ be the complete intersection of c hypersurfaces of degree $d_1, \ldots, d_c$. Then $K_X = \mathcal{O}_{\mathbf{P}^n}(-n-1+\sum d_i)$*

Proof Iterate adjunction formula. □

Definition 1.10 *A variety $X \subset \mathbf{P}^n$ is called subcanonical if $K_X = \mathcal{O}_{\mathbf{P}^n}(a)_{|X}$ for some $a \in \mathbf{Z}$.*

By Cor. 1.9 complete intersection varieties are subcanonical.

Remark We remark that the twisted cubic is not subcanonical, indeed $deg\ K_C = -2$ while $deg\ \mathcal{O}_{\mathbf{P}^3}(a)_{|C} = 3a$. Hence Cor. 1.9 gives another property that forbids C to be a complete intersection.

Remark The transverse intersection of two quadrics in $\mathbf{P}^3$ has $K = \mathcal{O}$, then it is an elliptic curve.

2 Degeneracy loci and the Koszul complex

Let E, F be vector bundle on the variety X and let $E \xrightarrow{\phi} F$ be a morphism.

Definition 2.1

$$D_k(\phi) := \{x \in X | rk(\phi_x) \leq k\}$$

is called the k-th degeneracy locus of ϕ.

We remark that $\forall x \in X$ ϕ_x is a vector space morphism, hence ϕ can be understood as a family of vector space morphisms. $D_k(\phi)$ is defined by the ideal generated by the $(k+1) \times (k+1)$ minors of X.

If $k = min\{rank\ E, rank\ F\} - 1$ we speak about maximal degeneracy loci.

The "universal" degeneracy locus is the following. Let

$$M := \mathbf{C}^{mn} = \text{vector space of } m \times n \text{ matrices}$$

$$M_k := \{m \in M | rk(m) \leq k\}$$

Let V, W be two vector spaces of dimension respectively n, m, let $\mathbf{V}, \mathbf{W}$ be the trivial vector spaces on the base $Hom(V, W)$. Consider the morphism $\mathbf{V} \xrightarrow{\psi} \mathbf{W}$ which at every $x \in Hom(V, W)$ takes v in $x(v)$, it follows that $M_k = D_k(\psi)$.

We have a filtration

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_{min(m,n)} = M$$

Theorem 2.2 *M_k is an algebraic irreducible subvariety of M of codimension $(m-k)(n-k)$. Moreover $SingM_k = M_{k-1}$*

Proof

The equations of M_k in M are given by the $(k+1) \times (k+1)$ minors. From Gauss elimination algorithm it follows that $M_k \setminus M_{k-1}$ are the orbits for the natural action of $GL(m) \times GL(n)$ su M. Hence $SingM_{k-1} \subset M_{k-1}$ and M_k is irreducible since it is the closure of an irreducible orbit. The dimension of M_k can be computed in a point where the first $k \times k$ minor is $\neq 0$. Then a local computation shows that M_k is given locally by $(m-k)(n-k)$ independent equations.

The last step is the check that at a point of M_{k-1} one of the partial derivatives of all $(k+1)\times(k+1)$ minors vanish (Laplace development along a line), hence the points of M_{k-1} are singular points for M_k. □

M_k is a cone, by abuse of notation we call with the same name the projectivization $M_k \subset \mathbf{P}(M) = \mathbf{P}^{mn-1}$. M_1 is isomorphic to the Segre variety $\mathbf{P}^{n-1} \times \mathbf{P}^{m-1}$, M_2 is the secant variety of M_1. In the same way M_k is the closure of the union of the linear spaces spanned by k-ples of points of M_1. It is classically known

$$deg\ M_k = \prod_{i=0}^{n-k-1} \frac{(m+i)!i!}{(k+i)!(m-k+i)!} \tag{2}$$

In particular

$$deg M_1 = \binom{m+n-2}{n-1}$$

Lemma 2.3 *Let E, F be vector bundles on a variety X such that $rkE = m$, $rkF = n$. Let $\phi: E \longrightarrow F$ be a morphism X. Then $codim_X D_k(\phi) \leq (m-k)(n-k)$.*

Proof Denote by $V(E^* \otimes F)$ the variety associated to the bundle $E^* \otimes F$, equipped with the projection $V(E^* \otimes F) \xrightarrow{\pi} X$, such that each fiber $\mathbf{C}^{mn}$ contains a subvariety isomorphic to M_k. This subvariety is well defined, it does not depend on the local coordinates. We get a global subvariety $\Sigma_k \subset V(E^* \otimes F)$ of codimension $(m-k)(n-k)$ whose fibers over X are isomorphic to M_k. ϕ defines a section of $E^* \otimes F$, hence a subvariety X' di $V(E^* \otimes F)$ isomorphic to X. We have $D_k(\phi) \simeq \Sigma_k \cap X'$, hence the thesis follows. □

Definition 2.4 *If $codim_X D_k(\phi) = (m-k)(n-k)$ we say that $D_k(\phi)$ has the expected codimension.*

Definition 2.5 *A vector bundle is called spanned (by global sections) if the natural evaluation map*

$$\mathcal{O} \otimes H^0(E) \longrightarrow E$$

is surjective

For example $\mathcal{O}(n)$ is spanned if and only if $n \geq 0$. The quotient of a spanned bundle is still spanned. In particular $T\mathbf{P}^n(m)$ is spanned if and only if $m \geq -1$ and $\Omega^1(m)$ is spanned if and only if $m \geq 2$.

Theorem 2.6 *(Bertini type) Let E, F be vector bundles on a variety X such that $rkE = m$, $rkF = n$. Let $E^* \otimes F$ be spanned by global sections. If $\phi: E \longrightarrow F$ is a generic morphism, then one of the following holds*

(i) $D_k(\phi)$ is empty

(ii) $D_k(\phi)$ has the expected codimension $(m-k)(n-k)$ and $Sing\ D_k(\phi) \subset D_{k-1}(\phi)$.

In particular if $dim\ X < (m-k+1)(n-k+1)$ then $D_k(\phi)$ is empty or smooth when ϕ is generic.

Proof By assumption we have the exact sequence

$$H^0(E^* \otimes F) \otimes \mathcal{O} \longrightarrow E^* \otimes F \longrightarrow 0$$

which induces the projection

$$X \times H^0(E^* \otimes F) \xrightarrow{p} V(E^* \otimes F) \supset \Sigma_k$$

which has maxinal rank everywhere.

We have

$$Z := p^{-1}(\Sigma_k) \xrightarrow{q} H^0(E^* \otimes F)$$

We remark that $Z = \{(x, \phi) | rk(\phi_x) \leq k-1\}$ then $q^{-1}(\phi_0) = \{(x, \phi_0) | rk((\phi_0)_x) \leq k-1\} \simeq D_k(\phi_0)$ Observe that $Sing\ Z = p^{-1}(Sing\ \Sigma_k)$ and consider

$$Z \setminus Sing\ Z \xrightarrow{p} H^0(E^* \otimes F)$$

We get

$$p^{-1}_{|Z \setminus Sing\ Z}(\phi) \simeq D_k(\phi) \setminus D_{k-1}(\phi)$$

There are two cases:

1) If $p_{|Z \setminus Sing\ Z}$ has dense image then $D_k(\phi)$ is smooth by generic smoothness theorem [Hart] cor. III 10.7, hence $Sing\ D_k(\phi) \subset D_{k-1}(\phi)$

2) If $p_{|Z \setminus Sing\ Z}$ has dense image then $D_k(\phi)$ is empty for generic ϕ . □

Remark Bertini proved the previous theorem in the case $E = \mathcal{O}$ and F is a spanned line bundle (put $k = 0$). The zero loci of $s \in H^0(E)$ were called "elements of the linear system".

Remark The inclusion of Thm. 2.6 can be strengthened to $Sing\ D_k(\phi) = D_{k-1}(\phi)$([Ban] 4.1), by using that M_k are Cohen-Macaulay varieties.

Corollary 2.7 *Let E be spanned of rank r, then*

(i) the zero locus of a generic section of E is empty or smooth of codimension r.

(ii) if dim $X \leq 5$ then the degeneracy locus of $r-1$generic sections of E is empty or smooth of codimension 2.

(iii) if dim $X \leq 7$ then the degeneracy locus of $r-2$generic sections of E is empty or smooth of codimension 3.

Exercise 2.8 *Compute the number of singular points of the 4-fold in $\mathbf{P}^6$ which is the degeneracy locus of a generic morphism $\mathcal{O}^2 \longrightarrow \mathcal{O}(1)^3$.*

Let E be a bundle of rank r on X and let $Z \subset X$ be a (irreducible) subvariety of codimension r which is the zero locus of $s \in H^0(E)$. We may construct a resolution of the sheaf $\mathcal{O}_Z$ which is named *Koszul complex*

$$0 \longrightarrow det\ E^* \longrightarrow \wedge^{r-1} E^* \longrightarrow \ldots \longrightarrow E^* \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0 \qquad (3)$$

The name "complex" is due to the fact that the sequence exists as a complex even when the codimension of Z is not the expected one, but with our assumptions the Koszul complex *is exact!*

Let K be a field. Let $S = K[x_0, \dots, x_n] = \oplus_q S^q(V)$, the resolution of K as S-module is the "universal" Koszul complex.

It is:

$$0 \longrightarrow \wedge^{n+1} V \otimes S(-n-1) \longrightarrow \wedge^n V \otimes S(-n) \longrightarrow \dots \longrightarrow V \otimes S(-1) \longrightarrow S \longrightarrow K \longrightarrow 0$$

All the morphisms are defined by the natural maps

$$\wedge^p(V) \otimes S^q(V) \longrightarrow \wedge^{p-1}(V) \otimes S^{q+1}(V)$$

defined (in coordinates) by

$$(z_{i_1} \wedge z_{i_2} \wedge \dots \wedge z_{i_p}) \otimes m \mapsto \sum_{j=1}^{p} (-1)^{j-1} \left(z_{i_1} \wedge \dots \wedge \hat{z}_{i_j} \wedge \dots \wedge z_{i_p} \right) \otimes (z_{i_j} \cdot m)$$

A proof of the exactness can be found in [Gre] or in [Ei].

Definition 2.9 *If $X \subset \mathbf{P}^n$ is a subvariety, its ideal sheaf is defined as the kernel of the restriction map, that is from the exact sequence*

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbf{P}^n} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Proposition 2.10 *Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^m$. Let $\chi(\mathcal{F}) = \sum_{i=0}^{m} h^i(\mathcal{F})$. Then*

(i) χ is additive, that is if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence we have

$$\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$$

(ii) $\chi(\mathcal{O}(t)) = \binom{t+m}{m}$

Exercise 2.11 *Let $X \subset \mathbf{P}^5$ be the intersection of two quadrics. X is the zero locus of a section of $\mathcal{O}(2)^2$. The resolution of the ideal sheaf $\mathcal{I}_X$ is:*

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-2)^2 \longrightarrow \mathcal{O} \longrightarrow \mathcal{I}_X \longrightarrow 0$$

hence we get the Hilbert polynomial

$$\chi(\mathcal{O}_X(t)) = \frac{1}{3}(t+1)(2t^2+4t+3)$$

Riemann-Roch theorem implies the following facts.

• If X is a curve in $\mathbf{P}^n$ then

$$\chi(\mathcal{O}_X(t)) = dt + (1-g)$$

where d is the degree and g is the genus.

• If X is a surface in $\mathbf{P}^n$ then

$$\chi(\mathcal{O}_X(t)) = \frac{d}{2}t^2 + (1 - g + \frac{d}{2})t + (1 + p_a)$$

where d is the degree and g is the genus of a generic hyperplane section and p_a is called the arithmetic genus. Rational surfaces have $p_a = 0$.
• If X has dimension m then

$$\chi(\mathcal{O}_X(t)) = \frac{d}{m!}t^m + \ldots + (1 + (-1)^m p_a)$$

A free resolution of a sheaf $\mathcal{F}$ on $\mathbf{P}^n$ is an exact sequence

$$0 \longrightarrow L_m \longrightarrow \ldots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where each L_i is a sum ofline bundles $\mathcal{O}(a_i)$. For more on free resolutions we refer to [Kunz] or to [Ei].

Exercise 2.12 *Write a free resolution of $\mathcal{O}_Q$ where Q is a point in $\mathbf{P}^3$.*

Solution: Q is the intersection of three hyperplanes, hence it is the zero locus of a section of $\mathcal{O}(1)^3$. Then the Koszul complex is

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-2)^3 \longrightarrow \mathcal{O}(-1)^3 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Q \longrightarrow 0$$

Exercise 2.13 *Write a free resolution of $\mathcal{O}_L$ where L is a line in $\mathbf{P}^3$.*

Exercise 2.14 *Let C be a smooth curve in $\mathbf{P}^n$ which is the zero locus of a section of a bundle E of rank $n-1$. Compute degree and genus of C in terms of $c_i := c_i(E)$*

Answer: $d = c_{n-1} \quad g = 1 + \frac{1}{2}(c_1 - n + 1)c_{n-1}$. In particular $c_1 c_{n-1} \equiv 0 \pmod 2$ if n is odd.

We will see more examples of computations of Hilbert polynomial in the next section.

3 k-normality and examples of resolutions

Definition 3.1 *$X \subset \mathbf{P}^n$ is called k-normal if one of the two following equivalent facts holds*
(i) $H^0(\mathbf{P}^n, \mathcal{O}(k)) \longrightarrow H^0(X, \mathcal{O}(k))$ is surjective.
(ii) $H^1(I_X(k)) = 0$.
X is called projectively normal if it is k-normal $\forall k \geq 0$.

The twisted cubic is projectively normal
Proof After the identification $C \simeq \mathbf{P}^1$ we have $\mathcal{O}(k)_{|C} \simeq \mathcal{O}_{\mathbf{P}^1}(3k)$. We have to show that if F is a homogeneous polynomial of degree $3k$ in s, t then there exists a homogeneous polynomial $G(x_0, x_1, x_2, x_3)$ of degree k such that

$$F(s,t) = G(s^3, s^2t, st^2, t^3)$$

Let $F = s^i t^{3k-i}$. Dividing by 3 we get:

$$i = 3p + q \quad 3k - i = 3p' + q' \text{with } 0 \leq q, q' \leq 2 \quad q + q' \equiv 0 \ (mod 3)$$

then $s^i t^{3k-i} = (s^3)^p (t^3)^{p'} s^q t^{q'}$ as we wanted. □

Exercise 3.2 *The rational normal curve in* $\mathbf{P}^n$ *is the curve* C *given by* $f: \mathbf{P}^1 \to \mathbf{P}^n$

$$f(s,t) = (s^n, s^{n-1}t, \ldots, st^{n-1}, t^n)$$

Prove that C *is projectively normal.*

Resolution of the twisted cubic We claim that the resolution of the twisted cubic is

$$0 \longrightarrow \mathcal{O}^2(-1) \xrightarrow{\phi} \mathcal{O}^3 \xrightarrow{\psi} \mathcal{I}_C(2) \longrightarrow 0 \tag{4}$$

where ψ is given by the three quadrics q_0, q_1, q_2 which are the minors of (1). Indeed developing along the first line the following

$$\det \begin{bmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix} = 0$$

we get

$$x_0 q_0 + x_1 q_1 + x_2 q_2 = 0$$

and in the same way

$$x_1 q_0 + x_2 q_1 + x_3 q_2 = 0$$

that is

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix} = 0$$

Hence if ϕ is defined by

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$$

we get that (4) is a complex, and a dimension count shows the exactness.

Remark that the resolution of the twisted cubic is not a Koszul complex. From the resolution it is easy to verify that

$$\chi(\mathcal{O}_C(t)) = 3t + 1$$

as we knew before.

We remark also that from Cor. 2.7 (ii) it follows that the curve obtained from a resolution of the form (4) with generic ϕ is smooth, and it has degree 3 and $g = 0$ by computing the Hilbert polynomial. It is not difficult to prove that it coincides with the twisted cubic after a linear change of coordinates.

Example 3.3 *Let C be the complete intersection of two quadrics in $\mathbf{P}^3$. We have the resolution*

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{I}_C(2) \longrightarrow 0$$

It follows

$$\chi(\mathcal{O}_C(t)) = 4t$$

$\mathbf{P}^2$ blown up in one point (Hirzebruch surface) We mimic the construction of the twisted cubic.

In $\mathbf{P}^4$ we have

$$0 \longrightarrow \mathcal{O}^2(-1) \xrightarrow{\phi} \mathcal{O}^3 \longrightarrow \mathcal{I}_X(2) \longrightarrow 0$$

where X is smooth if

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \end{bmatrix}$$

is generic (ϕ_{ij} are linear forms) by Cor. 2.7(ii). If $x \in X$ then the two rows of ϕ are dependent when computed at x, that is there are constants $(\lambda_1, \lambda_2) \neq (0,0)$ such that

$$\lambda_1 \phi_1(x) + \lambda_2 \phi_2(x) = 0$$

which give a morphism

$$X \longrightarrow \mathbf{P}^1$$

$$x \mapsto (\lambda_1, \lambda_2)$$

The fiber of such a morphism is given by

$$\lambda_1 \phi_1 + \lambda_2 \phi_2 = 0$$

which is the intersection of three hyperplanes, hence it is a line (it cannot be a plane because by Cor. 2.7(ii) X is smooth).

X is called a scroll.

Definition 3.4 *$X \subset \mathbf{P}^n$ is called a scroll if there exists a morphism $f \colon X \to Y$ and an integer m such that $f^{-1}(y) = \mathbf{P}^m$ $\forall y \in Y$.*

In the same way we have a morphism

$$X \longrightarrow \mathbf{P}^2$$

$$x \mapsto (\phi_{11}(x), \phi_{12}(x), \phi_{13}(x))$$

which is well defined because when ϕ_1 vanish we can use ϕ_2. The fiber of such a morphism over (μ_1, μ_2, μ_3) is given by the condition

$$rank \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \end{bmatrix} = 1$$

In particular the fiber is always a linear space, so it is generically a point and it is a line in x points (these lines are called exceptional divisors). So the surface is isomorphic to the plane blown up in x points. By the scroll structure it is easy to check that $x = 1$, indeed the exceptional divisors must be different from the fibers of the scroll.

This surface is obtained as the image of the system of plane conics through a point. If the point has coordinates $(0, 1, 0)$ we get the rational(means that it is not everywhere defined) morphism

$$\mathbf{P}^2 - - \to \mathbf{P}^4$$

$$(x_0, x_1, x_2) \mapsto (y_0, \ldots, y_4) := (x_0^2, x_0x_1, x_0x_2, x_1x_2, x_2^2)$$

(note that x_1^2 is missing) which fill the table

	x_0	x_1	x_2
x_0	x_0^2	x_0x_1	x_0x_2
x_2	x_0x_2	x_1x_2	x_2^2

Indeed the surface given by the equations

$$rank \begin{bmatrix} y_0 & y_1 & y_2 \\ y_2 & y_3 & y_4 \end{bmatrix} = 1 \qquad (5)$$

is filled by the lines

$$\lambda_1(y_0, y_1, y_2) + \lambda_2(y_2, y_3, y_4) = 0$$

Exercise 3.5 *Find the equation of the exceptional divisor of the surface (5)*

We can verify from the resolution

$$\chi(\mathcal{O}(t)) = \frac{3}{2}t^2 + \frac{5}{2}t + 1$$

hence $d = 3$, $g = p_a = 0$, indeed the hyperplane sections of X are twisted cubics!

In $\mathbf{P}^5$ the same resolution gives $\mathbf{P}^1 \times \mathbf{P}^2$. In $\mathbf{P}^n$ for $n \geq 6$ we get singular varieties.

The Bordiga surface

Let S be the surface given by

$$0 \longrightarrow \mathcal{O}(-4)^3 \xrightarrow{\phi} \mathcal{O}(-3)^4 \longrightarrow \mathcal{I}_S \longrightarrow 0$$

where ϕ is generic. ϕ can be seen as a $3 \times 4 \times 5$ hypermatrix. Again by Cor. 2.7(ii) S is smooth.

Let ϕ_0, ϕ_1, ϕ_2 be the three rows. We have $S := \{x \in \mathbf{P}^4 | rank\phi(x) \leq 2\}$. By Thm. 2.6 we have that the expected codimension of $D_1(\phi)$ is 6, hence the rank is never 1 (for generic ϕ), then $\forall x \in S$ the three rows of $\phi(x)$ are dependent and there are $\lambda_0(x), \lambda_1(x), \lambda_2(x)$ such that $\lambda_0(x)\phi_0(x) + \lambda_1(x)\phi_1(x) + \lambda_2(x)\phi_2(x) = 0$. Hence we have a morphism $S \longrightarrow \mathbf{P}^2$.

Its fiber in $(\lambda_0, \lambda_1, \lambda_2)$ is given by the equation

$$\lambda_0\phi_0 + \lambda_1\phi_1 + \lambda_2\phi_2 = 0 \tag{6}$$

that it is the intersection of 4 hyperplanes in $\mathbf{P}^4$. Hence the fiber is always a linear space, so that so it is generically a point and it is a line(exceptional divisor) in x points. So the surface is isomorphic to the plane blown up in x points.

Now we need a more subtle argument in order to show that $x = 10$.

The system (6) can be seen as a 4×5 matrix of linear polynomials in λ_i and when the λ_i vary we get a projective plane immersed in the projective space of 4×5 matrices. Hence the fiber is a line exactly for the values (λ_i) corresponding to the intersection of the plane with the variety M_3 of rank ≤ 3 matrices. By (2) we have *deg* $M_3 = \frac{5!0!}{3!2!} = 10$ as we wanted.

The minors of the 4×5 matrix give plane quartics, indeed it can be shown that the surface is given by the system of quartics through the 10 points.

From the resolution we compute

$$\chi(\mathcal{O}(t)) = 3t^2 + t + 1$$

hence $d = 6$, $g = 3$, $p_a = 0$. Note that $g = 3$ is, correctly, the genus of a plane quartic. The fact that $d = 6$ gives another argument to show that $x = 10$. Indeed the intersection of two hyperplane sections gives 6 points, and at the level of plane quartics this gives the equation $4^2 - x = 6$, hence $x = 10$.

Exercise 3.6 The Castelnuovo conic bundle

Let $S \subset \mathbf{P}^4$ be defined by the sequence

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{\phi} \mathcal{O}(1)^2 \oplus \mathcal{O}(2) \longrightarrow \mathcal{I}_S(4) \longrightarrow 0$$

with generic ϕ.

(i) prove that S is smooth.

(ii) check that $\chi(\mathcal{O}_S(t)) = \frac{1}{2}(5t^2 + 3t + 2)$, hence $d = deg\ S = 5$, $g = 2$, $p_a = 0$.

(iii) check that there is a morphism $f\colon S \to \mathbf{P}^1$ such that the fibers are conics. S is called a conic bundle.

It is possible to prove that there exactly 7 singular fibers.

Sections of $\Omega^1(2)$

Consider the last part of the Koszul complex on $\mathbf{P}(V)$

$$\begin{array}{ccccccc} \mathcal{O} \otimes \wedge^2 V & & \longrightarrow & & \mathcal{O}(1) \otimes V & \longrightarrow & \mathcal{O}(2) & \longrightarrow & 0 \\ & \searrow & & \nearrow & & & & & \\ & & \Omega^1(2) & & & & & & \end{array}$$

The fiber of Ω^1 at the point $x \in \mathbf{P}^n$ can be identified with $\{v \in V | vx^t = 0\}$. We have the identification

$$H^0(\Omega^1(2)) = \wedge^2 V \tag{7}$$

accordingly the global sections of $\Omega^1(2)$ can be identified with skew symmetric matrices A, in such a way on the fiber at x we have the vector $xA \in V$ (since $xAx^t = 0$).

If A is generic the zero locus is empty if n is odd and consists of a single point if n is even.

Remarks on the pfaffian Let A be a skew symmetric matrix of even order $2m$. If $e_1, \ldots e_{2m}$ is a basis of V, we identify A with the 2-form $\sum_{i<j} a_{ij} e_i \wedge e_j$.

We recall that we have the formula

$$A \wedge A \wedge \ldots \wedge A = \frac{Pf(A)}{m!} e_1 \wedge \ldots \wedge e_{2m}$$

which defines $Pf(A)$ which is called the pfaffian of A.

Theorem 3.7

$$Pf(BAB^t) = \det(B) Pf(A)$$

$$Pf(A)^2 = \det(A)$$

Proposition 3.8 *Let A be a skew symmetric matrix of odd order n. Let A_{ij} be obtained by deleting the i-th row and the j-th column. Let $C_i = Pf(A_{ii})$. The following holds*

(i) det $A_{ij} = C_i C_j$

(ii) If rank A is $n-1$ (maximal rank) then all the solutions of the homogeneous system $A \cdot x = 0$ are proportional to the vector $(C_1, \ldots, C_n)$.

Proof We may assume that $rank A = n-1$ also to prove (i). Consider the homogeneous system $AX^t = 0$. Deleting the i-th row we get a system $(n-1) \times n$, whose solutions are all proportional to the maximal minors $(\det A_{i1}, -\det A_{i2}, \det A_{i3}, \ldots)$

Hence the direction of the above vector does not depend on i. It follows that

$$(C_i C_j)^2 = \det A_{ij}^2$$

and the thesis follows by checking the sign of a distinguished monomial. □

Example

$$Pf \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + cd$$

Veronese surface in $\mathbf{P}^4$ Let $S \subset \mathbf{P}^4$ be defined by

$$0 \longrightarrow \mathcal{O}(-1)^3 \xrightarrow{f} \Omega^1(1) \longrightarrow \mathcal{I}_S(2) \longrightarrow 0$$

with generic f.

We follow an argument of Castelnuovo [Ca]. By (7)f defines a subspace $Span(A_0, A_1, A_2) \subset \wedge^2 V$, it has generically rank 3 and it has rank 2 at points x such that there exists $(\lambda_0, \lambda_1, \lambda_2) \neq (0,0,0)$ with

$$(\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2)\, x = 0$$

Hence by Prop. 3.8 (ii) the principal pfaffians of $\lambda_0 A_0 + \lambda_1 A_1 + \lambda_2 A_2$ give a regular parametrization of S in terms of $(\lambda_0, \lambda_1, \lambda_2)$. This shows that S is isomorphic to the projective plane. The 3×3 minors of the 5×3 matrix $(A_0 x | A_1 x | A_2 x)$ define equations for S (they are 10 cubics, with three relations, indeed $h^0(\mathcal{I}_S(3)) = 7$). Castelnuovo (by using Plücker coordinates for lines) proves that the plane

$$\pi_x := \{y | (y^t) A_0 x = 0, (y^t) A_1 x = 0, (y^t) A_2 x = 0\}$$

meets the surface in x and in a conic. These ∞^2 conics are the image of the lines in $\mathbf{P}^2$.

From the resolution it follows $H^1(I_S(1)) = H^1(\Omega^1) = \mathbf{C} \neq 0$, that is S is not 1-normal. Indeed $H^0(\mathbf{P}^4, \mathcal{O}(1)) = \mathbf{C}^5$ while $H^0(S, \mathcal{O}(1)) = \mathbf{C}^6$. The geometric explanation is that S is the projection from the celebrated Veronese surface in $\mathbf{P}^5$. This last can be seen as the locus of rank 1 symmetric matrices in the space $\mathbf{P}^5 = \mathbf{P}(S^2\mathbf{C}^3)$ of 3×3 symmetric matrices.

Exercise 3.9 *Consider C defined by*

$$0 \longrightarrow \mathcal{O}(-1)^3 \longrightarrow \Omega^1(1) \oplus \mathcal{O} \longrightarrow I_C(2) \longrightarrow 0$$

Prove (from the Hilbert polynomial) that $d = 4$, $g = 0$. C is a quartic rational curve in $\mathbf{P}^3$. The fact that $H^1(I_C(1)) \neq 0$ tells us that C is the projection from the quartic rational normal curve in $\mathbf{P}^4$.

A sample of a classification result is the following

Theorem 3.10 (Lanteri-Aure, see [La]) *Let $S \subset \mathbf{P}^4$ be a smooth scroll in $\mathbf{P}^4$. Then S is the plane blown up in one point of Example 3.3 or it is the elliptic quintic scroll.*

For recent results and a survey about constructions of projective varietiese with many examples we recommend [KMMNP].

References

[Ban] C. Banica, Smooth Reflexive Sheaves, Rev. Roumaine Math. Pures Appl., 36, 571-593 (1993)

[Ca] G.Castelnuovo, Ricerche di geometria della retta nello spazio a quattro dimensioni, Atti R. Ist. Veneto, ser, VII, 2, 855-901 (1891)

[Ei] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, GTM 150, Springer, New York, 1995

[Gre] M. Green, Koszul cohomology and geometry, in: Lectures on Riemann Surfaces, World Sc. Press 1989, M. Cornalba ed.

[Harr] J. Harris, Algebraic Geometry, A first course, Springer 1992, GTM 133

[Hart] R. Hartshorne, Algebraic Geometry, Springer 1977, GTM 52

[KMMNP] J. Kleppe, J. Migliore, R. Miró-Roig, U. Nagel, C. Peterson, Gorenstein liaison, complete intersection, liaison invariants and unobstructedness, Mem. AMS 154 (2001)

[Kunz] Introduction to Commutative Algebra and Algebraic Geometry, Birkhauser 1985

[La] A. Lanteri, On the existence of scrolls in $\mathbf{P}^4$, Atti Accad. Naz. Lincei, 69 (1980)

[OSS] C. Okonek, M. Schneider, H. Spindler, Vector bundles on complex projective spaces, Birkhauser 1980

[Ot] G. Ottaviani, Varietà proiettive di codimensione piccola, INDAM course, Aracne, Roma 1995, available at http://www.math.unifi.it/ottavian

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