APOLARITY THEORY

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Abstract. Apolarity Theory, starting from binary froms, including equations for secant varieties to rational normal curves.

1. The dual ring

Let V be an $n+1$ dimensional $\mathbb{C}$-vector space and denote its dual by $V^\vee$. We also consider the symmetric algebras $S = \mathbb{C}[x_0, \ldots, x_n] := \mathrm{Sym}(V)$ and $T = \mathbb{C}[\partial_0, \ldots, \partial_n] := \mathrm{Sym}(V^\vee)$. As the labeling of the variables suggests we let T act linearly on S by formal differentiation. This action will be indicated by a dot, e.g., $g \cdot f$ for $f \in S$ and $g \in T$. We note that $g_1 \cdot (g_2 \cdot f) = (g_1 g_2) \cdot f$ for all $f \in S$ and all $g_1, g_2 \in T$. Moreover, given multi-indices $\alpha = (\alpha_0, \ldots, \alpha_n) \in (\mathbf{Z}_{\geq 0})^{n+1}$ and $\beta = (\beta_0, \ldots, \beta_n) \in (\mathbf{Z}_{\geq 0})^{n+1}$ we introduce the shortcuts

$$\partial^\alpha := \partial_0^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \quad \text{and} \quad x^\beta := x_0^{\beta_0} x_1^{\beta_1} \cdots x_n^{\beta_n}$$

as well as

$$|\alpha| := \sum_{i=0}^n \alpha_i \,, \quad \alpha! := \prod_{i=0}^n \alpha_i! \quad \text{and} \quad \binom{d}{\alpha} := \frac{d!}{\alpha!} = \frac{d!}{\alpha_0! \cdots \alpha_n!} \,,$$

where in the latter $d = |\alpha|$.

Lemma 1.1. *Let α and β be multi-indices with $|\alpha| = |\beta|$, then*

$$\partial^\alpha \cdot x^\beta = \begin{cases} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

Proof. Clearly, $\partial^\alpha \cdot x^\alpha = \alpha!$ by the rules of formal differentiation. On the other hand, if $\alpha \neq \beta$ then $|\alpha| = |\beta|$ yields some j such that $\alpha_j > \beta_j$. The latter implies $\partial^\alpha \cdot x^\beta = 0$. □

Let us point out a direct consequence of Lemma 1.1. The bilinear map

$$\mathrm{Sym}^d(V^\vee) \times \mathrm{Sym}^d(V) \to \mathbb{C}, \quad (g, f) \mapsto g \cdot f \tag{1.1}$$

is a dual pairing. It gives an isomorphism $\mathrm{Sym}^d(V^\vee) \cong \big(\mathrm{Sym}^d(V)\big)^\vee$ under which $(\partial^\alpha)_{|\alpha|=d}$ becomes the dual basis of $\big((\alpha!)^{-1} x^\alpha\big)_{|\alpha|=d}$.

Lemma 1.2. *Let $g \in T_d = \mathrm{Sym}^d(V^\vee)$ and let $l = \sum_{i=0}^n c_i x_i \in S_1 = \mathrm{Sym}^1(V) = V$, where $c_i \in \mathbb{C}$. Then $g \cdot l^d = d!\, g(c_0, c_1, \ldots, c_n)$.*

Proof. The multinomial theorem gives

$$l^d = \sum_{|\beta|=d} \binom{d}{\beta} c^\beta x^\beta \quad \text{and we can write} \quad g = \sum_{|\alpha|=d} g_\alpha \partial^\alpha$$

with $g_\alpha \in \mathbb{C}$. Applying Lemma 1.1 we conclude

$$g \cdot l^d = \sum_{|\alpha|=d} g_\alpha c^\alpha \alpha! \binom{d}{\alpha} = \sum_{|\alpha|=d} g_\alpha c^\alpha d! = d!\, g(c_0, c_1, \ldots, c_n)\,,$$

which is the claim. □

2. Generalities and Apolarity Theory for binary forms

Definition 2.1. The **annihilator** or **apolar ideal** of $f \in \mathrm{Sym}^d(V)$ is the homogeneous ideal

$$f^\perp := \left\{ g \in \mathrm{Sym}(V^\vee) \mid g \cdot f = 0 \right\}$$

of $\mathrm{Sym}(V^\vee)$. Its d-th homogeneous part $(f^\perp)_d$ is called the **socle** of $f^\perp$. Moreover, as $f^\perp$ is homogeneous, we can consider the graded ring

$$A_f := \mathrm{Sym}(V^\vee)/(f^\perp) = \bigoplus_{e=0}^{\infty} \mathrm{Sym}^e(V^\vee)/(f^\perp)_e\,,$$

which is called the **apolar ring** of f.

The notation A_f is quite common in the literature. To avoid confusion, let us point out that the apolar ring is *not* related to localization at all.

Remark 2.2. *Let $f \in S_d = \mathrm{Sym}^d(V)$, $f \neq 0$.*

1. *The socle $(f^\perp)_d$ has codimension one in the $\mathbb{C}$-vector space $\mathrm{Sym}^d(V^\vee)$.*
2. *If $k > d$, then $(f^\perp)_k = \mathrm{Sym}^k(V^\vee)$.*
3. *By part (2) the graded $\mathbb{C}$-algebra A_f is Artinian, because $(A_f)_k = 0$ for all $k > d$ and $(A_f)_e$ is finite dimensional for all $e \leq d$.*

The following proposition will be needed to prove the Apolarity Lemma, Theorem 2.13.

Proposition 2.3. *Let $f \in \mathrm{Sym}^d(V)$. The apolar ideal $f^\perp$ is determined by its socle $(f^\perp)_d$, namely for all $e < d$*

$$(f^\perp)_e = \left[(f^\perp)_d : \mathcal{M}^{d-e} \right]_e := \left\{ g \in T_e \mid \forall h \in \mathcal{M}^{d-e}\colon\ (gh) \cdot f = 0 \right\},$$

where $\mathcal{M} := (\partial_0, \ldots, \partial_n)$ is the irrelevant ideal of T.

Proof. Since $f^\perp$ is an ideal, the inclusion $(f^\perp)_e \subseteq [(f^\perp)_d : \mathcal{M}^{d-e}]_e$ follows immediately. Conversely, for $g \in [(f^\perp)_d : \mathcal{M}^{d-e}]_e$ we have $(g\,\partial^\alpha) \cdot f = \partial^\alpha \cdot (g \cdot f) = 0$ for all multi-indices α with $|\alpha| = d - e$. Together with Lemma 1.1 this implies that all coefficients of $g \cdot f \in S_{d-e}$ are zero. Thus $g \cdot f = 0$, i.e., $g \in (f^\perp)_e$. □

The next proposition is equivalent to saying that A_f is a *Gorenstein* Artinian ring.

Proposition 2.4. *Let $f \in \mathrm{Sym}^d(V)$ and $e \in \{0, 1, \ldots, d\}$. The multiplication*

$$(A_f)_e \times (A_f)_{d-e} \to (A_f)_d \cong \mathbb{C}$$

is a perfect pairing, that is, given $[g] \in (A_f)_e$, if $[gt] = 0$ $\forall [t] \in (A_f)_{d-e}$ then $[g] = 0$. In particular, $\dim_{\mathbb{C}}(A_f)_e = \dim_{\mathbb{C}}(A_f)_{d-e}$.

Proof. We write $[g]$ for the equivalence class of $g \in T$ in $A_f = T/(f^\perp)$. By symmetry, it is enough to show that the pairing is non-degenerate in one component. Let $[t] \in (A_f)_e$ with $[tu] = 0$ in $(A_f)_d$ for all $[u] \in (A_f)_{d-e}$. In particular, $tu \in (f^\perp)_d$ for all $u \in \mathcal{M}^{d-e} \subseteq T_{d-e}$, i.e., $t \in [(f^\perp)_d : \mathcal{M}^{d-e}]_e$. Finally, Proposition 2.3 implies $t \in (f^\perp)_e$, i.e., $[t] = 0$ in $(A_f)_e$. □

The above proposition says that any graded Artinian ring of socle dimension one is Gorenstein. Macaulay Theorem states that any graded Artinian ring of socle dimension one is isomorphic to A_f for some f. Moreover A_f is isomorphic to A_g if and only if f and g differ by a scalar multiple. Indeed f can be recovered (up to scalar multiple) by the ring structure of A_f from the composition

$$\mathrm{Sym}^d V^\vee \longrightarrow \mathrm{Sym}^d (A_f)_1 \longrightarrow (A_f)_d \simeq \mathbb{C}$$

which corresponds to f, seen as multilinear map. In the same way any Artinian ring A of socle dimension one gives a polynomial f by the same construction, namely the composition

$$\mathrm{Sym}^d V^\vee \longrightarrow \mathrm{Sym}^d (A)_1 \longrightarrow (A)_d \simeq \mathbb{C}.$$

Example 2.5. Let $f = x^\alpha \in S$ be a monomial for some multi-index α. Then

$$f^\perp = \left(\partial_0^{\alpha_0+1}, \ldots, \partial_n^{\alpha_n+1}\right) \quad \text{and} \quad A_f = \mathbb{C}[\partial_0, \ldots, \partial_n]/(\partial_0^{\alpha_0+1}, \ldots, \partial_n^{\alpha_n+1}).$$

Since A_f has Krull dimension zero and is generated by homogeneous elements of degree one, it holds that $\deg A_f = \dim_{\mathbb{C}} A_f = (\alpha_0 + 1)(\alpha_1 + 1) \cdots (\alpha_n + 1)$.

Next, we turn to the main result of this section, the Apolarity Lemma. It was a smart idea due to Sylvester to link the differential operators killing f with the decompositions of f as sum of powers of linear forms.

To formulate the statement, recall that we identify $\mathrm{Sym}^d(V^\vee)$ with the space of homogeneous polynomials on V, compare Remark **??** b). Therefore, we view $\mathrm{Sym}(V^\vee)$ as the homogeneous coordinate ring of $\mathbb{P}(V)$. For a closed subscheme $Z \subseteq \mathbb{P}(V)$ let $I_Z \subseteq \mathrm{Sym}(V^\vee)$ denote the unique saturated ideal corresponding to Z; it is the vanishing ideal of Z.

Now, we state the reduced version of the Apolarity Lemma.

Theorem 2.6 (Apolarity Lemma, reduced version)**.**
Let $Z = \{[l_1], \ldots, [l_k]\} \subseteq \mathbb{P}(V)$ be a subscheme of closed reduced points with vanishing ideal $I_Z \subseteq \mathrm{Sym}(V^\vee)$. Then, for $f \in \mathrm{Sym}^d(V) \backslash \{0\}$,

$$I_Z \subseteq f^\perp \quad \Leftrightarrow \quad \exists\, c_i \in \mathbb{C}\colon\ f = \sum_{i=1}^{k} c_i l_i^d\,.$$

This is a particular case of next Theorem 2.13 and we wait for the proof until we prove Theorem 2.13.

Theorem 2.7 (Reduced Apolarity Lemma for binary forms)**.**
Let $f \in \mathbb{C}[x,y]_d$ *and pick distinct* $(\alpha_i : \beta_i) \in \mathbb{P}^1$ *for* $i = 1, \dots, k$. *Then*

$$\prod_{i=1}^{k}(\beta_i\partial_x - \alpha_i\partial_y)\cdot f = 0 \quad \Leftrightarrow \quad \exists\, c_i \in \mathbb{C}\colon\ f = \sum_{i=1}^{k} c_i(\alpha_i x + \beta_i y)^d\,.$$

Since $\mathbb{C}$ is algebraically closed, we may write $c_i l_i^d = (l_i')^d$, where l_i' a linear form with $[l_i] = [l_i']$, by taking a d-th root of c_i.

It is well known that the solutions of the wave equation

$$f_{xx} - f_{yy} = 0$$

have the form

$$f = g(x-y) + h(x+y) \tag{2.1}$$

for functions g, h. The key point to get these solutions is the factorization

$$\partial_{xx} - \partial_{yy} = (\partial_x + \partial_y)(\partial_x - \partial_y) \tag{2.2}$$

where the two factors of (2.2) correspond to the two summands of (2.1). When the space of solutions is given by homogeneous polynomials of degree d, we get the expression

$$f = c_1(x-y)^d + c_2(x+y)^d$$

The following result generalizes this fact to any polynomial homogeneous differential equation in two variables.

Theorem 2.8 (Apolarity Lemma for binary forms)**.**
Let $f \in \mathbb{C}[x,y]_d$, *pick distinct* $(\alpha_i : \beta_i) \in \mathbb{P}^1$ *and integers* $1 \le m_i \le d$ *for* $i = 1, \dots, k$. *Then*

$$\prod_{i=1}^{k}(\beta_i\partial_x - \alpha_i\partial_y)^{m_i}\cdot f = 0$$

$$\Longleftrightarrow \quad \exists\, c_i(x,y) \in \mathbb{C}[x,y]_{m_i-1}\colon\ f = \sum_{i=1}^{k} c_i(x,y)(\alpha_i x + \beta_i y)^{d-m_i+1}\,.$$

Proof. $\Longleftarrow$ is straightforward since the summand $c_i(x,y)(\alpha_i x + \beta_i y)^{d-m_i+1}$ is killed by $(\beta_i\partial_x - \alpha_i\partial_y)^{m_i}$, indeed

$(\beta_i\partial_x - \alpha_i\partial_y)\left[c_i(x,y)(\alpha_i x + \beta_i y)^{d-m_i+1}\right] = \left(\beta_i(c_i)_x - \alpha_i(c_i)_y\right)(\alpha_i x + \beta_i y)^{d-m_i+1}$ and so on for higher powers until the first parenthesis vanishes by degree reasons.

$\Longrightarrow$ follows by dimensional reasons.

Indeed, let $\delta = \sum_{i=1}^{k} m_i$. It is clear that the contraction

$$\begin{array}{ccc} \mathrm{Sym}^d\mathbb{C}^2 & \to & \mathrm{Sym}^{d-\delta}\mathbb{C}^2 \\ f & \mapsto & \prod_{i=1}^{k}(\beta_i\partial_x - \alpha_i\partial_y)^{m_i}\cdot f \end{array}$$

is surjective since every linear term $(\beta_i\partial_x - \alpha_i\partial_y)$ gives a surjective contraction. We need to prove that

$$\dim\left\{\sum_{i=1}^{k} c_i(x,y)(\alpha_i x + \beta_i y)^{d-m_i+1} \middle|\, \deg c_i = m_i - 1\right\} = \delta$$

***give above details on osculating spaces

The space corresponds to the osculating space $\sum T^{m_i-1}_{(\beta_i x-\alpha_i y)}$.

We compute its orthogonal

$$\left(\sum T^{m_i-1}_{(\beta_i x-\alpha_i y)}\right)^{\perp} = \bigcap \left(T^{m_i-1}_{(\beta_i x-\alpha_i y)}\right)^{\perp} = \left(\bigcap (\beta_i x-\alpha_i y)^{m_i}\right)_d$$

which has dimension $d-\delta+1$ since it is the space of polynomials of degree d having the root (α_i,β_i) of order $\geq m_i$. □

***Example Harmonic polynomials in two variables (give later with SO-action...)

Lemma 2.9. *Let ϕ_1, $\phi_2 \in \mathbb{C}[x,y] = S$ be polynomials without common factors.*

(1) *The ideal $I = (\phi_1,\phi_2)$ fills the polynomial ring in degree $\geq \deg\phi_1 + \deg\phi_2 - 1$ and has codimension 1 in degree $\deg\phi_1+\deg\phi_2-2 =: d$.*
(2) *The whole ideal can be reconstructed by its top piece I_d by the formula*

$$I_e = \left[I_d \colon \mathcal{M}^{d-e}\right]_e$$

where $\mathcal{M} := (x,y)$ is the irrelevant ideal of S.

Proof. Let $d_i = \deg\phi_i$ for $i = 1,2$. (1) is straightforward by the exact sequence

$$0 \longrightarrow R(-d_1-d_2) \longrightarrow R(-d_1)\oplus R(-d_2) \longrightarrow I \longrightarrow 0$$

(2) Let $g \in \left[I_d \colon \mathcal{M}^{d-e}\right]_e$, then $g\cdot\mathcal{M}^{d-e} \in I_d$ For $e = d-1$ the assumption amounts to $gx = \alpha\phi_1+\beta\phi_2$, $gy = \gamma\phi_1+\delta\phi_2$ for certain forms α, β, γ, δ. We get $(\alpha y-\gamma x)\phi_1 + (\beta y-\delta x)\phi_2 = 0$, this polynomial has degree $d+1 = d_1+d_2-1$, hence there are no syzygies of ϕ_1, ϕ_2 in such degree and we get $\alpha y-\gamma x = 0$, $\beta y-\delta x = 0$, hence x divides α and β, so that $g \in I_e$. We iterate this argument. For $e = d-2$ let g of degree e such that gx^2, gxy, gy^2 belong to I_{e+2}, hence gx and gy belong to I_{e+1} and $g \in I_e$. The same argument works for anyl smaller e. □

Remark 2.10. *Lemma 2.9 holds for any regular sequence of n elements $(\phi_1,\ldots,\phi_n) \in \mathbb{C}[x_1,\ldots,x_n]$, the socle degree is $\sum(\deg\phi_i - 1)$.*

Theorem 2.11. *The apolar ideal $f^{\perp}$ is the complete intersection (ϕ_1,ϕ_2) with ϕ_i without common factors, such that $\deg\phi_1+\deg\phi_2 = d+2$.*

Proof. Let $\phi_1 \in f^{\perp}$ of minimal degree, let $\phi_2 \in f^{\perp}\setminus(\phi_1)$, of minimal degree. we denote $d_i = \deg\phi_i$ for $i = 1,2$, hence by our construction $d_1 \leq d_2$. We first observe that the symmetry property of Proposition 2.4 implies

$$d_2 \leq d-d_1+2. \tag{2.3}$$

Indeed the Hilbert function of $f^{\perp}$ (namely the sequence $\dim(A_f)_i$) coincides with the Hilbert function of (ϕ_1) in degree $< d_2$ and the Hilbert function of (ϕ_1) (starting from degree 0 until degree d_2-1) is $1,2,\ldots,d_1,d_1,\ldots,d_1$ and if we have $d_2 > d-d_1+2$, the symmetry fails since the sequence of d_1 should continue only until degree $d-(d_1-1)$.

Assume now that ϕ_1, ϕ_2 have a common factor of positive degree, so we may write $\phi_1 = \phi_0 h_0$, $\phi_1 = \phi_0 h_1$ with $\deg\phi_0 = q \geq 1$ and h_0, h_1 without common factor. Note that $\phi_0 f$ is killed by both h_0 and h_1. By Lemma 2.9 all differential operators of degree $(d_1-q)+(d_2-q)-1$ kill $\phi_0 f$. Since $\phi_0 f$ is nonzero by the minimality of ϕ_1, it follows $(d_1-q)+(d_2-q)-1 > \deg(\phi_0 f) = d-q$ which implies $d_1+d_2 \geq d+q+2$ which contradicts (2.3), so we have that $q=0$ and ϕ_1, ϕ_2 have no common factors. Repeating the same argument with ϕ_0 scalar (so that $q=0$) gives the sought equality $d_1+d_2=d+2$.

By the Lemma (2.9) the ideal (ϕ_1,ϕ_2) has codimension 1 in degree d, and it is contained in $f^\perp$, which has also codimension 1 in degree d by Proposition 2.4. It follows that $(\phi_1,\phi_2)_d = f^\perp_d$, so that by Lemma 2.9 (2) and Proposition 2.3 the ideals (ϕ_1,ϕ_2) and $f^\perp$ coincide in any degree as we wanted. □

Corollary 2.12. *A general binary form of odd degree d has a canonical form* $\sum_{i=1}^{\frac{d+1}{2}} l_i^d$

Proof. The minimal generator of $f^\perp$ is unique by Theorem 2.11 since $d+2$ is odd. Then apply Theorem 2.8 to this minimal generator. □

*** Computation of rank, rank strata and closure
*** (k+1)-Minors of catalecticant as equations for k-secant
*** Singularities of k-secant
*** Exercise: polynomials in more variables such that $f^\perp$ is a complete intersection. Likely GL(V)-equivalent to monomials, that is split with only n+1 linear forms. But maybe something more, think at binary forms. —compare work by Peterson "apolar ideal of a product of linear forms" —Boij, Migliore, Mirò-Roig, NonLefschetz locus

2.1. **Apolarity theory in more variables.**

Theorem 2.13 (Apolarity Lemma, scheme version)**.**
Let $Z \subseteq \mathbb{P}(V)$ be a closed zero-dimensional subscheme with vanishing ideal $I_Z \subseteq Sym(V^\vee)$ and let $\nu_d \colon \mathbb{P}(V) \to \mathbb{P}(Sym^d(V))$, $[l] \mapsto [l^d]$ be the Veronese embedding. Then, for $f \in \mathrm{Sym}^d(V)\backslash\{0\}$,

$$I_Z \subseteq f^\perp \quad \Leftrightarrow \quad [f] \in \langle \nu_d(Z) \rangle \,.$$

In general, if $X \subseteq \mathbb{P}^N$ is a closed reduced subscheme, then $\langle X \rangle$ is given by the usual projective linear span of the closed points of X. Thus, if Z (and hence $\nu_d(Z)$) is reduced in Theorem 2.13 we obtain Theorem 2.6.

Proof of Theorem 2.13. First, we note that the coordinate ring of $\mathbb{P}(\mathrm{Sym}^d(V))$ is $R := \mathrm{Sym}\big((\mathrm{Sym}^d(V))^\vee\big)$. In particular, $R_1 = (\mathrm{Sym}^d(V))^\vee$. It is a property of the Veronese embedding that linear forms vanishing on $\nu_d(Z)$ correspond to homogeneous forms of degree d vanishing on Z, i.e.,

$$R_1 \cap I_{\nu_d(Z)} = (\mathrm{Sym}^d(V))^\vee \cap I_{\nu_d(Z)} \cong \mathrm{Sym}^d(V^\vee) \cap I_Z \,. \tag{2.4}$$

We can make this explicit as follows.

Viewing $g \in \mathrm{Sym}^d(V^\vee)$ as a linear form on $\mathrm{Sym}^d(V)$ via the dual pairing from Equation (1.1), we have that $\langle g, f' \rangle = g \cdot f'$, where $\langle g, f' \rangle$ denotes the function value of g at $f' \in \mathrm{Sym}^d(V)$. In particular, $\langle g, f \rangle = g \cdot f$. This identification is allowed, i.e., it respects (2.4), because for $l = \sum_i c_i x_i \in V$ we have $\langle g, l^d \rangle = g \cdot l^d = 0$ if and only if $g(l) = g(c_0, \ldots, c_n) = 0$, by Lemma 1.2.

Now, $[f] \in \langle \nu_d(Z) \rangle$ if and only if every linear form in $R_1 = (\mathrm{Sym}^d(V))^\vee$, that vanishes on $\nu_d(Z)$ also vanishes on f. Using the identification (2.4) via the action of differential forms, we have argued that $[f] \in \langle \nu_d(Z) \rangle$ if and only if $(I_Z)_d \subseteq (f^\perp)_d$.

We end the proof by showing that $(I_Z)_d \subseteq (f^\perp)_d$ is equivalent to $I_Z \subseteq f^\perp$. Clearly, the latter implies the former. For the converse recall that for all $e > d$, $(f^\perp)_e = \mathrm{Sym}^e(V^\vee)$ and hence $(I_Z)_e \subseteq (f^\perp)_e$. Using $(I_Z)_d \subseteq (f^\perp)_d$ and then Proposition 2.3 yields for all $1 \leq e < d$

$$(I_Z)_e \subseteq \left[(I_Z)_d : \mathcal{M}^{d-e}\right]_e \subseteq \left[(f^\perp)_d : \mathcal{M}^{d-e}\right]_e = (f^\perp)_e\,.$$

Altogether, we have $I_Z \subseteq f^\perp$ as desired. □

The scheme-theoretic version of the Apolarity Lemma is used to characterize a notion, which was of increasing importance during the last years. Namely, the *cactus rank* of a symmetric tensor $f \in \mathrm{Sym}^d(V)$ is the least length of any zero-dimensional subscheme $Z \subseteq \mathbb{P}^n$ with $I_Z \subseteq f^\perp$. Actually, cactus rank already appeared as *scheme length* in [2] and a generalization of the above definition is due to Buczynska and Buczynski in [1].

References

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