Tensor rank and eigenvectors

Mauro Maccioni

13 January 2017

Contents

List of Tables

List of Figures

Introduction

In algebraic tensor geometry, the problem of finding the minimal decomposition of a symmetric tensor T with coefficients in a field $\mathbb{K}$ in a sum of rank-1 terms over $\mathbb{K}$ is a classical problem. The minimal integer r such that T decomposes in a sum of r rank-1 terms is said to be the rank of T. If $\mathbb{K} = \mathbb{C}$, for complex or real tensors T, this problem is known as the *Waring problem* ([10], [20]).

For binary forms f of degree d with coefficients in $\mathbb{K}$ (that is, for homogeneous complex or real polynomials) the concept of rank of f over $\mathbb{K}$ turns into the research of the minimal integer r such that f decomposes into a sum of r d-th powers of linear binary forms $l_1, \dots, l_r$, multiplied by appropriate coefficients $c_1, \dots, c_r$.
If $\mathbb{K} = \mathbb{C}$, one can impose $c_j = 1$ for all coefficients. In the complex field, the rank of a general binary form f of odd degree $d = 2n+1$ is $n+1$. The Sylvester Theorem asserts that the decomposition of such general form f as a sum of $n+1$ powers of linear forms is unique and gives also a way to determine it. The rank of a general binary form f of even degree $d = 2n$ is $n+1$, but in this case such decompositions form an infinite set, which can be identified with the projective line. Given $S^{\mathbb{K}}_{d,r} = \left\{ f \in Sym^d(\mathbb{K}^2) \,|\, rank f = r \right\}$, we know that if $\mathbb{K} = \mathbb{C}$, $S^{\mathbb{C}}_{d,r}$ has not empty interior (i.e. is dense) if and only if $r = \left\lfloor \frac{d}{2} \right\rfloor + 1$.
If $\mathbb{K} = \mathbb{R}$, the coefficients c_j can be imposed to belong to $\{-1, 1\}$ and, moreover, in the real field the ranks r such that $S^{\mathbb{R}}_{d,r}$ has not empty interior are those between $\left\lfloor \frac{d}{2} \right\rfloor + 1$ and d (see Theorem 2.4 in [3]).

By Sylvester (see [12]), being $l^{\perp} = b\partial_x - a\partial_y$ the differential operator such that kills the linear form $l = a\partial_x + b\partial_y$, we have that $f = \sum_{j=1}^{r} l_j^d$ is killed by $g = \prod_{j=1}^{r} l_j^{\perp}$. Then, assigned a form f of degree d and complex rank $rk_{\mathbb{C}}(f) = k$, we have to consider the kernel K of the catalecticant matrix (or Henkel matrix) of size $(d-k+1) \times (k+1)$. Then K is the kernel of the linear map $A_f : D_k \longrightarrow R_{d-k}$, which is the set of differential operators of degree k that kill f of degree d. Therefore, we search in K a differential operator g with all real roots. If it does not exist, we search at degree $k+1$ and so on. When we find the above operator g with all real roots of a certain degree h, then we have that the real rank of f is h.

In the first part of this P.H.D. thesis, we give a complete classification of real ranks of real binary quartic and quintic forms, given their complex ranks. The main results are in section 1.4 of chapter 1, while in sections 1.2 and 1.3 we effectively compute the real ranks of quartic and quintic forms respectively, starting from their complex ranks.

In the second part of this work we consider eigenvectors of real symmetric tensors.

Given a real homogeneous polynomial f of degree d in n variables, its eigenvectors are $x \in \mathbb{C}^n$ such that $\nabla f(x) = \lambda x$.
In alternative way, the eigenvectors are the critical points of the euclidean distance function from f to the Veronese variety of polynomials of rank one (see [14]).
In the quadratic case ($d = 2$) the eigenvectors defined in this way coincide with the usual eigenvectors of the symmetric matrix associated to f. By the Spectral Theorem, the eigenvectors of a quadratic polynomial are all real. So a natural question is to investigate the reality of the eigenvectors of a polynomial f of any degree d. The number of complex eigenvectors of a polynomial f of degree d in n variables, when it is finite, is given by

$$\begin{cases} ((d-1)^n - 1)/(d-2), & d \geq 3 \\ (d-1)^{n-1} + (d-1)^{n-2} + ... + (d-1)^0 = n, & d = 2 \end{cases} \quad (1)$$

The value obtained in this formula has to be counted with multiplicities. The general polynomial has all eigenvectors of multiplicity one. The formula (1) is a result by Cartwright and Sturmfels in [7].

Our picture is quite complete in the case $n = 2$ of binary forms. We show that

Theorem 1: *The number of real eigenvectors of a real homogeneous polynomial in* 2 *variables is greater or equal than the number of its real roots.*

Moreover, we show that the inequality of Theorem 1 is sharp and it is the only essential constraint about the reality of eigenvectors, in the sense that the set of polynomials in $Sym^d\mathbb{R}^2$ with exactly k real roots contains subsets of positive volume consisting of polynomials with exactly t real eigenvectors, for any t such that $k \leq t \leq d$, $k \equiv t \equiv d \mod 2$, $t \geq 1$. The congruence $\mod 2$ is an obvious necessary condition on the pair (k, t) which comes from the complex conjugation. Note that all extremes cases are possible, so there are polynomials with the maximum number d of real eigenvectors. On the other side there are polynomials with one real eigenvectors for odd d (with only one real root by Theorem 1) and there are polynomials with two real eigenvectors for even d (with zero or two real roots by Theorem 1). There are no polynomials with zero real eigenvectors, this is due to the interpretation of the eigenvectors as critical points of the euclidean distance function, which attains always a real minimum.
We can summarise the inequality of Theorem 1 by saying that the topological type of f prescribes the possible cases for the number of real eigenvectors.

The next case we investigate is the one of ternary forms $n = 3$. In this case the topological type of f depends on the number of ovals in the real projective plane and on their mutual position (nested or not nested). Again we prove an inequality which follows the same philosophy of Theorem 1. Precisely we have

Theorem 2: *Let t be the number of real eigenvectors of a real homogeneous polynomial in* 3 *variables with c ovals. Then $t \geq 2c+1$, if d is odd and $t \geq \max(3, 2c+1)$, if d is even.*

We give evidence that the inequality of Theorem 2 is the best possible, by showing that in the cases $d = 3$ and $d = 4$ the set of polynomials in $Sym^d(\mathbb{R}^3)$ with exactly c real ovals contains subsets of positive volume that consist of polynomials with exactly t real eigenvectors, for any t such that t is odd and $2c+1 \leq t \leq 7$ ($d = 3$) and $\max(3, 2c+1) \leq t \leq 13$ ($d = 4$). Again the condition that t is odd is a necessary condition which follows from the fact that the values in (1) are odd for $n = 3$ (as for any odd n).

In Section 2.1 we give some preliminaries and a general result (Lemma 34) on the nature of real eigenvectors of a real symmetric tensor.

In Section 2.2 we investigate on binary forms. We give some examples in which it is evident that there are some prohibited values for the number of real eigenvectors of a form conditioned to the number of its real roots. Also we give the main Theorem 49, that shows that the number of real eigenvectors of a real homogeneous polynomial in two variables is greater or equal than the number of its real roots and this constraint is sharp.

In Section 2.3 we investigate on ternary forms. In primis, we give some computational examples of ternary cubics in which is evident that there are some prohibited values for t conditioned to c. Moreover, all possible numbers of real eigenvectors are possible for a cubic, according with the main Theorem 62. It shows that t is greater or equal than $2c+1$, if d is odd and t is greater or equal than $\max(3, 2c+1)$, if d is even. Moreover, we show how to find ternary forms of degree d with a certain number c of ovals and always with the maximum number of real eigenvectors. Then, we give examples of cubics and quartics with the minimum and the maximum number of real eigenvectors in all possible topological cases, showing that for $d = 3, 4$ the constraint of Theorem 12 is again the best possible (Propositions 68 and 69).

In Section 2.4, we give some computational examples of ternary quintics and sextics with all possible values of t conditioned to the value c in some topological cases.

Chapter 1

Real rank of binary forms

1.1 Preliminaries

Definition 1. *Let X be an algebraic projective variety. The k-secant variety of X is* $\mathrm{Sec}^k(X) = J(\underbrace{X, ..., X}_{k-times})$*, where $J(X, ..., X)$ is the join of k copies of X. The join of s algebraic projective varieties, $X_1, ..., X_s$, is the Zariski closure of the set of the projective subspaces generated by general points $p_1 \in X_1, ..., p_s \in X_s$.*

Definition 2. *([3]) Let $f \in \mathrm{Sym}^d(\mathbb{R}^2)$. The apolar ideal of f, $f^\perp$, is the ideal of all differential homogeneous operator h such that h kills f, that is*

$$f^\perp = \{h \in \mathbb{R}[\partial_x, \partial_y] \,|\, h(f) = 0\}.$$

Definition 3. *([11]) Let f be a real binary form of degree d. The real (complex) rank of f is the minimum integer r such that*

$$f = \sum_{j=1}^{r} l_j^d$$

where l_j are real (complex) linear binary forms.

Theorem 4. *([11]) For all $k \in [1, \lfloor d/2 \rfloor + 1]$ we have $\bar{S}_{d,k} - \bar{S}_{d,k-1} = S_{d,k} \cup S_{d,d-k+2}$, with $S_{d,k} = \left\{f \in \mathrm{Sym}^d\mathbb{C}^2 \mid \mathrm{rk}_{\mathbb{C}} f = k\right\}$. In particular, $f \in \bar{S}_{d,k} - \bar{S}_{d,k-1}$ has rank k if and only if $[f]$ lies in a k-secant plane of the Veronese curve X, otherwise f has rank $d - k + 2$.*

Assigned the complex rank of a real binary form of degree four or five, our goal is to classify this forms to respect to their real rank.

Proposition 5. *([12]) Any binary real form of degree d has real rank less or equal than d.*

Proof. The points of the projective space $\mathbb{P}^d = \mathbb{P}(Sym^d(\mathbb{R}^2))$ correspond to forms $f = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i$, which have coordinates $(a_0, \ldots, a_d)$. The rational normal curve X, corresponds to polynomials which are d-th powers of linear forms. From the expansion $(t_0 x + t_1 y)^d = \sum_{i=0}^{d} \binom{d}{i} t_0^{d-i} t_1^i x^{d-i} y^i$ we get that the curve X can be parametrized by $a_i = t_0^{d-i} t_1^i$. Pick $d-1$ general points on X corresponding to $l_i^d = (l_{i,0}x + l_{i,1}y)^d$ for $i = 1, \ldots, d-1$. The linear span of f and these points is a hyperplane, whose equation $\sum \binom{d}{i} a_i c_i$ restricts to X to the binary form $\sum \binom{d}{i} c_i t_0^{d-i} t_1^i$ of degree d with the $d-1$ real roots $(t_0, t_1) = (l_{i,0}, l_{i,1})$ (because $\sum \binom{d}{i} c_i l_{i,0}^{d-i} l_{i,1}^i = 0$) hence also the last root is real, corresponding to a last linear form l_d^d, which can be chosen different from the other linear form l_i^d, because the $d-1$ points on X are general, then we have a general hyperplane that meets the curve in d (real) distinct roots. This means that f is a projective linear combination of the the powers l_i^d for $i = 1, \ldots, d$, or equivalently, f has rank less or equal than d. □

Proposition 6. *([12]) $S_{3,r}^{\mathbb{R}}$ has non empty interior only for $r = 2, 3$. Precisely, let f be a polynomial of third degree without multiple roots. Then*

1. *f has rank two if and only if $\Delta(f) < 0$, or equivalently, if and only if f has one real root.*
2. *f has rank three if and only if $\Delta(f) > 0$, or equivalently, if and only if f has three real roots.*

Moreover, if $\Delta(f) = 0$ we have that f has complex and real rank one.

Proof. The differential operators of degree two which annihilate f consist of the kernel of the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{pmatrix}.$$

The discriminant of the quadratic generator of the kernel coincides with $-\Delta(f)$; thus the operators have two real roots if $\Delta(f) < 0$ and this means that the rank-2 complex decomposition is actually real. Note also that a cubic of real rank two can have only one real root. Indeed the equation $l_1^3 + l_2^3 = 0$ reduces to the three linear equations $l_1 - \exp^{\frac{n\pi i}{3}} l_2 = 0$ for $n = 0, 1, 2$. This proves the first statement. If $\Delta(f) > 0$, the quadratic generator has no real root and by Proposition 5 we have the second statement. □

Proposition 7. *([12]) Let $f \in \mathrm{Sym}^d(\mathbb{R}^2)$ such that f has d real distinct roots. Then $\mathrm{rk}_{\mathbb{R}} f = d$.*

Proof. The proof is by induction on d. If $d = 1$, it's trivial. If $d = 2$, we have that a real form f of degree two corresponds to a 2×2 symmetric matrix and then we have that f has two real distinct roots if and only if $rk_{\mathbb{R}} f = 2$. Then let $d \geq 3$. Assume the rank is less or equal than $d-1$. Then we get $f = \sum_{i=1}^{d-1} l_i^d$. We may assume that l_{d-1} does not divide f, because:

1. if $d = 3$ we have two summands for f. If the second summand divides f, then it necessarily divides the first summand and hence the two summands are proportional. Then we may assume that $l_{d-1} = l_2$ does not divide f;

2. if $d \geq 4$, the fibers from abstract secant variety to the secant variety of X have positive dimension. Hence there are infinitely many decompositions of f. Then we may assume that l_{d-1} does not divide f.

Consider the rational function

$$F = \frac{f}{l_{d-1}}$$

Under a linear (real) change of projective coordinates $\phi(x, y) = (x', y')$ with $y' = l_{d-1}$ we get $G(x', y') = F(\phi^{-1}(x', y')) = \frac{f(\phi^{-1}(x',y'))}{y'^d}$. Then the polynomial $G(x', 1) = \sum_{i=1}^{d-2} n_i(x')^d + 1$ has d distinct real roots since f had (where $\deg n_i = 1$) and its derivative $\frac{d}{dx'}G(x', 1) = \sum_{i=1}^{d-2} dn_i(x')^{d-1}\frac{d}{dx'}(n_i(x'))$ has $d-1$ distinct real roots. Now $\frac{d}{dx'}G(x', 1)$ has rank less or equal than $d-2$, indeed $\frac{d}{dx'}(n_i(x'))$ are constants. This contradicts the inductive assumption. Hence the assumption was false and the rank of f must exceed $d-1$. The rank of f must eventually be equal to d by Proposition 5. □

Lemma 8. *([12]) The following are canonical forms for general forms, under the action of the Möbius transformation group*
$Aut(\mathbb{R}) = \left\{x \mapsto \frac{ax+b}{cx+d} \mid ad - bc \neq 0\right\} = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det A \neq 0\right\}$*:*
$d = 4$*:*

1. $(x^2 + y^2)(x^2 + ay^2)$*, with* $a > 0$ *(four complex roots) or with* $a < 0$ *(two complex roots and two real roots);*

2. $(x^2 - y^2)(x^2 + ay^2)$*, with* $a < 0$ *(four real roots).*

$d = 5$*:*

1. $x(x^2 + y^2)(x^2 + 2axy + by^2)$*, with* $b - a^2 > 0$ *(one real roots and four complex roots) or with* $b - a^2 < 0$ *(three real roots and two complex roots);*

2. $x(x^2 - y^2)(x^2 + 2axy + by^2)$*, with* $b - a^2 < 0$ *(five real roots).*

Proof. We prove just the first case for $d = 4$, the other ones being analogous. When there are two pairs of conjugate roots, they lie in the complex plane on a circle with real center, then a convenient circle inversion makes the four roots on a vertical line. A translation and a homothety centered at zero conclude the argument. When there is one pair of conjugate roots, assume that they are $\pm\sqrt{-1}$. Then consider the transformations $x \mapsto \frac{x+c}{-cx+1}$, which preserve $\pm\sqrt{-1}$ and it is easy to show that a convenient choice of c makes the sum of the other two roots equal to zero. □

Proposition 9. *([12]) Let f be a real binary form of degree d with distinct roots. Then:*

1. *if f has d real roots then for every $(a,b) \neq (0,0)$ the binary form $af_x + bf_y$ has $d-1$ real roots.*

2. *Conversely, if for every $(a,b) \neq (0,0)$ the binary form $af_x + bf_y$ has $d-1$ real roots and $3 \leq d \leq 5$, then f has d real roots.*

Proof. 1. Consider that for any substitution $x = at+c$, $y = bt+d$ with $ad-bc \neq 0$ we have that $F(t) = f(at+c, bt+d)$ has d real roots, then $\frac{d}{dt}f(at+c, bt+d) = af_x + bf_y$ has $d-1$ real roots corresponding to the $d-1$ extremal points of F.

2. Assume that f has a number of real roots less or equal than $d-1$ (hence to $d-2$) and let us show that there exist (a,b) such that $af_x + bf_y$ has a number of real roots less or equal than $d-2$ (hence to $d-3$).
For $d = 3$, after a Möbius transformation, we may assume that $f = x^3 + 3xy^2$. Then $f_x = 3(x^2 + y^2)$ has no real roots.
For $d = 4$ we may assume by the Lemma 8 that $f = (x^2 + y^2)(x^2 + ay^2)$. For $a > -1$, we consider $f_x = x(4x^2 + 2(a+1)y^2)$ which has only one real root. For $a < -1$ we consider $f_y = y(4ay^2 + 2(a+1)x^2)$ which has only one real root. For $a = -1$ then $f_x - f_y$ has only one real root.
For $d = 5$ we may assume by the Lemma 8 that $f = x(x^2 + y^2)(x^2 + 2axy + by^2)$. The discriminant of f_x is (up to a positive scalar multiple) $D(a,b) = -540a^2 - 1584a^4 + 830b^3 - 180b^4 - 180b^2 - 8192a^6 + 405b^5 + 405b - 7476a^2b^2 + 1548a^2b + 14784a^4b - 396a^2b^3 + 576a^4b^2 - 432b^4a^2$. It can be shown that f_x has zero real roots if $D(a,b) > 0$ and two real roots if $D(a,b) < 0$. This concludes the proof. □

Corollary 10. *([12]) Let $d \in [3,5]$ and $f \in \mathrm{Sym}^d(\mathbb{R}^2)$ with distinct roots. Then $\mathrm{rk}_{\mathbb{R}} f = d \Longrightarrow f$ has d real roots.*

Proof. The proof is by induction on d. For $d = 3$ it follows from the Proposition 6. Let $4 \leq d \leq 5$. If f has a number of real roots less or equal than $d-2$ then by Proposition 9, there exists $(a,b) \neq (0,0)$ such that the binary form $af_x + bf_y$ has a number of real roots less or equal than $d-3$. Then by the inductive assumption $af_x + bf_y$ has rank less or equal than $d-2$. So we get $af_x + bf_y = \sum_{i=1}^{d-2} l_i^{d-1}$. Choose c, d such that $ad - bc \neq 0$. Let $F(t) = f(at+c, bt+d)$. We get that $F'(t) = \sum_{i=1}^{d-2} n_i(t)^{d-1}$ for some degree one polynomials n_i and by integration there is a constant K and degree one polynomials m_i such that $\frac{F(t)}{(bt+d)^d} = \sum_{i=1}^{d-2} \frac{m_i(t)^d}{(bt+d)^d} + \frac{K}{(bt+d)^d}$. With the substitution $t = \frac{dx-yc}{-bx+ay}$ we get that the rank of f is less or equal than $d-1$, which is against the assumption. □

Proposition 11. *([12]) Let $f \in \mathrm{Sym}^d(\mathbb{R}^2)$ such that $\mathrm{rk}_{\mathbb{C}} f = k$, for $k \in [2, \lfloor d/2 \rfloor + 1]$. Then we can have only the following two situations:*

1. $\mathrm{rk}_{\mathbb{R}} f = k$,

2. $\mathrm{rk}_{\mathbb{R}} f \geq d - k + 2$, *where equality holds for $k = 2$.*

Proof. Assume that the first statement does not hold. This means that the contraction from the space of the homogeneous differential operator of degree k to the space of the homogeneous polynomial of degree $d-k$

$$D_k \longrightarrow R_{d-k}$$

has rank k and that the one dimensional kernel is generated by one operator with at least two complex conjugate roots. It follows that also the transpose operator

$$D_{d-k} \longrightarrow R_k$$

has rank k and the operators in the kernel are given exactly by the previous operator times every operator of degree $d-2k$. In particular no operator in the kernel has all real roots. This argument works also for the next contraction

$$D_{d-k+1} \longrightarrow R_{k-1}$$

which has again rank k. At the next step it is possible to find an operator in the kernel with all real roots. This concludes the proof. When $k=2$ the equality holds by Proposition 5. $\square$

Theorem 12. *([3]) Let f be a binary form of degree d. Then $f^{\perp}$ is a complete intersection ideal over $\mathbb{C}$, i.e. $f^{\perp}$ is generated by two real binary forms, g_1, g_2, such that* $\deg g_1 + \deg g_2 = d+2$ *and* $\{g_1 = 0\} \cap \{g_2 = 0\} = \emptyset$. *Moreover, for any pairs of forms g_1, g_2 of this type, they generate an ideal $f^{\perp}$, for some real binary form f of degree* $d = \deg g_1 + \deg g_2 - 2$.

Remark 13. Given a binary form f of degree d, evidently the Kernel of a its catalecticant matrix of any dimension is contained in $f^{\perp}$. Moreover, we say that the apolar ideal of f is generated in generic degree when its two generators have degrees $\left(\frac{d+2}{2}, \frac{d+2}{2}\right)$ or $\left(\frac{d+1}{2}, \frac{d+3}{2}\right)$ for d respectively even or odd. This situation occurs exactly when the rank of the catalecticant matrix of f is maximum.

Theorem 14. *([3]) All ranks between $\lfloor d/2 \rfloor + 1$ and d are typical for binary forms of degree d.*

Proof. We use induction on the degree d. The base case $d=2$ is just bivariate quadratic forms and the real rank corresponds to the usual rank of the matrix. Therefore there is only one typical rank, which is 2.
Inductive Step: $d \Longrightarrow d+1$. We first note that it was already shows in [12] that rank $d+1$ is typical for forms in $Sym^{d+1}(\mathbb{R}^2)$. Suppose that $f \in Sym^{d+1}(\mathbb{R}^2)$ is a typical form of rank $\left\lfloor \frac{d+3}{2} \right\rfloor \leq m \leq d$. By perturbing f we may assume that the apolar ideal $f^{\perp}$ is generated in generic degrees.
Suppose $d = 2k$ is even. Then $f^{\perp}$ is generated by forms p_1, p_2 with $\deg p_1 = \deg p_2 = k+1$. First suppose that $m = k+1$. We may choose a generator $p_1 \in (f^{\perp})_m$ such that p_1 has all real distinct roots and let p_2 be a form in $(f^{\perp})_m$ linearly independent

from p_1. Now let l a linear real binary form such that the zero of l is not one of the zeroes of p_1 and consider the ideal $I = \langle p_1, lp_2 \rangle$. The forms p_1 and lp_2 form a complete intersection over $\mathbb{C}$. By Theorem 12, I is the apolar ideal of some form $g \in Sym^{d+1}(\mathbb{R}^2)$. Since we have $g^\perp \subset f^\perp$, by Lemma 2.3 in [3] we know that g is a typical form of rank m. Now suppose that $m > k+1$. By Apolarity Lemma there exists $s \in (f^\perp)_m$ such that s has all real distinct roots and by Lemma 2.3 in [3] we know that all forms in $(f^\perp)_{m-1}$ have at least 2 complex roots. Since $s \in (f^\perp)_m$ we can write $s = p_1q_1 + p_2q_2$ for $q_1, q_2 \in Sym^{m-k-1}(\mathbb{R}^2)$. We now claim that we may choose two generators p_1 and p_2 of $f^\perp$ so that the multiplier q_2 has a real root distinct from the roots of p_1. If this does not hold then we may pick a different set of generators of $f^\perp$: let $p_1' = p_1 + \alpha p_2$ with some $\alpha \in \mathbb{R}$. Then $s = p_1'q_1 + p_2(q_2 - \alpha q_1)$. We can easily adjust α so that $q_2 - \alpha q_1$ has a real root, and we need to argue that we can also make this root distinct from the roots of $p_1' = p_1 + \alpha p_2$. Suppose not, then for any $(a, b) \in \mathbb{R}^2$ that is not a root of q_1 we may set $\alpha = -q_2(a, b)/q_1(a, b)$ and make (a, b) a root of $q_2 - \alpha q_1$. Therefore we must have $\frac{p_1}{p_2} = -\frac{q_2}{q_1}$ which implies that $s = p_1q_1 + p_2q_2 = 0$ and that is a contradiction. Thus we have $q_2 - \alpha q_1 = lq$, with $q \in Sym^{k-m-2}(\mathbb{R}^2)$ and l does not divide p_1'. Let $I = \langle p_1', lp_2 \rangle$. As before, p_1' and lp_2 form a complete intersection over $\mathbb{C}$ and by Theorem 12 I is the apolar ideal of some form $g \in Sym^{d+1}(\mathbb{R}^2)$. Since $s \in I$ we know that the rank of g is at most m and since $I \subset f^\perp$ we know that the rank of g is at least m. Therefore the rank of g is m. Further, $g^\perp \subset f^\perp$ has no forms of degree $m-1$ with all real roots and $g^\perp$ is generated in generic degrees. Therefore g is a typical form of rank m.

Now suppose that $d = 2k+1$ is odd. Then $f^\perp$ is generated by forms p_1, p_2 with $\deg p_1 = k+1$ and $\deg p_2 = k+2$. We note that we only need to deal with the cases $m \geq k+2$. By Apolarity Lemma there exists $s \in (f^\perp)_m$ such that s has all real distinct roots and by Lemma 2.3 in [3] all forms in $(f^\perp)_{m-1}$ have at least 2 complex roots. Since $s \in (f^\perp)_m$ we can write $s = p_1q_1 + p_2q_2$ for $q_1 \in Sym^{m-k-1}(\mathbb{R}^2)$ and $q_2 \in Sym^{m-k-2}(\mathbb{R}^2)$. The generator p_1 is uniquely determined, but p_2 is unique only modulo the ideal generated by p_1. We now claim that we may choose generators of $f^\perp$ so that the multiplier q_1 has a real root distinct from the roots of p_2. If this does not hold then let $p_2' = p_2 + lp_1$ for some linear form l. We have $s = p_1(q_1 - lq_2) + p_2'q_2$. We can adjust l so that $q_1 - lq_2$ has a real root, and we need to argue that we may also make this root distinct from the roots of $p_2' = p_2 + lp_1$. Arguing as before we must have $\frac{p_2}{p_1} = -\frac{q_1}{q_2}$ which implies that $s = p_1q_1 + p_2q_2 = 0$ and that is a contradiction. Let $I = \langle lp_1, p_2' \rangle$. Since lp_1 and p_2' form a complete intersection over $\mathbb{C}$ by Theorem 12 I is the apolar ideal of some form $g \in Sym^{d+1}(\mathbb{R}^2)$. Since $s \in I$ we know that the rank of g is at most m and since $I \subset f^\perp$ we know that the rank of g is at least m. Therefore the rank of g is m. Further, $g^\perp \subset f^\perp$ has no forms of degree $m-1$ with all real roots and $g^\perp$ is generated in generic degrees. Therefore g is a typical form of rank m. □

Remark 15. By Proposition 7 and Corollary 10, if $f \in Sym^d(\mathbb{R}^2)$ has d distinct roots, with $d \in [3, 5]$, then are equivalent:

- f has real rank d,
- f has d real roots.

Remark 16. Let $f \in Sym^d(\mathbb{R}^2)$ such that f has τ real roots (counted with multiplicity). Then $\tau \leq rk_{\mathbb{R}} f$ (see Theorem 3.1 and 3.2 in [31]).

1.2 Quartic forms

In this section we want to give a general classification of real ranks for real binary forms of degree $d = 4$.

Proposition 17. *Let* $f \in \mathrm{Sym}^4(\mathbb{R}^2)$ *be such that* $\mathrm{rk}_{\mathbb{C}} f = r$*, with* $r \in [1, 4]$*. Then we have:*

1. $r = 1 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 1$.
2. $r = 2 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 2$ *or* $\mathrm{rk}_{\mathbb{R}} f = 4$.
3. $r = 3 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 3$ *or* $\mathrm{rk}_{\mathbb{R}} f = 4$.
4. $r = 4 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 4$.

Proof. Let $f \in Sym^4(\mathbb{R}^2)$ be with complex rank r:

1. trivial.
2. By Proposition 11.
3. Being $rk_{\mathbb{R}} f \geq rk_{\mathbb{C}} f$, by Proposition 5 and Theorem 14 we have the conclusion.
4. Trivial.

□

Let $f \in Sym^4(\mathbb{R}^2)$. Starting from Proposition 17 and fixing the complex rank of f, we see precisely how occur the real ranks of f. Moreover, we give a method to compute the real rank without having to search a homogeneous differential operator of degree $r = rk_{\mathbb{C}} f$ with all real roots. In the case they do not exist, we will find them without going into degree $r + 1$ and so on.

Remark 18. Let $f \in Sym^4(\mathbb{R}^2)$:

1. if $rk_{\mathbb{C}} f = 1$, trivially the real rank of f is 1 and conversely.
2. If $rk_{\mathbb{C}} f = 2$, we consider a quartic form with real coefficients

$$f = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$$

   Then, being $rk_{\mathbb{C}} f = 2$, the catalecticant matrix of dimension 3×3 (i.e. we consider the linear map from the space of the homogeneous differential operator of degree

$2 = rk_{\mathbb{C}}f$, D_2, to the space of the homogeneous polynomial of degree $4 - 2 = \deg f - rk_{\mathbb{C}}f$, R_2) of f

$$J = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}$$

has rank 2. Then $\ker J$ has dimension one and it is generated by a polynomial g of degree two. Therefore if g has two real distinct roots we have $rk_{\mathbb{R}}f = 2$, otherwise $rk_{\mathbb{R}}f = 4$.

3. If $rk_{\mathbb{C}}f = 3$, we consider a generic quartic with real coefficients

$$f = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4.$$

Depending on the type of roots of f, we have the following cases:

(a) f has four real distinct roots. In this case, we can rewrite f in the canonical form $(x^2 - y^2)(x^2 + ay^2) = x^4 + (a-1)x^2y^2 - ay^4$, with $a < 0$ (and $a \neq -1$) and we have that $rk_{\mathbb{R}}f = 4$ by Remark 15.

(b) f has four distinct roots but not all real. In this case, we can rewrite f in the canonical form $(x^2 + y^2)(x^2 + ay^2) = x^4 + (a+1)x^2y^2 + ay^4$, with $a \neq 0$ (and $a \neq -1$) and we have that $rk_{\mathbb{R}}f = 3$ by Remark 15 and by the fact that there are only two possibilities (3 or 4) for the real rank of f.

(c) f has three real roots, two distinct and one with multiplicity 2. In this case, we can rewrite f in the canonical form $x^2(x^2 - y^2) = x^4 - x^2y^2$ and we have that $rk_{\mathbb{R}}f = 4$. In fact, consider the catalecticant matrix of size 2×4 of f

$$J = \begin{pmatrix} 1 & 0 & -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} & 0 & 0 \end{pmatrix}.$$

Computing the two relative equations, we have that $\ker J$ is generated by the following two cubics

$$f_1 = x^3 + 6xy^2,\ f_2 = y^3$$

Then a generic element of $\ker J$ is of the type $f_1 + mf_2 = x^3 + 6xy^2 + my^3$ with discriminant the following polynomial in m of degree two

$$4\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}\begin{vmatrix} 0 & 2 \\ 2 & m \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 0 & m \end{vmatrix}^2 = -32 - m^2$$

which is always negative.

(d) f has two complex roots and two real coincident roots. In this case, we can rewrite f in the canonical form $x^2(x^2 + y^2) = x^4 + x^2y^2$ and we have that $rk_{\mathbb{R}}f = 3$. In fact, consider the catalecticant matrix of size 2×4 of f

$$J = \begin{pmatrix} 1 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & 0 \end{pmatrix}.$$

Computing the two relative equations, we have that $\ker J$ is generated by the following two cubics

$$f_1 = x^3 - 6xy^2, \ f_2 = y^3.$$

Then a generic element of $\ker J$ is of the type $f_1 + mf_2 = x^3 - 6xy^2 + my^3$ with discriminant the following polynomial in m of degree two

$$4\begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix}\begin{vmatrix} 0 & -2 \\ -2 & m \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 0 & m \end{vmatrix}^2 = 32 - m^2$$

that changes sign.

(e) f is the square of a quadratic form. In this case, we can rewrite f in the following two forms:

- $(x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4$ and $rk_{\mathbb{R}} f = 3$. In fact, consider the catalecticant matrix of size 2×4 of f

  $$J = \begin{pmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 \end{pmatrix}.$$

  Computing the two relative equations, we have that $\ker J$ is generated by the following two cubics

  $$f_1 = x^3 - 3xy^2, \ f_2 = -3x^2y + y^3.$$

  Then a generic element of $\ker J$ is of the type $f_1 + mf_2 = x^3 - 3mx^2y - 3xy^2 + my^3$ with discriminant the following polynomial in m of degree four

  $$4\begin{vmatrix} 1 & -m \\ -m & -1 \end{vmatrix}\begin{vmatrix} -m & -1 \\ -1 & m \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ -m & m \end{vmatrix}^2 = 4(-1 - m^2)^2$$

  which is always positive.

- $(x^2 - y^2)^2 = x^4 - 2x^2y^2 + y^4$ and $rk_{\mathbb{R}} f = 4$. In fact, consider the catalecticant matrix of size 2×4 of f

  $$J = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \end{pmatrix}.$$

  Computing the two relative equations, we have that $\ker J$ is generated by the following two cubics

  $$f_1 = x^3 + 3xy^2, \ f_2 = +3x^2y + y^3.$$

  Then a generic element of $\ker J$ is of the type $f_1 + mf_2 = x^3 + 3mx^2y + 3xy^2 + my^3$ with discriminant the following polynomial in m of degree four

  $$4\begin{vmatrix} 1 & m \\ m & 1 \end{vmatrix}\begin{vmatrix} m & 1 \\ 1 & m \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ m & m \end{vmatrix}^2 = -4(m^2 - 1)^2$$

  that vanishes in $m = \pm 1$ and is negative otherwise.

4. If $rk_{\mathbb{C}} f = 4$, we have that f has three coincident real roots and another one. Then we can write f as $f = xy^3$ and we have $rk_{\mathbb{R}} f = 4$.

1.3 Quintic forms

Proposition 19. *Let $f \in \mathrm{Sym}^5(\mathbb{R}^2)$ be such that $\mathrm{rk}_{\mathbb{C}} f = r$, with $r \in [1,5]$. Then we have:*

1. $r = 1 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 1$.
2. $r = 2 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 2$ *or* $\mathrm{rk}_{\mathbb{R}} f = 5$.
3. $r = 3 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 3$ *or* $\mathrm{rk}_{\mathbb{R}} f = 4$ *or* $\mathrm{rk}_{\mathbb{R}} f = 5$.
4. $r = 4 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 4$ *or* $\mathrm{rk}_{\mathbb{R}} f = 5$.
5. $r = 5 \Longrightarrow \mathrm{rk}_{\mathbb{R}} f = 5$.

Proof. Let $f \in Sym^5(\mathbb{R}^2)$ be with complex rank r:

1. trivial.
2. By Proposition 11.
3. Being $rk_{\mathbb{R}} f \geq rk_{\mathbb{C}} f$, by Proposition 5 and Theorem 14 we have the conclusion.
4. Being $rk_{\mathbb{R}} f \geq rk_{\mathbb{C}} f$, by Proposition 5 and Theorem 14 we have the conclusion.
5. Trivial.

□

Let $f \in Sym^5(\mathbb{R}^2)$. Starting from Proposition 19 and fixing the complex rank of f, we see precisely how occur the real ranks of f. Moreover, we give a method for compute the real rank without having to search homogeneous differential operator of degree $r = rk_{\mathbb{C}} f$ with all real roots. In the case they do not exist, we will find them without going into degree $r + 1$ and so on.

Remark 20. Let $f \in Sym^5(\mathbb{R}^2)$:

1. if $rk_{\mathbb{C}} f = 1$, trivially the real rank of f is 1 and conversely.
2. If $rk_{\mathbb{C}} f = 2$, we consider a quintic form with real coefficients

   $$f = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4.$$

   Then, being $rk_{\mathbb{C}} f = 2$, the catalecticant matrix of size 4×3 (i.e. we consider the linear application from the space of the homogeneous differential operator of degree $2 = rk_{\mathbb{C}} f$, D_2, to the space of the homogeneous polynomial of degree $5 - 2 = \deg f - rk_{\mathbb{C}} f$, R_3) of f

   $$J = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{pmatrix}$$

has rank 2. Then $\ker J$ has dimension one and it is generated by a polynomial g of degree two. Therefore if g has two real distinct roots we have $rk_{\mathbb{R}} f = 2$, otherwise $rk_{\mathbb{R}} f = 5$.

3. If $rk_{\mathbb{C}} f = 3$, the method appears in [12] and we must compute the sign of $\Delta(\beta)$, where $\beta = \sum_{i=0}^{3} \beta_i x^{3-i} y^i$ is the generator of the kernel of the (3×4)-catalecticant matrix J of f and β_i the appropriate (3×3)-determinants of J. In particular, let f be any quintic of complex rank three. Then $rk_{\mathbb{R}} f = 5$ if and only if f has only real roots not all coincident. On the other hand, in case f has some complex roots, $\Delta(\beta) > 0$ if and only if $rk_{\mathbb{R}} f = 3$ and $\Delta(\beta) < 0$ if and only if $rk_{\mathbb{R}} f = 4$.

4. If $rk_{\mathbb{C}} f = 4$, we consider a quintic f with real coefficients. Then f is not general and we have some cases that depend on the type of the roots of f.

   (a) f has five real roots. In this case, we can write f as $f = x(x^2 - y^2)(x^2 + 2axy + by^2)$, with $b - a^2 \leq 0$ and, by Remark 16, we have that $rk_{\mathbb{R}} f = 5$.

   (b) f has five roots not all real. In this case, we have the following situations:
   f has two coincident real roots. Then, we can rewrite f as $f = x^2(x^2 + y^2)(x - ay) = x^2(-ax^2y - ay^3 + x^3 + xy^2)$, with $a \neq 0$. Consider the catalecticant matrix of size 2×5 (i.e. we consider the linear application from the space of the homogeneous differential operator of degree $4 = rk_{\mathbb{C}} f$, D_4, to the space of the homogeneous polynomial of degree $5 - 4 = \deg f - rk_{\mathbb{C}} f$, R_1) of f

$$J = \begin{pmatrix} 1 & -\frac{a}{5} & \frac{1}{10} & -\frac{a}{10} & 0 \\ -\frac{a}{5} & \frac{1}{10} & -\frac{a}{10} & 0 & 0 \end{pmatrix}.$$

   Computing the two related equations, we have that $\ker J$ is generated by three quartics, with parameter a, f_1, f_2, f_3. Then a generic element of $\ker J$ is of the type $f_1 + mf_2 + nf_3$. Therefore it is difficult to continue as in the case of the quartic of complex rank 3, by computational problems. Then we consider the apolar ideal of f, $f^{\perp}$. Being the maximum rank (i.e. 3) of the (3×4)-catalecticant matrix of f, we know that $f^{\perp}$ is generated as in Theorem 12 and in Remark 13, that is precisely by a cubic g_2 and a quartic g_1. In particular g_2 has coefficients equal to the appropriate (3×3)-minors of the (3×4)-catalecticant matrix of f

$$\begin{pmatrix} 1 & -\frac{a}{5} & \frac{1}{10} & -\frac{a}{10} \\ -\frac{a}{5} & \frac{1}{10} & -\frac{a}{10} & 0 \\ \frac{1}{10} & -\frac{a}{10} & 0 & 0 \end{pmatrix}$$

   and two coincident roots, because $rk_{\mathbb{C}} f = 4$. Then, we have

$$g_2 = a_0 x^3 + a_1 x^2 y + a_2 xy^2 + a_3 y^3$$

   where

$$a_0 = \begin{vmatrix} -\frac{a}{5} & \frac{1}{10} & -\frac{a}{10} \\ \frac{1}{10} & -\frac{a}{10} & 0 \\ -\frac{a}{10} & 0 & 0 \end{vmatrix} = \frac{a^3}{1000}$$

$$a_1 = \begin{vmatrix} 1 & \frac{1}{10} & -\frac{a}{10} \\ -\frac{a}{5} & -\frac{a}{10} & 0 \\ -\frac{1}{10} & 0 & 0 \end{vmatrix} = -\frac{a^2}{1000}$$

$$a_2 = \begin{vmatrix} 1 & -\frac{a}{5} & -\frac{a}{10} \\ -\frac{a}{5} & \frac{1}{10} & 0 \\ \frac{1}{10} & -\frac{a}{10} & 0 \end{vmatrix} = \frac{a(-2a^2+1)}{1000}$$

$$a_3 = \begin{vmatrix} 1 & -\frac{a}{5} & \frac{1}{10} \\ -\frac{a}{5} & \frac{1}{10} & -\frac{a}{10} \\ \frac{1}{10} & -\frac{a}{10} & 0 \end{vmatrix} = \frac{-6a^2-1}{1000}$$

and with discriminant equal to zero. The discriminant of g_2 (up to scalar factors) is the following polynomial in a of degree twelve $a^6(2a^6 - 77a^4 - 16a^2 - 1)$ that vanishes in $a = 0$ and $a = \pm c_1$, being its factors a^6 and $2a^6 - 77a^4 - 16a^2 - 1$, where c_1 is

$$\frac{\sqrt{(467675 + 1200\sqrt{6})^{\frac{1}{3}}((467675 + 1200\sqrt{6})^{\frac{2}{3}} + 77(467675 + 1200\sqrt{6})^{\frac{1}{3}} + 6025)}}{\sqrt{6}(467675 + 1200\sqrt{6})^{\frac{1}{3}}}$$

. Then we have necessarily $a = \pm c_1$ and therefore the following two cases:

- $a = c_1$. Then we have $f = x^2(x^2+y^2)(x - c_1 y)$. The (2×5)-catalecticant matrix of f is

  $$J = \begin{pmatrix} 1 & -\frac{c_1}{5} & \frac{1}{10} & -\frac{c_1}{10} & 0 \\ -\frac{c_1}{5} & \frac{1}{10} & -\frac{c_1}{10} & 0 & 0 \end{pmatrix}.$$

  Computing the two related equations, we have that $\ker J$ is generated by the following three quartics:

  $$f_1 = \frac{1}{2c_1}x^4 + x^3y + \frac{5-2c_1^2}{c_1^2}xy^3, \ f_2 = -\frac{1}{2}x^4 + x^2y^2 - \frac{4}{c_1}xy^3, \ f_3 = y^4.$$

  Then a generic element of $\ker J$ is of the type $g = f_1 + mf_2 + nf_3 = \left(\frac{1}{2c_1} - \frac{m}{2}\right)x^4 + x^3y + mx^2y^2 + \left(\frac{5-2c_1^2}{c_1^2} - \frac{4m}{c_1}\right)xy^3 + ny^4$ with discriminant equal to the following polynomial in m, n of degree six $(-32c_1^{10}m^5n - 128c_1^{10}m^4n^2 + 32c_1^{10}m^4 - 128c_1^{10}m^3n^3 + 240c_1^{10}m^3n + 96c_1^{10}m^2n^2 - 128c_1^{10}m^2 - 96c_1^{10}mn - 108c_1^{10}n^2 + 128c_1^{10} + 128c_1^9m^5 + 1696c_1^9m^4n + 1024c_1^9m^3n^2 - 1696c_1^9m^3 + 384c_1^9m^2n^3 - 928c_1^9m^2n - 480c_1^9mn^2 + 1344c_1^9m - 48c_1^9n + 128c_1^8m^6 + 2304c_1^8m^5n - 7136c_1^8m^4 - 5856c_1^8m^3n - 2624c_1^8m^2n^2 + 8800c_1^8m^2 - 384c_1^8mn^3 + 504c_1^8mn + 384c_1^8n^2 - 1392c_1^8 - 11968c_1^7m^5 - 10368c_1^7m^4n + 3584c_1^7m^3 + 6592c_1^7m^2n + 2688c_1^7mn^2 - 13776c_1^7m + 128c_1^7n^3 + 240c_1^7n - 6912c_1^6m^6 + 69064c_1^6m^4 + 17424c_1^6m^3n - 68108c_1^6m^2 - 2100c_1^6mn - 960c_1^6n^2 + 6720c_1^6 + 48384c_1^5m^5 - 163064c_1^5m^3 - 12960c_1^5m^2n + 57720c_1^5m - 300c_1^5n -$

$140832c_1^4m^4 + 193140c_1^4m^2 + 3600c_1^4mn - 18200c_1^4 + 218160c_1^3m^3 - 114300c_1^3m - 189675c_1^2m^2 + 27000c_1^2 + 87750c_1m - 16875)\frac{1}{4c_1^{10}}$.
The companion matrix and the Bezoutiant of g are respectively

$$M = \begin{pmatrix} 0 & 0 & 0 & -\frac{2nc_1}{1-c_1m} \\ 1 & 0 & 0 & -\frac{2(5-2c_1^2-4mc_1)}{c_1(1-c_1m)} \\ 0 & 1 & 0 & -\frac{2mc_1}{1-c_1m} \\ 0 & 0 & 1 & -\frac{2c_1}{1-c_1m} \end{pmatrix}, \quad B = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{pmatrix}$$

where $s_0 = 4$, $s_1 = Tr(M) = \frac{2c_1}{c_1m-1}$, $s_2 = Tr(M^2) = \frac{4((c_1m-1)m+c_1)c_1}{(c_1m-1)^2}$, $s_3 = Tr(M^3) = \frac{2(4c_1^4-12*c_1^3m^3+6c_1^3m+39c_1^2m^2-6c_1^2-42c_1m+15)}{(c_1m-1)^3c_1}$, $s_4 = Tr(M^4) = \frac{8(c_1^4m^4+c_1^4m^3n+2c_1^4-10c_1^3m^3-3c_1^3m^2n+4c_1^3m+27c_1^2m^2+3c_1^2mn-4c_1^2-28c_1m-c_1n+10)}{(c_1m-1)^4}$, $s_5 = Tr(M^5) = \frac{4(5c_1^5m^3n+8c_1^5-20c_1^4m^5-30c_1^4m^3-15c_1^4m^2n+20c_1^4m+85c_1^3m^4+}{(c_1m-1)^5}$ $\frac{110c_1^3m^2+15c_1^3mn-20c_1^3-135c_1^2m^3-130c_1^2m-5c_1^2n+95c_1m^2+50c_1-25m)}{(c_1m-1)^5}$ and $s_6 = Tr(M^6) = \frac{4(4c_1^8m^6+6c_1^8m^5n+12c_1^8m^3n+16c_1^8-60c_1^7m^5-24c_1^7m^4n-72c_1^7m^3-}{(c_1m-1)^6c_1^2}$ $\frac{36c_1^7m^2n+48c_1^7m+48c_1^6m^6+168c_1^6m^4+36c_1^6m^3n+276c_1^6m^2+36c_1^6mn-48c_1^6-312c_1^5m^5-}{(c_1m-1)^6c_1^2}$ $\frac{124c_1^5m^3-24c_1^5m^2n-336c_1^5m-12c_1^5n+843c_1^4m^4-96c_1^4m^2+6c_1^4mn+132c_1^4-1212c_1^3m^3+}{(c_1m-1)^6c_1^2}$ $\frac{168c_1^3m+978c_1^2m^2-60c_1^2-420c_1m+75)}{(c_1m-1)^6c_1^2}$. Then the principal minors of B are the discriminant of g and the following polynomials in m, n of degree 4 and 2

$$d_1 = 16(4c_1^6m^4 + 8c_1^6m^3n - 6c_1^6m^2 + 6c_1^6mn + 12c_1^6 - 92c_1^5m^3 - 16c_1^5m^2n + 68c_1^5m - 6c_1^5n - 144c_1^4m^4 + 342c_1^4m^2 + 8c_1^4mn - 66c_1^4 + 648c_1^3m^3 - 434c_1^3m - 1089c_1^2m^2 + 180c_1^2 + 810c_1m - 225)$$

$$d_2 = 4(4c_1m^2 + 3c_1 - 4m)c_1$$

whose signs are both positive in some regions of the real plane (see Figures 1.1 and 1.2), whence $rk_{\mathbb{R}}f = 4$, because if the Bezoutiant of a polynomial is positive definite, then the polynomial has all real roots (see Corollary 4.49 in [15]). For example, in $m = 0$ and $n = 0$ we have that the discriminant of g is $\frac{128c_1^{10}-1392c_1^8+6720c_1^6-18200c_1^4+27000c_1^2-16875}{4c_1^{10}}$, d_1 is $8(2c_1^4-6c_1^2+15)(2c_1^2-5)$ and d_2 is $12c_1^2$, all positive for the above fixed c_1.

- $a = -c_1$. Then we have $f = x^2(x^2+y^2)(x+c_1y)$. The (2×5)-catalecticant matrix of f is

$$J = \begin{pmatrix} 1 & \frac{c_1}{5} & \frac{1}{10} & \frac{c_1}{10} & 0 \\ \frac{c_1}{5} & \frac{1}{10} & \frac{c_1}{10} & 0 & 0 \end{pmatrix}.$$

Computing the two related equations, we have that $\ker J$ is generated by the following three quartics:

$$f_1 = -\frac{1}{2c_1}x^4 + x^3y + \frac{5-2c_1^2}{c_1^2}xy^3,\ f_2 = -\frac{1}{2}x^4 + x^2y^2 + \frac{4}{c_1}xy^3,\ f_3 = y^4.$$

Then a generic element of $\ker J$ is of the type $g = f_1 + mf_2 + nf_3 = \left(-\frac{1}{2c_1} - \frac{m}{2}\right)x^4 + x^3y + mx^2y^2 + \left(\frac{5-2c_1^2}{c_1^2} + \frac{4m}{c_1}\right)xy^3 + ny^4$ with discriminant equal to the following polynomial in m, n of degree six $(-32c_1^{10}m^5n - 128c_1^{10}m^4n^2 + 32c_1^{10}m^4 - 128c_1^{10}m^3n^3 + 240c_1^{10}m^3n + 96c_1^{10}m^2n^2 - 128c_1^{10}m^2 - 96c_1^{10}mn - 108c_1^{10}n^2 + 128c_1^{10} - 128c_1^9m^5 - 1696c_1^9m^4n - 1024c_1^9m^3n^2 + 1696c_1^9m^3 - 384c_1^9m^2n^3 + 928c_1^9m^2n + 480c_1^9mn^2 - 1344c_1^9m + 48c_1^9n + 128c_1^8m^6 + 2304c_1^8m^5n - 7136c_1^8m^4 - 5856c_1^8m^3n - 2624c_1^8m^2n^2 + 8800c_1^8m^2 - 384c_1^8mn^3 + 504c_1^8mn + 384c_1^8n^2 - 1392c_1^8 + 11968c_1^7m^5 + 10368c_1^7m^4n - 35584c_1^7m^3 - 6592c_1^7m^2n - 2688c_1^7mn^2 + 13776c_1^7m - 128c_1^7n^3 - 240c_1^7n - 6912c_1^6m^6 + 69064c_1^6m^4 + 17424c_1^6m^3n - 68108c_1^6m^2 - 2100c_1^6mn - 960c_1^6n^2 + 6720c_1^6 - 48384c_1^5m^5 + 163064c_1^5m^3 + 12960c_1^5m^2n - 57720c_1^5m + 300c_1^5n - 140832c_1^4m^4 + 193140c_1^4m^2 + 3600c_1^4mn - 18200c_1^4 - 218160c_1^3m^3 + 114300c_1^3m - 189675c_1^2m^2 + 27000c_1^2 - 87750c_1m - 16875)\frac{1}{4c_1^{10}}$.

The companion matrix and the Bezoutiant of g are respectively

$$M = \begin{pmatrix} 0 & 0 & 0 & \frac{2nc_1}{1+c_1m} \\ 1 & 0 & 0 & \frac{2(5-2c_1^2+4mc_1)}{c_1(1+c_1m)} \\ 0 & 1 & 0 & \frac{2mc_1}{1+c_1m} \\ 0 & 0 & 1 & \frac{2c_1}{1+c_1m} \end{pmatrix},\ B = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{pmatrix}$$

where $s_0 = 4$, $s_1 = Tr(M) = \frac{2c_1}{c_1m+1}$, $s_2 = Tr(M^2) = \frac{4((m^2+1)c_1+m)c_1}{(c_1m+1)^2}$, $s_3 = Tr(M^3) = \frac{2(4c_1^4+12c_1^3m^3-6c_1^3m+39c_1^2m^2-6c_1^2+42c_1m+15)}{(c_1m+1)^3c_1}$, $s_4 = Tr(M^4) = \frac{8(c_1^4m^4+c_1^4m^3n+2c_1^4+10c_1^3m^3+3c_1^3m^2n-4c_1^3m+27c_1^2m^2+3c_1^2mn-4c_1^2+28c_1m+c_1n+10)}{(c_1m+1)^4}$,
$s_5 = Tr(M^5) = \frac{4(5c_1^5m^3n+8c_1^5+20c_1^4m^5+30c_1^4m^3+15c_1^4m^2n-20c_1^4m+85c_1^3m^4+}{(c_1m+1)^5}$
$\frac{110c_1^3m^2+15c_1^3mn-20c_1^3+135c_1^2m^3+130c_1^2m+5c_1^2n+95c_1m^2+50c_1+25m)}{(c_1m+1)^5}$ and
$s_6 = Tr(M^6) = \frac{4(4c_1^8m^6+6c_1^8m^5n+12c_1^8m^3n+16c_1^8+60c_1^7m^5+24c_1^7m^4n+72c_1^7m^3+}{(c_1m+1)^6c_1^2}$

$\frac{36c_1^7m^2n-48c_1^7m+48c_1^6m^6+168c_1^6m^4+36c_1^6m^3n+276c_1^6m^2+36c_1^6mn-48c_1^6+312c_1^5m^5+}{(c_1m+1)^6c_1^2}$
$\frac{124c_1^5m^3+24c_1^5m^2n+336c_1^5m+12c_1^5n+843c_1^4m^4-96c_1^4m^2+6c_1^4mn+132c_1^4+1212c_1^3m^3-}{(c_1m+1)^6c_1^2}$
$\frac{168c_1^3m+978c_1^2m^2-60c_1^2+420c_1m+75)}{(c_1m+1)^6c_1^2}$. Then the principal minors of B are the discriminant of g and the following polynomials in m, n of degree 4 and 2

$$d_1 = 16(4c_1^6m^4 + 8c_1^6m^3n - 6c_1^6m^2 + 6c_1^6mn + 12c_1^6 + 92c_1^5m^3 + 16c_1^5m^2n-$$

$$68c_1^5m + 6c_1^5n - 144c_1^4m^4 + 342c_1^4m^2 + 8c_1^4mn - 66c_1^4 - 648c_1^3m^3 + 434c_1^3m-$$

$$1089c_1^2m^2 + 180c_1^2 - 810c_1m - 225)$$

$$d_2 = 4(4c_1m^2 + 3c_1 + 4m)c_1$$

whose signs are both positive in some regions of the real plane (see Figures 1.3 and 1.4), whence $rk_{\mathbb{R}}f = 4$. For example, in $m = 0$ and $n = 0$ we have that the discriminant of g is $(128c_1^{10} - 1392c_1^8 + 6720c_1^6 - 18200c_1^4 + 27000c_1^2 - 16875)$, d_1 is $48(2c_1^4 - 6c_1^2 + 15)(2c_1^2 - 5)$ and d_2 is $12c_1^2$, all positive for the above fixed c_1.

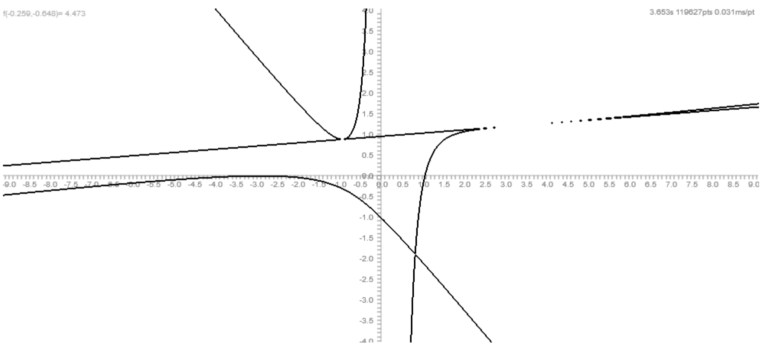

Figure 1.1: Discriminant of $g = \left(\frac{1}{2c_1} - \frac{m}{2}\right)x^4 + x^3y + mx^2y^2 + \left(\frac{5-2c_1^2}{c_1^2} - \frac{4m}{c_1}\right)xy^3 + ny^4$.

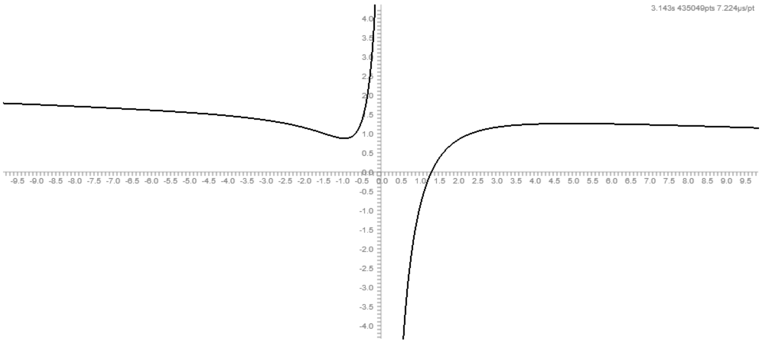

Figure 1.2: Minor $d_1 = 16(4c_1^6m^4+8c_1^6m^3n-6c_1^6m^2+6c_1^6mn+12c_1^6-92c_1^5m^3-16c_1^5m^2n+68c_1^5m-6c_1^5n-144c_1^4m^4+342c_1^4m^2+8c_1^4mn-66c_1^4+648c_1^3m^3-434c_1^3m-1089c_1^2m^2+180c_1^2+810c_1m-225)$.

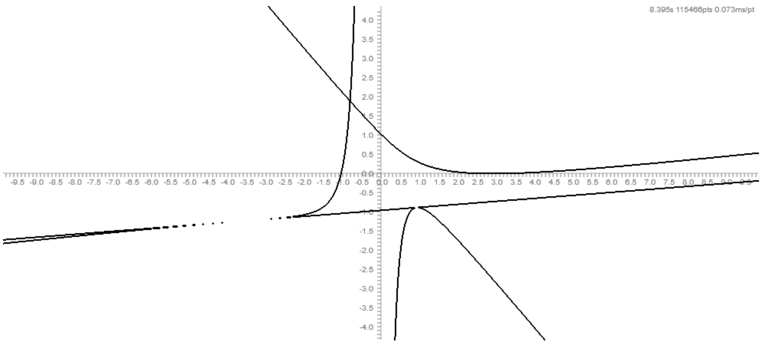

Figure 1.3: Discriminant of $g = \left(-\frac{1}{2c_1}-\frac{m}{2}\right)x^4+x^3y+mx^2y^2+\left(\frac{5-2c_1^2}{c_1^2}+\frac{4m}{c_1}\right)xy^3+ny^4$.

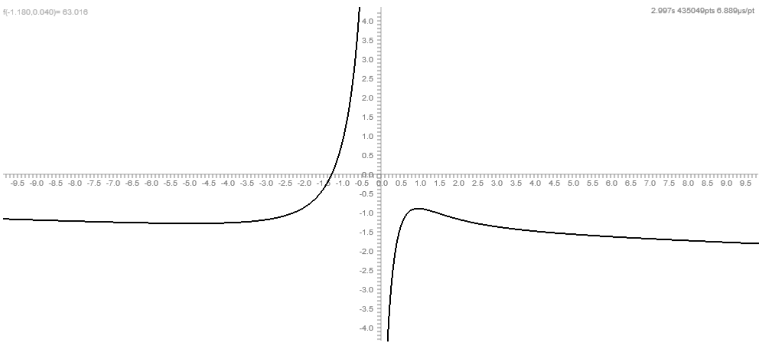

Figure 1.4: Minor $d_1 = 16(4c_1^6m^4+8c_1^6m^3n-6c_1^6m^2+6c_1^6mn+12c_1^6+92c_1^5m^3+16c_1^5m^2n-68c_1^5m+6c_1^5n-144c_1^4m^4+342c_1^4m^2+8c_1^4mn-66c_1^4-648c_1^3m^3+434c_1^3m-1089c_1^2m^2+180c_1^2-810c_1m-225)$.

f has five distinct roots. In this case, we can rewrite f as $f = x(x^2 +$

$y^2)(x^2 + 2axy + by^2)$, with $b - a^2 \neq 0$ (and $(a, b) \neq (0, 1)$, $b \neq 0$) and we have that $rk_{\mathbb{R}} f = 4$ by Remark 15 and by the fact that there are only two possibilities (4 or 5) for the real rank of f.

f has three real coincident roots. Then we can rewrite f as $x^3(x^2 + a^2y^2) = x^5 + a^2x^3y^2$, with $a \neq 0$. In this case, by the change of variables $x' = x$, $y' = ay$ and rename, f becomes $f = x^3(x^2 + y^2)^2$. Consider the catalecticant matrix of size 2×5 (i.e. we consider the linear application from the space of the homogeneous differential operator of degree $4 = rk_{\mathbb{C}} f$, D_4, to the space of the homogeneous polynomial of degree $5 - 4 = \deg f - rk_{\mathbb{C}} f$, R_1) of f

$$J = \begin{pmatrix} 1 & 0 & \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 & 0 & 0 \end{pmatrix}.$$

Computing the two related equations, we have that $\ker J$ is generated by the following three quartics:

$$f_1 = -x^4 + 10x^2y^2,\ f_2 = xy^3,\ f_3 = y^4.$$

Then a generic element of $\ker J$ is of the type $g = f_1 + mf_2 + nf_3 = -x^4 + 10x^2y^2 + mxy^3 + ny^4$ with discriminant equal to the following polynomial in m, n of degree four $-27m^4 + 1440m^2n + 4000m^2 - 256n^3 - 12800n^2 - 160000n$. The companion matrix and the Bezoutiant of g are respectively

$$M = \begin{pmatrix} 0 & 0 & 0 & n \\ 1 & 0 & 0 & m \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{pmatrix}$$

where $s_0 = 4$, $s_1 = Tr(M) = 0$, $s_2 = Tr(M^2) = 20$, $s_3 = Tr(M^3) = 3m$, $s_4 = Tr(M^4) = 4(n + 50)$, $s_5 = Tr(M^5) = 50m$ and $s_6 = Tr(M^6) = 3m^2 + 60n + 2000$. Then the principal minors of B are the discriminant of g and the following polynomials in m, n of degree two and zero

$$d_1 = 4(-9m^2 + 80n + 2000),\ d_2 = 80$$

whose signs are both positive in some regions of the real plane (see Figures 1.5 and 1.6), whence $rk_{\mathbb{R}} f = 4$. For example, in $m = -20$ and $n = -10$ we have that the discriminant of g is 576000, d_1 is 4800 and d_2 is 80.

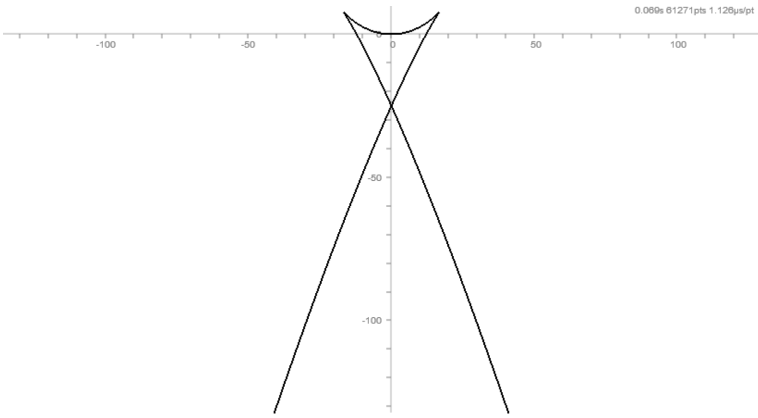

Figure 1.5: Discriminant of $g = -x^4 + 10x^2y^2 + mxy^3 + ny^4$.

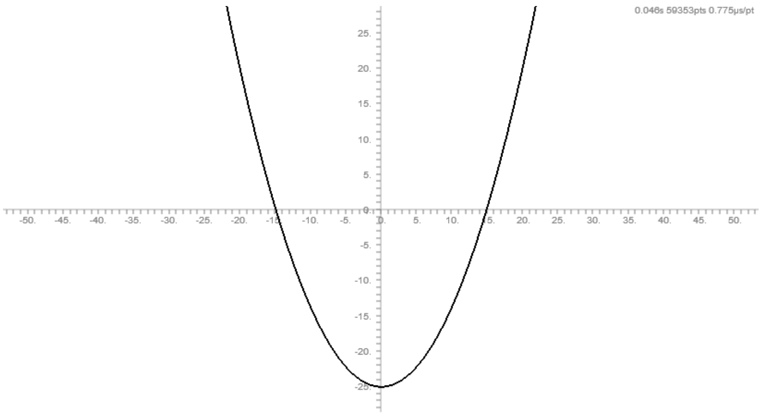

Figure 1.6: Minor $d_1 = 4(-9m^2 + 80n + 2000)$.

5. If $rk_{\mathbb{C}} f = 5$, we have that f has four coincident real roots and another one. Then we can write f as $f = xy^4$ and we have $rk_{\mathbb{R}} f = 5$.

1.4 Conclusions

Let f be a binary form of degree four or five. Then we have:

1. $d = 4$:

   - f has real rank one if and only if f has complex rank one (thus if and only if f has four coincident roots).
   - f has real rank two if and only if f has complex rank two and the quadratic generator of the kernel of its (3×3)-catalecticant matrix of rank 2 has two real distinct roots.

- f has real rank three if and only if f has complex rank three (i.e. f has generic rank) and it has not only real roots.
- f has real rank four if and only if one of the following possibilities holds:
  f has complex rank four,
  f has complex rank three and only real roots (not all coincident),
  f has complex rank two and the quadratic generator of the kernel of its (3×3)-catalecticant matrix of rank 2 has two complex roots.

2. $d = 5$:

- f has real rank one if and only if f has complex rank one (thus if and only if f has five coincident roots).
- f has real rank two if and only if f has complex rank two and the quadratic generator of the kernel of its (4×3)-catalecticant matrix of rank 2 has two real distinct roots.
- f has real rank three if and only if f has complex rank three (i.e. f has generic rank) and the cubic generator of the kernel of its (3×4)-catalecticant matrix has three real distinct roots.
- f has real rank four if and only if one of the following possibilities holds:
  f has complex rank four and not all real roots,
  f has complex rank three and the cubic generator of the kernel of its (3×4)-catalecticant matrix has two complex roots.
- f has real rank five if and only if one of the following possibilities holds:
  f has complex rank five,
  f has complex rank three or four and only real roots (not all coincident),
  f has complex rank two and the quadratic generator of the kernel of its (4×3)-catalecticant matrix of rank 2 has two complex roots.

Remark 21. In Propositions 17, 19, we show a classification of the real rank of $f \in Sym^d(\mathbb{R}^2)$ given the complex rank r of f, that is given a real form $f \in S_{d,r}(\mathbb{R})$ such that $d \in [4, 5]$ and $r \in [1, 5]$. Now we want to give, bearing in mind Remarks 18 and 20, another classification of the real ranks for quartic and quintic forms, using the secant varieties. Then we have, by Theorem 4:

1. $d = 4$. The secant varieties that are of interest to us are:

$$Sec^1(X) = \{[\phi] \in \mathbb{P}(S_4^*) \ st \ rk_{\mathbb{C}}\phi = 1\} \cup \emptyset$$

$$Sec^2(X) = \{[\phi] \in \mathbb{P}(S_4^*) \ st \ rk_{\mathbb{C}}\phi \leq 2\} \cup \{[\phi] \in \mathbb{P}(S_4^*) \ st \ rk_{\mathbb{C}}\phi = 4\}$$

$$Sec^3(X) = \{[\phi] \in \mathbb{P}(S_4^*) \ st \ rk_{\mathbb{C}}\phi \leq 3\} \cup \{[\phi] \in \mathbb{P}(S_4^*) \ st \ rk_{\mathbb{C}}\phi \geq 3\}.$$

Then $f \in \bar{S}_{4,2}$ can have complex rank 1, 2 and 4, therefore real rank 1, 2 and 4. Again, $f \in \bar{S}_{4,3}$ can have complex rank 1, 2, 3 and 4, therefore all real rank between 1 and 4.

2. $d = 5$. The secant varieties that are of interest to us are:

$$Sec^1(X) = \{[\phi] \in \mathbb{P}(S_5^*) \ st \ rk_{\mathbb{C}}\phi = 1\} \cup \emptyset$$

$$Sec^2(X) = \{[\phi] \in \mathbb{P}(S_5^*) \ st \ rk_{\mathbb{C}}\phi \leq 2\} \cup \{[\phi] \in \mathbb{P}(S_5^*) \ st \ rk_{\mathbb{C}}\phi = 5\}$$

$$Sec^3(X) = \{[\phi] \in \mathbb{P}(S_5^*) \ st \ rk_{\mathbb{C}}\phi \leq 3\} \cup \{[\phi] \in \mathbb{P}(S_5^*) \ st \ rk_{\mathbb{C}}\phi \geq 4\}.$$

Then $f \in \bar{S}_{5,2}$ can have complex rank 1, 2 and 5, therefore real rank 1, 2 and 5. Again, $f \in \bar{S}_{5,3}$ can have complex rank 1, 2, 3, 4 and 5, therefore all real rank between 1 and 5.

Finally, we have written two software with Macaulay2 for the calculation of the real and complex rank of a real binary quartic and quintic form.

Chapter 2

Real eigenvectors of real symmetric tensors

2.1 Preliminaries

Definition 22. *([7],[21],[28]) Let $x \in \mathbb{C}^n$ be and let $A = (a_{i_1,i_2,...,i_d})$ be a tensor of order d and format $n \times n \times \cdots \times n$. We define Ax^{d-1} to be the vector in $\mathbb{C}^n$ whose j-th coordinate is the scalar*

$$(Ax^{d-1})_j = \sum_{i_2=1}^{n} \cdots \sum_{i_d=1}^{n} a_{j,i_2,...,i_d} x_{i_2} \cdots x_{i_d}$$

Then, if $\lambda \in \mathbb{C}$ and $\tilde{x} \in \mathbb{C}^n \setminus \{0\}$ are elements such that $Ax^{d-1} = \lambda x$, we say that λ is an eigenvalue of A, $\tilde{x}$ is an eigenvector of A and $(\tilde{x}, \lambda)$ is an eigenpair. Two eigenpairs $(\lambda, \tilde{x})$ and $(\lambda\prime, \tilde{x}')$ of the same tensor A are considered to be equivalent if there exists a complex number $t \neq 0$ such that $t^{m-2}\lambda = \lambda'$ and $t\tilde{x} = \tilde{x}'$. Moreover, the fixed points of the rational map $\psi_A : \mathbb{P}^{n-1}(\mathbb{C}) \longrightarrow \mathbb{P}^{n-1}(\mathbb{C})$, $[x] \longmapsto [Ax^{d-1}]$ are exactly the eigenvectors of the tensor A with non-zero eigenvalue and the base locus of ψ_A is the set of eigenvectors with eigenvalue zero. In particular, the map ψ_A is defined everywhere if and only if 0 is not an eigenvalue of A. Finally, we say that A is nilpotent if and only if some iterate of ψ_A is nowhere defined.

Remark 23. *([7],[33])* Consider $f(x) \equiv f(x_1, ..., x_n)$ the homogeneous polynomial in $\mathbb{C}[x_1, ..., x_n]$ of degree d associated to the symmetric tensor A by the relation

$$f(x_1, ..., x_n) = A \cdot x^d = \sum_{i_1}^{n} \cdots \sum_{i_d}^{n} a_{i_1,i_2,...,i_d} x_{i_1} \cdots x_{i_d} = x \cdot Ax^{d-1}$$

Then $\tilde{x} \in \mathbb{C}^n$ is an eigenvector of A with eigenvalue $\lambda \in \mathbb{C}$ if and only if

$$\nabla f(\tilde{x}) = \lambda \tilde{x}$$

Moreover, the eigenvectors of A are precisely the fixed points of the projective map

$$\nabla f : \mathbb{P}^{n-1}(\mathbb{C}) \longrightarrow \mathbb{P}^{n-1}(\mathbb{C}),\ [x] \longmapsto [\nabla f(x)]$$

well-defined provided the hypersurface $\{f = 0\}$ has no singular points. Then $\tilde{x} \in \mathbb{C}^n$ is a representative of a $[\tilde{x}] \in \mathbb{P}^{n-1}(\mathbb{C})$ eigenvector of A if and only if $[\tilde{x}] = [\nabla f(\tilde{x})]$, that is $\tilde{x} \in \mathbb{C}^n$ must satisfy the system

$$\begin{cases} f_{x_1}(x) = \lambda x_1 \\ f_{x_2}(x) = \lambda x_2 \\ \qquad \vdots \\ f_{x_n}(x) = \lambda x_n \end{cases}$$

Evidently, an eigenvectors of A is geometrically a line through the origin of $\mathbb{C}^n$, because it is a point of $\mathbb{P}^{n-1}(\mathbb{C}) = \mathbb{P}(\mathbb{C}^n)$. Finally, all previous characterizations are equivalent to say that $\tilde{x} \in \mathbb{C}^n$ is an eigenvector of A if and only if all 2×2-minors of $2 \times n$-matrix

$$\begin{pmatrix} f_{x_1}(\tilde{x}) & f_{x_2}(\tilde{x}) & \dots & f_{x_n}(\tilde{x}) \\ x_1 & x_2 & \dots & x_n \end{pmatrix}$$

vanish on $\tilde{x}$, or obviously the vector $\nabla f(\tilde{x})$ and $\tilde{x}$ are proportional.

Theorem 24. *([7]) If a tensor A has finitely many equivalence classes of eigenpairs over $\mathbb{C}$ then their number, counted with multiplicity, is equal to $((d-1)^n - 1)/(d-2)$. If the entries of A are sufficiently generic, then all multiplicities are equal to 1, so there are exactly $((d-1)^n - 1)/(d-2)$ equivalence classes of eigenpairs.*

Proof. For $d = 2$, the expression $((d-1)^n - 1)/(d-2)$ simplifies to n, which is the number of eigenvalues of an ordinary $n \times n$-matrix. Hence we shall now assume that $d \geq 3$. For a fixed tensor A, the n equations determined by $Ax^{d-1} = \lambda x$ correspond to n homogeneous polynomials of degree $d-1$ in the graded polynomial ring R, where $R = \mathbb{C}[x_1, ..., x_n, \lambda]$, with $x_1, ..., x_n$ having degree 1 and λ having degree $d-2$. Since R is generated in degree $d-2$, the line bundle $\vartheta_X(d-2)$ is very ample, where $X = \mathbb{P}(1, 1, ..., d-2)$ (see [18], pag. 35). The corresponding lattice polytope Δ is an n-dimensional simplex with vertices at $(d-2)e_i$ for $1 \leq i \leq n$ and e_{n+1}, where e_i are the basis vectors in $\mathbb{R}^{n+1}$. The affine hull of Δ is the hyperplane $x_1 + ... + x_n + (d-2)\lambda = d-2$. The normalized volume of this simplex equals

$$\mathbb{V}(\Delta) = (d-2)^{n-1}. \tag{2.1}$$

The lattice polytope Δ, is smooth, except at the vertex e_{n+1} where it is simplicial with index $d-2$. Therefore, the projective toric variety X is simplicial, with precisely one isolated singular point corresponding to the vertex e_{n+1}. By [18], pag. 100, the variety X has a rational Chow ring $A^*(X)_{\mathbb{Q}}$, which we can use to compute intersection numbers of divisors on X. Our system of equations $Ax^{d-1} = \lambda x$ consists of n polynomials of degree $d-1$ in R. Let D be the divisor class corresponding to $\vartheta_X(d-1)$ and let H be the very ample divisor class corresponding to $\vartheta_X(d-2)$. The volume formula 2.1 is equivalent to $(d-2)^{n-1}$ in $A^*(X)_{\mathbb{Q}}$ and we compute the self-intersection number of D as the following rational number:

$$D^n = \left(\frac{d-1}{d-2} \cdot H\right)^n = \left(\frac{d-1}{d-2}\right)^n \cdot (d-2)^{n-1} = \frac{(d-1)^n}{d-2}$$

From this count we must remove the trivial solution $\{x = 0\}$ of $Ax^{d-1} = \lambda x$. That solution corresponds to the singular point e_{n+1} on X. Since that point has index $d-2$, the trivial solution counts for $1/(d-2)$ in the intersection computation, as shown in [18] pag. 100. Therefore the number of non-trivial solutions in X is equal to

$$D^n - \frac{1}{d-2} = \frac{(d-1)^n - 1}{d-2}. \tag{2.2}$$

Therefore, when the tensor A admits only finitely many equivalence classes of eigenpairs, then their number, counted with multiplicities, coincides with the positive integer in 2.2. □

Corollary 25. *([7]) If A has real entries and either d or n is odd, then A has a real eigenpair.*

Proof. When either d or n is odd, then one can check that the integer $((d-1)^n-1)/(d-2)$ in Theorem 24 is odd. This implies that A has a real eigenpair by Corollary 13.2 in [17]. □

Definition 26. *([29]) The* characteristic polynomial $\Phi_A(\lambda)$ *of a generic tensor A is defined as follows: consider the univariate polynomial in λ that arises by eliminating the unknowns $x_1, ..., x_n$ from the system of equations $Ax^{d-1} = \lambda x$ and $x \cdot x = 1$. If d is even, then this polynomial equals $\Phi_A(\lambda)$; if d is odd, then this polynomial has the form $\Phi_A(\lambda^2)$.*

Then Theorem 24 implies the following:

Proposition 27. *([7]) The set of normalized eigenvalues of a tensor is either finite or it consists of all complex numbers in the complement of a finite set.*

Proof. The set $\epsilon(A)$ of normalized eigenvalues λ of the tensor A is defined by the condition

$$\exists\, x \in \mathbb{C}^n \ s.t. \ Ax^{d-1} = \lambda x \ and \ x \cdot x = 1$$

Hence $\epsilon(A)$ is the image of an algebraic variety in $\mathbb{C}^{n+1}$ under the projection $(x, \lambda) \mapsto \lambda$. Chevalley's Theorem states that the image of an algebraic variety under a polynomial map is constructible, that is, defined by a Boolean combination of polynomial equations and inequations. We conclude that the set $\epsilon(A)$ of normalized eigenvalues is a constructible subset of $\mathbb{C}$. This means that $\epsilon(A)$ is either a finite set or the complement of a finite set. □

Proposition 28. *([7]) For a tensor A, each of the following conditions implies the next:*

1. *the set $\epsilon(A)$ of all normalized eigenvalues consists of all complex numbers.*
2. *The set $\epsilon(A)$ is infinite.*
3. *The characteristic polynomial $\Phi_A(\lambda)$ vanishes identically.*

Proof. Clearly, the first statement implies the second. By the projection argument in the proof above, the zero set in $\mathbb{C}$ of the characteristic polynomial $\Phi_A(\lambda)$ contains the set $\epsilon(A)$. Hence the second statement implies the third. $\square$

Proposition 29. *([7]) If a tensor A is nilpotent then 0 is the only eigenvalue of A. The converse is not true: there exist tensors with only eigenvalue 0 that are not nilpotent.*

Proof. Suppose $\lambda \neq 0$ is an eigenvalue and $x \in \mathbb{C}^n - \{0\}$ a corresponding eigenvector. Then x represents a point in $\mathbb{P}^{n-1}(\mathbb{C})$ that is fixed by ψ_A. Hence it is fixed by every iterate $\psi_A^{(r)}$ of ψ_A. In particular, $\psi_A^{(r)}$ is defined at (an open neighborhood) of $x \in \mathbb{P}^{n-1}(\mathbb{C})$ and A is not nilpotent. Let A be the $2 \times 2 \times 2$-tensor with $a_{111} = a_{211} = a_{212} = 1$ and the other five entries zero. The eigenpairs of A are the solutions to $x_1^2 = \lambda x_1$ and $x_1^2 + x_1 x_2 = \lambda x_2$. Up to equivalence, the only eigenpair is $x = (0, 1)$ and $\lambda = 0$. However, the self-map ψ_A on $\mathbb{P}^1$ is dominant. To see this, note that ψ_A acts by translation on the affine line $\mathbb{A}^1 = \{x_1 \neq 0\}$ because $[(x_1^2, x_1^2 + x_1 x_2)] = [(x_1, x_1 + x_2)]$. All iterates of ψ_A are defined on $\mathbb{A}^1$, i.e. there are no base points with $x_1 \neq 0$, and hence A is not nilpotent. $\square$

Then, for a symmetric tensor follow that

Corollary 30. *([7]) The singular points of the projective hypersurface*

$$\left\{x \in \mathbb{P}^{n-1} | f(x) = 0\right\}$$

are precisely the eigenvectors of the corresponding symmetric tensor A which have eigenvalue 0.

Proposition 31. *([7]) Fix a non-zero $\lambda \in \mathbb{C}$ and suppose $d \geq 3$. Then $\bar{x} \in \mathbb{C}^n$ is a normalized eigenvector of A with eigenvalue λ if and only if $\bar{x}$ is a singular point of the affine hypersurface defined by the polynomial*

$$f(x) - \frac{\lambda}{2} x \cdot x - \left(\frac{1}{d} - \frac{1}{2}\right) \lambda. \tag{2.3}$$

Proof. The gradient of the hypersurface is $\nabla f - \lambda x = A x^{d-1} - \lambda x$, so every singular point x is an eigenvector with eigenvalue λ. Furthermore, if we substitute $f(x) = \frac{1}{d} x \cdot \nabla f = \frac{\lambda}{d} x \cdot x$ into the hypersurface, then we obtain $x \cdot x = 1$. This argument is reversible: if $\bar{x}$ is a normalized eigenvector of A, then $x \cdot x = 1$ and $\nabla f(x) = \lambda x$ and this implies that the hypersurface and its derivatives vanish. $\square$

Corollary 32. *([7]) The characteristic polynomial $\Phi_A(\lambda)$ is a factor of the discriminant of 2.3.*

Theorem 33. *Every symmetric tensor A has at most*

$$\begin{cases} ((d-1)^n - 1)/(d-2), & d \geq 3 \\ (d-1)^{n-1} + (d-1)^{n-2} + ... + (d-1)^0 = n, & d = 2 \end{cases}$$

distinct normalized eigenvalues. This bound is attained for general symmetric tensors A.

Proof. It suffices to show that the number of normalized eigenvalues of every symmetric tensor A is finite. Recall from the proof of Theorem 24 in [7] that the set of eigenpairs is the intersection of n linearly equivalent divisors on a weighted projective space. Since these divisors are ample, each connected component of the set of eigenpairs contributes at least one to the intersection number. Therefore, the number of connected components of eigenpairs can be no more than $((m-1)^n - 1)/(m-2)$. We conclude that the number of normalized eigenvalues of A, if finite, must be bounded above by that quantity as well. Finally, Example 2.2 in [7] shows that the bound is tight.
We now prove that the number of normalized eigenvalues of a symmetric tensor A is finite. Let S be the affine hypersurface in C^n defined by the equation $x_1^2 + \ldots + x_n^2 = 1$. We claim that a point $x \in S$ is an eigenvector of A if and only if x is a critical point of f restricted to S, in which case, the corresponding eigenvalue λ equals $\frac{1}{m}f(x)$. By definition, a point $x \in S$ is a critical point of $f|S$ if and only if the gradient $\nabla(f|S)$ is zero at x. The latter condition is equivalent to the gradient ∇f being a multiple of $\nabla(x_1^2+\ldots+x_n^2-1) = 2x$. This is exactly the definition of an eigenvector. Finally, if $x \in S$ is a critical point of $f|S$, then $mf(x) = x \cdot \nabla f(x) = \lambda x \cdot x = \lambda$, and hence $\lambda = \frac{1}{m}f(x)$. Finally, to prove this Theorem, we note that, by generic smoothness (Corollary iI.10.7 in [19]), a polynomial function on a smooth variety has only finitely many critical values. Equivalently, Sard's Theorem in differential geometry says that the set of critical values of a differentiable function has measure zero, so, by Proposition 27, that set must be finite. □

The above Theorem is a result by D. Cartwright and B. Sturmfels in [7] (Theorem 5.5), although in [1] it has been remarked that it was already known by Sibony and Fornaess ([16]) in the setting of dynamical systems.

Lemma 34. *A vector $v \in \mathbb{R}^n$ is a real eigenvector of $f \in \mathrm{Sym}^d(\mathbb{R}^n)$ if and only if v is a critical point of $f|_{S^{n-1}}$, where $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.*

Proof. By Remark 23, finding (real) eigenvectors of f is equivalent to finding (real) fixed points of the projective application ∇f, or also to solving the system

$$Sys_1 = \begin{cases} f_{x_1}(x_1, x_2, \ldots, x_n) - \lambda x_1 = 0 \\ f_{x_2}(x_1, x_2, \ldots, x_n) - \lambda x_2 = 0 \\ \vdots \\ f_{x_n}(x_1, x_2, \ldots, x_n) - \lambda x_n = 0 \end{cases}$$

with $\lambda \in \mathbb{C}$ ($\lambda \in \mathbb{R}$).
Consider the Lagrangian map

$$L(x_1, x_2, \ldots, x_n, \lambda) = f(x_1, x_2, \ldots, x_n) - \lambda g(x_1, x_2, \ldots, x_n)$$

where $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1$. Then, the solutions of system

$$Sys_2 = \begin{cases} L_{x_1}(x_1, x_2, \ldots, x_n, \lambda) \equiv f_{x_1}(x_1, x_2, \ldots, x_n) - \lambda x_1 = 0 \\ L_{x_2}(x_1, x_2, \ldots, x_n, \lambda) \equiv f_{x_2}(x_1, x_2, \ldots, x_n) - \lambda x_2 = 0 \\ \vdots \\ L_{x_n}(x_1, x_2, \ldots, x_n, \lambda) \equiv f_{x_n}(x_1, x_2, \ldots, x_n) - \lambda x_n = 0 \\ L_{\lambda}(x_1, x_2, \ldots, x_n, \lambda) \equiv g(x_1, x_2, \ldots, x_n) = 0 \end{cases}$$

are all solutions of Sys_1. But solving Sys_2 gives critical points $v = (v_1, v_2, \ldots, v_n, \lambda_0)$ of L, that is critical points of $f|_{S^{n-1}}$ (it is the method of Lagrange multipliers), that is the solutions of the system
$Sys_3 = \nabla(f|_{S^{n-1}})(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$. □

Remark 35. The real eigenvectors of f have an important role, because they are the critical points of the Euclidean distance function of $[f]$ from the Veronese variety (the rational normal curve in the case of dimension two) X. Among them there is the point such that the function attains a minimum and then always exists at least a real eigenvector.

Our goal is study the number of real eigenvectors of f, supposing that $\{f = 0\}$ has a certain number of real connected components.

2.2 Binary forms

Let $f \in Sym^d(\mathbb{R}^2)$ be a binary form, that is a homogeneous polynomial of degree d in two variables x, y. In this case, the question of the number of real eigenvectors of f in relation with the number of real connected components of $\{f = 0\}$ simply means that we must compare the real roots of f with the real roots of the discriminant $yf_x - xf_y$ (also known as critical real roots of f) of the matrix

$$\begin{pmatrix} f_x(x,y) & f_y(x,y) \\ x & y \end{pmatrix}.$$

Remark 36. Consider the linear operator

$$D : Sym^d(\mathbb{R}^2) \longrightarrow Sym^d(\mathbb{R}^2),\ D(f) = xf_y - yf_x$$

such that:

- $D(fg) = D(f)g + fD(g)$, $\forall\, f, g \in Sym^d(\mathbb{R}^2)$ (Product rule or Leibniz's rule),
- $D(gf) = gD(f)$, $\forall\, g \in SO(2)$, $\forall\, f \in Sym^d(\mathbb{R}^2)$ ($SO(2)$-invariance), where

$$SO(2, \mathbb{R}) \equiv SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ |\, \theta \in [0, 2\pi) \right\}.$$

Remark 37. Let $f \in Sym^d(\mathbb{R}^2)$. Then f has d roots in $\mathbb{P}(\mathbb{C}^2)$ and, in particular, the real ones are in $\mathbb{P}(\mathbb{R}^2)$. Therefore the real roots of f are lines through the origin of $\mathbb{R}^2$. For example, the polynomial $f = x(x^2 - y^2)$ has three real roots in $\mathbb{P}(\mathbb{C}^2)$, then in $\mathbb{P}(\mathbb{R}^2)$ and these roots correspond to the three lines through the origin in Figure 2.1.

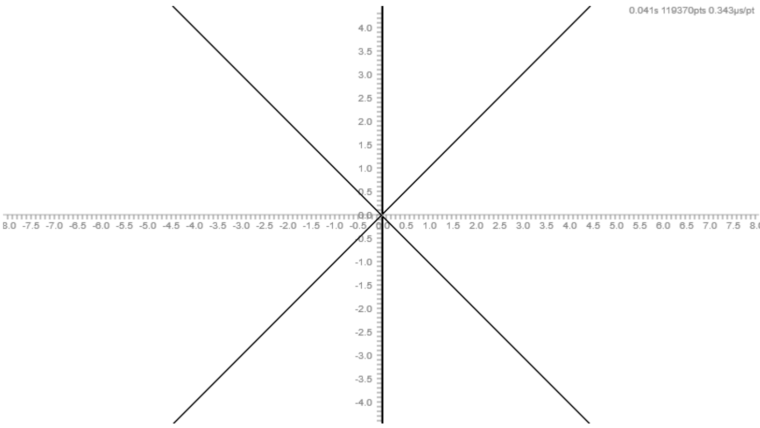

Figure 2.1: Roots of $f = x(x^2 - y^2)$.

Lemma 38. *Let $f = \left(x^2 + y^2\right)^n$, with $n \in \mathbb{N}$. Then $D(f) \equiv 0$; conversely, if $D(f) \equiv 0$, then $f = \left(x^2 + y^2\right)^n$. Furthermore, we have that $D\left(\left(x^2 + y^2\right)^n f\right) = \left(x^2 + y^2\right)^n D(f)$, $\forall\, n \in \mathbb{N}$ and $\forall\, f \in S_d\mathbb{R}^2$.*

Proof. If $f = \left(x^2 + y^2\right)^n$, then, by direct computation, $D(f) = xf_y - yf_x \equiv 0$. Conversely, consider $D(f) \equiv 0$. We have that ∇f is radial, hence $x^2 + y^2 = k$ are level lines orthogonal to the gradient, then the thesis. □

Lemma 39. *Let $d \geq 1$. Consider $P_d = \sum_{j=0}^{\lfloor d/2 \rfloor} (-1)^j \binom{d}{2j} x^{d-2j} y^{2j}$ and $Q_d = \sum_{j=0}^{\lfloor (d-1)/2 \rfloor} (-1)^j \binom{d}{2j+1} x^{d-2j-1} y^{2j+1}$. Then we have:*

- *$D(P_d) = -dQ_d$,*
- *$D(Q_d) = dP_d$,*
- *the subspace $S_d = \langle P_d, Q_d \rangle$ of $\mathrm{Sym}^d(\mathbb{R}^2)$ is D-invariant and $D^2 + d^2 I$, with I the identity, vanish on S_d.*

Proof. If $d > 1$, we get $\frac{\partial P_d}{\partial x} = dP_{d-1}$ and $\frac{\partial P_d}{\partial y} = -dQ_{d-1}$. We have two cases:
d even. Then $D(P_d) = x\frac{\partial P_d}{\partial y} - y\frac{\partial P_d}{\partial x} = -d(xQ_{d-1} + yP_{d-1}) =$
$-d\left(\sum_{j=0}^{\lfloor (d-2)/2 \rfloor} (-1)^j \binom{d-1}{2j+1} x^{d-2j-1} y^{2j+1} + \sum_{j=0}^{\lfloor (d-1)/2 \rfloor} (-1)^j \binom{d-1}{2j} x^{d-2j-1} y^{2j+1}\right) =$
$-d\sum_{j=0}^{\lfloor (d-1)/2 \rfloor} (-1)^j \left(\binom{d-1}{2j+1} + \binom{d-1}{2j}\right) x^{d-2j-1} y^{2j+1} = -dQ_d.$
d odd. Then $D(P_d) = x\frac{\partial P_d}{\partial y} - y\frac{\partial P_d}{\partial x} = -d(xQ_{d-1} + yP_{d-1}) =$
$-d\left(\sum_{j=0}^{\lfloor (d-2)/2 \rfloor} (-1)^j \binom{d-1}{2j+1} x^{d-2j-1} y^{2j+1} + \sum_{j=0}^{\lfloor (d-1)/2 \rfloor} (-1)^j \binom{d-1}{2j} x^{d-2j-1} y^{2j+1}\right) =$

$-d\left(\sum_{j=0}^{\lfloor (d-2)/2\rfloor}(-1)^j\left(\binom{d-1}{2j+1}+\binom{d-1}{2j}\right)x^{d-2j-1}y^{2j+1}+\binom{d-1}{d-1}y^d\right)=-dQ_d$.
Moreover, we get $\frac{\partial Q_d}{\partial x}=dQ_{d-1}$ and $\frac{\partial Q_d}{\partial y}=dP_{d-1}$. Then we have also $D(Q_d)=dP_d$ by the abovementioned computation reasons.
If $d=1$, we have $P_d=x$ and $Q_d=y$. Then $D(P_d)=x\frac{\partial P_d}{\partial y}-y\frac{\partial P_d}{\partial x}=-y=-Q_d$ and $D(Q_d)=x=P_d$. Hence we have the D-invariance of S_d. Finally, we have $(D^2+d^2I)(S_d)=D^2(S_d)+d^2I(S_d)=D(D(S_d))+d^2S_d=D(\langle -dQ_d,dP_d\rangle)+d^2S_d=-d^2\langle P_d,Q_d\rangle+d^2S_d=0$. □

Remark 40. We can extend the previous construction for $d=0$. In fact we can say that S_0 is generated by the constant polynomial 1 and then S_0 is $\mathbb{R}$. Furthermore, in $\mathbb{P}\left(Sym^d(\mathbb{C}^2)\right)$ the line $S_d=\langle P_d,Q_d\rangle$ is secant to the rational normal curve at the two points $(x\pm\sqrt{-1}y)^d$ and then we can write $S_d=\langle (x+\sqrt{-1}y)^d,(x-\sqrt{-1}y)^d\rangle$, $\forall\, d\geq 0$.

Proposition 41. *Every nonzero polynomial in the subspace S_d has d real distinct roots.*

Proof. We get $\frac{\partial P_d}{\partial x}=dP_{d-1}$, $\frac{\partial P_d}{\partial y}=-dQ_{d-1}$ and in general $\frac{d}{dt}P_d(\alpha+\beta t,\gamma+\delta t)=d(\beta P_{d-1}-\delta Q_{d-1})$. Moreover $\frac{\partial Q_d}{\partial x}=dQ_{d-1}$, $\frac{\partial Q_d}{\partial y}=dP_{d-1}$ and in general $\frac{d}{dt}Q_d(\alpha+\beta t,\gamma+\delta t)=d(\beta Q_{d-1}+\delta P_{d-1})$. The thesis follows now by induction on d from Theorem 1 in [8]. □

Proposition 42. *Let $f\in\mathrm{Sym}^d(\mathbb{R}^2)$, with $d\in\mathbb{N}$. Consider D the linear operator such that $D(f)=xf_y-yf_x$. Then* $\ker\left(D^2+(d-2j)^2i\right)=(x^2+y^2)^jS_{d-2j}$, $\forall\, j:0,\ldots,\left\lfloor\frac{d}{2}\right\rfloor$ *and each polynomial belonging to these kernels has exactly $d-2j$ real distinct roots. Moreover, we have the following decomposition of* $\mathrm{Sym}^d(\mathbb{R}^2)$*:*

$$\mathrm{Sym}^d(\mathbb{R}^2)=\oplus_{j=0}^{\lfloor d/2\rfloor}\ker\left(D^2+(d-2j)^2i\right)=\oplus_{j=0}^{\lfloor d/2\rfloor}(x^2+y^2)^jS_{d-2j}.$$

Proof. By Lemma 38 and by Lemma 39, we have that $(D^2+(d-2j)^2i)(S_{d-2j})=0$ and $D((x^2+y^2)^nf)=(x^2+y^2)^nD(f)$. Then $(x^2+y^2)^jS_{d-2j}\subseteq\ker(D^2+(d-2j)^2i)$. Moreover, for dimension reasons, we have that $\oplus_{j=0}^{\lfloor d/2\rfloor}(x^2+y^2)^jS_{d-2j}=Sym^d\mathbb{R}^2$ and then $(x^2+y^2)^jS_{d-2j}\supseteq\ker(D^2+(d-2j)^2i)$. □

Corollary 43. *The complex eigenvalues of D are $\lambda=\pm\sqrt{-1}j$, for $j:d,d-2,\ldots$. All of them are simple. Moreover, 0 is an eigenvalue of D if and only if d is even.*

Corollary 44. *Let D be the linear operator such that $D(f)=xf_y-yf_x$, with $f\in$* $\mathrm{Sym}^d(\mathbb{R}^2)$*. Then* $\mathrm{rk}(D)=d+1$*, if d is odd, while* $\mathrm{rk}(D)=d$*, if d is even, with one dimensional kernel. In particular D is invertible if and only if d is odd.*

Remark 45. The decomposition in Proposition 42 is orthogonal with respect to the scalar product

$$\left(\sum_{k=0}^{d}\binom{d}{k}a_kx^{d-k}y^k,\sum_{j=0}^{d}\binom{d}{j}b_jx^{d-j}y^j\right)=\sum_{k=0}^{d}\binom{d}{k}a_kb_k=$$

$$\left(\sum_{k=0}^{d}\binom{d}{k}a_k\partial_x^{d-k}\partial_y^k\right)\sum_{j=0}^{d}\binom{d}{j}b_jx^{d-j}y^j$$

and the scalar product is $SO(2)$-invariant. Finally, we have that D is antisymmetric with respect to this scalar product.

In [1], Theorem 2.7 gives, in a language different from ours, some information that we also found in this thesis, about the linear operator $D(f) = xf_y - yf_x$. In particular, Theorem 2.7 says that D is an isomorphism if d is odd and D has a one-dimensional kernel if d is even. The Theorem is the following:

Theorem 46. *[1] A set of d points $(u_i : v_i) \in \mathbb{P}^1$ is the eigenconfiguration of a symmetric tensor if and only if either d is odd, or d is even and the operator $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{d/2}$ annihilates the corresponding binary form $\prod_{i=1}^{d}(v_ix - u_iy)$.*

Now we conjecture that the number of the real roots of a binary form is less than or equal to the number of its real critical roots. In the following two Remarks 47 and 48, we consider some different approaches to prove this conjecture, but the result is effectively shown in Theorem 49.

Remark 47. Let $f \in Sym^d(\mathbb{R}^2)$, with $d \in [1, 4]$. We wonder if the number of the real roots of f is less than or equal to the number of the real critical roots of f. On the other hand, we wonder if this statement is true for $d \in \mathbb{N}$. The answer to the first question is positive, because we have the following:

1. $d = 1$. In this case, it is trivial.

2. $d = 2$. In this case, let $f = ax^2 + 2bxy + cy^2$ be. Then we have $D(f) \equiv g = x(2bx + 2cy) - y(2ax + 2by) = 2(bx^2 + (c - a)xy - by^2$. The discriminant of g is $\Delta(g) = 4(c-a)^2 + 4b^2$, which is a sum of two squares. Therefore it is always grater or equal than zero and the thesis trivially follows.

3. $d = 3$. In this case, let $f = x^3 + 3bx^2y + 3cxy^2 + dy^3$ be. By the action of $SO(2)$, we can rewrite f as $f = x^3 + 3cxy^2 + dy^3$. Then we have the discriminant $\Delta(f) = -4c^3-d^2$ and $g = x(6cxy+3dy^2)-y(3x^2+3cy^2) = 3y((2c-1)x^2+dxy-cy^2) = 3yg_1$. Evidently, the cubic g has at least a real root, because it is the product of a linear factor, $3y$, and a quadric, g_1. Then if f has only a real root, i.e. $\Delta(f) < 0 \iff d^2 > -4c^3$, we have the thesis. Moreover, if $\Delta(f) \geq 0 \iff d^2 \leq -4c^3$, hence necessarily $c \leq 0$. The discriminant of g_1 is $d^2 + 8c^2 - 4c$ which is always grater or equal than zero for $c \leq 0$ and we have the thesis.

4. $d = 4$. In this case, let $f = x^4+bx^3y+cx^2y^2+dxy^3+ey^4$ be. By the action of $SO(2)$, we can rewrite f as $f = x^4 + cx^2y^2 + dxy^3 + ey^4$. Then we have the discriminant $\Delta(f) = 16c^4e - 4c^3d^2 - 128c^2e^2 + 144cd^2e - 27d^4 + 256e^3$. The companion matrix

and the Bezoutiant of f are

$$M = \begin{pmatrix} 0 & 0 & 0 & -e \\ 1 & 0 & 0 & -d \\ 0 & 1 & 0 & -c \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{pmatrix}$$

where $s_0 = 4$, $s_1 = 0$, $s_2 = -2c$, $s_3 = -3d$, $s_4 = 2\left(c^2 - 2e\right)$, $s_5 = 5cd$ e $s_6 = -\left(2c^3 - 6ce - 3d^2\right)$. Then the principal minors of B are, up to scalar factor, the discriminants of f and the following polynomial in b, c, d of degree three and one

$$d_1 = 4\left(-2c^3 + 8ce - 9d^2\right), \; d_2 = -8c.$$

By Jacobi's criterion, B is positive definite if and only if the principal minors of B are all positive. Moreover, we know that f has four real (distinct) roots if and only if B is semidefinite (definite) positive. Consider now $g = xf_y - yf_x = x(2cx^2y + 3dxy^2 + 4ey^3) - y(4x^3 + 2cxy^2 + dy^3) = y(2(c-2)x^3 + 3dx^2y + 2(2e-c)xy^2 - dy^3) = yg_1$. Evidently, the quartic g has at least two real roots, because it is the product of a linear factor, y, and a cubic, g_1. Then if f has zero or two real roots, we have the thesis. Now, being $d_2 = -8c$, if $c > 0$ f has not four real roots and then we must investigate only for $c \leq 0$. Hence let $c \leq 0$ be. The discriminant of g_1 is, up to scalar factor, $\Delta(g_1) = 16c^4 - 96c^3e - 32c^3 + 36c^2d^2 + 192c^2e^2 + 192c^2e - 144cd^2e$ $-128ce^3 - 384ce^2 + 27d^4 + 36d^2e^2 + 216d^2e - 108d^2 + 256e^3$. The sets of solutions of the inequality $\Delta(g_1) < 0$ there are

$$\left\{d = 0,\, c < 2,\, e < \frac{c}{2}\right\}, \left\{d = 0,\, c > 2,\, e > \frac{c}{2}\right\}$$

that is g has exactly two real roots in these two sets. Adding the condition $c \leq 0$, we have the following set of solutions

$$S = \left\{d = 0,\, c \leq 0,\, e < \frac{c}{2}\right\}$$

where g has again two real roots. Computing the signs of $\Delta(f)$, d_1 and d_2, we have trivially that $\Delta(f)$ is negative, then f has zero or two real roots in S. Finally, we observe that g has four real roots in the complement of S under the condition $c \leq 0$ and hence we have the thesis.

As it regards the answer to the second question, the point is more complicated. In fact, already working for $d = 5$, it is not possible to follow the proof method used in the previous cases, because there are too many parameters, four. Then we try to use the decomposition of $Sym^d(\mathbb{R}^2)$ as in Proposition 42, at least for the degree 5, trying to find counterexamples or trying to look for polynomials verifying the thesis. First of all, we observe that, by Proposition 42, if f of degree $d > 4$ belongs to a D-invariant addend of the direct sum decomposition of $Sym^d(\mathbb{R}^2)$, then we have the thesis. Moreover, by Lemma 38, if f of degree $d > 4$, with at least two complex roots, is of the form $f = (x^2 + y^2)h$,

$\deg h = d - 2$, then we have again the thesis, by induction. Now, for $d = 5$ we have: Let f be a quintic. Then, we have

$$Sym^5(\mathbb{R}^2) = S_5 \oplus (x^2 + y^2)S_3 \oplus (x^2 + y^2)^2 S_1 =$$

$$\langle P_5, Q_5 \rangle \oplus (x^2 + y^2)\langle P_3, Q_3 \rangle \oplus (x^2 + y^2)^2 \langle P_1, Q_1 \rangle$$

where $P_5 = x^5 - 10x^3y^2 + 5xy^4$, $Q_5 = 5x^4y - 10x^2y^3 + y^5$, $P_3 = x^3 - 3xy^2$, $Q_3 = 3x^2y - y^3$, $P_1 = x$ and $Q_1 = y$. We want investigate in the case that f belongs to the sum of an any pair of the three addends of the decomposition of $Sym^5(\mathbb{R}^2)$.

1. $f \in (x^2 + y^2)S_3 \oplus (x^2 + y^2)^2 S_1$. Then we can write f as $f = (x^2 + y^2)h$, with $\deg h = 3$, hence the thesis.

2. $f \in S_5 \oplus (x^2 + y^2)S_3$. Then we can rewrite f, by the action of $SO(2)$, in one of the following two forms

$$(P_5 + aQ_5) + (x^2 + y^2)bQ_3,\ aQ_5 + (x^2 + y^2)\,(P_3 + bQ_3)$$

with $a, b \in \mathbb{R}$.
In the first case, we have $f = x^5 + (5a + 3b)x^4y - 10x^3y^2 + 2(-5a + b)x^2y^3 + 5xy^4 + (a - b)y^5$, then $g = (5a + 3b)x^5 - 25x^4y - 2(25a + 3b)x^3y^2 + 50x^2y^3 + (25a - 9b)xy^4 - 5y^5$. The discriminants of f and g are respectively the following two polynomials in a, b of degree 8 $\Delta(f) = 4096(3125a^8 - 2500a^6b^2 + 12500a^6 - 50a^4b^4 - 7500a^4b^2 + 18750a^4 - 512a^3b^5 - 36a^2b^6 - 100a^2b^4 - 7500a^2b^2 + 12500a^2 + 1536ab^5 - 27b^8 - 36b^6 - 50b^4 - 2500b^2 + 3125)$ e $\Delta(g) = 4096(1220703125a^8 - 351562500a^6b^2 + 4882812500a^6 - 2531250a^4b^4 - 1054687500a^4b^2 + 7324218750a^4 + 15552000a^3b^5 - 656100a^2b^6 - 5062500a^2b^4 - 1054687500a^2b^2 + 4882812500a^2 - 46656000ab^5 - 177147b^8 - 656100b^6 - 2531250b^4 - 351562500b^2 + 1220703125)$. As in Figure 2.3, the graphic of $\Delta(g)$ divides the upper half-plane (a,b) in two connected components, in each of which the polynomial g has the same number of real roots. Then, we can take two pairs of values (a,b) in the two connected components, for example $a = 0$, $b = 2$ and $a = 0$, $b = 0$. Computing g in these two pairs of values, we have the quintics

$$6x^5 - 25x^4y - 12x^3y^2 + 50x^2y^3 - 18xy^4 - 5y^5$$

and

$$-25x^4y + 50x^2y^3 - 5y^5$$

which have respectively 3 and 5 real roots and then we have the thesis on the connected component outside of the graphic of $\Delta(g)$. Moreover, as in Figure 2.2, $\Delta(f)$ again divides the upper half-plane (a,b) in two connected components. The region in which $\Delta(g)$ is negative is strictly contained in the connected components in which $\Delta(f)$ is negative. Hence, computing also f in the pair of values $a = 0$, $b = 2$, we obtain the quintic

$$x^5 + 6x^4y - 10x^3y^2 + 4x^2y^3 + 5xy^4 - 2y^5$$

with 3 real roots and then we have the thesis. Finally, for symmetric reasons we have the same results in the lower half-plane (a,b).
In the second case, we have $f = x^5+(5a+3b)x^4y-2x^3y^2+2(-5a+b)x^2y^3-3xy^4+(a-b)y^5$, then $g = (5a+3b)x^5-9x^4y+2(-25a-3b)x^3y^2-6x^2y^3+(25a-9b)xy^4+3y^5$. The discriminants of f and g are respectively the following two polynomials in a, b of degree 8 $\Delta(f) = 4096(3125a^8 - 2500a^6b^2 - 2500a^6 - 50a^4b^4 - 100a^4b^2 - 50a^4 - 512a^3b^5 + 5120a^3b^3 - 2560a^3b - 36a^2b^6 - 108a^2b^4 - 108a^2b^2 - 36a^2 - 27b^8 - 108b^6 - 162b^4 - 108b^2 - 27)$ e $\Delta(g) = 4096(1220703125a^8 - 351562500a^6b^2 - 351562500a^6 - 2531250a^4b^4 - 5062500a^4b^2 - 2531250a^4 + 15552000a^3b^5 - 155520000a^3b^3 + 77760000a^3b - 656100a^2b^6 - 1968300a^2b^4 -$
$1968300a^2b^2 - 656100a^2 - 177147b^8 - 708588b^6 - 1062882b^4 - 708588b^2 - 177147)$.
As in Figures 2.4 and 2.5, we note that, for example in the right half-plane (a,b), all the arguments of the previous case are valid and again for symmetric reasons we have the same results in the left half-plane. Therefore, we can take the two pairs of values $a = 1$, $b = 0$ and $a = 0$, $b = 0$. Then, computing g in the first pairs of values, we have the quintic

$$5x^5 - 9x^4y - 50x^3y^2 - 6x^2y^3 + 25xy^4 + 3y^5$$

with 5 real roots and the thesis, while in the second pairs of values f and g are respectively the quintics

$$x^5 - 2x^3y^2 - 3xy^4$$

$$-9x^4y - 6x^2y^3 + 3y^5$$

both with 3 real roots and we have again the thesis.

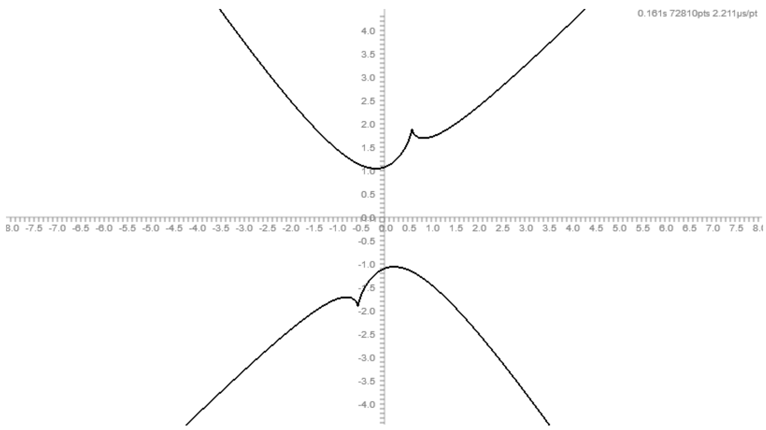

Figure 2.2: Discriminant of $f = x^5 + (5a + 3b)x^4y - 10x^3y^2 + 2(-5a + b)x^2y^3 + 5xy^4 + (a - b)y^5$.

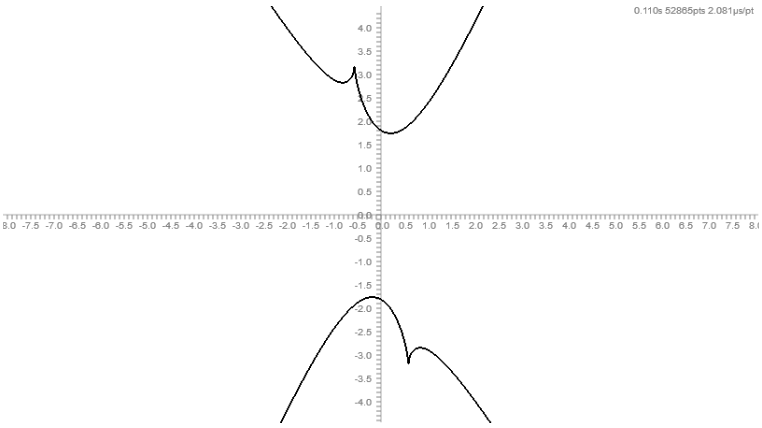

Figure 2.3: Discriminant of $g = (5a+3b)x^5 - 25x^4y - 2(25a+3b)x^3y^2 + 50x^2y^3 + (25a-9b)xy^4 - 5y^5$.

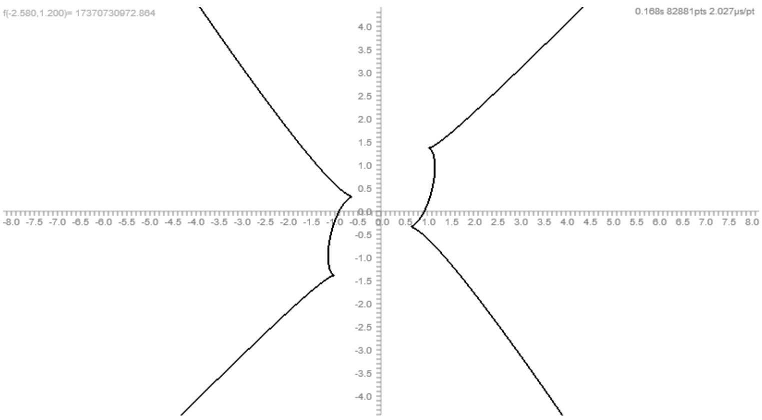

Figure 2.4: Discriminant of $f = x^5+(5a+3b)x^4y-2x^3y^2+2(-5a+b)x^2y^3-3xy^4+(a-b)y^5$.

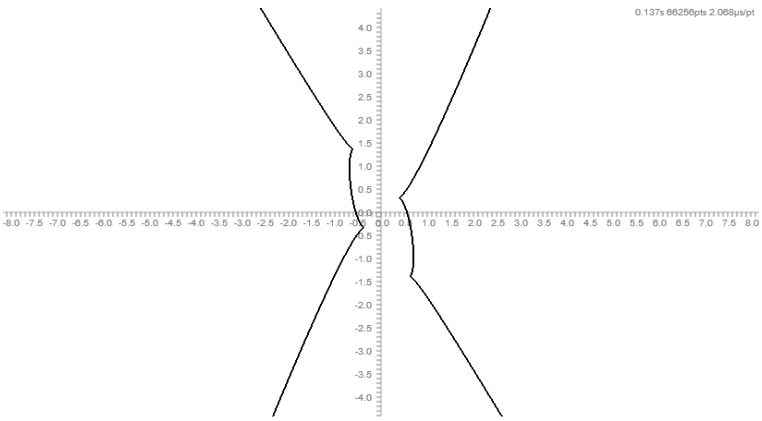

Figure 2.5: Discriminant of $g = (5a+3b)x^5 - 9x^4y + 2(-25a-3b)x^3y^2 - 6x^2y^3 + (25a-9b)xy^4 + 3y^5$.

3. $f \in S_5 \oplus (x^2+y^2)^2 S_1$. Then we can rewrite f, by the action of $SO(2)$, in one of the following two forms

$$(P_5 + aQ_5) + (x^2+y^2)^2 bQ_1,\ aQ_5 + (x^2+y^2)^2\,(P_1 + bQ_1)$$

with $a, b \in \mathbb{R}$.
In the first case, we have $f = x^5 + (5a+b)x^4y - 10x^3y^2 + 2(-5a+b)x^2y^3 + 5xy^4 + (a+b)y^5$, then $g = (5a+b)x^5 - 25x^4y + 2(-25a+b)x^3y^2 + 50x^2y^3 + (25a+b)xy^4 - 5y^5$. The discriminants of f and g are respectively the following two polynomials in a, b of degree 8 $\Delta(f) = 4096(3125a^8 - 3750a^6b^2 + 12500a^6 + 825a^4b^4 - 11250a^4b^2 + 18750a^4 + 216a^3b^5 + 16a^2b^6 + 1650a^2b^4 - 11250a^2b^2 + 12500a^2 + 216ab^5 + 16b^6 + 825b^4 - 3750b^2 + 3125)$ e $\Delta(g) = 102400(48828125a^8 - 2343750a^6b^2 + 195312500a^6 + 20625a^4b^4 - 7031250a^4b^2 + 292968750a^4 + 1080a^3b^5 + 16a^2b^6 + 41250a^2b^4 - 7031250a^2b^2 + 195312500a^2 + 1080ab^5 + 16b^6 + 20625b^4 - 2343750b^2 + 48828125)$. As in Figure 2.7, the graphic of $\Delta(g)$ divides the upper half-plane (a,b) in three connected components, in each of which the polynomial g has the same number of real roots. Then, we can take three pairs of values (a,b) in the three connected components, for example (0,10), (0,6) and (0,0). Computing g in these three pairs of values, we have the quintics

$$10x^5 - 25x^4y + 20x^3y^2 + 50x^2y^3 + 10xy^4 - 5y^5$$

$$6x^5 - 25x^4y + 12x^3y^2 + 50x^2y^3 + 6xy^4 - 5y^5$$

$$-25x^4y + 50x^2y^3 - 5y^5$$

which have respectively 1, 3 and 5 real roots and then we have the thesis in particular on the connected component outside of the graphic of $\Delta(g)$. Moreover, as in Figure 2.6, $\Delta(f)$ again divides the upper half-plane (a,b) in three connected components. The innermost of these contains strictly the regions in which g has

1 or 3 real roots. Hence computing f in a pair of values (a,b) in this region, for example (0,3), we obtain the quintic

$$x^5 + 3x^4y - 10x^3y^2 + 6x^2y^3 + 5xy^4 + 3y^5$$

with one real root and the thesis. Finally, for symmetric reasons we have the same results in the lower half-plane (a,b).
In the second case, we have $f = x^5+(5a+b)x^4y+2x^3y^2+2(-5a+b)x^2y^3+xy^4+(a+b)y^5$, then $g = (5a+b)x^5 - x^4y + 2(-25a+b)x^3y^2 - 2x^2y^3 + (25a+b)xy^4 - y^5$. The discriminants of f and g are respectively the following two polynomials in a, b of degree 8 $\Delta(f) = 4096a^2(3125a^6 - 3750a^4b^2 - 3750a^4 + 825a^2b^4 + 1650a^2b^2 + 825a^2 + 216ab^5 - 2160ab^3 + 1080ab + 16b^6 + 48b^4 + 48b^2 + 16)$ e $\Delta(g) = 102400a^2(48828125a^6 - 2343750a^4b^2 - 2343750a^4 + 20625a^2b^4 + 41250a^2b^2 + 20625a^2 + 1080ab^5 - 10800ab^3 + 5400ab + 16b^6 + 48b^4 + 48b^2 + 16)$. As in Figure 2.9, the graphic of $\Delta(g)$ divides the right half-plane (a,b) in five connected components, in each of which the polynomial g has the same number of real roots. Then, we can take five pairs of values (a,b) in the five connected components, for example $(\frac{1}{5},\frac{3}{5})$, $(\frac{1}{10},-\frac{1}{5})$, $(\frac{3}{10},-\frac{17}{10})$, (1,0) e (0,0). Computing g in these five pairs of values, we have the quintics

$$\frac{8x^5 - 5x^4y - 44x^3y^2 - 10x^2y^3 + 28xy^4 - 5y^5}{5}$$

$$\frac{3x^5 - 10x^4y - 54x^3y^2 - 20x^2y^3 + 23xy^4 - 10y^5}{10}$$

$$\frac{-x^5 - 5x^4y - 92x^3y^2 - 10x^2y^3 + 29xy^4 - 5y^5}{5}$$

$$5x^5 - x^4y - 50x^3y^2 - 2x^2y^3 + 25xy^4 - y^5$$

$$-y(x^4 + 2x^2y^2 + y^4)$$

which have respectively 3, 3, 3, 5 and 1 real roots and then we have the thesis in particular on the connected component outside at the right of the graphic of $\Delta(g)$. Moreover, as in Figure 2.8, $\Delta(f)$ again divides the right half-plane (a,b) in five connected components. The left-most of these contains strictly the first three regions of the graphic of $\Delta(g)$. Hence computing f in a pair of values (a,b) in this region, for example in (0,0), we obtain the quintic

$$x(x^4 + 2x^2y^2 + y^4)$$

with one real root. Moreover, the remaining connected components of the graphic of $\Delta(f)$ are strictly contained in the region in which g has five real roots and we have the thesis. Finally, for symmetric reasons we have the same results in the left half-plane (a,b).

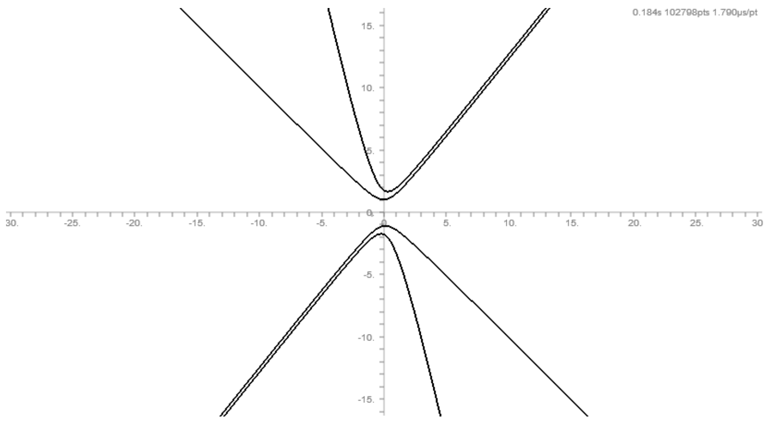

Figure 2.6: Discriminant of $f = x^5 + (5a+b)x^4y - 10x^3y^2 + 2(-5a+b)x^2y^3 + 5xy^4 + (a+b)y^5$.

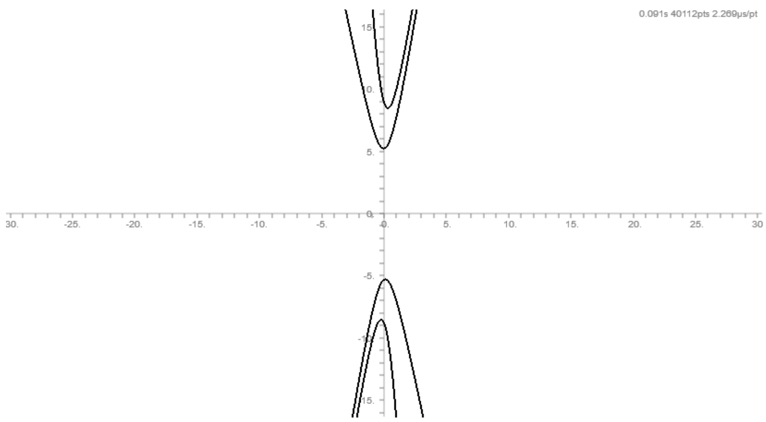

Figure 2.7: Discriminant of $g = (5a+b)x^5 - 25x^4y + 2(-25a+b)x^3y^2 + 50x^2y^3 + (25a+b)xy^4 - 5y^5$.

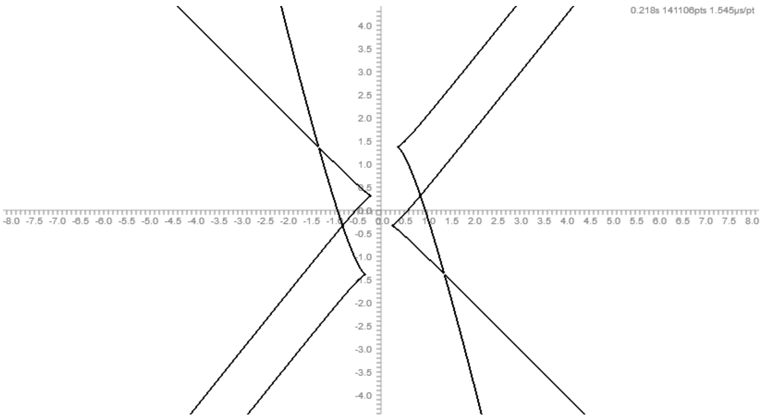

Figure 2.8: Discriminant of $f = x^5+(5a+b)x^4y+2x^3y^2+2(-5a+b)x^2y^3+xy^4+(a+b)y^5$.

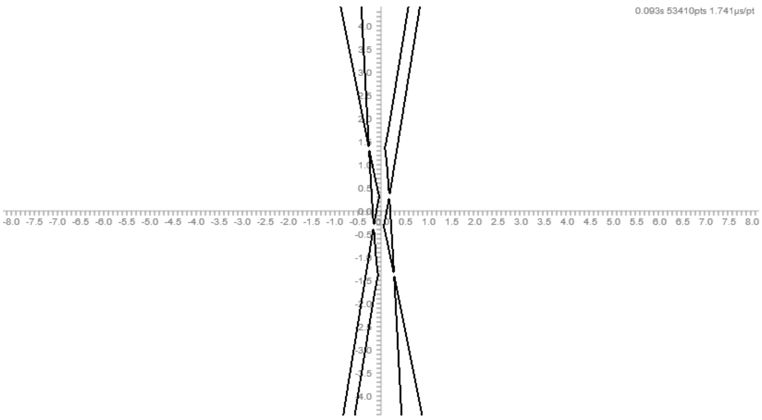

Figure 2.9: Discriminant of $g = (5a + b)x^5 - x^4y + 2(-25a + b)x^3y^2 - 2x^2y^3 + (25a + b)xy^4 - y^5$.

In Remark 47 we show that our conjecture is true if f belongs to one of the subspaces S_5, $(x^2 + y^2)S_3$, $(x^2 + y^2)^2S_1$ or if f belongs to the direct sum of any two of these subspaces. But we have not the answer if $f \in Sym^5(\mathbb{R}^2) = S_5 \oplus (x^2+y^2)S_3 \oplus (x^2+y^2)^2S_1$. Moreover, already for $d = 6$, the same investigation is not possible because there are many computing problems.

Remark 48. Let f be a cubic. By Remark 40 and by Proposition 42, we can rewrite f as $f = a(x+\sqrt{-1}y)^3+\bar{a}(x-\sqrt{-1}y)^3+b(x+\sqrt{-1}y)(x^2+y^2)+\bar{b}(x-\sqrt{-1}y)(x^2+y^2) = 2((h+l)x^3 - (m+3z)x^2y + (l-3h)xy^2 + (z-m)y^3)$, with $a, b \in \mathbb{C}$, $a = h + \sqrt{-1}z$, $b = l+\sqrt{-1}m$. The discriminant of f is $\Delta(f) = \frac{64}{27}(27h^4 - 18h^2l^2 - 18h^2m^2 + 54h^2z^2 + 8hl^3 - 24hlm^2 - l^4 - 2l^2m^2 + 24l^2mz - 18l^2z^2 - m^4 - 8m^3z - 18m^2z^2 + 27z^4)$ that we can rewrite as $64(h^2+z^2)^2 - \frac{64}{27}(l^2+m^2)^2 - \frac{128}{3}((l^2+m^2)(h^2+z^2)) + \frac{512}{27}(hl(l^2 - 3m^2) + zm(3l^2 - m^2)) = 64\,|a|^4 - \frac{64}{27}\,|b|^4 - \frac{128}{3}\,|a|^2\,|b|^2 + \frac{512}{27}Re(b^3\bar{a})$. Now consider

$D(f) \equiv g = 2(-(m+3z)x^3 - (l+9h)x^2y + (9z-m)xy^2 + (3h-l)y^3)$. The discriminant of g is $\Delta(g) = \frac{64}{27}(2187h^4 - 162h^2l^2 - 162h^2m^2 + 4374h^2z^2 - 24hl^3 + 72hlm^2 - l^4 - 2l^2m^2 - 72l^2mz - 162l^2z^2 - m^4 + 24m^3z - 162m^2z^2 + 2187z^4)$ that we can rewrite as $5184(h^2+z^2)^2 - \frac{64}{27}(l^2+m^2)^2 - 384((l^2+m^2)(h^2+z^2)) - \frac{512}{9}(hl(l^2-3m^2)+zm(3l^2-m^2)) = 5184\,|a|^4 - \frac{64}{27}\,|b|^4 - 384\,|a|^2\,|b|^2 - \frac{512}{9}Re(b^3\bar{a})$. Then we can easily obtain $\Delta(g)$ from $\Delta(f)$ by the real affinity

$$|a|^2 \longmapsto 9\,|a|^2\,,\ |b|^2 \longmapsto |b|^2\,,\ Re(b^3\bar{a}) \longmapsto -3Re(b^3\bar{a}).$$

In practice, we use Corollary 43 on the complex coefficients a, $\bar{a}$, b, $\bar{b}$, obtaining g $(\Delta(g))$ from f $(\Delta(f))$ by the transformation

$$a \longmapsto 3\sqrt{-1}a,\ \bar{a} \longmapsto -3\sqrt{-1}\bar{a},\ b \longmapsto \sqrt{-1}b,\ \bar{b} \longmapsto -\sqrt{-1}\bar{b},\ b^3\bar{a} \longmapsto -3b^3\bar{a}$$

where $\pm 3\sqrt{-1}$, $\pm\sqrt{-1}$ are the complex (simple) eigenvalues of D. We hope that this process is useful for prove the conjecture if $d \geq 5$, in the sense that we can try to write the discriminants of f and g as module functions of the real (imaginary) parts of the complex coefficients of $f \in Sym^d(\mathbb{R}^2)$. Then we can find the real affinity such that we obtain g from f and this affinity give us the reciprocal behavior of the discriminants $\Delta(f)$ and $\Delta(g)$, that is of the number of the roots of f and g. Now, this method gives certainly an alternative proof for the case $d = 3$, as follow: consider $|a|^2 = x$, $|b|^2 = y$ and $Re(b^3\bar{a}) = \pm t^2$. Depending on the sign of $Re(b^3\bar{a})$, we have two cases:

1. $Re(b^3\bar{a}) > 0$ (i.e. $Re(b^3\bar{a}) = t^2$). The we have $\Delta(f) = 64x^2 - \frac{64}{27}y^2 - \frac{128}{3}xy + \frac{512}{27}t^2$ and $\Delta(g) = 5184x^2 - \frac{64}{27}y^2 - 384xy - \frac{512}{9}t^2$. Hence, by the change of variables $x' = \frac{x}{t}$, $y' = \frac{y}{t}$ and renaming, we obtain the curves

$$64x^2 - \frac{64}{27}y^2 - \frac{128}{3}xy + \frac{512}{27}$$

$$5184x^2 - \frac{64}{27}y^2 - 384xy - \frac{512}{9}$$

which graphic are in Figures 2.11, 2.12. Then we have the thesis, remembering to work under the condition $xy^3 - 1 \geq 0$ (Figure 2.10), because we have that $t^2 = Re(b^3\bar{a}) \leq \left|b^3\bar{a}\right| \Rightarrow t^4 \leq \left|b^3\bar{a}\right|^2 = |b|^6\,|a|^2 = y^3x \Rightarrow 1 \leq \frac{y^3}{t^3}\frac{x}{t}$, that is, renaming, $1 \leq xy^3$.

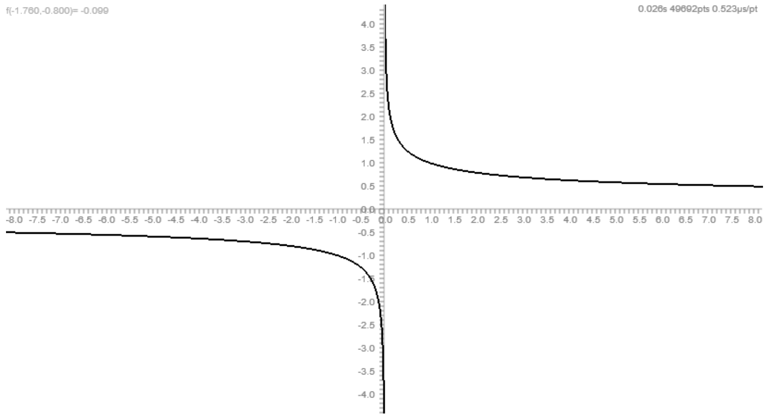

Figure 2.10: $xy^3 = 0$.

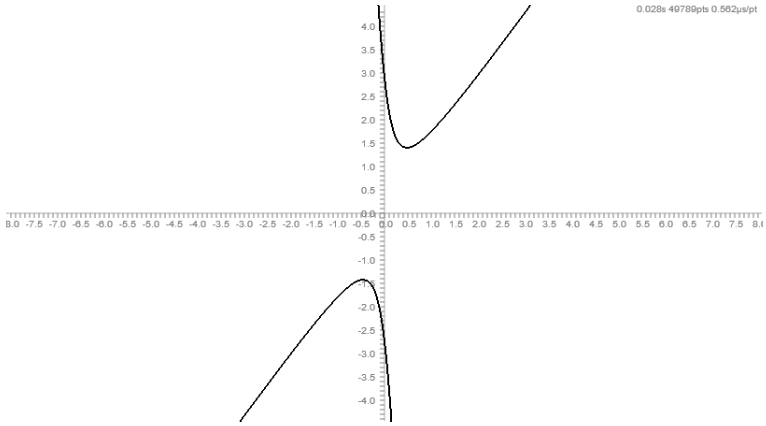

Figure 2.11: Discriminant of $f = 2((h+l)x^3 - (m+3z)x^2y + (l-3h)xy^2 + (z-m)y^3)$ if $Re(b^3\bar{a}) > 0$, with $a = h + \sqrt{-1}z$ and $b = l + \sqrt{-1}m$.

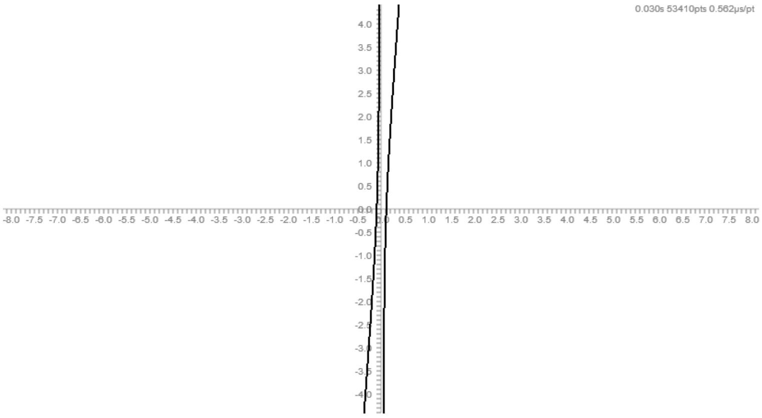

Figure 2.12: Discriminant of $g = 2(-(m+3z)x^3 - (l+9h)x^2y + (9z-m)xy^2 + (3h-l)y^3)$ if $Re(b^3\bar{a}) > 0$, with $a = h + \sqrt{-1}z$ and $b = l + \sqrt{-1}m$.

2. $Re(b^3\bar{a}) < 0$ (i.e. $-Re(b^3\bar{a}) = t^2$). Then we have $\Delta(f) = 64x^2 - \frac{64}{27}y^2 - \frac{128}{3}xy - \frac{512}{27}t^2$ and $\Delta(g) = 5184x^2 - \frac{64}{27}y^2 - 384xy + \frac{512}{9}t^2$. Hence, by the change of variables $t' = \frac{t}{y}$, $x' = \frac{x}{y}$ and renaming, we obtain the curves

$$64x^2 - \frac{64}{27} - \frac{128}{3}x - \frac{512}{27}t^2$$

$$5184x^2 - \frac{64}{27} - 384x + \frac{512}{9}t^2$$

which graphic are in Figures 2.13, 2.14. Then we have the thesis.

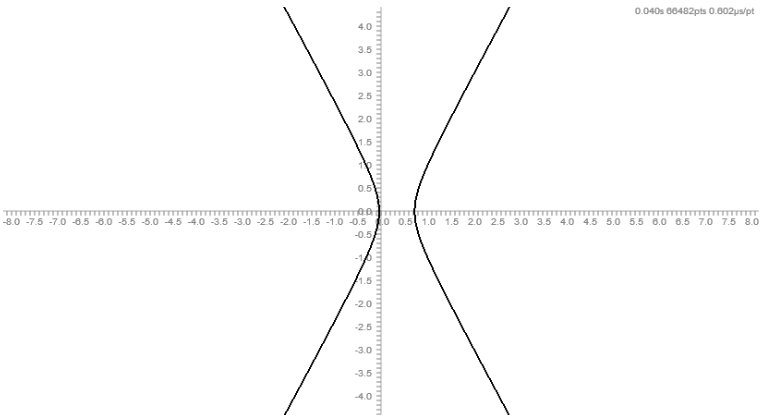

Figure 2.13: Discriminant of $f = 2((h+l)x^3 - (m+3z)x^2y + (l-3h)xy^2 + (z-m)y^3)$ if $Re(b^3\bar{a}) < 0$, with $a = h + \sqrt{-1}z$ and $b = l + \sqrt{-1}m$.

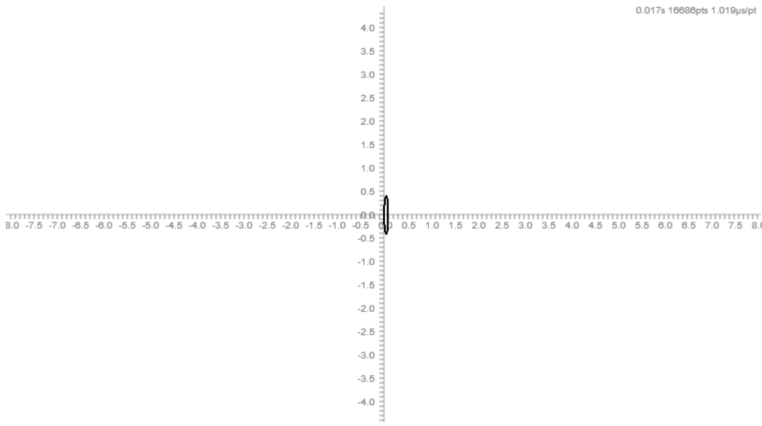

Figure 2.14: Discriminant of $g = 2(-(m+3z)x^3 - (l+9h)x^2y + (9z-m)xy^2 + (3h-l)y^3)$ if $Re(b^3\bar{a}) < 0$, with $a = h + \sqrt{-1}z$ and $b = l + \sqrt{-1}m$.

Unfortunately, if we go to the next degree $d = 4$, we can write explicitly the real affinity such that it gives g $(\Delta(g))$ from f $(\Delta(f))$, but for computational reasons we can not to repeat the proof of Remark 48. In fact, $\Delta(f)$ is too complicated in terms of the number of parameters, like modules, real or imaginary parts of the complex numbers a, b. Then we must to change approach. We have the following

Theorem 49. *Let $f \in \mathrm{Sym}^d(\mathbb{R}^2)$, with $d \in \mathbb{N}$. Then $\max(\#\text{real roots of } f, 1) \leq \#\text{real eigenvectors of } f$ and this relation is the only constraint for the number q of real roots of f, in the sense that for any pair (q,t) such that $q \equiv t \equiv d \mod 2$ and $\max(q,1) \leq t \leq d$ the set*

$$\left\{f \in \mathrm{Sym}^d(\mathbb{R}^2) \,|\, \#\text{real roots of } f = q,\ \#\text{real eigenvectors of } f = t\right\}$$

has positive volume.

Proof. Let q be the number of real roots of f. If $q = 0$, the thesis follows immediately; therefore, consider $q \geq 1$.
There are q lines through the origin of $\mathbb{R}^2$ corresponding to the q roots of f and each of these lines meets the circle $x^2 + y^2 = 1$ in two real points, that is in $2q$ total real points. Consider the following parametrization of the circle

$$S^1 : \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases},\ \theta \in [0, 2\pi)$$

and the function $F(\theta) = f(\cos\theta, \sin\theta)$, that is F is the restriction of f on S^1; evidently, the number of real roots of F is twice the number of real roots of f, or for each real root of f in $\mathbb{P}(\mathbb{R}^2)$, we have a uniquely determined pair of real roots of F. In particular, if for a given $\bar{\theta}$ we have $F(\bar{\theta}) = 0$, then $F(\bar{\theta}+\pi) = 0$ and the line through the points $(\cos\bar{\theta}, \sin\bar{\theta})$, $(\cos(\bar{\theta}+\pi), \sin(\bar{\theta}+\pi)) = (-\cos\theta, -\sin\theta)$ corresponds to a real root of f in $\mathbb{P}(\mathbb{R}^2)$ and

conversely. Now consider $F'(\theta) = -\sin\theta f_x(\cos\theta, \sin\theta) + \cos\theta f_y(\cos\theta, \sin\theta)$. By Rolle's Theorem, between two real roots of F there exists at least one real root of F' and then F' has at least $2q$ real roots. Consider $G(\theta) = g(\cos\theta, \sin\theta)$, where $g = -yf_x + xf_y$, that is G is the restriction of the polynomial g on S^1; then obviously $G(\theta) = F'(\theta)$, hence G has at least $2q$ real roots and therefore g has at least q real roots. We get $t \geq q$ as we wanted.
Finally, we must prove the following:

$$\forall\, n \in \mathbb{N}_0,\ \forall\, h \in \{h \in \mathbb{N}_0 \,|\, h = 2m\},\ \exists\, f \in Sym^d(\mathbb{R}^2)\ s.t.\ q = n,\ t = n + h$$

It is sufficient to consider binary forms of even degree t as Fourier polynomials

$$g(\cos\theta, \sin\theta) = \left(1 + \frac{\cos(2\theta)}{2}\right) + s(\cos(t\theta) + \sin(t\theta))$$

and binary forms of odd degree as Fourier polynomials

$$g(\cos\theta, \sin\theta) = \cos(\theta)\left(\left(1 + \frac{\cos(2\theta)}{2}\right) + s(\cos(t\theta) + \sin(t\theta))\right)$$

where $s \in \mathbb{R}$. Then we can choose s such that the corresponding Fourier polynomial g of degree t has q real roots in $[0, \pi)$ and its derivative with respect to θ has exactly t real roots in $[0, \pi)$ (see Figures 2.15, 2.16, 2.17); hence, taking $f = g(x^2 + y^2)^{\frac{d}{2} - \frac{t}{2}}$, we have a polynomial f of degree d with exactly q real roots and t real eigenvectors. $\square$

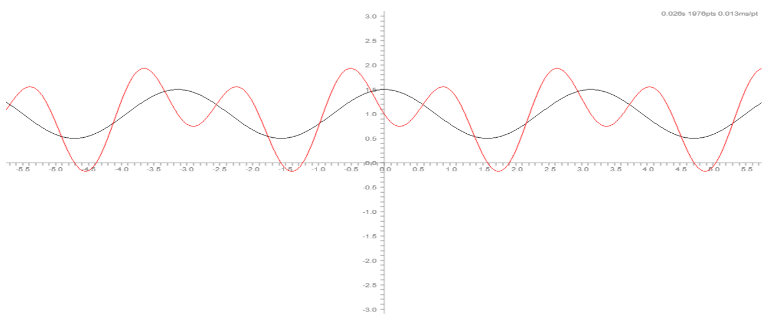

Figure 2.15: The two graphics of g respectively for $s = 0$ (the central one) and $s = -\frac{1}{2}$ (its perturbation). The second one has $q = 2$ real roots and its derivative has $t = 4$ real roots.

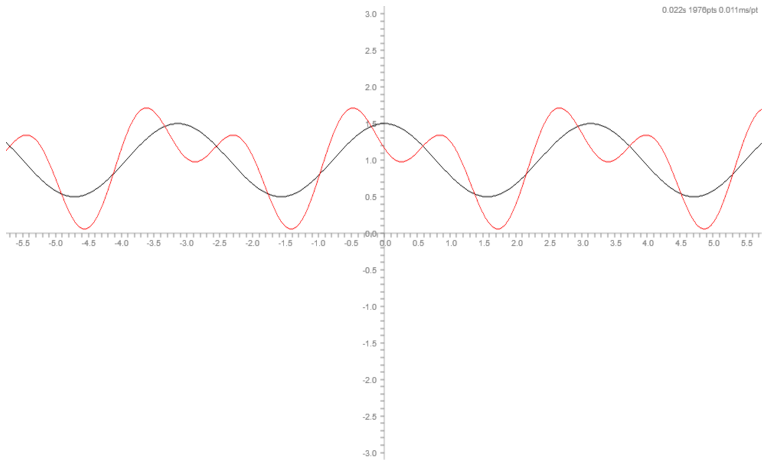

Figure 2.16: The two graphics of g respectively for $s = 0$ (the central one) and $s = -\frac{1}{3}$ (its perturbation). The second one has $q = 0$ real roots and its derivative has $t = 4$ real roots.

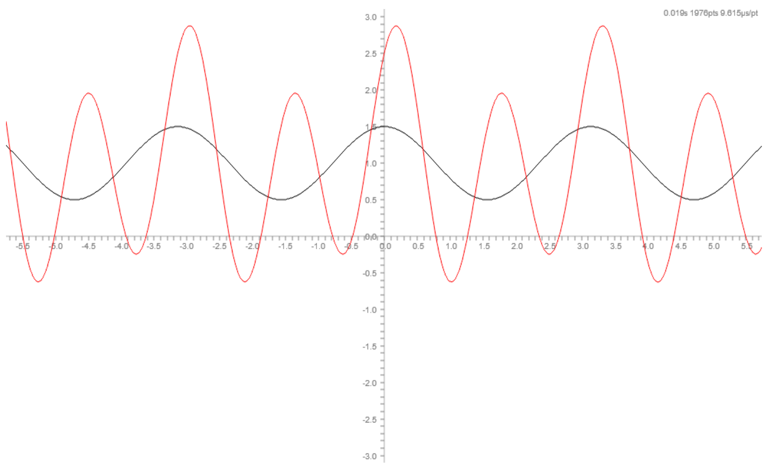

Figure 2.17: The two graphics of g respectively for $s = 0$ (the central one) and $s = 2$ (its perturbation). The second one has $q = 4$ real roots and its derivative has $t = 4$ real roots.

Corollary 50. *If f of degree d has exactly d real roots, then f has exactly d real eigenvectors.*

Corollary 50 is found also in [1] by H. Abo, A. Seigal and B. Sturmfels in Remark 6.7, as a consequence of Corollary 6.5.

Remark 51. Consider a sample of 100000 forms f of degree 4, 5, where

$$f = \sum_{i=0}^{d} \sqrt{\binom{d}{i}} a_i x^{d-i} y^i, \ a_i \approx N(0,1)$$

and $N(0,1)$ is the normal distribution of mean 0 and variance 1. Then we have estimated the probabilities of the aleatory variables $X_f = (0,2,4)$ for $d = 4$, $Y_f = (1,3,5)$ for $d = 5$ and respectively $X_{yf_x - xf_y} = (0,2,4)$, $Y_{yf_x - xf_y} = (1,3,5)$ with respect to f and $yf_x - xf_y$ and then relative expected values and we expect that $\mathbb{E}(X_f) \approx \sqrt{d}$ and $\mathbb{E}(X_{yf_x - xf_y}) \approx \sqrt{3d-2}$ and the same for $\mathbb{E}(Y_f)$ and $\mathbb{E}(Y_{yf_x - xf_y})$ (see Example 1.6 in [13] and Example 4.8 in [14]):

X_f	0	2	4
$\approx$ **probability**	0.1350	0.7307	0.1342

Table 2.1: $d = 4$.

Y_f	1	3	5
$\approx$ **probability**	0.4167	0.5491	0.0343

Table 2.2: $d = 5$.

whence $\mathbb{E}(X_f) = 1.9984 \approx \sqrt{4} = 2$ and $\mathbb{E}(Y_f) = 2.2352 \approx \sqrt{5}$.

$X_{yf_x - xf_y}$	0	2	4
$\approx$ **probability**	0	0.4190	0.5810

Table 2.3: $d = 4$.

$Y_{yf_x - xf_y}$	1	3	5
$\approx$ **probability**	0.0224	0.6569	0.3207

Table 2.4: $d = 5$.

whence $\mathbb{E}(X_{yf_x - xf_y}) = 3.1620 \approx \sqrt{10}$ and $\mathbb{E}(Y_{yf_x - xf_y}) = 3.5966 \approx \sqrt{13}$. Consider the following test: let p be expected probability such that we have quartics with two real roots. Then if we take $\mathbb{E}(X) = \sqrt{10}$ as expected value of X, we have $2*p+4*(1-p) = \sqrt{10}$, whence $p = 0.4188 \approx 0.4190$. This is very good, because there is a connection between the values up to two decimal digits. Now let p be expected probability such that we have quartics with four real roots. Then the same computation is satisfactory, because we have $p = 0.5811 \approx 0.5810$.
Again for a sample of 10000 forms f of degree 4, 5 we have estimated the probabilities for the real rank of f, i.e. the probabilities of the aleatory variables $X = (3,4)$ for $d = 4$ and $Y = (3,4,5)$ for $d = 5$ and them relative expected values:
if $d = 4$ we have only the real ranks 3 and 4, because the our forms are all general (i.e. $rk_{\mathbb{C}}(f) = 3$) and holds Proposition 17. Then we have

X	3	4
$\approx$ **probability**	0.8660	0.1340

Table 2.5: $d = 4$, f.

whence $\mathbb{E}(X) = 3.1340$.
If $d = 5$ we have only the real ranks 3, 4 and 5, because the our forms are all general (i.e. $rk_{\mathbb{C}} = 3$) and holds Proposition 17. Then we have

Y	3	4	5
$\approx$ **probability**	0.3844	0.5824	0.0332

Table 2.6: $d = 5$, f.

whence $\mathbb{E}(Y) = 3.6488$.
Again for a sample of 100000 forms f of degree 4, 5 we have estimated the probabilities for the variable t conditioned to the values of q, where q is the number of real roots of f and t is the number of real roots of $yf_x - xf_y$:

q	$t = 0$	$t = 2$	$t = 4$
4	0	0	1
2	0	0.5160	0.4840
0	0	0.3038	0.6962

Table 2.7: $d = 4$.

q	$t = 1$	$t = 3$	$t = 5$
5	0	0	1
3	0	0.7186	0.2814
1	0.0516	0.6234	0.3250

Table 2.8: $d = 5$.

Hence, we note that there are some prohibited values of t in relation to the value of q, in according with Theorem 49.
Again for a sample of 100000 forms f of degree 4, 5 we have estimated the probabilities for the variable q conditioned to the values of the $rk_{\mathbb{R}}(f)$:

$rk_{\mathbb{R}}(f)$	$q = 0$	$q = 2$	$q = 4$
4	0	0	1
3	0.1568	0.8432	0

Table 2.9: $d = 4$.

$rk_{\mathbb{R}}(f)$	$q = 1$	$q = 3$	$q = 5$
5	0	0	1
4	0.4903	0.5097	0
3	0.2882	0.7118	0

Table 2.10: $d = 5$.

2.3 Ternary forms

In Remark 51 we give the statistical estimates of the expected values of some assigned aleatory variable. On the other hand, we give also a statistical confirmation of Theorem 49. For example, in Tables 2.7 and 2.8 we can see that there are some prohibited values for the number of real roots of $D(f)$ conditioned to the number of real roots of f. We would like to do the same statistical survey for the ternary cubics, hoping to be able to generalize Theorem 49 for the ternary forms.

Remark 52. Let $f \in Sym^d(\mathbb{R}^3)$ be a ternary form, that is f is a homogeneous polynomial of degree d in three variables x, y, z. Then $\{f = 0\}$ has at most $\frac{(d-1)(d-2)}{2} + 1$ real connected components in $\mathbb{P}(\mathbb{R}^3)$ and, by Theorem 4, f has $((d-1)^3 - 1)/(d-2) = (d-1)^2 + (d-1) + 1$ distinct eigenvectors in the general case (note that the number $(d-1)^2+(d-1)+1$ is odd, $\forall\, d \in \mathbb{N}$). By Proposition 11.6.1 in [4], if d is odd, $\{f = 0\}$ has a finite number $c+1$ of connected components in $\mathbb{P}(\mathbb{R}^3)$, c ovals and one pseudo-line. Then the complement $S^2 \setminus \{f = 0\}$ consists of $2c + 2$ connected components (regions) which are symmetric in pairs. f has constant sign on each region and the signs are opposite for symmetric regions. Again by Proposition 11.6.1 in [4], if d is even, $\{f = 0\}$ has only a finite number c of connected components in $\mathbb{P}(\mathbb{R}^3)$, all ovals. Then the complement $S^2 \setminus \{f = 0\}$ consists of $2c+1$ connected components (regions), $2c$ of them are symmetric in pairs. Again f has constant sign on each region and the sign is the same for symmetric regions.

Theorem 53 (Harnack's curve)**.** *([4]) For any algebraic curve of degree d in the real projective plane, the number of connected components w is bounded by*

$$\frac{1-(-1)^d}{2} \leq w \leq \frac{(d-1)(d-2)}{2} + 1$$

The maximum number is one more than the maximum genus of a curve of degree d and it is attained when the curve is nonsingular. Moreover, any number of components in this range can be attained.

Definition 54. *A curve which attains the maximum number of real connected components is called an M-curve.*

Theorem 55 (Stickelberger)**.** *([15]) Let $I = (f_1, ..., f_k)$ be an ideal of $K[x_1, ..., x_n]$, with $K = \mathbb{C}$ or $K = \mathbb{R}$ e let $M_{x_i} : K[x_1, ..., x_n]/I \longrightarrow K[x_1, ..., x_n]/I$ linear applications (companions) induced by x_i multiplication. Then exists at least a common eigenvector v for all M_{x_i}, with eigenvalues λ_i, that is $M_{x_i}v = \lambda_i v$, if and only if $(\lambda_1, ..., \lambda_n) \in V(I)$.*

Proof. Let v be an eigenvector such that $M_{x_i}v = \lambda_i v$, $\forall\, i : 1...n$. If $f \in I$, $M_{f(x_1,...,x_n)} = 0$, then $0 = M_{f(x_1,...,x_n)}v = f(M_{x_1}, ..., M_{x_n})v = f(\lambda_1, ..., \lambda_n)v$, where the last equals follow from Lemma 4.2 in [15], hence $f(\lambda_1, ..., \lambda_n) = 0$.
Conversely, we must prove that coordinates of all $p_i \in V(I)$ are eigenvalues of a common eigenvector for matrices M_{x_j}. Decompose $\mathbb{C}[x_1, ..., x_n]/I = \oplus_{i=1}^k A_i$. A_i is M_{x_j}-invariant for $j = 1, ..., n$ and $M_{x_1}, ..., M_{x_n}$ commutate, by Proposition 1.17 in [15], exist a common eigenvector for M_{x_j} with eigenvalues the p_i's j-th coordinate. $\square$

Lemma 56. *([15]) Let $V(I) \subset \mathbb{C}^n$, $h \in \mathbb{C}[x_1, ..., x_n]$. Then eigenvalues of $M_h : K[x_1, ..., x_n]/I \longrightarrow K[x_1, ..., x_n]/I$ coincide with values $h(p_i) \in \mathbb{C}$, $p_i \in V(I)$.*

Proposition 57. *([15]) Consider a monomial order (e.g. lexicographical order) and let $x^{\alpha(1)}, ..., x^{\alpha(m)}$ be monomials not in $LT(I)$ that generate $K[x_1, ..., x_n]/I$. Then for all points $p \in V(I)$ and for all polynomial $h \in K[x_1, ..., x_n]$, the vector $p^{\alpha(1)}, ..., p^{\alpha(m)}$ (obtained computing monomials on p) is an eigenvector of M_h^t with eigenvalues $h(p)$.*

Proof. Let m_{ij} coefficients of M_h. we have $[x^{\alpha(j)}h] = M_h([x^{\alpha(j)}]) = \sum_{i=1}^m m_{ij}[x^{\alpha(i)}]$. Evaluating on p we obtain $p^{\alpha(j)}h(p) = \sum_{i=1}^m m_{ij}p^{\alpha(i)}$, that is the thesis. $\square$

Remark 58. Given a sample of real ternary forms f, we can compute eigenvectors of f with Macaulay2, since the eigenvectors of f are the solutions of the system associated to the ideal $I = (yf_x - xf_y, zf_y - yf_z, zf_x - xf_z)$, that is are elements of $V(I)$ (Remark 23). Then we can compute them by the Eigenvectors Method, that is we can compute the companions matrix M_x, M_y with respect to I and by Stickelberger's Theorem we can take their eigenvalues relative of their common eigenvectors as elements of $V(I)$. But by Proposition 57, we can compute the companion matrix M_x (or M_h, for any polynomial h), the normalized eigenvectors v_i of M_x^t (or of M_h^t) and hence, if entries of v_i corresponding to monomials x, y of the normalized base of $\mathbb{R}[x, y, 1]/I$ are real, we have a real eigenvectors $(x, y, 1)$ of f. Moreover, for a general real ternary cubic form f the base of monomials not in $LT(I)$ of $\mathbb{R}[x, y, 1]/I$ is composed from seven monomials, then $\mathbb{R}[x, y, 1]/I$ has finite dimension seven, then $V(I)$ has seven distinct elements (i.e. eigenvectors of f), according with Theorem 33.
For a sample of 1000 real ternary cubic forms f, where

$$f = \sum_{j_0+j_1+j_2=3} \sqrt{\binom{3}{j_0\, j_1\, j_2}} a_{j_0 j_1 j_2} x_0^{j_0} x_1^{j_1} x_2^{j_2},\ a_i \approx N(0, 1)$$

with c ovals, we have estimated the probabilities for the variable t conditioned to variable c in the following table:

t	1	3	5	7
$c = 1$	0	0,026	0,51	0,464
$c = 0$	0,038	0,186	0,422	0,354

Table 2.11: $d = 3$.

where t is the number of real eigenvectors of f; given $\Delta(f) = -T^2 + 64S^3$ the discriminant of degree 12 of f (see Proposition 4.4.7 pag. 167, Example 4.5.3 pag. 171 and Formula (4.5.8) pag. 173 in [34]), in particular, if $\Delta(f) > 0$ then f has two components ($c = 1$), while if $\Delta(f) < 0$ one ($c = 0$).
Again for a sample of 1000 ternary cubic forms f, we have estimated the probabilities of aleatory variables $X = (0, 1)$, $Y = (1, 3, 5, 7)$ and then their relative expected values and we expect that $\mathbb{E}(Y) \approx 1 + \frac{8}{7}\sqrt{14} \approx 5,276$ (see [13], the last Table in subsection 5.2):

X	0	1
$\approx$ **probability**	$0,735$	$0,265$

Table 2.12: $d = 3$.

whence $\mathbb{E}(X) = 0,265$.

Y	1	3	5	7
$\approx$ **probability**	$0,028$	$0,144$	$0,445$	$0,383$

Table 2.13: $d = 3$.

whence $\mathbb{E}(Y) = 5.366 \approx 5.276$.
Now let $f \in Sym^3(\mathbb{R}^3)$ such that

$$f = y^2z - \sum_{i=0}^{3} \sqrt{\binom{3}{i}} a_i x^{3-i} z^i = y^2 z - p(x, z),\ a_i \approx N(0, 1)$$

that is f is a cubic in the Weierstrass form. If we set $z = 1$, we have an univocal classification of the ternary cubic form in conics with one connected component ($c = 0$) or two connected components ($c = 1$), respectively if the discriminant of p, $\Delta(p)$, is less than zero or it is greater than zero (see Figures 2.18, 2.19 and 2.20).

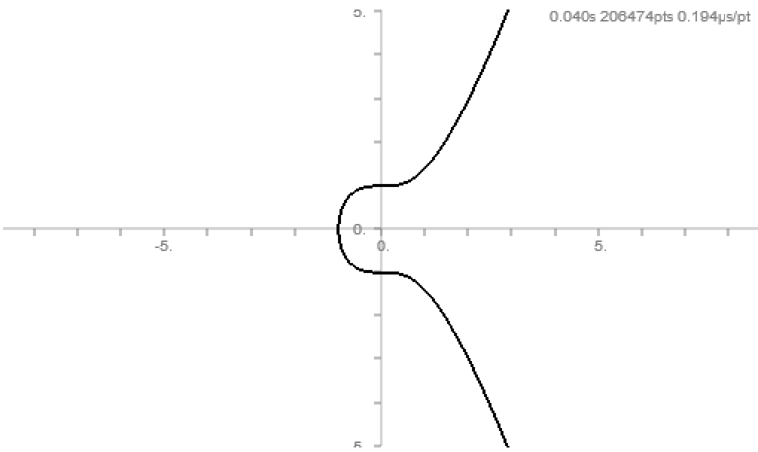

Figure 2.18: $\Delta(p) < 0$.

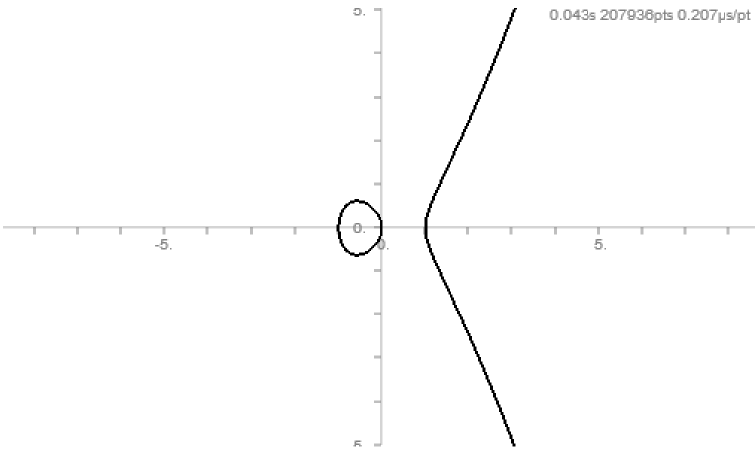

Figure 2.19: $\Delta(p) > 0$.

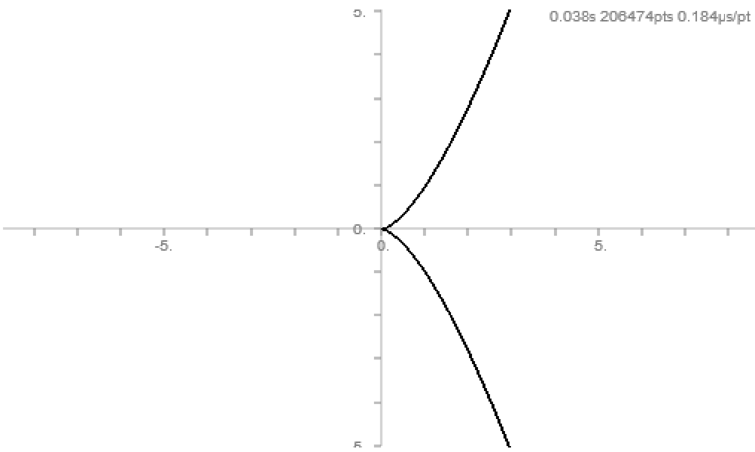

Figure 2.20: $\Delta(p) = 0$.

For a sample of 1000 ternary cubic forms f, we have estimated the probabilities of aleatory variables $X = (0, 1)$ and the probabilities for the variable t conditioned to variable c in the following tables:

X	0	1
$\approx$ **probability**	$0,625$	$0,375$

Table 2.14: $d = 3$.

whence $\mathbb{E}(X) = 0,375$

t	1	3	5	7
$c=1$	0	0,048	0,544	0,408
$c=0$	0,184	0,312	0,2384	0,2656

Table 2.15: $d = 3$.

We have the following

Theorem 59. *Let f be a ternary cubic. If f is in the Weierstrass form, then f has at least three real eigenvectors.*

Proof. Let f be in the Weierstrass form, that is

$$f = y^2 z - p(x, z),\ p(x, z) = x^3 + axz^2 + bz^3$$

If $\Delta(p) \geq 0$, we have that the inequality $-4a^3 - 27b^2 \geq 0$ is satisfied inside and along the graphic in Figure 2.21. Let $V(I) = \{(x, y, z) \in \mathbb{R}^3 | h_1 = h_2 = h_3 = 0\}$ be, with $h_1 = yf_x - xf_y$, $h_2 = zf_y - yf_z$, $h_3 = zf_x - xf_z$. Then $(1, 0, 0) \in V(I)$ (but $(0, 1, 0) \notin V(I)$). Setting $z = 1$, the system $h_1 = h_2 = h_3 = 0$ has the following six solutions with parameters a, b:

$$\left\{x_1 = \frac{\sqrt{-3a+1}-1}{3}, y_1 = \frac{\sqrt{2\sqrt{-3a+1}a - 2a + 9b + 6}}{\sqrt{3}}\right\}$$

$$\left\{x_2 = \frac{\sqrt{-3a+1}-1}{3}, y_2 = \frac{-\sqrt{2\sqrt{-3a+1}a - 2a + 9b + 6}}{\sqrt{3}}\right\}$$

$$\left\{x_3 = \frac{-\left(\sqrt{-3a+1}+1\right)}{3}, y_3 = \frac{\sqrt{-2\sqrt{-3a+1}a - 2a + 9b + 6}}{\sqrt{3}}\right\}$$

$$\left\{x_4 = \frac{-\left(\sqrt{-3a+1}+1\right)}{3}, y_4 = \frac{-\sqrt{-2\sqrt{-3a+1}a - 2a + 9b + 6}}{\sqrt{3}}\right\}$$

$$\left\{x_5 = \frac{\sqrt{8a^2 - 12a + 9b^2} - 3b}{2(2a-3)}, y_5 = 0\right\}$$

$$\left\{x_6 = \frac{-\sqrt{8a^2 - 12a + 9b^2} - 3b}{2(2a-3)}, y_6 = 0\right\}$$

The last two are reals if and only if $\Phi = 8a^2 - 12a + 9b^2 \geq 0$ and this is true outside and along the ellipse in Figure 2.22. Then if $\Delta(p) \geq 0$, we have that (x_5, y_5), (x_6, y_6) are real solutions and the thesis. □

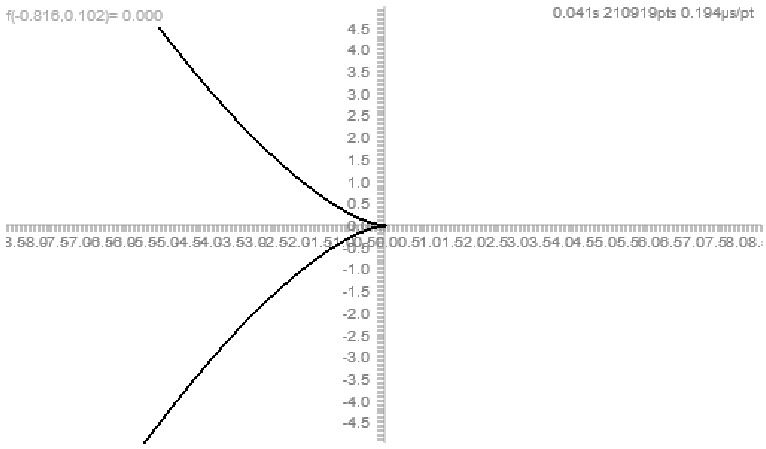

Figure 2.21: $\Delta(p) = -4a^3 - 27b^2 = 0$.

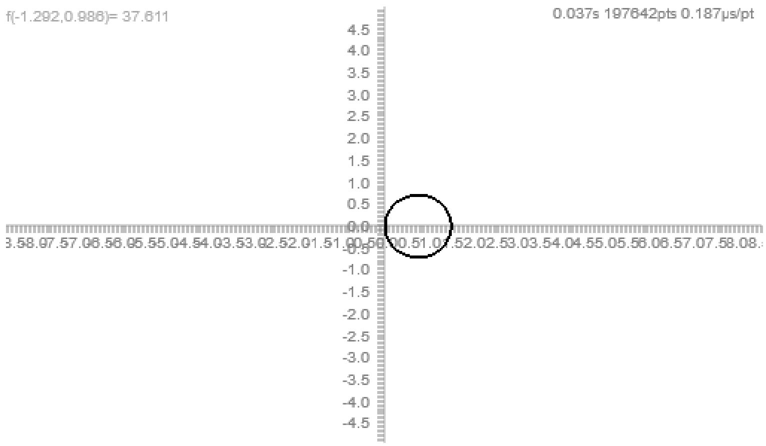

Figure 2.22: $\Phi = 8a^2 - 12a + 9b^2 = 0$.

Remark 60. In the proof of Theorem 59, we prove our conjecture only for the subset of cubic forms in the Weierstrass form, because our problem is not invariant by the action of $SL(3)$ group. In fact, we have that already for binary forms the problem is not invariant by the action of $SL(2)$. On the other hand, we have a valid counterexample when we write f in the Hesse form ([2]), that is $f = x^3 + y^3 + z^3 + 6\lambda xyz$. In this case, we have that the points $(1, 0, 0)$ and $(0, 1, 0)$ belong to $V(I)$. Then all cubic ternary forms have at least three real eigenvectors and this is not possible.

Remark 61. Consider a ternary cubic

$$f = a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xz^2 + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3.$$

We can take f with $a_0 = 1$. Setting $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, we obtain a subfamily $\mathcal{F}$ of $Sym^3(\mathbb{R}^3)$. Let $I = \langle p_1, p_2, p_3 \rangle$ be the ideal with $p1 = yf_x - xf_y$, $p2 = zf_y - yf_z$,

$p3 = zf_x - xf_z$. Then the system $p_1 = p_2 = p_3 = 0$ gives $V(I)$. Setting $z = 1$, we have that the system $p_1 = p_2 = p_3 = 0$ becomes

$$S = \begin{cases} x(-a_1x^2 + 2a_1y^2 + 2a_2y - 3a_6y^2 - 2a_7y - a_8 + 3xy) = 0 \\ a_1x^2 - a_2x^2y + 3a_6y^2 - a_7y^3 + 2a_7y - 2a_8y^2 + a_8 - 3a_9y = 0 \\ x(2a_1y - a_2x^2 + 2a_2 - a_7y^2 - 2a_8y - 3a_9 + 3x) = 0 \end{cases}$$

By direct computation, we have $p_1 = yp_3 - xp_2$, then to solve S means to solve the system $p_2 = p_3 = 0$. Therefore we have the three aligned solution points $(0, y_1, 1)$, $(0, y_2, 1)$, $(0, y_3, 1)$, where y_i are the solutions of the cubic equation in y

$$-3a_6y^2 - a_7y^3 + 2a_7y - 2a_8y^2 + a_8 - 3a_9y = 0.$$

Theorem 62. *Let f be a ternary form of degree d and suppose that f has c ovals. Then, if d is odd, we have $2c + 1 \leq$ #real eigenvectors of f and if d is even, we have $\max(2c + 1, 3) \leq$ #real eigenvectors of f.*

Proof. By Lemma 34, finding real eigenvectors of f means finding classes $[(x_0, y_0, z_0)] \in \mathbb{P}(\mathbb{R}^3)$ such that $(x_0, y_0, z_0) \in S^2$ is a critical point of f on the sphere, that is a maximum, minimum or saddle point of f on S^2. By Remark 52, we have that the complement $S^2 \setminus \{f = 0\}$ is divided at least into $2c$ pairs of symmetric regions, in which f has constant sign and f attains a non zero maximum inside any region where f is positive, and a non zero minimum inside any region where f is negative. Then, for any non zero maximum v there is an antipodal $-v$ which is a non zero minimum if f has odd degree, while for any non zero maximum (minimum) v there is an antipodal $-v$ which is a non zero maximum (minimum) if f has even degree; in conclusion, we have at least $2c$ critical points on the sphere corresponding to maxima or minima of f and then f has at least c real eigenvectors. Consider now the following situations:

1. $f \in Sym^d(\mathbb{R}^3)$, d odd. In this case, by Remark 52 there are $2c + 2$ regions on the sphere, then $2c + 2$ total maxima and minima and hence f has at least $c + 1$ real eigenvectors.
2. $f \in Sym^d(\mathbb{R}^3)$, d even. In this case, by Remark 52 there are $2c + 1$ regions on the sphere, then $2c + 2$ total maxima and minima and hence f has at least c real eigenvectors and at least another one, given by a non zero maximum (minimum) v and by its antipodal $-v$ which is a non zero maximum (minimum) of f in the internal of the complement on S^2 of the union of all other $2c$ symmetric regions, that is f has at least $c + 1$ real eigenvectors.

We must consider also the saddle points of f on S^2. By Morse's equation (see Theorem 5.2 pag. 29 in [23])

$$\sum_{\gamma} (-1)^{\gamma} C_{\gamma} = \chi(S^2) \tag{2.4}$$

where $\gamma \in \{0, 1, 2\}$ is the index of critical points of f on S^2 (respectively, we have a maximum, saddle or minimum point if γ is 0, 1 or 2), C_{γ} is the number of critical points with index γ of $f|_{S^2}$ and $\chi(S^2) = 2$ is the Euler's characteristic of S^2, we have the following equation:

$$C_0 - C_1 + C_2 = 2$$

We have seen that if f has c ovals we have at least $2c + 2$ total maxima and minima of f on S^2 and then

$$C_0 + C_2 = C_1 + 2 \geq 2c + 2 \Longrightarrow C_1 \geq 2c.$$

Hence, the total number of critical points of f on the sphere is at least $2c+2+2c = 4c+2$ and then f has at least $2c + 1$ real eigenvectors.
Finally, note that if d is even and if $c = 0$, by Weierstrass's Theorem we have that f attains at least a pair of absolute maxima and a pair of absolute minima on S^2, then f has at least 2 real eigenvectors, hence 3 because the total number of eigenvectors of f is always odd and therefore, if d is even, f has at least $\max\{2c + 1, 3\}$ real eigenvectors. □

Remark 63. Equation (2.4) can be seen in an equivalent way as a consequence of Poincaré-Hopf's Theorem as in [24], pag. 35.

Corollary 64. *Consider $f \in \mathrm{Sym}^3(\mathbb{R}^3)$. Then, according to Remark 18, if f has two components it has at least three real eigenvectors (see Figure 2.23, 2.24).*

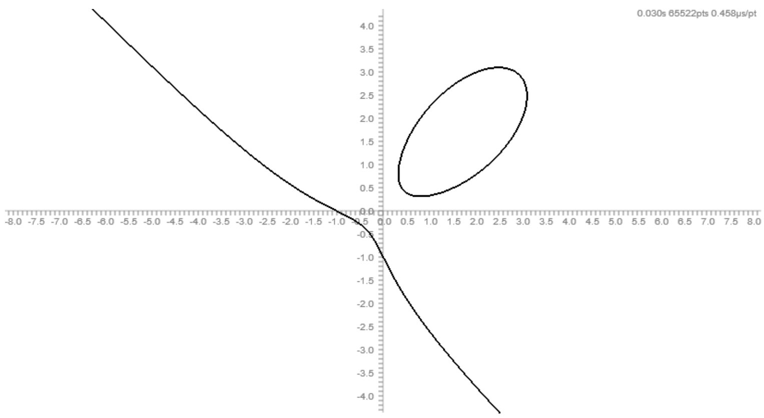

Figure 2.23: $x^3 + y^3 + 1 + 6axy = 0$, $\lambda < -\frac{1}{2}$.

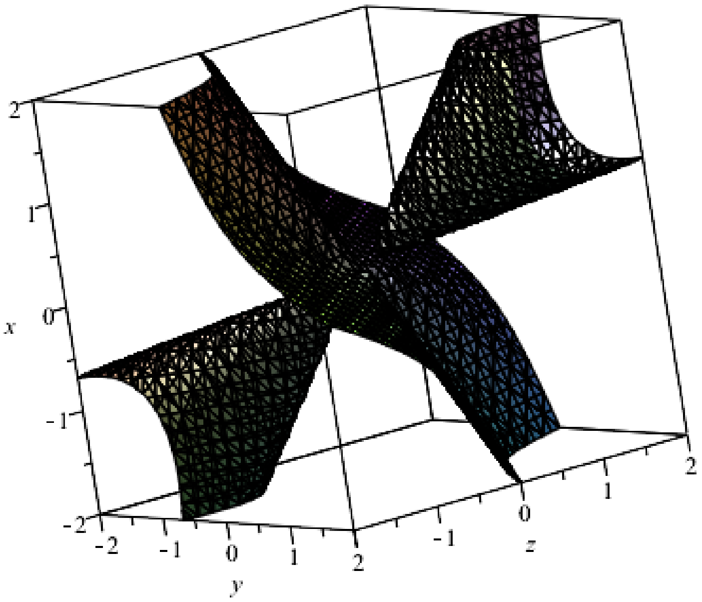

Figure 2.24: $x^3 + y^3 + z^3 + 6axyz = 0$, $\lambda < -\frac{1}{2}$.

Remark 65. For an M-curve we have the following:

1. $f \in Sym^d(\mathbb{R}^3)$, d odd. In this case, by Theorem 53 we have that an M-curve has $\frac{(d-1)(d-2)}{2} + 1$ components, $\frac{(d-1)(d-2)}{2}$ ovals and one pseudo-line and then by Theorem 62 f has at least $(d-1)(d-2) + 1 = d^2 - 3d + 3$ real eigenvectors.

2. $f \in Sym^d(\mathbb{R}^3)$, d even. In this case, by Theorem 53 we have that an M-curve has $\frac{(d-1)(d-2)}{2} + 1$ components, all ovals and then by Theorem 62 f has at least $(d-1)(d-2) + 3 = d^2 - 3d + 5$ real distinct eigenvectors.

Remark 66. Having fixed the topological type of a form $f \in Sym^d(\mathbb{R}^3)$, $d = 3, 4$, i.e. having fixed the kind (nested or not) and the number c of ovals of f, the set of all forms such that they have the same number c of f is connected (see Theorem 1.7 in [27]).

Remark 67. Consider a form $f \in Sym^d(\mathbb{R}^3)$ such that $f = l_1 l_2 \cdots l_d$, where l_i are linear ternary forms, that is f is a singular form of degree d such that its real locus of zeros consists of d lines in $\mathbb{R}^2$. If we choose all l_i such that $\forall\, i : 1, \ldots, d$ the set $\{l_i = 0\} \cap (\cup_{i \neq j} \{l_j = 0\})$ consists of $d-1$ distinct points $P_{i,j}$ in $\mathbb{R}^2$, i.e. each line meets all the others in $d-1$ distinct points, f has always the maximum number t of real eigenvectors with multiplicity 1. Then, we can perturb f by ϵg, $g \in Sym^d\mathbb{R}^3$, $\epsilon \in \mathbb{R}_+$ small enough and obtain a nonsingular quartic, smooth in $P_{i,j}$ depending on the sign of

g in $P_{i,j}$, with the maximum t. These results are in [1], precisely see Theorem 6.1 and Corollary 6.2.

Now we show that the inequalities of Theorem 62 are sharp for ternary cubics and quartics:

Proposition 68. *Let f be a cubic with $c \in \{0,1\}$ ovals and let t be odd such that $2c+1 \leq t \leq 7$. Then the set*

$$\left\{f \in \mathrm{Sym}^3(\mathbb{R}^3) \,|\, f \textit{ has } c \textit{ ovals}, \, \#\textit{real eigenvectors of } f = t\right\}$$

has positive volume.

Proof. By Remark 66, we must show examples of ternary cubic forms such that $c \in \{0,1\}$ and t attains the maximum and the minimum value. We have the following examples:

- t maximum. By Remark 67, we can take $f = xy(x+y+1)$, $\epsilon = \frac{1}{1000}$, $g_1 = x^3+y^3-2$ and $g_2 = -x^3 - y^3 + 2$ to obtain $f_1 = f + \epsilon g_1$ and $f_2 = f + \epsilon g_2$ with, respectively, 1 and 0 ovals and 7 real eigenvectors (see Figures 2.25, 2.26, 2.27).
- t minimum. Then we have:
  - f has 0 ovals. In this case, we can find the Weierstrass form $f = y^2 - x^3 - \frac{1}{9}x^2 - x - 1$ (see Figure 2.28) with 1 real eigenvectors.
  - f has 1 oval. In this case, we can find the Weierstrass form $f = y^2 - \frac{2}{100}x^3 + \frac{45}{100}x^2 + \frac{303}{100}x + \frac{29}{100}$ (see Figure 2.29) with 3 real eigenvectors.

□

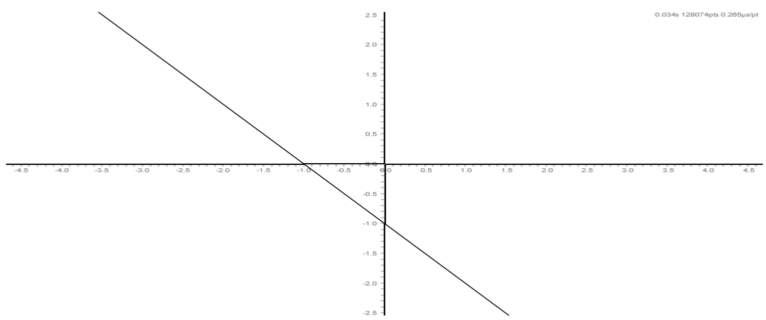

Figure 2.25: $d = 3$, $f = xy(x + y + 1)$.

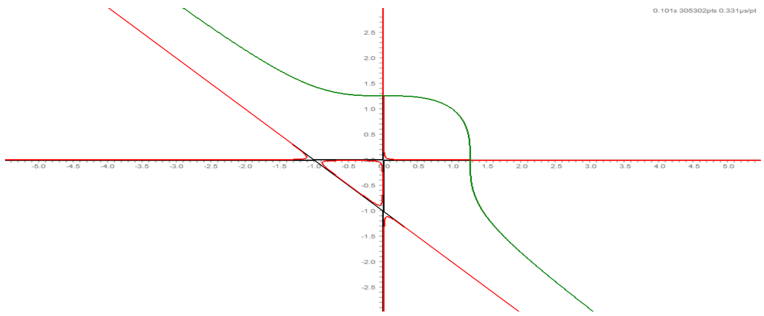

Figure 2.26: $d = 3$, $f = xy(x + y + 1)$, $g_1 = x^3 + y^3 - 2$ which is negative on the three singular points of f, $f_1 = f + \frac{1}{1000}g_1$ which has 1 oval.

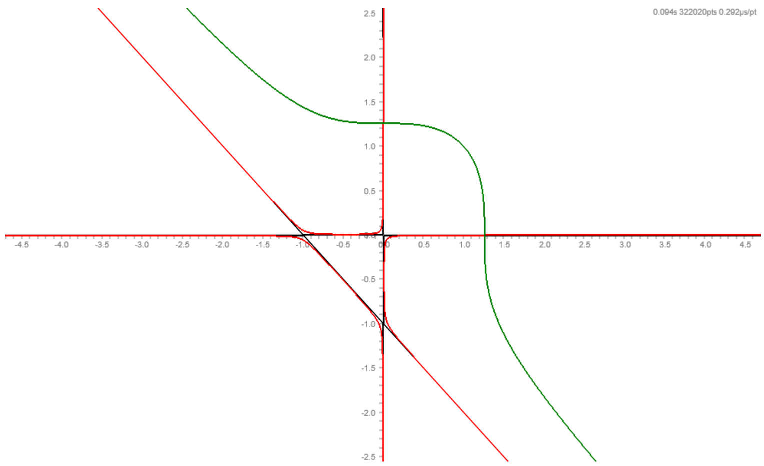

Figure 2.27: $d = 3$, $f = xy(x + y + 1)$, $g_2 = -x^3 - y^3 + 2$ which is positive on the three singular points of f, $f_2 = f + \frac{1}{1000}g_2$ which has 0 ovals.

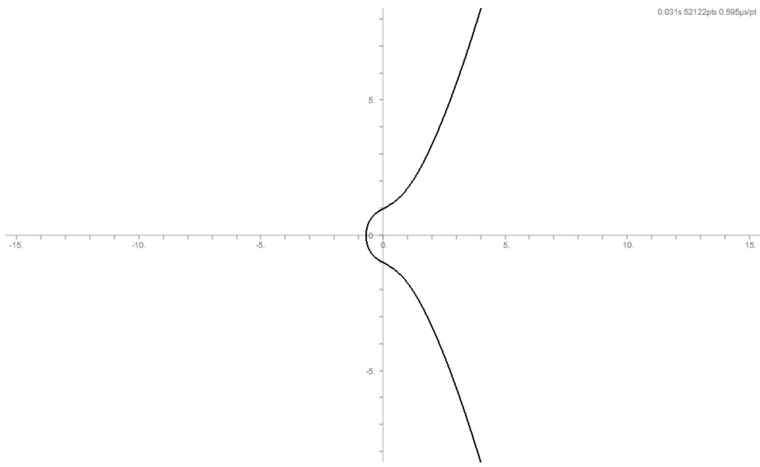

Figure 2.28: $d = 3$, $f = y^2 - x^3 - \frac{1}{9}x^2 - x - 1$ which has $c = 0$ ovals and $t = 1$ real eigenvector.

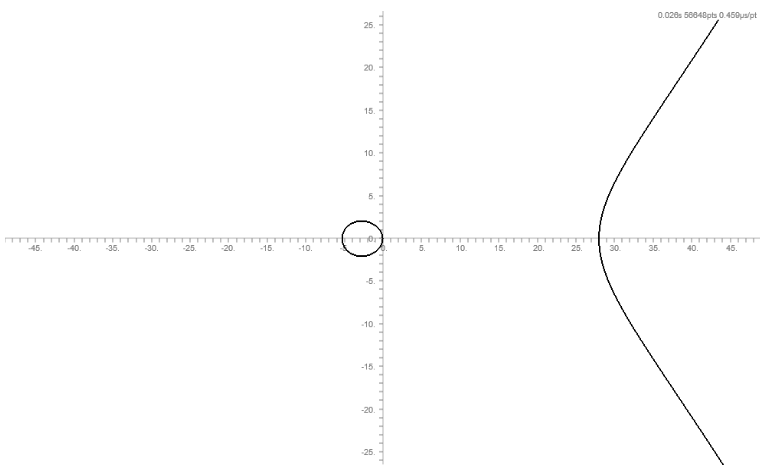

Figure 2.29: $d = 3$, $f = y^2 - \frac{2}{100}x^3 + \frac{45}{100}x^2 + \frac{303}{100}x + \frac{29}{100}$ which has $c = 1$ oval and $t = 3$ real eigenvectors.

Proposition 69. *Let f be a quartic with $c \in \{0, 1, 2\,nested, 2\,non\,nested, 3, 4\}$ ovals and let t be odd such that $\max(3, 2c+1) \leq t \leq 13$. Then the set*

$$\left\{f \in \mathrm{Sym}^4(\mathbb{R}^3) \,|\, f \text{ has } c \text{ ovals}, \#\text{real eigenvectors of } f = t\right\}$$

has positive volume.

Proof. By Remark 66, we must show examples of ternary quartic forms such that $c \in \{0, 1, 2\,nested, 2\,non\,nested, 3, 4\}$ and t assumes the maximum and the minimum value. We have the following examples:

- t maximum. By Remark 67, we can take $f = xy(x+y+\frac{1}{3})(-3x+y+1)$, $\epsilon = \frac{1}{1000}$, $g_1 = x^4 + y^4 - 1$, $g_2 = -x^4 - y^4 + \frac{5}{2}$, $g_3 = 7x^4 + 6y^4 - 1 - 5x$ and $g_4 = 7x^4 + 6y^4 - 1 - 5x - 9y$ to obtain $f_1 = f + \epsilon g_1$, $f_2 = f + \epsilon g_2$, $f_3 = f + \epsilon g_3$ and $f_4 = f + \epsilon g_4$ with, respectively, 4, 3, 2 non nested and 1 ovals and 13 real eigenvectors (see Figures 2.30, 2.31, 2.32, 2.33, 2.34). Moreover, we can take the hyperbolic quartic $f_5 = \det(I + xM_1 + yM_2)$, where

$$M_1 = \begin{pmatrix} \frac{2}{9} & 5 & 10 & \frac{7}{4} \\ 5 & 1 & 1 & \frac{3}{8} \\ 10 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{7}{4} & \frac{3}{8} & \frac{1}{2} & \frac{5}{3} \end{pmatrix}, \; M_2 = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & \frac{4}{5} \\ 1 & 8 & \frac{1}{3} & 8 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 8 \\ \frac{4}{5} & 8 & 8 & \frac{7}{8} \end{pmatrix}$$

are symmetric matrices, with 2 nested ovals and $t = 13$ (see Figure 2.35) and the Fermat quartic $f_6 = x^4 + y^4 + 1$ with 0 ovals and $t = 13$.

- t minimum. Then we have:
  - f has 0 ovals. In this case, we can find the SOS form $f = q_1^2 + q_2^2 + q_3^2 = (6x^2 + \frac{9}{8}xy + \frac{4}{9}y^2 + \frac{1}{6}x + \frac{2}{9}y + \frac{4}{9})^2 + (4x^2 + \frac{1}{2}xy + \frac{7}{9}y^2 + \frac{6}{7}x + \frac{3}{4}y + 2)^2 + (\frac{7}{3}x^2 + \frac{2}{5}xy + \frac{1}{10}y^2 + x + \frac{1}{2}y + \frac{1}{5})^2$ with 3 real eigenvectors.
  - f has 1 oval. In this case, we can find the form $f = \frac{9}{5}x^4 + \frac{4}{5}x^3y + \frac{1}{3}x^2y^2 + \frac{4}{9}xy^3 + \frac{5}{4}y^4 + x^3 + \frac{8}{7}x^2y + \frac{8}{5}xy^2 + \frac{1}{5}y^3 + x^2 + \frac{3}{8}xy + 2y^2 + \frac{5}{2}x + \frac{5}{9}y + \frac{3}{10}$ (see Figure 2.36) with 3 real eigenvectors.
  - f has 2 ovals non nested. In this case, we can find the form $f = q_1q_2 = (8x^2 + 3y^2 - \frac{1}{10}xy + 3x - 10y - 9)(7x^2 + 3y^2 + 5xy - 7x + 12y + 15)$ (see Figure 2.37) with 5 real eigenvectors.
  - f has 2 nested ovals. In this case, we can find the determinantal form $f = \det(I + xM_1 + yM_2)$ (see Figure 2.38), where

$$M_1 = \begin{pmatrix} \frac{5}{2} & \frac{5}{3} & 2 & \frac{9}{10} \\ \frac{5}{3} & \frac{7}{2} & \frac{1}{4} & \frac{2}{5} \\ 2 & \frac{1}{4} & \frac{10}{7} & \frac{1}{3} \\ \frac{9}{10} & \frac{2}{5} & \frac{1}{3} & 1 \end{pmatrix}, \; M_2 = \begin{pmatrix} \frac{4}{5} & \frac{5}{3} & 1 & \frac{5}{8} \\ \frac{5}{3} & \frac{1}{2} & 1 & 1 \\ 1 & 1 & 2 & \frac{8}{7} \\ \frac{5}{8} & 1 & \frac{8}{7} & \frac{10}{7} \end{pmatrix}$$

    are symmetric matrices, with 5 real eigenvectors.
  - f has 3 ovals. In this case, we have the quartic $f = (x^2 + y^2)^2 + p(x^2 + y^2) + q(x^3 - 3xy^2) + r$, where $p = \frac{16}{3}$, $q = \frac{80}{9}$, $r = \frac{2624}{9}$ in Figure 2.39 (see [9], pag. 116, 123), with 7 real eigenvectors.

– f has 4 ovals. In this case, we have the singular form $f = (y^2 - \frac{2}{100}x^3 + \frac{45}{100}x^2 + \frac{303}{100}x + \frac{29}{100})(x - 45)$, with 9 real eigenvectors and then we can perturb f by ϵg, where g is a quartic such that $f_6 = f + \epsilon g$ has 4 ovals and ϵ is small enough, to obtain a form with $c = 4$ and again $t = 9$; we can take $\epsilon = \frac{1}{1000}$ and $g = -x^4 - y^4 - 1$ (see Figure 2.40, 2.41).

□

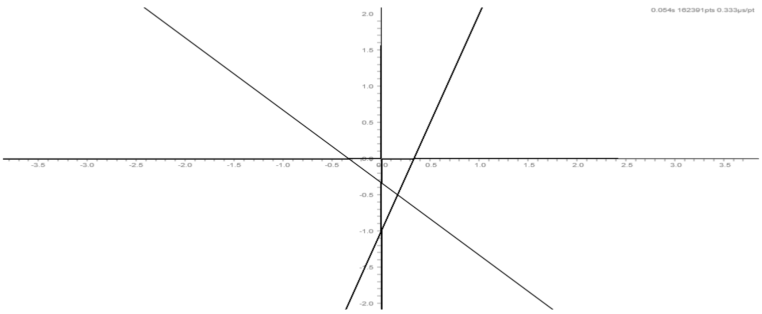

Figure 2.30: $d = 4$, $f = xy(x + y + \frac{1}{3})(-3x + y + 1)$.

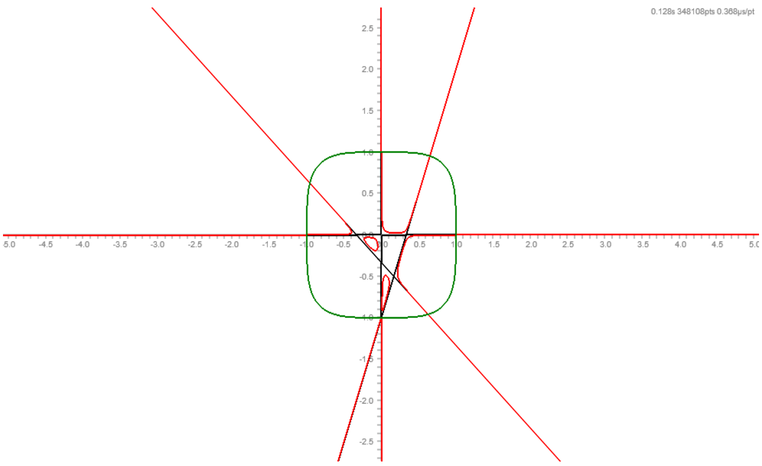

Figure 2.31: $d = 4$, $f = xy(x + y + \frac{1}{3})(-3x + y + 1)$, $g_1 = x^4 + y^4 - 1$ which is negative on the six singular points of f, $f_1 = f + \frac{1}{1000}g_1$ which has 4 ovals.

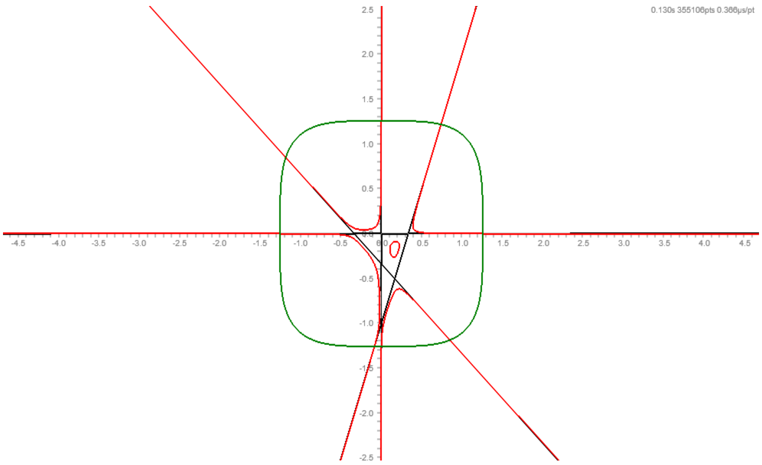

Figure 2.32: $d = 4$, $f = xy(x + y + \frac{1}{3})(-3x + y + 1)$, $g_2 = -x^4 - y^4 + \frac{5}{2}$ which is positive on the six singular points of f, $f_2 = f + \frac{1}{1000}g_2$ which has 3 ovals.

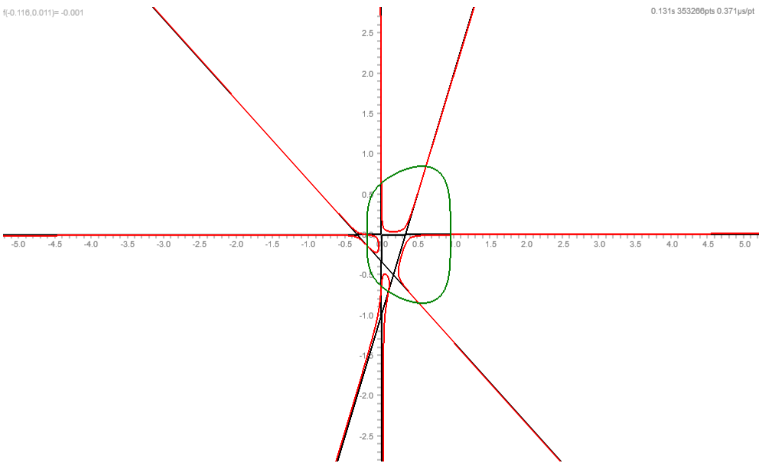

Figure 2.33: $d = 4$, $f = xy(x + y + \frac{1}{3})(-3x + y + 1)$, $g_3 = 7x^4 + 6y^4 - 1 - 5x$ which is negative on four of the six singular points of f and it is positive on the other two, $f_3 = f + \frac{1}{1000}g_3$ which has 2 non nested ovals.

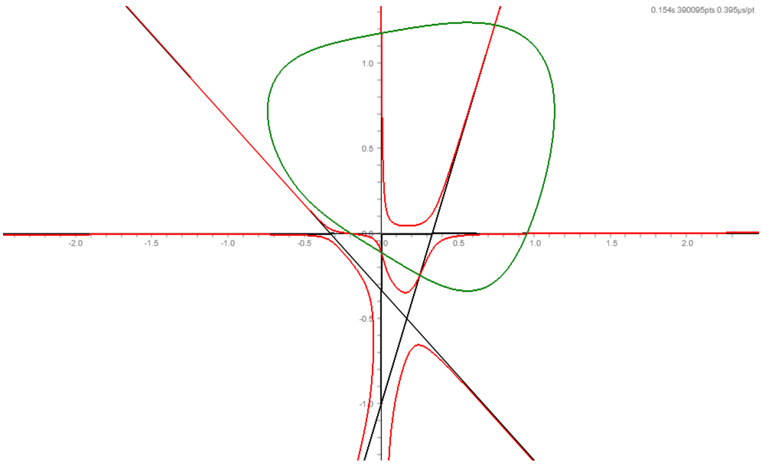

Figure 2.34: $d = 4$, $f = xy(x+y+\frac{1}{3})(-3x+y+1)$, $g_4 = 7x^4 + 6y^4 - 1 - 5x - 9y$ which is positive on four of the six singular points of f and it is negative on the other two, $f_4 = f + \frac{1}{1000}g_4$ which has 1 oval.

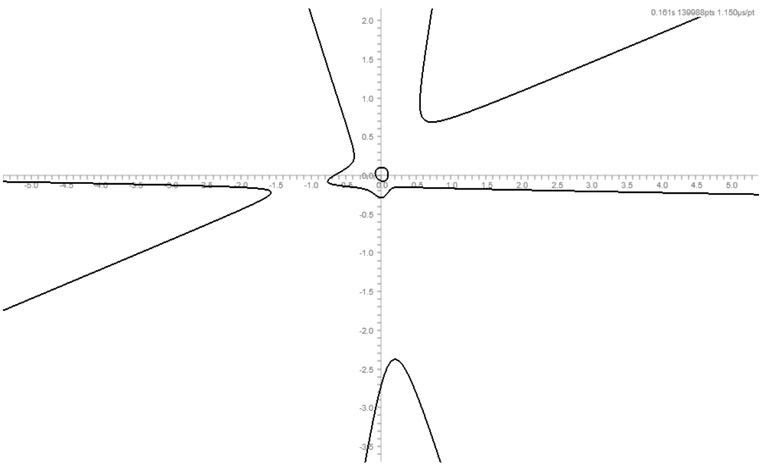

Figure 2.35: $d = 4$, $f_5 = \det(I + xM_1 + yM_2)$ which has 2 nested ovals and 13 real eigenvectors.

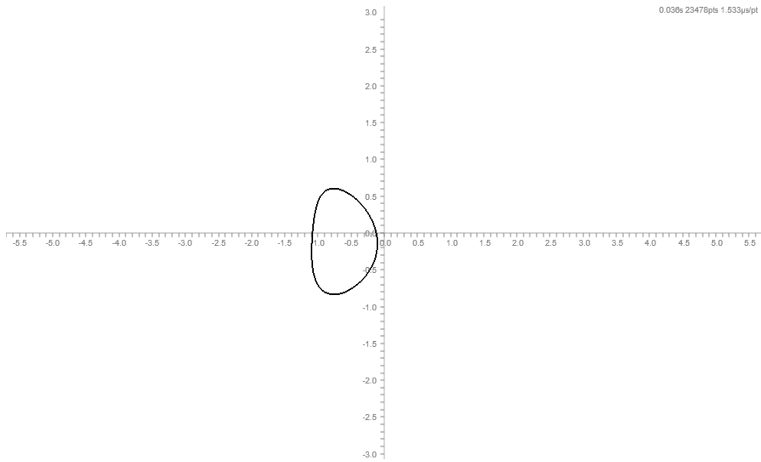

Figure 2.36: $d = 4$, $f = \frac{9}{5}x^4 + \frac{4}{5}x^3y + \frac{1}{3}x^2y^2 + \frac{4}{9}xy^3 + \frac{5}{4}y^4 + x^3 + \frac{8}{7}x^2y + \frac{8}{5}xy^2 + \frac{1}{5}y^3 + x^2 + \frac{3}{8}xy + 2y^2 + \frac{5}{2}x + \frac{5}{9}y + \frac{3}{10}$ which has $c = 1$ oval and $t = 3$ real eigenvectors.

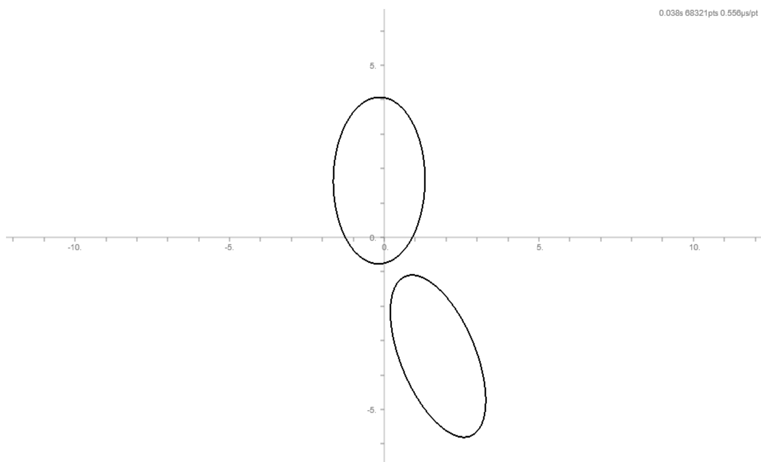

Figure 2.37: $d = 4$, $f = (8x^2+3y^2-\frac{1}{10}xy+3x-10y-9)(7x^2+3y^2+5xy-7x+12y+15)$ which has $c = 2$ non nested ovals and $t = 5$ real eigenvectors.

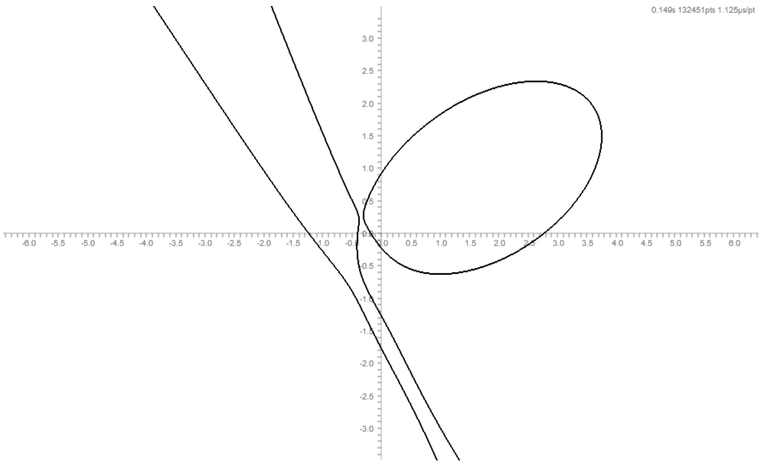

Figure 2.38: $d = 4$, $f = \det(I + xN_1 + yN_2)$ which has $c = 2$ nested ovals and $t = 5$ real eigenvectors.

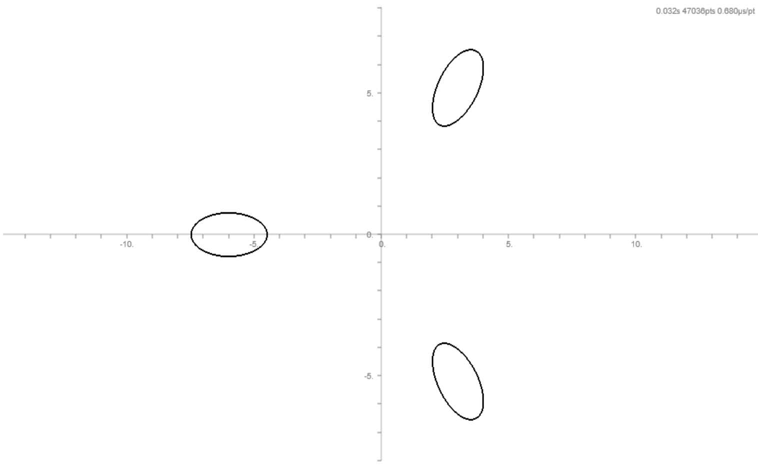

Figure 2.39: $d = 4$, $f = (x^2 + y^2)^2 + \frac{16}{3}(x^2 + y^2) + \frac{80}{9}(x^3 - 3xy^2) + \frac{2624}{9}$ which has $c = 3$ ovals and $t = 7$ real eigenvectors.

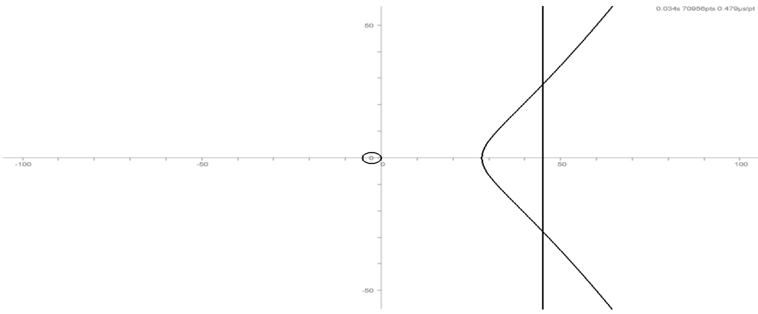

Figure 2.40: $d = 4$, $f = (y^2 - \frac{2}{100}x^3 + \frac{45}{100}x^2 + \frac{303}{100}x + \frac{29}{100})(x - 45)$ which has $t = 9$ real eigenvectors.

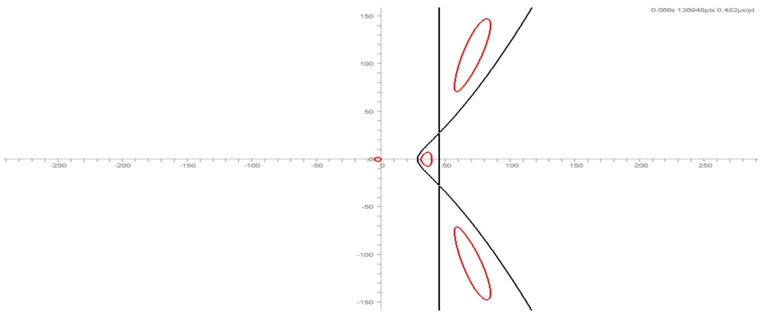

Figure 2.41: $d = 4$, $f = (y^2 - \frac{2}{100}x^3 + \frac{45}{100}x^2 + \frac{303}{100}x + \frac{29}{100})(x - 45)$ which has $t = 9$ real eigenvectors, $g = -x^4 - y^4 - 1$, $f_6 = f + \frac{1}{1000}g$ which has $c = 4$ ovals and $t = 9$ real eigenvectors.

2.4 Examples, partial results and open problems

Using Macaulay2, if $d \geq 4$ we show some computational examples of the possible values of t for some fixed c. We do this because we want to try to generalize Propositions 68, 69 for $d > 4$, but we can not use Remark 66 and then we can not repeat the proofs of those same Propositions.

Remark 70. Having fixed the topological type of a ternary quartic f, for a sample of 1000 forms we give the occurrences of all possible values of t in some topological cases:

1. f nonnegative, i.e. $c = 0$. In this case, we can write f as a sum of squares of 3 ternary quadratic forms q_1, q_2, q_3 (f is SOS) and we have the following table:

t	3	5	7	9	11	13
occurrences	458	240	215	79	6	2

Table 2.16: $d = 4$ and f nonnegative.

Note that if $c = 0$ all possible number of real eigenvectors can occur, also 3, according with Theorem 62.

2. f has one oval, i.e. $c = 1$. In this case, we can write f as a product of two quadratic forms q_1, q_2, where q_1 or q_2 has empty real locus of zeros and we have the following table:

t	3	5	7	9	11	13
occurrences	399	397	141	42	16	5

Table 2.17: $d = 4$ and $f = q_1 q_2$.

Note that if $c = 1$ all possible number of real eigenvectors can occur, also 3, according with Theorem 62.

3. f hyperbolic, i.e. $c = 2$ and the ovals are nested if $\{f = 0\}$ is smooth in $\mathbb{P}^2(\mathbb{C})$. In this case, we can write f as $\det(xI + yM_2 + zM_3)$, where M_i are 4×4 Hermitian matrices and I is the identity matrix, that is symmetric matrices in this case, because f has real coefficients and we have the following table:

t	3	5	7	9	11	13
occurrences	0	17	161	315	401	106

Table 2.18: $d = 4$ and $f = \det(xI + yM_2 + zM_3)$.

Note that if $c = 2$ (and the ovals are nested in this case) all possible number of real eigenvectors can occur except 3, according with Theorem 62.

Remark 71. Having fixed the topological type of a ternary quintic f, for a sample of 1000 forms we give the occurrences of all possible values of t in some topological cases:

1. f has only the pseudoline, i.e. $c = 0$. In this case, we can write f as a product of a line l and a nonnegative quartic g_1 or as a product of a cubic g_2, with $c = 0$ and a nonnegative quadric q_1. We have the following tables:

t	1	3	5	7	9	11	13	15	17	19	21
occurrences	346	282	207	100	48	13	4	0	0	0	0

Table 2.19: $d = 5$ and $f = lg_1$.

t	1	3	5	7	9	11	13	15	17	19	21
occurrences	20	91	330	399	121	25	3	1	0	0	0

Table 2.20: $d = 5$ and $f = q_1 g_2$.

2. f has one oval, i.e. $c = 1$. In this case we can write f as a product of a cubic g_1, with $c = 1$ and a nonnegative quadric q_1. We have the following table:

t	1	3	5	7	9	11	13	15	17	19	21
occurrences	0	22	96	380	348	120	33	1	0	0	0

Table 2.21: $d = 5$ and $f = g_1 q_1$.

3. f hyperbolic, i.e. $c = 2$ and the ovals are nested if $\{f = 0\}$ is smooth in $\mathbb{P}^2(\mathbb{C})$. In this case, we can write f as $\det(xI + yM_2 + zM_3)$, where M_i are 5×5 Hermitian matrices and I is the identity matrix, that is symmetric matrices in this case, because f has real coefficients and we have the following table:

t	1	3	5	7	9	11	13	15	17	19	21
occurrences	0	0	1	2	41	119	259	306	198	71	3

Table 2.22: $d = 5$ and $f = \det(xI + yM_2 + zM_3)$.

Then we have the following

Lemma 72. *Let f be a quintic with $c = 2$ nested ovals and let t be odd such that $2c + 1 \leq t \leq 21$. Then the set*

$$\left\{f \in \mathrm{Sym}^5(\mathbb{R}^3) \,|\, f \textit{ has } c \textit{ ovals}, \#\textit{real eigenvectors of } f = t\right\}$$

has positive volume.

Remark 73. Having fixed the topological type of a ternary sextic f, for a sample of 1000 forms we give the occurrences of all possible values of t in some topological cases:

1. f nonnegative, i.e. $c = 0$. In this case, we have two possibilities for our form: f is a sum of squares of 4 ternary cubic forms q_1, q_2, q_3, q_4 (f is SOS) or not.
In the first case, we have the following table:

t	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
occurrences	71	373	33	168	42	11	3	2	0	0	0	0	0	0	0

Table 2.23: $d = 6$ and f SOS.

In the second case, f is nonnegative but is not a sum of squares and then, taking known sextic with this property, for example $f_1 = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$ (the

Motzkin's sextic, [30]), $f_2 = x^6 + y^6 + z^6 - x^4y^2 - x^2y^4 - x^4z^2 - y^4z^2 - x^2z^4 - y^2z^4 + 3x^2y^2z^2$ (the Robinson's sextic, [30]), $f_3 = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$ (the Choi-Liu's sextic, [30]), we perturb them without changing their topological type, adding ϵg, where g is a random SOS sextic. We have the following tables:

t	3	5	7	9	11	13	15	17
occurrences	0	0	1	11	36	61	200	525

t	19	21	23	25	27	29	31
occurrences	156	10	0	0	0	0	0

Table 2.24: $d = 6$ and f_1.

t	3	5	7	9	11	13	15	17
occurrences	0	0	0	1	2	7	35	28

t	19	21	23	25	27	29	31
occurrences	47	84	186	610	0	0	0

Table 2.25: $d = 6$ and f_2.

t	3	5	7	9	11	13	15	17
occurrences	0	0	0	0	0	0	2	5

t	19	21	23	25	27	29	31
occurrences	13	20	70	701	173	14	2

Table 2.26: $d = 6$ and f_3.

2. f hyperbolic, i.e. $c = 3$ and the ovals are nested if $\{f = 0\}$ is smooth in $\mathbb{P}^2(\mathbb{C})$. In this case, we can write f as $\det(xI + yM_2 + zM_3)$, where M_i are 6×6 Hermitian matrices and I is the identity matrix, that is symmetric matrices in this case, because f has real coefficients and we have the following table:

t	3	5	7	9	11	13	15	17
occurrences	0	0	1	2	11	23	91	174

t	19	21	23	25	27	29	31
occurrences	261	207	163	49	16	1	1

Table 2.27: $d = 6$ and $f = \det(xI + yM_2 + zM_3)$.

3. f is obtained by slightly perturbing six lines, i.e. we perturb the product of six linear forms $l_1, \ldots, l_6$ by adding ϵg, where g is a random sextic and we have the following table:

t	3	5	7	9	11	13	15	17
occurrences	0	0	0	2	7	9	17	35

t	19	21	23	25	27	29	31
occurrences	49	65	75	97	145	218	281

Table 2.28: $d = 6$.

Then we have the following

Lemma 74. *Let f be a sextic with $c \in \{0, 3\,nested\}$ ovals and let t be odd such that* $\max(3, 2c + 1) \leq t \leq 31$. *Then the set*

$$\left\{f \in \mathrm{Sym}^6(\mathbb{R}^3) \,|\, f \text{ has } c \text{ ovals}, \#\text{real eigenvectors of } f = t\right\}$$

has positive volume.

By Remarks 71, 73, it is evident that already for $d = 5, 6$ the generalization of Propositions 68, 69 is very hard. In fact, we have trouble to writing the forms f of degree five or six in all possible topological cases. When we can to do this, the choice to use a reducible form or a specific form for f constrains the range of t. For example, to get all the possible values of t in the case of a nonnegative sextic, it is not sufficient to consider SOS forms and we must use also perturbations of some known irreducible nonnegative forms (e.g. Motzkin's sextic). Again, we have $t = 15$ as maximum value of t in the cases of a quintic with $c = 0, 1$ and not $t = 21$. Moreover, there are many difficulties for obtain the two forms with the minimum value of t in the nested cases of degree 5, 6. Then we do not know if the inequality of Theorem 62 is sharp and if it is the only essential constraint about the reality of eigenvectors for $f \in Sym^d(\mathbb{R}^3)$, $d > 4$. Moreover, we do not know how to extend Theorem 62 in higher dimension. These are open problems.
If you want to see the software with which we have done the Examples and Tables in this thesis, you can use the following link:

https://drive.google.com/drive/folders/0B0Z3u5Ct9E6Vbl9HMHZnSG1Vdzg?usp=sharing

Bibliography

[1] H. Abo, A. Seigal and B. Sturmfels. *Eigenconfigurations of Tensors.* Algebraic and Geometric Methods in Discrete Mathematics, to appear.

[2] M. Banchi. *Rank and border rank of real ternary cubics.* Boll. Unione Mat. Ital. 8 (2015), no. 1, 65-80. 14N05 (14H50 14Q05).

[3] G. Blekherman. *Typical real ranks of binary forms.* Found. Comput. Math. 15 (2015), no. 3, 793-798. 14N05 (15A69).

[4] J. Bochnak, M. Coste, M. F. Roy. *Real Algebraic Geometry.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 36, Springer-Verlag, Berlin, 1998.

[5] A. Boralevi, J. Draisma, E. Horobeţ, E. Robeva. *Orthogonal and Unitary Tensor Decomposition from an Algebraic Perspective.* Israel Journal of Mathematics, to appear.

[6] P. Breiding. *The expected number of Z-eigenvalues of a real gaussian tensor.* arXiv:1604.03910v1 [math.AG] (2016).

[7] D. Cartwright, B. Sturmfels. *The Number of Eigenvalues of a Tensor.* Linear Algebra Appl. 438 (2013), no. 2, 942-952. 15A69.

[8] A. Causa, R. Re. *On the maximum rank of a real binary form.* Ann. Mat. Pura Appl. (4) 190 (2011), no. 1, 55-59. 12D10 (14P99).

[9] E. Ciani *Scritti Geometrici Scelti, Volume primo.* Cedam, Padova. 1937.

[10] C. Ciliberto. *Geometric aspects of polynomial interpolation in more variables and of Waring's problem.* In European Congress of Mathematics, Vol. I (Barcelona, 2000), volume 201 of Progr. Math., pages 289-316. Birkähuser, Basel, 2001.

[11] G. Comas and M. Seiguer. *On the rank of a binary form.* Found. Comput. Math. 11 (2011), no. 1, 65-78. 15A69 (14N05 15A03).

[12] P. Comon and G. Ottaviani. *On the typical rank of real binary forms.* Linear Multilinear Algebra 60 (2012), no. 6, 657-667. 15A72 (15A69).

[13] J. Draisma, E. Horobeţ. *The average number of critical rank-one approximations to a tensor*. The average number of critical rank-one approximations to a tensor, Linear Multilinear Algebra (2016), vol. 64, no. 12, 2498-2518.

[14] J. Draisma, E. Horobeţ, G. Ottaviani, B. Sturmfels and R. R. Thomas. *The Euclidean Distance Degree of an Algebraic Variety*. Found. Comput. Math. 16 (2016), no. 1, 99-149.

[15] M. Elkadi, B. Mourrain. *Introduction à la résolution de systèmes polynomiaux*. Mathématiques et Applications 59, Springer, Berlin, 2007.

[16] J.E. Fornaess and N. Sibony. *Complex dynamics in higher dimensions*. I, Astérisque 222 (1994), 201-231.

[17] W. Fulton. *Intersection Theory, Second edition*. Springer, New York, NY, 1998.

[18] W. Fulton. *Introduction to Toric Varieties*. Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[19] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, 22, Springer. New York, 1977.

[20] A. Iarrobino and V. Kanev. *Power sums. Gorenstein algebras and determinantal loci*. Springer, 1999.

[21] L.H. Lim. *Singular values and eigenvalues of tensors: a variational approach*. Proc. IEEE Inter-nat. Workshop on Comput. Advances in Multi-Sensor Adaptive Processing (CAMSAP 2005), 129-132.

[22] M. Maccioni. *The number of real eigenvectors of a real polynomial*. BUMI (2017), 1972-6724, 10.1007/s40574-016-0112-y, 1-21.

[23] J. W. Milnor. *Morse Theory*. Princeton Univ. Press, Princeton, 1963.

[24] J. W. Milnor. *Topology from the differentiable viewpoint*. The University Press of Virginia, Charlottesville, Va. 1965.

[25] G. Ottaviani, R. Paoletti. *A Geometric Perspective on the Singular Value Decomposition*. Rend. Istit. Mat. Univ. Trieste, Volume 47, 107-125, 2015.

[26] D. Plaumann, C. Vinzant. *Determinantal representations of hyperbolic plane curves: an elementary approach*. J. Symbolic Comput. 57 (2013), 48-60. 14M12 (14H50 15B57).

[27] D. Plaumann, B. Sturmfels and C. Vinzant. *Quartic Curves and their Bitangents*. J. Symbolic Comput. 46 (2011), no. 6, 712-733. 14H50 (14N05).

[28] L. Qi. *Eigenvalues of a real supersymmetric tensor.* J. of Symbolic Comput. 40 (2005), 1302-1324.

[29] L. Qi. *Eigenvalues and invariants of tensors*, J. Math. Anal. Appl. 325 (2007), 1363-1377.

[30] B. Reznick. *On Hilbert's construction of positive polynomials.* arXiv:0707.2156v1 [math.AG] (2007).

[31] B. Reznick. *On the lenght of binary forms.* Quadratic and higher degree forms, 207-232, Dev. Math., 31, Springer, New York, 2013. 11E76 (11P05 14N10).

[32] B. Reznick. *Sums of even powers of real binary forms.* Mem. Amer. Math. Soc. 96 (1992), no. 463, viii+155 pp. 11E76 (11P05 52A21).

[33] E. Robeva. *Orthogonal Decomposition of Symmetric Tensors.* SIAM Journal on Matrix Analysis and Applications, 37 (2016), pp. 86-102.

[34] B. Sturmfels. *Algorithms in Invariant Theory, Second edition.* SpringerWien-NewYork, Berkeley, 2008.