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Contents

Chapter 1

Introduction

Tensors are the natural extension of matrices to higher dimensions, indeed matrices are order two tensors. In practical terms this means that they can be used to store data that relies in more than two parameters. Therefore, tensors naturally appear in a vast field of applications: for instance they are utilised for plant biodiversity estimation [BIR21], data analysis [Com94], computer vision [Pan+21].

The *singular value decomposition* of matrices is one of the main tools in applied mathematics, it states that for a real valued matrix $A \in \mathbb{R}^m \otimes \mathbb{R}^n$ there exists orthogonal matrices $U \in \mathrm{O(m)}$ and $V \in \mathrm{O(n)}$ such that $A = U\Sigma V^t$, where Σ is quasi-diagonal, that is, $\Sigma_{i,j} = 0$ if $i \neq j$ and $\Sigma_{ii} = \sigma_i$, with $\sigma_1 \geq \cdots \geq \sigma_{\min\{m,n\}}$.

One of its main application is that it can be utilised to compute the best given rank approximation of A. This is the subject of the famous *Eckart-Young Theorem* that states that for a matrix A with SVD given by $A = U\Sigma V^t$, where $\Sigma_{ii} = \sigma_i$, the best approximation of A by a matrix of rank k is given by $U\Sigma_k V^t$, where Σ_k is the quasi-diagonal matrix with the first k entries in the diagonal equal to $\sigma_i,\ i = 1, \ldots, k$ and all the others equal to zero. In particular, the rank one tensors $\sigma_i u_i \otimes v_i$ are critical points of the distance function from A to the variety of rank one matrices.

We remark that the SVD is usually considered over the real numbers, a similar result exists over the complex numbers with U and V unitary matrices and the hermitian transposition, called complex SVD. However, our interest is to study the real version of SVD and its generalizations. Therefore, we are going to extend this version to complex values, in other words, we consider $U,\ V \in \mathrm{O}_{\mathbb{C}}(m), \mathrm{O}_{\mathbb{C}}(n)$ together with the usual notion of transposition, where $\mathrm{O}_{\mathbb{C}}(m)$ means the Zariski closure of orthogonal matrices over the field of complex numbers. To differentiate it from the complex version we are going to call such decomposition the *algebraic SVD*. The main motivation to such construction is that this allows the use of robust techniques of algebraic geometry that were not possible over the reals, for instance, the algebraic SVD has been studied in [Dru+17]. In particular it is interesting to notice that not every complex valued matrix admits an algebraic SVD decomposition, differently from the complex SVD that can be obtained to every matrix in $\mathbb{C}^m \otimes \mathbb{C}^n$.

A similar decomposition as the SVD is obtained for tensors via the *high order singular value decomposition*, such decomposition is obtained by considering the SVD decomposition of the flattenings of the tensors and reconstructing a tensor from them. However, the HOSVD does not lead to the best rank approximation of the tensor. In 2005 [Lim05; Qi05] introduced independently the notion of singular vector tuples of tensors, or eigentensors. Let q_i be a quadratic form on V_i that is nondegenerated over the reals, and q be the Bombieri-Weyl product on $V = V_1 \otimes \cdots \otimes V_k$ obtained from the q_i, we utilise them to identify V_i with V_i^* and V with V^*. A tensor $v_1 \otimes \cdots \otimes v_k$, where $q_i(v_i, v_i) = 1$, is an eigentensor of a tensor $T \in V$ if for each flattening

$$T : V_1 \otimes \cdots \otimes V_{i-1} \otimes V_{i+1} \otimes \cdots \otimes V_k \to V_i$$

it holds that

$$T(v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_k) = \lambda v_i,$$

where $\lambda \in \mathbb{C}$ and $\widehat{v_i}$ denotes that such term is omitted. We associate to each eigentensor $v_1 \otimes \cdots \otimes v_k$ a tuple, called singular vector tuple, $([v_1], \ldots, [v_k]) \in \times_{i=1}^k \mathbb{P}V_i$, see Definition 3.1.2. We notice that for the case of matrices we have that the eigentensors $u_i \otimes v_i$ of a matrix A consists exactly of the pairs (u_i, v_i), called singular pairs, where u_i and v_i are the columns of the orthogonal matrices U and V giving the SVD of A [OP15]. Moreover, [Lim05] shows that the singular vector tuples of a tensor T are the critical points of the distance function between T and the Segre variety of rank one tensors, see Theorem 3.1.4, thus giving an extended notion of the *Eckart-Young Theorem* to higher order tensors. Later [DOT17] defined the critical space associated to a tensor T, such space contains all the best rank approximations of the tensor T.

The study of singular vector tuples has received extensive attention in the past years, for instance [FO14] have computed the ED-degree of the Segre-Veronese variety, Theorem 3.1.10, this is an invariant of the variety that, in simple terms, counts the number of singular vector tuples of a general tensor, see [Dra+13] for further comprehension. Later [OP15] have introduced the *critical space* H_T of a tensor T, this is the space obtained by considering the tensors, not necessarily of rank one, that satisfies (3.1.3). Following this work [DOT17] have shown that the critical space contains all the best given rank approximations of the tensor, moreover they demonstrated that H_T is generated by the singular vector tuples of T under the boundary format condition, see Definition 3.1.13. This also implies that the tensor T itself lies in the span of its singular vector tuples, we will refer to this as the *membership problem*.

Question 1.0.1 (Membership problem)**.** *Given* $T \in \bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i$*, denote* Z_T *the set of singular vector tuples of* T*, does* $T \in \langle Z_T \rangle$*?*

As mentioned before, this question is answered for general tensors in [DOT17] where the tensor space is assumed to be of a format named boundary format. In Chapter 5 we will work on some specific cases without the assumption of such format where our partial results indicate that a positive answer can be expected in general.

Moreover, it is noteworthy that although T is a linear combination of the singular vector tuples, in general such sum of rank one tensors is not minimal and, at the present moment, there is not a clear connection between tensor decomposition and singular vector tuples except for particular formats, for instance *orthogonally decomposable tensors* [Bor+15; RS17].

The question that motivated the presented work on singular vector tuples was: *Are tensors determined by their singular vector tuples?* This has been addressed first in [ASS17] where a positive answer was given for polynomials in three variables and odd degree. Later, [BGV21] extended this result to three variables and even degree, where it was shown that the property does not holds true anymore. In Chapter 4 we will generalise such result to any degree and any number of variables.

Theorem 4.1.4. *[Tur22, Theorem 1.1] Let V be a vector space of dimension $m+1$. Let $d \geq 3$ be an integer, and $f \in \mathbb{P}(\mathrm{Sym}^d V)$ be a general polynomial. Let*

$$\tau : \mathbb{P}\big(\mathrm{Sym}^d V\big) \dashrightarrow \mathbb{P}V^{(\mathrm{ed}_X)}, \; f \mapsto \mathrm{Eig}(f)$$

be the map that associates to f its eigentensors locus $\mathrm{Eig}(f)$*. Then*

$$\tau^{-1}(\tau(f)) = \begin{cases} [f], & \text{if } d \text{ is odd;} \\ \{[f + cq^{\frac{d}{2}}] | c \in \mathbb{C}\}, & \text{if } d \text{ is even.} \end{cases}$$

Moreover, the image of the map τ has dimension

$$\dim(\mathrm{Im}(\tau)) = \begin{cases} \binom{d+m}{d} - 1, & \text{if } d \text{ is odd;} \\ \binom{d+m}{d} - 2, & \text{if } d \text{ is even.} \end{cases}$$

Furthermore, we extend such result to partially symmetric tensors under the boundary format condition.

Theorem 4.2.10. *[Tur22, Theorem 1.2] Let $V_1, \dots, V_k$ be vector spaces of dimension $m_1+1, \dots, m_k+1$. Let $d_1, \dots, d_k$ be positive integers, and $T \in \mathbb{P}\big(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k\big)$ be a general tensor. Let*

$$\tau : \mathbb{P}\big(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k\big) \dashrightarrow (\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k)^{(\mathrm{ed}_X)}, \; T \mapsto \mathrm{Eig}(T),$$

be the map that associates a tensor T to its singular tuples locus $\mathrm{Eig}(T)$*. If $k \geq 3$ and suppose that $m_l \leq \sum_{j \neq l} m_j$ whenever $d_l = 1$, and for $k = 2$ we include the hypothesis that $(d_1, d_2) \neq (1,1)$, then*

$$\tau^{-1}(\tau(T)) = \begin{cases} [T], & \text{if } d_i \text{ is odd for some } i; \\ \{[T + cq_1^{\frac{d_1}{2}} \otimes \cdots \otimes q_k^{\frac{d_k}{2}}] | c \in \mathbb{C}\}, & \text{if } d_l \text{ is even for all } l. \end{cases}$$

Moreover, the image of the map τ has dimension

$$\dim(\operatorname{Im}(\tau)) = \begin{cases} \prod_{l=1}^{k} \binom{d_l+m_l}{d} - 1, & \textit{if } d_i \textit{ is odd for some } i; \\ \prod_{l=1}^{k} \binom{d_l+m_l}{d} - 2, & \textit{if } d_l \textit{ is even for all } l. \end{cases}$$

It is important to remark that Theorem 4.2.10 does not cover the matrix case ($k = 2$ and $d_1 = d_2 = 1$). Such case is treated separately in Example 4.0.1, where we show that it has a completely different behavior than the higher order tensor case, in particular the fiber has a bigger dimension.

We remark that both main results obtained in [DOT17; Tur22] must assume that the tensor space satisfy the boundary format condition. This is connected with the defectiveness of the dual variety of the Segre-Veronese variety [OSV21]. The same problems studied in [DOT17; Tur22] are tackled in Chapter 5 for some formats beyond boundary format. We will show that for several different formats the membership problem still holds true. We use this fact to demonstrate this implies that the tensors will still be determined by their singular vector tuples. Moreover, for small formats we explicitly describe the new relations among the singular vector tuples. This chapter consists of a joint work with Luca Sodomaco.

Another fundamental concept of matrices is the notion of rank, it consists of the dimension of the image of the associated linear map. A natural question is to find a minimal rank decomposition of a matrix $A \in \mathcal{M}_{m\times n} = \mathbb{R}^m \otimes \mathbb{R}^n$, in other words, find the smallest number r and rank one matrices $A_1 = a_1 \otimes b_1, \dots, A_r = a_r \otimes b_r$, where $a_i \otimes b_i = a_i b_r^t$ and $a_i \in \mathbb{C}^m, b_i \in \mathbb{C}^n$, such that $A = \sum_{i=1}^{r} A_i$. The singular value decomposition of A plays a major role in this question as well, we write $A = U\Sigma V^t$, with U, V orthogonal matrices and Σ quasi-diagonal, we can obtain such decomposition via the SVD:

$$A = \sigma_1 u_1 v_1^t + \cdots + \sigma_r u_r v_r^t,$$

where u_i, v_i are the columns of U, V respectively, $\sigma_i = \Sigma_{i,i}$ and r is the last index such that $\sigma_i = \Sigma_{i,i} \neq 0$. Notice that the complex SVD also gives the answer over the complex numbers, however the algebraic SVD fails in some cases.

The same question for tensors, or also for polynomials, is considerably more difficult. One of the first methods to compute decomposition of higher order tensors has been developed by Sylvester in the XIX century for binary forms, that is, homogeneous polynomials in two variables. The first modern version of such method is presented in [CS11]. Moreover, in such work it has been studied the behavior of the strata of the given rank varieties of binary forms of subgeneric rank. Those varieties are defined as the Zariski closure of the set of all binary forms of a given rank r, that is, sum of powers of r linear forms. When the rank r is smaller than the generic rank it follows that this variety corresponds to the r-secant variety of the Veronese variety $\nu_d(\mathbb{P}^1)$, see Definitions 2.1.1 and 2.1.4. In chapter 7, in a joint work with Alejandro Gonazález Nevado, we will describe the strata of those varieties for the case of suprageneric rank, see Theorem 7.3.4. Furthermore, we will study the singular locus of such varieties and show that a behavior similar to the subgeneric rank case happens also in the suprageneric case: the previous step in the stratification is contained in the singular locus, see Theorem 7.3.5.

Although the celebrated algorithm of Sylvester gives an efficient answer for binary forms, the question for higher dimensions is not fully answered. Efficient algorithms for the decomposition of low rank symmetric tensors have been recently developed [Bra+10; OO13], however a general method is still unknown. We will discuss the algorithm developed by [OO13] in section 6.1 and exploit the work developed in [CCO17] on the Waring locus of plane cubics to extend this algorithm to a general plane cubic. We present an implementation of the algorithm for general plane cubics in the language Macaulay2 in section 6.3.

Chapter 2

Preliminaries

2.1 Preliminaries on tensors

In this section we introduce the main concepts regarding tensors, as for example the Veronese and the Segre varieties, Terracini lemma. We suggest [Lan12] as an introductory book on this area. Let V be a complex vector space of dimension $\dim(V) = m + 1$.

2.1.1 Symmetric tensors and the Veronese variety

Definition 2.1.1. Let $d \in \mathbb{Z}_{\geq 1}$ be an integer. We define the d-Veronese embedding ν_d by

$$\begin{aligned} \nu_d : & \mathbb{P}V \to \mathbb{P}\mathrm{Sym}^d V \\ & [v] \mapsto [v^d] \end{aligned}$$

The d-Veronese embedding is a closed map, thus its image is a variety. We call the variety $\nu_d(\mathbb{P}V)$ the Veronese variety (d-Veronese variety).

Definition 2.1.2. Let $f \in \mathrm{Sym}^d V$ be a homogeneous polynomial of degree d. f is said to be a decomposable polynomial if there exists a linear form $v \in V$ such that $f = v^d$.

Observe that the Veronese variety, by definition, consists exactly of the decomposable polynomials in $\mathrm{Sym}^d V$.

Definition 2.1.3. A polynomial $f \in \mathrm{Sym}^d V$ has Waring rank (also named symmetric rank) one if it is decomposable. f has rank r if r is the minimal number such that f is a linear combination of decomposable polynomials, i.e.

$$\mathrm{rank}(f) = \min\{r \in \mathbb{Z}_{\geq 1} \mid f = \sum_{i=1}^{r} v_i^d,\ v_i \in V\}.$$

Such minimal decomposition is called a Waring decomposition of f.

Consider the set $S_{d,r} = \{f \in \mathrm{Sym}^d V \mid \mathrm{rank}(f) = r\}$ of polynomials of a given rank r. This set is not Zariski closed, thus it is not a variety. In order to be able to have geometrical techniques to study this rank we define the secant variety.

Definition 2.1.4. Let $X \subset \mathbb{P}V$ be a variety. The r-secant variety of X, denoted $\Sigma_r(X)$ is defined as

$$\Sigma_r(X) = \overline{\bigcup_{x_1,\dots,x_r \in X} \{x \in \langle x_1, \dots, x_r\rangle\}}$$

Observe that if we let $X = \nu_d(\mathbb{P}V)$ we have that the general point on the r-secant variety of the Veronese variety is a polynomial of rank r. The secant varieties of the Veronese variety is the main tool to study the rank of a polynomial.

We recall the classic Terracini lemma that gives a powerful tool to understand the secant varieties.

Lemma 2.1.5 (Terracini)**.** *Let $X \subset \mathbb{P}V$ be a subvariety, then*

$$\langle \mathrm{T}_{x_1}X, \dots, \mathrm{T}_{x_r}X\rangle = \mathrm{T}_x\Sigma_r(X),$$

where $x_1, \dots, x_r \in X$ and $x \in \langle x_1, \dots, x_r\rangle$ are general.

Moreover,

$$\dim(\Sigma_r(X)) = \dim\langle \mathrm{T}_{x_1}X, \dots, \mathrm{T}_{x_r}X\rangle$$

Observe that to compute the dimension of the secants varieties a naive approach consists of counting the number of parameters appearing in a rank r decomposition, by doing so we have that the expected dimension of the r-secant variety is

$$\dim(\Sigma_r(X)) = \min\{r(\dim(X)+1)-1, \dim(\mathbb{P}V)\}.$$

In particular for the r-secant variety of the Veronese variety $\Sigma_r(\nu_d(\mathbb{P}V))$ we have that the expected dimension is

$$\dim(\Sigma_r(\nu_d(\mathbb{P}V))) = \min\{r(m+1)-1, \dim(\mathbb{P}\mathrm{Sym}^d V)\}.$$

Moreover, the expected minimum is obtained at $r(m+1)-1$ if

$$r < \left\lceil \frac{\binom{m+d}{d}}{m+1} \right\rceil.$$

When $r = g = \left\lceil \frac{\binom{m+d}{d}}{m+1} \right\rceil$ it is expected that the r-secant variety of the Veronese variety fulfils the ambient space, such number is called the generic rank of $\mathrm{Sym}^d V$ and a general polynomial in $\mathrm{Sym}^d V$ has rank g. Moreover, for $r \geq g$ it is trivial to see that the secant variety is equal to the ambient space $\mathrm{Sym}^d V$, thus the secant variety techniques are powerful for the subgeneric rank.

We say that the dimension is expected to be given by this value because the celebrated Alexander-Hirschowitz theorem shows that there exists finitely many cases where the dimension is not the expected one through our naive approach. Indeed, by utilising Terracini's lemma it is easy to verify the first defective examples.

Theorem 2.1.6 (Alexander-Hirschowitz)**.** *Let V be a complex vector space of dimension $m+1$. Then the generic rank in $\mathbb{P}\mathrm{Sym}^d V$ is given by*

$$g = \left\lceil \frac{\binom{m+d}{d}}{m+1} \right\rceil$$

with the exception of the following cases:

- *$d = 2$ where the generic rank is $m+1$;*
- *$2 \leq m \leq 4$, $d = 4$, where the generic rank is $\binom{m+2}{2}$;*
- *$(m, d) = (4, 3)$ where the generic rank is 8.*

Moreover, the dimension of the secant variety $\Sigma_r(\nu_d(\mathbb{P}V))$ is equal to the expected value $\min\{r(m+1) - 1, \dim(\mathbb{P}\mathrm{Sym}^d V)\}$ *for $r \leq g$ with the exception of the following cases:*

- *$d = 2$, $2 \leq r \leq m$;*
- *$2 \leq m \leq 4$, $d = 4$, $r = \binom{m+2}{2} - 1$;*
- *$(m, d) = (4, 3)$, $r = 7$.*

We refer to [BO08] as a reference to a clear explanation of the importance and the proof of Alexander-Hirschowitz theorem.

If we let $S_{d,r} = \{f \in \mathrm{Sym}^d V \mid \mathrm{rank}(f) = r\}$ be the set of polynomials of rank r, we have that $\overline{S_{d,r}} = \Sigma_r(\nu_d(\mathbb{P}V))$ when $r \leq g$. For rank $r > g$ the variety $\overline{S_{d,r}}$ cannot be related to the r-secant variety of the Veronese variety since the last coincides with the ambient space. We study the variety $\overline{S_{d,r}}$ for binary forms of suprageneric rank on section 7.

We notice that the secant variety itself describe a different notion of rank. Since $\Sigma_g(\nu_d(\mathbb{P}V)) = \mathrm{Sym}^d V$, it means that also the polynomials of rank higher than g belong to some r-secant variety of the Veronese variety.

Definition 2.1.7. The border Waring rank, denoted rank_B, of a polynomial $f \in \mathrm{Sym}^d V$ is defined as the minimal number $1 \leq r \leq g$ such that

$$f \in \Sigma_r(\nu_d(\mathbb{P}V)).$$

Equivalently, the border rank of f may be defined as the minimal number r such that there exists a sequence of polynomials $\{f_n\}$ converging to f such that $\mathrm{rank}(f_n) = r$.

Example 2.1.8. Consider the rank 3 polynomial $f = x^2y$, indeed it can be shown that its minimal decomposition is given by

$$f = x^2y = \frac{1}{6}\left((x+y)^3 - (x-y)^3 - 2y^3\right).$$

On the other hand, consider the sequence $f_n = \frac{1}{3}\left(n\left(x + \frac{y}{n}\right)^3 - nx^3\right)$. Clearly f_n has rank 2 for every $n \geq 1$. Expanding the polynomial we obtain

$$f_n = x^2y + \frac{1}{n}xy^2 + \frac{1}{3n^2}y^3,$$

thus

$$\lim_{n\to\infty} f_n = f$$

and f has border rank 2.

2.1.2 Tensors and the Segre variety

We recall that a homogeneous polynomial in $n+1$ variables can be associated to a symmetric tensor. Now we extend the notions described above to tensors. Let $V_1, \dots, V_k$ be complex vector spaces of dimension $\dim(V_i) = m_i + 1$, $i = 1, \dots, k$.

Definition 2.1.9. We define the Segre embedding σ as the map

$$\begin{aligned} \sigma :& \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k \to \mathbb{P}(V_1 \otimes \cdots \otimes V_k) \\ & ([v_1], \dots, [v_k]) \longmapsto [v_1 \otimes \cdots \otimes v_k]. \end{aligned}$$

The Segre embedding is a closed map, therefore its image is a variety. We call the variety $\sigma(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k)$ the Segre variety.

If there is no risk of confusion between the variety and the map we will denote $\sigma(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k) = \sigma$ for the sake of simplicity.

Definition 2.1.10. A tensor $T \in V_1 \otimes \cdots \otimes V_k$ is said to be decomposable if

$$T = v_1 \otimes \cdots \otimes v_k,$$

for some $v_i \in V_i$, $i = 1, \dots, k$.

The Segre variety is the analogous of the Veronese variety for tensors, it consists exactly of the decomposable tensors of $V_1 \otimes \cdots \otimes V_k$.

Definition 2.1.11. Let $v_1 \otimes \cdots \otimes v_k \in V_1 \otimes \cdots \otimes V_k$ be a decomposable tensors, then we say it has rank one. If $T \in V_1 \otimes \cdots \otimes V_k$ and r is the minimal number such that T is a linear combination of r decomposable tensors, then we say that $\operatorname{rank}(T) = r$. More precisely

$$\operatorname{rank}(T) = \min\{r \in \mathbb{Z}_{\geq 1} \mid T = \sum_{i=1}^{r} v_{i,1} \otimes \cdots \otimes v_{i,k}\}.$$

Similar to the polynomial case, we are able to study the varieties of a given rank r by means of the r-secant variety of the Segre variety. Observe that, by utilising again the naive approach of counting the number of parameters appearing in a rank r decomposition we obtain that the expected dimension of $\Sigma_r(\sigma)$ is given by

$$\dim(\Sigma_r(\sigma)) = \min\left\{ r\left(\left(\sum_{i=1}^{k} m_i\right) + 1\right) - 1, \prod_{i=1}^{k}(m_i + 1) - 1\right\}.$$

Moreover, this implies that the expected generic rank g in the tensor space $V_1 \otimes \cdots \otimes V_k$ is given by

$$g = \left\lceil \frac{\prod_{i=1}^{k}(m_i + 1)}{\left(\sum_{i=1}^{k} m_i + 1\right)} \right\rceil$$

In the case of tensors there is not a counterpart of Alexander-Hirschowitz theorem 2.1.6. At the current moment only few exceptions are known for when the generic rank and thus the expected dimension of the r-secant variety of the Segre variety are defective. The next result enumerates the state-of-art result on this defectiveness.

Example 2.1.12. The known examples where a general $T \in V_1 \otimes \cdots \otimes V_k$, with $k \geq 3$, has defective generic rank, that is, different from

$$g = \left\lceil \frac{\prod_{i=1}^{k}(m_i + 1)}{\left(\sum_{i=1}^{k} m_i\right) + 1} \right\rceil,$$

happens in the cases:

1. The unbalanced case, i.e., when it holds

$$m_k \geq \prod_{i=1}^{k-1}(m_i + 1) - \left(\sum_{i=1}^{k-1} m_i\right) + 1.$$

2. $k = 3$, $(m_1, m_2, m_3) = (2, 2, m)$ with even m [Lic85].

3. $k = 3$, $(m_1, m_2, m_3) = (2, 3, 3)$, sporadic case [AOP06].

4. $k = 4$, $(m_1, m_2, m_3, m_4) = (1, 1, n, n)$ [AOP06].

Moreover, the asymptotically behavior of the case of general rank for the case $\mathbb{C}^{m+1} \otimes \cdots \otimes \mathbb{C}^{m+1}$ has been studied.

Theorem 2.1.13. *[AOP06] Let $g(m^k)$ be the generic rank of the tensor space $\bigotimes_{i=1}^k \mathbb{C}^m$. Then*

$$g(m^k) \sim \frac{(m+1)^k}{mk+1}$$

when $m \to \infty$ or $k \to \infty$.

As mentioned previously, a polynomial $f \in \mathrm{Sym}^d V$ can be considered as a symmetric tensor inside of $\bigotimes_{i=1}^d V$. A natural question is to compare the rank of f as a polynomial and as a tensor.

In order to separate the notion of Waring rank and tensor rank of f, when needed we will denote the Waring rank of f by $\mathrm{rank}_W(f)$ and the tensor rank simply by $\mathrm{rank}(f)$.

Notice that since the Waring decomposition is a particular decomposition of a tensor, that is, it is a decomposition restricted to symmetric tensors, we have that

$$\mathrm{rank}(f) \leq \mathrm{rank}_W(f).$$

Many cases and examples where such equality holds true have been found, this led to the famous Comon's conjecture: *The Waring rank of a symmetric tensor is equal to its tensor rank.* Such conjecture has been thought to be true for several years.

However, recently the first example of when such equality does not hold has been found [Shi18] for a symmetric tensor f of order $800 \times 800 \times 800$ and $\mathrm{rank}(f) \leq 903$, but that cannot be written as a sum of at most 903 decomposable symmetric tensors.

2.1.3 Partially symmetric tensors and the Segre-Veronese variety

The last tensor format that we are going to be interested throughout the text is the so called partially symmetric tensors.

Definition 2.1.14. We say that a tensor $T \in \bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i$ is a partially symmetric tensor. We define the Segre-Veronese embedding $\nu_{d_1 \ldots d_k}$ as the map

$$\begin{aligned} \nu_{d_1 \ldots d_k} : & \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k \to \mathbb{P}\left(\bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i\right) \\ & ([v_1], \ldots, [v_k]) \longmapsto [v_1^{d_1} \otimes \cdots \otimes v_k^{d_k}]. \end{aligned}$$

Definition 2.1.15. A partially symmetric tensor $T \in \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$ is decomposable if

$$T = v_1^{d_1} \otimes \cdots \otimes v_k^{d_k},$$

for some $v_i \in V_i$.

Notice that the Segre-Veronese variety consists exactly of the partially symmetric tensors that are decomposable. Moreover, the notion of decomposable as well the Segre-Veronese embedding, as the name indicates, is a combination of both the Segre and the Veronese embeddings.

Definition 2.1.16. A tensor $T \in \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$ has rank one if it is decomposable. T has rank r if r is the minimal number of decomposable tensors $v_{i,1} \otimes \cdots \otimes v_{i,k}$ such that T is a linear combination of them. That is,

$$\mathrm{rank}(T) = \min\{r \in \mathbb{Z}_{\geq 1} \mid T = \sum_{i=1}^{r} v_{i,1}^{d_1} \otimes \cdots \otimes v_{i,k}^{d_k}\}.$$

We will work with partially symmetric tensors on section 4.2.

2.2 Preliminaries on Cohomology and Bott Theorem

This section follows mainly the lectures notes of Giorgio Ottaviani [Ott95]. Another suggestions for an introduction on this topic are the books [Wey03] and [FH91].

During this section let V be a $m+1$-dimensional complex vector space.

2.2.1 Lie groups and Lie algebras

Let $v \neq 0 \in V$, we consider the map

$$\varphi_i : \bigwedge^{i} V \to \bigwedge^{i+1} V,\ \varphi_i(w) = w \wedge v.$$

Such map is called the Koszul maps.

Definition 2.2.1. The exact sequence

$$0 \to \bigwedge^{0} V \cong \mathbb{C} \xrightarrow{\varphi_0} \bigwedge^{1} V \xrightarrow{\varphi_1} \bigwedge^{2} V \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} \bigwedge^{n+1} V \to 0$$

Is called the Koszul complex of V. The Koszul complex is an exact sequence.

First notice that the Koszul complex is a complex. Indeed $\varphi_i \circ \varphi_{i-1}(w) = \varphi_i(w \wedge v) = w \wedge v \wedge v = 0$. Furthermore the Koszul complex is an exact sequence: Let $\{e_1, \ldots, e_m, e_{m+1}\}$ be a basis of V with $e_{m+1} = v$. Let $w =$

$\sum_{i_1<\dots<i_r} a_{i_1\dots i_r} e_{i_1} \wedge \dots \wedge e_{i_r} \in \bigwedge^r V$ such that $\varphi_r(w) = w \wedge v = 0$. This implies that all the non-zero coefficients $a_{i_1\dots i_r}$ have the last index $i_r = m+1$. Therefore $\varphi_{r-1}(u) = u \wedge v = w$ for $u = \sum_{i_1<\dots<i_{r-1}} a_{i_1\dots i_{r-1}} e_{i_1} \wedge \dots \wedge e_{i_{r-1}}$.

The Koszul complex can be extended for vector bundles $\mathcal{F}$ over a projective variety X.

Definition 2.2.2. Let $\mathcal{F}$ be a rank m vector bundle over X and let $s \in H^0(X, \mathcal{F})$ be a global section such that the zero locus of the section $Z_s = \{x \in X | s(x) = 0\}$ is zero-dimensional, that is $\mathrm{codim}(Z_s) = m$. Let $\varphi_i : \bigwedge^i \mathcal{F} \to \bigwedge^{i+1} \mathcal{F}$ be defined by $\varphi_i(w) = w \wedge s$ and the dual map $\varphi_i^t : \bigwedge^{i+1} \mathcal{F}^* \to \bigwedge^i \mathcal{F}^*$. The exact sequence

$$0 \to \bigwedge^m \mathcal{F}^* \xrightarrow{\varphi_{m-1}^t} \bigwedge^{m-1} \mathcal{F}^* \xrightarrow{\varphi_{m-2}^t} \dots \xrightarrow{\varphi_1^t} \mathcal{F}^* \to \mathcal{I}_{Z_s} \to 0$$

is called the Koszul complex of the ideal sheaf $\mathcal{I}_{Z_s}$.

Notice that the classical presentation of the Koszul complex is using the coordinate ring of Z_s:

$$0 \to \bigwedge^m \mathcal{F}^* \xrightarrow{\varphi_{m-1}^t} \bigwedge^{m-1} \mathcal{F}^* \xrightarrow{\varphi_{m-2}^t} \dots \xrightarrow{\varphi_1^t} \mathcal{F}^* \xrightarrow{\varphi_0^t} \mathcal{O}_X \to \mathcal{O}_{Z_s} \to 0.$$

The sequence presented in the Definition 2.2.2 is obtained by considering the short exact sequence $0 \to I_{Z_s} \to \mathcal{O}_X \to \mathcal{O}_{Z_s} \to 0$.

Definition 2.2.3. A complex manifold G which is also a group and such that the map

$$G \times G \to G,\ (x, y) \mapsto xy^{-1}$$

is holomorphic is called a complex Lie group. In the category of Lie groups the morphisms are holomorphic maps that are group homomorphisms.

Definition 2.2.4. A complex vector space V with a map

$$[\cdot,\cdot] : V \times V \to V$$

that satisfies:

1. Bilinearity:

$$[\alpha x+y, \beta z+w] = \alpha\beta[x, z]+\alpha[x, w]+\beta[y, z]+[y, w]\ \forall\, x, y, z, w \in V,\ \alpha, \beta \in \mathbb{C};$$

2. Skew-symmetry:

$$[x, y] = -[y, x],\ \forall\ x, y \in V;$$

3. Jacobi-identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,\ \forall\ x, y, z \in V;$$

is called a Lie algebra. In the category of Lie algebras the morphisms are vector space morphisms that preserves the bracket $[\cdot,\cdot]$.

Example 2.2.5. On a complex Lie group G consider the holomorphic vector fields that are invariant by the left translation $L_g : g' \mapsto gg'$. The left translation is completely characterized by the value that assume at any point, in particular on the identity element $e \in G$. The vector space of left invariant vector fields is therefore isomorphic to the tangent space $\mathrm{T}_e G$.

Let $X, Y \in \mathrm{T}_e G$ be two left invariant vector fields, then $XY - YX$ is still a left invariant vector field. Therefore $\mathrm{T}_e G$ equipped with the the bracket $[X, Y] = XY - YX$ is a Lie algebra.

Definition 2.2.6. The Lie algebra associated to $\mathrm{T}_e G$, where G is a Lie group, is denoted $\mathrm{Lie}(G)$.

Example 2.2.7. The Lie algebra of $\mathrm{GL}(n)$, denoted $\mathcal{GL}(n)$, consist of the $n \times n$ matrices with the bracket defined by $[A, B] = A \cdot B - B \cdot A$.

Definition 2.2.8. Let V be a Lie algebra. A subalgebra $I \subset V$ is called an ideal if for all $v \in V$ and $i \in I$ it holds

$$[v, i] \in I.$$

The quotient space V/I has the natural structure of quotient Lie algebra.

Definition 2.2.9. A Lie algebra V is called solvable if the derived series

$$V_1 = [V, V],\ V_2 = [V_1, V_1],\ \ldots, V_i = [V_{i-1}, V_{i-1}]$$

terminates to zero. A Lie group G is solvable if $\mathrm{Lie}(G)$ is solvable.

Theorem 2.2.10. *Let G be a Lie group and let $\mathcal{H} \subset \mathrm{Lie}(G)$ be a subalgebra. Then there exists a connected Lie subgroup $H \subset G$ such that $\mathrm{Lie}(H) = \mathcal{H}$.*

Proof. [FH91, Proposition 8.41] □

Theorem 2.2.11. *Let $H \subset G$ be a closed subgroup. Then H is normal if and only if the subalgebra $\mathrm{Lie}(H)$ is an ideal.*

Proof. [NS82, IX §3]. □

Definition 2.2.12. A Lie algebra V is simple if $\dim V > 1$ and it contains only the trivial ideals 0 and V. A Lie algebra is semi-simple if it contains no nonzero solvable ideals. A Lie group G is semi-simple if and only if $\mathrm{Lie}(G)$ is semi-simple.

Notice that if I_1 and I_2 are solvable ideals of V, then also $I_1 + I_2$ is a solvable ideal. This implies that V has a unique maximal solvable ideal $\mathrm{rad}(V)$ that is called the radical ideal of V. Another characterization for semi-simple algebras is: V is semi-simple if and only if $\mathrm{rad}(V) = 0$.

We give a purely algebraic definition of $[\cdot,\cdot]$. For this we must define the adjoint representation that will be used later to define the Cartan subalgebras.

Definition 2.2.13. Let $\rho_g : G \to G$ be the inner automorphism, defined by an element g of the Lie group G, given by

$$\rho_g : h \mapsto ghg^{-1}.$$

We get a morphism

$$G \to \mathrm{Aut}(G),\ g \mapsto \rho_g.$$

Definition 2.2.14. Consider the derivative at the identity of ρ_g, that is,

$$(\partial\rho_g)_e : \mathrm{Lie}(G) \to \mathrm{Lie}(G).$$

We define the adjoint representation of G as

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G)),\ g \mapsto (\partial\rho_g)_e.$$

Definition 2.2.15. We define the adjoint representation of the Lie algebra $\mathrm{Lie}(G)$, denoted ad, by

$$\mathrm{ad} = (\partial\mathrm{Ad})_e : \mathrm{Lie}(G) \to \mathcal{GL}(\mathrm{Lie}G).$$

Definition 2.2.16. We may define the bracket $[\cdot,\cdot]$ of Definition 2.2.3 for a Lie group G in terms of the adjoint representation of $\mathrm{Lie}(G)$ as

$$[X, Y] = \mathrm{ad}(X)(Y).$$

2.2.2 Cartan subalgebra and the Killing form

We now have the basis to define the Cartan subalgebra. This is one of the main objects to understand Bott's Theorem 2.2.71, as it will allow us to define the notion of weights and roots.

Definition 2.2.17. Let $\mathcal{G}$ be a Lie algebra. A subalgebra $\mathcal{H} \subset \mathcal{G}$ is called abelian if

$$[h_1, h_2] = 0$$

for all $h_1, h_2 \in \mathcal{H}$.

Definition 2.2.18. A subalgebra $\mathcal{H} \subset \mathcal{G}$ is called a Cartan subalgebra if

1. $\mathcal{H}$ is abelian and $\mathrm{ad}|_{\mathcal{H}} : \mathcal{H} \to \mathcal{GL}(\mathcal{G})$ acts diagonally.
2. $\mathcal{H}$ is maximal with respect to (1).

Theorem 2.2.19. *In any semi-simple Lie algebra $\mathcal{G}$ there exists Cartan subalgebras $\mathcal{H}$.*

Proof. [FH91, Appendix D]. □

From now onwards we will denote $\mathcal{H}$ to be a fixed Cartan subalgebra of a Lie algebra $\mathcal{G}$.

Definition 2.2.20. For any $\alpha \in \mathcal{H}^*$, where $\mathcal{H}^*$ denotes the dual space of $\mathcal{H}$, we define

$$\mathcal{G}_\alpha = \{X \in \mathcal{G} \mid \mathrm{ad}(H)(X) = \alpha(H)(X),\ \forall\ H \in \mathcal{H}\}.$$

Notice that from the Definition 2.2.18 item (1) we have that $\mathcal{G}$ is decomposed as direct sum of the eigenspaces $\mathcal{G}_\alpha$.

Theorem 2.2.21. *Let $\alpha,\ \beta \in \mathcal{H}^*$. Then*

$$[\mathcal{G}_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta}.$$

Proof. Let $X \in \mathcal{G}_\alpha$, $Y \in \mathcal{G}_\beta$ and $H \in \mathcal{H}$. From the property (3) we have that

$$\begin{aligned}[H,[X,Y]] &= -[X,[Y,H]] - [Y,[H,X]] \\ &= [X, \beta(H)Y] - [Y, \alpha(H)X] \\ &= [\alpha(H) + \beta(H)[X,Y].\end{aligned}$$

□

The next proposition gives a description of $\mathcal{H}$ in terms of the eigenspaces.

Proposition 2.2.22.

$$\mathcal{H} = \mathcal{G}_0.$$

Proof. The inclusion $\mathcal{H} \subset \mathcal{G}_0$ is trivial. Indeed, let $H' \in \mathcal{H}$, then

$$\mathrm{ad}(H')(H) = [H, H'] = 0,\ \forall H \in \mathcal{H}$$

since $\mathcal{H}$ is abelian (Definition 2.2.18).

On the other hand, the inclusion can not be strict, otherwise we could enlarge $\mathcal{H}$ by adding the elements of $\mathcal{G}_0$ while satisfying (1) and this violates the maximality of $\mathcal{H}$. □

The last result implies that the decomposition of $\mathcal{G}$ mentioned after Definition 2.2.20 is given by

$$\mathcal{G} = \mathcal{H} \bigoplus_{\alpha \neq 0 \in \mathcal{H}^*} \mathcal{G}_\alpha.$$

Such decomposition is called the Cartan decomposition of the Lie algebra $\mathcal{G}$.

Definition 2.2.23. Let $\alpha \in \mathcal{H}^*$ be a non-zero element such that $\mathcal{G}_\alpha \neq 0$. Such α is called a *root* with respect to the Cartan subalgebra $\mathcal{H}$.

The set of roots is denoted by $\Phi \subset \mathcal{H}^*$. The eigenspaces $\mathcal{G}_\alpha$ are called the *root spaces* of $\mathcal{G}$.

Theorem 2.2.24. *If α is a root the also $-\alpha$ is a root.*

Proof. Suppose $-\alpha$ is not a root, then for all $X \in \mathcal{G}_\alpha$ and for some $H \in \mathcal{H}$ we have that $[-X, H] \neq -\alpha(H)(X)$, therefore $[X, H] \neq \alpha(H)(X)$, this is a contradiction.

□

By choosing a direction in $\mathcal{H}^*$ that is irrational with respect to the lattice generated by the roots, the previous result leads to a decomposition

$$\Phi = \Phi^+ \cup \Phi^-.$$

Such decomposition is called an ordering of the roots. Moreover it is trivial to see that $-\Phi^+ = \Phi^-$.

We introduce a bilinear form on the Lie algebra $\mathcal{G}$.

Definition 2.2.25. The bilinear form B is called the Killing form and defined as

$$B : \mathcal{G} \times \mathcal{G} \to \mathbb{C}, \ (X, Y) \mapsto \operatorname{tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)),$$

where $\operatorname{ad}(X) \circ \operatorname{ad}(Y)(Z) = [X, [Y, Z]] \in \mathcal{G}$.

From the definition of the trace operator and $[\cdot, \cdot]$ we have that B is both bilinear and symmetric.

Lemma 2.2.26. *Let α, β, $\gamma \in \mathcal{H}^*$ be roots. Then*

1. *Let $X \in \mathcal{G}_\alpha$ and $Y \in \mathcal{G}_\beta$, then*

$$\operatorname{ad}(X) \circ \operatorname{ad}(Y)(\mathcal{G}_\gamma) \subset \mathcal{G}_{\alpha+\beta+\gamma}.$$

2. *Let $\mathcal{Q}_\alpha = \mathcal{G}_\alpha \oplus \mathcal{G}_{-\alpha}$. Then the decomposition*

$$\mathcal{G} = \mathcal{H} \bigoplus_{\alpha \in \Phi^+} \mathcal{Q}_\alpha$$

is an orthogonal decomposition with respect to the Killing form.

Proof. For (1) we have that

$$\operatorname{ad}(X) \circ \operatorname{ad}(Y)(\mathcal{G}_\gamma) = [X, [Y, \mathcal{G}_\gamma]].$$

From Proposition 2.2.21, it follows that the containment holds.

For (2), suppose that $\alpha \neq -\beta$, from (1) we have that $\operatorname{ad}(X) \circ \operatorname{ad}(Y)(\mathcal{G}_\gamma)$ has no component coming from $\mathcal{G}_\gamma$, so the trace must be zero. □

Lemma 2.2.27. 1. *Let $X, Y, Z \in \mathcal{G}$,, then*

$$B([X, Y], Z) = B(X, [Y, Z]).$$

2. *For any ideal $\mathcal{I} \subset \mathcal{G}$ the orthogonal subspace*

$$\mathcal{I}^{\perp} = \{X \in \mathcal{G} \mid B(X,Y) = 0 \ \forall \ Y \in \mathcal{I}\}$$

is an ideal of $\mathcal{G}$.

Proof. For (1) we have

$$\begin{aligned} B([X,Y],Z) &= \mathrm{tr}(\mathrm{ad}(XY - YX) \circ \mathrm{ad}(Z)) \\ &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z) - \mathrm{ad}(Y) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Z)) \\ &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z) - \mathrm{ad}(X) \circ \mathrm{ad}(Z) \circ \mathrm{ad}(Y)), \end{aligned}$$

where on the last equality it was used that $\mathrm{tr}(CD) = \mathrm{tr}(DC)$.

On the other hand we have

$$\begin{aligned} B(X,[Y,Z]) &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(YZ - ZY)) \\ &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z) - \mathrm{ad}(X) \circ \mathrm{ad}(Z) \circ \mathrm{ad}(Y)) \\ &= B([X,Y],Z). \end{aligned}$$

Therefore the desired equality holds.

We prove (2) now. Let $X \in \mathcal{I}^{\perp}$, we wish to show that for all $Y \in \mathcal{G}$ we have that $[X,Y] \in \mathcal{I}^{\perp}$. Let $Z \in \mathcal{I}$, then

$$B([X,Y],Z) = B(X,[Y,Z]) = 0,$$

since $[Y,Z] \in \mathcal{I}$ because $\mathcal{I}$ is an ideal. □

Theorem 2.2.28 (Cartan's Criterion)**.** *Let $\mathcal{L} \subset \mathcal{G}$ be a subalgebra of the Lie algebra $\mathcal{G}$. Suppose that $B(X,Y) = 0$ for all $X,\ Y \in \mathcal{L}$, then $\mathcal{L}$ is solvable.*

Proof. [FH91, Theorem C5]. □

Theorem 2.2.29. *The Lie algebra $\mathcal{G}$ is semi-simple if and only if the Killing form B is non-degenerate.*

Proof. Suppose $\mathcal{G}$ is semi-simple, then $\mathrm{rad}(\mathcal{G}) = 0$. Let $\mathcal{G}^{\perp} = \{X \in \mathcal{G} \mid B(X,Y) = 0 \ \forall \ Y \in \mathcal{G}\}$, this is is an ideal by Lemma 2.2.27. From Theorem 2.2.28 we have that $\mathcal{G}^{\perp}$ is solvable. This implies that $\mathcal{G}^{\perp} = 0$.

On the other hand, suppose that $\mathcal{G}^{\perp} = 0$ but $\mathrm{rad}(\mathcal{G}) \neq 0$. This means that the last term $\mathcal{G}_r$ in the sequence $\mathcal{G}_i = [\mathcal{G}_{i-1}, \mathcal{G}_{i-1}]$ is an abelian ideal, since $[\mathcal{G}_r, \mathcal{G}_r] = 0$.

Notice that from the assumption that $\mathcal{G}$ is not semi-simple we have that it is non-trivial. Let $X \in \mathcal{G}_r$ and $Y \in \mathcal{G}$, then $\mathrm{ad}(X) \circ \mathrm{ad}(Y) : \mathcal{G} \to \mathcal{G}_r$. This means that the image of $\mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y)$ belongs to $[\mathcal{G}_r, \mathcal{G}_r] = 0$.

This implies that $\mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) = 0$, therefore $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ is nilpotent thus $\mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = 0$. This implies that $\mathcal{G}_r \subset \mathcal{G}^{\perp}$, however it was trivial by hypothesis and this is a contradiction, so $\mathcal{G}$ must be semi-simple. □

Theorem 2.2.30. *A semi-simple algebra is a direct sum of simple Lie subalgebras.*

Proof. Let $\mathcal{I}$ be an ideal of $\mathcal{G}$. From Lemma 2.2.27 we have that $\mathcal{I}^{\perp}$ is also an ideal. We have that $\mathcal{I} \cap \mathcal{I}^{\perp}$ is solvable by Theorem 2.2.28, thus the intersection is trivial. It follows $\mathcal{G} = \mathcal{I} \oplus \mathcal{I}^{*}$. The result follows by considering a minimal ideal $\mathcal{I}$, so that we have no non-trivial ideals on $\mathcal{I}$ and proceed in the same manner on $\mathcal{I}^{\perp}$. □

Lemma 2.2.31. *The root system $\Phi \subset \mathcal{H}^{*}$ spans $\mathcal{H}^{*}$.*

Proof. Suppose it is not true. This means that there exists some $X \in \mathcal{H}$ such that $\alpha(X) = 0$ for all the roots $\alpha \in \mathcal{H}^{*}$. This is the same to say that $[X, \mathcal{G}_{\alpha}] = 0$ for all $\mathcal{G}_{\alpha}$. This means that X is in the center $Z(\mathcal{G}_{\alpha})$ that is solvable. However, since $\mathcal{G}$ is semi-simple we have that $Z(\mathcal{G}_{\alpha}) = 0$, therefore $X = 0$ that is a contradiction. □

We now show that there exists a copy of $\mathcal{SL}(2)$ inside of any semi-simple Lie algebra $\mathcal{G}$. In order to prove that we recall that Lie's theorem say that a representation ρ of a solvable linear algebraic group G can be written as an upper triangular for all $g \in G$, that is, $\rho(g)$ is an upper triangular matrix.

Lemma 2.2.32. *Let $X \in \mathcal{G}_{\alpha}$, $Y \in \mathcal{G}_{-\alpha}$ such that $B(X, Y) \neq 0$. Then $[X, Y]$, X and Y span a subalgebra $\mathcal{S} \subset \mathcal{G}$ isomorphic to $\mathcal{SL}(2)$.*

Proof. The first thing is to notice that from Lemma 2.2.26 and Theorem 2.2.30 we have that such X and Y exist.

From Lemma 2.2.27 it holds that for all $H \in \mathcal{H}$

$$
\begin{aligned}
B\left(H, [X, Y]\right) &= B\left([H, X], Y\right) \\
&= B\big(\mathrm{ad}(H)(X), Y\big) \\
&= \alpha(H) B(X, Y).
\end{aligned}
$$

Moreover, by definition we have that

$$
[[X, Y], X] = \mathrm{ad}([X, Y])(X) = \alpha([X, Y])X
$$

For those two identities we are using the fact that $[\mathcal{G}_{\alpha}, \mathcal{G}_{-\alpha}] \subset \mathcal{G}_0 = \mathcal{H}$. We now wish to show that $\alpha([X, Y]) \neq 0$.

Suppose that $[X, Y] = 0$, then $\mathcal{S} \cong \mathrm{ad}(\mathcal{S}) \subset \mathcal{SL}(\mathcal{G})$ is a solvable subalgebra. Using Lie's theorem we have that all the elements of $\mathrm{ad}(\mathcal{S})$ are of upper triangular, thus $\mathrm{ad}[X, Y]$ is an upper triangular matrix. On the other hand the elements of $\mathcal{H}$ are diagonalisable, thus $\mathrm{ad}[X, Y] = 0$ that is a contradiction with our assumption that $[X, Y] \neq 0$.

Let $H = [X, Y]$, the multiplication table of $\mathcal{SL}(2)$ is given by $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. Adjusting by scalars to obtain this same multiplication table on $\mathcal{S}$ we have $\alpha([X, Y]) = 2$. □

Lemma 2.2.33. *Let α be a root.*

1. *For $k \in \mathbb{Z}$, $k \neq 1, -1$, then $k\alpha$ is not a root.*

2. $\dim \mathcal{G}_\alpha = 1$.

Proof. [Ott95, Lemma 6.16]. □

The Lemmas 2.2.32 and 2.2.33 imply that the subalgebra $\mathcal{S}$ isomorphic to $\mathcal{SL}(2)$ is determined by α. Therefore, when we refer to the copies of $\mathcal{SL}(2)$ in $\mathcal{G}$ in the future we will denote it by $\mathcal{S}_\alpha$. Furthermore, the elements H, X, Y in the proof of Lemma 2.2.32 describing the multiplication table of $\mathcal{S}_\alpha$ will be denoted H_α, X_α, Y_α.

We now explore the relations between the Cartan subalgebra $\mathcal{H}$ and the representations of $\mathcal{G}$.

Definition 2.2.34. Let $\rho : \mathcal{G} \to \mathcal{GL}(V)$ be a representation of $\mathcal{G}$. For any $\lambda \in \mathcal{H}^*$, denote

$$V_\lambda = \{v \in V | \rho(H)(v) = \lambda(H)v \ \forall \ H \in \mathcal{H}\}.$$

Theorem 2.2.35. *Let $\alpha \in \Phi$ be a root, $\lambda \in \mathcal{H}^*$ and $\rho : \mathcal{G} \to \mathcal{GL}(V)$. We have that*

$$\rho(\mathcal{G}_\alpha)V_\lambda \subset V_{\alpha+\lambda}.$$

Proof. Let $X \in \mathcal{G}_\alpha$, $v \in V_\lambda$ and $H \in \mathcal{H}$. We wish to show that $\rho(X)v \in V_{\alpha+\lambda}$.

$$\begin{aligned}\rho(H)(\rho(X)v) &= \rho([H,X])v + \rho(X)\rho(H)v \\ &= \alpha(H)\rho(X)v + \lambda(H)\rho(X)v \in V_{\alpha+\lambda}.\end{aligned}$$

□

Definition 2.2.36. An element $\lambda \in \mathcal{H}^*$ such that $V_\lambda \neq 0$ is called a weight of the representation ρ such that $\rho(H)v = \lambda(H)v$. The spaces V_λ are called the weight spaces.

Theorem 2.2.37. *The vector space V is a direct sum of its weight spaces, that is,*

$$V = \bigoplus V_\lambda.$$

Proof. [Ott95, Theorem 6.22] □

Corollary 2.2.38. *Let $\rho : \mathcal{G} \to \mathcal{GL}(V)$ be a representation and λ a weight of ρ. Then $\lambda(H_\alpha) \in \mathbb{Z}$ for every root α.*

Definition 2.2.39. The set

$$\Lambda_W = \{\beta \in \mathcal{H}^* | \beta(H_\alpha) \in \mathbb{Z}\}$$

is called the weight lattice of $\mathcal{G}$.

2.2.3 Weyl Group

The Weyl Group will play a major role in the Bott's Theorem 2.2.71. The cohomological spaces that are nonvanishing will correspond to the spaces associated to weights in the fundamental Weyl chambers.

Proposition 2.2.40. *The hyperplane Ω_α defined as*

$$\Omega_\alpha = \{\beta | \beta(H_\alpha) = 0\}$$

is the hyperplane orthogonal to α.

Proof. Observer that by duality we may state the proposition as: H_α is orthogonal to $\ker \alpha$. Let $H \in \ker \alpha$. Using Lemma 2.2.27 we have

$$\begin{aligned} B(H_\alpha, H) &= B([X_\alpha, Y_\alpha], H) \\ &= B(X_\alpha, [Y_\alpha, H]) \\ &= B(X_\alpha, \alpha(H) Y_\alpha) \\ &= B(X_\alpha, 0) = 0. \end{aligned}$$

□

Definition 2.2.41. The Weyl group is defined as the subgroup in $GL(\mathcal{H}^*)$ generated by the orthogonal reflections w_α with respect to Ω_α

$$w_\alpha(\beta) = \beta - 2\frac{B(\alpha,\beta)}{B(\alpha,\alpha)}\alpha.$$

Observe that we may write such product as well as

$$w_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha.$$

Indeed to see this equality we have to show that $\beta - \frac{1}{2}\beta(H_\alpha)\alpha \in \Omega_\alpha$. We have that

$$\beta(H_\alpha) - \frac{1}{2}\beta(H_\alpha)\alpha(H_\alpha) = \beta(H_\alpha)(1 - \frac{1}{2}\alpha(H_\alpha)) = 0$$

since $\alpha(H_\alpha) = 2$. With this we get the following corollary.

Corollary 2.2.42.

$$\beta(H_\alpha) = 2\frac{B(\alpha,\beta)}{B(\alpha,\alpha)}.$$

Theorem 2.2.43.

1. *The set of weights of any representation of $\mathcal{G}$ is invariant under the action of the Weyl group.*

2. *Let α be a root. If λ is a weight for some $\mathcal{G}$-module V, then in the infinite sequence*

$$\dots, -\alpha + \lambda, \lambda, \lambda + \alpha, \dots$$

the string of weights for V is a connected set. If λ' is the right extreme of this string, then the string has length $\lambda'(H_\alpha) + 1$. In other words, after replacing λ by $\lambda + k\alpha$ for a convenient k the string of weights is

$$w_\alpha(\lambda) = \lambda - \lambda(H_\alpha)\alpha, \dots, \lambda - \alpha, \lambda.$$

Proof. [Ott95, Theorem 6.30] □

Definition 2.2.44. The fundamental Weyl chamber $\mathcal{C}$ is the convex set

$$\mathcal{C} = \{\gamma \in \mathcal{H}^* | B(\gamma, \alpha) \geq 0 \ \forall \ \alpha \in \Phi^+\}.$$

Theorem 2.2.45. *The Weyl group acts simply and trasitively on the set of orderings and likewise on the set of Weyl chambers.*

Proof. [FH91, Proposition D.29 and Corollary D.32]. □

Definition 2.2.46. Let $\rho : \mathcal{G} \to \mathcal{GL}(V)$ be a representation of $\mathcal{G}$. A nonzero vector $v \in V$ is called highest weight vector of ρ if it satisfies two properties:

1. $\rho(\mathcal{G}_\alpha)(v) = 0$ for all $\alpha \in \Phi^+$.

2. v is an eigenvector for the the action of $\mathcal{H}$. If $\rho(H)(v) = \lambda(H)v$ for $\lambda \in \mathcal{H}^*$, then λ is called a highest weight.

Proposition 2.2.47. *All the representations of $\mathcal{G}$ have a highest weight vector.*

Proof. [Ott95, Proposition 6.34] □

Theorem 2.2.48. *A representation of $\mathcal{G}$ is irreducible if and only if it has a unique highest weight vector.*

Proof. [Ott95, Theorem 6.36] □

Theorem 2.2.49. *Let $\rho : \mathcal{G} \to \mathcal{GL}(V)$, $\rho' : \mathcal{GL}(V')$ be two irreducible representations. Let λ and λ' be the respective highest weights. Then $\rho \cong \rho'$ if and only if $\lambda = \lambda'$.*

Proof. One side of the statement is trivial. For the other, let $v \in V$ and $v' \in V'$ be the two highest weight vectors. Then $(v, v') \in V \oplus V'$ is the highest weight vector with weight λ for $\rho \oplus \rho'$. Let $U \subset V \oplus V'$ be the irreducible representation generated by (v, v'). The projections $\pi_1 : U \to V$ and $\pi_2 : U \to V'$ are both nonzero, thus by Schur's lemma they must be isomorphisms. It follows that $V \cong V' \cong U$ and $\rho \cong \rho'$. □

Proposition 2.2.50. *The highest weight λ of an irreducible representation lies in the fundamental Weyl chamber $\mathcal{C}$.*

Proof. Suppose that it is not the case. Then there exists a positive root $\alpha \in \Phi^+$ such that $B(\alpha, \lambda) < 0$. By Theorem 2.2.43 we have that $\lambda' = w_\alpha(\lambda) = \lambda - \lambda(H)\alpha$ is a weight too. Since $\lambda(H_\alpha) < 0$ we have that λ is not the highest weight, a contradiction. □

Theorem 2.2.51. *For all weight $\lambda \in \mathcal{C} \cap \Lambda_W$ there exists an irreducible representation $V_\lambda \in \mathcal{G}$ with highest weight λ.*

Proof. [Ott95, Theorem 6.40]. □

Definition 2.2.52. A positive root $\alpha \in \Phi^+$ is simple if α is not the sum of two positive roots.

Notice that if α_1 and α_2 are two distinct positive simple roots, then $B(\alpha_1, \alpha_2) \leq 0$. Indeed, if it was not the case we have that $\alpha_1(H_{\alpha_2}) > 0$, thus by Theorem 2.2.43 we have that $\alpha_1 - \alpha_2$ is a root. It cannot be positive since we have $\alpha_1 = \alpha_2 + (\alpha_1 - \alpha_2)$ and α_1 is simple. It cannot be negative either since $\alpha_2 = -\alpha_1 + (\alpha_1 - \alpha_2)$ and α_2 is simple.

Proposition 2.2.53. *Let $\alpha_1, \ldots, \alpha_{m+1}$ be the simple positive roots. Then $\{H_{\alpha_i}\}_{i=1}^m$ generates $\mathcal{H}$.*

Proof. We recall that $H_{\alpha_i} = [X_{\alpha_i}, Y_{\alpha_i}]$. Moreover, $B(H_{\alpha_i}, H) = \alpha_i(H)B(X_{\alpha_i}, Y_{\alpha_i})$ for every $H \in \mathcal{H}$. The isomorphism $\mathcal{H}^* \to \mathcal{H}$ induced by B is defined by

$$\alpha \mapsto T_\alpha,$$

where $B(T_\alpha, H) = \alpha(H)$ for every $H \in \mathcal{H}$. Using the above relations we have that

$$T_\alpha = \frac{H_\alpha}{B(X_\alpha, Y_\alpha)}.$$

This is a multiple of H_α that proves the result. □

Definition 2.2.54. The fundamental weights $\lambda_i \in \mathcal{H}^*$ are the dual basis of H_{α_i}, where α_i are the simple roots. Thus this is equivalent to say that $\lambda_i \in \mathcal{H}^*$ are the weights such that

$$\lambda_i(H_{\alpha_j}) = \delta_{i,j}.$$

2.2.4 Borel and Parabolic subgroups

Proposition 2.2.55. *Let $\mathcal{B}$ be a Lie subalgebra of $\mathcal{G}$ defined as*

$$\mathcal{B} = \mathcal{H} \bigoplus \left(\oplus_{\alpha \in \Phi^+} \mathcal{G}_\alpha \right).$$

Then $\mathcal{B}$ is a maximal solvable Lie algebra.

Proof. From Theorem 2.2.21 we have that $\mathcal{B}$ is solvable. Suppose that there exists another solvable subalgebra $\mathcal{B}' \supset \mathcal{B}$. This means that $\mathcal{B}'$ has to contain some subalgebra associated to a negative root $\mathcal{G}_{-\alpha}$. Hence $\mathcal{B}' \supset \mathcal{S}_\alpha \cong \mathcal{SL}(2)$ which implies

$$[\mathcal{S}_\alpha, \mathcal{S}_\alpha] = \mathcal{S}_\alpha.$$

Therefore, $\mathcal{B}'$ cannot be solvable because $\mathcal{S}_\alpha \neq 0$. □

A subalgebra as described in the last proposition is called a *Borel* subalgebra. If there exists a Lie group G such that $\mathcal{G} = \mathrm{Lie}(G)$ and a subgroup B such that $\mathrm{Lie}(B)$ is a Borel subalgebra, then B is called a Borel subgroup.

Proposition 2.2.56. *Let $B \subset G$ be a Borel subgroup. Then B is closed and G/B is a projective variety. Furthermore, all the Borel subgroups are conjugates.*

Proof. [Ott95, Proposition 7.3]. □

Definition 2.2.57. A closed subgroup $P \subset G$ is called parabolic if it contains some Borel subgroup.

Theorem 2.2.58. *Let $P \subset G$ be a closed subgroup. P is a parabolic subgroup if and only if G/P is a projective variety.*

Proof. [Ott95, Theorem 7.5]. □

Definition 2.2.59. Let $\Delta = \{\alpha_1, \dots, \alpha_m\}$ be the set of simple positive roots of $\mathcal{G}$ and let $\Sigma \subset \Delta$. Denote

$$\Phi^-(\Sigma) = \{\alpha \in \Phi^- | \alpha = \sum_{\alpha_i \notin \Sigma} p_i \alpha_i\}.$$

We denote by $\mathcal{P}(\Sigma)$ the subalgebra

$$\mathcal{P}(\Sigma) = \mathcal{H} \bigoplus (\oplus_{\alpha \in \Phi^+} \mathcal{G}_\alpha) \bigoplus (\oplus_{\alpha \in \Phi^-(\Sigma)} \mathcal{G}_\alpha).$$

The subgroup $P(\Sigma)$ is the subgroup such that Lie $P(\Sigma) = \mathcal{P}(\Sigma)$.

Theorem 2.2.60. *Let G be a semisimple and simply connected Lie group. Let P be a parabolic subgroup of G, then there exists $g \in G$ and $\Sigma \subset \Delta$ such that*

$$g^{-1} P g = P(\Sigma).$$

Proof. [Ott95, Theorem 7.8]. □

Corollary 2.2.61. *Let G be semisimple and simply connected and $G = G_1 \times \cdots \times G_k$ the decomposition into direct product of simple simply connected Lie groups. Let $P \subset G$ be a parabolic group. Then there are parabolic groups $P_i \subset G_i$ such that $P = P_1 \times \cdots \times P_k$, moreover*

$$G/P = G_1/P_1 \times \cdots \times G_k/P_k.$$

Observe that Theorem 2.2.60 and Corollary 2.2.61 implies that rational homogeneous varieties are classified. They correspond exactly to the product of varieties $G/P(\Sigma)$, where G is simple and simply connected and Σ a subset of the positive simple roots.

Definition 2.2.62. Let E be a bundle over G/P. E is called homogeneous if there exists an action of G over E such that the following diagram commutes

$$\begin{tikzcd} G \times E \arrow[r] \arrow[d] & E \arrow[d] \\ G \times G/P \arrow[r] & G/P \end{tikzcd}$$

Lemma 2.2.63. *$G \xrightarrow{\pi} G/P$ is a principal bundle with fiber P.*

Proof. [Ott95, Lemma 9.4]. □

Definition 2.2.64. Let $\rho : P \to GL(r)$ be a representation of the parabolic subgroup $P \subset G$. We define the vector bundle E_ρ on G/p associated to this representation as the bundle with fiber $\mathbb{C}^r$ coming from the principal bundle $G \xrightarrow{\pi} G/P$ via ρ.

An equivalent definition for E_ρ is via the quotient of $G \times \mathbb{C}^r / \sim$, where the relation $\sim$ is defined by $(g, v) \sim (g', v')$ if there exists $p \in P$ such that $g = g'p$ and $v = \rho(p^{-1})v'$.

Theorem 2.2.65 (Matsushima)**.** *A vector bundle E of rank r over G/P is homogeneous if and only if there exists a representation $\rho : P \to \mathrm{GL}(r)$ such that $E \cong E_\rho$.*

Proof. [Ott95, Theorem 9.7]. □

We recall that an irreducible representation $\rho : P \to GL(r)$ has an unique highest weight λ associated to it (Theorem 2.2.48). This allows us to associate the vector bundle E_ρ with the highest weight λ of ρ. Depending on which object we want to emphasize, we may denote the bundle also as E_λ.

Lemma 2.2.66. *Let Σ be a subset of the simple roots and $P \subset G$ a parabolic subgroup. We have the decomposition*

$$\begin{aligned} \text{Lie } P &= \mathcal{H} \bigoplus \left(\oplus_{\alpha>0}\mathcal{G}_\alpha\right) \bigoplus \left(\oplus_{\alpha\in\Phi^-(\Sigma)}\mathcal{G}_\alpha\right) \\ &= \text{Lie } S_p \bigoplus \left(\oplus_{i=1}^k[\mathcal{G}_{\alpha_i,\mathcal{G}_{-\alpha_i}}]\right) \bigoplus \left(\oplus_{\alpha\notin\Phi^+(\Sigma)}\mathcal{G}_\alpha\right), \end{aligned}$$

where S_P is semisimple and it is called the semisimple part of P.

Proof. [Ott95, Lemma 10.3]. □

Definition 2.2.67. Let Σ be a subset of the simple roots and $P \subset G$ a parabolic subgroup. Let $U \subset P$ be the subgroup such that

$$\mathrm{Lie}\, U = \left(\bigoplus_{\alpha \notin \Phi^+(\Sigma)} \mathcal{G}_\alpha \right).$$

U is called the unipotent part of P.

Proposition 2.2.68 (Ise)**.** *A representation $\rho : P \to \mathrm{GL}(V)$ is completely reducible if and only if $\rho|_U$ is trivial*

Proof. [Ott95, Proposition 10.5]. □

Proposition 2.2.69. *Let $\Sigma = \{\alpha_1, \ldots, \alpha_k\}$ be a subset of simple roots. Let $\lambda_1, \ldots, \lambda_k$ be the corresponding set of fundamental weights. Then all the irreducible representations of $P(\Sigma)$ are*

$$V \otimes L_{\lambda_1}^{m_1} \otimes \cdots \otimes L_{\lambda_k}^{m_k},$$

where V is a representation of the weight lattice S_p, $m_i \in \mathbb{Z}$ and L denotes a line bundle.

Proof. [Ott95, Proposition 10.9] □

2.2.5 Bott Theorem

Definition 2.2.70. Let λ be a weight and α a root of the semisimple group G.

1. The weight λ is singular if there exists a positive root α such that $B(\lambda, \alpha) = 0$.

2. The weight λ is regular of index p if it is not singular and there exists exactly p roots $\alpha_1, \ldots, \alpha_p \in \Phi^+$ such that $B(\lambda, \alpha) < 0$.

Theorem 2.2.71. *Let G be a semisimple connected group and let $P \subset G$ be a parabolic subgroup. Let λ be a weight and E_λ be the homogeneous bundle arising from the irreducible representation P with highest weight λ. Denote $\delta = \sum_{i=1}^m \lambda_i$ be the sum of all fundamental weights.*

1. If $\lambda + \delta$ is regular of index p, then

$$H^i(G/P, E_\lambda) = \begin{cases} 0 & \text{if } i \neq p \\ G_{w(\lambda+\delta)-\delta} & \text{if } i = p. \end{cases}$$

2. If $\lambda + \delta$ is singular, then

$$H^i(G/P, E) = 0.$$

Where weight $w(\lambda+\delta)$ is the unique element of the Weyl chamber of G which is congruent to $\lambda+\delta$ modulo the action of the Weyl group.

Proof. [Ott95, Section 11]. □

An application of Bott's theorem that is going to be useful later on sections 3.2 and 4 is the computation of the cohomologies of the sheaf of differential r-forms denoted $\Omega^r_{\mathbb{P}^m}$.

Lemma 2.2.72 (Bott's Formula). *The q-cohomology group of $\Omega^r_{\mathbb{P}^m}(t)$ is non vanishing in the following cases and has dimension given by:*

$$h^q\big(\Omega^r_{\mathbb{P}^m}(t)\big) = \begin{cases} \binom{t+r-m}{t}\binom{t-1}{r} & \text{if } q=0\leq r\leq m \text{ and } t>r; \\ 1 & \text{if } 0\leq q=r\leq m \text{ and } t=0; \\ \binom{-t+r}{-t}\binom{-t-1}{m-r} & \text{if } q=m\geq r\geq 0 \text{ and } t<r-m. \end{cases} \tag{2.2.1}$$

Computing the nonvanishing of the above cohomologies is simple using Bott's theorem. Denote by Q the universal quotient bundle in $\mathbb{P}^m$. We have

$$\begin{aligned} \Omega^r_{\mathbb{P}^m}(t) &= \left(\bigwedge^r \Omega^1_{\mathbb{P}^m}(1)\right)\otimes\mathcal{O}(t-r) = \left(\bigwedge^r Q^*\right)\otimes\mathcal{O}(t-r) \\ &= \bigwedge^{m-r} Q(-1)\otimes\mathcal{O}(t-r) = \bigwedge^{m-r} Q(t-r-1). \end{aligned}$$

The associated weight to this bundle is $\lambda=\lambda_{r+1}+(t-r-1)\lambda_1$. The next step is to calculate when such weight is singular or regular depending on the value of t.

$$B(\lambda+\delta,\alpha_1+\cdots+\alpha_s) = \begin{cases} t-r-1+s & 1\leq s\leq r, \\ t-r+s & r+1\leq s\leq m. \end{cases}$$

This gives the following:

1. For $t>r$ the weight is regular of index 0.
2. For $r\geq t\geq 1$ the weight is singular.
3. For $t=0$ the weight is regular of index r.
4. For $-1\geq t\geq m-r$ the weight is singular.
5. For $t<m-r$ the weight is regular of index m.

This means that the cohomology $H^q(\Omega_{\mathbb{P}^m})$ is non vanishing in the following cases:

1. $q=0$ and $t>r$.
2. $q=r$ and $t=0$.

3. $q = m$ and $t < m - r$.

We recall another useful theorem for the following chapters.

Theorem 2.2.73 (Künneth's formula)**.** *Let $\mathcal{B}_i$ be vector bundles on $\mathbb{P}V_i$, for $i = 1, \ldots, k$, $X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k$ and q a nonnegative integer, then*

$$H^q\left(X, \bigotimes_{i=1}^{k} \pi_i^* \mathcal{B}_i\right) \cong \bigoplus_{q_1+\cdots+q_k=q} \bigotimes_i H^{q_i}(\mathbb{P}V_i, \mathcal{B}_i).$$

where the sum goes over all tuples of nonnegative integers summing to q.

$\mathrm{Tor}_p^S(M, \underline{k})_q$ has rank equal to $\beta_{p,q}$.

Chapter 3

Singular vector tuples

3.1 Introduction to singular vector tuples

We suggest both [Lan12] and [QL17] as references for a deeper understanding of the notions presented in this chapter.

We will denote the dual space of a vector space V by V^* and by $[n]$ the set of numbers $\{1, \ldots, n\}$.

Let V_i be a $m_i + 1$-dimensional vector space over $\mathbb{C}$. Let $q_i^{\mathbb{R}} : (V_i^{\mathbb{R}}) \times (V_i^{\mathbb{R}}) \to \mathbb{R}$ be a real inner product, we permit a small abuse of notation and denote $q_i \in \mathrm{Sym}^2 V_i^*$ the homogeneous quadratic polynomial associated to the inner product extended over the complex numbers. We consider each space V_i with an associated action of the special orthogonal group $\mathrm{SO}(V_i)$ that respect the inner product defined by q_i. Moreover, we are going to consider each V_i with a q_i-orthornormal basis $\{x_{0,i}, \ldots, x_{m_i,i}\}$. In a simpler terms this means that, if $\mathbf{x}_i = \sum_{j=0}^{m_i} \alpha_j x_{j,i} \in V_i$, $\alpha_j \in \mathbb{C}$, then $q_i(\mathbf{x}_i, \mathbf{x}_i) = \sum_{j=0}^{m_i} \alpha_j^2$. We will identify all the vector spaces V_i with their dual V_i^* using their respective inner products.

Definition 3.1.1. The Bombieri-Weyl inner product q of two complex decomposable tensors $T = \bigotimes_{i=1}^k t_i^{d_i}$, $S = \bigotimes_{i=1}^k s_i^{d_i} \in \bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i$ is given by

$$q(T, S) = \bigotimes_{i=1}^{k} q_i(t_i, s_i)^{d_i}.$$

It is naturally extended by linearity to every vector.

Definition 3.1.2. Let $T \in V = \bigotimes_{i=1}^k V_i$. A rank-one tensor $\lambda \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_k \in V$, $q_i(\mathbf{x}_i, \mathbf{x}_i) = 1$, is an eigentensor of T if and only if

$$q(T, \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{k-1} \otimes v_i \otimes \mathbf{x}_{k+1} \otimes \cdots \otimes \mathbf{x}_k) = \lambda q(\mathbf{x}_i, v_i)$$

for every $v_i \in V_i$ and all $i \in [k]$. The number $\lambda \in \mathbb{C}$ is the singular value of the eigentensor. As we present soon in Theorem 3.1.4, denoting $\mathbf{x} = \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_k$, this condition is equivalent to say that $T - \lambda \mathbf{x} \perp \mathrm{T}_{\mathbf{x}} \widehat{X}$, where $\widehat{X}$ is the cone over the Segre variety.

We associate the tuple $([\mathbf{x}_1], \dots, [\mathbf{x}_k])$, named singular vector k-tuple, to the eigentensor $\mathbf{x}_1 \otimes \dots \mathbf{x}_k$. We will commit an abuse of notation and use the singular vector k-tuples to refer to the eigentensor. Since for a general T there is a unique representative of the tuple that is an eigentensor, this passage is well defined.

Notice that from the definition, we can obtain the equations of the locus of singular vector k-tuples of a tensor T from the minors of the following matrix:

$$\operatorname{rank}\begin{pmatrix} T(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{i-1} \otimes \mathbf{x}_{i+1} \otimes \cdots \otimes \mathbf{x}_k) \\ \mathbf{x}_i \end{pmatrix} \leq 1 \quad \forall\, i \in [k]\,, \tag{3.1.1}$$

where

$$T(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{i-1} \otimes \mathbf{x}_{i+1} \otimes \cdots \otimes \mathbf{x}_k) \coloneqq \sum_{j_\ell \in [n_\ell]} t_{j_1 \cdots j_i \cdots j_k}\, x_{j_1,1} \cdots \widehat{x_{j_i,i}} \cdots x_{j_k,k} \tag{3.1.2}$$

is the tensor contraction of $T = (t_{j_1,\dots,j_k})$ with respect to $\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{i-1} \otimes \mathbf{x}_{i+1} \otimes \cdots \otimes \mathbf{x}_k$. The symbol $\widehat{x_{j_i,i}}$ in (3.1.2) means that the variable $x_{j_i,i}$ is omitted in the product.

We remark that being a singular vector k-tuple, up to the value of the singular value λ, is a projective property. Although some geometrical interpretation is missed, in many cases it is better to work over projective spaces and disregard the value of λ.

These definitions can be extended to partially symmetric tensors, in particular to polynomials.

Definition 3.1.3. Let $T \in \bigotimes_{i=1}^k \left(\mathrm{Sym}^{d_i} V_i\right)$. An eigentensor of T is a rank-one partially symmetric tensor $\mathbf{x} = \lambda \mathbf{x}_1^{d_1} \otimes \cdots \otimes \mathbf{x}_k^{d_k}$, $\mathbf{x}_i \in V_i$ with $q_i(\mathbf{x}_i, \mathbf{x}_i) = 1$, such that

$$q(T, \mathbf{x}_1^{d_1} \otimes \cdots \otimes \mathbf{x}_{i-1}^{d_{i-1}} \otimes \mathbf{x}_i^{d_i - 1} v_i \otimes \mathbf{x}_{i_{i+1}}^{d_{i+1}} \otimes \cdots \otimes \mathbf{x}_k^{d_k}) = \lambda q_i(\mathbf{x}_i, v_i)$$

for every $v_i \in V_i$ and all $i \in [k]$. The number λ is the singular value of the eigentensor $\mathbf{x}$. Similar to the tensor case, this is equivalent to say that $T - \lambda \mathbf{x} \perp \mathrm{T}_{\mathbf{x}} \widehat{X}$, where $\widehat{X}$ is the cone over the Segre-Veronese variety.

From the definition we have that the equations defining the locus of eigentensors of the tensor T is defined by the 2×2-minors of the matrices

$$\begin{bmatrix} T(\mathbf{x}_1^{d_1} \otimes \cdots \otimes \mathbf{x}_{i-1}^{d_{i-1}} \otimes \mathbf{x}_i^{d_i - 1} \otimes \mathbf{x}_{i+1}^{d_{i+1}} \otimes \cdots \otimes \mathbf{x}_k^{d_k}) \\ \mathbf{x}_i \end{bmatrix} \tag{3.1.3}$$

for all $i \in [k]$. Considering T as a multi-homogeneous polynomial, the flattening on the first row may be understood as the gradient $\nabla_i T$ with respect to the vector $\mathbf{x}_i = (x_{0,i}, \dots, x_{m_i,i})$.

As in the tensor case, for a general tensor T, we will identify the eigentensor $\lambda \mathbf{x}_1^{d_1} \otimes \cdots \otimes \mathbf{x}_k^{d_k}$ to the tuple $([\mathbf{x}_1], \dots, [\mathbf{x}_k])$, named a singular vector k-tuple of

T. The condition that T is general implies that there is a unique representative of the singular vector k-tuple that is an eigentensor.

A singular vector tuple $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is said to be isotropic if $q_i(\mathbf{x}_i) = 0$ for some i.

Singular vector k-tuples have an important significance, they are the critical points of the distance function between the tensor T and the cone $\widehat{X}$ over the Segre-Veronese variety X. Since the Segre-Veronese variety is the variety of rank one tensors, this result has a greater significance, singular vector tuples consists of the best rank one approximation of T [Lim05].

We give a precise statement and proof in the case of the Segre variety. The generalised version follow by a similar argument.

Theorem 3.1.4. *[Lim05] Let $T \in V = \bigotimes_{i=1}^k V_i$ and q_i be a real inner product in V_i extended to the complex numbers. Then the singular vector tuples of T are the critical points of the distance function defined by the Bombieri-Weyl product q from the tensor T to the cone $\widehat{X}$ of the Segre variety X.*

Proof. Let $\mathbf{x} = \lambda \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_k$ be a critical point of the distance function between T and $\widehat{X}$, with $q_i(\mathbf{x}_i, \mathbf{x}_i) = 1$. Notice that $\mathbf{x}$ being a critical point of the distance function from T to $\widehat{X}$ is equivalent to say that $T - \mathbf{x} \perp \mathrm{T}_{\mathbf{x}}\widehat{X}$. This means that $T - \mathbf{x} \perp \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{i-1} \otimes V_i \otimes \mathbf{x}_{i+1} \otimes \cdots \otimes \mathbf{x}_k$ for every i. From $q(T - \mathbf{x}, \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{i-1} \otimes V_i \otimes \mathbf{x}_{i+1} \otimes \ldots \mathbf{x}_k) = 0$, it follows

$$q(T, \mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{i-1} \otimes v_i \otimes \mathbf{x}_{i+1} \otimes \ldots \mathbf{x}_k) = \lambda q_i(\mathbf{x}_i, v_i),$$

for every $v_i \in V_i$ and all $i \in [k]$. This is equivalent to say

$$T(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_{i-1} \otimes - \otimes \mathbf{x}_{i+1} \otimes \ldots \mathbf{x}_k) = \lambda q_i(\mathbf{x}_i, -),$$

for every $i \in [k]$. This means that $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is a singular vector tuple of T with singular value λ. $\square$

Definition 3.1.5. Consider a tensor $T \in V$. Then

$$Z_T := \{[\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_k] \in \mathbb{P}(V) \mid (\mathbf{x}_1, \ldots, \mathbf{x}_k) \text{ is a singular } k\text{-tuple of } T\}. \quad (3.1.4)$$

For a general $T \in V$ we have $\dim(Z_T) = 0$ and its cardinality $|Z_T|$ equals the ED degree of the Segre variety $X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k$ computed by the Friedland-Ottaviani formula of Theorem 3.1.10.

Definition 3.1.6. We define the pairing $[\cdot|\cdot] : \mathrm{Sym}^{d_i} V_i \times \mathrm{Sym}^{d_i} V_i \to \bigwedge^2 V_i$ on decomposable tensors by

$$[v^d | w^d] = q_i(v, w)^{d-1} v \wedge w.$$

Extending this to nondecomposable tensors we obtain

$$[v_1 \dots v_d | w_1 \dots w_d] = \frac{1}{d \cdot d!} \sum_{i',j' \in [d]} \sum_{\pi : [d] \setminus i' \to [d] \setminus j'} \left(\prod_{i \neq i'} q_i(v_i, w_{\pi(i)}) \right) v_{i'} \wedge w_{j'}.$$

Definition 3.1.7. Using the Definition 3.1.6, we construct for each $l \in [k]$ the pairing $[\cdot|\cdot]_l : \left(\bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i \right) \times \left(\bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i \right) \to \bigwedge^2 V_l$ by

$$[f_1 \otimes \cdots \otimes f_k | g_1 \otimes \cdots \otimes g_k]_l = \left(\prod_{i \neq l} q_i(f_i, g_i) \right) [f_l | g_l], \ \ f_i, g_i \in \mathrm{Sym}^{d_i} V_i.$$

Definition 3.1.8. Let $X = \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k$ be the Segre-Veronese variety of rank 1 tensors embedded with $\mathcal{O}(d_1, \dots, d_k)$ in $\mathbb{P}(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k)$. Let $\pi_l : X \to \mathbb{P}V_l$ be the projection on the l-th component, and let Q_l be the quotient bundle, whose fibers over a point $v_l \in V_l$ are $V_l / \langle v_l \rangle$. Let $\mathcal{E}_l = \pi_l^* Q_l \otimes \mathcal{O}(d_1, \dots, d_l - 1, \dots, d_k)$, we can construct the vector bundle

$$\mathcal{E} = \bigoplus_{l=1}^k \mathcal{E}_l.$$

A tensor $T \in \mathbb{P}(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k)$ leads to a global section of $\mathcal{E}_l$ which over a point $v = (v_1, \dots, v_k)$ is the map sending $v_1^{d_1} \otimes \cdots \otimes v_l^{d_l - 1} \otimes \cdots \otimes v_k^{d_k}$ to the natural pairing of T with $(v_1^{d_1}) \cdots (v_l^{d_l - 1}) \cdots (v_k^{d_k})$ modulo $\langle v_l \rangle$, that is a vector in $V_l / \langle v_l \rangle$. In other words, the map defined by the section s_T is given by

$$s_T : v_1^{d_1} \otimes \cdots \otimes v_l^{d_l - 1} \otimes \cdots \otimes v_k^{d_k} \mapsto T(v_1^{d_1} \otimes \cdots \otimes v_l^{d_l - 1} \otimes \cdots \otimes v_k^{d_k}) + \langle v_l \rangle.$$

Such bundle has been introduced in [FO14]. The main property that makes this bundle useful for our purposes is that the zero locus of the global section $s_T \in H^0(\mathcal{E})$ associated with a tensor T is given exactly by Z_T. In other words, the zero locus of the sections corresponds exactly to the singular vector tuples of T. This can be obtained clearly by spelling out the sections of the bundles $\mathcal{E}_i$, they are given exactly by

$$\begin{bmatrix} \nabla_i T(\mathbf{x}) \\ \mathbf{x}_i \end{bmatrix}$$

thus the zero locus corresponds exactly to the conditions on the i-th component for $\mathbf{x}$ to be a singular vector tuple.

Definition 3.1.9. We define the ED-degree of a subvariety $X \subset \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k$ as the number of critical points of the function $d_T : X \to \mathbb{C}$, where $d_T(x) = q(x - T)$ is the distance function from $x \in X$ to a general tensor $T \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k$ defined by the Bombieri-Weyl inner product, see [DOT17]. More precisely, for any real tensor we have a squared distance function defined on the real part of X. It is convenient to complexify such function to the complex part of X, and by abuse of notation we call this complex function again a distance function.

The ED-degree has been studied in [Dra+13], and we suggest it as a reference for a better comprehension. In particular, if we consider the variety X to be the Segre-Veronese variety, we have that the ED-degree counts the number of singular tuples of a general tensor. We are going to denote the ED-degree of the Segre-Veronese variety by ed_X. This particular ED-degree has been studied before in [FO14], where the next theorem is presented.

Theorem 3.1.10. *[FO14, Theorem 15] Let $V_1, \ldots, V_k$ be vector spaces of dimension $m_1+1, \ldots, m_k+1$. The number of singular tuples of a general tensor $T \in \mathbb{P}(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k)$, is equal to the coefficient of $t_1^{m_1} \cdots t_k^{m_k}$ in the polynomial*

$$\prod_{l=1}^{k} \frac{\hat{t}_l^{\,m_l+1} - t_l^{m_l+1}}{\hat{t}_l - t_l},$$

where $\hat{t}_l = \left(\sum_i^k d_i t_i\right) - t_l$.

The technique employed by the authors to obtain such result consists on the computation of the highest Chern class of the bundle $\mathcal{E}$ previously defined in Definition 3.1.8.

The stabilization of the ED-degree of the Segre-Veronese variety has been studied in [OSV21], where the next two results have been obtained.

Proposition 3.1.11. *[OSV21, Corollary 4.14] Let $X \subset \left(\times_{i=1}^{k-1}\mathbb{P}^{m_i}\right) \times \mathbb{P}^m$ be the Segre variety and $N = \sum_{i=1}^{k-1} m_i$. For all $m \geq N$, we have*

$$\mathrm{ED}\left(\left(\times_{i=1}^{k-1}\mathbb{P}^{m_i}\right) \times \mathbb{P}^m\right) = \mathrm{ED}_X\left(\left(\times_{i=1}^{k-1}\mathbb{P}^{m_i}\right) \times \mathbb{P}^N\right).$$

Proposition 3.1.12. *[OSV21, Corollary 4.16] Let $X \subset \left(\times_{i=1}^{k-1}\mathrm{Sym}^{d_i}\mathbb{P}^{m_i}\right) \times \mathbb{P}^m$ be the Segre-Veronese variety and $N = \sum_{i=1}^{k} m_i$. For all $m \geq N$, we have*

$$\mathrm{ED}\left(\left(\times_{i=1}^{k-1}\mathrm{Sym}^{d_i}\mathbb{P}^{m_i}\right) \times \mathbb{P}^m\right) = \mathrm{ED}_X\left(\left(\times_{i=1}^{k-1}\mathbb{P}^{m_i}\right) \times \mathbb{P}^N\right).$$

The triangular inequality on the dimensions considered in the previous results is connected to the dual-defectiveness of the Segre-Veronese variety and the existence of the hyperdeterminant, as we see in Theorem 3.1.15.

Definition 3.1.13. Consider a tensor space $V = \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$, let $\mathbf{m} = (m_1+1, \ldots, m_k+1)$ be the tensor format of V. Then $\mathbf{m}$ satisfies the boundary format condition if for every $i \in [k]$ such that $d_i = 1$ we have

$$m_i \leq \sum_{j \neq i} m_j. \tag{3.1.5}$$

When (3.1.5) is satisfied and for some $i \in [k]$ the equality holds we say that $\mathbf{m}$ is of boundary format. We say that the format $\mathbf{m}$ is beyond boundary format if it does not satisfy (3.1.5).

Definition 3.1.14. Let $X \subset \mathbb{P}(W)$ be a projective variety, where $\dim(W) = m+1$. Its dual variety $X^* \subset \mathbb{P}(W^*)$ is the closure of all hyperplanes tangent to X at some smooth point [GKZ94, Chapter 1]. The dual defect of X is the natural number $\delta_X \coloneqq m - 1 - \dim(X^*)$. A variety X is said to be dual defective if $\delta_X > 0$. Otherwise, it is dual non-defective. When $X = \mathbb{P}(W)$, taken with its tautological embedding into itself, $X^* = \emptyset$ and $\operatorname{codim}(X^*) = m+1$.

Of particular interest are dual varieties of Segre-Veronese varieties, whose nondefectiveness is characterized by the following result.

Theorem 3.1.15. *[GKZ94, Corollary 5.11] Suppose X_l for $l = 1, \dots, k$ is the projective space $\mathbb{P}^{m_l}$ in the Veronese embedding into $\mathbb{P}(\mathrm{Sym}^{d_l} V_l)$. Then the dual variety $(X_1 \times \cdots \times X_k)^*$ is a hypersurface if and only if the format $\boldsymbol{m} = (m_1 + 1, \dots, m_k + 1)$ satisfies the boundary format condition (3.1.5).*

The geometry of the fact that the dual variety of the Segre variety is no longer a hypersurface beyond boundary format and how it influences singular vector tuples has been understood recently in [OSV21].

Theorem 3.1.16. *[OSV21, Theorem 4.13] Let $N = 1 + \sum_{i=1}^{k-1}(n_i - 1)$ and $m \geq N$. Let Det be the hyperdeterminant in the boundary format $(n_1, \dots, n_{k-1}, N)$. Consider a tensor $T \in \bigotimes_{i=1}^{k-1} V_i \otimes \mathbb{C}^{N+1} \subset \bigotimes_{i=1}^{k-1} V_i \otimes \mathbb{C}^{m+1}$ with $\mathrm{Det}(T) \neq 0$. Then the critical points of T on the Segre variety $\prod_{i=1}^{k-1} \mathbb{P}(V_i) \times \mathbb{P}(\mathbb{C}^{m+1})$ lie in the subvariety $\prod_{i=1}^{k-1} \mathbb{P}(V_i) \times \mathbb{P}(\mathbb{C}^{N+1})$.*

Definition 3.1.17. Given sets $Y_1, \dots, Y_k$ will denote the unordered cartesian product by

$$\left(\times_{i=1}^{k} Y_i\right)^{\times r} / S_r = \left(\times_{i=1}^{k} Y_i\right)^{(r)},$$

where S_r is the symmetric group of order r.

Let $T \in V = \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$ be a general tensor and $X \in \mathbb{P}V$ the Segre-Veronese variety, we define

$$\mathrm{Eig}(T) = \{(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(\mathrm{ed}_X)}) \mid \mathbf{x}^{(i)} \text{ is a singular tuple of } T, \forall\, i \in [k]\} \subset X^{(\mathrm{ed}_X)}\}$$

the set of unordered singular vector k-tuples of T.

3.2 The critical space and linear relation among the singular vector tuples

We start this section by exploring the results on [DOT17] and the construction of the *critical space*. We will prolong in this section and show most results since the technique utilised was the inspiration for the proofs of Theorems 4.1.4 and 4.2.10.

The critical space H_T of a tensor $T \in V$ has been first introduced in [OP15], where it was called the singular space of T, and later renamed to critical space in [DOT17]. In simple terms, it was defined in [OP15; DOT17] as the set of all tensors U such that $[T, U]_l = 0$ for each $l \in [k]$, where the bracket $[,]$ is the paring introduced in Definition 3.1.7. Recently in [Ott22] this space has been redefined once again, now in the optics of group action. This definition is more sophisticated than the previous and we choose to utilise it, for instance one of its advantages is that it allows the extension of the critical space to Grassmannians in a very natural manner. Denote by $\mathrm{SO}(V) = \mathrm{SO}(V, q)$ the special orthogonal group that leaves the quadratic form q invariant and let d_T be the distance function from T defined by q.

Definition 3.2.1. Let $T \in V$ and $G \subset \mathrm{SO}(V)$ be a group, we define the critical space H_T^G of T as the orthogonal space to the orbit of the action of the Lie group of G on T

$$H_T^G = (\mathrm{Lie}\ G \cdot T)^\perp = \{v \in V \mid q(v, w) = 0\ \forall\ w \in \mathrm{Lie}\ G \cdot T\}.$$

The next result presents the most important properties of H_T and enlightens its relation with the singular vector tuples of T.

Theorem 3.2.2. *[Ott22, Theorems 1.2 and 1.3] Let $X \subset V$ be a subvariety that is G-invariant for the action of $G \subset \mathrm{SO}(V)$. Let $x \in X \cap H_T^G$. Then:*

1. *The critical points of d_T lie on H_T^G.*
2. *When T is real, any closest point to T in $X_{\mathbb{R}}$ belongs to H_T^G.*
3. *$T \in H_T^G$.*
4. *If the orbit $G \cdot x$ is dense in X then x is a critical point of d_T restricted to X.*
5. *If X is a cone, x is non-isotropic and the orbit $G \cdot [x]$ is dense in $\mathbb{P}X$, then there exists $\lambda \in \mathbb{C}$ such that λx is a critical point of d_T restricted to X.*

Theorem 3.2.3. *[Ott22, Section 2.2] Let q_i be a real inner product in V_i extended to the complex numbers and consider q the Bombieri-Weyl inner product defined by them in $V = \bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i$. Let $G = \times_{i=1}^k \mathrm{SO}(V_i, q_i) \subset \mathrm{SO}(V, q)$. Let $\{x_{i,0}, \ldots, x_{i,m_i}\}$ be the set of variables in V_i. Then*

$$\mathrm{Lie}\ G \cdot T = \left\langle x_{p,j} \frac{\partial T}{\partial x_{p,i}} - x_{p,i} \frac{\partial T}{\partial x_{p,j}} \right\rangle_{0 \leq i \leq j \leq m_p;\ p = 1, \ldots, k}.$$

We remark that Theorem 3.2.3 shows that for the choice of $G = \times_{i=1}^k \mathrm{SO}(V_i, q_i)$ the critical space H_T^G coincides with the critical space defined in [DOT17] that we recall below.

Theorem 3.2.4. *[DOT17] For a tensor $T \in \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$, the critical space $H_T \subset \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$ of T is given by*

$$H_T = \{U \in \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i \mid [T|U]_l = 0 \text{ for all } l \in [k]\},$$

where $[\cdot,\cdot]$ is the bracket product of Definition 3.1.7.

From now on for $G = \times_{i=1}^{k} \mathrm{SO}(V_i, q_i)$ we are going to denote $H_T = H_T^G$.

Theorem 3.2.5. *[Ott22, Theorem 2.4] Let $T \in V = \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$ be a general tensor, let $X \subset \mathbb{P}V$ be the Segre-Veronese variety, then $H_T \cap X$ consists exactly of the singular vector tuples of T.*

Definition 3.2.6. Let $T \in \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$, X be the Segre-Veronese variety and r be a non-negative integer. A critical rank-at-most r tensor for T is a tensor $U \in \mathrm{Sec}_r(X)$ such that $T - U \perp \mathrm{T}_U \mathrm{Sec}_r(X)$.

Notice that the last condition is equivalent to say that U is a critical point of the distance function defined by the Bombieri-Weyl inner product between T and the r-secant variety of the Segre-Veronese variety X, this is the geometrical motivation for the name of such variety. Moreover, this also means that U is a best rank-at-most r approximation of T.

Lemma 3.2.7. *[Dru+17, Lemma 4.2] Let $Y \subset \mathbb{R}^m$ be a variety and $W \subset Y$ be an dense open subset of $Y_{\mathbb{C}}$. Then all the critical points of the distance function between a general $y \in \mathbb{C}^n$ and Y lie in W.*

Observe that Lemma 3.2.7 implies that all the critical rank-at-most r tensors for a general $T \in \bigotimes_{i=1}^{k} \mathrm{Sym}^{d_i} V_i$ are smooth points of $\mathrm{Sec}_r(X)$ and can be written as sum of r non-isotropic rank-one tensors. If we assume that r is at most the generic rank, then those critical rank-at-most r tensors have rank equal to r. If r is larger than the generic rank, we have that the general tensor T itself is the unique critical rank-at-most r tensor.

Proposition 3.2.8. *[DOT17, Lemma 2.12] Let $T \in V = \bigotimes_{i=1}^{k} V_i$ be a general tensor and r a non-negative integer. Then all the critical rank-at-most k tensors of T belong to the critical space H_T.*

Notice that Proposition 3.2.8 is obtained again as a direct corollary of the previous result. This exemplifies the strength of such techniques and the new perspectives that these results creates for the near future.

Let $\mathrm{Gr}(k, V) \subset \bigwedge^k V$ be the Grassmannian variety of k-dimensional vector subspaces of V. If q_V is a quadratic form on V, we may extend it to $\bigwedge^k V$ by $q(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) = \det((q_V(v_i, w_j))_{1 \le i,j \le k})$.

Proposition 3.2.9. *[Ott22, section 3.1] Let $T \in \bigwedge^k V_i$, then*

$$\mathrm{Lie}(\mathrm{SO}(V)) \cdot T = \left\langle \frac{\partial T}{\partial x_i} \wedge x_j - \frac{\partial T}{\partial x_j} \wedge x_i \right\rangle_{0 \le i \le j \le m}.$$

This proposition allows us to construct the critical space $H_T = \mathrm{Lie}(\mathrm{SO}(V))\cdot T$ for Grassmann varieties and to find the best rank one approximation of skew-symmetric tensors.

Theorem 3.2.10. *[Ott22, Theorem 3.1] Let $T \in \bigwedge^k V$ be a general skew-symmetric tensor, then $H_T \cap \mathrm{Gr}(k, V)$ consists exactly of the critical points of d_T restricted to $\mathrm{Gr}(k, V)$.*

We recall that the ED-degree of Grassmannians varieties is still unknown.

The techniques utilised by the authors in the next results were the main inspiration for the cohomology techniques utilised in the results obtained in Chapter 4. The main idea is to use the fact that the bundle $\mathcal{E}$ has the zero locus of its global sections associated to a tensor T equals to Z_T, as discussed after Definition 3.1.8. We then use this to correlate $H^0(\mathcal{E}^*(f_1 \dots, d_k))$ with the equations defining the critical space. The main goal is to prove that $\langle Z_T \rangle = \mathbb{P}H_T$ for a general tensor T. This result also implies that the tensor T is a linear combination of its singular tuples. We start the path for this result with two technical lemmas.

Lemma 3.2.11. *[DOT17, Lemma 3.4] Let X be the Segre-Veronese variety in $\mathbb{P} \bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i$, then*

$$\left(\bigwedge^r \mathcal{E}^*\right) \otimes \mathcal{O}(d_1, \dots, d_k) \cong \bigoplus_{r_1+\cdots+r_k=r} \bigotimes_{l=1}^k \pi_l^* \Omega^{r_l}_{\mathbb{P}^{m_l}}(-d_l(r-1) + 2r_l).$$

Proof. Taking the dual bundle of $\mathcal{E}$ leads to

$$\mathcal{E}^* = \bigoplus_{l=1}^k (\pi_l^* Q_l^*) \otimes \mathcal{O}(-d_1, \dots, -(d_l - 1), \dots, -d_k)).$$

Applying the wedge product gives

$$\bigwedge^r \mathcal{E}^* = \bigoplus_{r_1+\cdots+r_k=r} \bigotimes_{l=1}^k \bigwedge^{r_l} (\pi_l^* Q_l^* \otimes \mathcal{O}(-d_1, \dots, -d_l + 1, \dots, -d_k)).$$

From the definition of Ω we have that $Q_l^* = \Omega^1_{\mathbb{P}^{m_l}}(1)$ and $\bigwedge^r(\Omega^1_{\mathbb{P}^{m_l}}(1)) = \Omega^r_{\mathbb{P}^{m_l}}(r)$, together with the fact that $\bigwedge^r(\mathcal{F} \otimes \mathcal{O}(t)) = \bigwedge^r \mathcal{F}(rt)$ we obtain that

$$\left(\bigwedge^r \mathcal{E}^*\right) \otimes \mathcal{O}(d_1, \dots, d_k) \cong \bigoplus_{r_1+\cdots+r_k=r} \bigotimes_{l=1}^k \pi_l^* \Omega^{r_l}_{\mathbb{P}^{m_l}}(-d_l(r-1) + 2r_l).$$

□

Lemma 3.2.12. *[DOT17, Lemma 3.2] Suppose the boundary format condition holds for $\bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i$, let $r \geq 2$ be an integer and $q_1, \dots, q_k$ non-negative integers such that $q = \sum_{i=1}^k q_i$. Then for all $q < r$ and $r_1, \dots, r_k$ such that*

$r = \sum_{i=1}^k r_i$ it holds

$$\bigotimes_{i=1}^{k} H^{q_i}\left(\mathbb{P}V_l, \Omega^{r_l}_{\mathbb{P}^{m_l}}(-d_l(r-1)+2r_l)\right) = 0$$

The proof of this theorem consists in analysing the possible choices of non-zero cohomologies in the equation 2.2.1 and noticing that not all cohomologies in the tensor product may be non-zero simultaneously under those hypothesis.

Corollary 3.2.13. *Under the hypothesis of Lemma 3.2.12 we have*

$$H^q\left(\left(\bigwedge^r \mathcal{E}^*\right) \otimes \mathcal{O}(d_1,\dots,d_k)\right) = 0.$$

Proof. The result is a combination of Lemmas 3.2.12, 3.2.11 and Theorem 2.2.73. □

We remark that the boundary format condition is crucial for such result, indeed in general the vanishing of the cohomologies will no longer hold. For instance, in the tensor format $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$ we have that $H^q\left(\left(\bigwedge^2 \mathcal{E}^*\right) \otimes \mathcal{O}(1,1,1)\right)$ is a 1-dimensional space. This will imply that the isomorphism (3.2.4) does not hold anymore. Actually we obtain that $h^0(\mathcal{I}_{Z_T} \otimes \mathcal{O}(1,1,1)) = h^0(\mathcal{E}^* \otimes \mathcal{O}(1,1,1)) + 1$, implying that $\langle Z_T \rangle$ has codimension one in H_T.

We are able to prove the main required tool for our objective.

Lemma 3.2.14. *[DOT17, Lemma 3.5] Assume the boundary format condition for $\bigotimes_{i=1}^k \mathrm{Sym}^{d_i} V_i$, let T be a general tensor and $m = \sum_{i=1}^k m_i$. Then*

$$H^0(\mathcal{E}^* \otimes (d_1,\dots,d_k)) \cong H^0(\mathcal{I}_{Z_T} \otimes \mathcal{O}(d_1,\dots,d_k)).$$

Proof. Consider the Koszul complex

$$0 \to \bigwedge^m \mathcal{E}^* \xrightarrow{\varphi_{m-1}} \dots \xrightarrow{\varphi_2} \bigwedge^2 \mathcal{E}^* \to \mathcal{E}^* \to \mathcal{I}_{Z_T} \to 0. \tag{3.2.1}$$

Divide the complex into short exact sequences

$$0 \to \mathcal{F}_2 \to \mathcal{E}^* \to \mathcal{I}_{Z_T} \to 0; \tag{3.2.2}$$

$$0 \to \mathcal{F}_{r+1} \to \bigwedge^r \mathcal{E}^* \to \mathcal{F}_r \to 0, \tag{3.2.3}$$

where $\mathcal{F}_r = \bigwedge^r \mathcal{E}^* / \mathrm{Im}\varphi_r$.

Tensoring the short exact sequence on (3.2.3) with $\mathcal{O}(d_1,\dots,d_k)$ and taking the long exact sequence of cohomologies leads to

$$\cdots \to H^{r-2}\left(\bigwedge^r \mathcal{E}^* \otimes \mathcal{O}(d_1,\dots,d_k)\right) \to H^{r-2}(\mathcal{F}_r \otimes \mathcal{O}(d_1,\dots,d_k)) \to$$
$$\to H^{r-1}(\mathcal{F}_{r+1} \otimes \mathcal{O}(d_1,\dots,d_k)) \to H^{r-1}\left(\bigwedge^r \mathcal{E}^* \otimes \mathcal{O}(d_1,\dots,d_k)\right) \to$$
$$\to H^{r-1}(\mathcal{F}_r \otimes \mathcal{O}(d_1,\dots,d_k)) \to H^r(\mathcal{F}_{r+1} \otimes \mathcal{O}(d_1,\dots,d_k)) \to \dots$$

Since we have the vanishings for the cohomologies

$$H^{r-2}\left(\bigwedge^r \mathcal{E}^* \otimes \mathcal{O}(d_1,\dots,d_k)\right) = H^{r-1}\left(\bigwedge^r \mathcal{E}^* \otimes \mathcal{O}(d_1,\dots,d_k)\right) = 0$$

we obtain the following chains

$$\begin{aligned} H^0(\mathcal{F}_2 \otimes \mathcal{O}(d_1,\dots,d_k)) \cong H^1(\mathcal{F}_3 \otimes \mathcal{O}(d_1,\dots,d_k)) \cong \\ \cong \dots \cong H^{m-1}(\mathcal{F}_{m+1} \otimes \mathcal{O}(d_1,\dots,d_k)) = 0 \end{aligned}$$

$$\begin{aligned} H^1(\mathcal{F}_2 \otimes \mathcal{O}(d_1,\dots,d_k)) \subset H^2(\mathcal{F}_3 \otimes \mathcal{O}(d_1,\dots,d_k)) \subset \\ \subset \cdots \subset H^{m-1}(\mathcal{F}_m \otimes \mathcal{O}(d_1,\dots,d_k)) = 0 \end{aligned}$$

Considering the long exact sequence of (3.2.2) we have that

$$H^0(\mathcal{I}_{Z_T} \otimes \mathcal{O}(d_1,\dots,d_k)) \cong H^0(\mathcal{E}^* \otimes \mathcal{O}(d_1,\dots,d_k)). \tag{3.2.4}$$

□

Theorem 3.2.15. *[DOT17, Proposition 3.6] Assume the boundary format and that T is general, then $\langle Z_T \rangle = \mathbb{P}H_T$ and $\operatorname{codim}(H_T) = \sum_{i=1}^r \binom{m_i+1}{2}$.*

Proof. The space of linear forms of $\bigotimes_{i=1}^k \operatorname{Sym}^{d_i} V_i$ vanishing on Z_T is given exactly by $H^0(Z_T \otimes \mathcal{O}(d_1,\dots,d_k))$. From Lemma 3.2.14 we have the isomorphism

$$H^0(Z_T \otimes \mathcal{O}(d_1,\dots,d_k)) \cong H^0(\mathcal{E}^* \otimes \mathcal{O}(d_1,\dots,d_k)).$$

Notice that

$$\begin{aligned}\mathcal{E}^* \otimes \mathcal{O}(d_1, \ldots, d_k) &= \bigoplus_{i=1}^{k} \pi_l^* Q_l \otimes \mathcal{O}(-d_1, \ldots, -d_i + 1, \ldots, -d_k) \otimes (d_1, \ldots, d_k) \\ &= \bigoplus_{i=1}^{k} \pi_l^* Q_l \otimes \mathcal{O}(0, \ldots, \overset{i}{\sqcap}\!\!1, \ldots, 0) \\ &= \bigoplus_{i=1}^{k} \pi_i^* (\Omega^1_{\mathbb{P}^{m_i}}(2)).\end{aligned}$$

From the first line of (2.2.1) we have that $H^0(\mathcal{E}^* \otimes \mathcal{O}(d_1, \ldots, d_k))$ has dimension $\sum_{i=1}^k \binom{m_i+1}{2}$. Since those are the linear equations defining $\langle Z_T \rangle$ we have that $\operatorname{codim}(Z_T) = \sum_{i=1}^k \binom{m_i+1}{2}$.

To prove the isomorphism with the critical space we study the map

$$H^0(\Omega^1_{\mathbb{P}^{m_l}}(2)) \to H^0(\mathcal{I}_{Z_T} \otimes \mathcal{O}(d_1, \ldots, d_k))$$

in more details. The left side is canonically isomorphic to $\left(\bigwedge^2 V_l\right)^*$ via Theorem 2.2.71. An element ξ in this space is mapped to the linear form

$$\bigotimes_{i=1}^{k} \operatorname{Sym}^{d_i} V_i \to \mathbb{C}, U \mapsto \xi([T|U]_i).$$

Letting i vary we obtain the linear forms that cuts H_T. Therefore $\mathbb{P}H_T = \langle Z_T \rangle$. $\square$

Chapter 4

Tensors determined by their singular vector tuples

In this chapter the goal is to answer the question: *Given a general tensor T, are there other tensors $S \neq T$ such that* $\mathrm{Eig}(S) = \mathrm{Eig}(T)$*?* Another way of interpreting it is asking: *is the tensor T determined by its eigenscheme?* This question had been studied before in [ASS17] where it was shown that for a general polynomial f in three variables and odd degree it holds that f is determined by $\mathrm{Eig}(f)$. Later [BGV21] has shown that for a general polynomial f in three variables and even degree such behavior is no longer seen. It was proven that the polynomials h with same singular vector tuples as f are given by $h = f + \mu(x_0^2 + x_1^2 + x_2^2)^{\frac{d}{2}}$, where $d = \deg(f)$ and $\mu \in \mathbb{C}$. In this section we extend such result to polynomials of any degree in Theorem 4.1.4. We further generalise this result to partially symmetric tensors (or multisymmetric tensor) in Theorem 4.2.10. This is the work developed by the author in [Tur22].

A first example regarding if a tensor is determined by their singular tuples is the case of matrices. The next example answers the question and highlights how different the behavior of singular vector tuples of matrices is from the behavior of higher order tensors.

Example 4.0.1. The relation between a given set of singular tuples and the matrices that have such singular tuple locus configuration is described by the *Singular Value Decomposition*, since the singular tuples of a matrix are given by the first columns of the orthogonal matrices on the decomposition. We briefly describe it next.

Let $A \in \mathrm{Hom}(Y, W)$, where Y, W are real vector spaces of dimensions $\dim Y = n$, $\dim W = m$, we recall that the singular value decomposition tells us that for a general matrix A we have $A = U\mathrm{diag}(\sigma_1, \dots, \sigma_{\min\{m,n\}})V^t$, where U, V are orthogonal matrices, diag is a quasi-diagonal matrix with nonzero entries $\sigma_i \neq 0$ for all i only in the main diagonal, i.e. $\mathrm{diag}_{(\mathrm{i,j})} = \begin{cases} \sigma_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.
If we let u_i and v_i be the columns of U and V, as described by Ottaviani and Paoletti in [OP15], we have that for $1 \leq i \leq m = \min\{m, n\}$, $Av_i = \sigma_i u_i$ and

$A^t u_i = \sigma_i v_i$, in other words, the pairs (u_i, v_i) are the singular pairs of A. Let $\tau : \mathrm{Hom}(Y, W) \dashrightarrow (Y \times W)^{(m)}$, $A \mapsto \mathrm{Eig}(A)$, where $\mathrm{Eig}(A)$ is the set consisting of the singular tuples of A. Note that τ is not defined if A has two singular values that are equal. Given a singular tuple locus $Z = \{(u_i, v_i)\}_{i=1}^m$, and orthogonal matrices U, V such that the first m columns are u_i and v_i we have that

$$\tau^{-1}(Z) = \{B \in \mathrm{Hom}(Y, W) | B = U\mathrm{diag}(\sigma_1, \dots, \sigma_m)V^t, \sigma_i \in \mathbb{C}\}.$$

is the subset obtained by varying the σ_i.

We can formulate the problem better in mathematical terms as we did in the Example 4.0.1 by means of the map τ.

Definition 4.0.2. Let $T \in \bigotimes_{i=1}^k \mathrm{Sym}^{d_i}\mathbb{C}^{m_i+1}$ be a general tensor and X the Segre-Veronese variety, we define the map τ that associates a tensor to its eigenscheme

$$\tau : \mathbb{P}\left(\bigotimes_{i=1}^k \mathrm{Sym}^{d_i}\mathbb{C}^{m_i+1}\right) \dashrightarrow \left(\times_{i=1}^k \mathbb{P}^{m_i}\right)^{(\mathrm{ed}_X)}, \; T \mapsto \mathrm{Eig}(T). \tag{4.0.1}$$

We remark that the map τ is not defined for all tensors, since the number of singular tuples may vary. However for a general tensor the number is always equal to ed_X.

Studying the map τ is difficult in general, but exploiting the relation of the vector bundle $\mathcal{E}$ allows us to decompose τ through this vector bundle, and this proves to be quite advantageous. We decompose τ in the following manner

$$\begin{tikzcd}
\mathbb{P}\big(\mathrm{Sym}^{d_1}V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k}V_k\big) \arrow[rr, dashed, "\tau"] \arrow[dr, dashed, "\varphi"'] & & \big[\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_k)\big]^{(\mathrm{ed}_X)} \\
& \mathbb{P}(H^0(\mathcal{E})) \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}.$$

In this diagram the map φ associates a tensor T to the global section s_T described before in the definition 3.1.8. The map ψ sends a global section $s \in H^0(\mathcal{E})$ to its zero locus $Z(s)$, in particular the codomain is well defined for a section s_T when the singular tuples of T consists of exactly ed_X points.

The idea of the proofs of Theorems 4.1.4 and 4.2.10 is to first describe when the map φ is injective, or otherwise what is the kernel. The next step is to show that for a section s_T associated to the general tensor T it holds that $\psi(s_T) = \psi(s_U)$ only if $s_T = \lambda s_U$.

In sections 4.1 and 4.2 we fix the basis of the vector spaces $V_1, \dots, V_k$ to be $\{x_{0,i}, \dots, x_{m_i,i}\}$ and we choose the inner product $q_i : V_i \times V_i \to \mathbb{R}$ such that the quadratic form associated is given in coordinates by $q_i(x) = x_{0,i}^2 + \cdots + x_{m_i,i}^2$.

4.1 Symmetric Tensors

Lemma 4.1.1. *[Tur22, Lemma 3.1] Let V be a vector space of dimension $m+1$, $q = x_0^2 + \cdots + x_m^2$ a quadratic form on V, consider the action of $\mathrm{SO}(V)$ that respects q, and a positive integer d. If d is odd, the map*

$$\varphi : \mathrm{Sym}^d V \to H^0(Q(d-1)),\ \ f \mapsto s_f = \begin{bmatrix} \frac{\partial f}{\partial x_0} & \cdots & \frac{\partial f}{\partial x_k} \\ x_0 & \cdots & x_k \end{bmatrix},$$

is injective. If d is even, φ has a 1-dimensional kernel, namely, $\ker \varphi = \langle q^{d/2} \rangle$.

Proof. We recall that $\mathrm{Sym}^d V$ splits as $\mathrm{SO}(V)$-modules as

$$\mathrm{Sym}^d V = H_d \oplus H_{d-2} \oplus \cdots \oplus \begin{cases} H_1 \text{ if } d \text{ is odd;} \\ H_0 \text{ if } d \text{ is even,} \end{cases}$$

where $H_{d-2j} = \{fq^j | f \text{ is a harmonic polynomial of degree } d-2j\}$ is an irreducible $\mathrm{SO}(V)$-module.

Therefore we can restrict φ to each H_j, in such way we have

$$\varphi : H_j \to W_j \subset H^0(Q(d-1)),$$

where $W_j = \mathrm{Im}(\varphi|_{H_j})$. This map is either an isomorphism or zero by Schur's Lemma. Let j be such that $d-2j \geq 1$, then we have that for $g = (x_0+ix_1)^{d-2j} q^j \in H_{d-2j}$ it is mapped by φ to

$$s_g = \begin{bmatrix} \frac{\partial g}{\partial x_0} & \cdots & \frac{\partial g}{\partial x_m} \\ x_0 & \cdots & x_m \end{bmatrix},$$

that does not have rank 1 everywhere. Indeed

$$\frac{\partial g}{\partial x_0} x_1 - \frac{\partial g}{\partial x_1} x_0 = \left((d-2j)(x_0+ix_1)^{d-2j-1}\right)(x_1 - ix_0) q^j \not\equiv 0.$$

On the other hand, $H_0 = \{\lambda q^{\frac{d}{2}} | \lambda \in \mathbb{C}\}$. In such case we have for an element of H_0 that

$$\frac{\partial \lambda q^{\frac{d}{2}}}{\partial x_i} x_j - \frac{\partial \lambda q^{\frac{d}{2}}}{\partial x_j} x_i = \lambda(2x_i x_j q^{\frac{d}{2}-1} - 2x_i x_j q^{\frac{d}{2}-1}) = 0,\ \forall\, i,j \in \{0, \ldots, m\}.$$

We conclude that if d is odd, the map φ is an isomorphism in each irreducible representation; if d is even, it is an isomorphism in each of them, with the exception of H_0, as we wished. □

We recall that the bundle Q is a simple bundle, in other words, $H^0(\mathrm{End}(Q)) \cong \mathbb{C}$. With this fact in mind we may state the next result.

Lemma 4.1.2. *[Tur22, Lemma 3.2] Let Z be the zero locus of a section in $Q(d-1)$, and assume that $d \geq 3$. Then the natural map from the Koszul complex $H^0(\mathrm{End}(Q)) \to H^0(\mathcal{I}_Z \otimes Q(d-1))$ is an isomorphism of 1-dimensional spaces.*

Proof. Indeed, consider the Koszul complex

$$0 \xrightarrow{\varphi_m} \bigwedge^m Q^*(m(1-d)) \xrightarrow{\varphi_{m-1}} \dots \xrightarrow{\varphi_2} \bigwedge^2 Q^*(2(1-d)) \to Q^*(1-d) \to \mathcal{I}_Z \to 0,$$

tensoring it by $Q(d-1)$ we obtain the exact sequence

$$\begin{aligned} 0 \to \bigwedge^m Q^* \otimes Q((m-1)(1-d)) \to \dots \to \\ \to \bigwedge^2 Q^* \otimes Q(1-d) \to \mathrm{End}(Q) \to \mathcal{I}_Z \otimes Q(d-1) \to 0. \end{aligned}$$

Let $\mathcal{F}_r$ to be defined as the quotient $\mathcal{F}_r = \bigwedge^r Q^*(r(1-d))/\mathrm{Im}\varphi_r$. Thus we obtain short exact sequences

$$\begin{gathered} 0 \to \mathcal{F}_2 \to Q^*(1-d) \to \mathcal{I}_Z \to 0, \\ 0 \to \mathcal{F}_{r+1} \to \bigwedge^r Q^*(r-rd) \to \mathcal{F}_r \to 0, \end{gathered}$$

for $r = 2, \dots, m$.

Tensoring the second short exact sequence by $Q(d-1)$ we obtain the long exact sequence of cohomologies

$$\begin{aligned} \dots \to H^{r-2}(\bigwedge^r Q^* \otimes Q((r-1)(1-d))) \to H^{r-2}(\mathcal{F}_r \otimes Q(d-1)) \to \\ \to H^{r-1}(\mathcal{F}_{r+1} \otimes Q(d-1)) \to H^{r-1}(\bigwedge Q^* \otimes Q((r-1)(1-d))) \to \\ \to H^{r-1}(\mathcal{F}_r \otimes Q(d-1)) \to H^r(\mathcal{F}_{r+1} \otimes Q(d-1)) \to \dots \end{aligned}$$

We have that $\bigwedge^r Q^* \otimes Q((r-1)(1-d)) = \bigwedge^{m-r} Q \otimes Q((r-1)(1-d)-1)$, by Lemma 4.2.3 if we have that $r \geq 2$, we obtain that $H^{r-2}(\bigwedge^{m-r} Q \otimes Q((r-1)(1-d)-1)) = 0$. Also, if $d \geq 3$, $H^{r-1}(\bigwedge^{m-r} Q \otimes Q((r-1)(1-d)-1)) = 0$.

This means that

$$\begin{gathered} H^0(\mathcal{F}_2 \otimes Q(d-1)) \cong H^1(\mathcal{F}_3 \otimes Q(d-1)) \cong \dots \cong H^{m-1}(\mathcal{F}_{m+1} \otimes Q(d-1)) = 0, \\ H^1(\mathcal{F}_2 \otimes Q(d-1)) \subset H^2(\mathcal{F}_3 \otimes Q(d-1)) \subset \dots \subset H^m(\mathcal{F}_{m+1} \otimes Q(d-1)) = 0. \end{gathered}$$

Applying the long exact sequence of cohomologies to

$$0 \to \mathcal{F}_2 \otimes Q(d-1) \to \mathrm{End}(Q) \to \mathcal{I}_Z \otimes Q(d-1) \to 0$$

gives the desired result. □

We would like to add a remark that, although already utilised, the vanishing of the cohomology $H^q(\bigwedge^r Q^* \otimes Q(t))$ is carefully done in the next section in Lemma 4.2.3. We decide in favour of postponing those computations because the full usefulness of such cohomologies appears in the partially symmetric case.

Corollary 4.1.3. *[Tur22, Corollary 3.3] Let $f, g \in \mathrm{Sym}^d V$ be two general polynomials such that $Z(s_f) = Z(s_g)$, $d \geq 3$. Then $s_f = \alpha s_g$ for some $\alpha \in \mathbb{C}^*$.*

Proof. The hypothesis that $Z(s_f) = Z(s_g)$ implies that $s_f \in H^0(\mathcal{I}_{Z(s_g)} \otimes Q(d-1))$. Since this space is one-dimensional we have that $s_f = \alpha s_g$. □

We conclude this section observing that since $\tau = \psi \circ \varphi$, then the Theorem 4.1.4 is obtained just as the combinination of the Lemma 4.1.1 with the Corollary 4.1.3.

Theorem 4.1.4. *[Tur22, Theorem 1.1] Let V be a vector space of dimension $m+1$. Let $d \geq 3$ be an integer, and $f \in \mathbb{P}(\mathrm{Sym}^d V)$ be a general polynomial. Let*

$$\tau : \mathbb{P}\big(\mathrm{Sym}^d V\big) \dashrightarrow \mathbb{P}V^{(\mathrm{ed}_X)}, \; f \mapsto \mathrm{Eig}(f)$$

be the map that associates to f its eigentensors locus $\mathrm{Eig}(f)$. Then

$$\tau^{-1}(\tau(f)) = \begin{cases} [f], & \text{if } d \text{ is odd;} \\ \{[f + cq^{\frac{d}{2}}] | c \in \mathbb{C}\}, & \text{if } d \text{ is even.} \end{cases}$$

Moreover, the image of the map τ has dimension

$$\dim(\mathrm{Im}(\tau)) = \begin{cases} \binom{d+m}{d} - 1, & \text{if } d \text{ is odd;} \\ \binom{d+m}{d} - 2, & \text{if } d \text{ is even.} \end{cases}$$

Remark 4.1.5. *Let $f \in \mathrm{Sym}^d V$ be a general polynomial of even degree d. Consider $v \in V$ and eigentensor of f with $q(v,v) = 1$ and singular value λ. Then v is an eigentensor of $f + \mu(x_0^2 + \cdots + x_m^2)^{\frac{d}{2}}$, we compute the associated singular value on the cone. For every $w \in V$ we have:*

$$\begin{aligned} q(f + \mu(x_0^2 + \cdots + x_m^2)^{\frac{d}{2}} - \alpha v^d, v^{d-1}w) &= 0 \\ q(f, v^{d-1}w) + \mu q((x_0^2 + \cdots + x_m^2)^{\frac{d}{2}}, v^{d-1}w) &= \alpha q(v,w) \\ \nabla f(v) + (\mu d) q(v,v)^{\frac{d}{2}-1} v &= \alpha v \\ \lambda v + (\mu d) v &= \alpha v \\ \lambda + \mu d &= \alpha. \end{aligned}$$

Therefore the singular value of v in the polynomial $f + \mu(x_0^2 + \cdots + x_m^2)^{\frac{d}{2}}$ is $\lambda + \mu d$. In particular, for $\mu \in \mathbb{R}_{\geq 0}$, it does not change which eigentensor gives the best rank one approximation.

4.2 Partially symmetric tensors

Now that the pre-image of the map τ is completely analysed for symmetric tensors, we can go through to the next step, that is, we consider the Segre-Veronese variety $\mathrm{Sym}^{d_1}V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k}V_k$ and we analyse the map $\tau : \mathbb{P}\big(\mathrm{Sym}^{d_1}V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k}V_k\big) \dashrightarrow \big[\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_k)\big]^{(\mathrm{ed}_X)}$ that associates a tensor T to its singular tuples $\mathrm{Eig}(T)$. We begin the partially symmetric case with the generalization of the Lemma 4.1.1 to Segre-Veronese varieties.

Theorem 4.2.1. *[Tur22, Theorem 4.1] Let $V_1, \ldots, V_k$ be vector spaces of dimension $m_1+1, \ldots, m_k+1$, and we recall that $q_i = x_{0,i}^2 + \cdots + x_{m_i,i}^2$ is the quadratic form on V_i that defines the distance function for $i = 1 \ldots, k$ and fix an action of the special orthogonal group $\mathrm{SO}(V_i)$. We consider the map*

$$\varphi : \mathrm{Sym}^{d_1}V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k}V_k \to H^0(\mathcal{E}),$$

where $\mathcal{E}$ is defined in the Definition 3.1.8. Then φ is injective if at least one d_i is odd. In the case that all the d_i are even, we have that the kernel of φ is one dimensional and it is given by

$$\ker \varphi = \langle q_1^{\frac{d_1}{2}} \rangle \otimes \cdots \otimes \langle q_k^{\frac{d_k}{2}} \rangle.$$

Proof. Since we have that

$$\mathrm{Sym}^{d_l}V_l \cong H_{d_l} \oplus H_{d_l-2} \oplus \cdots \oplus \begin{cases} H_1 \text{ if } d_l \text{ is odd;} \\ H_0 \text{ if } d_l \text{ is even,} \end{cases}$$

and that each $H_{d_j-2t_j}$ is an irreducible $\mathrm{SO}(V_l)$-representation, then also $H_{d_1-2t_1} \otimes \cdots \otimes H_{d_k-2t_k}$ is an irreducible $\mathrm{SO}(V_1) \times \cdots \times \mathrm{SO}(V_k)$-representation, we need to show that φ is non-zero when $d_j - 2t_j > 0$ for at least one j, and that it is zero when we have $d_j - 2t_j = 0$ for all j.

Indeed, in the first case we consider the element

$$g = g_1 \otimes \cdots \otimes g_k, \;\; g_j = (x_{0,j} + ix_{1,j})^{d_j - 2t_j} q_j^{t_j},$$

then $\varphi(g) = s_g = (s_{g_1} \otimes \mathbb{1}) \oplus \cdots \oplus (\mathbb{1} \otimes s_{g_k})$, where $s_{g_j} \otimes \mathbb{1} \in \mathcal{E}_j$ is non-zero as seen before in the symmetric tensor case. Therefore by Schur's Lemma we have that in this restriction the map is an isomorphism, thus if $d_j - 2t_j > 0$ for some j, s_g does not belong to the kernel of φ.

On the other hand, if all $d_j - 2t_j = 0$, then $g_j = cq_j^{\frac{d_j}{2}}$, where $c \in \mathbb{C}$, then $s_{g_j} = 0$, therefore summing all together we obtain that $s_g = 0$, so by Schur's Lemma the restriction of φ on this subrepresentations is the zero map, as wished. $\square$

With this result we understand the first map φ in the decomposition $\tau = \psi \circ \varphi$. Now we can aim to understand better the map ψ, we will show that, under the hypothesis of Theorem 4.2.10, when two section s, t have the same image under the map ψ, where s, t are sections coming from tensors S, $T \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k$, then $s = \lambda t$.

The first step to achieve this goal is to prove a similar result to Lemma 4.1.2 for the case of partially symmetric tensors, in order to do that we prove a series of technical lemmas. We are going to denote by $\Omega^r_{\mathbb{P}^m}(k)$ the $\mathcal{O}(k)$-twisted sheaf of differential r-forms.

Lemma 4.2.2. *[Tur22, Lemma 4.2] Let* $\mathcal{E}^* = \bigoplus_{l=1}^k Q_l^*(-d_1, \dots, -d_l+1, \dots, -d_k)$*, then, for* $j = 1, \dots, k$*, then*

$$\bigwedge^r \mathcal{E}^* \otimes Q_j(d_1, \dots, d_j - 1, \dots, d_k) =$$
$$= \bigoplus_{r_1+\cdots+r_k=r} \bigotimes_{l=1, l\neq j}^{k} \Omega^{r_l}_{\mathbb{P}^{m_l}}(2r_l - d_l(r-1)) \otimes \bigwedge^{m_j - r_j} Q_j \otimes Q_j(-d_j(r-1) + r_j - 2).$$

Proof. From the definition of $\mathcal{E}$ we have that

$$\bigwedge^r \mathcal{E}^* = \bigoplus_{r_1+\cdots+r_k=r} \bigotimes_{l=1}^{k} \Big(\bigwedge^{r_l} Q_l^*\Big)\Big(-r_l d_1, \dots, -r_l(d_l-1), \dots, -r_l d_k\Big),$$

by separating the terms we obtain that

$$\bigwedge^r \mathcal{E}^* = \bigoplus_{r_1+\cdots+r_k=r} \bigotimes_{l=1}^{k} \bigwedge^{r_l} Q_l^*(-r d_l + r_l).$$

We now tensor it by $Q_j(d_1, \dots, d_j - 1, \dots, d_k)$, so we have that $\bigwedge^r \mathcal{E}^* \otimes Q_j(d_1, \dots, d_j - 1, \dots, d_k)$ is equal to

$$\bigoplus_{r_1+\cdots+r_k=r} \bigotimes_{l=1, l\neq j}^{k} \bigwedge^{r_l} Q_l^*(-r d_l + r_l + d_l) \otimes \bigwedge^{r_j} Q_j^* \otimes Q_j(-r d_j + r_j + d_j - 1).$$

We now utilise the facts that $\Omega^{r_l}(r_l) = \bigwedge^{r_l}(\Omega^1(1))$, $\Omega^1(1) = Q^*$ and $\bigwedge^{r_j} Q_j^* = \bigwedge^{m_j - r_j} Q_j(-1)$, to obtain that $\bigwedge^r \mathcal{E}^* \otimes Q_j(d_1, \dots, d_j - 1, \dots, d_k)$ is equal to

$$\bigoplus_{r_1+\cdots+r_k=r} \bigotimes_{l=1, l\neq j}^{k} \Omega^{r_l}_{\mathbb{P}^{m_l}}(2r_l - d_l(r-1)) \otimes \bigwedge^{m_j - r_j} Q_j \otimes Q_j(-d_j(r-1) + r_j - 2).$$

□

Lemma 4.2.3 (Bott's Formula)**.** *[Tur22, Lemma 4.3] The cohomology*

$$H^q\left(\bigwedge^{m_j - r_j} Q_j \otimes Q_j(t)\right)$$

is non-vanishing for the following cases

$$H^q\Big(\bigwedge^{m_j-r_j} Q_j\otimes Q_j(t)\Big)\neq 0,\ \ if\ \begin{cases} q=0,\ t\geq 0;\\ q=r_j-1,\ t=-r_j,\ 1\leq r_j\leq m_j;\\ q=r_j,\ t=-r_j-1,\ 0\leq r_j\leq m_j-1;\\ q=m_j-1,\ t=-m_j-1,\ 0\leq r_j\leq m_j-1;\\ q=m_j,\ t\leq -m_j-2.\end{cases} \tag{4.2.1}$$

Proof. The proof is an application of Bott's Theorem 2.2.71, the technique consists on computing if either the associated weights are singular or regular and their index as described in definition 2.2.70, we remark that in this proof the product (,) represents the Killing form. We denote by $\delta=\sum_{i=1}^{m_i}\lambda_i$ the sum of all fundamental weights.

The associated weight will be calculated in three cases depending on the r_j; the cases are $r_j=0$, $1\leq r_j\leq m_j-1$ and $r_j=m_j$.

For the case $1\leq r_j\leq m_j-1$, we have that $\bigwedge^{m_j-r_j} Q_j\otimes Q_j(t)$ is not irreducible, therefore we have that the associated weight λ is given by two parts

$$\lambda=\lambda_{(1)}\oplus\lambda_{(2)}.$$

where $\lambda_{(1)}=\lambda_{r_j+1}+\lambda_{m_j}+t\lambda_1$ and $\lambda_{(2)}=\lambda_{r_j}+t\lambda_1$. We remind from [Wey03] that the Killing product satisfies $(\lambda_i,\alpha_j)=\begin{cases}1, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases}$

For $\lambda_{(1)}$ we have that

$$(\lambda_{(1)}+\delta,\alpha_1+\cdots+\alpha_s)=\begin{cases} s+t \text{ if } s\leq r_j;\\ s+t+1 \text{ if } r_j+1\leq s\leq m_j-1;\\ s+t+2 \text{ if } s=m_j.\end{cases}$$

This implies the following cases:

1. $t\geq 0$, then the weight is regular of index 0.

2. $-1\geq t\geq -r_j$, then the weight is singular ($s=-t$).

3. If $t=-r_j-1$, then the weight is regular of index r_j.

4. If $-r_j-2\geq t\geq -m_j$, then the weight is singular ($s=-t-1$).

5. if $t=-m_j-1$, then the weight is regular of index m_j-1.

6. if $t=-m_j-2$, then the weight is singular ($s=m_j$).

7. if $t\leq -m_j-3$, then the weight is regular of index m_j.

For $\lambda_{(2)}$ we have that

$$(\lambda_{(2)} + \delta, \alpha_1 + \cdots + \alpha_s) = \begin{cases} s + t \text{ if } s \leq r_j - 1; \\ s + t + 1 \text{ if } s \geq r_j. \end{cases}$$

That implies the following cases:

1. If $t \geq 0$, then the weight is regular of index 0.
2. If $-1 \geq t \geq -(r_j - 1)$, then the weight is singular ($s = -t$).
3. If $t = -r_j$, then the weight is regular of index $(r_j - 1)$.
4. If $-r_j - 1 \geq t \geq -m_j - 1$, then the weight is singular ($s = -t - 1$).
5. If $t \leq -m_j - 2$, then the weight is regular of index m_j.

For $r_j = m_j$ we have $Q_j(t)$, therefore the associated weight λ is $\lambda = \lambda_{m_j} + t\lambda_1$, thus we have

$$(\lambda + \delta, \alpha_1 + \cdots + \alpha_s) = \begin{cases} s + t \text{ if } s \leq m_j - 1; \\ s + t + 1 \text{ if } s = m_j. \end{cases}$$

This implies the following cases

1. If $t \geq 0$, then the weight is regular of index 0.
2. If $-1 \geq t \geq -m_j + 1$, then the weight is singular ($s = -t$).
3. If $t = -m_j$, then the weight is regular of index $m_j - 1$.
4. If $t = -m_j - 1$, then the weight is singular ($s = m_j$).
5. If $t \leq -m_j - 2$, then the weight is regular of index m_j.

The final case is when $r_j = 0$, then we have $Q_j(t + 1)$ and the associated weight λ is $\lambda_{m_j} + (t + 1)\lambda_1$, therefore

$$(\lambda + \delta, \alpha_1 + \cdots + \alpha_s) = \begin{cases} s + t + 1 \text{ if } s \leq m_j - 1; \\ s + t + 2 \text{ if } s = m_j. \end{cases}$$

This implies the following cases

1. If $t \geq -1$, then the weight is regular of index 0.
2. If $-2 \geq t \geq -m_j$, then the weight is singular ($s = -t - 1$).
3. If $t = -m_j - 1$, then the weight is regular of index $m_j - 1$.

4. If $t = -m_j - 2$, then the weight is singular ($s = m_j$).

5. If $t \leq -m_j - 3$, then the weight is regular of index m_j.

□

Lemma 4.2.4. *[Tur22, Lemma 4.5] Let $m_l = \dim \mathbb{P}V_l$ and $k \geq 3$. Suppose that $m_l \leq \sum_{i \neq l} m_i$ holds for every l such that $d_l = 1$. Let $r \geq 2$ be an integer, $q_1, \dots, q_k$ be non negative integers such that $\sum q_l \leq r - 1$, and let $r_1, \dots, r_k$ be non negative integers such that $\sum r_l = r$, then*

$$\bigotimes_{l=1, l \neq j}^{k} H^{q_l}(\Omega^{r_l}_{\mathbb{P}^{m_l}}(2r_l - d_l(r-1))) \bigotimes H^{q_j}\Big(\bigwedge^{m_j - r_j} Q_j \otimes Q_j(-d_j(r-1) + r_j - 2)\Big) = 0,$$

for every $j \in \{1, \dots, k\}$.

Furthermore, if $k = 2$ and we add the hypothesis that $(d_1, d_2) \neq (1, 1)$ the result still holds.

Proof. Suppose that the cohomology of the tensor product is non-vanishing. We fix that the index j will associated to the unique case coming from the cohomology table (4.2.1), if not said otherwise.

Not all the cases can come from the third, fourth or fifth line of (4.2.1) and from the second and third lines of (2.2.1). Suppose that one case comes from either the third, fourth or fifth lines of (4.2.1), and all the remaining cases come from the second and third line (2.2.1), this means that $q_l \geq r_l$, and we have that $r > q = \sum q_l \geq \sum r_l = r$.

So at least one cohomology case must come from the other lines in (4.2.1) or (2.2.1).

No case can come from the first line of (4.2.1). Suppose that the only case of (4.2.1) comes from the first line, this means that $-d_j(r-1) + r_j - 2 \geq 0$, so we obtain that

$$r_j \geq (r-1)d_j + 2 > d_j(r-1) + 1 \geq (r-1) + 1 = r,$$

that is $r_j > r$, a contradiction.

No case can come from the first line of (2.2.1). Suppose that we have one case coming from the first line of (2.2.1) for a fixed l, we have that $r_l > d_l(r-1)$, then the only possiblity is that $r_l = r$ and all other $r_i = 0$, for $i \neq l$ and $d_l = 1$. In such case, for $i \neq l$ we have that the other cohomologies can not be on the first line, otherwise it would be 0. Let j be the only case coming from (4.2.1), then for $i \neq l, j$ we have that it can not be on the second line of (2.2.1), because $0 = r_i = q_i = -d_i(r-1)$ and $r-1,\ d_i > 0$. For j we have that the second line of (4.2.1) does not apply since $q_j = r_j - 1 = -1$ and the third line of (4.2.1) implies $0 = q_j = r_j$ and $-d_j(r-1) - 2 = -1$, then $d_j(r-1) = -1$, that is a contradiction since both terms on the left side are non negative. So in those cases we have the vanishing of the cohomology, therefore we have that one case is either on the fourth or fifth line of (4.2.1) and all the remaining cases are on the third line of (2.2.1). If one case is on the fifth line of (4.2.1) and all the others on the third line of (2.2.1), we have that $q_i = m_i$ and $q_j = m_j$ for $i \neq l$. This means

$$m_l \geq r_l = r > \sum_{i \neq l} q_i = \sum_{i \neq l} m_i,$$

this implies that $m_l > \sum_{i \neq l} m_i$, that is a contradiction since $d_l = 1$. The case coming from (4.2.1) can not be on the fourth line of (4.2.1), and all the others coming from the third line of (2.2.1) either, because in such case we have that $-d_j(r-1) - 2 = -m_j - 1$, that is, $m_j - 1 = d_j(r-1)$, but since $r > \sum_{i \neq l} q_i = \sum_{i \neq l,j}(m_i) + m_j - 1 \geq m_j$, we have that $r > m_j$, and the equality can not be satisfied since $d_j \geq 1$. In the case $k = 2$, notice that $r \geq m_j$ and since $d_l = 1$, we must have $d_j \geq 2$. Again the wished equality $m_j - 1 = d_j(r-1)$ can not hold. This implies that no cohomology can come from the first line of (2.2.1).

No case can come from the second line of (4.2.1). The last remaining possibility is to have the only case of (4.2.1) coming from the second line. In such case we notice that we have $q_j = r_j - 1$ and no case on (2.2.1) comes from the first line, thus $q_l \geq r_l$ for $l \neq j$. This, together with the fact that $\sum_{i=1}^{k} q_l < r$, implies that $q_l = r_l$ for $l \neq j$. We have that $-2(r_j - 1) = -d_j(r-1)$, therefore $r_j = r$ and $d_j = 2$, or $r_j < r$ and $d_j = 1$.

In the first case we have that $r_j = r$ implies that $r_i = 0$ for every $i \neq j$. This means that we have $\Omega^{r_i}(2r_i - d_i(r-1)) = \mathcal{O}_{\mathbb{P}^{m_i}}(-d_i(r-1))$. Since $-d_i(r-1) < 0$, we have that the cohomology $H^{q_i}(\mathcal{O}_{\mathbb{P}^{m_i}}(-d_i(r-1)))$ does not vanish just for $q_i = m_i$, but since $m_i > 0$, we have that $q = \sum_{i \neq j} q_i + q_j = \sum_{i \neq j} q_i + r - 1 \geq r$, therefore our cases of interest have vanishing cohomology.

The second possibility for this cohomology to be non-vanishing is that we have $r_j < r$ and $d_j = 1$. Suppose that one of these non-vanishing cohomologies comes from the second line of (2.2.1) for some l. From the conditions on (4.2.1) and (2.2.1) respectively, we have that $2(r_j - 1) = r - 1$ and $2r_l - d_l(r-1) = 0$, since $d_l \geq 1$ this implies that

$$r_j = \frac{r}{2} + \frac{1}{2}, \; r_l \geq \frac{r}{2} - \frac{1}{2},$$

therefore $r_j + r_l \geq r$. Since k is at least 3, we have another case i, that comes either from the second or third line of (2.2.1), and we must have $r_i = 0$. If it is on the second line we have that $2r_i - d_i(r-1) = 0$, thus $d_i(r-1) = 0$, that is a contradiction. It can not be on the third line either, since $r_i = m_i = 0$ is also a contradiction. Otherwise, in case $k = 2$, we assume, without loss of generality, that $j = 1$ and $l = 2$, then $d_1 = 1$ and $d_2 \geq 2$, thus $r_2 \geq r - 1$, so we obtain

$$r_1 + r_2 \geq \frac{r}{2} + \frac{1}{2} + r - 1 \geq r + \frac{1}{2} > r,$$

that is a contradiction. Therefore, no case can come from the second line on (2.2.1).

This means that all the other cases must come from the third line of (2.2.1), that is $q_l = m_l = r_l$ for $l \neq j$. We notice that $r_j > \frac{r}{2}$ implies that $r_j > \sum_{l \neq j} r_l$, thus

$$m_j \geq r_j > \sum_{l \neq j} r_l = \sum_{l \neq j} m_l,$$

this is a contradiction since $d_j = 1$.

□

Corollary 4.2.5. *[Tur22, Corollary 4.6] On the hypothesis of Lemma 4.2.4 we have that*

$$H^q((\bigwedge^r \mathcal{E}^*) \otimes \mathcal{E}) = 0.$$

Theorem 4.2.6. *[Tur22, Theorem 4.7] On the hypothesis of Lemma 4.2.4, the induced homomorphism*

$$\mathcal{E}^* \otimes \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{E}$$

induces an isomorphism at the level of global sections, where Z is the zero locus of a section $s \in \mathcal{E}$.

Proof. We have the following Koszul complex

$$0 = \bigwedge^{N+1} \mathcal{E}^* \xrightarrow{\varphi_N} \bigwedge^{N} \mathcal{E}^* \xrightarrow{\varphi_{N-1}} \ldots \xrightarrow{\varphi_2} \bigwedge^{2} \mathcal{E}^* \to \mathcal{E}^* \to \mathcal{I}_Z \to 0.$$

Let $\mathcal{F}_r$ to be defined as the quotient $\mathcal{F}_r = \bigwedge^r \mathcal{E}^* / \mathrm{Im}\varphi_r$. Thus we obtain short exact sequences

$$0 \to \mathcal{F}_2 \to \mathcal{E}^* \to \mathcal{I}_Z \to 0,$$
$$0 \to \mathcal{F}_{r+1} \to \bigwedge^r \mathcal{E}^* \to \mathcal{F}_r \to 0,$$

for $r = 2, \ldots, N$.

Tensoring the second short exact sequence by $\mathcal{E}$, we obtain the long exact sequence of cohomologies

$$\begin{aligned} \cdots \to H^{r-2}(\bigwedge^r \mathcal{E}^* \otimes \mathcal{E}) \to H^{r-2}(\mathcal{F}_r \otimes \mathcal{E}) \to H^{r-1}(\mathcal{F}_{r+1} \otimes \mathcal{E}) \to \\ \to H^{r-1}(\bigwedge^r \mathcal{E}^* \otimes \mathcal{E}) \to H^{r-1}(\mathcal{F}_r \otimes \mathcal{E}) \to H^{r}(\mathcal{F}_{r+1} \otimes \mathcal{E}) \to \ldots \end{aligned} \tag{4.2.2}$$

By Corollary 4.2.5 we have that both terms on the left are zero, therefore we have that

$$H^{r-2}(\mathcal{F}_r \otimes \mathcal{E}) \cong H^{r-1}(\mathcal{F}_{r+1} \otimes \mathcal{E}), \; H^{r-1}(\mathcal{F}_r \otimes \mathcal{E}) \subset H^{r}(\mathcal{F}_{r+1} \otimes \mathcal{E}).$$

This implies that

$$H^0(\mathcal{F}_2 \otimes \mathcal{E}) \cong \ldots \cong H^{N-1}(\mathcal{F}_{N+1} \otimes \mathcal{E}) = 0,$$
$$H^1(\mathcal{F}_2 \otimes \mathcal{E}) \subset \cdots \subset H^N(\mathcal{F}_{N+1} \otimes \mathcal{E}) = 0.$$

If we consider now the long exact sequence of cohomologies from $0 \to \mathcal{F}_2 \otimes \mathcal{E} \to \mathcal{E}^* \otimes \mathcal{E} \to \mathcal{I}_Z \otimes \mathcal{E} \to 0$, we obtain

$$H^0(\mathcal{F}_2 \otimes \mathcal{E}) \to H^0(\mathcal{E}^* \otimes \mathcal{E}) \to H^0(\mathcal{I}_Z \otimes \mathcal{E}) \to H^1(\mathcal{F}_2 \otimes \mathcal{E}),$$

since the end terms are zero, we obtain the desired isomorphism. $\square$

To make a comparison with the symmetric case, our next objective is to prove the extension of Corollary 4.1.3 to the partially symmetric case. That is, we will show if two sections s, t, that arise from the respective tensors S, $T \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k$, have the same image under ψ, that is, we have the equality of the zero locus $Z(s) = Z(t)$, then $s = \lambda t$.

Lemma 4.2.7. *[Tur22, Lemma 4.8] Let $\mathcal{E}_i = \pi_i^* Q_i(d_1, \ldots, d_i - 1, \ldots, d_k)$. If $\dim \mathbb{P}V_j \geq 2$ for all j, then*

$$H^0(\mathrm{Hom}(\mathcal{E}, \mathcal{E}_j)) = H^0(\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_j)) = \mathbb{C}.$$

Moreover, if we assume that $i \neq j$, then

$$H^0(\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)) = 0.$$

Proof. For the second equality we have that

$$\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = \pi_i^* Q_i^* \otimes \pi_j^* Q_j(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0) =$$
$$\mathcal{O}_{\mathbb{P}V_1} \otimes \cdots \otimes \pi_i^* Q_i^*(1) \otimes \cdots \otimes \pi_j^* Q_j(-1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}V_k}.$$

We notice that, for all i, we have that $H^0(Q_i^*(1)) = H^0(\Omega^1(2)) \neq 0$. Meanwhile, if $\dim \mathbb{P}V_j \geq 2$, then $H^0(Q_j(-1)) = 0$, thus by the Künneth's Formula we have that

$$H^0(\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)) = 0.$$

On the other hand,

$$\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_j) = \mathcal{O}_{\mathbb{P}V_1} \otimes \cdots \otimes \left(\pi_j^*(Q_j^* \otimes Q_j)\right) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}V_k} = \mathrm{Hom}(Q_j, Q_j),$$

since the bundle Q_j is simple we obtain the desired result.

$\square$

Lemma 4.2.8. *[Tur22, Lemma 4.9] Let $\rho \in \mathrm{End}(H^0(\mathcal{E}))$ be a endomorphism of $H^0(\mathcal{E})$, suppose that f, $g \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k$ are tensors such that $\rho(s_f) = \rho(s_g)$, then $s_f = \lambda s_g$ for $\lambda \in \mathbb{C}$.*

Proof. Let I_1 be the set of indices such that $\dim \mathbb{P}V_i = 1$ and I_2 be the set of indices such that $\dim \mathbb{P}V_i \geq 2$. By the previous lemma, we have that $H^0(\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)) = 0$, whenever $j \in I_2$, that is, no map can act there besides its own endomorphism.

Now we consider a section s_f coming from a tensor $f \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k$. We recall that the map φ associates f to the diagonal map of its flattenings in each coordinate l, that is, $f : \mathrm{Sym}^{d_1} V_1 \otimes \mathrm{Sym}^{d_l - 1} V_l \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k \to V_l$. This means that $\varphi(f) = s_f$ can be interpreted as the diagonal element $s_f = (f, \ldots, f)$, where f in the l entry of this vector means the section of $\mathcal{E}_l$ corresponding to f.

Suppose that the first l indices are in I_1 and the others are in I_2, both nonempty. Applying ρ to $\varphi(f)$ we obtain that

$$\rho(\varphi(f)) = (M_1(f), \ldots, M_l(f), M_{l+1}(f), \ldots, M_k(f)) = (g, \ldots, g) = s_g,$$

where $g \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k$ is a tensor and $M_i(f) \in H^0(\mathcal{E}_i)$ is a global section. From the Lemma 4.2.7 we have that $M_i(f) = \lambda f$ for any $i \geq l+1$ and $\lambda_i \in \mathbb{C}$. Since $M_i(f) = g$ for all $i = 1, \ldots, k$, we obtain that $M_i(f) = \lambda f$ with $\lambda \in \mathbb{C}$, thus $g = \lambda f$.

It remains the case when $I_2 = \emptyset$. In such case we notice that

$$Hom(\mathcal{E}_i, \mathcal{E}_j) = (0, \ldots, 0, Q_i^*(1), 0, \ldots, 0, Q_j(-1), 0, \ldots, 0);$$

since the dimension of each $\mathbb{P}V_i$ is 1, we have $Q_j(-1) = \mathcal{O}_{\mathbb{P}^1}$, moreover $Q_i^*(1) = \Omega^1_{\mathbb{P}^1}(2) = \mathcal{O}_{\mathbb{P}^1}$. We recall that both of those bundles are 1-dimensional at the level of global sections, that is, $\dim H^0(\mathcal{O}_{\mathbb{P}^1}) = 1$, therefore $\dim H^0(Hom(\mathcal{E}_i, \mathcal{E}_j)) = 1$. This implies that if $\rho(s_g) = s_f$, then $s_f = \lambda s_g$ is the only possible image. $\square$

Combining the previous results together, we obtain the next theorem.

Theorem 4.2.9. *[Tur22, Theorem 4.10] Let $S, T \in \mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k$ be two general tensors. Assume that $m_l \leq \sum_{i \neq l} m_i$ holds for every l such that $d_l = 1$, $k \geq 3$, and that $m_j \geq 1$ for all j. Let $s, t \in \mathcal{E}$ be the sections coming from the tensors, S and T, and assume that $Z(s) = Z(t)$, then $s = \lambda t$, for $\lambda \in \mathbb{C}^*$.*

Additionally, if $k = 2$ and we also consider the hypothesis that $(d_1, d_2) \neq (1, 1)$, then the result still holds.

Proof. The Theorem 4.2.6 says that the map $H^0(\mathrm{End}(\mathcal{E})) \to H^0(\mathcal{I}_Z \otimes E)$ defined by $\rho \mapsto \rho(s_g)$ is an isomorphism. This means that if s, t are two tensors such that $Z(s) = Z(t)$, then there exists a morphism $\rho \in \mathrm{End}\mathcal{E}$ such that

$$\rho(t) = s.$$

Furthermore, from the Lemma 4.2.8 we obtain that $s = \lambda t$. $\square$

With all those results in mind, we can conclude with a note that combining together Theorem 4.2.1 with the Theorem 4.2.9 we obtain the Theorem 4.2.10.

Theorem 4.2.10. *[Tur22, Theorem 1.2] Let $V_1, \dots, V_k$ be vector spaces of dimension $m_1+1, \dots, m_k+1$. Let $d_1, \dots, d_k$ be positive integers, and $T \in \mathbb{P}\big(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k\big)$ be a general tensor. Let*

$$\tau : \mathbb{P}\big(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k\big) \dashrightarrow (\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_k)^{(\mathrm{ed}_X)}, \ T \mapsto \mathrm{Eig}(T),$$

be the map that associates a tensor T to its singular tuples locus $\mathrm{Eig}(T)$. If $k \geq 3$ and suppose that $m_l \leq \sum_{j \neq l} m_j$ whenever $d_l = 1$, and for $k = 2$ we include the hypothesis that $(d_1, d_2) \neq (1, 1)$, then

$$\tau^{-1}(\tau(T)) = \begin{cases} [T], & \text{if } d_i \text{ is odd for some } i; \\ \{[T + cq_1^{\frac{d_1}{2}} \otimes \cdots \otimes q_k^{\frac{d_k}{2}}] | c \in \mathbb{C}\}, & \text{if } d_l \text{ is even for all } l. \end{cases}$$

Moreover, the image of the map τ has dimension

$$\dim(\mathrm{Im}(\tau)) = \begin{cases} \prod_{l=1}^{k} \binom{d_l + m_l}{d} - 1, & \text{if } d_i \text{ is odd for some } i; \\ \prod_{l=1}^{k} \binom{d_l + m_l}{d} - 2, & \text{if } d_l \text{ is even for all } l. \end{cases}$$

4.3 A remark on sections coming from tensors

We make a brief remark that the Lemma 4.2.8 does not mean that all the morphisms in $\mathrm{End}(\mathcal{E})$ are multiplication by scalars, since the map φ is not surjective in general. Indeed, for each space V_l, we can compute what is the image of $\varphi_l : \mathrm{Sym}^{d_l} V_l \to H^0(Q_l(d_l - 1))$.

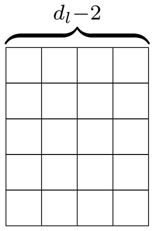
We have the Euler exact sequence

$$0 \to H^0(\mathcal{O}(d_l - 2)) \to H^0(\mathcal{O}(d_l - 1) \otimes V_l) \to H^0(Q(d_l - 1)) \to 0.$$

Moreover we have an isomorphism

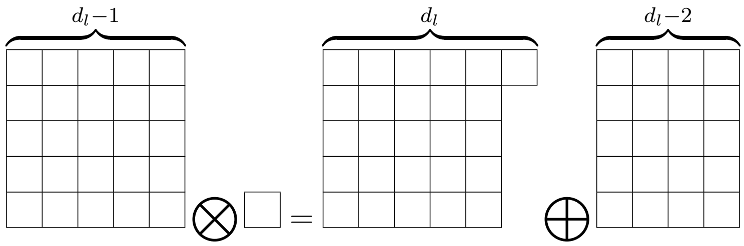
$$H^0(\mathcal{O}(d_l - 1) \otimes V_l) \cong \mathrm{Sym}^{d_l - 1} V_l^* \otimes V_l \text{ and } H^0(\mathcal{O}(d_l - 2)) \cong \mathrm{Sym}^{d_l - 2} V_l.$$

In terms of Young diagrams, $\mathrm{Sym}^{d_l-2} V_l^*$ has the representation

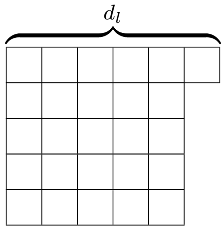


where the tableaux has $d_l - 2$ columns and $\dim V_l - 1$ rows. Moreover, $\mathrm{Sym}^{d_l-1} V_l^* \otimes$

V_l is represented by



via Pieri's formula, where all the diagrams have $\dim V_i - 1$ rows. Thus $H^0(Q_l(d_l - 1))$ diagram is given by



where the diagram has $\dim V_l - 1$ rows.

We can compute what exactly is $H^0(Q(d_l - 1))$ in terms of irreducible $\mathrm{SO}(m_l)$-representations by restricting the product $\mathrm{Sym}^{d_l-1} V_l \otimes V$ as $SL(m_l)$-representations to $\mathrm{SO}(m_l)$-representations. Indeed, we have that as $SL(m_l)$-representations

$$\mathrm{Sym}^{d_l-1} V_l \otimes V_l = \mathrm{Sym}^{d_l} V_l \oplus \Gamma^{(d_l-2,1,0,\dots,0)},$$

the first summand restricts to

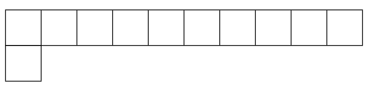
$$Res^{SL(m_l)}_{\mathrm{SO}(m_l)}(\mathrm{Sym}^{d_l} V_l) = H_{d_l} \oplus H_{d_l-2} \oplus \cdots \oplus \begin{cases} H_1 & \text{if } d_l \text{ is odd;} \\ H_0 & \text{if } d_l \text{ is even.} \end{cases} \tag{4.3.1}$$

To compute the restriction of $\Gamma^{(d_l-2,1,0,\dots,0)}$ we now utilise the fact that

$$Res^{SL(m_l)}_{\mathrm{SO}(m_l)}(\Gamma^{\lambda}) = \bigoplus_{\overline{\lambda}} N_{\lambda\overline{\lambda}} \Gamma^{\overline{\lambda}},$$

with $N_{\lambda\overline{\lambda}} = \sum_{\delta} N_{\delta,\overline{\lambda}\lambda}$, where $N_{\delta\overline{\lambda}\lambda}$ is the Littlewood-Richardson coefficient, $\overline{\lambda} = (\overline{\lambda}_1 \geq \overline{\lambda}_2 \geq \cdots \geq \overline{\lambda}_t \geq 0)$, $m_l = 2t$ or $2t+1$, and $\delta = (\delta_1 \geq \delta_2 \geq \cdots \geq 0)$, with δ_i even for all i. For more details about the restriction and Littlewood-Richardson coefficients we suggest [FH91] section 25.3 and appendix A respectively.

In our setting $\lambda = (d_l - 2, 1, \dots, 0)$ is represented as the Young tableaux given by



with $d_l - 1$ boxes in the first line and 1 box on the second.

To compute the Littlewood-Richardson coefficient we have to go through all the possible Young tableaux δ with even number of box in each rows and complete it into λ via a $\overline{\lambda}$-expansion. The procedure is to fill each box in the i-th row of $\overline{\lambda}$ with the number i and from left to right and from top to bottom add each row to the Young tableaux of δ following the Pieri's rule; in simple terms, boxes coming from the same row in $\overline{\lambda}$ can not be attached in the same column and boxes coming from the same column in $\overline{\lambda}$ can not be attached in the same row. For a precise description of the algorithm we refer to the appendices (A7) and (A8) in [FH91].

If $\delta = (0)$, the only possible tableaux for $\overline{\lambda}$ is the tableaux of λ itself, that is, $\overline{\lambda} = (d_l - 2, 1, 0, \dots, 0)$.

If $\delta = 2h$, for $h \geq 1$ and $d_l - 2h - 2 \geq 0$, then there are two other possibilities for $\overline{\lambda}$, indeed $\overline{\lambda} = (d_l - 2h, 0, \dots, 0)$ or $\overline{\lambda} = (d_l - 2h - 2, 1, 0, \dots, 0)$, indeed this results respectively in the following two $\overline{\lambda}$-expansion:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline & & & & & & 1 & 1 & 1 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}$$

and

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline & & & & & & 1 & 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Notice that, if we tried to add more columns to the second line, we would have a box of index 1 and a box of index 2 in the first row, that is forbidden by the Littlewood-Richardson rule since they come from the same column. Similarly, if we tried to add more rows, we would have the same problem. Therefore those are all the possible $\overline{\lambda}$. Moreover, the first possibility for $\overline{\lambda}$ corresponds to H_{d_l-2h} and the second to $\Gamma^{(d_l-2h-2,1,0,\dots,0)}$. Let

$$\Gamma = \Gamma^{(d_l-2,1,0,\dots,0)} \oplus \Gamma^{(d_l-4,1,0,\dots,0)} \oplus \dots \oplus \begin{cases} \Gamma^{(1,1,0,\dots,0)} & \text{if } d_l \text{ is odd;} \\ \Gamma^{(0,1,0,\dots,0)} & \text{if } d_l \text{ is even,} \end{cases}$$

and

$$\mathcal{H}_{d_l-2} = H_{d_l-2} \oplus H_{d_l-4} \oplus \dots \oplus \begin{cases} H_1 & \text{if } d_l \text{ is odd;} \\ H_2 & \text{if } d_l \text{ is even.} \end{cases}$$

We obtain that

$$Res^{SL(m_l)}_{\mathrm{SO}(m_l)}(\Gamma^{(d_l-2,1,0,\dots,0)}) = \mathcal{H}_{d_l-2} \oplus \Gamma, \tag{4.3.2}$$

moreover, by assembling together the equations 4.3.1 and 4.3.2 we have found that

$$Res^{SL(m_l)}_{\mathrm{SO}(m_l)}(\mathrm{Sym}^{d_l-1} V_l \otimes V_l) = \mathcal{H}_{d_l-2} \oplus \Gamma \oplus H_{d_l} \oplus H_{d_l-2} \oplus \dots \oplus \begin{cases} H_1 & \text{if } d_l \text{ is odd;} \\ H_0 & \text{if } d_l \text{ is even.} \end{cases}$$

Now we can compute the $H^0(Q(d_l - 1))$, from the Euler exact sequence we have that it is given by the difference $\mathrm{Sym}^{d_l-1}V_l \otimes V_l - \mathrm{Sym}^{d_l-2}V_l$, that is,

$$\begin{aligned} H^0(Q(d_l-1)) =& \mathcal{H}_{d_l-2} \oplus \Gamma \oplus H_{d_l} \oplus H_{d_l-2} \oplus \cdots \oplus \begin{cases} H_1 \text{ if } d_l \text{ is odd;} \\ H_0 \text{ if } d_l \text{ is even;} \end{cases} \\ & - H_{d_l-2} \oplus H_{d_l-4} \oplus \cdots \oplus \begin{cases} H_1 \text{ if } d_l \text{ is odd;} \\ H_0 \text{ if } d_l \text{ is even,} \end{cases} \end{aligned}$$

therefore we obtain that,

$$H^0(Q(d_l-1)) = \mathcal{H}_{d_l} \oplus \Gamma,$$

where $\mathcal{H}_{d_l}$ is the subrepresentation of the sections coming from tensors by the Lemma 4.1.1, thus it is the image of the φ_l that we sought at the beggining of this section. We conclude that φ_l is not surjective.

Chapter 5

Singular vector tuples beyond boundary format

Notice that during the previous chapter the boundary format condition has been assumed for all the results. The results presented in this chapter comes from the work developed by the author together with Luca Sodomaco [ST22] on the same problem disregarding the boundary format condition. The first non-trivial example beyond boundary format is for the tensor format $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$. We explore this example and other formats on this chapter.

Since in this chapter we deal with several different tensor formats we introduce a new notation, a tensor space $V = \mathbb{C}^{m_1+1} \otimes \cdots \otimes \mathbb{C}^{m_k+1}$ will said to be a tensor space of order k and format $\mathbf{m} = (m_1 + 1, \ldots, m_k + 1)$. We recall that the dimension of the critical space has been given beyond boundary format for tensors in [OP15].

Proposition 5.0.1. *[OP15, Proposition 5.6] Let $T \in V = \bigotimes_{i=1}^k \mathbb{C}^{m_i+1}$. Assume $m_1 \leq \cdots \leq m_k$ and let $D = \prod_{i=1}^{k-1}(m_i + 1)$. The dimension of the critical space $H_T \subset V$ is*

$$\begin{cases} \prod_{i=1}^k (m_i+1) - \sum_{i=1}^k \binom{m_i+1}{2} & \text{for } m_k + 1 \leq D \\ \binom{D+1}{2} - \sum_{i=1}^{k-1} \binom{m_i+1}{2} & \text{for } m_k + 1 \geq D\,. \end{cases} \tag{5.0.1}$$

An important concept that will be used in this chapter is the notion of concise tensor spaces. A tensor space $V = \bigotimes_{i=1}^k \mathbb{C}^{m_i+1}$ is concise if there exists a tensor $T \in V$ such that does not exist a proper linear subspace $L_i \subset \mathbb{C}^{m_i+1}$ with $T \in V_{L_i} = \mathbb{C}^{m_1+1} \otimes \cdots \otimes L_i \otimes \cdots \otimes \mathbb{C}^{m_k+1}$, for every $i \in [k]$. The tensor space V is non-concise if every tensor belong to some subspace of the form of V_{L_i}. The tensor T is said to be concise if it does not belong to any V_{L_i}, and non-concise otherwise.

The conciseness of the tensor space is correlated to the dimension of the tensor space. Suppose that $m_1 \leq \cdots \leq m_k$, then the tensor space V is concise if and only if $m_k + 1 \leq \prod_{i=1}^{k-1}(m_i + 1)$.

In this chapter we work in two main problems beyond the boundary format:

1. Does $T \in \langle Z_T \rangle$?

2. Is a generic T determined by its singular vector tuples?

We observe that if we prove (1), (2) for the last concise format $(m_1+1, \ldots, m_{k-1}+1, \prod_{i=1}^{k-1}(m_i+1))$, then the same results follow immediately to all non-concise formats $(m_1+1, \ldots, m_k+1)$, $m_k+1 > \prod_{i=1}^{k-1}(m_i+1)$, by considering the concise subspace that T belongs.

Therefore, in this chapter we will be interested in tensors spaces of format $(m_1+1, \ldots, m_k+1)$ such that

$$\sum_{i=1}^{k-1} m_i < m_k \leq \prod_{i=1}^{k-1}(m_i+1) - 1.$$

For the sake of simplicity we will denote $\bigwedge^r \mathcal{E}^* \otimes \mathcal{O}(d_1, \ldots, d_k)$ by $\mathcal{E}^{(r)}$ in this chapter.

5.1 First examples beyond boundary format

We start this section by analysing the behavior of particular formats beyond boundary format.

Proposition 5.1.1. *[ST22, Proposition 5.6] Let T be a general tensor of format $(2,2,4)$. Then $\langle Z_T \rangle$ has dimension six in $\mathbb{P}(V) \cong \mathbb{P}^{15}$ and codimension one in $\mathbb{P}(H_T)$. This is the last concise format $(2,2,N+1)$.*

Proof. Following the similar cohomology computation in [DOT17, Lemma 3.5], we have that the vanishing of the cohomologies $H^q(\mathcal{E}^{(r)})$, $q = r-1, r-2$, does not hold anymore. Moreover, this means that computing $H^r(\mathcal{E}^{(r)})$ is useful in many cases. In this case one computes that the only non-zero dimensions $h^q(\mathcal{E}^{(r)})$ are

$$h^2(\mathcal{E}^{(3)}) = 1\,, \quad h^3(\mathcal{E}^{(3)}) = 1\,.$$

Consider the short exact sequence in (3.2.2). The corresponding long exact sequence in cohomology is

$$\begin{aligned} \cdots \to H^{r-2}(\mathcal{E}^{(r)}) \to H^{r-2}(\mathcal{F}_r(\mathbf{1})) \to H^{r-1}(\mathcal{F}_{r+1}(\mathbf{1})) \to H^{r-1}(\mathcal{E}^{(r)}) \to \\ \to H^{r-1}(\mathcal{F}_r(\mathbf{1})) \to H^r(\mathcal{F}_{r+1}(\mathbf{1})) \to H^r(\mathcal{E}^{(r)}) \to \cdots \end{aligned} \tag{5.1.1}$$

The sequence (5.1.1) yields the following inclusions and isomorphisms:

- $H^{r-2}(\mathcal{F}_r(\mathbf{1})) \cong H^{r-1}(\mathcal{F}_{r+1}(\mathbf{1}))$ and $H^{r-1}(\mathcal{F}_r(\mathbf{1})) \cong H^r(\mathcal{F}_{r+1}(\mathbf{1}))$ for $r \neq 3$
- $H^1(\mathcal{F}_3(\mathbf{1})) \subset H^2(\mathcal{F}_4(\mathbf{1}))$ and $H^2(\mathcal{F}_3(\mathbf{1})) \cong H^2(\mathcal{E}^{(3)})$ for $r = 3$.

In turn, we get that

- $H^0(\mathcal{F}_2(\mathbf{1})) \cong H^1(\mathcal{F}_3(\mathbf{1})) \subset H^2(\mathcal{F}_4(\mathbf{1})) \cong H^3(\mathcal{F}_5(\mathbf{1})) \cong H^4(\mathcal{F}_6(\mathbf{1})) = 0$

- $H^1(\mathcal{F}_2(\mathbf{1})) \cong H^2(\mathcal{F}_3(\mathbf{1}))$ and $H^3(\mathcal{F}_4(\mathbf{1})) \cong H^4(\mathcal{F}_5(\mathbf{1})) \cong H^5(\mathcal{F}_6(\mathbf{1})) = 0$

Therefore, if we take the second short exact sequence in (3.2.3) and we compute the corresponding long exact sequence in cohomology, we get that

$$0 = H^0(\mathcal{F}_2(\mathbf{1})) \to H^0(\mathcal{E}^{(1)}) \to H^0(\mathcal{I}_{Z_T}(\mathbf{1})) \to H^1(\mathcal{F}_2(\mathbf{1})) \to H^1(\mathcal{E}^{(1)}) = 0\,,$$

thus $h^0(\mathcal{I}_{Z_T}(\mathbf{1})) = h^0(\mathcal{E}^{(1)}) + h^1(\mathcal{F}_2(\mathbf{1})) = 8 + 1$. This means that $\langle Z_T\rangle$ has codimension 9 in $\mathbb{P}(V) \cong \mathbb{P}^{15}$, that is $\dim\langle Z_T\rangle = 15 - 9 = 6$. $\square$

Example 5.1.2. Experimental computations suggest to consider the following linear relation. Let $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ be a singular triple of U. By definition the two vectors $U(\mathbf{x}_1 \otimes \mathbf{x}_2)$ and $\mathbf{x}_3$ are proportional. From this fact we build the 4×4 matrix

$$A \coloneqq \begin{bmatrix} U(\mathbf{x}_1 \otimes \mathbf{x}_2) & \mathbf{x}_3 & U_{(1,1)} & U_{(1,2)} \end{bmatrix}^T ,$$

where $U_{(i,j)} = \begin{bmatrix} u_{ij1} & \dots & u_{ij4} \end{bmatrix}^T$ for all $(i,j) \in \{1,2\} \times \{1,2\}$. If U is sufficiently general, we have that $\operatorname{rank}(A) = 3$. Now let $\mathbf{x}_1' = \begin{bmatrix} x_{1,2} & x_{1,1} \end{bmatrix}^T$ and consider the matrix

$$A' \coloneqq \begin{bmatrix} U(\mathbf{x}_1' \otimes \mathbf{x}_2) & \mathbf{x}_3 & U_{(2,1)} & U_{(2,2)} \end{bmatrix}^T .$$

In this case the first two rows of A' are not proportional. We checked symbolically that still $\operatorname{rank}(A') = 3$, hence the determinant of A', which is linear in the coordinates z_{ijk} of $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4)$, is contained in the ideal of $\langle Z_U\rangle$. We verified also that $\det(A')$ is linearly independent from the equations of H_T. Hence $\det(A')$ can be considered as "the" unknown additional relation among the singular triples of U.

Developing $\det(A')$ using the Laplace expansion along the first two rows of A' and taking into account the relations $z_{ijk} = x_{1,i}x_{2,j}x_{3,k}$, we get that (we omit the computation)

$$\det(A') = \begin{vmatrix} z_{211} & z_{212} & z_{213} & z_{214} \\ u_{111} & u_{112} & u_{113} & u_{114} \\ u_{211} & u_{212} & u_{213} & u_{214} \\ u_{221} & u_{222} & u_{223} & u_{224} \end{vmatrix} + \begin{vmatrix} z_{221} & z_{222} & z_{223} & z_{224} \\ u_{121} & u_{122} & u_{123} & u_{124} \\ u_{211} & u_{212} & u_{213} & u_{214} \\ u_{221} & u_{222} & u_{223} & u_{224} \end{vmatrix} .$$

From this expression, we immediately observe that this additional relation is satisfied by the tensor U itself, since substituting z_{ijk} by u_{ijk} implies that in the first summand the first and the third rows are the same, as well the first and fourth rows in the second summand are the same, meaning that $[U] \in \langle Z_U\rangle$.

Note the change of indices with respect to

$$\det(A) = \begin{vmatrix} z_{211} & z_{212} & z_{213} & z_{214} \\ u_{211} & u_{212} & u_{213} & u_{214} \\ u_{111} & u_{112} & u_{113} & u_{114} \\ u_{121} & u_{122} & u_{123} & u_{124} \end{vmatrix} + \begin{vmatrix} z_{221} & z_{222} & z_{223} & z_{224} \\ u_{221} & u_{222} & u_{223} & u_{224} \\ u_{111} & u_{112} & u_{113} & u_{114} \\ u_{121} & u_{122} & u_{123} & u_{124} \end{vmatrix} .$$

Note that both determinants may be seen as bihomogeneous polynomials in the variables u_{ijk} and z_{ijk} of bidegree $(3, 1)$. What is more, observe that in the construction of A we have made a choice for the last two rows. In general, there are $6 = \binom{4}{2}$ possibilities to complete the matrix A using the vectors $(u_{ijk})_k$. Consider also the vector $\mathbf{x}'_2 = \begin{bmatrix} x_{2,2} & x_{2,1} \end{bmatrix}^T$ and build the 9×4 matrix

$$\begin{bmatrix} U(\mathbf{x}'_1 \otimes \mathbf{x}_2) & U(\mathbf{x}_1 \otimes \mathbf{x}'_2) & U(\mathbf{x}_2 \otimes \mathbf{x}_1) & U(\mathbf{x}'_2 \otimes \mathbf{x}'_1) & \mathbf{x}_3 & U_{(1,1)} & U_{(1,2)} & U_{(2,1)} & U_{(2,2)} \end{bmatrix}^T . \tag{5.1.2}$$

We computed symbolically all maximal minors of the previous matrix. There are exactly 6 of them which belong to the ideal of $\langle Z_U \rangle$. One of them is exactly the determinant of A' studied above. The other five are obtained considering all remaining choices of pairs of rows $(U_{(i_1,j_1)}, U_{(i_2,j_2)})$ among the last four rows, the row of $\mathbf{x}_3$ and one of the first four rows (according to symmetries of the pair $(U_{(i_1,j_1)}, U_{(i_2,j_2)})$ chosen).

With similar techniques, we are able to prove the following result.

Theorem 5.1.3. *[ST22, Theorem 5.8] Let T be a general tensor of format $(2, 3, N+1)$.*

- *(i) If $N + 1 = 5$, then $\langle Z_T \rangle$ has either dimension 13 or 14 in $\mathbb{P}(V) \cong \mathbb{P}^{29}$. The expected dimension is 13, hence there are 2 more linear relations among the singular tuples of T.*
- *(ii) If $N + 1 = 6$, $\langle Z_T \rangle$ has either dimension 13 or 14 in $\mathbb{P}(V) \cong \mathbb{P}^{35}$. The expected dimension is 13, hence there are 3 more linear relations among the singular tuples of T. This is the last concise case of format $(2, 3, N+1)$.*

Proof. (i) Suppose that $n = 5$. In this case the non-zero dimensions $H^q(\mathcal{E}^{(r)})$ are

$$h^3(\mathcal{E}^{(3)}) = 1\,, \quad h^3(\mathcal{E}^{(4)}) = 2\,, \quad h^5(\mathcal{E}^{(5)}) = 3\,.$$

These equalities immediately imply the relations

1. $H^0(\mathcal{F}_2(\mathbf{1})) \cong \cdots \cong H^2(\mathcal{F}_4(\mathbf{1})) \subset H^3(\mathcal{F}_5(\mathbf{1})) \cong \cdots \cong H^6(\mathcal{F}_8(\mathbf{1})) = 0$.
2. $H^1(\mathcal{F}_2(\mathbf{1})) \cong H^2(\mathcal{F}_3(\mathbf{1})) \subset H^3(\mathcal{F}_4(\mathbf{1}))$ and $H^4(\mathcal{F}_5(\mathbf{1}) \subset \cdots \subset H^7(\mathcal{F}_8(\mathbf{1})) = 0$.

We study again the sequence (5.1.1) where the vanishing does not hold, namely for $r \in \{3, 4, 5\}$:

Case $r = 5$: we get $h^5(\mathcal{F}_5(\mathbf{1})) = h^5(\mathcal{E}^{(5)}) = 3$.
Case $r = 4$: it is again straightforward to see that $h^3(\mathcal{F}_4(\mathbf{1})) = h^3(\mathcal{E}^{(4)}) = 2$ and $h^4(\mathcal{F}_4(\mathbf{1})) = h^5(\mathcal{F}_5(\mathbf{1})) = 3$.
Case $r = 3$: we have

$$0 \to H^2(\mathcal{F}_3(\mathbf{1})) \to H^3(\mathcal{F}_4(\mathbf{1})) \to H^3(\mathcal{E}^{(3)}) \to H^3(\mathcal{F}_3(\mathbf{1})) \to H^4(\mathcal{F}_4(\mathbf{1})) \to 0\,.$$

Since $h^3(\mathcal{E}^{(3)}) = 1$, we have either $H^2(\mathcal{F}_3(\mathbf{1})) \cong H^3(\mathcal{F}_4(\mathbf{1}))$ or $H^2(\mathcal{F}_3(\mathbf{1})) \to H^3(\mathcal{F}_4(\mathbf{1})) \to H^3(\mathcal{E}^{(3)})$ is exact. In the first case we get $h^2(\mathcal{F}_3(\mathbf{1})) = 2$ and in the second $h^2(\mathcal{F}_3(\mathbf{1})) = 1$.

Plugging this information in the sequence

$$0 = H^0(\mathcal{F}_2(\mathbf{1})) \to H^0(\mathcal{E}^{(1)}) \to H^0(\mathcal{I}_{Z_T}(\mathbf{1})) \to H^1(\mathcal{F}_2(\mathbf{1})) \to 0$$

we get that $\mathrm{codim}(\langle Z_T\rangle) \in \{15, 16\}$, therefore $\dim(\langle Z_T\rangle) \in \{13, 14\}$. This implies that we have at least one new relation among the singular tuples and at most two.

(ii) Suppose that $n = 6$. The only non-zero dimensions $h^q(\mathcal{E}^{(r)})$ are

$$h^3(\mathcal{E}^{(3)}) = 1\,, \quad h^3(\mathcal{E}^{(4)}) = 12\,, \quad h^3(\mathcal{E}^{(5)}) = 9\,, \quad h^5(\mathcal{E}^{(5)}) = 3\,.$$

This implies the relations

1. $H^0(\mathcal{F}_2(\mathbf{1})) \cong H^1(\mathcal{F}_3(\mathbf{1})) \cong H^2(\mathcal{F}_4(\mathbf{1})) \subset H^3(\mathcal{F}_5(\mathbf{1}))$ and $H^4(\mathcal{F}_6(\mathbf{1})) \cong \cdots \cong H^7(\mathcal{F}_9(\mathbf{1})) = 0$.
2. $H^1(\mathcal{F}_2(\mathbf{1})) \cong H^2(\mathcal{F}_3(\mathbf{1})) \subset H^2(\mathcal{F}_4(\mathbf{1}))$ and $H^5(\mathcal{F}_6(\mathbf{1})) \cong \cdots \cong H^8(\mathcal{F}_9(\mathbf{1})) = 0$.
3. $H^6(\mathcal{F}_6(\mathbf{1})) \cong \cdots \cong H^9(\mathcal{F}_9(\mathbf{1})) = 0$.

We analyze the long exact sequence (5.1.1) where the vanishing does not hold, namely for $r \in \{3, 4, 5\}$:
Case $r = 5$: we have

$$0 \to H^3(\mathcal{E}^{(5)}) \to H^3(\mathcal{F}_5(\mathbf{1})) \to 0 \to H^4(\mathcal{F}_5(\mathbf{1})) \to 0 \to H^5(\mathcal{E}^{(5)}) \to H^5(\mathcal{F}_5(\mathbf{1})) \to 0\,.$$

This implies $h^3(\mathcal{F}_5(\mathbf{1})) = 9$, $h^4(\mathcal{F}_5(\mathbf{1})) = 0$, $h^5(\mathcal{F}_5(\mathbf{1})) = 3$.
Case $r = 4$: we have

$$\begin{aligned} 0 \to H^2(\mathcal{F}_4(\mathbf{1})) \to H^3(\mathcal{F}_5(\mathbf{1})) \to H^3(\mathcal{E}^{(4)}) \to H^3(\mathcal{F}_4(\mathbf{1})) \to \\ \to 0 \to H^4(\mathcal{F}_4(\mathbf{1})) \to H^5(\mathcal{F}_5(\mathbf{1})) \to 0\,. \end{aligned}$$

This yields $h^2(\mathcal{F}_4(\mathbf{1})) + 3 = h^3(\mathcal{F}_4(\mathbf{1}))$ and $h^4(\mathcal{F}_4(\mathbf{1})) = 3$.
Case $r = 3$: we have

$$\begin{gathered} 0 \to H^1(\mathcal{F}_3(\mathbf{1})) \to H^2(\mathcal{F}_4(\mathbf{1})) \to 0 \to H^2(\mathcal{F}_3(\mathbf{1})) \to H^3(\mathcal{F}_4(\mathbf{1})) \to \\ \to H^3(\mathcal{E}^{(3)}) \to H^3(\mathcal{F}_3(\mathbf{1})) \to H^4(\mathcal{F}_4(\mathbf{1})) \to 0\,. \end{gathered}$$

We get that $h^1(\mathcal{F}_3(\mathbf{1})) = h^2(\mathcal{F}_4(\mathbf{1}))$. Notice that $h^3(\mathcal{E}^{(3)}) = 1$, thus either $h^2(\mathcal{F}_3(\mathbf{1})) = h^3(\mathcal{F}_4(\mathbf{1}))$ or $H^2(\mathcal{F}_3(\mathbf{1})) \to H^3(\mathcal{F}_4(\mathbf{1})) \to H^3(\mathcal{E}^{(3)})$ is exact. This implies $h^2(\mathcal{F}_3(\mathbf{1})) = h^3(\mathcal{F}_4(\mathbf{1}))$ (Case 1) or $h^2(\mathcal{F}_3(\mathbf{1})) = h^3(\mathcal{F}_4(\mathbf{1})) - 1$ (Case 2).

Using these equalities on the long exact in cohomology of the short exact sequence

$$0 \to \mathcal{F}_2(\mathbf{1}) \to \mathcal{E}^{(1)} \to \mathcal{I}_{Z_T}(\mathbf{1}) \to 0$$

leads to $h^0(\mathcal{F}_2(\mathbf{1})) - 19 + \dim(\mathcal{I}_{Z_T}) - h^1(\mathcal{F}_2(\mathbf{1})) = 0$.

In the first case we get $h^2(\mathcal{F}_4(\mathbf{1})) - 19 + h^0(\mathcal{I}_{Z_T}) - h^3(\mathcal{F}_4(\mathbf{1})) = 0$, since $h^2(\mathcal{F}_4(\mathbf{1})) + 3 = h^3(\mathcal{F}_4(\mathbf{1}))$ we get $h^0(\mathcal{I}_{Z_T}) = 22$.

For the second case, in a similar manner we get $h^0(\mathcal{I}_{Z_T}) = 21$. Those two cases imply that $\dim(\langle Z_T \rangle) \in \{13, 14\}$. In turn, we have either two or three extra relations among the singular tuples of T. $\square$

Example 5.1.4. Consider a general tensor $U = (u_{ijk})$ of format $\mathbf{m} = (2, 3, 5)$. It admits 18 singular triples, and by Proposition 5.0.1 the projectivized critical space $\mathbb{P}(H_U) \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5) \cong \mathbb{P}^{29}$ has dimension 15. Let $Z_U \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5)$. By Theorem 3.1.10, we have that $|Z_U| = 18$ for a general U. The projective span $\langle Z_U \rangle$ is strictly contained in $\mathbb{P}(H_U)$: indeed we showed in Theorem 5.1.3(i) that $13 \leq \dim(\langle Z_U \rangle) \leq 14$. We verified symbolically that there exist two new relations among the singular triples, thus proving that $\dim(\langle Z_U \rangle) = 13$. We write them as determinants of 5×5 matrices:

$$\det(A_1) = \begin{vmatrix} U(\mathbf{x}_1' \otimes \mathbf{x}_2) & \mathbf{x}_3 & U_{(1,1)} & U_{(1,2)} & U_{(1,3)} \end{vmatrix} = \begin{vmatrix} T_{(1,1)} \\ U_{(2,1)} \\ U_{(1,1)} \\ U_{(1,2)} \\ U_{(1,3)} \end{vmatrix} + \begin{vmatrix} T_{(1,2)} \\ U_{(2,2)} \\ U_{(1,1)} \\ U_{(1,2)} \\ U_{(1,3)} \end{vmatrix} + \begin{vmatrix} T_{(1,3)} \\ U_{(2,3)} \\ U_{(1,1)} \\ U_{(1,2)} \\ U_{(1,3)} \end{vmatrix}$$

$$\det(A_2) = \begin{vmatrix} U(\mathbf{x}_1' \otimes \mathbf{x}_2) & \mathbf{x}_3 & U_{(2,1)} & U_{(2,2)} & U_{(2,3)} \end{vmatrix} = \begin{vmatrix} T_{(2,1)} \\ U_{(1,1)} \\ U_{(2,1)} \\ U_{(2,2)} \\ U_{(2,3)} \end{vmatrix} + \begin{vmatrix} T_{(2,2)} \\ U_{(1,2)} \\ U_{(2,1)} \\ U_{(2,2)} \\ U_{(2,3)} \end{vmatrix} + \begin{vmatrix} T_{(2,3)} \\ U_{(1,3)} \\ U_{(2,1)} \\ U_{(2,2)} \\ U_{(2,3)} \end{vmatrix}.$$

Also in this case we have chosen specific vectors $(u_{ijk})_k$ to form the matrices A_1 and A_2, but there are of course other choices and all possibilities can be obtained by computing all maximal minors of a large matrix similar to the one in (5.1.2).

Example 5.1.5. Consider a general tensor $U = (u_{ijk})$ of format $\mathbf{m} = (2, 3, 6)$. It admits 18 singular triples, and by Proposition 5.0.1 the projectivized critical space $\mathbb{P}(H_U) \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^6) \cong \mathbb{P}^{35}$ has dimension 16. Let $Z_U \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^6)$. By Theorem 3.1.10, we have that $|Z_U| = 18$ for a general U. Also in this case the projective span $\langle Z_U \rangle$ is strictly contained in $\mathbb{P}(H_U)$. By Theorem 5.1.3(ii) we have that $13 \leq \dim(\langle Z_U \rangle) \leq 14$, hence there are at least two and at most three new relations among singular triples. We computed symbolically the new three linear relations in this way. Consider $\mathbf{x}_1' = \begin{bmatrix} x_{1,2} & x_{1,1} \end{bmatrix}^T$ and the 6×6 matrices

$$
\begin{aligned}
A_1 &= \begin{bmatrix} U(\mathbf{x}_1' \otimes \mathbf{x}_2) & \mathbf{x}_3 & U_{(0,0)} & U_{(0,1)} & U_{(0,2)} & U_{(1,0)} \end{bmatrix}^T \\
A_2 &= \begin{bmatrix} U(\mathbf{x}_1' \otimes \mathbf{x}_2) & \mathbf{x}_3 & U_{(0,0)} & U_{(0,1)} & U_{(0,2)} & U_{(1,1)} \end{bmatrix}^T \\
A_3 &= \begin{bmatrix} U(\mathbf{x}_1' \otimes \mathbf{x}_2) & \mathbf{x}_3 & U_{(0,0)} & U_{(0,1)} & U_{(0,2)} & U_{(1,2)} \end{bmatrix}^T
\end{aligned}
$$

where $U_{(i,j)} = \begin{bmatrix} u_{ij1} & \dots & u_{ij6} \end{bmatrix}^T$ for all $(i,j) \in [2]\times[3]$. Each determinant $\det(A_j)$, after the substitutions $z_{ijk} = x_{1,i}x_{2,j}x_{3,k}$, gives a linear relation among the 18 singular triples of the general tensor U. Each linear relation can be seen as a sum of 2 determinants of 6×6 matrices.

The next proposition generalizes the observations made in Examples 5.1.2, 5.1.4 and 5.1.5, and provides a method to check easily that the new relations among singular k-tuples of a tensor U are satisfied by U itself.

Proposition 5.1.6. *[ST22, Proposition 6.3] Consider a tensor $U = (u_{i_1\cdots i_k})$ of format $\boldsymbol{m} = (m_1+1,\dots,m_k+1)$, where $m_k \geq \sum_{i=1}^{k-1}(m_i)$. Consider the $(m_k+1)\times(m_k+1)$ matrix*

$$A = \begin{bmatrix} U(\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_{k-1}) & \boldsymbol{y}_k & U_{I_1} & \cdots & U_{I_{m_k-1}} \end{bmatrix}^T$$

where $I_l \in \prod_{i=1}^{k-1}[m_i+1]$, with $[m_i+1] = \{1,\dots,m_i+1\}$ and $U_{I_l} = (u_{j_1\cdots j_{k-1}j_k} \mid (j_1,\dots,j_{k-1}) \in I_l)$ for all $l \in [m_k-1]$, while using (3.1.2),

$$U(\boldsymbol{y}_1 \otimes \cdots \otimes \boldsymbol{y}_{k-1})_s = \sum_{j_\ell \in [m_\ell+1]} u_{j_1\cdots j_{k-1}s}\, y_{1,j_1}\cdots y_{k-1,j_{k-1}} \quad \forall\, s \in [m_k+1]\,.$$

Then $\det(A)$ contains only terms in $y_{1,j_1}\cdots y_{k,j_k}$ with $(j_1,\dots,j_{k-1}) \in \prod_{i=1}^{k-1}[m_i+1] \setminus \{I_1,\dots,I_{m_k-1}\}$.

Proof. We compute $\det(A)$ by applying the generalized Laplace formula with respect to the first two rows of A. We use the shorthand $U_{I_l}^{(p,q)}$ to denote the row vector obtained after removing the columns p and q from U_{I_l}. We also denote by

$\sigma_{p,q}$ the permutation of $[m_k+1]$ sending 1 to p and 2 to q.

$$
\begin{aligned}
\det(A) &= \sum_{1\le p<q\le m_k+1} \operatorname{sign}(\sigma_{p,q}) \begin{vmatrix} U(\mathbf{y}_1\otimes\cdots\otimes\mathbf{y}_{k-1})_p & U(\mathbf{y}_1\otimes\cdots\otimes\mathbf{y}_{k-1})_q \\ y_{k,p} & y_{k,q} \end{vmatrix} \cdot \begin{vmatrix} U^{(p,q)}_{I_1} \\ \vdots \\ U^{(p,q)}_{I_{m_k-1}} \end{vmatrix} \\
&= \sum_{1\le p<q\le m_k+1} \operatorname{sign}(\sigma_{p,q}) \sum_{j_\ell\in[m_\ell+1]} (u_{j_1\cdots j_{k-1}p} z_{j_1\cdots j_{k-1}q} - u_{j_1\cdots j_{k-1}q} z_{j_1\cdots j_{k-1}p}) \begin{vmatrix} U^{(p,q)}_{I_1} \\ \vdots \\ U^{(p,q)}_{I_{m_k-1}} \end{vmatrix} \\
&= \sum_{j_\ell\in[m_\ell+1]} \sum_{1\le p<q\le m_k+1} \operatorname{sign}(\sigma_{p,q}) \begin{vmatrix} u_{j_1\cdots j_{k-1}p} & u_{j_1\cdots j_{k-1}q} \\ z_{j_1\cdots j_{k-1}p} & z_{j_1\cdots j_{k-1}q} \end{vmatrix} \cdot \begin{vmatrix} U^{(p,q)}_{I_1} \\ \vdots \\ U^{(p,q)}_{I_{m_k-1}} \end{vmatrix} \\
&= \sum_{j_\ell\in[m_\ell+1]} \det(\tilde{A}(j_1,\dots,j_{k-1}))\,,
\end{aligned}
\tag{5.1.3}
$$

where in the second equality in (5.1.3) we plugged in the relations $u_{j_1\cdots j_k} = y_{1,j_1}\cdots y_{k,j_k}$ and

$$
\tilde{A}(j_1,\dots,j_{k-1}) := \begin{bmatrix} U_{(j_1,\dots,j_{k-1})} & z_{(j_1,\dots,j_{k-1})} & U_{I_1} & \cdots & U_{I_{m_k-1}} \end{bmatrix}^T .
$$

Hence $\det(\tilde{A}(j_1,\dots,j_{k-1})) \neq 0$ only if $(j_1,\dots,j_{k-1}) \in \prod_{i=1}^{k-1}[m_i+1]\setminus\{I_1,\dots,I_{m_k-1}\}$, giving the desired result. $\square$

Note that equation (5.1.3) tells us that $\det(A)$ may be written as a sum of determinants of the matrices $\tilde{A}(j_1,\dots,j_{k-1})$. The number of non-zero summands is equal to the cardinality of $\prod_{i=1}^{k-1}[m_i+1]\setminus\{I_1,\dots,I_{m_k-1}\}$, that is $(m_1+1)\cdots(m_{k-1}+1)-(m_k+1)+2$. For example, we have seen in Example 5.1.2 that the unknown relations among singular triples of a $2\times2\times4$ tensor can be written as the sum of $2\cdot2-4+2=2$ determinants and in Example 5.1.4 that the unknown relations among singular triples of a $2\times3\times5$ tensor can be written as the sum of $2\cdot3-5+2=3$ determinants.

With this we understand the membership problem $T\in\langle Z_T\rangle$ in the following cases.

Theorem 5.1.7. *[ST22, Theorem 6.4] Let $T\in V$ be a general tensor of order k of the following formats:*

1. *$k=3$, $\boldsymbol{m}=(2,2,N+1)$, $N\ge3$;*
2. *$k=3$, $\boldsymbol{m}=(2,3,N+1)$, $N\ge4$;*

Then $T\in\langle Z_T\rangle$.

Proof. The first two items are done in the Examples 5.1.2, 5.1.4 and 5.1.5. □

The next result extend Theorem 4.2.10 beyond of boundary format for some tensor formats.

Theorem 5.1.8. *[ST22, Theorem 6.6] Let $T \in V = \mathbb{C}^{m_1+1} \otimes \cdots \otimes \mathbb{C}^{m_k+1}$ be a general tensor of order k of the formats as in Theorem 5.1.7. Then the fiber of the rational map $\tau\colon T \mapsto \operatorname{Eig}(T)$ is T itself. Furthermore, T is determined by its singular vector tuples.*

Proof. Let $T \in \bigotimes_{i=1}^{k-1} \mathbb{C}^{m_i+1} \otimes L \subset V$ be a general tensor of boundary format $\mathbf{m}$, that is $\dim(L) = \sum_{i=1}^{k-1} m_i$. By Theorem 3.1.16 we have that $\langle Z_T \rangle \subset \bigotimes_{i=1}^{k-1} \mathbb{C}^{m_i+1} \otimes L$. Moreover Theorem 4.2.10 says that the fiber of the map $\tau\colon T \mapsto \operatorname{Eig}(T)$ introduced in (4.0.1) is one point for tensors in spaces satisfying the boundary format. Suppose that $U \in V$ is not contained in any subspace satisfying boundary format. Examples 5.1.2, 5.1.4 and 5.1.5 show that in such case $U \in \langle Z_U \rangle$ and $\langle Z_U \rangle$ is not contained in any subspace of boundary format, thus $Z_U \neq Z_T$ and the fiber of the map at $\operatorname{Eig}(T)$ is a single point.

We now proceed by using the fact that the rank of the map τ satisfies semicontinuity, therefore the map is generically finite-to-one. Furthermore, since the fibers are linear spaces, we obtain that the general fiber is a single point. □

Chapter 6

Waring problem and tensor decomposition

In this chapter we describe the algorithm for Waring decomposition presented in [OO13]. Utilizing the results obtained in [CCO17] on the Waring loci of plane cubics, we are able to give an algorithm for general plane cubics over the complex numbers.

6.1 Algorithm for Waring decomposition

Let V be an $m+1$-dimensional vector space over $\mathbb{C}$.

Definition 6.1.1. Let $f \in \mathrm{Sym}^d V$, fix $0 \le a \le m$, $1 \le l \le d-1$. We construct the linear map

$$P_f : \mathrm{Hom}\left(\mathrm{Sym}^l V, \bigwedge^a V\right) \to \mathrm{Hom}\left(\bigwedge^{m-a} V, \mathrm{Sym}^{d-l-1} V\right).$$

Such map is defined on decomposable polynomials as

$$P_{v^d}(M)(w) = (M(v^l) \wedge v \wedge w)(v^{d-l-1}),$$

where $M \in \mathrm{Hom}\left(\mathrm{Sym}^l V, \bigwedge^a V\right)$, $w \in \bigwedge^{m-a} V$ and we identify $\bigwedge^{m+1} V \cong \mathbb{C}$. The definition of P_f for any element $f \in \mathrm{Sym}^d V$ is extended via linearity.

From the definition it seems quite hard to compute directly the matrix of the map P_f, however the next result from [LO13] gives a better manner to compute it.

Lemma 6.1.2. *[LO13] Let $f \in \mathrm{Sym}^d V$. The matrix of*

$$P_f : \mathrm{Hom}\left(\mathrm{Sym}^l V, \bigwedge^a V\right) \to \mathrm{Hom}\left(\bigwedge^{m-a} V, \mathrm{Sym}^{d-l-1} V\right)$$

can be computed using the matrix k_{m+1-a} of the Koszul complex

$$0 \to R(-m-1) \xrightarrow{k_{m+1}} \dots \xrightarrow{k_2} R(-1) \xrightarrow{k_1} R \to \mathbb{C} \to 0,$$

that has size $\binom{m+1}{a} \times \binom{m+1}{a+1}$, where at the place of the indeterminate x_i we substitute the catalecticant matrix $C^l_{f_i}$ of size $\binom{m+d-l-1}{m} \times \binom{m+l}{m}$, where $f_i = \frac{\partial f}{\partial x_i}$.

Proof. [LO13, Section 8.3] □

We give now a more general definition of eigenvectors of a tensor.

Definition 6.1.3. Given $M \in \mathrm{Hom}(\mathrm{Sym}^l V, \bigwedge^a V)$, a vector $v \in V$ is an eigenvector of the tensor M if

$$M(v^m) \wedge v = 0.$$

Notice that for $a = 1$ such definition coincides with the definition of singular vector tuples utilised throughout chapters 3 and 4.

Lemma 6.1.4. *[OO13, Lemma 3.3] Let $M \in \mathrm{Hom}(\mathrm{Sym}^l V, \bigwedge^a V)$.*

1. *A vector $v \in V$ is an eigenvector of M if and only if $M \in \ker(P_{v^d})$.*
2. *Let $f = \sum_{i=1}^r v_i^r$. If each v_i is an eigenvector of M, then $M \in \ker(P_f)$.*

Proof. To prove the first item recall the definition of P_{v^d}:

$$P_{v^d}(M)(w) = (M(v^l) \wedge v \wedge w)(v^{d-l-1}).$$

This is equal to 0 for all w if and only if $M(v^l) \wedge v = 0$.

The second part follows from the first item by using linearity. □

This generalized notion of eigenvectors of tensors can be understood by means of the universal quotient bundle Q on $\mathbb{P}V$ in the same fashions as in chapter 3.

Lemma 6.1.5. *[OO13, Lemma 3.7]*

1. *The fiber of $\bigwedge^a Q(l)$ at $x = \langle v \rangle$ is isomorphic to $\mathrm{Hom}(\langle v^l \rangle, \bigwedge^a V / \langle v \wedge \bigwedge^{a-1} V \rangle)$.*
2. *The section s_M vanishes in $\langle v \rangle$ if and only if v is an eigenvector of the tensor M.*

We now introduce the linear map A_f that was first introduced in [LO13]. Such map is interesting since the zero loci of $\ker A_f$ can be utilised to describe the summands of a Waring decomposition of f, moreover it can be related to P_f, that can be easily computed, allowing to have an effective algorithm (see Algorithm 2).

Let L be a line bundle on an irreducible subvariety X that gives the embedding $X \subset \mathbb{P}(H^0(X, L)^*) = \mathbb{P}W$. In particular, if we take $L = \mathcal{O}(d)$ on $\mathbb{P}V$, this gives the embedding of the Veronese variety $X = \nu_d(\mathbb{P}^n)$ and $W = \mathrm{Sym}^d V$.

Let E be a vector bundle on X. The linear map A_f was constructed on [LO13], it depends linearly on $f \in W$. It comes naturally from the contraction map

$$H^0(E) \otimes H^0(E^* \otimes L) \to H^0(L).$$

This induces a linear map

$$H^0(E) \otimes H^0(L)^* \to H^0(E^* \otimes L)^*.$$

Therefore, if we fix an element $f \in H^0(L)^*$, we can see such map as

$$A_f : H^0(E) \to H^0(E^* \otimes L)^*.$$

Let $f = \sum_{i=1}^r v_i \in W$. The ideal sheaf of the decomposition of f, that is, the ideal defining the points $\{[v_1], \ldots, [v_r]\}$, is related with the map A_f by the following result.

Proposition 6.1.6. *[LO13, Proposition 5.4.1] Let $Z = \{[v_i]\}_{i=1}^r$. Then*

$$H^0(\mathcal{I}_Z \otimes E) \subset \ker(A_f), \;\; H^0(\mathcal{I}_Z \otimes E^* \otimes L) \subset (\mathrm{Im} A_f)^\perp.$$

The first inclusion is an equality if $H^0(E^* \otimes L) \to H^0(E^* \otimes L|_Z)$ is surjective. The second one is an equality if $H^0(E) \to H^0(E|_Z)$ is surjective.

Notice that the meaning of such result is that with an appropriate bundle E, we may study the set $\{[v_1], \ldots, [v_r]\}$ by considering the zero locus of $H^0(\mathcal{I}_Z \otimes E)$. When we have equality of the inclusions in the proposition, this means that we can obtain it directly from the zero locus of the kernel of A_f.

The next two results of [OO13] shows how to choose an appropriated bundle E so that the equality holds.

Proposition 6.1.7. *[OO13, Proposition 4.3] Assume that $\mathrm{rank}(A_f) = k{\cdot}\mathrm{rank}(E)$. Then the equalities hold on Proposition 6.1.6.*

Theorem 6.1.8. *[OO13, Theorem 4.4] Assume that $\mathrm{rank}(A_f) = k \cdot \mathrm{rank}(E)$ and*

$$H^0(\mathcal{I}_Z \otimes E) \otimes H^0(\mathcal{I}_Z \otimes E^* \otimes L) \to H^0(\mathcal{I}_Z^2 \otimes L)$$

is surjective.

Assume that X is not k-weakly defective, then the common zero locus of $\ker(A_f)$ and of $(\mathrm{Im}(A_f))^\perp$ is given by Z itself, hence Z can be reconstructed from f.

Such results gives the following algorithm.

As previously mentioned, describing the map A_f is difficult. The next step is to relate it to P_f. We now make a summary of the main points of [OO13, section 4.1]. In order to show the relation of P_f and A_f, consider the minimal resolution of the bundle E:

$$\cdots \to L_2 \to L_1 \to E \to 0,$$

Algorithm 1 [OO13, Algorithm 4]

Input: $f \in \mathrm{Sym}^d V$ and a convenient vector bundle E.

1. Construct the map A_f with $L = \mathcal{O}(d)$.
2. Compute $\ker(A_f)$. If $\ker(A_f)$ is trivial the algorithm fails.
3. Find the base locus $Z = \{[v_1], \ldots, [v_s]\}$ of $\ker A_f$. If it does not consist of finitely many points the algorithm fails.
4. Solve the linear system defined by $f = \sum_{i=1}^{s} c_i v_i^d$.

Output: The Waring decomposition of f.

where each L_i is a direct sum of line bundles and has the property that the induced map on global sections $H^0(L_1) \to H^0(E)$ is surjective.

The minimal resolution of E^* is given by

$$\cdots \to L_{-1} \to L_0 \to E^* \to 0,$$

where once again L_i is a direct sum of line bundles and for any line bundle L we have $H^0(L_0 \otimes L) \to H^0(E \otimes L)$ is surjective.

Dualising this sequence and bringing them together and we obtain the resolution

$$\cdots \to L_2 \to L_1 \xrightarrow{p} L_0 \to L_{-1} \to \ldots,$$

where $\mathrm{Im} p = E$. The map p gives the presentation of E. We have the composition

$$P_f = \beta \circ A_f \alpha : H^0(L_1) \xrightarrow{\alpha} H^0(E) \xrightarrow{A_f} H^0(E^* \otimes L)^* \xrightarrow{\beta} H^0(L_0^* \otimes L)^*,$$

where α is surjective and β is injective. Hence, we have that $\mathrm{rank} P_f = \mathrm{rank} A_f$, but the most important fact coming from this is that the zero locus of $\ker(P_f)$ and $\ker(A_f)$ are the same, thus instead of utilising A_f on Algorithm 1, we may use P_f instead. The details of the previous argument are described in [LO13, Section 8.3] and [OO13, Section 4.1].

With this we are able to present a practical algorithm for Waring decompositions of low rank polynomials by techniques of vector bundles (Algorithm 2).

6.2 Waring and forbidden loci

In this section we follow closely the work in [CCO17]. Let V be a complex vector space of dimension $m+1$ with basis $\{x_0, \ldots, x_m\}$. Denote $S = \mathbb{C}[x_0, \ldots, x_m]$ the

Algorithm 2 [OO13, Algorithm 5]

Input: $f \in \mathrm{Sym}^d V$, where V has basis $\{x_0, \ldots, x_m\}$.

1. Compute $\delta_- = \lfloor \frac{d-1}{2} \rfloor$, $\delta_+ = \lceil \frac{d-1}{2} \rceil$ and choose $a = \lceil \frac{m}{2} \rceil$.
2. Construct the Koszul matrix k_{m+1-a}.
3. Construct the catalecticants $C_{f_i} : \mathrm{Sym}^{\delta_+} V \to \mathrm{Sym}^{\delta_-} V$ of $f_i = \frac{\partial f}{x_i}$ for each i.
4. Construct the matrix $P_f : \mathrm{Hom}(\mathrm{Sym}^{\delta_+} V, \bigwedge^a V) \to \mathrm{Hom}(\bigwedge^{m-a} V, \mathrm{Sym}^{\delta_-} V)$.
5. Compute a basis $\{M_1, \ldots, M_t\}$ of $\ker(P_f)$ and associate vectors of polynomials w_i to each M_i. If $\ker(P_f)$ is trivial stop, the method fails.
6. Compute the eigenvectors $\{v_1, \ldots, v_s\}$ of a general element in $\ker(P_f)$.
7. Solve the linear system $f = \sum_{i=1}^{s} c_i v_i^d$ in the unknowns c_i.

Output: The unique Waring decomposition of f.

polynomial ring over the basis of V, in other words,

$$S = \bigoplus_{d \geq 0} \mathrm{Sym}^d V,$$

and the d-graded part of S corresponds exactly to $\mathrm{Sym}^d V$. Let $R = \mathbb{C}[y_0, \ldots, y_m]$ be another polynomial ring. We can define a structure of R-module on S via differentiation, that is, we interpret the variables y_i as the differential form ∂x_i. This action is called the apolarity action and defined by

$$g \circ f = g(y_0, \ldots, y_m) f = g(\partial x_0, \ldots, \partial x_m) f,$$

where $g \in R$ and $f \in S$.

Definition 6.2.1. Let $f \in \mathrm{Sym}^d V = S_d$, the apolar ring of f, denoted $(f)^\perp$, is defined as

$$(f)^\perp = \{g \in R | g \circ f = 0\}.$$

In other words, the apolar ring of f consists of the polynomials that under the act of differentiation annihilates f.

Lemma 6.2.2 (Apolarity lemma)**.** *Let $X = \{p_1, \ldots, p_r\} \subset \mathbb{P}^m$ be a set of reduced points. Denote $p_i = \left(p_i^{(0)}, \ldots, p_i^{(m)}\right)$, and consider the associated linear form $l_i = p_i^{(0)} x_0 + \cdots + p_i^{(m)} x_m$ for all $i = 1, \ldots, r$. Let $f \in \mathrm{Sym}^d V$, then the following are equivalent:*

1. *$I_X \subset (f)^\perp$.*

Description	Normal form	Result
three non concurrent lines	$x_0x_1x_2$	Theorem 6.2.6
line + conic meeting transversely	$x_0(x_1x_2+x_0^2)$	Theorem 6.2.7
nodal	$x_0x_1x_2-(x_1+x_2)^3$	Theorem 6.2.8
cusp	$x_0^3-x_1^2x_2$	Theorem 6.2.10
general smooth $(a^3\neq -27,0,6^3)$	$x_0^3+x_1^3+x_2^3+ax_0x_1x_2$	Theorem 6.2.9

Table 6.1: Rank 4 plane cubics and their normal form.

2. *$f=\sum_{i=1}^{r} c_i l_i^d$ for some $c_i\in\mathbb{C}$, $i=1,\dots,r$.*

A set of points X such that the apolarity lemma holds is said to be an apolar set to f.

In the light of Lemma 6.2.2, we can define the Waring locus of a polynomial.

Definition 6.2.3. Given a homogeneous polynomial $f\in\mathrm{Sym}^d V$, the Waring locus of f, denoted $\mathcal{W}_f$, is defined as

$$\mathcal{W}_f=\{p\in\mathbb{P}^m \mid p\in X,\ I_X\subset(f)^{\perp} \text{ and } |X|=\mathrm{rank}(f)\}.$$

The forbidden locus of $\mathcal{F}_f$ is the complement of $\mathcal{W}_f$; $\mathcal{F}_f=\mathbb{P}^m\setminus\mathcal{W}_f$

In other words, the Waring locus of a homogeneous polynomial f consists of the linear forms that appear, up to scalar multiplication, in a Waring decomposition of f. If the rank of f is equal to r, another equivalent definition is

$$\mathcal{W}_f=\{[l]\in S_1 \mid \exists\, l_1,\dots,l_{r-1}\in S_1,\ f\in\langle l^d,l_1^d,\dots,l_{r-1}^d\rangle\}.$$

The forbidden locus on the other hand consists of the linear forms that cannot appear in any minimal decomposition of f.

The goal in this section is to describe the Waring locus of a rank 4 plane cubic f. We follow the results obtained in [CCO17] to show that the forbidden locus in such case is a closed subset for the general plane cubic, this implies that a general linear form l is on the Waring locus of f, thus it is part of a Waring decomposition. This is the main fact utilised in section 6.3 to extended Algorithm 2 to plane cubics of generic rank.

We start by recalling the next two results that will be needed to prove the results on the description of the Waring locus of rank four cubics.

Proposition 6.2.4. *[CCG12, Proposition 3.1] Let $1\leq d_0\leq d_1\leq\cdots\leq d_m$ and $f=x_0^{d_0}\cdots x_m^{d_m}$ be a monomial in $m+1$-variables. Then*

$$\mathrm{rank}(f)=\prod_{i=1}^{m}(d_i+1).$$

Proposition 6.2.5. *[BBT13, Corollary 19] Let $1 \leq d_0 \leq d_1 \leq \cdots \leq d_m$, $d = \sum_{i=0}^m d_i$ and $f = x_0^{d_0} \cdots x_m^{d_m}$ be a monomial in $m+1$-variables. Then for any Waring decomposition*

$$f = l_1^d + \cdots + l_r^d$$

the linear forms l_i have x_0 appearing with a nonzero coefficient

Theorem 6.2.6. *[CCO17, Theorem 3.3] If $f = x_0^{d_0} \cdots x_m^{d_m}$, with $d_0 \leq d_1 \leq \cdots \leq d_m$, then*

$$\mathcal{F}_f = V(y_0 \cdots y_n) \subset \mathbb{P}^m,$$

where $n = \max\{i \mid d_i = d_0\}$.

Proof. The apolar ideal of f is $(f)^\perp = (y_0^{d_0+1}, \ldots, y_m^{d_m+1})$. Given a point $p = [(p_0, \ldots, p_m)] \notin V(y_0 \cdots y_m)$, we have that $p_i \neq 0$ for $i \leq n$, thus assume that $p_0 = 1$. We prove that $p \in \mathcal{W}_f$. For $i = 1, \ldots, m$, construct the following hypersurfaces on $\mathbb{P}^m$:

$$H_i = \begin{cases} y_i^{d_i+1} - p_i^{d_i+1} y_0^{d_i+1} & \text{if } p_i \neq 0; \\ y_i^{d_i+1} - y_i y_0^{d_i} & \text{if } p_i = 0. \end{cases}$$

The hypersurfaces H_i are the union of $d_i + 1$ hyperplanes.

The ideal $I = (H_1, \ldots, H_n)$ is contained in $(f)^\perp$. Indeed, if $d_i = d_0$, $H_i = y_i^{d_i+1} - p_i^{d_i+1} y_0^{d_i+1}$ and it is trivial. On the other hand, if $d_i > d_0$, it could happen that $H_i = y_i^{d_i+1} - y_i y_0^{d_i}$, thus $x_0^{d_0}$ is annihilated by $y_0^{d_i}$.

Moreover, $V(I)$ is the set of reduced points $[(1, q_1, \ldots, q_m)]$ where

$$q_i \in \begin{cases} \{\xi_i^j p_i \mid j = 0, \ldots, d_i\} & \text{if } p_i \neq 0, \text{ where } \xi_i^{d_i+1} = 1; \\ \{\xi_i^j \mid j = 0, \ldots, d_i - 1\} \cup \{0\} & \text{if } p_i = 0, \text{ where } \xi_i^{d_i} = 1. \end{cases}$$

Therefore, it is a set of $\operatorname{rank}(f)$ distinct points apolar to f and containing the point p itself, this implies that $p \in \mathcal{W}_f$ and $V(y_0 \cdots y_n) \supset \mathcal{F}_f$. The equality follows from Proposition 6.2.5. □

Notice that if f is a rank four plane cubic that is not a cusp, it holds that $\mathcal{L} = (f)_2^\perp$ the degree 2 part of the apolar ideal of f is a net of conics, thus $\mathcal{L} = \langle C_1, C_2, C_3 \rangle$, moreover all the sets of four points apolar to f are the complete intersection of two conics.

This means that fixing a point $p \in \mathbb{P}^2$, we can consider the pencil $\mathcal{L}(-p)$ of plane conics passing through p. If $\mathcal{L}(-p)$ consists of 4 distinct points, then $p \in \mathcal{W}_f$, otherwise $p \in \mathcal{F}_f$. In the plane $\mathbb{P}(\mathcal{L})$ we can consider the degree three curve Δ of reducible conics in $\mathcal{L}$. Moreover, a pencil of conics $\mathcal{L}'$ has four distinct base points, no three collinear, if and only if the pencil consists of three irreducible conics. Therefore, the main technique utilised in the next proofs consists in considering a fixed point $p \in \mathbb{P}^2$ and the line $\mathbb{P}(\mathcal{L}(-p)) \subset \mathbb{P}(\mathcal{L})$. If the line is a proper secant line of Δ and cuts it in three distinct points, then $p \in \mathcal{W}_f$.

Otherwise, $p \in \mathcal{F}_f$. Therefore we have to analyse the dual curve $\tilde{\Delta} \subset \mathbb{P}(\mathcal{L})^*$ of lines not intersecting Δ in three distinct points.

To find $\mathcal{F}_f$ we can consider the map

$$\begin{aligned} \phi : \mathbb{P}(S_1) &\to \mathbb{P}(\mathcal{L})^* \\ ([a,b,c]) &\mapsto [(C_1(a,b,c), C_2(a,b,c), C_3(a,b,c))]. \end{aligned}$$

This map is generically $4:1$, in particular $\mathcal{F}_f = \phi^{-1}(\tilde{\Delta})$.

Theorem 6.2.7. *[CCO17, Theorem 3.12] If $f = x_0(x_1x_2 - x_0^2)$, then*

$$\mathcal{F}_f = V(y_0y_1y_2(y_0^2 - 12y_1y_2)).$$

Proof. Let $\mathcal{L} = (f)_2^{\perp}$ and $C_1 = y_0^2 - 6y_1y_2$, $C_2 = y_1^2$ and $C_3 = y_2^2 = 0$ be the conics that generates $\mathcal{L}$. In the plane $\mathbb{P}(\mathcal{L})$ with coordinates α, β, γ let Δ be the cubic of irreducible conics in $\mathcal{L}$. We have that the equation for Δ is given by

$$\det \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & -3\alpha \\ 0 & -3\alpha & \gamma \end{bmatrix} = \alpha\beta\gamma - 9\alpha^2 = 0,$$

□

Therefore Δ is the union of the conic $C = 9\alpha^2 - \beta\gamma$ and the line $r = \alpha = 0$. The line r corresponds to $\mathcal{L}([(-1,0,0)])$, thus $([(1,0,0)]) \in \mathcal{F}_f$.

To understand the forbidden locus $\mathcal{F}_f$ we have to study all the lines on $\mathbb{P}(\mathcal{L})$ that do not intersect Δ in three distinct points. The remaining options are the lines that cuts C in two distinct points and the lines that go through the intersection points of r and C: $[(0,1,0)]$, $[(0,0,1)]$. More precisely, $p \in \mathcal{F}_f$ if and only if the line

$$L : C_1(P)\alpha + C_2(P)\beta + C_3(p)\gamma = 0$$

satisfies one of the following:

1. L is tangent to the conic C.
2. L passes through $[(0,1,0)]$.
3. L passes through $[(0,0,1)]$.

The cases (2) and (3) corresponds exactly to $y_1^2 = 0$ and $y_2^2 = 0$, thus $V(y_1y_2) \subset \mathcal{F}_f$. If we assume that $p \notin \{y_1y_2 = 0\}$ it gives that L is tangent to the conic C if $y_0^2(y_0^2 - 12y_1y_2) = 0$. Thus

$$\mathcal{F}_f = V(y_0y_1y_2(y_0^2 - 12y_1y_2)).$$

Theorem 6.2.8. *[CCO17, Theorem 3.13] If $f = x_1^2x_2 - x_0^3 - x_0x_1^2$, then*

$$\mathcal{F}_f = (g_1g_2),$$

where $g_1 = y_0^3 - 6y_1^2y_2 + 3y_0y_2$ *and* $g_2 = 9y_0^4y_1^2 - 4y_1^6 - 24y_0y_1^4y_2 - 30y_0^2y_1^2y_2^2 + 4y_0^3y_2^3 - 3y_1^2y_2^4 - 12y_0y_2^5$.

Proof. We start noticing that $[(1,0,0)] \in \mathcal{F}_f$ since $f + x^3 = z(y^2 - xz)$ that has rank 5. Let $\mathcal{L} = (f)_2^{\perp}$, $C_1 = y_0y_1$, $C_2 = y_0^2 - 3y_2^2 = 0$ and $C_3 = y_1^2 - y_0y_2 = 0$ the generators of $\mathcal{L}$.

In the plane $\mathbb{P}(\mathcal{L})$ with coordinates α, β, γ let Δ be the cubic of irreducible conics in $\mathcal{L}$. We have that the equation of Δ is given by

$$\det \begin{bmatrix} \beta & \frac{1}{2}\alpha & \frac{1}{2}\gamma \\ \frac{1}{2}\alpha & \gamma & 0 \\ \frac{1}{2}\gamma & 0 & -3\beta \end{bmatrix} = 3\alpha^2\beta - 12\beta^2\gamma - \gamma^3 = 0.$$

Δ is an smooth and irreducible cubic, this implies that the only possibility of a line $\mathbb{P}\mathcal{L}(-p)$ to not cut Δ in three distinct points is if the line is tangent. This gives

$$\mathcal{F}_f = \{p \in \mathbb{P}^2 \mid \mathbb{P}(\mathcal{L}(-p) \text{ is tangent to } \Delta \subset \mathbb{P}(\mathcal{L})\}.$$

The line defined by $\mathbb{P}\mathcal{L}(-p)$ is given by

$$L : C_1(P)\alpha + C_2(P)\beta + C_3(P)\gamma = 0,$$

and we wish for it to be tangent to Δ. Consider two cases:

1. $C_1(P) \neq 0$.
2. $C_1(P) = 0$.

For (1), we can compute α from the equation of the line and substitute in the equation of Δ. The next step is to compute the discriminant D of the form in β and γ

$$3(C_2\beta + C_3\gamma)^2\beta - 12C_1^2\beta^2\gamma - C_1^2\gamma^3,$$

we get that $D = 27C_1^4g_1^2g_2$. This means that if $C_1(P) \neq 0$, then $p \in \mathcal{F}_f$ if and only if $P \in V(g_1g_2)$.

On the other hand for (2) we obtain that $\mathcal{F}_f \cap V(C_1) = V(g_1g_2) \cap V(C_1)$. □

Theorem 6.2.9. *[CCO17, Theorem 3.16] If* $f = x_0^3 + x_1^3 + x_2^3 + ax_0x_1x_2$, *then:*

1. *If* $\left(\frac{a^3-54}{9a}\right)^3 \neq 27$, *then* $\mathcal{F}_f = \phi^{-1}(\tilde{\Delta})$, *where* $\tilde{\Delta}$ *is the dual curve of the smooth plane cubic*

$$\alpha^3 + \beta^3 + \gamma^3 - \left(\frac{a^3-54}{9a}\right)^3 \alpha\beta\gamma = 0.$$

2. *Otherwise,* $\mathcal{F}_f$ *is the union of three lines pairwise intersecting in three distinct points.*

Proof. Let $\mathcal{L} = (f)_2^\perp$ and $C_1 = ay_0^2 - 6y_1y_2 = 0$, $C_2 = ay_1^2 - 6y_0y_2 = 0$ and $C_3 = ay_2^2 - 6y_0y_1$ be its generators. The cubic curve Δ of reducible conics has equation in the plane $\mathbb{P}(\mathcal{L})$ with coordinates α, β and γ given by

$$\det \begin{bmatrix} a\alpha & -3\gamma & -3\beta \\ -3\gamma & a\beta & -3\alpha \\ -3\beta & -3\alpha & a\gamma \end{bmatrix} = (a^3 - 54)\alpha\beta\gamma - 9a\alpha^3 - 9a\beta^3 - 9a\gamma^3 = 0.$$

Assuming that $\left(\frac{a^3-54}{9a}\right)^3 \neq 27$ it holds that Δ is a smooth cubic curve. Thus, we have that

$$\mathcal{F}_f = \{p \in \mathbb{P}^2 \mid \mathbb{P}(\mathcal{L}(-p)) \text{ is tangent to } \Delta \subset \mathbb{P}\mathcal{L}\}.$$

This gives that $\mathcal{F}_f$ is obtained as mentioned before Theorem 6.2.7.

Otherwise Δ is given by the union of three lines intersecting in three distinct points Q_1, Q_2, Q_3. Hence

$$\mathcal{F}_f = \{p \in \mathbb{P}^2 \mid Q_i \in \mathbb{P}(\mathcal{L}(-p)) \text{ for some } i\}.$$

□

Theorems 6.2.6, 6.2.7, 6.2.8 and 6.2.9 imply that for those types of normal form for the plane cubic the forbidden locus is a closed subvariety, in particular it is for the generic plane cubic. The last case of a cusp in Theorem 6.2.10 is the only case where the rank 4 plane cubic does not have a closed forbidden locus and, in such case, the algorithm will not work.

Theorem 6.2.10. *[CCO17, Theorem 5.1] Let $f = x_0^3 - x_1^2x_2$ be a cusp, then*

$$\mathcal{W}_f = \mathcal{W}_{x_0^3} \cup \mathcal{W}_{x_1^2x_2} \subset \mathbb{P}^2,$$

where $\mathcal{W}_{x_0^3} = \mathbb{P}^0$ and $\mathcal{W}_{x_1^2x_2} \subset \mathbb{P}^1$.

The proof of such theorem is long thus we skip it. It can be found given for a more general format in [CCO17, Section 5].

Observe that $\mathcal{W}_{x_1^2x_2} \subset \mathbb{P}^1$ is described in Theorem 6.2.6, it consists of the line $\mathbb{P}^1$ without two points: $(1:0)$ and $(0:1)$. Therefore, we can give a geometrical interpretation for the Waring locus $\mathcal{W}_f$ a cusp f: it is the union of a line with two points removed and a single point sitting inside of the plane $\mathbb{P}^2$. This makes easy to visualise that $\mathcal{W}_f$ is not an open subset of $\mathbb{P}^2$, moreover it is not dense either.

By joining together theorems 6.2.6, 6.2.7, 6.2.8, 6.2.9, 6.2.10 we obtain the following corollary.

Corollary 6.2.11. *Let f be a cubic in $\mathbb{P}^2$, then $\mathcal{F}_f$ is closed subvariety if and only if f is not a cusp.*

6.3 Numeric algorithm for general plane cubics decomposition

In this section we present a numerical algorithm in Macaulay2 [GS] for the Waring decomposition of a general plane cubic. The main idea is that given a general plane cubic $f \in \mathrm{Sym}^3\mathbb{C}^3$, the results on section 6.2 guarantees that a general linear form l is part of a Waring decomposition of f, in other words $l \in \mathcal{W}_f$. A general linear form can be understood in this context as a random linear form, thus we exploit this idea to utilise Algorithm 2. Such algorithm a piori does not work for a general plane cubic, but we reduce the rank of f by adding αl and computing which should be the coefficient α such that $p = f + \alpha l$ has rank 3. We proceed to apply the Algorithm 2 to p.

Notice that actually the algorithm works not only for a general plane cubic, but it works for any plane cubic of rank 4 except the cusp (see table 6.1). The reason is that if f is a plane cubic of rank 4, then the forbidden locus of f is a closed subvariety of $\mathbb{P}^2$ if and only if f is not a cusp. The forbidden locus being a closed subvariety of $\mathbb{P}^2$ is equivalent to say that the random linear form l chosen will be on the Waring locus with probability one, thus we are able to reduce the rank. Otherwise, if f is a cusp, then $l \in \mathcal{F}_f$ with probability one, in such case we cannot reduce the rank of f via a random linear form l and the algorithm fails. We remark that although we have not implemented, it is still possible to write an algorithm for the case that f is a cusp, since the Waring locus of the cusp is completely described in Theorem 6.2.10. The only adaptation necessary in the code presented next is that instead of choosing a random linear form we would have to compute a linear form $l \in W_f$ and then proceed in the same manner.

Below we present the Macaulay2 code.

```
n=2;
d=3;
s=4;
R=QQ[x_0..x_n];
b=basis(1,R);
ff=random(3,R) --the general plane cubic
gg=random(1,R) --a random linear form is part of the decomposition
↪  of ff
Q=QQ[x_0..x_n,a_0];
h=sub(ff,Q)-a_0*(sub(gg,Q))^3 --the new polynomial to be
↪  decomposed
m=diff(sub(b,Q),diff(transpose sub(b,Q),
diff(matrix{{0,x_2,-x_1},{-x_2,0,x_0},{x_1,-x_0,0}},h))) --the
↪  catalecticant of h
I=saturate pfaffians(8,m) --computing the pfaffians is faster than
↪  the determinant
```

```
xa=sub(a_0%I,R) --the coefficient such that gg^3 appears on the
 ↪  decomposition of ff
p=ff-(xa)*(gg)^3
use R
H= diff(b,diff(transpose
 ↪  b,diff(matrix{{0,x_2,-x_1},{-x_2,0,x_0},{x_1,-x_0,0}},p)))
--H is the substitution of the catalecticant on the Koszul matrix
HM = generators kernel H
r=rank HM --3 is the expected value
m1=sub(sub(random(QQ),R)*transpose
 ↪  matrix{{HM_(0,0),HM_(1,0),HM_(2,0)},
 ↪  {HM_(3,0),HM_(4,0),HM_(5,0)},{HM_(6,0),HM_(7,0),HM_(8,0)}},QQ)
--The matrix associated to the first generator of the kernel of H
m2=sub(sub(random(QQ),R)*transpose
 ↪  matrix{{HM_(0,1),HM_(1,1),HM_(2,1)},
 ↪  {HM_(3,1),HM_(4,1),HM_(5,1)},{HM_(6,1),HM_(7,1),HM_(8,1)}},QQ)
m3=sub(sub(random(QQ),R)*transpose
 ↪  matrix{{HM_(0,2),HM_(1,2),HM_(2,2)},
 ↪  {HM_(3,2),HM_(4,2),HM_(5,2)},{HM_(6,2),HM_(7,2),HM_(8,2)}},QQ)
--A random combination of the matrices coming from the kernel of H
MM=m1+m2+m3
eig=(eigenvectors(MM))_1 --The eigenvectors of MM give the linear
 ↪  forms of the decomposition of H
S =CC[x_0..x_n,c_0..c_n]
bS = sub(b,S);
for i from 0 to n do l_i=(sum(n+1,j->eig_(j,i)*x_j))^3
--the linear forms obtained from the eigenvectors
mm=matrix{{l_0},{l_1},{l_2},{sub(p,S)}}
coe=sub( transpose contract(mm,symmetricPower(3,bS)),CC)
--the matrix of the coefficients of l0,l1,l2,p
A=submatrix(coe,0..2)
B=submatrix(coe,3)
C=solve(A,B, ClosestFit=>true)
--the coefficients of l0,l1,l2 for the best approximation of p
fc=C_(0,0)*l_0+C_(1,0)*l_1+C_(2,0)*l_2 --the numerical solution
 ↪  for a decomposition of p
fc-sub(p,S) biggest error noticed was of order 10^(-13)
f=fc+sub(xa*gg^3,S) --the numerical solution for the decomposition
 ↪  of ff
f-sub(ff,S) biggest error noticed was of order 10^(-13)
```

Chapter 7

Binary forms of suprageneric rank

In this chapter we present the work done by the author in collaboration with Alejandro Gonzáles Nevado [NT20].

7.1 The variety of given rank

Let V be a $m+1$-dimension complex vector space. Denote by $S_{d,r}$ the set of polynomials in $\mathrm{Sym}^d V$ of rank r, that is,

$$S_{d,r} = \{f \in \mathrm{Sym}^d V \mid \mathrm{rank}(f) = r\}.$$

Let $\nu_d(\mathbb{P}V) \subset \mathbb{P}\mathrm{Sym}^d V$ be the d-th Veronese variety. If the number r is smaller or equal to the generic rank

$$g = \left\lceil \frac{\binom{m+d}{d}}{m+1} \right\rceil$$

of $\mathrm{Sym}^d V$, then the r-th secant variety of the Veronese variety, denoted $\Sigma_r \nu_d(\mathbb{P}V)$ coincides with the Zariski closure of $S_{d,r}$, i.e.,

$$\Sigma_r \nu_d(\mathbb{P}V) = \overline{S_{d,r}}.$$

On the other hand, if $r \geq g$, the r-th secant variety fulfils the ambient space, thus $\overline{S_{d,r}}$ cannot be expressed as a secant variety of the Veronese variety. The connection with the secant variety allows the use of a larger spectre of techniques to tackle the problems related to $S_{d,r}$, such as the Terracini lemma, thus the subgeneric case has been vastly studied in the past. The first goal in this section is to relate the suprageneric case with the multiple root loci (section 7.2). This allows a better understanding of the suprageneric case.

The second goal is related with the result obtained in [CS11, Theorem 2], that, for binary forms, describes the strata of the variety $S_{d,r}$ for $r \leq g$.

Theorem 7.1.1. *[CS11, Theorem 2] Let $0 \leq r \leq \lceil \frac{d+1}{2} \rceil$ be an integer, then*

$$\overline{S_{d,r+1}} = \left(\cup_{i=1}^{r+1}\right) \bigcup \left(\cup_{i=0}^{r} S_{d,d-i+1}\right),$$

where $\overline{S_{d,0}} = \overline{S_{d,d+1}} = \emptyset$. Furthermore, this implies that

$$\overline{S_{d,k+1}} \setminus \overline{S_{d,k}} = S_{d,k+1} \cup S_{d,d-k+1}.$$

We extend such result in Theorem 7.3.4 for binary forms of suprageneric rank.

7.2 The multiple root loci

We follow basically the notation in [LS16]. Given an integer m, we say that a vector $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a partition of m with d parts if $\lambda_1 \geq \cdots \geq \lambda_d > 0$ and $|\lambda| := \lambda_1 + \cdots + \lambda_d = m$. Apart from this notation, we may also write a partition as a multiset $\lambda = \{1^{m_1}, \ldots, p^{m_p}\}$, where $m_i \geq 0$ is an integer for $i = 1, \ldots, p$, and represents that there are m_i elements in the partition that are equal to i.

The set of homogeneous binary forms of degree m corresponds to a variety on $\mathbb{P}^m$ associating the points to the coefficient of each monomial in the polynomial expansion. The multiple root locus Δ_λ associated to a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ of m is a subvariety of $\mathbb{P}^m$ associated to the polynomials that have d roots with multiplicity $\lambda_1, \ldots, \lambda_d$. The dimension of this variety is $\dim(\Delta_\lambda) = d$ and its singular locus is a subset of the union

$$\bigcup_{\lambda \text{ properly refines } \mu} \Delta_\mu,$$

as described in [Kur12, Section 3], and in [Chi03].

We are particularly interested in the dual varieties Δ_λ^*. These are studied in [Oed12] and [LS16]. In particular, Hilbert found that the degree of Δ_λ is $\deg(\Delta_\lambda) = \frac{d}{m_1! \cdots m_p!} \lambda_1 \cdots \lambda_d$ and, when the dual Δ_λ^* is a hypersurface (i.e., $m_1 = 0$), [Oed12, Theorem 5.3] establishes that its degree is $\deg(\Delta_\lambda^*) = \frac{(d+1)!}{m_2! \cdots m_p!}(\lambda_1 - 1) \cdots (\lambda_d - 1)$.

Notice that, given a partition λ as above, we have another definition for Δ_λ, it also is the image of

$$(\mathbb{P}^1)^d \longrightarrow \mathbb{P}^n, \ (l_1, \ldots, l_d) \longmapsto l_1^{\lambda_1} \ldots l_d^{\lambda_d}.$$

It follows that the dimension of Δ_λ is d and its smooth points are those in which all the linear forms l_i are pairwise different.

The following lemma gives an expression of the tangent space of a multiple root locus at a smooth point.

Lemma 7.2.1. *[LS16, Lemma 2.1, Lemma 2.2] Let $f = l_1^{\lambda_1} \dots l_d^{\lambda_d} \in \Delta_\lambda$ be a smooth point and $g \in (\mathbb{P}^n)^*$. Then the tangent space at f is given by*

$$\mathrm{T}_f \Delta_\lambda = \{h(x, y) \prod_{i=1}^{d} l_i^{\lambda_i - 1} \mid h \in \mathbb{P}(K[x, y]_d)\}.$$

Furthermore, $g \perp \mathrm{T}_f \Delta_\lambda$ if and only if

$$\prod_{i=1}^{d} l_i^{\lambda_i - 1}(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$$

annihilates $g(u, v)$.

As we want to obtain a good description of the dual variety, we need to introduce the conormal variety. The conormal variety of Δ_λ is given by the closure of the set

$$\{(f, g) \mid f \in \Delta_\lambda \text{ is a smooth point and } g \perp \mathrm{T}_f \Delta_\lambda\}.$$

The dual variety Δ_λ^* is the image of the projection of the conormal variety onto the second factor. Therefore it is the variety of binary forms that are annihilated by some f. This observation leads to a parametrization of the conormal variety: it can be seen as the set of points (f, g) of the form

$$f(x, y) = \prod_{i=1}^{d} (t_i x - s_i y)^{\lambda_i}, \;\; g(u, v) = \sum_{i=1, \lambda_i \neq 1}^{d} (s_i u + t_i v)^{m - \lambda_i + 2} g_i(u, v),$$

where $g_i(u, v)$ are binary forms of degree $\lambda_i - 2$, and $(s_i, t_i) \in \mathbb{P}^1$. The dimension of the dual variety to Δ_λ is given, using [Kat03, Corollary 7.3], by

$$\dim \Delta_\lambda^* = m - m_1 - 1.$$

The inclusions between multiple root loci can be characterized in terms of refinements of the partitions that define them. Hence we have that $\Delta_\lambda \subset \Delta_\mu$ if and only if μ refines λ.

In addition, for a partition $\lambda = \{1^{m_1}, \dots, p^{m_p}\}$, we denote its derived partition $\lambda' := (1^{m_2}, \dots, (p-1)^{m_p})$, and this is a partition of $m - d$, where $d = \sum m_i$ is the number of parts. The next proposition gives a result similar to the one in the previous paragraph for inclusions between dual varieties. These inclusions are also characterized via refinements of partitions although it is not as direct as the previous one: the equivalent condition for the inclusion of duals involves refinements of derived partitions. Expressing this new condition requires thus the related partitions that we have just introduced.

Proposition 7.2.2. *[LS16, Proposition 3.4] Given two partitions λ, μ of m, then*

$\Delta_\lambda^* \subset \Delta_\mu^*$ holds if and only if $|\lambda'| \leq |\mu'|$ and, by adding to the parts, λ' can be transformed into a partition $\tilde{\lambda}$ that is refined by μ'.

7.3 The variety of rank k forms and the multiple root loci

We are now able to prove the main results obtained in [NT20]. We start by analysing the strata of the varieties of given rank r for $r \geq g$, where g is the generic rank in $\mathrm{Sym}^d\mathbb{P}^1$. A final important remark concerns the good description of the apolar ideal that we have in the case of a binary form. This description is fundamental in the proof of the result bridging multiple root loci and varieties generated by forms of fixed rank, which is itself an intermediate result towards our main theorem.

Remark 7.3.1. *If f is a binary form of degree d, then $(f)^\perp = (g_1, g_2)$ with $\deg(g_1) + \deg(g_2) = n + 2$. In addition, if $\deg(g_1) \leq \deg(g_2)$, then $\operatorname{rank}(f) = \deg(g_1)$ if g_1 is square free and $\operatorname{rank}(f) = \deg(g_2)$ otherwise.*

The relations between the variety $\overline{S_{d,k}}$ was well know for degrees smaller than 6. Therefore the first interesting example is the case where the degree is $d = 6$. We explore this case for ranks bigger than the generic rank $r = 4$. In the particular case of $f \in S_{6,6}$, we have that the ideal $(f)^\perp = (g_1, g_2)$ with $d_1 + d_2 = 8$, where d_1 and d_2 are the respective degrees. Since the rank of f is 6, we must have $d_1 = 2, d_2 = 6$, and g_1 has a double root. Therefore the only possibility is that $g_1 = l^2$, where l is a linear form. In such case, by an immediate application of [LS16, Lemma 2.2], we know that $f \in \Delta^*_{3,1^3}$. The other inclusion follows from dimensional count. We can use such idea to compute any $S_{d,r}$. For example, proceeding similarly for the rank 5 we have that $d_1 = 3, d_2 = 5$ and therefore we have that g_1 has two possible cases: either l_1^3 or $l_1^2 l_2$. In such case, $f \in \Delta^*_{4,1^2}$ or $f \in \Delta^*_{3,2,1}$, respectively. We can see that the first is contained in the second, and therefore $f \in \Delta^*_{3,2,1}$. The other side follows again by dimensional count.

In [Buc+17, Proposition 19] it was obtained that the dimension of $\overline{S_{d,r}}$ for r bigger than the generic rank is given by

$$\dim \overline{S_{d,r}} = \dim \Sigma_{d-k+2}(\nu_d(\mathbb{P}^1)) - 1 = 2(d - r + 1).$$

Using this fact together with the preceding idea developed in the example, we can obtain the following argument.

Proposition 7.3.2. *[NT20, Proposition 2] Let k be an integer and suppose that $d \geq d - k > \lceil \frac{d+1}{2} \rceil$, then*

$$\overline{S_{d,d-k}} = \Delta^*_{3,2^k,1^{d-2k-3}}.$$

Proof. Let $f \in S_{d,d-k}$ be a homogeneous polynomial of degree d and rank $d - k$. We know that the apolar ideal $(f)^\perp$ is generated by (g_1, g_2), such that $d_1 + d_2 =$

$d+2$, with $d_1 \leq d_2$ the respective degrees, and $\operatorname{rank}(f) = d_2$, if g_1 is not square free, or $\operatorname{rank}(f) = d_1$ otherwise. So we may assume that $d_2 = d-k, d_1 = k+2$ and g_1 has a double root. Thence g_1 has the following form $l_0^2 l_1 \dots l_k$ and $f \in \Delta^*_{3,2^k,1^{d-2k-3}}$. (Notice that all other possibilities for g_1, that is, with more than a single double root, lead to a different partition λ but all of those are such that λ' is refined by $(2,1^k)$ and therefore we have $\Delta^*_\lambda \subset \Delta^*_{3,2^k,1^{d-2k-3}}$ in such case.) It follows that $S_{d,d-k} \subseteq \Delta^*_{3,2^k,1^{d-2k-3}}$. On the other hand, by the proof of [Buc+17, Proposition 19], we have that $\dim \overline{S_{d,d-k}} = \dim \Sigma_{k+2}(\nu_d(\mathbb{P}^1)) - 1 = (2k+3) - 1 = 2k+2$, and $\dim \Delta^*_{3,2^k,1^{d-2k-3}} = d - m_1 - 1 = 2k+2$, so equality holds. □

Following [CS11, Theorem 2], we obtain a similar result for the varieties of rank r bigger than the generic rank. Furthermore, we also give another description for $\overline{S_{d,d-k}}$. In order to do that we first recall the description of the Waring locus of binary forms. In particular, the forbidden locus of a suprageneric-rank binary form is a closed subvariety.

Theorem 7.3.3. *[CCO17, Theorem 3.5] Let f be a degree d binary form and let $g \in (f)^\perp$ be an element of minimal degree. Then:*

1. *If $\operatorname{rank}(f) < \lceil \frac{d+1}{2} \rceil = g$, then $\mathcal{W}_f = V(g)$.*
2. *If $\operatorname{rank}(f) > g$, then $\mathcal{F}_f = V(g)$.*
3. *If $\operatorname{rank}(f) = g$ and d is even, then $\mathcal{F}_f$ is finite and non empty.*
4. *If $\operatorname{rank}(\mathrm{f}) = g$ and d is odd, then $\mathcal{W}_f = V(g)$.*

Proof. For (1) it is enough to use that the Waring decomposition of f is unique, thus the apolar set is $V(g)$.

To prove (2) we use remark 7.3.1. The apolar ideal is given by $(f)^\perp = (g_1, g_2)$, with $d_1 = \deg g_1 < d_2 = \deg g_2$, $d_1 + d_2 = d+2$, g_1 is not square free and $\operatorname{rank}(f) = d_2$. Notice that the reason this case holds is that f has suprageneric rank and $d_1 \leq g$. We first show that $\mathcal{F}_f \supset V(g_1)$. Let $p = V(l) \in V(g_1)$ for some linear form l that divides g_1, we will show that there is no apolar set of points to f containing p. This is equivalent to show that there is no square free element of degree d_2 in $(f)^\perp$ divisible by l. Since g_1 and g_2 have no common factors and l divides g_1, thus l does not divides g_2. It follows that the only elements of degree d_2 in $(f)^\perp$ divisible by l are multiples of g_1, thus not square free. Hence $p \in \mathcal{F}_f$. On the other hand, if $p = V(l) \notin V(g_1)$, then l does not divide g_1. Consider

$$(f)^\perp : (l) = (l \circ f)^\perp = (h_1, h_2),$$

where $c_1 = \deg h_1 \leq c_2 = \deg h_2$ and $c_1 + c_2 = d+1$. Since h_1 is a minimal degree element in $(f)^\perp$ and l does not divide g_1, we have $h_1 = g_1$ and $c_2 = d_2 - 1$. Thus $\operatorname{rank}(f) = \operatorname{rank}(l \circ f) + 1$. Since $((f)^\perp : (l))_{d_2-1}$ is a base point free, we can choose $h \in (f)^\perp : (l)$ to be a degree $d_2 - 1$ square free element not divisible by l. Hence $P \in V(hl)$ and $V(hl)$ is a set of d_2 points apolar to f.

For (3), let $(f)^\perp = (g_1, g_2)$, with $d_1 = \deg g_1$ and $d_2 = \deg g_2$. Since d is even, then $d_1 = d_2 = \mathrm{rank}(f)$ and f has finitely many apolar sets of $\mathrm{rank}(f)$ distinct points. However, for each $p \in \mathbb{P}^1$, there is a unique set of $\mathrm{rank}(f)$ points apolar to f and containing p. Thus, there is a unique element $g \in (f)^\perp_d$ vanishing at p. Thus $p \in \mathcal{F}_f$ if and only if g is not square free. There are finitely many not square free elements in $(f)^\perp_d$, because they correspond to the intersection of the line given by $(f)^\perp_d$ in $\mathbb{P}(T_d)$ (as in the notation of section 6.2) with the hypersurface given by the discriminant and this line is not contained in the hypersurface since $(f)^\perp$ contains square free elements.

To prove (4) we have that $(f)^\perp = (g_1, g_2)$ with $d_1 = \deg g_1$ and $d_2 = \deg g_2$. Since d is odd we have $d_2 = d_1 + 1$ and rankf $= d_1$. This means that g_1 is a square free element of minimal degree and f has a unique apolar set of d_1 distinct points given by $V(g_1)$.

□

Theorem 7.3.4. *[NT20, Theorem 1] Let k be an integer and suppose that $d \geq d - k > \lceil(\frac{d+1}{2})\rceil$, then $\overline{S_{d,d-k}}$ is the union*

$$\overline{S_{d,d-k}} = (\cup_{i=1}^{k+1} S_{d,i}) \bigcup (\cup_{i=0}^{k} S_{d,d-i}).$$

In particular $\overline{S_{d,d-k}} \setminus \overline{S_{d,d-k+1}} = S_{d,k+1} \cup S_{d,d-k}$.

Proof. First we notice that, since $\overline{S_{d,d-k}} = \Delta^*_{3,2^k,1^{d-2k-3}}$, then it can be described as the set of sums of powers of linear forms

$$\{l_0^{d-1} g + l_1^d + \cdots + l_k^d | l_i, g \text{ are linear forms, for } i = 0, \ldots k\}.$$

It is trivial that $\cup_{i=0}^{k+1} S_{d,i}$ is contained on $\overline{S_{d,d-k}}$ by just considering $g = l_0$. If f has rank $d-i$, with $i \leq k$, then $f = \sum_{j=1}^{d-i} l_j^d$. It is possible to choose $l_{d-i+1}, \ldots, l_d$ linear forms such that

$$f + l_{d-i+1}^d + \cdots + l_d^d = \sum_{j=1}^{d} l_j^d$$

has rank d. This can be done by choosing the first linear form l_{d-i+1} in the forbidden locus $\mathcal{F}_f$ of f, and repeating this procedure inductively to the new polynomial obtained. Since the obtained polynomial has rank d, we have that it can be written as $l^{d-1} g$, for some l, g linear forms. Substituting it on the right side of the equation, we have that

$$f = l^{d-1} g - \sum_{j=1}^{i} l_{d-i+j}^d,$$

which is an element of $\Delta^*_{3,2^k,1^{d-2k-3}}$. So we obtained that

$$\overline{S_{d,d-k}} \supset (\cup_{i=0}^{k+1} S_{d,i}) \bigcup (\cup_{i=0}^{k} S_{d,d-i}).$$

For the other inclusion, suppose that $f \in \Delta^*_{3,2^k,1^{d-2k-3}}$, then $f = l_0^{d-1}g + l_1^d + \cdots + l_k^d$ for some linear forms $l_0, \dots, l_k, g$. We analyse two cases. If $g = l_0$, it is clear that $\mathrm{rank}(f) \leq k+1$. Otherwise, suppose that $g \neq l_0$, then $l_0^{d-1}g$ has rank d. Since all the other summands are power of linear forms, each of them can either decrease the rank by 1, if they are on the Waring locus of $l_0^{d-1}g$, or it does not change the rank, and it remains equal to d, if they are in the forbidden locus; in both cases, we have that $\mathrm{rank}(f) \geq d-k$, hence we have the equality. □

Theorem 7.3.5. *[NT20, Theorem 3] Let $d-k > \lceil \frac{d+1}{2} \rceil$, then the singular locus of $\overline{S_{d,d-k}}$ contains the subvariety $\overline{S_{d,k+1}} \cup \overline{S_{d,d-k+1}}$.*

Proof. Let $f = l_0^{d-1}g + l_1^d + \cdots + l_k^d$ be a point of $\overline{S_{d,d-k}}$. We compute the tangent space at f, by considering $l_i = a_i x + b_i y$ and $g = \alpha x + \beta y$. We can consider a curve

$$f(t) = \sum_{i=1}^{k} (a_i(t)x + b_i(t)y)^d + (a_0(t)x + b_0(t)y)^{d-1}(\alpha(t)x + \beta(t)y),$$

with $f(0) = f$, then taking the derivatives on the a_i, b_i, α, β we have that the tangent space is generated by

$$\mathrm{T}_f \overline{S_{d,d-k}} = \langle y l_i^{d-1}, x l_i^{d-1}, x l_0^{d-2} g, y l_0^{d-2} g, x l_0^{d-1}, y l_0^{d-1} \rangle, \ i = 1, \dots, k.$$

This space has $2k+4$ generators, but we notice that the last four of them span a 3-dimensional space, so it has projective dimension $2k+2$ in a general point, as expected. We consider two cases now, first if g is equal to l_0, in other words, the case that f is a general element of $\overline{S_{d,k+1}}$. We notice that instead of a 3-dimensional space, the last four elements on the span generates a 2-dimenensional space, this means that the projective dimension of $\mathrm{T}_f \overline{S_{d,d-k}}$ is at most $2k+1$, therefore f is a singular point. Now instead, assume that $l_i = l_j$ for some $i, j \neq 0$ and $i \neq j$, then f is a general element of $\overline{S_{d,d-k+1}}$ and the dimension of $\mathrm{T}_f \overline{S_{d,d-k}}$ is less than $2k+2$, again this gives that f is a singular element of $\overline{S_{d,d-k}}$. □

7.4 The hypersurface $\overline{S_{2k+1,k+2}}$

Let $f \in \overline{S_{2k+1,k+2}}$, the maximal catalecticant matrix C_f associated to f has size $(k+1) \times (k+2)$. In [LS16, Theorem 4.1] it is proven that this hypersurface has degree $2k(k+1)$ and its equation is computed, namely, the defining polynomial is the discriminant of

$$q(u,v) = \det \begin{bmatrix} u^{k+1} & u^k v & \dots & uv^k & v^{k+1} \\ a_0 & a_1 & \dots & a_k & a_{k+1} \\ a_1 & a_2 & \dots & a_{k+1} & a_{k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_k & a_{k+1} & \dots & a_{2k} & a_{2k+1} \end{bmatrix}.$$

With this description we can obtain the following result.

Theorem 7.4.1. *[NT20, Theorem 11]* $\overline{S_{2k+1,k}}$ *is an irreducible component of* $\mathrm{Sing}(\overline{S_{2k+1,k+2}})$.

Proof. Let $(a_0, \dots, a_d)$ be the coefficients of polynomials in S_d. The equation of $\overline{S_{2k+1,k+2}}$ has degree $2k$ in the $(k+1)$-minors b_j (for $j = 0, \dots, k$) of the maximal catalecticant matrix of size $(k+1)\times(k+2)$. Each b_j is a homogeneous polynomial of degree $(k+1)$ in the a_i. Let $b_0^{\alpha_0} \dots b_k^{\alpha_k}$ be a monomial with $|\alpha| = 2k$. The derivative with respect to a_i of such monomial is $\sum_j \alpha_j b_0^{\alpha_0} \dots b_j^{\alpha_j - 1} \dots b_k^{\alpha_k} \frac{\partial b_j}{\partial a_i}$. Evaluated at a point $(a_0, \dots, a_d)$ where all b_j vanishes (this is a point in $\overline{S_{2k+1,k}}$) this monomial vanishes. This concludes the proof. □

The case $k = 2$ was studied before in [CO12] by Comon and Ottaviani, it is known as the *apple invariant*, in such case the singular locus has two irreducible components, one is $\overline{S_{5,2}}$, that comes from the minors of the catalecticant, and the other comes from the pullback from the locus of cubics with a triple root $\Delta_{3,1,1}$, that is the dual of the tangent variety $\mathrm{T}(\overline{S_{5,5}}) = \overline{S_{5,4}}$. For $k \geq 3$, $\mathrm{Sing}(\overline{S_{2k+1,k+2}})$ has at least three irreducible components, one is $\overline{S_{2k+1,k}}$, that is obtained from the minors of the catalecticant, the other two components arise from the two irreducible components of the singular locus of the discriminant of $\sum_{i=0}^{k+1} a_i t^i$, it comes as the pullback from the locus of degree $k+1$ polynomials with two double roots and with a triple root. For $k = 3$ the components can be computed in Macaulay2 [GS], one is $\overline{S_{7,3}}$, that has codimension 2 and degree 10. The other two components have codimension 2 and degree respectively 24 (8 generators of degree 7, this component comes as pullback from locus of quartics with two double roots) and 36 (55 generators of degree among 8 and 12, this component comes as pullback from locus of quartics with a triple root), this case was named as the *big apple invariant* in [LS16].

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