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The Hermitian Killing form and the Hermitian Distance degree

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List of symbols

We list below a collection of useful symbols subdivided by mathematical areas. These notations are adopted throughout the work and, the majority of the times, will not be recalled when used.

General notations

$\mathbb{N}$	is the set of natural numbers including 0
$\mathbb{R}$	is the set of real numbers
$\mathbb{C}$	is the set of complex numbers
e	is the Napier's constant
i	is the imaginary unit $\sqrt{-1} \in \mathbb{C}$
$\bar{\lambda}$	indicates the conjugate of a number $\lambda \in \mathbb{C}$
$\lambda^{\Re}/\lambda^{\Im}$	denotes the real/imaginary part of a number $\lambda \in \mathbb{C}$
$\mathbf{0}$	is the generic zero element
π_U	denotes component (orthogonal/unitary) projection in a set (subspace) U
$\dim / \operatorname{codim}$	is the dimension/codimension of an object, a subscript emphasizes the field
Id	denotes the identity map
$\circ$	denotes the composition of maps
$\deg p$	denotes the total degree of a polynomial p
$\deg_z p$	denotes the degree of a polynomial p with respect to the variable z
$\lvert\lambda\rvert$	is the modulus of a number $\lambda \in \mathbb{C}$
$\lVert\ \rVert_{\mathbb{C}}/\lVert\ \rVert_{\mathbb{R}}$	is the norm induced by the canonical Hermitian/Euclidean inner product

$\\| \ \\|_p$	is the p-norm for vectors with $1 \leq p \in \mathbb{R}$, in particular $\\| \ \\|_2 = \\| \ \\|_{\mathbb{C}}$
$\\| \ \\|_\infty$	is the ∞-norm for vectors
$[n]$	is the subset of natural numbers from one up to the argument $1 \leq n \in \mathbb{N}$
$\binom{n}{\alpha}$	is the multinomial coefficient with α vector of naturals such that $\\|\alpha\\|_1 = n \in \mathbb{N}$
∂	indicates the partial derivative, a subscript emphasizes the variable
∇ / J	indicates the gradient/Jacobian, a subscript emphasizes the variables
f'	denotes the derivative of a map f

Linear algebra

$\otimes$	denotes the tensor product
$\langle \ , \ \rangle_{\mathbb{C}} / \langle \ , \ \rangle_{\mathbb{R}}$	is the canonical Hermitian/Euclidean inner product
$\perp_{\mathbb{C}} \ / \ \perp_{\mathbb{R}}$	indicates the perpendicularity condition $\langle \ , \ \rangle_{\mathbb{C}} = 0 / \langle \ , \ \rangle_{\mathbb{R}} = 0$
I_n	denotes the identity matrix of size $n \in \mathbb{N}$
A^T / A^H	denotes the transpose/conjugate transpose of a matrix A
$A_{k_1 k_2}$	indicates the (k_1, k_2) entry of a matrix A
$A_{k,\cdot} / A_{\cdot,k}$	denotes the k-th row/column of a matrix A
diag	is a (rectangular) matrix with diagonal a given vector and elsewhere zero
det	indicates the determinant of a square matrix
Tr	indicates the trace of a square matrix
rk	indicates the rank of a matrix
span	denotes the spanned subspace of a vectors set, a subscript emphasizes the field
Row / Col	indicates the space spanned by the rows/columns of a matrix

Algebraic geometry

$X_{\text{sing}} / X_{\text{reg}}$	is the singular locus/regular set of an algebraic variety X
$\overline{U}^Z$	denotes the Zariski closure of a set U
$\langle \quad \rangle$	is the ideal generated by the polynomials in angle brackets
LT( )	denotes the set of leading terms of the ideal in brackets

$V(\quad)$	is the set of common zeros in an ambient space of the polynomials in brackets
$\mathbb{P}\mathbb{V}$	is the projectivization of a vector space $\mathbb{V}$
$\mathbb{P}^n$	is the projectivization of the complex vector space $\mathbb{C}^{n+1}$ for $n \in \mathbb{N}$
Δp	is the discriminant of a polynomial p, a subscript emphasizes the variable
$\mathbb{S}^n$	is the n-sphere in $\mathbb{R}^{n+1}$ for $n \in \mathbb{N}$

Introduction

The notion of polynomials is ubiquitous in mathematics. In any field, the fundamental importance of these objects is established from ancient times. Across the different areas of study, polynomials have a primary role. Their great versatility stands in the property to approximate a large family of problems and, since they are very widely known in comparison to other maps, polynomials can be used in a variety of contexts.

In particular, there are various situations in which problems that have a physical meaning can be totally described by a collection of polynomial equations.

A more general and more complicated family of similar functions are polynomials where the variables could be conjugated, meaning that the conjugate operation in the complex set is applied after evaluation.

In case the polynomial is sum of terms with only non conjugated or only conjugated variables, then it is called a harmonic polynomial. These objects are much less investigated and a complete theory still needs to be developed.

The general case is even less treated, sometimes in literature it is referred to such type of functions as polyanalitic functions or, more specifically, polyanalitic polynomials. However, since this nomenclature is not very established, we introduce a definition of our convenience in the very first pages.

In this work we want to give a little contribution to this last, not very explored, line of reasearch. In order to pursue this achievement, we develop machineries to study the general problem of finding the complex solutions of a system of equations of complex polynomials, where the conjugated variables appear. Moreover, we apply these results on a variety of examples, in particular we obtain new non trivial results for the case of harmonic polynomials.

In this thesis we also try to address the problem of finding the closest point of an algebraic variety from a fixed position, with respect to a distance function in the vector space containing the variety. There are many different distances we can choose, our first aim is to choose one in such a way so that the problem can be purely described in terms of polynomial equations where the conjugated variables could appear.

The first natural instance of the distance problem from algebraic varieties is the Euclidean distance problem. Consider a real algebraic variety and the distance to be the Euclidean distance or, in a certain sense equivalently, the distance induced by any positive definite symmetric bilinear form. It is well known that this problem can be entirely represented by a system of real polynomial equations.

As natural as it is, the first results in this direction can be found already in the 300 b.C. by Apollonius, who studied the Euclidean distance from the conics. Several examples have been investigated from that time and many related notions have arised since.

More recently, as anticipated, the investigation was carried on using tools from mostly Analysis and Algebraic Geometry. Belonging to the latter field, a fundamental concept in this study is represented by the Euclidean Distance degree, which is a number associated to any real algebraic variety that bounds the number of critical points for the positive definite symmetric bilinear distance problem. This approach was introduced in order to develop a general tool to handle this specific problem.

From this idea, various others came through and the Euclidean Distance degree was computed for many examples along with other achievements, such as sharpness results, explicit computations of the critical points and thus of the points of minimal distance for smooth varieties.

Most current works generalize the ideas to account distance functions that are not induced by a positive definite symmetric bilinear form.

In this work we try to generalize the study in a different direction and consider the distance to be the Hermitian distance, or similarly, as above, the distance induced by any Hermitian form, from a complex algebraic variety. This modification leads to a much harder problem that will be challenging even for simple instances.

In particular, as expected, the results we obtained on systems of polynomial equations with conjugate operation apply to the Hermitian distance problem and show the ability to solve many of the examples we present.

We associate to any algebraic variety a natural number, that we call virtual Hermitian Distance degree and we denote vHDdeg, and a set of natural numbers, that we call Hermitian Distance degree and we denote HDdeg, that quantify the complexity of the problem. The first value is the number of solutions of a system of polynomial equations in the classical sense and bounds from above the possible number of critical points. The second is the set of possible numbers of critical points from a fixed position and we need to solve systems of polynomial equations with conjugated variables to compute these numbers. Then, we proceed to explore the problem and introduce tools and related notions developing a detailed introduction on the subject.

We compute the first value for a variety of cases. The set of natural numbers is much harder to find, we solve the problem for a smaller family of examples, nonetheless of interest. In the end, we prove results for a few non trivial tensor varieties that are of particular importance.

The content of this thesis is collected in three different papers. The first of them has been already published, while the remaining two are almost ready for submission at the time of redaction.

Throughout the work, the necessary and related scientific literature will be mentioned. When needed, classical results will be recalled. The thesis is subdivided as follows.

Section 1 deals with preliminaries and formally presents polynomials with conjugated variables. It sets the first notations. We present a completely solved preparatory example to highlight the intricacy of the problem.

Section 2 collects the first results and properties of these polynomials. We propose a few strategies to simplify the problem and examine the topological degree of such maps and how it relates to their number of zeros. Moreover, we focus the analysis on degree two equations for which we completely describe the case of infinitely many solutions.

In Section 3 we present a fundamental object for our study that encodes information about solutions of a system of polynomials with conjugated variables. Several properties of this tool are proved, and a few examples are discussed. In particular, its relevance for our problem is established.

The ideas are applied to the case of degree two equations and to the case of harmonic equations. We obtain an upper bound for the number of the zeros in the latter case, which sharpens previously known bounds.

Section 4 presents the main aim of the thesis. We introduce the setting of the Hermitian distance problem and characterize it in several ways.

The set of critical points is investigated in algebraic geometric terms to define the Hermitian Distance degree. We recall the notion of Euclidean Distance degree and provide different relations about the two concepts. The case of affine and projective varieties are both considered. We prove general results and present introductory and non trivial examples.

We treat the case of hypersurfaces and of parametrized varieties proving some upper bounds, which are shown to be sharp in some cases. Then we focus on conics, developing more precise and complete results.

At the end of the section, we discuss how the notion of dual variety relates to the Hermitian distance problem.

In Section 5 a generalization of the better known focal loci that we call Hermitian Distance discriminant is considered. This object relates to the number of critical points and to how this number changes when the reference point moves in the ambient space. In particular, we perform a comprehensive study for the case of curves.

Special attention is reserved to the real plane. We introduce an interesting construction that possesses similar properties to the evolute of a real curve.

In Section 6 we introduce the last basic tool for our analysis. This is a polynomial in the classical sense called Hermitian Distance polynomial, which encodes all the information on the

Hermitian distance problem such as the number of critical points and their distance. The properties of this object are related to the previous definitions and a few computed examples are presented.

Section 7 is reserved to the study of the problem on determinantal varieties. We connect the results given by our approach to the results already present in the literature, focusing on the general case and to the case of symmetric matrices. We prove that the theory allows to reobtain classical results. In the end we discuss other consequences and related concepts.

In Section 8 we consider the case of tensor varieties, which are among the most interesting for us. For the case of the Segre and the Veronese varieties, we provide a different characterization of the critical points of the Hermitian distance problem. This description leads to several notions. In particular, we discuss a generalization of the singular values for matrices that carries many of the classical properties.

We apply the concepts presented in the dissertation to investigate the firsts non trivial examples of Segre variety and Veronese variety. We provide some explicit computations for both cases. This is an initial step in the hope of opening the way for more challenging tasks.

1 Preliminaries

A basic background in mathematics, in particular familiarity with the field of Algebraic Geometry is suggested for the full comprehension of this thesis.

However, along the dissertation there will be various recalling to the literature of classical notions and results.

Let $\mathbf{z}$ denote a collection of variables $\{z_1, \ldots, z_r\}$ and let $\mathbb{C}[\mathbf{z}]$ be the ring of polynomials with variables $\mathbf{z}$. For a multi-index $\alpha \in \mathbb{N}^r$ we denote with $\mathbf{z}^\alpha$ the monomial $z_1^{\alpha_1} \cdots z_r^{\alpha_r}$. Consider a system of equations

$$\begin{cases} p_1(\mathbf{z}) = 0 \\ \vdots \\ p_c(\mathbf{z}) = 0 \end{cases}$$

where $p_1, \ldots, p_c \in \mathbb{C}[\mathbf{z}]$. If the number of solutions of the system is finite, the sum of the multiplicities of the solutions is bounded by $\prod_{k=1}^{c} \deg p_k$ using the Bézout Theorem and if $r = c$ equality holds in the generic case. In particular, it is well known that a univariate polynomial $p(z)$ always possesses a finite number of zeros and the sum of their multiplicities is equal to $\deg p$.

We introduce our objects of interest.

Definition 1.0.1. A *generalized polynomial* is an expression of the form

$$p(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{(\alpha,\beta) \in \mathbb{N}^{2r}} a_{\alpha,\beta} \mathbf{z}^\alpha \bar{\mathbf{z}}^\beta$$

where the coefficients $a_{\alpha,\beta} \in \mathbb{C}$ are non zero only for finitely many multi-indices $(\alpha, \beta) \in \mathbb{N}^{2r}$. We denote its conjugate $\bar{p}(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{(\alpha,\beta)} \bar{a}_{\alpha,\beta} \mathbf{z}^\alpha \bar{\mathbf{z}}^\beta$.

As classical polynomials, a generalized polynomial p can be interpreted as a map

$$\begin{aligned} p \colon \mathbb{C}^r &\to \mathbb{C} \\ \mathbf{z} &\mapsto p(\mathbf{z}, \bar{\mathbf{z}}) \end{aligned}$$

by evaluation. With a slight abuse of notation we say that the *degree* of a generalized polynomial is the maximum of the degrees of its monomials, where $\deg \mathbf{z}^\alpha \bar{\mathbf{z}}^\beta = \|\alpha\|_1 + \|\beta\|_1$. Similarly for the degree with respect to a variable, e.g. $\deg_{z_k} \mathbf{z}^\alpha \bar{\mathbf{z}}^\beta = \alpha_k$ and $\deg_{\bar{z}_k} \mathbf{z}^\alpha \bar{\mathbf{z}}^\beta = \beta_k$. Moreover, the *leading terms* of a generalized polynomial p will be the terms $a_{\alpha,\beta} \mathbf{z}^\alpha \bar{\mathbf{z}}^\beta$ of p such that $\|\alpha\|_1 + \|\beta\|_1 = \deg p$.

Mainly, we are interested in solving systems of equations of the form

$$\begin{cases} p_1(\mathbf{z}, \bar{\mathbf{z}}) = 0 \\ \vdots \\ p_c(\mathbf{z}, \bar{\mathbf{z}}) = 0 \end{cases} \tag{1.0.1}$$

where $p_1, \ldots, p_c$ are generalized polynomials.

Even in the simplest case of a univariate generalized polynomial $p(z, \bar{z})$, the analysis is very different from the case of classical polynomials in $\mathbb{C}[z]$. First of all we could get no zeros, e.g. consider the generalized polynomial $z\bar{z}+1$, and infinitely many zeros, e.g. consider the generalized polynomial $z + \bar{z}$. Secondly, we will see that even if the set of zeros is finite, its cardinality could be greater than $\deg p$, see for example Proposition 2.3.1.

Let us recall that a *harmonic polynomial* is a generalized polynomial $h(z, \bar{z})$ that satisfies the equation $\partial_{\bar{z}} \partial_z h = 0$, equivalently it is of the form $h(z, \bar{z}) = p(z) + q(\bar{z})$ where p and q are polynomials in the variables z and $\bar{z}$ respectively, see [She02] for an introduction.

Consider the degree n harmonic polynomial equation

$$h(z, \bar{z}) = z^n + a\bar{z}^n + b = 0 \tag{1.0.2}$$

in which we have assumed the coefficient of z^n to be non zero to avoid triviality. If we introduce a new variable w that substitutes $\bar{z}$, we can search solutions of the equation (1.0.2) among the solutions of the system $h(z, w) = \bar{h}(w, z) = 0$. The Bézout Theorem predicts $(\deg h)^2 = n^2$ solutions for this system and this is a classical known bound for this type of equations. Firstly we solve the case $n = 1$ with the next lemma.

Lemma 1.0.2. *The harmonic equation $z + a\bar{z} + b = 0$ where $a, b \in \mathbb{C}$ admits one solution iff $|a| \neq 1$, in this case if $b = 0$ the solution is $z = 0$.*

If $|a| = 1$ the equation admits infinitely many solutions if $b/\sqrt{a} \in \mathbb{R}$ and zero solutions otherwise.

Proof. Using real coordinates $z = x + iy$ we divide the equation into its real and imaginary part to get the linear system

$$\begin{bmatrix} 1 + a^{\Re} & a^{\Im} \\ a^{\Im} & 1 - a^{\Re} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -b^{\Re} \\ -b^{\Im} \end{bmatrix},$$

which, by the Rouché–Capelli Theorem, admits one solution iff $|a| \neq 1$. The case $b = 0$ follows directly.

If $|a| = 1$, using polar coordinates $z = \rho e^{i\theta}$ and $a = e^{i\varphi}$, we write

$$\begin{aligned} z + a\bar{z} + b = \rho(e^{i\theta} + e^{i(\varphi-\theta)}) + b &= \rho(e^{i(\theta-\varphi/2)} + e^{i(\varphi/2-\theta)})e^{i\varphi/2} + b \\ &= \rho f(\theta)\sqrt{a} + b, \end{aligned}$$

where $f(\theta) = e^{i(\theta-\varphi/2)} + e^{i(\varphi/2-\theta)}$ is a real-valued function with image the set $[-2, 2]$. Thus, the map $\rho f(\theta)$ has image $\mathbb{R}$ and the equation $z + a\bar{z} + b = 0$ admits infinitely many solutions if $b/\sqrt{a} \in \mathbb{R}$ and no solutions otherwise. □

Thus in general for a degree 1 generalized polynomial equation, we could get 3 different situations for the number of solutions, for example:

$z + \bar{z} + i$	has no zeros.
$z + 2\bar{z}$	has one zero that is 0.
$z - \bar{z}$	has as zeros all the points in $\mathbb{R}$.

The general solution for equation (1.0.2) now follows from the previous lemma.

Proposition 1.0.3. *The harmonic equation $z^n + a\bar{z}^n + b = 0$ where $a, b \in \mathbb{C}$ admits n solutions iff $|a| \neq 1$ and $b \neq 0$. If $|a| \neq 1$ and $b = 0$ the only solution is $z = 0$.*

If $|a| = 1$ the equation admits infinitely many solutions if $b/\sqrt{a} \in \mathbb{R}$ and zero solutions otherwise.

Proof. Note that, a solution of the equation $z+a\bar{z}+b = 0$ generates for the equation $z^n+a\bar{z}^n+b = 0$ the solutions given by the n-th roots of unity.

On the other hand, a solution of the equation $z^n + a\bar{z}^n + b = 0$ generates for the equation $z + a\bar{z} + b = 0$ the solutions given by the n-th power.

The assertion follows from this argument and Lemma 1.0.2. □

Counting the number of zeros of harmonic polynomials is an open problem of particular interest. We give more information and partial results on this problem in Subection 3.4.

In the presence of generalized polynomial equations, the present literature suggests the use of Gröbner basis as in [Kal87], or to introduce collections of variables $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ and subdivide the polynomials of the system (1.0.1) into their real and imaginary parts to obtain the system of real polynomial equations

$$\begin{cases} p_1^{\Re}(\mathbf{x}, \mathbf{y}) = 0, & p_1^{\Im}(\mathbf{x}, \mathbf{y}) = 0 \\ \vdots & \vdots \\ p_c^{\Re}(\mathbf{x}, \mathbf{y}) = 0, & p_c^{\Im}(\mathbf{x}, \mathbf{y}) = 0 \end{cases} \tag{1.0.3}$$

where $p_k^{\Re}(\mathbf{x}, \mathbf{y}) + ip_k^{\Im}(\mathbf{x}, \mathbf{y}) = p_k(\mathbf{z}, \bar{\mathbf{z}})$ for any $k = 1, \ldots, c$. In order to obtain the solutions of the system (1.0.1), we search for solutions of the system (1.0.3) with $\mathbf{x}$ and $\mathbf{y}$ real vectors. Thus, we can study the problem in the real setting, for example by using the Killing form, see Section 3. A first work on the Killing form is [PRS93], consult [PV21] for a comprehensive overview.

Aside from the simple example of equation (1.0.2) treated above, dividing the polynomials into their real and imaginary parts requires a non trivial computational effort and we would like to avoid this step.

A simple observation we already hinted is that a generalized polynomial $p(\mathbf{z}, \bar{\mathbf{z}})$ vanishes exactly on the points where the generalized polynomial $\bar{p}(\bar{\mathbf{z}}, \mathbf{z})$ vanishes. We will consider this fact throughout the work to develop the theory. Using this fact and treating $\bar{\mathbf{z}}$ as a new collection of independent variables $\mathbf{w}$, we can bound the number of solutions of the system (1.0.1) with the number of solutions of the system

$$\begin{cases} p_1(\mathbf{z}, \mathbf{w}) = 0, & \bar{p}_1(\mathbf{w}, \mathbf{z}) = 0 \\ \vdots & \vdots \\ p_c(\mathbf{z}, \mathbf{w}) = 0, & \bar{p}_c(\mathbf{w}, \mathbf{z}) = 0 \end{cases} \tag{1.0.4}$$

and in order to obtain the solutions of system (1.0.1), we search for solutions of the system (1.0.4) such that $\mathbf{w}$ is the conjugate of $\mathbf{z}$, in symbols $\mathbf{w} = \bar{\mathbf{z}}$.

Theoretically the two approaches are pretty much equivalent. It follows by considering that the condition $\mathbf{w} = \bar{\mathbf{z}}$ is equivalent to $\mathbf{w} = \mathbf{x} - i\mathbf{y}$ and it is straightforward to check that for any $k = 1, \dots, c$ there hold the equalities

$$p_k^{\Re}(\mathbf{x}, \mathbf{y}) + ip_k^{\Im}(\mathbf{x}, \mathbf{y}) = p_k(\mathbf{z}, \mathbf{w}) \qquad \text{and} \qquad p_k^{\Re}(\mathbf{x}, \mathbf{y}) - ip_k^{\Im}(\mathbf{x}, \mathbf{y}) = \bar{p}_k(\mathbf{w}, \mathbf{z}).$$

More precisely we have the following result.

Lemma 1.0.4. *The map between* $\mathbb{C}[\mathbf{z}, \mathbf{w}]$ *and* $\mathbb{R}[\mathbf{x}, \mathbf{y}] \times \mathbb{R}[\mathbf{x}, \mathbf{y}]$ *given by*

$$p(\mathbf{z}, \mathbf{w}) \mapsto \left(\frac{p(\mathbf{z}, \mathbf{w}) + \bar{p}(\mathbf{w}, \mathbf{z})}{2}, \frac{p(\mathbf{z}, \mathbf{w}) - \bar{p}(\mathbf{w}, \mathbf{z})}{2i} \right) = \left(p^{\Re}(\mathbf{x}, \mathbf{y}), p^{\Im}(\mathbf{x}, \mathbf{y}) \right)$$

where $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ *and* $\mathbf{w} = \mathbf{x} - i\mathbf{y}$ *is a bijection that remains valid if we restrict to polynomials with fixed degree.*

Proof. The statement follows by noting that monomials $\mathbf{x}^\alpha \mathbf{y}^\beta$ are image of the map only of monomials with variables $\mathbf{z}^{\tilde{\alpha}} \mathbf{w}^{\tilde{\beta}}$ such that $\tilde{\alpha} + \tilde{\beta} = \alpha + \beta$. Thus, we obtain any desired such monomial by a linear combination

$$\mathbf{x}^\alpha \mathbf{y}^\beta = \left(\frac{\mathbf{z} + \mathbf{w}}{2} \right)^\alpha \left(\frac{\mathbf{z} - \mathbf{w}}{2i} \right)^\beta = \sum_{\tilde{\alpha} + \tilde{\beta} = \alpha + \beta} a_{\tilde{\alpha}, \tilde{\beta}} \mathbf{z}^{\tilde{\alpha}} \mathbf{w}^{\tilde{\beta}}$$

for suitable $a_{\tilde{\alpha}, \tilde{\beta}} \in \mathbb{C}$ and by considering any linear combination with complex coefficients the claim follows. $\square$

We resume the behavior of the solutions of the two approaches in the next result.

Lemma 1.0.5. *The endomorphism of* $\mathbb{C}^{2r}$ *given by*

$$(\mathbf{z}, \mathbf{w}) \mapsto \left(\frac{\mathbf{z} + \mathbf{w}}{2}, \frac{\mathbf{z} - \mathbf{w}}{2i} \right) \qquad \textit{with inverse} \qquad (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x} + i\mathbf{y}, \mathbf{x} - i\mathbf{y}),$$

sends the solutions of system (1.0.4) *to the solutions of system* (1.0.3)*. Moreover, solutions such that* $\mathbf{w} = \bar{\mathbf{z}}$ *are sent to solutions such that* $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2r}$*.*

The theory on system of real polynomial equations is widely established. In particular, from Lemma 1.0.4 and Lemma 1.0.5 we can infer various information on the number of solutions of the system (1.0.1). Especially, if the number of solutions of the system (1.0.4), or equivalently of the system (1.0.3), is finite, then the sum of the multiplicities of the solutions is bounded by $\prod_{k=1}^{c} (\deg p_k)^2$ using the Bézout Theorem and if $r = c$ equality holds in the generic case. More specifically, for univariate generalized polynomial, we have the next result.

Proposition 1.0.6. *Let* $p(z, \bar{z})$ *be a generalized polynomial, if* $\deg p$ *is even (odd) then* p *generically admits any even (odd) number between* 0 (1) *and* $(\deg p)^2$ *of zeros. For general generalized polynomials we could have any number between* 0 *and* $(\deg p)^2$ *or infinitely many zeros.*

However, even if the number of solutions of the system (1.0.4) is finite, this value will not always be a sharp bound for the number of solutions of the system (1.0.1). In some cases, starting with a generalized polynomial that admits a finite amount of zeros, introducing the variables $\mathbf{w}$ could yield infinitely many solutions, for example consider the generalized polynomial $z\bar{z}$.

A reason why to prefer the approach with variables $\mathbf{w}$ is that dividing the equations into their real and imaginary parts often requires non trivial computational efforts.

2 Basics on generalized polynomials

Consider an univariate generalized polynomial equation $p(z, \bar{z}) = 0$. The system we obtain introducing the variable w is

$$\begin{cases} p(z, w) = 0 \\ \bar{p}(w, z) = 0 \end{cases}$$

and in general there can be various different cases.

Sometimes such an equation admits no solutions and so does the system obtained introducing w, e.g. consider the generalized polynomial $z + \bar{z} + i$.

Moreover, if the system admits infinitely many solutions, it could happen both that equation has only a finite number of solutions, e.g. consider the generalized polynomial $z\bar{z} + 1$, or that it has an infinite number of solutions, e.g. consider the generalized polynomial $z\bar{z} - 1$.

We characterize a family of polynomials for which introducing the variable w does not help.

Lemma 2.0.1. *Let $p(z, \bar{z})$ be a generalized polynomial such that $\deg p \geq 1$ and for which it holds $p(z, \bar{z}) = \bar{p}(\bar{z}, z)$, then the system $p(z, w) = \bar{p}(w, z) = 0$ admits infinitely many solutions.*

Proof. Firstly note that the equality $p(z, \bar{z}) = \bar{p}(\bar{z}, z)$ is equivalent to $p(z, w) = \bar{p}(w, z)$, in particular it suffices to show that $p(z, w)$ admits infinitely many zeros. Now, from the hypotheses $\deg p \geq 1$ and $p(z, \bar{z}) = \bar{p}(\bar{z}, z)$, it follows $\deg_z p(z, w), \deg_w p(z, w) \geq 1$. Thus, for almost any fixed $w \in \mathbb{C}$ the equation $p(z, w) = 0$ admits a solution $z \in \mathbb{C}$. □

The contrary of this last result is not valid, for example consider the generalized polynomials $(z+\bar{z})(z+2\bar{z}) = z^2+3z\bar{z}+2\bar{z}^2$ and $z^2\bar{z}$ for which the systems obtained introducing w have both infinitely many solutions and that have infinitely many zeros and only one zero respectively.

2.1 Simplifying strategies

The higher the degree of a generalized polynomial, the more difficult it is to find its zeros. We suggest a way to modify the problem in order to lower the degrees of the polynomials involved in a system.

To solve the equation $p(z, \bar{z}) = 0$, we can use the system of polynomial equations

$$\begin{cases} p_1(z, w, \rho) = 0 \\ \bar{p}_1(w, z, \rho) = 0 \\ zw - \rho = 0 \end{cases}$$

where p_1 is the polynomial $p(z, w)$ in which we have replaced any instance of zw with the variable ρ. The solution of this system are triples of the form (z, w, zw) where (z, w) is a solution of the system $p(z, w) = \bar{p}(w, z) = 0$. The advantage in introducing this new variable is that when searching solutions such that $z = \bar{w}$ we can assume that ρ must be a nonnegative real number.

We will see how this idea could be practically applied in Example 4.2.16.

Another strategy to simplify our problem is to modify it by composing the generalized polynomial with the translation in the complex plane

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z + \mu \end{aligned}$$

where $\mu \in \mathbb{C}$, so that the zeros of p minus μ are exactly the zeros of the generalized polynomial $\tilde{p}(z, \bar{z}) = p(z + \mu, \bar{z} + \bar{\mu})$. For example, if we consider the degree two generalized polynomial $p(z, \bar{z}) = az^2 + bz\bar{z} + c\bar{z}^2 + dz + e\bar{z} + f$ we get

$$\begin{aligned} \tilde{p}(z, \bar{z}) &= az^2 + bz\bar{z} + c\bar{z}^2 + (2a\mu + b\bar{\mu} + d)z + (2c\bar{\mu} + b\mu + e)\bar{z} + p(\mu, \bar{\mu}) \\ &= az^2 + bz\bar{z} + c\bar{z}^2 + d'z + e'\bar{z} + f'. \end{aligned}$$

In particular, we can eliminate one of the monomials under certain hypothesis. For example, if μ is a zero of p we can pose $f' = 0$. On the other hand, assuming that equation $2az + b\bar{z} + d = 0$ is solvable, see Lemma 1.0.2, if μ is a solution of it we can pose $d' = 0$. Similarly for e'.

This is a first naive way to use transformations of the complex plane in order to simplify our problem. Other possibility could be given by the *Möbius transformation*, that we recall are the transformations of the extended complex plane

$$\begin{aligned} \mathbb{C} \cup \{\infty\} &\to \mathbb{C} \cup \{\infty\} \\ z &\longmapsto \frac{\alpha z + \beta}{\delta z + \gamma} \end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and ∞ is the point at infinity.

2.2 Topological degree

In this section we relate topological properties of the generalized polynomials with the cardinality of their set of zeros. We start by recalling some classical results that can be found for example in [Hat02, Section 2.2].

Denote with $H_n(\mathbb{S}^n)$ the n-th homology group of the sphere $\mathbb{S}^n$. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map, the *topological degree* or simply *degree* of f is the number $\deg' f \in \mathbb{N}$ such that the induced pushforward in homology $f_*\colon H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$ is the multiplication by $\deg' f$.

Let $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ then:

i) $\deg' (f \circ g) = \deg' f \cdot \deg' g$.

ii) f and g are homotopic if and only if it holds the equality $\deg' f = \deg' g$.

Suppose now f has the property that for some point $w \in \mathbb{S}^n$, the preimage $f^{-1}(w)$ consists of finitely many points, say $z^{(1)}, \dots, z^{(d)}$ and let $U_1, \dots, U_d$ be mutually disjoint neighborhood of those points respectively. For any $k \in [d]$ the *local degree* of f at $z^{(k)}$ is the number $\deg' f|_{z^{(k)}} \in \mathbb{N}$ such that the pushforward in homology of $f\colon U_k \setminus \{z^{(k)}\} \to f(U_k) \setminus \{w\}$ is the multiplication by $\deg' f|_{z^{(k)}}$. In particular:

iii) It holds the equality $\deg' f = \sum_{k=1}^{d} \deg' f|_{z^{(k)}}$.

iv) If f maps homeomorphically U_k into $f(U_k)$ then it holds $\deg' f|_{z^{(k)}} = \pm 1$ depending on the orientation induced by the mapping of f.

It is well known that a univariate polynomial $p(z)$ can be extended to a continuous map from the sphere $\mathbb{S}^2$ to itself and that it holds $\deg' p = \deg p$. Moreover, a Möbius transformation f is a continuous map from the sphere to itself such that $\deg' f = 1$.

We now apply these concepts for generalized polynomials on the sphere. Many of the following results are known, however they are not very present in the literature and could be useful to recall them. We start by proving a basic lemma about the conjugation map.

Lemma 2.2.1. *The conjugation map* $^-$ *can be interpreted as a continuous map from* $\mathbb{S}^2$ *to itself such that* $\deg' {}^- = -1$.

Proof. Consider $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ where ∞ is the point at infinity. We just define the map to be classical conjugation on $\mathbb{C}$ and to send ∞ to itself, in particular this map is orientation reversing and by properties iii) and iv) of the topological degree the assertion follows. □

The next result characterizes the set of generalized polynomials that can be extended as maps on the sphere $\mathbb{S}^2$.

Proposition 2.2.2. *A generalized polynomial $p(z,\bar{z})$ is constant or such that*

$$\lim_{|z|\to\infty} |p(z,\bar{z})| = \infty$$

if and only if it can be interpreted as a continuous map from $\mathbb{S}^2$ to itself.

Proof. Consider $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ where ∞ is the point at infinity.

For the only if part we define the map to be equal to p on $\mathbb{C}$ and to send ∞ to itself if $\deg p \geq 1$ and to p otherwise, i.e. if $\deg p = 0$ we can regard p as a constant. We need to check the continuity for the case $\deg p \geq 1$.

For the open subset $\mathbb{C}$ we take the identity as coordinate map and obtain $\mathrm{Id} \circ p \circ \mathrm{Id} = p$. For a neighborhood of ∞ not containing the origin a coordinate map ψ is given by $z \mapsto 1/z$ and the composition

$$\psi \circ p \circ \psi = \frac{1}{p(1/z, 1/\bar{z})}$$

can be continuously extended since by hypothesis $\lim_{z\to 0} |p(1/z, 1/\bar{z})| = \infty$ and thus $\lim_{z\to 0} \psi \circ p \circ \psi = 0$.

For the contrary assume p is non constant, otherwise we have finished. Moreover, assume that the limit of the statement does not hold. We prove the limit is thus non existent and then the map can not be continuously extended obtaining a contradiction.

Collect the monomial with same degree and rewrite the generalized polynomial as

$$p(z,\bar{z}) = \sum_{s=0}^{\deg p} \left(z^s \sum_{\ell=0}^{s} a_{s-\ell,\ell} \left(\frac{\bar{z}}{z}\right)^{\ell} \right).$$

Now, use the polar coordinates $z = \rho e^{i\theta}$ to write

$$p(\rho e^{i\theta}, \rho e^{-i\theta}) = \sum_{s=0}^{\deg p} \left(\rho^s e^{si\theta} \sum_{\ell=0}^{s} a_{s-\ell,\ell} (e^{-2i\theta})^{\ell} \right). \tag{2.2.1}$$

Note that the polynomial

$$q(\lambda) = \sum_{\ell=0}^{\deg p} a_{\deg p-\ell,\ell} \lambda^{\ell}$$

has a finite number of solutions on $\mathbb{S}^1 \subseteq \mathbb{C}$. In particular, the modulus of the coefficient of the monomial $\rho^{\deg p}$ of the polynomial in ρ of equation (2.2.1), that is

$$|e^{\deg p i\theta} \sum_{\ell=0}^{\deg p} a_{s-\ell,\ell} (e^{-2i\theta})^{\ell}| = |q(e^{-2i\theta})|,$$

is different from 0 for some $\theta_1 \in [0, 2\pi]$. From this, it holds the limit

$$\lim_{\rho\to\infty} |p(\rho e^{i\theta_1}, \rho e^{-i\theta_1})| = \lim_{\rho\to\infty} \rho^{\deg p} |q(e^{-2i\theta_1})| = \infty$$

and hence the limit of the statement has to be undefined. □

In particular, it is not hard to show, for example by dividing the map into real and imaginary parts, that a generalized polynomial $p(z, \bar{z})$ is orientation preserving or orientation reversing at a point z_0 if the determinant

$$\det \begin{bmatrix} (\partial_z p)(z_0, \bar{z}_0) & (\partial_{\bar{z}} p)(z_0, \bar{z}_0) \\ (\partial_z \bar{p})(\bar{z}_0, z_0) & (\partial_{\bar{z}} \bar{p})(\bar{z}_0, z_0) \end{bmatrix} = |(\partial_z p)(z_0, \bar{z}_0)|^2 - |(\partial_{\bar{z}} p)(z_0, \bar{z}_0)|^2$$

is positive or negative respectively. On the other hand, z_0 is said to be *singular* if this determinant vanishes.

By topological arguments, if $\lim_{|z|\to\infty} |p(z, \bar{z})| = \infty$ then the set of the zeros of p is contained inside a compact subset of the complex plane, however the contrary is not valid. For example, consider the polynomial $p(z, \bar{z}) = z + \bar{z} + i$ which has no zeros and the sequence $\{b_k = ki\}_{k\in\mathbb{N}}$ such that $|b_k| \to \infty$ for $k \to \infty$. Then it is easy to compute

$$\lim_{k\to\infty} p(b_k, \bar{b}_k) = i.$$

From Lemma 2.2.1 follows that if $p(z, \bar{z})$ is a generalized polynomial that satisfies the condition of Proposition 2.2.2 then $\deg' \bar{p}(\bar{z}, z) = \deg' {}^- \cdot \deg' p(z, \bar{z}) = -\deg' p(z, \bar{z})$ by property i) of the topological degree.

With the next result we compute the degree of a monomial on the sphere.

Corollary 2.2.3. *The topological degree of the continuous map $z^n \bar{z}^m \colon \mathbb{S}^2 \to \mathbb{S}^2$ is $n - m$.*

Proof. We let $n \geq m$, the cases $n < m$ follows from $\deg' z^n \bar{z}^m = -\deg' \bar{z}^n z^m$.

One method to see the validity of the statement is to consider the continuous homotopy

$$\begin{aligned} F \colon \mathbb{C} \times [0, 1] &\to \mathbb{C} \\ (z, t) &\mapsto t z^n \bar{z}^m + (1 - t) z^{n-m} = z^{n-m}(t|z|^{2m} + (1 - t)) \end{aligned}$$

of $z^n \bar{z}^m$ and z^{n-m}. Now, $F(z, t)$ satisfies the hypothesis of Proposition 2.2.2 uniformly in t, thus the homotopy can be extended on $\mathbb{S}^2$ and the claim follows since $\deg' z^n \bar{z}^m = \deg' z^{n-m} = n - m$ by property ii) of the topological degree.

A simpler way, is to note that the map $z^n \bar{z}^m = |z|^{2m} z^{n-m}$ is a local homeomorphism on $\mathbb{C} \setminus \{0\}$, orientation preserving and the cardinality of the preimage of any element in $\mathbb{C} \setminus \{0\}$ is equal to $n - m$ thus the assertion follows by properties iii) and iv) of the topological degree. □

The proposition we present now is of principal importance for our aim, it links the topological degree of a generalized polynomial on the sphere to the number of its zeros.

Proposition 2.2.4. *The number of zeros of a generic generalized polynomial $p(z, \bar{z})$ that satisfies the hypothesis of Proposition 2.2.2 is bounded from below by $|\deg' p|$.*

Proof. If p has an infinite number of zeros the statement is trivial. If not, since the map induced by a generic generalized polynomial is locally a homeomorphism in some neighborhoods of the preimages of 0, the assertion follows from the properties iii) and iv) of the topological degree by the inequalities

$$\sum_{z\in p^{-1}(0)} 1 = \sum_{z\in p^{-1}(0)} \left|\deg' p|_z\right| \geq \left|\sum_{z\in p^{-1}(0)} \deg' p|_z\right| = \left|\deg' p\right| .$$

□

We compute the degree of a family of generalized polynomials that will be useful to us.

Proposition 2.2.5. *Let $p(z,\bar{z}) = \sum a_{k,j} z^k \bar{z}^j$ be a generalized polynomial and let $a_{n,m} z^n \bar{z}^m$ be the only term of p such that $n+m = \deg p$, then $\deg' p = n-m$. In particular, the equation $p = 0$ generically admits at least $|n-m|$ solutions.*

Proof. The continuous homotopy

$$\begin{aligned} F\colon \mathbb{C}\times[0,1] &\to \mathbb{C} \\ (z,t) &\mapsto tp + (1-t)a_{n,m} z^n \bar{z}^m = t \sum_{k+j<\deg p} a_{k,j} z^k \bar{z}^j + a_{n,m} z^n \bar{z}^m \end{aligned}$$

of p and $a_{n,m} z^n \bar{z}^m$ satisfies the hypothesis of Proposition 2.2.2 uniformly in t, then the homotopy can be extended on $\mathbb{S}^2$ and thus $\deg' p = \deg' a_{n,m} z^n \bar{z}^m = n-m$ by property ii) of the topological degree and Corollary 2.2.3.

The last part follows from Proposition 2.2.4. □

The next result gives a sufficient condition for a generalized polynomial to be extended on the sphere.

Lemma 2.2.6. *Let $p(z,\bar{z}) = \sum a_{k,j} z^k \bar{z}^j$ be a non constant generalized polynomial, if the polynomial*

$$q(\lambda) = \sum_{\ell=0}^{\deg p} a_{\deg p-\ell,\ell} \lambda^\ell$$

has no zeros in $\mathbb{S}^1 \subseteq \mathbb{C}$ then $\lim_{|z|\to\infty} |p(z,\bar{z})| = \infty$.

Proof. Using the polar coordinates $z = \rho e^{i\theta}$, by the same steps of the second part of the proof of Proposition 2.2.2, the limit in the statement reads

$$\lim_{\rho\to\infty} |p(\rho e^{i\theta}, \rho e^{-i\theta})| = \lim_{\rho\to\infty} \left| \sum_{s=0}^{\deg p} \left(\rho^s e^{si\theta} \sum_{\ell=0}^{s} a_{s-\ell,\ell} (e^{-2i\theta})^\ell \right) \right| = \infty$$

uniformly for $\theta \in [0,2\pi]$. Since the module of the quantity multiplying $\rho^{\deg p}$ is $|q(e^{-2i\theta})|$ which by hypothesis is bounded from below by a positive constant independent from θ this limit holds. □

The contrary of Lemma 2.2.6 is not valid, for example consider the polynomial

$$p(z,\bar{z}) = a_{2,0}z^2 + a_{0,2}\bar{z}^2 + a_{1,0}z = z^2 + \bar{z}^2 + z$$

for which it holds

$$|p(z,\bar{z})|^2 = p^{\Re}(z,\bar{z})^2 + p^{\Im}(z,\bar{z})^2 = \left(z^2 + \bar{z}^2 + \frac{z+\bar{z}}{2}\right)^2 + \left(\frac{z-\bar{z}}{2i}\right)^2$$

and thus $\lim_{|z|\to\infty} |p(z,\bar{z})| = \infty$. In this case, the polynomial $q(\lambda) = a_{2,0} + a_{0,2}\lambda = 1 + \lambda$ of Lemma 2.2.6 vanishes for $\lambda = -1 \in \mathbb{S}^1$.

The condition of Lemma 2.2.6 is trivial if there is only one leading term. While, if there are two leading terms, the condition is equivalent to have leading coefficients with different norms, see the example below.

Moreover, from the proof of Lemma 1.0.2, follows that the condition of Lemma 2.2.6 is also necessary if the generalized polynomial is of degree one.

Example 2.2.7. Consider the generalized polynomial

$$p(z,\bar{z}) = a_{n,\deg p-n} z^n \bar{z}^{\deg p-n} + a_{m,\deg p-m} z^m \bar{z}^{\deg p-m} + \sum_{k+j<\deg p} a_{k,j} z^k \bar{z}^j.$$

with $n \geq m$. The polynomial of Lemma 2.2.6 is

$$q(\lambda) = a_{n,\deg p-n}\lambda^{\deg p-n} + a_{m,\deg p-m}\lambda^{\deg p-m} = \lambda^{\deg p-n}\left(a_{n,\deg p-n} + a_{m,\deg p-m}\lambda^{n-m}\right)$$

and it has no solution in $\mathbb{S}^1 \subseteq \mathbb{C}$ if and only if $|a_{n,\deg p-n}| \neq |a_{m,\deg p-m}|$. In particular, if we assume $|a_{n,\deg p-n}| > |a_{m,\deg p-m}|$, then the continuous homotopy

$$F\colon \mathbb{C} \times [0,1] \to \mathbb{C}$$

$$(z,t) \mapsto a_{n,\deg p-n} z^n \bar{z}^{\deg p-m} + t\left(a_{m,\deg p-m} z^m \bar{z}^{\deg p-m} + \sum_{k+j<\deg p} a_{k,j} z^k \bar{z}^j\right)$$

that satisfies the hypothesis of Proposition 2.2.2 uniformly in t and thus

$$\deg' p = \deg' a_{n,\deg p-n} z^n \bar{z}^{\deg p-n} = 2n - \deg p$$

by property ii) of the topological degree and Corollary 2.2.3.

Similarly, if we assume $|a_{n,\deg p-n}| < |a_{m,\deg p-m}|$ then $\deg' p = 2m - \deg p$.

2.3 Degree two generalized polynomial I

We examine here the degree two univariate generalized polynomial equation

$$p(z,\bar{z}) = az^2 + 2\cdot bz\bar{z} + c\bar{z}^2 + 2\cdot dz + 2\cdot e\bar{z} + f = 0. \tag{2.3.1}$$

We translate this equation into the problem of finding some intersections of curves. To be more clear in our purpose, we first briefly recall a method to study the intersections of two conics that can be found for example in [Ric11, Section 11].

It is very classical to make a correspondence between any conic and a symmetric bilinear form in the following way

$$ax^2 + 2 \cdot bxy + cy^2 + 2 \cdot dx + 2 \cdot ey + f = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

where the form is represented by a unique 3×3 symmetric matrix. Let $A, B \in \mathbb{C}^{3\times 3}$ be two symmetric matrices representing two conics. We can find their intersection by applying the following steps.

1. Find a solution $(\lambda, \mu) \in \mathbb{C}^2$ of the cubic homogeneous equation $\det(\lambda A + \mu B) = 0$.
2. Split the degenerate conic $C = \lambda A + \mu B$ into two lines g and h.
3. Intersect the lines g and h with one of the two conics.

This is a classical method to solve this problem.

The next result is also a consequence of Proposition 1.0.6.

Proposition 2.3.1. *The degree* 2 *generalized polynomial*

$$az^2 + 2 \cdot bz\bar{z} + c\bar{z}^2 + 2 \cdot dz + 2 \cdot e\bar{z} + f$$

with $a, b, c, d, e, f \in \mathbb{C}$ *generically admits* 0, 2 *or* 4 *zeros. For special parameters it could have* 0, 1, 2, 3, 4 *or infinitely many zeros.*

Proof. Using real coordinates x and y for solving equation (2.3.1) we get the system

$$\begin{cases} p^{\Re}(x,y) = (a^{\Re} + 2b^{\Re} + c^{\Re})x^2 + 2(c^{\Im} - a^{\Im})xy + (2b^{\Re} - a^{\Re} - c^{\Re})y^2 + 2(d^{\Re} + e^{\Re})x + 2(e^{\Im} - d^{\Im})y + f^{\Re} = 0 \\ p^{\Im}(x,y) = (a^{\Im} + 2b^{\Im} + c^{\Im})x^2 + 2(a^{\Re} - c^{\Re})xy + (2b^{\Im} - a^{\Im} - c^{\Im})y^2 + 2(d^{\Im} + e^{\Im})x + 2(d^{\Re} - e^{\Re})y + f^{\Im} = 0 \end{cases}$$

and we are lead to find the real intersections of the two conics

$$p^{\Re}(x,y) = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a^{\Re} + 2b^{\Re} + c^{\Re} & c^{\Im} - a^{\Im} & d^{\Re} + e^{\Re} \\ c^{\Im} - a^{\Im} & 2b^{\Re} - a^{\Re} - c^{\Re} & e^{\Im} - d^{\Im} \\ d^{\Re} + e^{\Re} & e^{\Im} - d^{\Im} & f^{\Re} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} A_{\Re} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and

$$p^{\Re}(x,y) = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a^{\Im} + 2b^{\Im} + c^{\Im} & a^{\Re} - c^{\Re} & d^{\Im} + e^{\Im} \\ a^{\Re} - c^{\Re} & 2b^{\Im} - a^{\Im} - c^{\Im} & d^{\Re} - e^{\Re} \\ d^{\Im} + e^{\Im} & d^{\Re} - e^{\Re} & f^{\Im} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} A_{\Im} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

It is straightforward to see by imposing a linear systems, that the two matrices $A_{\Re}$ and $A_{\Im}$ are unrelated in the sense that they could be any symmetric real matrices depending on the parameters. The assertion now follows from the established theory on intersection of real conics. $\square$

In particular, using the notations of the proof of Proposition 2.3.1, the number of solutions of the equation

$$\det(\lambda A_{\Re} + \mu A_{\Im}) = 0$$

of step 1 and the type of conics obtained in step 2 are sufficient to characterizes the number of real intersections of the two conics. Avoiding trivial cases of two linearly dependent conics, this widely established behavior of intersections of projective conics is described by a total of 8 different cases up to complex projective equivalence and, more importantly to us, 13 different cases up to real projective equivalence. These types are presented in [Lev64] and are classically enumerated by roman numerals and lowercase letters to denote subclasses of a complex class as I, Ia, Ib, II, IIa, III, IIIa, IV, V, VI, VIa, VII, VIII.

We provide a collection of examples that covers all possible cases of Proposition 2.3.1.

$z + \bar{z} - 1 + iz\bar{z}$	has no zeros.
$z + \bar{z} + iz\bar{z}$	has 1 zero that is 0.
$(z + \bar{z})^2 + i(z\bar{z} - 1)$	has 2 zeros that are $i, -i$.
$(z + \bar{z})(z + \bar{z} - 2) + i(z\bar{z} - 1)$	has 3 zeros that are $1, i, -i$.
$(z + \bar{z})(z - \bar{z}) + z\bar{z} - 1$	has 4 zeros that are $\pm 1, \pm i$.
$z\bar{z} - 1$	has as zeros all the points in $\mathbb{S}^1 \subseteq \mathbb{C}$.

For all these examples the solution can be easily computed since their simple division into real and imaginary parts.

About second degree harmonic equations we have the following result.

Proposition 2.3.2. *The degree* 2 *harmonic polynomial*

$$az^2 + c\bar{z}^2 + 2 \cdot dz + 2 \cdot e\bar{z} + f$$

where $a, c, d, e, f \in \mathbb{C}$ *generically admits* 2 *or* 4 *zeros. For special parameters it could have* 0*,* 1*,* 2*,* 3*,* 4 *or infinitely many zeros.*

Proof. The assertion follows using Proposition 2.2.4, since by Example 2.2.7 the modulus of the topological degree of the generalized polynomial we are considering is equal to 2.

For the second statement we provide a collection of harmonic polynomial that covers all possible cases.

$z^2 + \bar{z}^2 + i$	has no zeros.
$z^2 + 2\bar{z}^2$	has 1 zero that is 0.
$z^2 + 2\bar{z} - 3$	has 2 zeros that are $1, -3$.
$z^2 + 2\bar{z} + 1$	has 3 zeros that are $-1, 1 + 2i, 1 - 2i$.
$z^2 + 2\bar{z}$	has 4 zeros that are $0, -2, 1 + \sqrt{3}i, 1 - \sqrt{3}i$.
$z^2 - \bar{z}^2$	has as zeros all the points in $\mathbb{R} \cup i\mathbb{R} \subseteq \mathbb{C}$.

The results of Subsection 3.1 about the Hermitian Killing form were used to compute the number of zeros of these examples. □

Note that, using the notations of the proof of Proposition 2.3.1, since for degree two harmonic equation we set $b = 0$, the two matrices $A_{\Re}$ and $A_{\Im}$ have two opposite diagonal entries, in particular these matrices are never positive or negative definite.

Introducing w for solving equation (2.3.1) we get the system

$$\begin{cases} p(z,w) = az^2 + 2bzw + cw^2 + 2dz + 2ew + f = 0 \\ \bar{p}(w,z) = \bar{a}w^2 + 2\bar{b}zw + \bar{c}z^2 + 2\bar{d}w + 2\bar{e}z + \bar{f} = 0 \end{cases}$$

and we are led to find the intersections for which $z = \bar{w}$ of the two conics

$$p(z,w) = \begin{bmatrix} z & w & 1 \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} z \\ w \\ 1 \end{bmatrix} = \begin{bmatrix} z & w & 1 \end{bmatrix} B \begin{bmatrix} z \\ w \\ 1 \end{bmatrix}$$

and

$$\bar{p}(w,z) = \begin{bmatrix} w & z & 1 \end{bmatrix} \bar{B} \begin{bmatrix} w \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} z & w & 1 \end{bmatrix} \begin{bmatrix} \bar{c} & \bar{b} & \bar{e} \\ \bar{b} & \bar{a} & \bar{d} \\ \bar{e} & \bar{d} & \bar{f} \end{bmatrix} \begin{bmatrix} z \\ w \\ 1 \end{bmatrix} = \begin{bmatrix} z & w & 1 \end{bmatrix} B_1 \begin{bmatrix} z \\ w \\ 1 \end{bmatrix}.$$

This time the two matrices representing the conics are related, in particular it can be shown $\det(B_1) = \det(\bar{B})$. Thus, if $\det(B) = 0$ then both conics $p(z,w) = 0$ and $\bar{p}(w,z) = 0$ are degenerate and the generalized polynomial decomposes

$$p(z,\bar{z}) = (z + \alpha_1\bar{z} + \alpha_2)(z + \beta_1\bar{z} + \beta_2)$$

for some $\alpha_k, \beta_k \in \mathbb{C}$ for $k = 1, 2$. Moreover, it is well known that these two factors are proportional iff $\mathrm{rk}(B) = 1$. Thus, since we already studied degree one generalized polynomial equations in Lemma 1.0.2, generically we have 1 zero if $\mathrm{rk}(B) = 1$ and 2 zeros if $\mathrm{rk}(B) = 2$.

To perform step 1 of the procedure for intersecting two conics, assume $\det(B) \neq 0$. From the properties of the two matrices exploited above we obtain

$$\det(\lambda B + \mu B_1) = \det(B)\lambda^3 + \xi\lambda^2\mu + \bar{\xi}\lambda\mu^2 + \overline{\det(B)}\mu^3$$

where

$$\xi = (af - d^2)\bar{a} + 2(de - bf)\bar{b} + (cf - e^2)\bar{c} + 2(bd - ae)\bar{d} + 2(be - cd)\bar{e} + (ac - b^2)\bar{f}.$$

From the hypothesis on the determinant, it is truly equivalent to consider $\mu = 1$ to solve the equation $\det(\lambda B + \mu B_1) = 0$. Moreover, if $\lambda \in \mathbb{C}$ is a solution of this equation, then so it is also $1/\bar{\lambda} \in \mathbb{C}$. Now, since the polynomial $\det(\lambda B + B_1)$ has at most three zeros then there exists a solution of the equation such that $|\lambda| = 1$. In particular, if B is real then so are the values $\det(B)$ and ξ, thus $\lambda = -1$ is a solution.

We want to emphasize the relations between the matrices $A_{\Re}$ and $A_{\Im}$ and the matrix B in the formulas

$$A_{\Re} = \frac{1}{2}\left(\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} B \begin{bmatrix} U^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} + \overline{\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} B \begin{bmatrix} U^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}}\right)$$

and

$$A_{\Im} = \frac{1}{2i}\left(\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} B \begin{bmatrix} U^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} - \overline{\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} B \begin{bmatrix} U^T & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}}\right).$$

where $U = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. These two matrices are real symmetric and we can consider their signature to solve the problem. On the other hand the matrix B is only complex symmetric.

We now associate the generalized polynomial to two unique matrices C_1 and C_2 in the following way

$$\begin{aligned} az^2 + 2 \cdot bz\bar{z} + c\bar{z}^2 + 2 \cdot dz + 2 \cdot e\bar{z} + f &= \begin{bmatrix} \bar{z} & z & 1 \end{bmatrix} \begin{bmatrix} b & c & e \\ a & b & d \\ d & e & f \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \bar{z} & z & 1 \end{bmatrix} \left(\begin{bmatrix} \frac{b+\bar{b}}{2} & \frac{c+\bar{a}}{2} & \frac{e+\bar{d}}{2} \\ \frac{\bar{c}+a}{2} & \frac{b+\bar{b}}{2} & \frac{\bar{e}+d}{2} \\ \frac{\bar{e}+d}{2} & \frac{e+\bar{d}}{2} & \frac{f+\bar{f}}{2} \end{bmatrix} + i \begin{bmatrix} \frac{b-\bar{b}}{2i} & \frac{c-\bar{a}}{2i} & \frac{e-\bar{d}}{2i} \\ \frac{a-\bar{c}}{2i} & \frac{b-\bar{b}}{2i} & \frac{d-\bar{e}}{2i} \\ \frac{d-\bar{e}}{2i} & \frac{e-\bar{d}}{2i} & \frac{f-\bar{f}}{2i} \end{bmatrix} \right) \begin{bmatrix} z \\ \bar{z} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \bar{z} & z & 1 \end{bmatrix} (C_1 + iC_2) \begin{bmatrix} z \\ \bar{z} \\ 1 \end{bmatrix}. \end{aligned}$$

Similarly to the real case, it is straightforward to see that the matrices C_1 and C_2 are unrelated. Moreover, these matrices are Hermitian and since one is multiplied by i, the expression above vanishes if and only if the products involving C_1 and C_2 respectively simultaneously vanish. For example a direct consequence of this correspondence is that if any of C_1 or C_2 is positive definite then equation (2.3.1) admits no solutions.

Note that there holds the equalities

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{B + B_1}{2} = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} B + \bar{B} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

and

$$C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{B - B_1}{2i} = \frac{1}{2i}\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} B - \bar{B} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right),$$

for the definition of B_1 see above.

With the next result we provide a converse for Lemma 2.0.1 in the case of degree two generalized polynomials.

Lemma 2.3.3. *Let $p(z,\bar{z})$ be a degree 2 generalized polynomial. If the system of polynomial equations $p(z,w) = \bar{p}(w,z) = 0$ admits infinitely many solutions then one of the following two mutually exclusive cases holds:*

i) *The polynomial $p(z,w)$ is irreducible and*

$$p(z,\bar{z}) = g(az^2 + 2bz\bar{z} + \bar{a}\bar{z}^2 + 2dz + 2\bar{d}\bar{z} + f)$$

with $a, d \in \mathbb{C}$, $b, f \in \mathbb{R}$ and $g \in \mathbb{S}^1 \subseteq \mathbb{C}$.

ii) *The polynomial $p(z,w)$ is reducible and p can be written in one of the two forms*

- $p(z,\bar{z}) = (a_1 z + \bar{a}_1\bar{z} + c_1)(a_2 z + b_2\bar{z} + c_2)$ *with $a_1, a_2, b_2, c_2 \in \mathbb{C}$ and $c_1 \in \mathbb{R}$,*
- $p(z,\bar{z}) = g(az + b\bar{z} + c)(\bar{b}z + \bar{a}\bar{z} + c)$ *with $a, b, g \in \mathbb{C}$ and $c \in \mathbb{R}$.*

Proof. We distinguish two cases in which the system $p(z,w) = \bar{p}(w,z) = 0$ has infinitely many solutions, that are whether $p(z,w)$ is irreducible or not.

If $p(z,w)$ is irreducible then by hypothesis must hold the equality $p(z,w) = \lambda \cdot \bar{p}(w,z)$ for some non zero $\lambda \in \mathbb{C}$. In particular it holds $p(z,\bar{z}) = \lambda \cdot \bar{p}(\bar{z},z)$ and if we write $p(z,\bar{z}) = az^2 + 2bz\bar{z} + c\bar{z}^2 + 2dz + 2e\bar{z} + f$, the last equivalence is satisfied iff the system

$$\begin{cases} \lambda\bar{a} = c, & \lambda\bar{c} = a \\ \lambda\bar{d} = e, & \lambda\bar{e} = d \\ \lambda\bar{b} = b, & \lambda\bar{f} = f \end{cases} \quad \text{that is equivalent to} \quad \begin{cases} c = \lambda\bar{a}, & e = \lambda\bar{d}, \\ b = \lambda\bar{b}, & f = \lambda\bar{f} \\ |\lambda| = 1 \end{cases}$$

is satisfied. Imposing these conditions we get the polynomial

$$p(z,\bar{z}) = az^2 + \sqrt{\lambda}2bz\bar{z} + \lambda\bar{a}\bar{z}^2 + 2dz + \lambda 2\bar{d}\bar{z} + \sqrt{\lambda}f$$

with $b, f \in \mathbb{R}$ and $\lambda \in \mathbb{S}^1 \subseteq \mathbb{C}$. Now, denote $\sqrt{\lambda}$ with g and collect it from the formula above so that the claim follows after relabeling the coefficients a and d.

If $p(z,w)$ is reducible then so it is $\bar{p}(w,z)$ and by hypothesis these two polynomials share at least a common factor. Thus we can write them as

$$\begin{aligned} p(z,w) &= (a_1 z + b_1 w + c_1)(a_2 z + b_2 w + c_2), \\ \bar{p}(w,z) &= (a_1 z + b_1 w + c_1) q(z,w) \end{aligned}$$

for some suitable coefficients $a_1, b_1, a_2, b_2, c_2 \in \mathbb{C}$, $c_1 \in \mathbb{R}$ and q linear factor. On the other hand, it holds $\bar{p}(w,z) = (\bar{b}_1 z + \bar{a}_1 w + c_1)(\bar{b}_2 z + \bar{a}_2 w + \bar{c}_2)$, then there has to be two possibilities.

Either it holds $(a_1 z + b_1 w + c_1) = \lambda(\bar{b}_1 z + \bar{a}_1 w + c_1)$ for some non zero $\lambda \in \mathbb{C}$ and there hold the equalities

$$\begin{cases} a_1 = \lambda\bar{b}_1 \\ c_1 = \lambda c_1 \end{cases} \quad \text{so that} \quad \begin{cases} p(z,\bar{z}) = (a_1 z + \bar{a}_1\bar{z} + c_1)(a_2 z + b_2\bar{z} + c_2) & \text{if } c_1 \neq 0, \\ p(z,\bar{z}) = (\sqrt{\lambda}a_1 z + \sqrt{\lambda}\bar{a}_1\bar{z})(\sqrt{\lambda}a_2 z + \sqrt{\lambda}b_2\bar{z} + \sqrt{\lambda}c_2) & \substack{\text{with } |\lambda| = 1 \\ \text{if } c_1 = 0,} \end{cases}$$

and we get the first bullet point by relabeling the coefficients if needed.

Either it holds $(a_1z + b_1w + c_1) = \lambda(\bar{b}_2z + \bar{a}_2w + \bar{c}_2)$ for some non zero $\lambda \in \mathbb{C}$ and there hold the equalities

$$\begin{cases} a_1 = \lambda\bar{b}_2 \\ b_1 = \lambda\bar{a}_2 \\ c_1 = \lambda\bar{c}_2 \end{cases} \qquad \text{so that} \qquad p(z,\bar{z}) = g(a_1z + b_1\bar{z} + c_1)(\bar{b}_1z + \bar{a}_1\bar{z} + c_1) \quad \text{where } g = 1/\bar{\lambda}.$$

□

Using this last result we can characterize the degree two generalized polynomials with an infinitely many zeros.

Corollary 2.3.4. *Let $p(z,\bar{z})$ be a degree 2 generalized polynomial and let A be its symmetric representative matrix. Then p admits infinitely many zeros iff one of the following two mutually exclusive cases holds:*

- *It holds $\det(A) \neq 0$, the generalized polynomial can be written in the form*
$$p(z,\bar{z}) = g(az^2 + 2bz\bar{z} + \bar{a}\bar{z}^2 + 2dz + 2\bar{d}\bar{z} + f)$$
where $a, d \in \mathbb{C}$, $b, f \in \mathbb{R}$ and $g \in \mathbb{S}^1 \subseteq \mathbb{C}$ and the real conic
$$(a + \bar{a} + 2b)x^2 + 2\frac{\bar{a} - a}{i}xy + (2b - a - \bar{a})y^2 + 2(d + \bar{d})x + 2\frac{\bar{d} - d}{i}y + f$$
has non trivial real locus.
- *It holds $\det(A) = 0$ and the generalized polynomial can be written in the form*
$$p(z,\bar{z}) = (az + b\bar{z} + c\sqrt{ab})(a_1z + b_1\bar{z} + c_1)$$
with $a, b, a_1, b_1, c_1 \in \mathbb{C}$, $c \in \mathbb{R}$ and $|a| = |b|$.

Proof. Recall that a polynomial $p(z,w) = az^2 + 2bzw + cw^2 + 2dz + 2ew + f$ is reducible iff it holds
$$\det\left(\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}\right) = 0.$$

If the generalized polynomial p has infinitely many zeros, then the system $p(z,w) = \bar{p}(w,z) = 0$ has infinitely many solutions and we reduce to the cases of Lemma 2.3.3.

In the form of case i), the equation $p = 0$ is equivalent to the equation $\tilde{p}(z,\bar{z}) = az^2 + 2bz\bar{z} + \bar{a}\bar{z}^2 + 2dz + 2\bar{d}\bar{z} + f = 0$. The polynomial $p(z,w)$ is irreducible iff $\tilde{p}(z,w)$ is irreducible. The condition on the conic follows with real coordinates $z = x+iy$ from the equality $\tilde{p}(z,\bar{z}) = \tilde{p}^{\Re}(x,y)$.

In the form of the second bullet point of case ii) that is for $p(z,\bar{z}) = g(az+b\bar{z}+c)(\bar{b}z+\bar{a}\bar{z}+\bar{c})$, from Lemma 1.0.2 we have have infinitely many zeros for p iff $|a| = |b|$ and $c/\sqrt{ab} \in \mathbb{R}$. On the other hand, we can retrieve the first bullet point of case ii) by setting $b = \bar{a}$.

The converse is straightforward since by construction these generalized polynomials have infinitely many zeros. □

Using this last result, it is not hard to check whether if we are in the presence of a degree two generalized polynomial with infinitely many zeros.

3 The Hermitian Killing form

The majority of the results contained in this section can be found in [Fur25]. We begin by recalling well known results about counting solutions of a real polynomial system of equations.

Let $c \in \mathbb{N}$ and consider a zero-dimensional system

$$\begin{cases} p_1(\mathbf{z}) = 0 \\ \vdots \\ p_c(\mathbf{z}) = 0 \end{cases}$$

where $p_k \in \mathbb{C}[\mathbf{z}]$ for any $k \in [c]$. Denote with $I = \langle p_1(\mathbf{z}), \ldots, p_c(\mathbf{z}) \rangle \subseteq \mathbb{C}[\mathbf{z}]$ the zero-dimensional ideal generated by this system. In particular, $V(I) = \{\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(d)}\} \subseteq \mathbb{C}^r$ for some $d \in \mathbb{N}$ and denoting with m_k the multiplicity of $\mathbf{z}^{(k)}$ for $k = 1, \ldots, d$, then it holds $\dim(\mathbb{C}[\mathbf{z}]/I) = \sum_{k=1}^{d} m_k$.

The multiplications by z_k for $k = 1, \ldots, r$ induce linear maps on the quotient $\mathbb{C}[\mathbf{z}]/I$, in other terms we define the maps

$$\begin{aligned} M_{z_k} \colon \mathbb{C}[\mathbf{z}]/I &\to \mathbb{C}[\mathbf{z}]/I \\ [f] &\mapsto [f z_k] \end{aligned}$$

The matrices representing these maps, with respect to the basis $\{[\mathbf{z}^\alpha] \mid \mathbf{z}^\alpha \notin \mathrm{LT}(I)\}$, are the *companion matrices* of I. More generally, for $[g] \in \mathbb{C}[\mathbf{z}]/I$ we define the product $M_g([f]) \coloneqq [fg]$.

We have the following results, see [CLO05, Chapter 2, Section 4].

Lemma 3.0.1. *Let $f, g \in \mathbb{C}[\mathbf{z}]$, then:*

i) *The map M_f is well-defined.*

ii) *It holds $M_f + M_g = M_{f+g}$.*

iii) *It holds $M_f M_g = M_g M_f = M_{fg}$.*

iv) *It holds $M_f = f(M_{z_1}, \ldots, M_{z_n})$.*

v) *The eigenvalues of* M_f *are the values* $f(\mathbf{z}^{(k)}) \in \mathbb{C}$ *of* f *at* $\mathbf{z}^{(k)} \in V(I)$ *for* $k = 1, \ldots, d$ *with relative multiplicities.*

The next proposition presents a decomposition of the quotien ring $\mathbb{C}[\mathbf{z}]/I$, see [CLO05, Chapter 4, Section 2].

Proposition 3.0.2. *Let* $\xi \in \mathbb{C}[\mathbf{z}]$ *be such that the evaluations* $\xi(\mathbf{z}^{(k)})$ *are all different for* $k = 1, \ldots, d$. *Consider for any* $k = 1, \ldots, d$ *the linear maps* $M_{\xi(\mathbf{z})-\xi(\mathbf{z}^{(k)})}$. *Then, setting* $\mathcal{A}_k = \bigcup_{n\in\mathbb{N}} \mathrm{Ker}(M_{\xi(\mathbf{z})-\xi(\mathbf{z}^{(k)})})^n$, *we have the decomposition*

$$\mathbb{C}[\mathbf{z}]/I = \oplus_{k=1}^{d} \mathcal{A}_k.$$

If $f \in \mathcal{A}_k$ *then* $f(\mathbf{z}^{(j)}) \neq 0$ *for* $k \neq j$. *Any subalgebras* $\mathcal{A}_k$ *admits a unit* e_k *such that* $e_k^2 = e_k$, $e_k e_j = 0$ *for* $k \neq j$ *and* $e_k(\mathbf{z}^{(j)}) = \delta_{k,j}$ *for any* k, j. *Moreover, it holds* $\bar{e}_k = e_j$ *if* $\bar{\mathbf{z}}^{(k)} = \mathbf{z}^{(j)}$.

We choosed the above definition for the subalgebras $\mathcal{A}_k$ for the sake of simplicity. After their introduction, in [CLO05, Chapter 4, Section 2] it is present a discussion on different equivalent definitions.

Let now the ideal I be generated by real-valued polynomials. We are ready to recall the Killing or Trace form.

Definition 3.0.3. We define for $\xi \in \mathbb{R}[\mathbf{x}]$ the symmetric bilinear form

$$\begin{aligned} \mathcal{K}_{\mathbb{R}}^{\xi} \colon \mathbb{C}[\mathbf{z}]/I \times \mathbb{C}[\mathbf{z}]/I &\to \mathbb{C} \\ ([f],[g]) &\longmapsto \mathrm{Tr}(M_\xi M_f M_g) \end{aligned}$$

The form $\mathcal{K}_{\mathbb{R}}^{1}$ is called the *Killing* or *Trace* form.

The proof of the following theorem is due to Hermite, it characterizes the properties of the Killing form and shows how it can be used to count the number of solutions of a real system of polynomial equations, see [CLO05, Chapter 2, Theorem 5.2].

Theorem 3.0.4. *Let* I *be generated by real polynomials, let* $\xi \in \mathbb{R}[\mathbf{z}]$ *and consider the restriction of* $\mathcal{K}_{\mathbb{R}}^{\xi}$ *on* $\mathbb{R}[\mathbf{z}]/I \times \mathbb{R}[\mathbf{z}]/I$, *then:*

i) *The variety* $V(I)$ *consists only of real points* $\mathbf{z}^{(k)}$ *with multiplicity* 1 *such that* $\xi(\mathbf{z}^{(k)}) > 0$ *iff* $\mathcal{K}_{\mathbb{R}}^{\xi}$ *is positive definite.*

ii) *The rank of* $\mathcal{K}_{\mathbb{R}}^{\xi}$ *is the number of distinct points* $\mathbf{z}^{(k)} \in V(I)$ *for which* $\xi(\mathbf{x}^{(k)}) \neq 0$.

iii) *The number of distinct real points* $\mathbf{z}^{(k)} \in V(I)$ *such that* $\xi(\mathbf{z}^{(k)}) > 0$ *minus the number of distinct real points* $\mathbf{z}^{(k)} \in V(I)$ *such that* $\xi(\mathbf{z}^{(k)}) < 0$ *equals the signature of* $\mathcal{K}_{\mathbb{R}}^{\xi}$.

If moreover ξ *is such that the evaluations* $\xi(\mathbf{z}^{(k)})$ *are all different for* $k = 1, \ldots, d$, *then:*

iv) *The number of distinct real points* $\mathbf{z}^{(k)} \in V(I)$ *such that* $\xi(\mathbf{z}^{(k)}) > 0$ *equals the number of positive eigenvalues of* $\mathcal{K}_{\mathbb{R}}^{\xi}$ *minus the number of negative eigenvalues of* $\mathcal{K}_{\mathbb{R}}^{1}$.

v) *The number of distinct real points* $\mathbf{z}^{(k)} \in V(I)$ *such that* $\xi(\mathbf{z}^{(k)}) < 0$ *equals the number of negative eigenvalues of* $\mathcal{K}_{\mathbb{R}}^{\xi}$ *minus the number of negative eigenvalues of* $\mathcal{K}_{\mathbb{R}}^{1}$.

3.1 Counting the number of zeros

We want to modify the ideas constituting the Killing form to our problem. In order to generalize these techniques we need to hide the conjugate operation.

Let $c \in \mathbb{N}$ and consider a system

$$\begin{cases} p_1(\mathbf{z}, \bar{\mathbf{z}}) = 0 \\ \vdots \\ p_c(\mathbf{z}, \bar{\mathbf{z}}) = 0 \end{cases}$$

where p_k is a generalized polynomial for any $k \in [c]$ with a finite number of solutions. When solving a system of generalized polynomial equations we can note that $p_k(\mathbf{z}, \bar{\mathbf{z}}) = 0$ iff $\bar{p}_k(\bar{\mathbf{z}}, \mathbf{z}) = 0$ for any $k \in [c]$. Thus, we introduce a new collection $\mathbf{w}$ of variables $\{w_1, \dots, w_r\}$ where w_k represents $\bar{z}_k$ for $k = 1, \dots, r$ and consider a zero-dimensional ideal

$$\tilde{I} = \langle p_1(\mathbf{z}, \mathbf{w}), \dots, p_c(\mathbf{z}, \mathbf{w}), \bar{p}_1(\mathbf{w}, \mathbf{z}), \dots, \bar{p}_c(\mathbf{w}, \mathbf{z}) \rangle \subseteq \mathbb{C}[\mathbf{z}, \mathbf{w}]. \tag{3.1.1}$$

In particular, we have

$$V(\tilde{I}) = \{(\mathbf{z}^{(1)}, \mathbf{w}^{(1)}), \dots, (\mathbf{z}^{(d)}, \mathbf{w}^{(d)})\} \subseteq \mathbb{C}^{2r}$$

for some $d \in \mathbb{N}$. Denote with m_k the multiplicity of $(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})$ for $k = 1, \dots, d$, then it holds $\dim(\mathbb{C}[\mathbf{z}, \mathbf{w}]/\tilde{I}) = \sum_{k=1}^{d} m_k$.

We now prove two lemmas that reflect the property of the solutions of a system of real polynomial equations of coming in conjugate pairs in our context.

Lemma 3.1.1. *Consider a polynomial $p(\mathbf{z}, \mathbf{w}) \in \mathbb{C}[\mathbf{z}, \mathbf{w}]$, a point $(\mathbf{z}^{(\lambda)}, \mathbf{w}^{(\lambda)}) \in \mathbb{C}^{2r}$ and $\lambda \in \mathbb{C}$, then it holds $p(\mathbf{z}^{(\lambda)}, \mathbf{w}^{(\lambda)}) = \bar{p}(\mathbf{w}^{(\lambda)}, \mathbf{z}^{(\lambda)}) = \lambda$ iff $p(\bar{\mathbf{w}}^{(\lambda)}, \bar{\mathbf{z}}^{(\lambda)}) = \bar{p}(\bar{\mathbf{z}}^{(\lambda)}, \bar{\mathbf{w}}^{(\lambda)}) = \bar{\lambda}$. Moreover, if $\mathbf{z}^{(\lambda)} = \bar{\mathbf{w}}^{(\lambda)}$ then it holds $p(\mathbf{z}^{(\lambda)}, \mathbf{w}^{(\lambda)}) = \bar{p}(\mathbf{w}^{(\lambda)}, \mathbf{z}^{(\lambda)}) = \lambda$ iff $\lambda \in \mathbb{R}$.*

Proof. By conjugation, $p(\mathbf{z}^{(\lambda)}, \mathbf{w}^{(\lambda)}) = \bar{p}(\mathbf{w}^{(\lambda)}, \mathbf{z}^{(\lambda)}) = \lambda$ iff

$$p(\bar{\mathbf{w}}^{(\lambda)}, \bar{\mathbf{z}}^{(\lambda)}) = \overline{\bar{p}(\mathbf{w}^{(\lambda)}, \mathbf{z}^{(\lambda)})} = \bar{\lambda} \qquad \text{and} \qquad \bar{p}(\bar{\mathbf{z}}^{(\lambda)}, \bar{\mathbf{w}}^{(\lambda)}) = \overline{p(\mathbf{z}^{(\lambda)}, \mathbf{w}^{(\lambda)})} = \bar{\lambda}.$$

The second statement follows again by conjugation since if $\mathbf{z}^{(\lambda)} = \bar{\mathbf{w}}^{(\lambda)}$ then

$$p(\mathbf{z}^{(\lambda)}, \mathbf{w}^{(\lambda)}) = \overline{\bar{p}(\bar{\mathbf{z}}^{(\lambda)}, \bar{\mathbf{w}}^{(\lambda)})} = \overline{\bar{p}(\mathbf{w}^{(\lambda)}, \mathbf{z}^{(\lambda)})}.$$

□

We define a linear extension of the conjugation map

$$\begin{aligned} *\colon \mathbb{C}[\mathbf{z},\mathbf{w}] &\to \mathbb{C}[\mathbf{z},\mathbf{w}] \\ f &\mapsto *(f) = f^* \end{aligned}$$

such that $z_k^* = w_k$, $w_k^* = z_k$ for $k = 1,\dots,r$ and $a^* = \bar{a}$ for any $a \in \mathbb{C}$ or more concisely $f(\mathbf{z},\mathbf{w})^* = \bar{f}(\mathbf{w},\mathbf{z})$. Clearly, it holds $*^2 = \mathrm{Id}$ and since the ideal $\tilde{I}$ is invariant under the action of $*$, this map passes to the quotient $\mathbb{C}[\mathbf{z},\mathbf{w}]/\tilde{I}$.

Lemma 3.1.2. *Let $\tilde{I} \subseteq \mathbb{C}[\mathbf{z},\mathbf{w}]$ be an ideal invariant under the action of $*$. If $(\mathbf{z}^{(k)},\mathbf{w}^{(k)}) \in V(\tilde{I})$ then $(\bar{\mathbf{w}}^{(k)},\bar{\mathbf{z}}^{(k)}) \in V(\tilde{I})$. Moreover, the two solutions possess the same multiplicity.*

Proof. Using Lemma 3.1.1 with $\lambda = 0$, if $p(\mathbf{z}^{(k)},\mathbf{w}^{(k)}) = \bar{p}(\mathbf{w}^{(k)},\mathbf{z}^{(k)}) = 0$ then $p(\bar{\mathbf{w}}^{(k)},\bar{\mathbf{z}}^{(k)}) = \bar{p}(\bar{\mathbf{z}}^{(k)},\bar{\mathbf{w}}^{(k)}) = 0$ for any $p \in \tilde{I}$. □

These last results motivate now the next definition.

Definition 3.1.3. Let $\mathcal{S}$ be any system of polynomial equations with polynomials in $\mathbb{C}[\mathbf{z},\mathbf{w}]$. We call a pair of distinct solutions $(\mathbf{z},\mathbf{w}),(\bar{\mathbf{w}},\bar{\mathbf{z}})$ of $\mathcal{S}$ an *associated pair.* We call a solution $(\mathbf{z},\bar{\mathbf{z}})$ of $\mathcal{S}$ a *conjugated single.*

A straightforward corollary already permits to limit the possible cases for the number of solutions of the system of generalized polynomial equations.

Corollary 3.1.4. *Let $\tilde{I} \subseteq \mathbb{C}[\mathbf{z},\mathbf{w}]$ be any ideal invariant under the action of $*$. The number of conjugated singles of $\tilde{I}$ has the same parity as the number of points in $V(\tilde{I})$.*

Proof. By definition and Lemma 3.1.2 the number of conjugated singles is $d - 2k$ where k is the number of associated pairs. □

We will often consider polynomials $\xi = \xi^* \in \mathbb{C}[\mathbf{z},\mathbf{w}]$. We prove a useful lemma.

Lemma 3.1.5. *It holds $\xi = \xi^* \in \mathbb{C}[\mathbf{z},\mathbf{w}]$ iff $\xi(\mathbf{x}+i\mathbf{y},\mathbf{x}-i\mathbf{y}) \in \mathbb{R}[\mathbf{x},\mathbf{y}]$.*

Proof. The assertion follows from the equalities

$$\begin{aligned} \xi(\mathbf{x}+i\mathbf{y},\mathbf{x}-i\mathbf{y}) &= \xi(\mathbf{z},\mathbf{w}) = \xi(\mathbf{z},\mathbf{w})^* = \bar{\xi}(\mathbf{w},\mathbf{z}) \\ &= \bar{\xi}(\mathbf{x}-i\mathbf{y},\mathbf{x}+i\mathbf{y}) = \overline{\xi(\mathbf{x}+i\mathbf{y},\mathbf{x}-i\mathbf{y})}. \end{aligned}$$

□

We will simply use the notation $\xi(\mathbf{x},\mathbf{y})$ to mean $\xi(\mathbf{x}+i\mathbf{y},\mathbf{x}-i\mathbf{y})$. We now define a new object that generalizes the classical Killing form.

Definition 3.1.6. We define for $\xi = \xi^* \in \mathbb{C}[\mathbf{z},\mathbf{w}]$ the sesquilinear forms

$$\begin{aligned} \mathcal{K}_{\mathbb{C}}^{\xi}\colon \mathbb{C}[\mathbf{z},\mathbf{w}]/\tilde{I} \times \mathbb{C}[\mathbf{z},\mathbf{w}]/\tilde{I} &\to \mathbb{C} \\ ([f],[g]) &\longmapsto \mathrm{Tr}(M_\xi M_f M_{g^*}) \end{aligned}$$

We call the form $\mathcal{K}_{\mathbb{C}}^{1}$ the *Hermitian Killing form.*

We prove that any form $\mathcal{K}_{\mathbb{C}}^{\xi}$ with $\xi = \xi^*$ is indeed Hermitian.

Proposition 3.1.7. *The form $\mathcal{K}_{\mathbb{C}}^{\xi}$ is Hermitian.*

Proof. The form is clearly sesquilinear.

We only need to show that for any $[f], [g] \in \mathbb{C}[\mathbf{z}, \mathbf{w}]/\tilde{I}$, it holds

$$\mathcal{K}_{\mathbb{C}}^{\xi}([f],[g]) = \mathrm{Tr}(M_{\xi f g^*}) = \overline{\mathrm{Tr}(M_{\xi g f^*})} = \overline{\mathcal{K}_{\mathbb{C}}^{\xi}([g],[f])}.$$

The second equality follows from

$$\begin{aligned}
\mathrm{Tr}(M_{\xi f g^*}) &= \sum_{k=1}^{d} m_k \xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) f(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) \bar{g}(\mathbf{w}^{(k)}, \mathbf{z}^{(k)}) && \text{(Lemma 3.0.1, v)} \\
&= \sum_{k=1}^{d} m_k \overline{\bar{\xi}(\bar{\mathbf{z}}^{(k)}, \bar{\mathbf{w}}^{(k)}) \bar{f}(\bar{\mathbf{z}}^{(k)}, \bar{\mathbf{w}}^{(k)}) g(\bar{\mathbf{w}}^{(k)}, \bar{\mathbf{z}}^{(k)})} && \text{(Conjugation)} \\
&= \sum_{k=1}^{d} m_k \overline{\bar{\xi}(\mathbf{w}^{(k)}, \mathbf{z}^{(k)}) \bar{f}(\mathbf{w}^{(k)}, \mathbf{z}^{(k)}) g(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})} && \text{(Lemma 3.1.2)} \\
&= \sum_{k=1}^{d} m_k \overline{\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) \bar{f}(\mathbf{w}^{(k)}, \mathbf{z}^{(k)}) g(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})} && (\xi = \xi^*) \\
&= \overline{\mathrm{Tr}(M_{\xi g f^*})}. && \text{(Lemma 3.0.1, v)}
\end{aligned}$$

□

Consider the decomposition of Proposition 3.0.2 with $\xi = \xi^* \in \mathbb{C}[\mathbf{z}, \mathbf{w}]$. The next lemma characterizes the action of the map $*$ on this decomposition.

Lemma 3.1.8. *The map $*$ fixes the subalgebras $\mathcal{A}_k$ corresponding to a conjugated single and switches two subalgebras $\mathcal{A}_k$ and $\mathcal{A}_j$ corresponding to an associated pair. In other terms, it holds $e_k^* = e_k$ in the first case and $e_k^* = e_j$, $e_j^* = e_k$ in the latter.*

Moreover, it holds $e_k e_k^* = e_k$ in the first case and $e_k e_k^* = 0$ in the latter.

Proof. From the equalities

$$\xi - \xi(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) = \xi^* - \bar{\xi}(\bar{\mathbf{z}}^{(k)}, \mathbf{z}^{(k)}) = (\xi - \xi(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}))^*$$

the part about conjugated singles follows.

From the equalities

$$\xi - \xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) = \xi^* - \bar{\xi}(\mathbf{w}^{(k)}, \mathbf{z}^{(k)}) = (\xi - \xi(\bar{\mathbf{w}}^{(k)}, \bar{\mathbf{z}}^{(k)}))^*$$

the part about associated pairs follows.

The last statement follows from Proposition 3.0.2. □

Let $d_1 \leq d$ be the number of conjugated singles. We reorder the decomposition in such a way that

$$\mathbb{C}[\mathbf{z}, \mathbf{w}]/\tilde{I} = (\oplus_{k=1}^{d_1} \mathcal{A}_k) \oplus (\oplus_{k=d_1+1}^{\frac{d-d_1}{2}} (\mathcal{A}_k \oplus \mathcal{A}_k^*)) \tag{3.1.2}$$

where the first d_1 summands are the subalgebras corresponding to conjugated singles and the second summands correspond to associated pairs.

Proposition 3.1.9. *For $\xi = \xi^* \in \mathbb{C}[\mathbf{z}, \mathbf{w}]$, the decomposition* (3.1.2) *is orthogonal with respect to $\mathcal{K}_{\mathbb{C}}^{\xi}$.*

Proof. The assertion follows since the elements of the decomposition are invariant under the action of the map $*$, and $\mathcal{K}_{\mathbb{C}}^{\xi}$ behaves as $\mathcal{K}_{\mathbb{R}}^{\xi}$ on this decomposition.

However, we prove this statement directly. Computing the form on the units is sufficient, we consider three cases:

i) If $k \neq j \leq d_1$, then $\mathcal{K}_{\mathbb{C}}^{\xi}(e_k, e_j) = \mathrm{Tr}(M_{\xi e_k e_j^*}) = \mathrm{Tr}(M_{\xi e_k e_j}) = \mathrm{Tr}(M_0) = 0$.

ii) If $k \leq d_1$ and $j > d_1$, then $\mathcal{K}_{\mathbb{C}}^{\xi}(e_k, e_j + e_j^*) = \mathrm{Tr}(M_{\xi(e_k e_j^* + e_k e_j)}) = \mathrm{Tr}(M_0) = 0$.

iii) If $k \neq j > d_1$, then $\mathcal{K}_{\mathbb{C}}^{\xi}(e_k + e_k^*, e_j + e_j^*) = \mathrm{Tr}(M_{\xi(e_k + e_k^*)(e_j^* + e_j)}) = \mathrm{Tr}(M_0) = 0$.

These equalities follow from the properties of the units e_k for $k = 1, \ldots, d$ exploited in Lemma 3.1.8. □

The following is the main result of this section, its proof is analogous to the proof of Theorem 3.0.4.

Theorem 3.1.10. *Let $\xi = \xi^* \in \mathbb{C}[\mathbf{z}, \mathbf{w}]$, then:*

i) *The variety $V(\tilde{I})$ consists only of conjugated singles $(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)})$ with multiplicity 1 such that $\xi(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) > 0$ iff $\mathcal{K}_{\mathbb{C}}^{\xi}$ is positive definite.*

ii) *The rank of $\mathcal{K}_{\mathbb{C}}^{\xi}$ is the number of distinct points $(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) \in V(\tilde{I})$ such that $\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) \neq 0$.*

iii) *The number of distinct conjugated singles $(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) \in V(\tilde{I})$ such that $\xi(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) > 0$ minus the number of distinct conjugated singles $(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) \in V(\tilde{I})$ such that $\xi(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) < 0$ equals the signature of $\mathcal{K}_{\mathbb{C}}^{\xi}$.*

If moreover ξ is such that the evaluations $\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})$ are all different for $k = 1, \ldots, d$, then:

iv) *The number of distinct conjugated singles in $(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) \in V(\tilde{I})$ such that $\xi(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) > 0$ equals the number of positive eigenvalues of $\mathcal{K}_{\mathbb{C}}^{\xi}$ minus the number of negative eigenvalues of $\mathcal{K}_{\mathbb{C}}^{1}$.*

v) *The number of distinct conjugated singles in $(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) \in V(\tilde{I})$ such that $\xi(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}) < 0$ equals the number of negative eigenvalues of $\mathcal{K}_{\mathbb{C}}^{\xi}$ minus the number of negative eigenvalues of $\mathcal{K}_{\mathbb{C}}^{1}$.*

Proof. From Proposition 3.1.9 we can study the signature of $\mathcal{K}_{\mathbb{C}}^{\xi}$ over the decomposition (3.1.2).

For $k = 1, \ldots, d$, consider a basis of $\mathcal{A}_k$ given by $\{[e_k f_{k,j}]\}_{j=0,\ldots,m_k-1}$, where $f_{k,j} \in \mathbb{C}[\mathbf{z}, \mathbf{w}]$ are polynomials such that $f_{k,0} \equiv 1$ and $f_{k,j}(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) = 0$ for $j = 1, \ldots, m_k - 1$.

If $k \leq d_1$, then $f_{k,j}(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)})^* = 0$ and thus the only non vanishing term is

$$\begin{aligned}\operatorname{Tr}(M_{\xi e_k e_k^*}) = \operatorname{Tr}(M_{\xi e_k^2}) = \operatorname{Tr}(M_{\xi e_k}) &= \sum_{j=1}^{d} m_j \xi(\mathbf{z}^{(j)}, \mathbf{w}^{(j)}) \delta_{k,j} \\ &= m_k \xi(\mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)})\end{aligned}$$

corresponding to the leading term.

If $k > d_1$, consider the basis of the subspace $\mathcal{A}_k \oplus \mathcal{A}_k^*$ given by

$$\{[e_k f_{k,j} + (e_k f_{k,j})^*], [(e_k f_{k,j} - (e_k f_{k,j})^*)/i]\}_{j=0,\ldots,m_k-1}.$$

Similarly as before, we can consider just the subspace spanned by $\{[e_k + e_k^*], [(e_k - e_k^*)/i]\}$ corresponding to the second leading principal minor that is

$$\begin{bmatrix} \operatorname{Tr}(M_{\xi(e_k+e_k^*)(e_k^*+e_k)}) & i\operatorname{Tr}(M_{\xi(e_k+e_k^*)(e_k^*-e_k)}) \\ -i\operatorname{Tr}(M_{\xi(e_k-e_k^*)(e_k^*+e_k)}) & \operatorname{Tr}(M_{\xi(e_k-e_k^*)(e_k^*-e_k)}) \end{bmatrix} = \begin{bmatrix} \operatorname{Tr}(M_{\xi(e_k+e_k^*)}) & -i\operatorname{Tr}(M_{\xi(e_k-e_k^*)}) \\ -i\operatorname{Tr}(M_{\xi(e_k-e_k^*)}) & -\operatorname{Tr}(M_{\xi(e_k+e_k^*)}) \end{bmatrix}$$

and from $e_k(\mathbf{z}^{(j)}, \mathbf{w}^{(j)}) = \delta_{k,j}$ and Lemma 3.1.2, this matrix is

$$m_k \begin{bmatrix} \xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) + \xi(\bar{\mathbf{w}}^{(k)}, \bar{\mathbf{z}}^{(k)}) & -i(\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) - \xi(\bar{\mathbf{w}}^{(k)}, \bar{\mathbf{z}}^{(k)})) \\ -i(\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) - \xi(\bar{\mathbf{w}}^{(k)}, \bar{\mathbf{z}}^{(k)})) & -(\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) + \xi(\bar{\mathbf{w}}^{(k)}, \bar{\mathbf{z}}^{(k)})) \end{bmatrix}.$$

Since from $\xi = \xi^*$ the equality $\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) = \overline{\xi(\bar{\mathbf{w}}^{(k)}, \bar{\mathbf{z}}^{(k)})}$ holds, we rewrite this matrix as

$$m_k \begin{bmatrix} \xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) + \overline{\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})} & -i(\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) - \overline{\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})}) \\ -i(\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) - \overline{\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})}) & -(\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) + \overline{\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})}) \end{bmatrix}$$

and note that it is real symmetric. We decompose it as

$$m_k U \begin{bmatrix} \xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)}) & 0 \\ 0 & \overline{\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})} \end{bmatrix} U^T$$

where $U = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. Using the Binet theorem the determinant is

$$(2i)^2 m_k^2 |\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})|^2 = -4m_k^2 |\xi(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})|^2$$

and the claim follows. □

Finally, we use the Hermitian Killing form to describe the number of solutions of a system of generalized polynomial equations.

Corollary 3.1.11. *The number of zeros of a system of generalized polynomial equations* $p_k(\mathbf{z}, \bar{\mathbf{z}}) = 0$ *for* $k = 1, \dots, c$*, is equal to the number of positive minus the number of negative eigenvalues of the Hermitian Killing form* $\mathcal{K}^1_{\mathbb{C}}$ *of the system*

$$\begin{cases} p_1(\mathbf{z}, \mathbf{w}) = 0, & \bar{p}_1(\mathbf{w}, \mathbf{z}) = 0 \\ \vdots & \vdots \\ p_c(\mathbf{z}, \mathbf{w}) = 0, & \bar{p}_c(\mathbf{w}, \mathbf{z}) = 0 \end{cases}$$

Moreover, the signature of $\mathcal{K}^1_{\mathbb{C}}$ *is equal to the signature of the Killing form* $\mathcal{K}^1_{\mathbb{R}}$ *considering coordinates* $\mathbf{z} = \mathbf{x} + i\mathbf{y}$*,* $\mathbf{w} = \mathbf{x} - i\mathbf{y}$*.*

Proof. The claim follows from Theorem 3.1.10 and Lemma 1.0.5. □

3.2 Properties of the Hermitian Killing form

Firstly, we want to consider the case of of a single univariate complex polynomial equation

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0.$$

The number of zeros counted with multiplicity is equal to $\deg p$. Let $\sigma_1, \dots, \sigma_n \in \mathbb{C}$ be the zeros of p, possibly repeated based on their multiplicity. The companion matrix of the ideal $I = \langle p \rangle \subseteq \mathbb{C}[z]$ with respect to the basis $\{[1], \dots, [z^{n-1}]\}$ of $\mathbb{C}[z]/I$ is

$$M_z = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix} \in \mathbb{C}^{n \times n}$$

and its eigenvalues are exactly the zeros of p, in particular it holds $\mathrm{Tr}(M_z^j) = \sum_{k=1}^n \sigma_k^j$ for any $j \in \mathbb{N}$. If the coefficients of the polynomial are real we can set the Vandermonde matrix

$$V = \begin{bmatrix} 1 & \sigma_1 & \cdots & \sigma_1^{n-1} \\ 1 & \sigma_2 & \cdots & \sigma_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \sigma_n & \cdots & \sigma_n^{n-1} \end{bmatrix} \in \mathbb{C}^{n \times n}$$

so that the matrix representing the Killing form decomposes $\mathcal{K}^1_{\mathbb{R}} = V^T V \in \mathbb{R}^{n \times n}$.

Let now $\otimes$ denote the Kronecker product. The companion matrices of the ideal $\tilde{I} = \langle p, p^* \rangle$ with respect to the basis

$$\{[1], \ldots, [z^{n-1}], [w], \ldots, [z^{n-1}w], \ldots, [w^{n-1}], \ldots, [z^{n-1}w^{n-1}]\}$$

of $\mathbb{C}[z,w]/\tilde{I}$ are

$$\tilde{M}_z = I_n \otimes M_z \in \mathbb{C}^{n^2 \times n^2} \qquad \text{and} \qquad \tilde{M}_w = \overline{M}_z \otimes I_n \in \mathbb{C}^{n^2 \times n^2}.$$

In particular, by the properties of the Kronecker product for any $k, j \in \mathbb{N}$ we get

$$\tilde{M}_z^k \tilde{M}_w^j = (I_n \otimes M_z)^k (\overline{M}_z \otimes I_n)^j = (I_n \otimes M_z^k)(\overline{M}_z^j \otimes I_n) = \overline{M}_z^j \otimes M_z^k \in \mathbb{C}^{n^2 \times n^2},$$

thus the eigenvalues of this matrix are $\{\sigma_r^k \bar{\sigma}_s^j\}_{r,s=1}^n \subseteq \mathbb{C}$ and

$$\operatorname{Tr}(\tilde{M}_z^k \tilde{M}_w^j) = \operatorname{Tr}(M_z^k) \operatorname{Tr}(\overline{M}_z^j) = \operatorname{Tr}(M_z^k) \overline{\operatorname{Tr}(M_z^j)}.$$

In the next example we want to compare the Hermitian Killing form and the Killing form applied to generalized polynomial equation.

Example 3.2.1. Consider the harmonic equation

$$h(z, \bar{z}) = z^2 + a\bar{z} + b = 0$$

with $a, b \in \mathbb{C}$. Dividing the equation into its real and imaginary parts we obtain the real system

$$\begin{cases} h^{\Re}(x,y) = x^2 - y^2 + a^{\Re}x + a^{\Im}y + b^{\Re} = 0 \\ h^{\Im}(x,y) = 2xy + a^{\Im}x - a^{\Re}y + b^{\Im} = 0 \end{cases}$$

We compute the matrix representing the Killing form associated to the real system with respect to the basis $\{[1], [x], [y], [y^2]\}$ of $\mathbb{R}[x,y]/\langle h^{\Re}, h^{\Im} \rangle$ to obtain the real symmetric matrix

$$\mathcal{K}_{\mathbb{R}}^1 = \begin{bmatrix} 4 & 0 & 0 & \frac{3|a|^2+4b^{\Re}}{2} \\ 0 & \frac{3|a|^2-4b^{\Re}}{2} & -2b^{\Im} & \frac{3|a|^2a^{\Re}+4(a^{\Re}b^{\Re}+a^{\Im}b^{\Im})}{4} \\ 0 & -2b^{\Im} & \frac{3|a|^2+4b^{\Re}}{2} & \frac{3|a|^2a^{\Im}+12(a^{\Im}b^{\Re}-a^{\Re}b^{\Im})}{4} \\ \frac{3|a|^2+4b^{\Re}}{2} & \frac{3|a|^2a^{\Re}+4(a^{\Re}b^{\Re}+a^{\Im}b^{\Im})}{4} & \frac{3|a|^2a^{\Im}+12(a^{\Im}b^{\Re}-a^{\Re}b^{\Im})}{4} & \frac{A}{8} \end{bmatrix}$$

where $A = 9|a|^4 + 8\left(3(a^{\Re})^2 b^{\Re} + 4(a^{\Im})^2 b^{\Re} - a^{\Re}a^{\Im}b^{\Im} + 2(b^{\Re})^2 + (b^{\Im})^2\right)$. This matrix is quite intricate, even if it comes from an apparently simple problem.

On the other hand, the Hermitian matrix representing the Hermitian Killing form of the system

$$\begin{cases} h(z,w) = z^2 + aw + b = 0 \\ \bar{h}(w,z) = w^2 + \bar{a}z + \bar{b} = 0 \end{cases}$$

with respect to the basis $\{[1], [z], [w], [zw]\}$ of $\mathbb{C}[z,w]/\langle h, h^*\rangle$ is

$$\mathcal{K}_{\mathbb{C}}^{1} = \begin{bmatrix} 4 & 0 & 0 & 3|a|^2 \\ 0 & 3|a|^2 & -4b & 4a\bar{b} \\ 0 & -4\bar{b} & 3|a|^2 & 4\bar{a}b \\ 3|a|^2 & 4\bar{a}b & 4a\bar{b} & 3|a|^4 + 4|b|^2 \end{bmatrix}.$$

The second, third and fourth leading principal minors are given by

i) $12|a|^2$,

ii) $4(3|a|^2 - 4|b|)(3|a|^2 + 4|b|)$,

iii) $27|a|^8 - 32|a|^4|b|^2 - 256|a^2\bar{b} + b^2|^2$,

respectively. If any of them is equal to zero, the equation admits at most 3 solutions since the matrix possesses at most 3 positive eigenvalues. If any of them is negative, the equation admits at most $3-1=2$ solutions since the matrix possesses at most 3 positive eigenvalues and at least a negative eigenvalue. On the other hand, if i), ii) and iii) are all positive, which is the case for $a \neq 0$ and $b = 0$, the Sylvester's Theorem assures that this matrix is positive definite and the equation admits exactly 4 solutions.

The Hermitian Killing form and the Killing form encode the same information on the problem, we want to understand how they are related depending on their entries.

We can pass from the matrix representing $K_{\mathbb{R}}^{\xi}$ to the matrix representing $K_{\mathbb{C}}^{\xi}$ by applying the map of Lemma 1.0.5, starting from a real symmetric matrix, this modification yields an Hermitian matrix and vice versa. Sometimes this procedure translates in a change of basis as exploited in the following proposition.

For the sake of simplicity, we will use a polynomial f also denoting the value $\mathrm{Tr}(M_f)$, when used as entry in a matrix. Since it is irrelevant for what follows in this section, from now on we will drop the dependence from ξ in the computations.

The following result uses the same techniques of Lemma 1.0.4.

Proposition 3.2.2. *Let* $\mathbf{n} \in \mathbb{N}^r$, $\mathbb{C}[\mathbf{z},\mathbf{w}]/\tilde{I} = \{[\mathbf{z}^\alpha \mathbf{w}^\beta] \mid \alpha_k + \beta_k \leq n_k\}$ *and* $\xi = \xi^* \in \mathbb{C}[\mathbf{z},\mathbf{w}]$, *so that* $\xi(\mathbf{x},\mathbf{y}) \in \mathbb{R}[\mathbf{x},\mathbf{y}]$. *Then* $\mathcal{K}_{\mathbb{C}}^{\xi}$ *and* $\mathcal{K}_{\mathbb{R}}^{\xi}$ *can be obtained one from the other by a change of basis.*

Moreover, this holds true for any restriction on $\{[\mathbf{z}^\alpha \mathbf{w}^\beta] \mid \alpha_k + \beta_k \leq \tilde{n}_k\}$ *with* $\tilde{n}_k \leq n_k$ *for any* $k = 1, \ldots, r$.

Proof. Using the map of Lemma 1.0.5 and Lemma 3.0.1 we obtain

$$\mathbf{z}^\alpha \mathbf{w}^\beta = (\mathbf{x} + i\mathbf{y})^\alpha (\mathbf{x} - i\mathbf{y})^\beta = \sum_{\tilde{\alpha}+\tilde{\beta}=\alpha+\beta} \gamma_{\tilde{\alpha},\tilde{\beta}} \mathbf{x}^{\tilde{\alpha}} \mathbf{y}^{\tilde{\beta}}$$

for suitable $\gamma_{\tilde{\alpha},\tilde{\beta}}$. Thus, if

$$I = \langle p_1^{\Re}(\mathbf{x},\mathbf{y}), p_1^{\Im}(\mathbf{x},\mathbf{y}), \ldots, p_c^{\Re}(\mathbf{x},\mathbf{y}), p_c^{\Im}(\mathbf{x},\mathbf{y})\rangle$$

then $\mathbb{C}[\mathbf{x},\mathbf{y}]/I = \{[\mathbf{x}^\alpha\mathbf{y}^\beta] \mid \alpha_k + \beta_k \leq n_k\}$ and vice versa. We can obtain one basis from the other by linear combinations, if $\Gamma = \left[\gamma_{\tilde{\alpha},\tilde{\beta}}\right]_{\tilde{\alpha}+\tilde{\beta}\leq n_k}$ then

$$\Gamma \begin{bmatrix} \vdots \\ \mathbf{x}^\alpha\mathbf{y}^\beta \\ \vdots \end{bmatrix} \begin{bmatrix} \cdots & \mathbf{x}^\alpha\mathbf{y}^\beta & \cdots \end{bmatrix} \Gamma^H = \begin{bmatrix} \vdots \\ \mathbf{z}^\alpha\mathbf{w}^\beta \\ \vdots \end{bmatrix} \begin{bmatrix} \cdots & \mathbf{w}^\alpha\mathbf{z}^\beta & \cdots \end{bmatrix}.$$

□

The next example highlights the change of bases we presented.

Example 3.2.3. We consider the simplest non trivial case of Proposition 3.2.2. Thus, let $r = 1$ and $n = 1$, they lead us to the bases $\{[1],[z],[w]\}$ and $\{[1],[x],[y]\}$. Using the notation of U from the proof of Theorem 3.1.10 we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & U^H \end{bmatrix} \begin{bmatrix} 1 & x & y \\ x & x^2 & xy \\ y & xy & y^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 1 & -i \end{bmatrix} \begin{bmatrix} 1 & x & y \\ x & x^2 & xy \\ y & xy & y^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -i & i \end{bmatrix}$$
$$= \begin{bmatrix} 1 & x-iy & x+iy \\ x+iy & x^2+y^2 & (x+iy)^2 \\ x-iy & (x-iy)^2 & x^2+y^2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & \frac{1}{2}U \end{bmatrix} \begin{bmatrix} 1 & w & z \\ z & zw & z^2 \\ w & w^2 & zw \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & \frac{1}{2}U^H \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2i} & -\frac{1}{2i} \end{bmatrix} \begin{bmatrix} 1 & w & z \\ z & zw & z^2 \\ w & w^2 & zw \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2i} \\ 0 & \frac{1}{2} & \frac{1}{2i} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \frac{w+z}{2} & \frac{z-w}{2i} \\ \frac{w+z}{2} & \frac{(z+w)^2}{4} & \frac{z^2-w^2}{4i} \\ \frac{z-w}{2i} & \frac{z^2-w^2}{4i} & \frac{-(z-w)^2}{4} \end{bmatrix}.$$

The change of basis of this last example is a special case, which is particularly simple and will be valid whenever we restrict the form on these subspaces. However, in general such a change of basis is not trivial, as in the case of Example 3.2.1 with bases $\{[1],[z],[w],[zw]\}$ and $\{[1],[x],[y],[y^2]\}$.

We have noticed that the Hermitian Killing form often results in a nicer form than the Killing form, it could be the case that this happens all the time.

A reason to think so is that, since for any polynomial f holds $\mathrm{Tr}(M_f) = \overline{\mathrm{Tr}(M_{f^*})}$, not only the matrix representing $\mathcal{K}_{\mathbb{C}}^{\xi}$ is Hermitian, moreover the entries above the diagonal could be conjugated one to another. In Example 3.2.1 we have checked that $\mathcal{K}_{\mathbb{C}}^{1}$ is simpler to be computed than $\mathcal{K}_{\mathbb{R}}^{1}$. This check extends to all the cases we have computed and it seems that the form $K_{\mathbb{C}}^{\xi}$, beyond its theoretical use, has also a computational advantage.

The next computations show a few examples of this fact.

Example 3.2.4. For the bases $\{[1],[z],[w],[zw]\}$ and $\{[1],[z],[w],[z^2]\}$ with matrices representing the Hermitian Killing form

$$\mathcal{K}_{\mathbb{C}}^1 = \begin{bmatrix} 1 & w & \underline{z} & \underline{zw} \\ z & zw & \underline{z^2} & \underline{z^2w} \\ w & w^2 & zw & \underline{zw^2} \\ zw & zw^2 & z^2w & \underline{z^2w^2} \end{bmatrix} \quad \text{and} \quad \mathcal{K}_{\mathbb{C}}^1 = \begin{bmatrix} 1 & w & \underline{z} & \underline{w^2} \\ z & \underline{zw} & z^2 & \underline{zw^2} \\ w & w^2 & zw & \underline{w^3} \\ z^2 & z^2w & z^3 & \underline{z^2w^2} \end{bmatrix},$$

we underline a set of 5 and 6 entries respectively, that are generically needed to complete each matrix.

For the bases $\{[1],[x],[y],[y^2]\}$ and $\{[1],[x],[y],[xy]\}$ with matrices representing the Killing form

$$\mathcal{K}_{\mathbb{R}}^1 = \begin{bmatrix} 1 & \underline{x} & \underline{y} & \underline{y^2} \\ x & \underline{x^2} & \underline{xy} & \underline{xy^2} \\ y & xy & \underline{y^2} & \underline{y^3} \\ y^2 & xy^2 & y^3 & \underline{y^4} \end{bmatrix} \quad \text{and} \quad \mathcal{K}_{\mathbb{R}}^1 = \begin{bmatrix} 1 & \underline{x} & \underline{y} & \underline{xy} \\ x & \underline{x^2} & xy & \underline{x^2y} \\ y & xy & \underline{y^2} & \underline{xy^2} \\ xy & x^2y & xy^2 & \underline{x^2y^2} \end{bmatrix},$$

we underline a set of 8 entries each, that are generically needed to complete each matrix.

If we consider the two left matrices, we are in the case of Example 3.2.1.

In the end, we note that we have more control on the diagonal. In fact, the entries in the diagonal of the matrix representing $K_{\mathbb{R}}^{\xi}$ are different in general, for two monomials $\mathbf{x}^{\alpha}\mathbf{y}^{\beta} \neq \mathbf{x}^{\tilde{\alpha}}\mathbf{y}^{\tilde{\beta}}$ it holds $(\mathbf{x}^{\alpha}\mathbf{y}^{\beta})^2 \neq (\mathbf{x}^{\tilde{\alpha}}\mathbf{y}^{\tilde{\beta}})^2$. On the other hand, in the matrix representing $K_{\mathbb{C}}^{\xi}$, it happens to have equal entries in the diagonal, for two monomials $\mathbf{z}^{\alpha}\mathbf{w}^{\beta} \neq \mathbf{z}^{\tilde{\alpha}}\mathbf{w}^{\tilde{\beta}}$ the equality $\mathbf{z}^{\alpha}\mathbf{w}^{\beta} \cdot (\mathbf{z}^{\alpha}\mathbf{w}^{\beta})^* = \mathbf{z}^{\tilde{\alpha}}\mathbf{w}^{\tilde{\beta}} \cdot (\mathbf{z}^{\tilde{\alpha}}\mathbf{w}^{\tilde{\beta}})^*$ holds true when $\alpha + \beta = \tilde{\alpha} + \tilde{\beta}$.

We now examine a few multivariate systems of generalized polynomial equations. We assume the ideals in $\mathbb{C}[\mathbf{z},\mathbf{w}]$ always to be zero-dimensional.

Example 3.2.5. Consider the system of generalized polynomial equations

$$\begin{cases} p_1(z_1,z_2,\bar{z}_1,\bar{z}_2) = z_1^2 + a\bar{z}_2 + b = 0 \\ p_2(z_1,z_2,\bar{z}_1,\bar{z}_2) = z_2 + cz_1 + d = 0 \end{cases}$$

with $a,b,c,d \in \mathbb{C}$. Using real coordinates we obtain the system

$$\begin{cases} p_1^{\Re}(x,y) = x_1^2 - y_1^2 + a_1x_2 + a_2y_2 + b_1 = 0 \\ p_1^{\Im}(x,y) = 2x_1y_1 + a_2x_2 - a_1y_2 + b_2 = 0 \\ p_2^{\Re}(x,y) = x_2 + c_1x_1 - c_2y_1 + d_1 = 0 \\ p_2^{\Im}(x,y) = y_2 + c_2x_1 + c_1y_1 + d_2 = 0 \end{cases}$$

and the entry with the highest number of terms in the matrix representing the Killing form w.r.t. the (non symmetric) basis $\{[1],[y_1],[y_2],[y_2^2]\}$ possesses 122 summands. On the other hand, the matrix representing the Hermitian Killing form with respect to the (symmetric) basis $\{[1],[z_2],[w_2],[z_2w_2]\}$ is

$$\mathcal{K}^1_{\mathbb{C}} = \begin{bmatrix} 4 & -4\bar{d} & -4d & 3|a|^2|c|^4+4|d|^2 \\ \vdots & 3|a|^2|c|^4+4|d|^2 & 4c^2(a\bar{d}-b)+4d^2 & -10|a|^2|c|^4d+4|c|^4a\bar{b}-4ac^2\bar{d}^2+4bc^2\bar{d}-4|d|^2d \\ & \vdots & 3|a|^2|c|^4+4|d|^2 & -10|a|^2|c|^4d+4|c|^4\bar{a}b-4\bar{a}\bar{c}^2d^2+4\bar{b}\bar{c}^2d-4|d|^2\bar{d} \\ & & \cdots & A \end{bmatrix}$$

where $A = 3|a|^4|c|^8 + 4|c|^4|3a\bar{d}-b|^2 - 4|ac^2\bar{d}-d^2|^2 - 4(\bar{b}\bar{c}^2d^3 + bc^2\bar{d}^3 - 2|d|^4)$.

Example 3.2.6. Consider the system of generalized polynomial equations

$$\begin{cases} p_1(z_1,z_2,\bar{z}_1,\bar{z}_2) = z_1^2 + a\bar{z}_2 = 0 \\ p_2(z_1,z_2,\bar{z}_1,\bar{z}_2) = z_2\bar{z}_1 + bz_1 = 0 \end{cases}$$

with $a,b\in\mathbb{C}$. Using real coordinates we obtain the system

$$\begin{cases} p_1^{\Re}(x,y) = x_1^2 - y_1^2 + a^{\Re}x_2 + a^{\Im}y_2 = 0 \\ p_1^{\Im}(x,y) = 2x_1y_1 + a^{\Im}x_2 - a^{\Re}y_2 = 0 \\ p_2^{\Re}(x,y) = x_1x_2 + y_1y_2 + b^{\Re}x_1 - b^{\Im}y_1 = 0 \\ p_2^{\Im}(x,y) = x_1y_2 - x_2y_1 + b^{\Im}x_1 + b^{\Re}y_1 = 0 \end{cases}$$

The matrix representing the Killing form with respect to the basis

$$\{[1],[x_1],[x_2],[y_1],[y_2],[x_1y_2],[x_2y_2],[y_1y_2],[y_2^2]\}$$

possesses 19 non zero entries and its characteristic polynomial $q_{\mathbb{R}}(\lambda)$ satisfy

$$\begin{aligned} |a|^2q_{\mathbb{R}}(\lambda) =& (\lambda-4|b|^2)^2(\lambda^2+2(a_1b_1-a_2b_2)|b|^2\lambda-4|a|^2|b|^4)^2 \\ & (|a|^2\lambda^3-(4|b|^4+9)|a|^2\lambda^2+4(A+5|a|^2|b|^4)\lambda-4A), \end{aligned}$$

where

$$\begin{aligned} A =& (a^{\Im})^2(b^{\Re})^8 + (a^{\Re})^2(b^{\Im})^8 + 2a^{\Re}a^{\Im}b^{\Re}b^{\Im}\left((b^{\Re})^6+(b^{\Im})^6\right) + 6a^{\Re}a^{\Im}(b^{\Re})^3(b^{\Im})^3\left((b^{\Re})^2+(b^{\Im})^2\right) \\ & + (b^{\Re})^2(b^{\Im})^2\left((a^{\Re})^2(b^{\Re})^4 + (a^{\Im})^2(b^{\Im})^4 + 3(a^{\Im})^2(b^{\Re})^4 + 3(a^{\Re})^2(b^{\Im})^4\right) + 3|a|^2(b^{\Re})^4(b^{\Im})^4. \end{aligned}$$

Hence, from the first factor we obtain two positive eigenvalues and from the second factor we get two positive and two negative eigenvalues. While the study of the third factor is not straightforward.

On the other hand, the matrix representing the Hermitian Killing form with respect to the basis

$$\{[1],[z_1],[z_2],[w_1],[w_2],[z_1z_2],[z_2w_2],[w_1w_2],[w_2^2]\}$$

possesses only 11 non zero entries. Moreover, the correspongin characteristic polynomial is

$$p(\lambda) = (\lambda-8|b|^2)^2(\lambda-8|b|^4)(\lambda^2-64|a|^2|b|^4)^2(\lambda^2-(9+8|b|^4)\lambda+8|b|^4)$$

and thus follows that the system possesses $7-2=5$ solutions if $ab\neq 0$ and so that the third factor of $|a|^2q_{\mathbb{R}}(\lambda)$ possesses 3 positive solutions.

Example 3.2.7. Consider the system of generalized polynomial equations

$$\begin{cases} p_1(z,\bar{z}) = z_1^2 + a\bar{z}_2 = 0 \\ p_2(z,\bar{z}) = z_2^2 + bz_1 = 0 \end{cases}$$

with $a, b \in \mathbb{C}$. Using real coordinates we obtain the system

$$\begin{cases} p_1^{\Re}(x,y) = x_1^2 - y_1^2 + a^{\Re}x_2 + a^{\Im}y_2 = 0 \\ p_1^{\Im}(x,y) = 2x_1y_1 + a^{\Im}x_2 - a^{\Re}y_2 = 0 \\ p_2^{\Re}(x,y) = x_2^2 - y_2^2 + b^{\Re}x_1 - b^{\Im}y_1 = 0 \\ p_2^{\Im}(x,y) = 2x_2y_2 + b^{\Im}x_1 + b^{\Re}y_1 = 0 \end{cases}$$

The matrix representing the Killing form with respect to the (non-symmetric) basis

$$\{[1],[x_1],[x_2],[y_1],[y_2],[x_1x_2],[x_1y_2],[x_2y_1],[y_1^2],[y_1y_2],[y_2^2],[x_1y_2^2],[x_2y_1^2],[y_1^2y_2],[y_1y_2^2],[y_1^2y_2^2]\}$$

possesses 42 non zero entries and computing its characteristic polynomial is very expensive.

On the other hand, the matrix representing the Hermitian Killing form with respect to the (symmetric) basis $\{[\mathbf{z}^\alpha\mathbf{w}^\beta] \mid \|\alpha\|_\infty, \|\beta\|_\infty \le 1\}$ possesses only 18 non zero entries. Moreover, the corresponding characteristic polynomial is

$$p(\lambda) = (\lambda - 15|ab|^2)^9(\lambda + 15|ab|^2)^5(\lambda^2 - (16 + 15|ab|^4)\lambda + 15|ab|^4)$$

and thus our system possesses exactly $9 + 2 - 5 = 6$ solutions if $ab \neq 0$.

3.3 Degree two generalized polynomial II

We study deeper the problem of Subsection 2.3 adopting the Hermitian Killing form. The general case

$$p(z,\bar{z}) = az^2 + 2 \cdot bz\bar{z} + c\bar{z}^2 + 2 \cdot dz + 2 \cdot e\bar{z} + f = 0.$$

is computationally expensive and the results are hard to read. Thus, we focus on the simpler cases in which at least one of the leading coefficients vanishes.

We have already seen what happens when we set $b = c = 0$ in Example 3.2.1. Thus we start by considering the case $a = c = 0$.

Example 3.3.1. Consider the equation

$$p(z,\bar{z}) = 2z\bar{z} + 2dz + 2e\bar{z} + f = 0.$$

We have assumed without loss of generality $b = 1$. The Hermitian matrix representing the Hermitian Killing form of the system

$$\begin{cases} p(z,w) = 0 \\ \bar{p}(w,z) = 0 \end{cases}$$

with respect to the basis $\{[1],[w]\}$ of $\mathbb{C}[z,w]/\langle p, p^*\rangle$ is

$$\mathcal{K}^1_{\mathbb{C}} = \begin{bmatrix} 2 & \frac{2(|e|^2-|d|^2)+\bar{f}-f}{2(d-\bar{e})} \\ \frac{2(|e|^2-|d|^2)-\bar{f}+f}{2(\bar{d}-e)} & \frac{2(|d|^2-|e|^2)^2-(|d|^2+|e|^2)(f+\bar{f})+2(\bar{d}\bar{e}f+de\bar{f})}{2|d-\bar{e}|^2} \end{bmatrix}.$$

In particular the equation admits no solutions if the determinant is negative, 1 solution if the determinant vanishes and 2 solutions if the determinant is positive.

If we only set $c = 0$, we have seen in Subsection 2.1 that by applying a translation of the form $z + \mu$ with $\mu \in \mathbb{C}$ we can set $d = 0$ under certain conditions, for example if $|a| \neq |b|$.

Example 3.3.2. Consider the equation

$$p(z,\bar{z}) = az^2 + 2z\bar{z} + 2e\bar{z} + f = 0.$$

We have assumed without loss of generality $b = 1$. The Hermitian matrix representing the Hermitian Killing form of the system

$$\begin{cases} p(z,w) = 0 \\ \bar{p}(w,z) = 0 \end{cases}$$

with respect to the basis $\{[1],[z],[w],[w^2]\}$ of $\mathbb{C}[z,w]/\langle p, p^*\rangle$ is of dimension 4, however it is too complicated to be reported here. We only present the 2×2 submatrix representing the restriction of the Hermitian Killing form to the subspace $\operatorname{span}\{[1],[z]\}$ that is

$$\begin{bmatrix} 4 & \frac{4(\bar{a}\bar{e}-2e)}{\bar{a}(|a|^2-4)} \\ \frac{4(ae-2\bar{e})}{a(|a|^2-4)} & \frac{4\left(2|a|^4|e|^2-8(ae^2+\bar{a}\bar{e}^2)+|a|^2(|a|^2-4)(|e|^2+f+\bar{f})\right)}{|a|^2(|a|^2-4)^2} \end{bmatrix}.$$

Using this information we can not infer anything about the number of solutions of the equation in general.

3.4 Harmonic polynomials

The case of harmonic polynomial equations is of particular interest, e.g. see [KLS18; LL15; Bri+20; Hau+15].

Set $r = 1$ and consider

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \qquad \text{and} \qquad q(\bar{z}) = b_m \bar{z}^m + \ldots + b_1 \bar{z} \tag{3.4.1}$$

complex univariate generalized polynomials. The Bézout Theorem gives the naive bound n^2 for the number of isolated roots of the harmonic polynomial $h(z, \bar{z}) = p(z) + q(\bar{z})$ where the degree zero term is given by the coefficient a_0 of p with $n = \deg p = n \geq \deg q = m$. In [Wil98] this bound was proved to be sharp for $m = n$ and the general bound

$$3n - 2 + m(m-1) \tag{3.4.2}$$

was conjectured for $m < n$.

For $m = n - 1$, formula (3.4.2) yields the bound n^2, whose sharpness was proved in [Wil98; BHS95].

For $m = 1$, we get the bound $3n - 2$, which was shown to be correct for any n and even sharp for $n = 2, 3$ in [KŚ02]. Then, this bound was shown to be sharp also for $n = 4, 5, 6, 8$ in [BL13] and for any n in [Gey08].

However, in [LLL15] it was shown that the bound (3.4.2) is in general false by providing counterexamples for $m = n - 3$ and was proposed the general bound

$$2m(n-1) + n. \tag{3.4.3}$$

Formula (3.4.3) is linear in n and is equal to the sharp bound $3n - 2$ for $m = 1$. However, if $m \geq n/2$, formula (3.4.3) is greater than the naive bound n^2.

Introduce the variable w representing $\bar{z}$. The ideal $\tilde{I} = \langle h, h^* \rangle$ is generically zero-dimensional and $\{[z^\alpha w^\beta] \mid \alpha, \beta < n\}$ is a basis of cardinality n^2 of the quotient space $\mathbb{C}[z, w]/\tilde{I}$. We want to bound the number of conjugated singles and we sharpen the current bound n^2 for $1 < m < n - 1$ with the following result.

Theorem 3.4.1. *Using the notations of equations* (3.4.1)*, if $n - 2 \geq m$ the harmonic equation $h(z, \bar{z}) = p(z) + q(\bar{z}) = 0$ admits at most $n^2 - 1$ solutions if $(n-1)a_{n-1}^2 - 2na_{n-2} = 0$ and at most $n^2 - 2$ solutions otherwise.*

We deal with two lemmas before proving Theorem 3.4.1. For our computations we consider the basis $\{[z^\alpha w^\beta] \mid \alpha, \beta < n\}$ of the quotient $\mathbb{C}[z, w]/\tilde{I}$.

Lemma 3.4.2. *If $n \geq 2$ with $n - 1 \geq m$, for the harmonic equation $h(z, \bar{z}) = p(z) + q(\bar{z}) = 0$ hold*

$$\operatorname{Tr}(M_w) = -n\bar{a}_{n-1} \qquad \textit{and} \qquad \operatorname{Tr}(M_{w^2}) = n(\bar{a}_{n-1}^2 - 2\bar{a}_{n-2}).$$

Proof. Compute $\operatorname{Tr}(M_w)$:

The multiplications $w \cdot z^k w^j$ with $0 \leq k \leq n-1$ and $0 \leq j \leq n-2$ do not give contribution.

Let $0 \leq k \leq n - 1$, in $\mathbb{C}[z, w]/\tilde{I}$ hold the equalities

$$\begin{aligned} w \cdot z^k w^{n-1} = z^k w^n = z^k (w^n - \bar{h}(w, z)) &= -\sum_{j=1}^{m} \bar{b}_j z^{j+k} - \sum_{s=0}^{n-1} \bar{a}_s z^k w^s \\ &= -\sum_{j=k+1}^{n-1} \bar{b}_{j-k} z^j - \sum_{\ell=n}^{m+k} \bar{b}_{\ell-k} z^\ell - \sum_{s=0}^{n-1} \bar{a}_s z^k w^s. \end{aligned}$$

The second sum appears if $n \leq m+k$, in this case the element z^ℓ with $n \leq \ell \leq m+k \leq 2n-2$ can be rewritten as $z^{\ell-n}(z^n - h(z,w))$, and the highest degree of z for monomials with variable w is $\ell - n \leq m+k-n < k$. In the end, for any k, the term $-\bar{a}_{n-1}z^k w^{n-1}$ is given by the third sum and then

$$\mathrm{Tr}(M_w) = \sum_{k=0}^{n-1}(-\bar{a}_{n-1}) = -n\bar{a}_{n-1}.$$

Compute $\mathrm{Tr}(M_{w^2})$:

The multiplications $w^2 \cdot z^k w^j$ with $0 \leq k \leq n-1$ and $0 \leq j \leq n-3$ do not give contribution. Let $0 \leq k \leq n-1$, for $w^2 \cdot z^k w^{n-2} = z^k w^n$ we consider the computations for $\mathrm{Tr}(M_w)$ and get the term $-\bar{a}_{n-2}z^k w^{n-2}$. Now, in $\mathbb{C}[z,w]/\tilde{I}$ hold the equalities

$$\begin{aligned} w^2 \cdot z^k w^{n-1} = z^k w^{n+1} = z^k w(w^n - \bar{h}(w,z)) &= -\sum_{j=1}^{m} \bar{b}_j z^{j+k} w - \sum_{s=0}^{n-1} \bar{a}_s z^k w^{s+1} \\ &= -\sum_{j=k+1}^{n-1} \bar{b}_{j-k} z^j w - \sum_{\ell=n}^{m+k} \bar{b}_{\ell-k} z^\ell w - \sum_{s=0}^{n-2} \bar{a}_s z^k w^{s+1} - \bar{a}_{n-1} z^k (w^n - \bar{h}(w,z)). \end{aligned}$$

The second sum appears if $n \leq m+k$, again we rewrite z^ℓ and the highest degree of z for monomials with variable w is $\ell - n \leq m+k-n < k$. Thus, adding the coefficients obtained before to the coefficients given by the third and fourth sum we get

$$\mathrm{Tr}(M_{w^2}) = \sum_{k=0}^{n-1}(-\bar{a}_{n-2}) + \sum_{j=0}^{n-1}\left(-\bar{a}_{n-2} + \bar{a}_{n-1}^2\right) = n(\bar{a}_{n-1}^2 - 2\bar{a}_{n-2}).$$

□

Lemma 3.4.3. *If $n \geq 2$, for the harmonic equation $h(z,\bar{z}) = q(z) + p(\bar{z}) = 0$ hold*

$$\mathrm{Tr}(M_{zw}) = \begin{cases} |a_{n-1}|^2 + (2n-1)|b_{n-1}|^2 & \text{if } n-1 = m \\ |a_{n-1}|^2 & \text{if } n-2 \geq m \end{cases}.$$

Proof. Compute $\mathrm{Tr}(M_{zw})$:

The multiplications $zw \cdot z^k w^j$ with $0 \leq k, j \leq n-2$ do not give contribution.

Let $1 \leq k \leq n-1$, for $zw \cdot z^{k-1}w^{n-1} = z^k w^n$, we consider the computations for $\mathrm{Tr}(M_w)$ in Lemma 3.4.2. Thus, if $m = n-1$, we get the term $|b_{n-1}|^2 z^{k-1} w^{n-1}$ given by the monomial $\bar{b}_m z^{m+k} = \bar{b}_m z^{n+k-1}$ in the second sum, otherwise we get no terms. By conjugation we get the information on the products $zw \cdot z^{n-1}w^{k-1} = z^n w^k$ with $1 \leq k \leq n-1$. Lastly, in $\mathbb{C}[z,w]/\tilde{I}$

hold the equalities

$$\begin{aligned} zw \cdot z^{n-1}w^{n-1} &= z^n w^n = (z^n - h(z,w))(w^n - \bar{h}(w,z)) \\ &= \left(\sum_{j=1}^{m} b_j w^j + \sum_{t=0}^{n-1} a_t z^t\right)\left(\sum_{k=1}^{m} \bar{b}_k z^k + \sum_{s=0}^{n-1} \bar{a}_s w^s\right) \\ &= \sum_{j,k=1}^{m} b_j \bar{b}_k z^k w^j + \sum_{\ell=1}^{n+m-1} (\gamma_\ell z^\ell + \bar{\gamma}_\ell w^\ell) + \sum_{s,t=0}^{n-1} a_t \bar{a}_s z^t w^s, \end{aligned}$$

for suitable γ_ℓ. The first sum yields the term $|b_{n-1}|^2 z^{n-1}w^{n-1}$ if and only if $m = n-1$. The last sum yields the term $|a_{n-1}|^2 z^{n-1}w^{n-1}$. Arguing as in Lemma 3.4.2, we rewrite the elements z^ℓ with $\ell \geq n$ and the highest degree of z for monomials with variable w is $\ell - n \leq n + m - 1 - n = m - 1 < n - 1$, similarly for the terms w^ℓ. In the end, adding the coefficients, if $m < n-1$ we get

$$\operatorname{Tr}(M_{zw}) = |a_{n-1}|^2,$$

while if $m = n-1$ we get

$$\operatorname{Tr}(M_{zw}) = |b_{n-1}|^2 + 2\sum_{k=1}^{n-1} |b_{n-1}|^2 + |a_{n-1}|^2 = |a_{n-1}|^2 + (2n-1)|b_{n-1}|^2.$$

□

Now, the proof of Theorem 3.4.1.

Proof. We consider the matrix representing the Hermitian Killing form on the subspace of quotient $\mathbb{C}[z,w]/\tilde{I}$ spanned by $\{[1],[z],[w]\}$. From the choice of the basis follows that to compute this matrix it is sufficient to calculate $\operatorname{Tr}(M_w), \operatorname{Tr}(M_{w^2})$ and $\operatorname{Tr}(M_{zw})$. Then, by Lemma 3.4.2 and 3.4.3 the matrix is

$$\begin{bmatrix} \operatorname{Tr}(M_1) & \operatorname{Tr}(M_z) & \operatorname{Tr}(M_w) \\ \operatorname{Tr}(M_w) & \operatorname{Tr}(M_{zw}) & \operatorname{Tr}(M_{w^2}) \\ \operatorname{Tr}(M_z) & \operatorname{Tr}(M_{z^2}) & \operatorname{Tr}(M_{zw}) \end{bmatrix} = \begin{bmatrix} n^2 & -na_{n-1} & -n\bar{a}_{n-1} \\ -n\bar{a}_{n-1} & |a_{n-1}|^2 & n(\bar{a}_{n-1}^2 - 2\bar{a}_{n-2}) \\ -na_{n-1} & n(a_{n-1}^2 - 2a_{n-2}) & |a_{n-1}|^2 \end{bmatrix}$$

and its determinant is $-n^2|(n-1)a_{n-1}^2 - 2na_{n-2}|^2$. In particular, the matrix possesses a negative eigenvalue if the determinant does not vanish and a non positive eigenvalue otherwise. Thus, by Corollary 3.1.11, the equation admits at most $(n^2-1)-1 = n^2-2$ solutions in the first case and n^2-1 in the latter. □

Remark 3.4.4. Note that the expression $(n-1)a_{n-1}^2 - 2na_{n-2}$ equals

$$\frac{n^2 \operatorname{Tr}(M_{w^2}) - \operatorname{Tr}(M_w)^2}{n^2} = \frac{n^2 \operatorname{Tr}(M_w^2) - \operatorname{Tr}(M_w)^2}{n^2}.$$

For $n = 2$ the formula $(n-1)a_{n-1}^2 - 2na_{n-2}$ in Theorem 3.4.1 is the discriminant $a_{n-1}^2 - 4a_{n-2}$ of the second degree polynomial $z^2 + a_{n-1}z + a_{n-2}$. For general $n \geq 2$, the formula is a multiple of the discriminant of the second degree polynomial given by the $(n-2)$-th derivative. In fact

$$\begin{aligned}\frac{\partial^{n-2}\left(z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0\right)}{\partial z^{n-2}} &= \frac{n!}{2}z^2 + (n-1)!a_{n-1}z + (n-2)!a_{n-2} \\ &= (n-2)!\left(\frac{n(n-1)}{2}z^2 + (n-1)a_{n-1}z + a_{n-2}\right)\end{aligned}$$

and the discriminant of the polynomial in the parenthesis is

$$(n-1)^2a_{n-1}^2 - 2n(n-1)a_{n-2} = (n-1)((n-1)a_{n-2}^2 - 2na_{n-2}).$$

Another natural appearance of this formula is in the third term of the logarithmic discriminant of a n-dimensional vector bundle $\mathcal{E}$. From the equalities

$$\mathrm{ch}(\mathcal{E}) = n + c_1 + \frac{c_1^2 - 2c_2}{2} + \ldots = n\left(1 + \frac{c_1}{n} + \frac{c_1^2 - 2c_2}{2n} + \ldots\right)$$

and the Maclaurin series $\log(1+z) = z - z^2/2 + \ldots$ we get

$$\begin{aligned}\log(\mathrm{ch}(\mathcal{E})) &= \log n + \log\left(1 + \frac{c_1}{n} + \frac{c_1^2 - 2c_2}{2n} + \ldots\right) \\ &= \log n + \frac{c_1}{n} + \frac{c_1^2 - 2c_2}{2n} - \frac{c_1^2}{2n^2} + \ldots \\ &= \log n + \frac{c_1}{n} + \frac{(n-1)c_1^2 - 2nc_2}{2n^2} + \ldots\end{aligned}$$

Let now h be a harmonic polynomial with $\deg_z h = n$ and $\deg_{\bar{z}} h = m$ such that it posses the highest possible finite number of zeros. Moreover, let $\nu_+(n,m)$ and $\nu_-(n,m)$ be the numbers of zeros in which the harmonic polynomial h is orientation preserving and orientation reversing respectively. Without loss of generality assume $n \geq m$, clearly the two numbers above satisfy the equation involving the total number of zeros $\nu_+(n,m) + \nu_-(n,m)$ and from the results presented in Subsection 2.2, we know that $\nu_+(n,m) - \nu_-(n,m) = \deg' h = n$. Thus, this numbers are well-defined and independent from the harmonic polynomial h. We have seen the only four well known cases are $m = 0, 1, n-1, n$ for which in particular if $m = n-1, n$ there hold

$$\begin{cases}\nu_+(n,m) + \nu_-(n,m) = n^2 \\ \nu_+(n,m) - \nu_-(n,m) = n\end{cases} \quad \ldots \quad \begin{cases}\nu_+(n,m) = \frac{n(n+1)}{2} \\ \nu_-(n,m) = \frac{n(n-1)}{2}\end{cases}$$

if $m = 1$ there hold

$$\begin{cases}\nu_+(n,1) + \nu_-(n,1) = 3n-2 \\ \nu_+(n,1) - \nu_-(n,1) = n\end{cases} \quad \ldots \quad \begin{cases}\nu_+(n,1) = 2n-1 \\ \nu_-(n,1) = n-1\end{cases}$$

and if $m = 0$ there holds

$$\begin{cases} \nu_+(n,0) + \nu_-(n,0) = n \\ \nu_+(n,0) - \nu_-(n,0) = n \end{cases} \quad \cdots \quad \begin{cases} \nu_+(n,0) = n \\ \nu_-(n,0) = 0 \end{cases} .$$

Assuming the values of $\nu_+(n,m)$ and $\nu_-(n,m)$ to be polynomials functions of n and m the systems above yield enough equations to state the following.

Conjecture 3.4.5. *Using the notations of equations* (3.4.1)*, if $n \geq m$ the harmonic equation $h(z,\bar{z}) = p(z) + q(\bar{z}) = 0$ admits at most $n(2m+1) - m(m+1)$ solutions. In particular*

$$\begin{cases} \nu_+(n,m) = \frac{(2n-m)(m+1)}{2} \\ \nu_-(n,m) = \frac{m(2n-m-1)}{2} \end{cases}$$

One direct drawback of the formula proposed in the conjecture above is that it is not symmetric in the degrees n and m.

Even if the Hermitian Killing form helps simplify the problem, computations remains complicated. Related to the problem of Proposition 1.0.3 we state the following.

Conjecture 3.4.6. *Some computed examples suggest that the characteristic polynomial of the matrix representing the Hermitian Killing form with respect to the basis $\{[z^\alpha w^\beta] \mid \alpha, \beta < n\}$ of the quotient $\mathbb{C}[z,w]/\langle h, h^* \rangle$ for $h(z,w) \equiv h(z,\bar{z}) = z^n + a\bar{z}^n + b$ with $|a| \neq 1$ is*

$$q_{\mathbb{C}}(\lambda) = \frac{(\lambda - n^2)((|a|^2-1)\lambda \pm n^2|b - a\bar{b}|)^{n-1}((|a|^2-1)^2\lambda - n^2|b-a\bar{b}|^2)^{\frac{n(n-1)}{2}}((|a|^2-1)^2\lambda + n^2|b-a\bar{b}|^2)^{\frac{(n-1)(n-2)}{2}}}{(|a|^2-1)^{2n(n-1)}}.$$

Consider the conjecture above.

If $b \neq 0$, the difference of the number of positive and the number of negative roots of $q_{\mathbb{C}}(\lambda)$ is

$$1 + (n-1) + \frac{n(n-1)}{2} - \left((n-1) + \frac{(n-1)(n-2)}{2}\right) = n,$$

in accordance to Proposition 1.0.3.

If $b = 0$, the polynomial simplifies $q_{\mathbb{C}}(\lambda) = \lambda^{n^2-1}(\lambda - n^2)$ and the difference of the number of positive and the number of negative roots is $1 - 0 = 1$, in accordance to Proposition 1.0.3.

4 The Hermitian Distance degree

Let $n \in \mathbb{N}$ and let $\mathbb{V}$ be an n-dimensional complex vector space with complex conjugation, that is an antilinear (or conjugate linear) map $^{-}: \mathbb{V} \to \mathbb{V}$ such that $^{-2} = \mathrm{Id}$. We recall that an antilinear map f is additive and conjugate homogeneus in the sense that $f(\lambda \mathbf{z}) = \bar{\lambda} f(\mathbf{z})$ for any $\mathbf{z} \in \mathbb{V}$ and $\lambda \in \mathbb{C}$.

Consider a differentiable function

$$\begin{aligned} q\colon \mathbb{V} \times \mathbb{V} &\to \mathbb{C} \\ (\mathbf{z}, \mathbf{w}) &\mapsto q(\mathbf{z}, \mathbf{w}) \end{aligned}$$

such that $q(\mathbf{z}, \mathbf{w}) = \overline{q(\mathbf{w}, \mathbf{z})}$. We are interested in the critical points of the induced real-valued function

$$\begin{aligned} q_{\mathbf{u}}\colon X \subseteq \mathbb{V} &\to \mathbb{R} \\ \mathbf{z} &\longmapsto q(\mathbf{u} - \mathbf{z}, \mathbf{u} - \mathbf{z}) \end{aligned}$$

where $\mathbf{u} \in \mathbb{V}$ and $X = V(f_1, \ldots, f_s) \subseteq \mathbb{V}$ is an algebraic variety with $f_1, \ldots, f_s \in \mathbb{C}[\mathbf{z}]$.

The conjugation map extends its definition on complex vector bundles over X. In particular, this holds true for the tangent bundle TX of X. The tangent space of X in a point $\mathbf{z} \in X$ will be denoted $T_{\mathbf{z}}X$.

We prove a basic lemma that characterizes critical points of this induced function.

Lemma 4.0.1. *Let X be an algebraic variety and $\mathbf{u}$ a point, then a regular critical point $\mathbf{z} \in X$ of the function $q_{\mathbf{u}}$ satisfies $\nabla_{\mathbf{z}} q_{\mathbf{u}}(\mathbf{z}) \perp_{\mathbb{R}} T_{\mathbf{z}}X$.*

Proof. The variety X is locally a subset of $\mathbb{C}^n$ defined by a finite number $c \leq n$, of polynomials $f_1, \ldots f_c \in \mathbb{C}[\mathbf{z}]$ where c is the local codimension of X. From the complex version of the implicit function theorem, without loosing generality, we can reorder the z_k in such a way that for a regular point

$$\mathbf{z} = (\mathbf{z}_a, \mathbf{z}_b) = (z_1, \ldots, z_{n-c}, z_{n-c+1}, \ldots, z_n) \in X,$$

it holds $\det J_{\mathbf{z}_b}(f_1, \ldots, f_c) \neq 0$ and, by the implicit function theorem, there exists a holomorphic map

$$h\colon U_1 \subseteq \mathbb{C}^{n-c} \to U_2 \subseteq \mathbb{C}^c$$

where U_1, U_2 are open sets and $U_1 \times U_2$ is a neighbourhood of $\mathbf{z}$ such that the map

$$(\mathbf{z}_a, h(\mathbf{z}_a))\colon U_1 \subseteq \mathbb{C}^{n-c} \to U_1 \times U_2 \subseteq \mathbb{C}^n$$

is a parametrization of X around $\mathbf{z}$. Consider now the composition

$$q_{\mathbf{u}}(\mathbf{z}_a, h(\mathbf{z}_a))\colon U_1 \subseteq \mathbb{C}^{n-c} \to \mathbb{R}.$$

Since it takes real values, if we see this map on an open set of $\mathbb{R}^{2(n-c)}$ with coordinates

$$z_k = x_k + iy_k \qquad \text{for } k = 1, \dots, n-c,$$

it holds that its partial derivatives ∂_{x_k} and ∂_{y_k} for $k = 1, \dots, n-c$ vanish in a critical point. Again, since this composition takes real values, for any $k = 1, \dots, n-c$ the equation

$$\partial_{x_k} q_{\mathbf{u}}(\mathbf{x}_a + i\mathbf{y}_a, h(\mathbf{x}_a + i\mathbf{y}_a)) = \partial_{y_k} q_{\mathbf{u}}(\mathbf{x}_a + i\mathbf{y}_a, h(\mathbf{x}_a + i\mathbf{y}_a)) = 0$$

holds if and only if holds the equation

$$\partial_{z_k} q_{\mathbf{u}}(\mathbf{z}_a, h(\mathbf{z}_a)) = \frac{\partial_{x_k} - i\partial_{y_k}}{2} q_{\mathbf{u}}(\mathbf{x}_a + i\mathbf{y}_a, h(\mathbf{x}_a + i\mathbf{y}_a)) = 0.$$

Thus, using the complex chain rule in a critical point we obtain

$$\mathbf{0} = \nabla_{\mathbf{z}_a} q_{\mathbf{u}}(\mathbf{z}_a, h(\mathbf{z}_a)) = \begin{bmatrix} \nabla_{\mathbf{z}_a} q_{\mathbf{u}} & \nabla_{\mathbf{z}_b} q_{\mathbf{u}} \end{bmatrix} \begin{bmatrix} I_{n-c} \\ J_{\mathbf{z}_a}(h) \end{bmatrix} = \nabla_{\mathbf{z}_a} q_{\mathbf{u}} + \nabla_{\mathbf{z}_b} q_{\mathbf{u}} \cdot J_{\mathbf{z}_a}(h) \tag{4.0.1}$$

and

$$\begin{aligned} \mathbf{0} &= \nabla_{\mathbf{z}_a} \left(f_1(\mathbf{z}_a, h(\mathbf{z}_a)), \dots, f_c(\mathbf{z}_a, h(\mathbf{z}_a)) \right) = \begin{bmatrix} J_{\mathbf{z}_a}(f_1, \dots, f_c) & J_{\mathbf{z}_b}(f_1, \dots, f_c) \end{bmatrix} \begin{bmatrix} I_{n-c} \\ J_{\mathbf{z}_a}(h) \end{bmatrix} \\ &= J_{\mathbf{z}_a}(f_1, \dots, f_c) + J_{\mathbf{z}_b}(f_1, \dots, f_c) \cdot J_{\mathbf{z}_a}(h) \end{aligned}$$

which implies $J_{\mathbf{z}_a}(h) = -J_{\mathbf{z}_b}(f_1, \dots, f_c)^{-1} \cdot J_{\mathbf{z}_a}(f_1, \dots, f_c)$. Substituting this last identity in (4.0.1) we obtain

$$\nabla_{\mathbf{z}_a} q_{\mathbf{u}} = \nabla_{\mathbf{z}_b} q_{\mathbf{u}} \cdot J_{\mathbf{z}_b}(f_1, \dots, f_c)^{-1} \cdot J_{\mathbf{z}_a}(f_1, \dots, f_c).$$

Thus, we can write

$$\begin{aligned} \nabla q_{\mathbf{u}} = \begin{bmatrix} \nabla_{\mathbf{z}_a} q_{\mathbf{u}} & \nabla_{\mathbf{z}_b} q_{\mathbf{u}} \end{bmatrix} &= \nabla_{\mathbf{z}_b} q_{\mathbf{u}} \cdot J_{\mathbf{z}_b}(f_1, \dots, f_c)^{-1} \cdot \begin{bmatrix} J_{\mathbf{z}_a}(f_1, \dots, f_c) & J_{\mathbf{z}_b}(f_1, \dots, f_c) \end{bmatrix} \\ &= \nabla_{\mathbf{z}_b} q_{\mathbf{u}} \cdot J_{\mathbf{z}_b}(f_1, \dots, f_c)^{-1} \cdot J(f_1, \dots, f_c). \end{aligned}$$

In the end, since $\mathrm{Row}(J(f_1, \dots, f_c)) = T_{\mathbf{z}}X^{\perp_{\mathbb{R}}}$, regular critical points satisfy $\nabla_{\mathbf{z}} q_{\mathbf{u}}(\mathbf{z}) \perp_{\mathbb{R}} T_{\mathbf{z}}X$.

Symmetrically, we can argue that for any $k = 1, \dots, n-c$ the derivatives ∂_{x_k} and ∂_{y_k} vanish if and only if $\partial_{\bar{z}_k}$ vanishes, which results in a conjugate operation to get the equivalent condition $\nabla_{\bar{\mathbf{z}}} q_{\mathbf{u}}(\mathbf{z}) \perp_{\mathbb{R}} \overline{T_{\mathbf{z}}X}$. □

From now we consider q to be a Hermitian form. Thus, if $\mathbf{z}$ is a regular critical point, another way to express the property $\nabla_{\mathbf{z}} q_{\mathbf{u}}(\mathbf{z}) \in \mathrm{Row}(J(f))$, or equivalently $\overline{\nabla_{\mathbf{z}} q_{\mathbf{u}}(\mathbf{z})} \perp_{\mathbb{C}} T_{\mathbf{z}}X$, is by saying $\mathbf{u} - \mathbf{z} \perp_q T_{\mathbf{z}}X$ where $\perp_q$ indicates the perpendicularity condition $q(\ ,\ ) = 0$.

Moreover, for the sake of simplicity we will refer to regular critical points just as critical points.

4.1 Hermitian critical set

Let $X \subseteq \mathbb{V}$ be an algebraic variety and let $G \subseteq \mathrm{GL}(\mathbb{V})$ be a Lie subgroup that leaves X invariant. We denote with $\mathfrak{g} = T_e G$ the complex Lie algebra of G where e is the identity element. The tangent space to the orbit $G \cdot \mathbf{u}$ at $\mathbf{u} \in \mathbb{V}$ is $\mathbf{u} + \mathfrak{g} \cdot \mathbf{u}$. Denoting by $G_{\mathbf{u}} = \{g \in G \mid g \cdot \mathbf{u} = \mathbf{u}\}$ the isotropy group of $\mathbf{u}$, we have $\dim \mathfrak{g} \cdot \mathbf{u} = \dim \mathfrak{g} - \dim G_{\mathbf{u}}$.

In [Ott22], assuming that the elements of G preserve a symmetric bilinear form $\hat{q}$ on $\mathbb{V}$ for any $\mathbf{u} \in \mathbb{V}$ is defined the *critical space* that is the vector subspace $\{\mathbf{v} \in \mathbb{V} \mid \hat{q}(\mathbf{v}, g \cdot \mathbf{u}) = 0 \quad \forall g \in \mathfrak{g}\}$. The key property of this space is that it contains the critical points of the distance function from X induced by $\hat{q}$. We try to adapt similar arguments in the Hermitian case.

We recall that with q we indicate a Hermitian form on $\mathbb{V}$ and let $\mathrm{U}(\mathbb{V}) \subseteq \mathrm{GL}(\mathbb{V})$ be the group of linear transformations of $\mathbb{V}$ which preserve q.

Lemma 4.1.1. *Let $G \subseteq \mathrm{U}(\mathbb{V})$, if $g \in \mathfrak{g}$ then $q(g \cdot \mathbf{u}, \mathbf{v}) = -q(\mathbf{u}, g \cdot \mathbf{v})$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, in particular $q(g \cdot \mathbf{u}, \mathbf{u})^{\Re} = 0$.*

Proof. Let $\psi(t) \colon [-1, 1] \to G$ be a path such that $\psi(0) = e$ and $\psi'(0) = g$. Taking the derivative at $t = 0$ of the constant function $q(\psi(t) \cdot \mathbf{u}, \psi(t) \cdot \mathbf{v}) = q(\mathbf{u}, \mathbf{v})$ the first assertion follows.

The second assertion follows from the chain of equalities

$$q(g \cdot \mathbf{u}, \mathbf{u}) = -q(\mathbf{u}, g \cdot \mathbf{u}) = -\overline{q(g \cdot \mathbf{u}, \mathbf{u})}.$$

□

If we consider a symmetric bilinear form $\hat{q}$ and a group G that preserves it, the same proof of the last lemma shows $\hat{q}(g \cdot \mathbf{u}, \mathbf{u}) = 0$. This is the main difference between the symmetric and the Hermitian cases resulting in a more complicated definition in the latter since the set we focus on is no more a vector subspace.

Definition 4.1.2. Let $\mathbf{u} \in \mathbb{V}$ we define the *Hermitian critical set* of $\mathbf{u}$ as the subset

$$H_{\mathbf{u}} \coloneqq \{\mathbf{v} \in \mathbb{V} \mid q(\mathbf{v}, g \cdot (\mathbf{v} - \mathbf{u})) = 0 \quad \forall g \in \mathfrak{g}\}.$$

Remark 4.1.3. We can notice that from Lemma 4.1.1 for $\mathbf{v} \in H_{\mathbf{u}}$ it holds

$$0 = q(\mathbf{v}, g \cdot (\mathbf{v} - \mathbf{u}))^{\Re} = q(\mathbf{v}, g \cdot \mathbf{v})^{\Re} - q(\mathbf{v}, g \cdot \mathbf{u})^{\Re} = -q(\mathbf{v}, g \cdot \mathbf{u})^{\Re}.$$

In particular, the condition $q(\mathbf{v}, g \cdot \mathbf{u}) = \rho e^{\pm \frac{\pi}{2} i}$ for some $0 \leq \rho \in \mathbb{R}$, or in other words the Kasner's pseudo-angle of $\mathbf{v}$ and $g \cdot \mathbf{u}$ is equal to $\pm\pi/2$, defines a linear subspace that contains the Hermitian critical set.

Theorem 4.1.4. *Let $X \subseteq \mathbb{V}$ be an algebraic variety and let $G \subseteq \mathrm{U}(\mathbb{V})$ be a subgroup that leaves X invariant, then the critical points of $q_{\mathbf{u}}$ lie in $H_{\mathbf{u}}$.*

Proof. Let $\mathbf{z} \in X$ be a critical point, from the fact that X is G invariant follows $\mathfrak{g} \cdot \mathbf{z} \subseteq T_{\mathbf{z}}X$. Thus the claim follows from the equalities

$$0 = q(g \cdot \mathbf{z}, \mathbf{z} - \mathbf{u}) = -q(\mathbf{z}, g \cdot (\mathbf{z} - \mathbf{u}))$$

where on the left we used the fact that $\mathbf{z}$ is critical and on the right Lemma 4.1.1. □

Theorem 4.1.5. *Let $X \subseteq \mathbb{V}$ be an algebraic variety and let $G \subseteq \mathrm{U}(\mathbb{V})$ be a subgroup that leaves X invariant. Let $\mathbf{z} \in H_{\mathbf{u}} \cap X$, then*

i) *if the orbit $G \cdot \mathbf{z}$ is dense in X then $\mathbf{z}$ is a critical point of $q_{\mathbf{u}}$,*

ii) *if X is an affine cone and the orbit $G \cdot [\mathbf{z}]$ is dense in $X \subseteq \mathbb{P}\mathbb{V}$ then there exists a unique $\lambda \in \mathbb{C}$ such that $\lambda\mathbf{z}$ is a critical point of $q_{\mathbf{u}}$.*

Proof. By assumption we have the equality $\mathfrak{g} \cdot \mathbf{z} = T_{\mathbf{z}}X$ and since the steps of the proof of Theorem 4.1.4 are invertible this proves i).

The assumption of ii) implies that $\mathfrak{g} \cdot \mathbf{z} + \mathrm{span}\{\mathbf{z}\} = T_{\mathbf{z}}X$. Since $q(\mathbf{z}, \mathbf{z})$ is non zero for $\lambda = q(\mathbf{u}, \mathbf{z})/q(\mathbf{z}, \mathbf{z})$ it holds $q(\lambda\mathbf{z}, \lambda\mathbf{z} - \mathbf{u}) = 0$, so that we get orthogonality on $\mathrm{span}\{\mathbf{z}\} \subseteq T_{\mathbf{z}}X$. The orthogonality to $\mathfrak{g} \cdot \mathbf{z}$ follows by replacing $\mathbf{z}$ with $\lambda\mathbf{z}$ into the same argument of i). □

4.2 HDdeg

For the sake of simplicity, we assume to have set a basis on the complex vector space $\mathbb{V}$ and that q is the canonical Hermitian inner product on $\mathbb{V}$, so that in particular $\nabla_{\mathbf{z}} q_{\mathbf{u}}(\mathbf{z}) = \bar{\mathbf{z}} - \bar{\mathbf{u}}$ and by Lemma 4.0.1 a critical point $\mathbf{z}$ must satisfy $\mathbf{u} - \mathbf{z} \perp_{\mathbb{C}} T_{\mathbf{z}}X$. We will mostly consider this situation from now on.

We face the problem to study the critical points of the Hermitian distance from an algebraic variety. The same problem was studied for the Euclidean distance in several papers that will be cited throughout the dissertation, for a foundational work see [Dra+16].

Let X be an algebraic variety, in what follows we will denote $X_\circ := X_{\mathrm{reg}} \times \overline{X}_{\mathrm{reg}} \subseteq \mathbb{V}^2$.

Fix a radical ideal $I_X := \langle f_1, \ldots, f_s \rangle \in \mathbb{C}[\mathbf{z}]$, for our purpose we assume that $X = V(I_X) \subseteq \mathbb{V}$ is irreducible and that I_X is a prime ideal.

Write $J(f)$ for the $s \times n$ Jacobian matrix of $(f_1, \ldots, f_s)$. The singular locus of X is defined by the ideal

$$I_{X_{\mathrm{sing}}} := I_X + \langle c\text{-minors of } J(f) \rangle$$

where c is the codimension of X. We now augment the matrix $J(f)$ with the row vector $\bar{\mathbf{u}} - \bar{\mathbf{z}}$ to get the $(s+1) \times n$-matrix

$$\begin{bmatrix} \bar{\mathbf{u}} - \bar{\mathbf{z}} \\ J(f) \end{bmatrix}.$$

This matrix has rank $\leq c$ on the critical points of $q_{\mathbf{u}}$ on X since it has to hold $\nabla_{\mathbf{z}} q_{\mathbf{u}}(\mathbf{z}) = \bar{\mathbf{z}} - \bar{\mathbf{u}} \in \mathrm{Row}(J(f))$.

We now introduce two new collections $\mathbf{w}$ and $\mathbf{v}$ of variables $\{w_1, \dots, w_n\}$ and $\{v_1, \dots, v_n\}$ respectively, where w_k and v_k represent the variables $\bar{z}_k$ and $\bar{u}_k$ respectively for $k = 1, \dots, n$. Moreover, with a slight abuse of notation, we use the same symbol $*$ to extend the map of Section 3 to

$$\begin{aligned} *\colon \mathbb{C}[\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}] &\to \mathbb{C}[\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}] \\ g \quad &\mapsto \quad *(g) = g^* \end{aligned}$$

such that $z_k^* = w_k$, $w_k^* = z_k$, $u_k^* = v_k$, $v_k^* = u_k$ for $k = 1, \dots, n$ and $a^* = \bar{a}$ for any $a \in \mathbb{C}$.

Thus, using the ideal

$$I'_X := \left\langle (c+1)\text{-minors of } \begin{bmatrix} \mathbf{v} - \mathbf{w} \\ J_{\mathbf{z}}(f) \end{bmatrix} \right\rangle,$$

we define the *Hermitian critical ideal* of $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ as the following saturation

$$\left(I_X + (I_X)^* + I'_X + (I'_X)^*\right) : \left(I_{X_{\mathrm{sing}}} \cdot (I_{X_{\mathrm{sing}}})^*\right)^\infty \subseteq \mathbb{C}[\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}]. \tag{4.2.1}$$

Note that the Hermitian critical ideal is invariant under the action of the map $*$, thus we can apply the results of Subsection 3.1. In particular, the Hermitian critical ideal is invariant also if we set points $\mathbf{v} = \bar{\mathbf{u}}$ and consider it as an ideal in the polynomial ring $\mathbb{C}[\mathbf{z}, \mathbf{w}]$.

Lemma 4.2.1. *Let $X \subseteq \mathbb{V}$ be an algebraic variety then the variety of the Hermitian critical ideal of a generic point $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ is finite. When $\mathbf{v} = \bar{\mathbf{u}}$ the first component of solutions with $\mathbf{w} = \bar{\mathbf{z}}$ consists precisely of the critical points of the function $q_{\mathbf{u}}$ on X_{reg}.*

Proof. For fixed $(\mathbf{z}, \mathbf{w}) \in X_\circ$, the Jacobian has rank c, so the $(c+1) \times (c+1)$-minors of $\begin{bmatrix} \mathbf{v} - \mathbf{w} \\ J_{\mathbf{z}}(f) \end{bmatrix}$ define an affine-linear subspace of dimension c in $\mathbf{v}$. Similarly for $\mathbf{u}$. Hence the variety of quadruples $(\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}) \in X_\circ \times \mathbb{V}^2$ that are zeros of the Hermitian critical ideal is irreducible of dimension $2n$. The fiber of the projection $\pi_{\mathbb{V}^2}$ over a generic point $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ must hence be finite.

The second assertion follows from the definition of critical points. □

The number of points in the zero locus of the Hermitian critical ideal is constant on an open dense subset of $\mathbb{V}^2$. This number takes account of points $(\mathbf{z}, \mathbf{w}) \in X_\circ$. The number of solutions with $\mathbf{w} = \bar{\mathbf{z}}$ can be lower and if $\mathbf{v} = \bar{\mathbf{u}}$ it is equal to the number of critical points of $q_{\mathbf{u}}$. This number is only locally constant on an open dense subset and so depends on the point $\mathbf{u} \in \mathbb{V}$. With a slight abuse of notation we also call these *critical points*, they coincide with the conjugated singles defined in Subsection 3.1. Since the definition is additive over the components of a variety, the assumption of irreducibility of X is not restrictive.

Definition 4.2.2. Let X be an algebraic variety, the *virtual Hermitian Distance degree* of X is the constant number of solutions, for $(\mathbf{u}, \mathbf{v})$ in a dense subset of $\mathbb{V}^2$, of the Hermitian critical ideal (4.2.1) and it is denoted $\mathrm{vHDdeg}(X)$. The *Hermitian Distance degree* of X is the subset of $\mathbb{N}$ consisting on the numbers of critical points of $q_{\mathbf{u}}$ which do not vary when $\mathbf{u}$ is taken in some open subset of $\mathbb{V}$ with respect to the topology induced by q, and it is denoted $\mathrm{HDdeg}(X)$.

By definition it holds the inequality $\max \mathrm{HDdeg}(X) \leq \mathrm{vHDdeg}(X)$. It will be common to get the strict inequality for an algebraic variety X, we will see this in the case of conics in Subsection 4.5.

In [Dra+16] for the Euclidean distance problem it is defined $\mathrm{EDdegree}(X)$ the *Euclidean Distance degree* of a real algebraic variety X.

The Euclidean case is, in a certain sense, simpler, in fact the perpendicularity condition for a positive symmetric bilinear form $\hat{q}$ simplifies $\mathbf{u} - \mathbf{z} \in (T_{\mathbf{z}}X)^{\perp_{\hat{q}}}$ and the conjugate operation, and thus the introduction of the variables $\mathbf{w}$ and $\mathbf{v}$ is no longer necessary to define the desired ideal. In particular the *critical ideal* of $\mathbf{u} \in \mathbb{V}$ do not need the ideals obtained by the map $*$ and simplifies to

$$\left(I_X + \left\langle (c+1)\text{-minors of } \begin{bmatrix} \mathbf{u} - \mathbf{z} \\ J(f) \end{bmatrix} \right\rangle\right) : \left(I_{X_{\mathrm{sing}}}\right)^{\infty} \subseteq \mathbb{C}[\mathbf{z}, \mathbf{u}]. \tag{4.2.2}$$

The $\mathrm{EDdegree}(X)$ is the constant number, on a dense subset, of solution of the critical ideal (4.2.2). However, to make a geometric sense of critical points for the Euclidean distance one has to restrict the attention to real points $\mathbf{u} \in \mathbb{V}$ and real solutions $\mathbf{z} \in X$, thus facing a similar problem to ours. To make this similarity clearer we introduce the following notion.

Definition 4.2.3. Let $X \subseteq \mathbb{V}$ be a real algebraic variety and $\hat{q}$ a positive symmetric bilinear form on $\mathbb{V}$, the *real Euclidean Distance degree* of X is the subset of $\mathbb{N}$ consisting on the numbers of real critical points of $\hat{q}_{\mathbf{u}}$ which do not vary when $\mathbf{u}$ is taken in some open real subset of $\mathbb{V}$ with respect to the topology induced by $\hat{q}$, and it is denoted $\mathrm{rEDdegree}(X)$.

Similarly as above, it holds $\max \mathrm{rEDdegree}(X) \leq \mathrm{EDdegree}(X)$, however in this context it is more common that the highest possible number of critical points is equal to the Euclidean Distance degree, or in other words that it holds the equality $\mathrm{EDdegree} = \max \mathrm{rEDdegree}$. As an example we recall the results on conics for which there holds the equalities

$$\begin{aligned}
&\mathrm{EDdegree}(X) = 3 \quad \text{and} \quad \mathrm{rEDdegree}(X) = \{1,3\} \quad \text{if } X \text{ is a parabola,}\\
&\mathrm{EDdegree}(X) = 2 \quad \text{and} \quad \mathrm{rEDdegree}(X) = \{2\} \quad \text{if } X \text{ is a circle,}\\
&\mathrm{EDdegree}(X) = 4 \quad \text{and} \quad \mathrm{rEDdegree}(X) = \{2,4\} \quad \text{if } X \text{ is an ellipse or a hyperbola.}
\end{aligned}$$

Some inequalities involving both the EDdegree and HDdeg of a real variety are presented in the following result.

Proposition 4.2.4. *Let X be a real algebraic variety, then*

$$\mathrm{vHDdeg}(X) \geq \mathrm{EDdegree}(X) \geq \min \mathrm{HDdeg}(X)$$

and

$$\max \mathrm{HDdeg}(X) \geq \max \mathrm{rEDdegree}(X).$$

In particular, when considering real points, real critical points of the Euclidean distance are real critical points of the Hermitian distance.

Proof. Setting $\mathbf{v} = \mathbf{u}$ the left hand inequality of the chain follows from the fact that if $\mathbf{z} \in X$ is in the zero locus of the critical ideal (4.2.2) of $\mathbf{u}$, since it holds $\overline{X} = X$ then $(\mathbf{z}, \mathbf{z}) \in X \times \overline{X}$ is in the zero locus of the Hermitian critical ideal (4.2.1) of $(\mathbf{u}, \mathbf{u})$.

For the right hand inequality of the chain note that the minimum on the right is bounded from above by the minimum on the subset in which $\mathbf{v} = \mathbf{u}$ is real. Now, if $(\mathbf{z}, \bar{\mathbf{z}}) \in X \times \overline{X}$ is in the zero locus of the Hermitian critical ideal (4.2.1) of $(\mathbf{u}, \mathbf{u})$, i.e. is a critical point of the Hermitian distance, since all the polynomials are real and in particular $X = \overline{X}$, then $\mathbf{z}, \bar{\mathbf{z}} \in X$ are in the zero locus of the critical ideal (4.2.2) of $\mathbf{u}$.

The remaining inequality and the last part follow setting $\mathbf{v} = \mathbf{u}$ real. Now, if $\mathbf{z} \in X$ real is in the zero locus of the critical ideal (4.2.2), i.e. is a real critical point of the Euclidean distance, then $(\mathbf{z}, \bar{\mathbf{z}}) = (\mathbf{z}, \mathbf{z}) \in X \times \overline{X}$ is in the zero locus of the Hermitian critical ideal (4.2.1), i.e. it is a real critical point of the Hermitian distance. □

Again, in general all the inequalities of this last proposition could be strict. We will see in Theorem 4.6.1 that for any non trivial algebraic variety X it holds $\min \mathrm{HDdeg}(X) > 0$. For many examples we will treat, this can also be seen by using the topological degree of Subsection 2.2 and Proposition 2.2.4.

We now briefly discuss the case of parametrized varieties.

Let X be parametrized by $\psi(\mathbf{z})\colon U \subseteq \mathbb{C}^m \to X \subseteq \mathbb{C}^n$ where U is an open subset. Thus, we can write

$$q_{\mathbf{u}}(\mathbf{z}) = \sum_{k=1}^{n} |\psi_k(\mathbf{z}) - u_k|^2$$

and, if we define the map

$$D_{\mathbf{u},\mathbf{v}}(\mathbf{z}, \mathbf{w}) := \sum_{k=1}^{n} (\psi_k(\mathbf{z}) - u_k)(\bar{\psi}_k(\mathbf{w}) - v_k),$$

the critical points must satisfy $\nabla_{\mathbf{z},\mathbf{w}} D_{\mathbf{u},\mathbf{v}}(\mathbf{z}, \mathbf{w}) = \mathbf{0}$. The critical locus of $D_{\mathbf{u},\mathbf{v}}(\mathbf{z}, \mathbf{w})$ in the subset $U \times \overline{U} \subseteq \mathbb{C}^{2m}$ is the relative set of points at which the Jacobians of $\psi(\mathbf{z})$ and $\bar{\psi}(\mathbf{w})$ have maximal rank. The closure of the image of this critical locus under $\psi \times \bar{\psi}$ coincides with the variety given by the zero locus of the Hermitian critical ideal (4.2.1) of $(\mathbf{u}, \mathbf{v})$. If the parametrization ψ is generically d-to-one then the critical locus in $U \subseteq \mathbb{C}^{2m}$ is finite and its cardinality equals $d^2 \cdot \mathrm{vHDdeg}(X)$.

In the next paragraph we want to give a different intuition of our problem, by rephrasing it in terms of a Euclidean distance problem.

Consider the decomposition $\mathbb{V} = \mathbb{V}_{\mathbb{R}} \oplus i\mathbb{V}_{\mathbb{R}}$ where $\mathbb{V}_{\mathbb{R}} \subseteq \mathbb{V}$ is the subspace invariant under the conjugation map, that is the realification of $\mathbb{V}$. Now consider coordinates $\mathbf{u} = \mathbf{u}^{\Re} + i\mathbf{u}^{\Im}$ and $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ to write

$$q_{\mathbf{u}}(\mathbf{z}) = q_{\mathbf{u}^{\Re}+i\mathbf{u}^{\Im}}(\mathbf{x} + i\mathbf{y}) = q_{\mathbf{u}^{\Re}}(\mathbf{x}) + q_{\mathbf{u}^{\Im}}(\mathbf{y}).$$

Split the polynomials $f_1, \dots, f_s$ defining X into their real and imaginary parts $f_k = f_k^{\Re} + i f_k^{\Im}$ such that $f_k^{\Re}, f_k^{\Im} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ for $k = 1, \dots, s$ and denote the real algebraic variety

$$\tilde{X} := \left\{ \left(\frac{\mathbf{z} + \mathbf{w}}{2}, \frac{\mathbf{z} - \mathbf{w}}{2i} \right) \in \mathbb{V}_{\mathbb{R}}^2 \otimes \mathbb{C} \mid \mathbf{z} \in X, \mathbf{w} \in \overline{X} \right\} = V(f_1^{\Re}, f_1^{\Im}, \dots, f_s^{\Re}, f_s^{\Im}) \subseteq \mathbb{V}_{\mathbb{R}}^2 \otimes \mathbb{C}.$$

The following proposition links our problem to the Euclidean distance problem.

Proposition 4.2.5. *Let $X \subseteq \mathbb{V}$ be an algebraic variety, then for any $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ the map*

$$(\mathbf{z}, \mathbf{w}) \mapsto \left(\frac{\mathbf{z} + \mathbf{w}}{2}, \frac{\mathbf{z} - \mathbf{w}}{2i} \right) \qquad \textit{with inverse} \qquad (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x} + i\mathbf{y}, \mathbf{x} - i\mathbf{y}),$$

from $\mathbb{V}^2$ to the complexification $\mathbb{V}_{\mathbb{R}}^2 \otimes \mathbb{C}$, sends the points in the zero set of the Hermitian critical ideal of X of $(\mathbf{u}, \mathbf{v})$ to the points in the zero set of the critical ideal of the EDdegree *of $\tilde{X}$ of $\left(\frac{\mathbf{u}+\mathbf{v}}{2}, \frac{\mathbf{u}-\mathbf{v}}{2i}\right)$, in particular*

$$\mathrm{vHDdeg}(X) = \mathrm{EDdegree}(\tilde{X}).$$

Moreover, if $\mathbf{v} = \bar{\mathbf{u}}$ this map sends critical points into critical points, in particular

$$\mathrm{HDdeg}(X) = \mathrm{rEDdegree}(\tilde{X}).$$

Proof. The first part follows from the linearity of the Wirtinger derivatives

$$\partial_{z_k} = \frac{\partial_{x_k} - i\partial_{y_k}}{2} \qquad \text{and} \qquad \partial_{\bar{z}_k} = \frac{\partial_{x_k} + i\partial_{y_k}}{2}$$

and the validity of the Cauchy-Riemann equations for holomorphic function in several variables, by unfolding the definitions of the critical ideals. In fact, the variety X is the zero locus of the s polynomials $f = (f_1, \dots, f_s)$ and of codimension c iff the variety $\tilde{X}$ is the zero locus of the $2s$ polynomials $(f^{\Re}, f^{\Im})$ and of codimension $2c$. In particular, the $(s+1) \times n$ matrices

$$\begin{bmatrix} \mathbf{v} - \mathbf{w} \\ J_{\mathbf{z}}(f) \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \mathbf{u} - \mathbf{z} \\ J_{\mathbf{w}}(f^*) \end{bmatrix}$$

have rank at most c if and only if the $(2s+1) \times 2n$ matrix

$$\begin{bmatrix} \mathbf{u}^{\Re} - \mathbf{x} & \mathbf{u}^{\Im} - \mathbf{y} \\ J_{\mathbf{x}}(f^{\Re}) & J_{\mathbf{y}}(f^{\Re}) \\ J_{\mathbf{x}}(f^{\Im}) & J_{\mathbf{y}}(f^{\Im}) \end{bmatrix}$$

has rank at most $2c$.

The second part follows by the definition of the map in the assertion. □

A similar argument applies when X is a parametrized variety.

Note that the same map of the last proposition yields a biholomoprhism between $X \times \overline{X}$ and $\tilde{X} \subseteq \mathbb{V}_{\mathbb{R}}^2 \otimes \mathbb{C}$.

The most important thing is that we can apply all the results known about the Euclidean distance problem to the variety $\tilde{X}$. However, we already pointed out that using this approach could often be tedious since dividing the polynomials $f_1, \dots, f_s$ into their real and imaginary parts requires a non trivial computational effort and the geometry of $\tilde{X}$ could be less clear. Moreover, the polyonomials $f_1^{\Re}, f_1^{\Im}, \dots, f_s^{\Re}, f_s^{\Im}$ are surely non generic since they satisfy the Cauchy-Riemann equations.

We recall an important result from [Dra+16].

Proposition 4.2.6 (Proposition 2.6, [Dra+16]). *Let $X \subseteq \mathbb{V}$ be a variety of codimension c that is cut out by real polynomials $f_1, f_2, \dots, f_s$ of degrees $d_1 \geq d_2 \geq \dots \geq d_s$. Then*

$$\text{EDdegree}(X) \leq d_1 \cdots d_c \sum_{k_1+k_2+\dots+k_c \leq n-c} (d_1-1)^{k_1} \cdots (d_c-1)^{k_c}$$

and equality holds for generic varieties.

As a consequence of the last result, if $X \subseteq \mathbb{V}$ is a variety of codimension c that is cut out by polynomials $f_1, f_2, \dots, f_s$ of degrees $d_1 \geq d_2 \geq \dots \geq d_s$, then

$$\begin{aligned} \text{vHDdeg}(X) = \text{EDdegree}(\tilde{X}) &\leq (d_1 \cdots d_c)^2 \sum_{k_1+j_1+\dots+k_c+j_c \leq 2n-2c} (d_1-1)^{k_1+j_1} \cdots (d_c-1)^{k_c+j_c} \\ &= (d_1 \cdots d_c)^2 \sum_{\ell_1+\dots+\ell_c \leq 2n-2c} (\ell_1+1) \cdots (\ell_c+1)(d_1-1)^{\ell_1} \cdots (d_c-1)^{\ell_c}. \end{aligned} \tag{4.2.3}$$

Even if we have obtained an upper bound for the value of vHDdeg(X), this bound is not sharp since as we discussed above the variety $\tilde{X}$ is far from being generic. We will sharpen the bound for the value of vHDdeg(X) of a generic hypersurface for several cases in Subsection 4.3. We will see that the true value is much lower.

In the Euclidean distance problem, since all equations are real the conjugated solutions come in pairs. Since the Hermitian distance problem is in particular a Euclidean distance problem as discussed above, something similar should happen and this is described in the following proposition.

Proposition 4.2.7. *Let X be an algebraic variety, if* vHDdeg(X) *is even (odd) then so are all the numbers in* HDdeg(X).

Proof. When $\mathbf{v} = \bar{\mathbf{u}}$ the Hermitian critical ideal is invariant under the action of the map $*$ and the assertion follows from Corollary 3.1.4. ☐

From this last result it follows that, when $\mathbf{v} = \bar{\mathbf{u}}$, two cases can occur for any solution of the Hermitian critical ideal, either it is a conjugated single or, equivalently, a critical point or belongs to an associated pair, see Definition 3.1.3.

Remark 4.2.8. A similar argument, when X is defined via real polynomials, shows that when $\mathbf{v} = \mathbf{u}$ then for any solution $(\mathbf{z}, \mathbf{w})$ of the Hermitian critical ideal we obtain also $(\mathbf{w}, \mathbf{z})$ as another solution. In particular, when $\mathbf{u}$ is real then for any solution $(\mathbf{z}, \mathbf{w})$ of the Hermitian critical ideal we obtain also the solutions $(\mathbf{w}, \mathbf{z}), (\bar{\mathbf{z}}, \bar{\mathbf{w}}), (\bar{\mathbf{w}}, \bar{\mathbf{z}})$. Thus, three cases can occur for any solution, either it is a real critical point or a complex critical point, which yields another complex critical point, or belongs to one of two associated pairs.

We will see that it is easy to construct examples for which the HDdeg does not contain all the numbers with the same parity between $\min \mathrm{HDdeg}(X)$ and $\max \mathrm{HDdeg}(X)$, see Lemma 4.2.12 below.

A natural first simple example is the case of X an affine subspace. As expected, in this setting we get $\mathrm{vHDdeg}(X) = 1$ and $\mathrm{HDdeg}(X) = \{1\}$.

Consider here $X = V(az_1 + bz_2 + c) \subseteq \mathbb{C}^2$ where $a, b, c \in \mathbb{C}$, it is not hard to see that the Hermitian critical ideal of the point $(\mathbf{u}, \mathbf{v}) \in \mathbb{C}^4$ is generated by the polynomials of the equations

$$\begin{cases} az_1 + bz_2 + c = 0 \\ \bar{a}w_1 + \bar{b}w_2 + \bar{c} = 0 \\ \bar{b}(z_1 - u_1) - \bar{a}(z_2 - u_2) = 0 \\ b(w_1 - v_1) - a(w_2 - v_2) = 0 \end{cases}$$

where the first two equations come from the definition of X and the second two from the perpendicularity condition $\mathbf{u} - \mathbf{z} \perp_{\mathbb{C}} T_{\mathbf{z}}X$. In particular this system admits only a solution for any choice of the parameters and if $\mathbf{v} = \bar{\mathbf{u}}$ then the solution satisfy $\mathbf{z} = \bar{\mathbf{z}}$.

A complete proof for affine subspace with much more details is given in Proposition 6.0.8.

We consider the case of two ambient spaces.

Proposition 4.2.9. *Let $\mathbb{V}, \mathbb{W}$ be complex vector spaces and X be an algebraic variety such that $X \subseteq \mathbb{V} \subseteq \mathbb{W}$. Then the solutions of the Hermitian critical ideal of $(\mathbf{u}, \mathbf{v}) \in \mathbb{W}^2$ are the solutions of the Hermitian critical ideal of $(\pi_{\mathbb{V}}(\mathbf{u}), \pi_{\mathbb{V}}(\mathbf{v})) \in \mathbb{V}^2$. In particular the critical points of $\mathbf{u} \in \mathbb{W}$ from X are the critical points of $\pi_{\mathbb{V}}(\mathbf{u}) \in \mathbb{V}$ from X.*

Proof. As before let n be the dimension of $\mathbb{V}$ and c be the codimension of X inside $\mathbb{V}$. Let m be the dimensions of $\mathbb{W}$ so that the codimension of X inside it is $c + n - m$. After a change of coordinates we can assume $X = V(f_1(\mathbf{z}), \ldots, f_s(\mathbf{z}), z_{n+1}, \ldots, z_m) \subseteq \mathbb{W}$. Let $\tilde{\mathbf{z}}$ be the collection of variables $\{z_{n+1}, \ldots, z_m\}$ and similarly for the parameters $\tilde{\mathbf{u}}$. In particular, before saturation the Hermitian critical ideal of $((\mathbf{u}, \tilde{\mathbf{u}}), (\mathbf{v}, \tilde{\mathbf{v}})) \in \mathbb{W}^2$ is of the form

$$I_X + I_X^* + \left\langle (c + n - m + 1)\text{-minors of } \begin{bmatrix} \mathbf{v} - \mathbf{w} & \tilde{\mathbf{v}} - \tilde{\mathbf{w}} \\ J(f) & \mathbf{0} \\ \mathbf{0} & I_{m-n} \end{bmatrix} \right\rangle + \left(I'_X\right)^*$$

where the ideal explicitated is I'_X. In the end, since polynomials defining I_X and I_X^* require the last $m - n$ coordinates $\tilde{\mathbf{z}}$ and $\tilde{\mathbf{w}}$ to vanish, the formula above simplifies to the Hermitian critical ideal of $(\pi_{\mathbb{V}}(\mathbf{u}, \tilde{\mathbf{u}}), \pi_{\mathbb{V}}(\mathbf{v}, \tilde{\mathbf{v}})) = (\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$. □

The proofs of the following three lemmas follow from the definitions and are straightforward.

Lemma 4.2.10. *Let $X \subseteq \mathbb{V}$ be an algebraic variety, $0 \neq c \in \mathbb{C}$ and $\mathbf{b} \in \mathbb{V}$, then the map $(\mathbf{z}, \mathbf{w}) \mapsto (c\mathbf{z} + \mathbf{b}, \bar{c}\mathbf{w} + \bar{\mathbf{b}})$ is a bijection from the set of solutions of the Hermitian critical ideal of X of $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ to the set of solutions of the Hermitian critical ideal of $cX + \mathbf{b}$ of $(c\mathbf{u} + \mathbf{b}, \bar{c}\mathbf{v} + \bar{\mathbf{b}}) \in \mathbb{V}^2$, also if restricted to critical points. In particular,* $\mathrm{vHDdeg}(X) = \mathrm{vHDdeg}(cX + \mathbf{b})$ *and* $\mathrm{HDdeg}(X) = \mathrm{HDdeg}(cX + \mathbf{b})$. *Moreover, this map is proximity preserving for critical points up to the scalar factor $|c|^2$.*

Lemma 4.2.11. *Let $X \subseteq \mathbb{V}$ be an algebraic variety, then the map $(\mathbf{z}, \mathbf{w}) \mapsto (\bar{\mathbf{z}}, \bar{\mathbf{w}})$ is a bijection from the set of solutions of the Hermitian critical ideal of X of $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ to the set of solutions of the Hermitian critical ideal of $\overline{X}$ of $(\mathbf{v}, \mathbf{u}) \in \mathbb{V}^2$, also if restricted to critical points. In particular,* $\mathrm{vHDdeg}(X) = \mathrm{vHDdeg}(\overline{X})$ *and* $\mathrm{HDdeg}(X) = \mathrm{HDdeg}(\overline{X})$. *Moreover, this map is proximity preserving for critical points.*

Lemma 4.2.12. *Let $\mathbb{W}$ denote a m-dimensional complex vector space and let $X \subseteq \mathbb{V}$, $Y \subseteq \mathbb{W}$ be algebraic varieties, then*

$$\mathrm{vHDdeg}(X \times Y) = \mathrm{vHDdeg}(X) \cdot \mathrm{vHDdeg}(Y)$$

and if $\mathrm{HDdeg}(X) = \{a_1, \dots, a_{d_1}\}$ *and* $\mathrm{HDdeg}(Y) = \{b_1, \dots, b_{d_2}\}$ *for some $d_1, d_2 \in \mathbb{N}$ then*

$$\mathrm{HDdeg}(X \times Y) = \{a_1 b_1, \dots, a_1 b_{d_2}, \dots, a_{d_1} b_1, \dots, a_{d_1} b_{d_2}\}.$$

We can use this last result to construct an example of algebraic variety such that the set HDdeg does not contain all the numbers with the same parity between $\min \mathrm{HDdeg}(X)$ and $\max \mathrm{HDdeg}(X)$.

Example 4.2.13. Let X and Y are both the variety of Example 4.4.3, then using Lemma 4.2.12 there hold $\mathrm{vHDdeg}(X \times Y) = 25$ and $\mathrm{HDdeg}(X \times Y) = \{1, 3, 9\}$.

Let $\mathrm{Iso}(\mathbb{V}) \subseteq \mathrm{GL}(\mathbb{V})$ be the group of affine linear transformations of $\mathbb{V}$ which preserve the Hermitian inner product. This group is equal to the sum of the group $\mathrm{U}(\mathbb{V})$ introduced in Subsection 4.1 and the group of translations of $\mathbb{V}$. Moreover, for any affine linear transformation $g \in \mathrm{GL}(\mathbb{V})$ we denote with $\bar{g}$ the affine linear transformation given by the composition $\bar{\ } \circ g \circ \bar{\ }$.

We provide a particularly useful lemma that characterizes the action of some elements in the group $\mathrm{Iso}(\mathbb{V})$.

Lemma 4.2.14. *Let $X \subseteq \mathbb{V}$ be an algebraic variety and let $g \in \mathrm{Iso}(\mathbb{V})$ be an element that leaves X invariant. If $(\mathbf{z}, \mathbf{w})$ is a solution of the Hermitian critical ideal of $(\mathbf{u}, \mathbf{v})$ then $(g \cdot \mathbf{z}, \bar{g} \cdot \mathbf{w})$ is a solution of the Hermitian critical ideal of $(g \cdot \mathbf{u}, \bar{g} \cdot \mathbf{v})$.*

Proof. The assertion follows from the equalities

$$\begin{aligned}\langle \mathbf{u} - \mathbf{z}, \mathbf{v} - \mathbf{w} \rangle_{\mathbb{R}} &= \langle \mathbf{u} - \mathbf{z}, \bar{\mathbf{v}} - \bar{\mathbf{w}} \rangle_{\mathbb{C}} = \langle g \cdot \mathbf{u} - g \cdot \mathbf{z}, g \cdot \bar{\mathbf{v}} - g \cdot \bar{\mathbf{w}} \rangle_{\mathbb{C}} \\ &= \langle g \cdot \mathbf{u} - g \cdot \mathbf{z}, \bar{g} \cdot \mathbf{v} - \bar{g} \cdot \mathbf{w} \rangle_{\mathbb{R}}.\end{aligned}$$

□

With the same proof of the lemma above we can prove the following result.

Lemma 4.2.15. *Let $X \subseteq \mathbb{V}$ be an algebraic variety and let $g \in \mathrm{Iso}(\mathbb{V})$. If $(\mathbf{z}, \mathbf{w})$ is a solution of the Hermitian critical ideal of X of $(\mathbf{u}, \mathbf{v})$ then $(g \cdot \mathbf{z}, \bar{g} \cdot \mathbf{w})$ is a solution of the Hermitian critical ideal of $g \cdot X$ of $(g \cdot \mathbf{u}, \bar{g} \cdot \mathbf{v})$.*

Now we present the first example considering the complex unit circle. We can already appreciate the properties of the Hermitian Killing form introduced in Subection 3.1.

Example 4.2.16. (Circle). The unit circle

$$X = V\left(z_1^2 + z_2^2 - 1\right) \subseteq \mathbb{C}^2$$

satisfies $\mathrm{vHDdeg}(X) = 6$ and $\mathrm{HDdeg} = \{2, 4\}$. The value of vHDdeg can be seen by computing the degree of the critical ideal of X for general parameters $(\mathbf{u}, \mathbf{v})$. This ideal is generated by the polynomials of the system

$$\begin{cases} p_1(\mathbf{z}, \mathbf{w}) = z_1^2 + z_2^2 - 1 = 0 \\ \bar{p}_1(\mathbf{w}, \mathbf{z}) = w_1^2 + w_2^2 - 1 = 0 \\ p_2(\mathbf{z}, \mathbf{w}) = w_2(z_1 - u_1) - w_1(z_2 - u_2) = 0 \\ \bar{p}_2(\mathbf{w}, \mathbf{z}) = z_2(w_1 - v_1) - z_1(w_2 - v_2) = 0 \end{cases}$$

where the upper two polynomials are given by the definition of X and the bottom two polynomials are given by the determinant of the 2×2 matrix obtained from the perpendicularity condition. In particular, directly follows the containment $\mathrm{HDdeg}(X) \subseteq \{2, 4, 6\}$. We now compute the true values.

Set $\mathbf{v} = \bar{\mathbf{u}}$. From the various possible bijective parametrizations of the unit circle consider

$$\begin{aligned} \psi \colon \mathbb{C}\backslash\{0\} &\to X \subseteq \mathbb{C}^2 \\ z &\mapsto \left(\frac{z^2+1}{2z}, \frac{i(z^2-1)}{2z}\right) \end{aligned}$$

so that in this case the critical points of the Hermitian distance satisfy the equation

$$\begin{aligned} \partial_z \|\psi(z) - \mathbf{u}\|_{\mathbb{C}}^2 &= \partial_z \left[\left(\frac{z^2+1}{2z} - u_1\right)\left(\frac{\bar{z}^2+1}{2\bar{z}} - \bar{u}_1\right) + \left(\frac{i(z^2-1)}{2z} - u_2\right)\left(\frac{-i(\bar{z}^2-1)}{2\bar{z}} - \bar{u}_2\right)\right] \\ &= \frac{1}{2z^2}\left((z^2-1)\left(\frac{\bar{z}^2+1}{2\bar{z}} - \bar{u}_1\right) + i(z^2+1)\left(\frac{-i(\bar{z}^2-1)}{2\bar{z}} - \bar{u}_2\right)\right) = 0 \end{aligned}$$

or equivalently, considering the numerator, satisfy the equation

$$\begin{aligned} &(z^2-1)\left(\bar{z}^2 + 1 - 2\bar{u}_1\bar{z}\right) + (z^2+1)\left(\bar{z}^2 - 1 - 2i\bar{u}_2\bar{z}\right) \\ &\quad = 2\left[z^2\bar{z}^2 - 1 - \bar{z}\left((\bar{u}_1 + i\bar{u}_2)z^2 - \bar{u}_1 + i\bar{u}_2\right)\right] = 0. \end{aligned}$$

Since the point $\mathbf{u}$ is generic, relabeling it as $\mathbf{a}$, we can solve the complex linear system

$$\begin{cases} \bar{a}_1 + i\bar{a}_2 = -\bar{u}_1 \\ -\bar{a}_1 + i\bar{a}_2 = -\bar{u}_2 \end{cases}$$

so that with a change of variables, solving the equation above is equivalent to solving an equation of the form

$$p(z,\bar{z}) = z^2\bar{z}^2 - 1 + \bar{u}_1 z^2\bar{z} + \bar{u}_2\bar{z} = 0.$$

By Lemma 4.2.14, this problem remains the same under the canonical action on $\mathbb{C}^2$ of the subgroup of the unitary group $\mathrm{U}_2 = \mathrm{U}(\mathbb{C}^2) \subseteq \mathrm{Iso}(\mathbb{C}^2)$ that leaves X invariant. This is the group of real orthogonal matrices and thus we can rotate a point $\mathbf{u} \in \mathbb{C}^2$ to a point $\mathbf{u} \in \mathbb{R}\times\mathbb{C}$, thus for simplicity we assume $u_1 \in \mathbb{R}$. We now use an idea discussed in Subsection 2.1 and consider the system

$$\begin{cases} zw = \rho \\ p(z,w) = 0 \\ \bar{p}(w,z) = 0 \end{cases} \quad \begin{cases} zw = \rho \\ u_1\rho z + \bar{u}_2 w = 1-\rho^2 \\ u_2 z + u_1\rho w = 1-\rho^2 \end{cases} \quad \cdots \quad \begin{cases} zw = \rho \\ z = \frac{(1-\rho^2)(u_1\rho-\bar{u}_2)}{u_1^2\rho^2-|u_2|^2} \\ w = \frac{(1-\rho^2)(u_1\rho-u_2)}{u_1^2\rho^2-|u_2|^2} \end{cases}$$

for which a solving triple (z,w,ρ) is required to satisfy $\rho > 0$ as a necessary condition for the first component z to be a critical point, so that we are led to search for positive real solutions of the equation

$$zw-\rho = \frac{(1-\rho^2)^2|u_1\rho-\bar{u}_2|^2}{(u_1^2\rho^2-|u_2|^2)^2} - \rho = \frac{u_1^2\rho^6 + c_5\rho^5 + c_4\rho^4 + c_3\rho^3 + c_2\rho^2 + c_1\rho + |u_2|^2}{(u_1^2\rho^2-|u_2|^2)^2} = 0$$

where

$$\begin{aligned} c_5 &= -u_1(u_1^3 + u_2 + \bar{u}_2), \\ c_4 &= -2u_1^2 + |u_2|^2, \\ c_3 &= 2u_1(u_1|u_2|^2 + u_2 + \bar{u}_2), \\ c_2 &= u_1^2 - 2|u_2|^2, \\ c_1 &= -|u_2|^4 - u_1(u_2 + \bar{u}_2). \end{aligned}$$

Now, we note that for the coefficients c_4 to be positive, it requires the coefficient c_2 to be negative. To conclude, with this choices the polynomial at the numerator has at most 5 changes of sign among its ordered coefficients and by the Descartes' rule of signs it admits at most 5 positive solutions. In particular, $\max \mathrm{HDdeg}(X) \leq 5$ and since by testing points $\mathbf{u}$ in the real plane we get both two or four critical points it follows $\mathrm{HDdeg}(X) = \{2,4\}$.

We now simplify more this problem by considering points $\mathbf{u} \in \mathbb{C}\times\{0\}$. In this case, the characteristic polynomial of the matrix representing the Hermitian Killing form $\mathcal{K}^1_{\mathbb{C}}$ with respect to the basis

$$\{[1],[z_1],[z_1w_1],[w_1],[w_2],[w_2^2]\}$$

of the quotient $\mathbb{C}[\mathbf{z},\mathbf{w}]/\langle p_1, p_1^*, p_2, p_2^*\rangle$ decomposes

$$(\lambda-4)^2\left(2|u_1|^6\lambda + 4|u_1|^4 - (|u_1|^4 - u_1^2 - \bar{u}_1^2)^2\right)\left(4|u_1|^{12}\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0\right)$$

where

$$\begin{aligned} c_2 &= -\left||u_1|^2 + u_1 - \bar{u}_1\right|^4 (|u_1|^8 + (u_1+\bar{u}_1)^4), \\ c_1 &= \dots, \\ c_0 &= 8\left||u_1|^2 + u_1 - \bar{u}_1\right|^4 (|u_1|^2 - u_1 - \bar{u}_1)^2(|u_1|^2 + u_1 + \bar{u}_1)^2 \geq 0. \end{aligned}$$

In particular, by the Descartes' rule of signs, since we generically have at least one permanence of signs among c_0, c_1, c_2, the third factor generically possesses at least one negative solution and thus this matrix has a negative eigenvalue.

We will continue to study this problem in Example 5.2.5.

Let $X \subseteq \mathbb{V}$ be an affine cone with $I_X \subseteq \mathbb{C}[\mathbf{z}]$ a homogeneous ideal. By a slight abuse of notation, we denote the projective variety in $\mathbb{P}\mathbb{V}$ as its affine cone X. The vHDdeg of a projective variety will be the vHDdeg of the relative affine cone and similarly for the HDdeg.

The singular locus is defined by the same homogeneous ideal $I_{X_{\text{sing}}}$. To take advantage of the homogeneity of the generators of I_X, we define

$$I'_{\mathbb{P}X} := \left\langle (c+2)\text{-minors of } \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \\ J(f) \end{bmatrix} \right\rangle,$$

and replace the Hermitian critical ideal in expression (4.2.1) with the following bi-homogeneous ideal

$$\left(I_X + (I_X)^* + I'_{\mathbb{P}X} + (I'_{\mathbb{P}X})^*\right) : \left(I_{X_{\text{sing}}} \cdot (I_{X_{\text{sing}}})^* \cdot \langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{R}}\right)^\infty \subseteq \mathbb{C}[\mathbf{z}, \mathbf{w}]. \tag{4.2.4}$$

We denote the projective variety $\tilde{Q} := V\left(\langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{R}}\right) \subseteq \mathbb{V}^2$. Note that the variety $X_\circ$ is never contained in $\tilde{Q}$. In particular, any non zero critical point $(\mathbf{z}, \bar{\mathbf{z}})$ does not lie in $\tilde{Q}$ since $\langle \mathbf{z}, \bar{\mathbf{z}} \rangle_{\mathbb{R}} > 0$.

The following lemma concerns the transition between affine cones and projective varieties.

Lemma 4.2.17. *Let $X \subseteq \mathbb{V}$ be an affine cone and $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ a point. Let $(\mathbf{z}, \mathbf{w})$ be such that the corresponding point $([\mathbf{z}], [\mathbf{w}])$ does not lie in $\tilde{Q}$, then $([\mathbf{z}], [\mathbf{w}])$ lies in the projective variety of the Hermitian critical ideal* (4.2.4) *if and only if for some unique scalars $\mu_{\mathbf{z}}, \mu_{\mathbf{w}} \in \mathbb{C}$ the point $(\mu_{\mathbf{z}}\mathbf{z}, \mu_{\mathbf{w}}\mathbf{w})$ lies in the affine variety of the Hermitian critical ideal* (4.2.1). *Moreover, if $\mathbf{v} = \bar{\mathbf{u}}$ then $[\mathbf{z}] = [\bar{\mathbf{w}}]$ iff $\mu_{\mathbf{z}}\mathbf{z} = \bar{\mu}_{\mathbf{w}}\bar{\mathbf{w}}$.*

Proof. Since both Hermitian critical ideals (4.2.1) and (4.2.4) are saturated with respect to $I_{X_{\text{sing}}} \cdot (I_{X_{\text{sing}}})^*$ and the definitions are symmetric under the operation $*$, it suffices to prove the assertion for $(\mathbf{z}, \mathbf{w}) \in X_\circ$ where the Jacobians $J(f)$ at $\mathbf{z}$ and $J(f^*)$ at $\mathbf{w}$ have rank c.

If $\mathbf{v} - \mu_{\mathbf{w}}\mathbf{w}$ lies in $\text{Row}(J(f))$ at $\mu_{\mathbf{z}}\mathbf{z}$, then the subspace $\text{span}\{\mathbf{v}, \mathbf{w}\} + \text{Row}(J(f))$ has dimension at most $c+1$. The rest of the proof of the if condition follows using a symmetric reasoning.

Conversely, suppose that $([\mathbf{z}], [\mathbf{w}])$ lies in the variety of the ideal (4.2.4). First assume that $\mathbf{w}$ lies in $\text{Row}(J(f))$. Then, $\mathbf{w} = \sum_{k=1}^{s} \mu_k \nabla f_k(\mathbf{z})$ for some $\mu_k \in \mathbb{C}$ with $k = 1, \ldots, s$. Now recall that if g is a homogeneous polynomial in $\mathbb{C}[\mathbf{z}]$ of degree d, then $\langle \mathbf{z}, \nabla g(\mathbf{z}) \rangle_{\mathbb{R}} = d \cdot g(\mathbf{z})$. Since $f_k(\mathbf{z}) = 0$ for any k, we find that $\langle \mathbf{z}, \nabla f_k(\mathbf{z}) \rangle_{\mathbb{R}} = 0$ for any k, which implies that $\langle \mathbf{z}, \mathbf{w} \rangle_{\mathbb{R}} = 0$. This contradicts our hypothesis, so the matrix $\begin{bmatrix} \mathbf{w} \\ J(f) \end{bmatrix}$ has rank $c+1$. But then $\mathbf{v} - \mu_{\mathbf{w}}\mathbf{w}$ lies in $\text{Row}(J(f))$ for a unique $\mu_{\mathbf{w}} \in \mathbb{C}$. The rest of the proof of the only if condition follows using a symmetric reasoning.

The last assertion can be proved using the argument above by adding the hypothesis of conjugation in each direction. □

From the last lemma and the fact $X_\circ \not\subseteq \tilde{Q}$, we directly obtain the following result.

Corollary 4.2.18. *Let $X \subseteq \mathbb{PV}$ be a projective variety and $(\mathbf{u}, \mathbf{v}) \in (\mathbb{PV})^2$ a point, then* $\mathrm{vHDdeg}(X)$ *is equal to the number of zeros of the ideal* (4.2.4) *in* $(\mathbb{PV})^2$ *and similarly for* $\mathrm{HDdeg}(X)$.

4.3 vHDdeg of hypersurfaces

We discuss the value of $\mathrm{vHDdeg}(X)$ when X is an algebraic variety of codimension one. The results obtained in this section are easily achieved thanks to the clear symmetries of the problem given by the complex approach.

We firstly introduce a notion that will help to our purpose, see [Stu02; CLO05; KK12]. The following concept will be useful also in Subsection 4.4.

Let $\triangle_1, \ldots, \triangle_n \subseteq \mathbb{R}^n$ be n *convex bodies*, that are non-empty compact convex sets. For nonnegative parameters $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, the Euclidean volume of the scaled Minkowski sum $\mathrm{Vol}_n(\lambda_1\triangle_1 + \ldots + \lambda_n\triangle_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$ with nonnegative coefficients.

The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ is called the *mixed volume* of $\triangle_1, \ldots, \triangle_n$ and we denoted it by $\mathrm{MV}_n(\triangle_1, \ldots, \triangle_n)$. We state here an explicit formula for the mixed volume that is sometimes used as a definition,

$$\mathrm{MV}_n(\triangle_1, \ldots, \triangle_n) = (-1)^n \sum_{\mathbf{k} \in \{0,1\}^n} (-1)^{\|\mathbf{k}\|_1} \mathrm{Vol}_n(k_1\triangle_1 + \ldots + k_n\triangle_n). \tag{4.3.1}$$

The mixed volume possesses three notable properties which uniquely characterize it:

i) For any convex body $\triangle$ it holds $\mathrm{MV}_n(\triangle, \ldots, \triangle) = n!\,\mathrm{Vol}_n(\triangle)$.

ii) Vol is symmetric in its arguments, in other terms for any $k, j \in [n]$ it holds

$$\mathrm{MV}_n(\triangle_1, \ldots, \triangle_k, \ldots, \triangle_j, \ldots, \triangle_n) = \mathrm{MV}_n(\triangle_1, \ldots, \triangle_j, \ldots, \triangle_k, \ldots, \triangle_n).$$

iii) Vol is multilinear, in other terms for nonnegative $\lambda, \lambda' \in \mathbb{R}$ it holds

$$\mathrm{MV}_n(\lambda\triangle_1 + \lambda'\triangle_1', \ldots, \triangle_n) = \lambda\,\mathrm{MV}_n(\triangle_1, \ldots, \triangle_n) + \lambda'\,\mathrm{MV}_n(\triangle_1', \ldots, \triangle_n).$$

Thanks to the Bernstein–Kushnirenko Theorem historically proved in [Ber75], the mixed volume is often used to count the number of solutions of systems of Laurent polynomials. In this subsection by using the notation conv we consider the convex hull of a subset of a vector space.

We start by considering the case of curves.

Let $X = V(f) \subseteq \mathbb{V}$ be a curve of degree d, the Hermitian critical ideal of X of $(\mathbf{u}, \mathbf{v})$ takes the form

$$\langle f,\ f^*,\ g,\ g^* \rangle \subseteq \mathbb{C}[z_1, z_2, w_1, w_2].$$

where

$$g \coloneqq (v_2 - w_2)\partial_{z_1} f - (v_1 - w_1)\partial_{z_2} f.$$

For generic $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ this ideal is zero-dimensional and the Bézout Theorem predicts at most d^4 points in the zero locus, in particular $\mathrm{vHDdeg}(X) \leq d^4$.

On the other hand if X is a real curve, or in other words f is a real polynomial, we can apply the same argument to prove $\mathrm{EDdegree}(X) \leq d^2$ and equality holds for generic curves, this result follows as a particular case of Proposition 4.2.6.

We find a new bound for $\mathrm{vHDdeg}(X)$ with the following result.

Proposition 4.3.1. *Let $X = V(f)$ be a generic curve of degree d, then*

$$\mathrm{vHDdeg}(X) \leq d^2\left((d-1)^2 + 1\right) = d^2|d - 1 + i|^2.$$

Proof. The number of solutions of the Hermitian critical ideal $\langle f, f^*, g, g^* \rangle$ is bounded using the Bernstein–Kushnirenko Theorem by the value $\mathrm{MV}_4(\triangle_f, \triangle_{f^*}, \triangle_g, \triangle_{g^*})$ where the arguments are the convex hulls

$$\begin{aligned} &\triangle_f \coloneqq \mathrm{conv}\{(\mathbf{k}, \mathbf{0}) \in (\mathbb{N}^2)^2 \mid \|\mathbf{k}\|_1 \leq d\},\ \triangle_g \coloneqq \mathrm{conv}\{(\mathbf{k}, \mathbf{j}) \in (\mathbb{N}^2)^2 \mid \|\mathbf{k}\|_1 \leq d-1,\ \|\mathbf{j}\|_1 \leq 1\},\\ &\triangle_{f^*} \coloneqq \mathrm{conv}\{(\mathbf{0}, \mathbf{j}) \in (\mathbb{N}^2)^2 \mid \|\mathbf{j}\|_1 \leq d\},\ \triangle_{g^*} \coloneqq \mathrm{conv}\{(\mathbf{k}, \mathbf{j}) \in (\mathbb{N}^2)^2 \mid \|\mathbf{k}\|_1 \leq 1,\ \|\mathbf{j}\|_1 \leq d-1\}. \end{aligned}$$

The left ones come from the equations $f = f^* = 0$ while the right ones come from the equations $g = g^* = 0$. Now, considering some clear symmetries the use of formula (4.3.1) yields

$$\begin{aligned} \mathrm{MV}_4(\triangle_f, &\triangle_{f^*}, \triangle_g, \triangle_{g^*}) = -2\,\mathrm{Vol}_4(\triangle_g) + 2\,\mathrm{Vol}_4(\triangle_f + \triangle_g) + 2\,\mathrm{Vol}_4(\triangle_f + \triangle_{g^*})\\ &+ \mathrm{Vol}_4(\triangle_f + \triangle_{f^*}) + \mathrm{Vol}_4(\triangle_g + \triangle_{g^*}) - 2\,\mathrm{Vol}_4(\triangle_f + \triangle_{f^*} + \triangle_g)\\ &- 2\,\mathrm{Vol}_4(\triangle_f + \triangle_g + \triangle_{g^*}) + \mathrm{Vol}_4(\triangle_f + \triangle_{f^*} + \triangle_g + \triangle_{g^*})\\ &= \frac{-(d-1)^2 + (2d-1)^2 + (d+1)^2(d-1)^2 + d^4 - d^2(2d)^2 - (2d-1)^2(d+1)^2 + 8d^4}{2} \end{aligned}$$

and the assertion follows. □

The symmetric degrees in $\mathbf{z}$ and $\mathbf{w}$ is not the only property shared by the polynomials of the system given by the Hermitian critical ideal. In fact, the Hermitian critical ideal is invariant under the action of the map $*$ thus the true bound of vHDdeg could be lower, see Remark 4.3.4. Nonetheless, we already said that max HDdeg could be lower than vHDdeg, see the next example.

Example 4.3.2. (Ellipse). The ellipse

$$X = V\left(z_1^2 + cz_2^2 - c\right) \subseteq \mathbb{C}^2$$

where $0 < c < 1$ satisfies $\mathrm{vHDdeg}(X) = 8$ as predicted by Proposition 4.3.1 and $\{2, 4\} \subseteq \mathrm{HDdeg}(X) \subseteq \{2, 4, 6\}$. The value of vHDdeg can be seen by computing the degree of the

Hermitian critical ideal of X for general parameters $(\mathbf{u}, \mathbf{v})$. This ideal is generated by the polynomials of the system

$$\begin{cases} p_1(\mathbf{z}, \mathbf{w}) = z_1^2 + cz_2^2 - c = 0 \\ \bar{p}_1(\mathbf{w}, \mathbf{z}) = w_1^2 + cw_2^2 - c = 0 \\ p_2(\mathbf{z}, \mathbf{w}) = cw_2(z_1 - u_1) - w_1(z_2 - u_2) = 0 \\ \bar{p}_2(\mathbf{w}, \mathbf{z}) = cz_2(w_1 - v_1) - z_1(w_2 - v_2) = 0 \end{cases}$$

where the upper two polynomials are given by the definition of X and the bottom two polynomials are given by the determinant of the 2×2 matrix obtained from the perpendicularity condition. In particular, directly follows the containment $\mathrm{HDdeg}(X) \subseteq \{2, 4, 6, 8\}$. We now show that 8 it is not an acceptable value.

Set $\mathbf{v} = \bar{\mathbf{u}}$. For the sake of simplicity, we multiply the polynomial defining the variety by $1/c$ and relabel it as a so that $X = V(az_1^2 + z_2^2 - 1)$ with $1 < a \in \mathbb{R}$. The matrix representing the Hermitian Killing form with respect to the basis

$$\{[1], [z_1], [z_2], [w_1], [w_2], [z_2 w_2], [w_1 w_2], [w_2^2]\}$$

of the quotient $\mathbb{C}[\mathbf{z}, \mathbf{w}]/\langle p_1, p_1^*, p_2, p_2^* \rangle$ is too big to be reported here. However, the determinant of the submatrix given by the subspace $\mathrm{span}\{[z_1], [z_2]\}$, is

$$-\frac{32a^2(a^2+1)|u_2^2 a + u_1^2|^2}{(a^2-1)^4} \leq 0.$$

In particular this matrix outside the set $V(u_2^2 a + u_2^2) \subseteq \mathbb{C}^2$ possesses a negative eigenvalue and from Corollary 3.1.11 we obtain $\max \mathrm{HDdeg}(X) \leq 7 - 1 = 6$. Moreover, since by testing points $\mathbf{u}$ in the real plane we get both two or four critical points it follows $\{2, 4\} \subseteq \mathrm{HDdeg}(X)$.

We will continue to study this problem in Example 5.2.6.

We now generalize Proposition 4.3.1 to hypersurfaces. Let $X = V(f) \subseteq \mathbb{V}$ be a hypersurface of degree d, using equation (4.2.3) we have already seen

$$\begin{aligned} \mathrm{vHDdeg}(X) &\leq d^2 \sum_{k=0}^{2n-2} (k+1)(d-1)^k \\ &= \frac{d^2\left((2n-2)(d-1)^{2n} - (2n+1-d)(d-1)^{2n-1} + 1\right)}{(d-2)^2}. \end{aligned} \tag{4.3.2}$$

In particular, to obtain the second formula we assume $d > 2$. On the other hand, if X is a real hypersurface, or in other words f is a real polynomial, from Proposition 4.2.6 we get $\mathrm{EDdegree}(X) \leq d \sum_{k=0}^{n-1} (d-1)^k$ and equality holds for generic hypersurface.

Proposition 4.3.3. *Let $X = V(f) \subseteq \mathbb{V}$ be a generic hypersurface of degree d, then*

$$\mathrm{vHDdeg}(X) \leq d^2 \sum_{k=0}^{n-1} \binom{n-1}{k}^2 (d-1)^{2k}. \tag{4.3.3}$$

Proof. The codimension of X is $c = 1$. The number of solutions of the Hermitian critical ideal of X is bounded using the Bernstein–Kushnirenko Theorem by the value

$$\mathrm{MV}_{2n}(\triangle_f, \triangle_{f^*}, \triangle_{g_1}, \dots, \triangle_{g_{n-1}}, \triangle_{g_1^*}, \dots, \triangle_{g_{n-1}^*})$$

where the first two arguments are the convex hulls

$$\triangle_f \coloneqq \mathrm{conv}\{(\mathbf{s}, \mathbf{0}) \in (\mathbb{N}^n)^2 \mid \|\mathbf{s}\|_1 \leq d\}, \qquad \triangle_{f^*} \coloneqq \mathrm{conv}\{(\mathbf{0}, \mathbf{t}) \in (\mathbb{N}^n)^2 \mid \|\mathbf{t}\|_1 \leq d\},$$

while the remaining are in the families of convex hulls

$$\begin{aligned} \triangle_{g_{\ell,r}} &\coloneqq \mathrm{conv}\{(\mathbf{s}, \mathbf{t}) \in (\mathbb{N}^n)^2 \mid \|\mathbf{s}\|_1 \leq d-1,\ \|\mathbf{t}\|_1 \leq 1,\ t_\xi = 0 \text{ if } \xi \neq \ell, r\}, \\ \triangle_{g_{\ell,r}^*} &\coloneqq \mathrm{conv}\{(\mathbf{s}, \mathbf{t}) \in (\mathbb{N}^n)^2 \mid \|\mathbf{s}\|_1 \leq 1,\ \|\mathbf{t}\|_1 \leq d-1,\ s_\xi = 0 \text{ if } \xi \neq \ell, r\} \end{aligned}$$

for $\ell < r \in [n]$ respectively. The first two come from the equations $f = f^* = 0$ while the other $2\binom{n}{2}$ come from the equations given by the order 2 minors of the matrices involved in the Hermitian critical ideal and only $2n-2$ of them are required since our matrices are $2 \times n$. Denote the convex hulls

$$\triangle_{1,0} \coloneqq \mathrm{conv}(\triangle \times \{\mathbf{0}\}) \subseteq \mathbb{R}^{2n} \qquad \text{and} \qquad \triangle_{0,1} \coloneqq \mathrm{conv}(\{\mathbf{0}\} \times \triangle) \subseteq \mathbb{R}^{2n},$$

where $\triangle \coloneqq \{\mathbf{s} \in \mathbb{N}^n \mid \|\mathbf{s}\|_1 \leq 1\} \subseteq \mathbb{R}^n$, so that

$$\triangle_f = d\triangle_{1,0} \subseteq \mathbb{R}^{2n} \quad \text{and} \quad \triangle_{f^*} = d\triangle_{0,1} \subseteq \mathbb{R}^{2n}.$$

Moreover, there hold the containments

$$\triangle_{g_{\ell,r}} \subseteq \triangle_{\mathcal{Z}} \coloneqq (d-1)\triangle_{1,0} + \triangle_{0,1} \subseteq \mathbb{R}^{2n} \quad \text{and} \quad \triangle_{g_{\ell,r}^*} \subseteq \triangle_{\mathcal{W}} \coloneqq \triangle_{1,0} + (d-1)\triangle_{0,1} \subseteq \mathbb{R}^{2n}.$$

It will be clear by the rest of the proof that the mixed volume we are computing is equal to

$$\mathrm{MV}_{2n}(\triangle_f, \triangle_{f^*}, \underbrace{\triangle_{\mathcal{Z}}, \dots, \triangle_{\mathcal{Z}}}_{n-1}, \underbrace{\triangle_{\mathcal{W}}, \dots, \triangle_{\mathcal{W}}}_{n-1}).$$

We make this change in order to simplify the notation. Since the mixed value is symmetric multilinear we rewrite this last value as

$$d^2 \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{j=0}^{n-1} \binom{n-1}{j} (d-1)^{n-1+k-j}\, \mathrm{MV}_{2n}(\triangle_{1,0}, \triangle_{0,1}\, \underbrace{\triangle_{1,0}, \dots, \triangle_{1,0}}_{k+j}, \underbrace{\triangle_{0,1}, \dots, \triangle_{0,1}}_{2n-2-k-j}).$$

Now, [Ewa96, Lemma 4.5] implies that the mixed volumes of the terms with $k+j \neq 2n-2-k-j$, or equivalently $k \neq n-1-j$, vanish, thus the formula simplifies

$$d^2 \sum_{k=0}^{n-1} \binom{n-1}{k}^2 (d-1)^{2k}\, \mathrm{MV}_{2n}(\underbrace{\triangle_{1,0}, \dots, \triangle_{1,0}}_{n}, \underbrace{\triangle_{0,1}, \dots, \triangle_{0,1}}_{n}).$$

Applying the same lemma for the remaining terms we get

$$\mathrm{MV}_{2n}(\underbrace{\triangle_{1,0},\dots,\triangle_{1,0}}_{n},\underbrace{\triangle_{0,1},\dots,\triangle_{0,1}}_{n}) = \mathrm{MV}_n(\triangle,\dots,\triangle)\,\mathrm{MV}_n(\triangle,\dots,\triangle) = 1.$$

If we do not change the convex bodies in the formula, the steps will be the same until the last equality. In this case, on the left hand side, instead of the standard simplex $\triangle \subseteq \mathbb{R}^n$, we could have instances of standard simplexes in pairwise distinct coordinate subspaces $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$. We can replace those convex bodies with $\triangle$ by means of [Che19, Theorem 2] and the claim follows. $\square$

The symmetric degrees in $\mathbf{z}$ and $\mathbf{w}$ is not the only property shared by the polynomials of the system given by the Hermitian critical ideal. In fact, the Hermitian critical ideal is invariant under the action of the map $*$, thus the true bound of vHDdeg could be lower, see Remark 4.3.4. Nonetheless, we already said that max HDdeg could be lower than vHDdeg, see Subsection 4.5.

Remark 4.3.4. The bound of equation (4.3.3) is not sharp in general. In fact the bound of equation (4.3.2) it is still better for n tending to infinity. For example, assume X to be a hypersurface of degree $d = 2$. In this case equation (4.3.3) yields the value

$$4\sum_{k=0}^{n-1}\binom{n-1}{k}^2 = 4\binom{2n-2}{n-1} \geq 4\cdot 2^{n-1},$$

where we used the Chu-Vandermonde identity and a well known inequality on the binomial coefficient $\binom{2n-2}{n-1}$. On the other hand, equation (4.3.2) yields the value $4\sum_{k=0}^{2n-2}(k+1) = 4\cdot n(2n-1)$. In particular, the value of (4.3.2) is is surely lower than the value of (4.3.3) for $n \geq 8$.

The last remark motivates the following result.

Corollary 4.3.5. *Let $X = V(f) \subseteq \mathbb{V}$ be a hypersurface of degree d, then*

$$\mathrm{vHDdeg}(X) \leq d^2 \min\left\{\sum_{k=0}^{n-1}\binom{n-1}{k}^2 (d-1)^{2k},\ \sum_{k=0}^{2n-2}(k+1)(d-1)^k\right\}. \tag{4.3.4}$$

We compute some values of the bound of the result above in Table 4.1. For the case $d = 1$ clearly always holds vHDdeg = 1.

d	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
2	$\hat{4}$	$\hat{8}$	24	80	<u>180</u>	<u>264</u>	<u>364</u>	<u>480</u>
3	$\hat{9}$	$\widehat{45}$	297	2205	17289	139725	<u>884745</u>	<u>4128777</u>
4	$\widehat{16}$	$\widehat{160}$	1888	24640	340576	4868800	71097280	1053289600
5	$\widehat{25}$	$\widehat{425}$	8025	163625	3513625	78064425	1774203225	40958848425

Table 4.1: Values of equation (4.3.4) for different d and n. The underlined values are the ones where the minimun is reached by the formula on the right. The values with a hat are the ones we checked to be sharp.

In the following example we present the vHDdeg of a curve of degree 3 and of the Fermat cubic.

Example 4.3.6. The curve

$$X = V(z_1^3 + z_2^3 - 1) \subseteq \mathbb{C}^2$$

satisfies $\mathrm{vHDdeg}(X) = 45 = 3^2(2 \cdot 2 + 1)$ as predicted by Proposition 4.3.1, this can be seen by computing the degree of the Hermitian critical ideal of different points $(\mathbf{u}, \mathbf{v}) \in \mathbb{C}^4$.

(Fermat cubic). The Fermat cubic

$$X = V(z_1^3 + z_2^3 + z_3^3 - 1) \subseteq \mathbb{C}^3$$

satisfies $\mathrm{vHDdeg}(X) = 189 = 3^3(2 \cdot 3 + 1)$, this can be seen by calculating the degree of the Hermitian critical ideal of different points $(\mathbf{u}, \mathbf{v}) \in \mathbb{C}^6$. This value is lower than $3^3 \cdot 11 = 297$ predicted by Proposition 4.3.3.

Related to these last computations we state the following.

Conjecture 4.3.7. *Let* $2 \leq n \in \mathbb{N}$ *and*

$$X = V(z_1^3 + z_2^3 + \ldots + z_n^3 - 1) \subseteq \mathbb{C}^n$$

then $\mathrm{vHDdeg}(X) = 3^n(2n + 1)$.

4.4 vHDdeg of parametrized varieties

We discuss the value of $\mathrm{vHDdeg}(X)$ when X is a parametrized variety. As in Subsection 4.3, the results obtained in this section are easily achieved thanks to the clear symmetries of the problem given by the complex approach. Again, by using the notation conv we consider the convex hull of a subset of a vector space.

Let $X \subseteq \mathbb{C}^n$ be an algebraic variety parametrized by polynomials where m of them are generic of degree d and the remaining are generic of degree less or equal to d.

We recall that if X is real then it holds $\mathrm{EDdegree}(X) \leq (2d - 1)^m$ and equality holds for generic parametrization.

On the other hand, for a complex variety the Bézout Theorem applied to the system of polynomial equations $\nabla_{\mathbf{z},\mathbf{w}} D_{\mathbf{u},\mathbf{v}}(\mathbf{z}, \mathbf{w}) = \mathbf{0}$ yields the bound

$$\mathrm{vHDdeg}(X) \leq (2d - 1)^{2m}. \tag{4.4.1}$$

We find a new bound for $\mathrm{vHDdeg}(X)$ with the following result.

Proposition 4.4.1. *Let X be a generic algebraic variety parametrized by n polynomials of degree d in m variables, then*

$$\mathrm{vHDdeg}(X) \leq \sum_{k=0}^{m} \binom{m}{k}^2 d^{2(m-k)}(d-1)^{2k}. \tag{4.4.2}$$

Proof. Let m polynomials be generic of degree d and the remaining be generic of degree less or equal d. The number of solutions of the system $\nabla_{\mathbf{z},\mathbf{w}} D_{\mathbf{u},\mathbf{v}}(\mathbf{z},\mathbf{w}) = \mathbf{0}$ is bounded using the Bernstein–Kushnirenko Theorem by the value

$$\mathrm{MV}_{2m}(\underbrace{\triangle_{\mathcal{Z}}, \ldots \triangle_{\mathcal{Z}}}_{m}, \underbrace{\triangle_{\mathcal{W}}, \ldots, \triangle_{\mathcal{W}}}_{m}).$$

The first m arguments of this mixed volume come from the m equations $\nabla_{\mathbf{z}} D_{\mathbf{u},\mathbf{v}}(\mathbf{z},\mathbf{w}) = \mathbf{0}$ and are all the same convex hull $\triangle_{\mathcal{Z}} \coloneqq \operatorname{conv} \mathcal{Z} \subseteq \mathbb{R}^{2m}$ and the second m arguments come from the m equations $\nabla_{\mathbf{w}} D_{\mathbf{u},\mathbf{v}}(\mathbf{z},\mathbf{w}) = \mathbf{0}$ and are all the same convex hull $\triangle_{\mathcal{W}} \coloneqq \operatorname{conv} \mathcal{W} \subseteq \mathbb{R}^{2m}$ where we used the two sets $\mathcal{Z} \coloneqq \{(\mathbf{s},\mathbf{t}) \in (\mathbb{N}^m)^2 \mid \|\mathbf{s}\|_1 \leq d-1, \|\mathbf{t}\|_1 \leq d\}$ and $\mathcal{W} \coloneqq \{(\mathbf{t},\mathbf{s}) \mid (\mathbf{s},\mathbf{t}) \in \mathcal{Z}\}$, respectively. Denote the convex hulls

$$\triangle_{1,0} \coloneqq \operatorname{conv}(\triangle \times \{\mathbf{0}\}) \subseteq \mathbb{R}^{2m} \qquad \text{and} \qquad \triangle_{0,1} \coloneqq \operatorname{conv}(\{\mathbf{0}\} \times \triangle) \subseteq \mathbb{R}^{2m},$$

where $\triangle \coloneqq \{\mathbf{s} \in \mathbb{N}^m \mid \|\mathbf{s}\|_1 \leq 1\} \subseteq \mathbb{R}^m$, so that

$$\triangle_{\mathcal{Z}} = (d-1)\triangle_{1,0} + d\triangle_{0,1} \quad \text{and} \quad \triangle_{\mathcal{W}} = d\triangle_{1,0} + (d-1)\triangle_{0,1}.$$

Since the mixed volume is symmetric multilinear we rewrite $\mathrm{MV}_{2m}(\triangle_{\mathcal{Z}}, \ldots \triangle_{\mathcal{Z}}, \triangle_{\mathcal{W}}, \ldots, \triangle_{\mathcal{W}})$ as

$$\sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{m} \binom{m}{j} (d-1)^{m+k-j} d^{m+j-k} \, \mathrm{MV}_{2m}(\underbrace{\triangle_{1,0}, \ldots, \triangle_{1,0}}_{k+j}, \underbrace{\triangle_{0,1}, \ldots, \triangle_{0,1}}_{2m-k-j}).$$

Now, [Ewa96, Lemma 4.5] implies that the mixed volumes of the terms with $k+j \neq 2m-k-j$, or equivalently $k \neq m-j$, vanish, thus the formula simplifies

$$\sum_{k=0}^{m} \binom{m}{k}^2 (d-1)^{2k} d^{2(m-k)} \, \mathrm{MV}_{2m}(\underbrace{\triangle_{1,0}, \ldots, \triangle_{1,0}}_{m}, \underbrace{\triangle_{0,1}, \ldots, \triangle_{0,1}}_{m}).$$

Applying the same lemma for the remaining terms we get

$$\mathrm{MV}_{2m}(\underbrace{\triangle_{1,0}, \ldots, \triangle_{1,0}}_{m}, \underbrace{\triangle_{0,1}, \ldots, \triangle_{0,1}}_{m}) = \mathrm{MV}_m(\triangle, \ldots, \triangle)\, \mathrm{MV}_m(\triangle, \ldots, \triangle) = 1$$

and the claim follows. □

Remark 4.4.2. The bound of equation (4.4.2) is always better than the bound of equation (4.4.1). In fact, we bound the value of equation (4.4.2) by

$$\begin{aligned}\sum_{k=0}^{m} \binom{m}{k}^2 d^{2(m-k)}(d-1)^{2k} &\leq \left(\sum_{k=0}^{m} \binom{m}{k} d^{m-k}(d-1)^k \right)^2 \\ &= ((d+(d-1))^m)^2 = (2d-1)^{2m},\end{aligned}$$

where in particular, the last formula of the chain above is exactly equation (4.4.1).

We compute some values of equation (4.4.2) in Table 4.2. For the case $d = 1$ clearly always holds vHDdeg $= 1$.

	$m=1$	$m=2$	$m=3$
d	$2d^2-2d+1$	$6d^4-12d^3+10d^2-4d+1$	$20d^6-60d^5+78d^4-56d^3+24d^2-6d+1$
2	$\hat{5}$	33	245
3	$\hat{13}$	241	5005
4	$\hat{25}$	913	37225
5	$\hat{41}$	2481	167321

Table 4.2: Values of equation (4.4.2) for different d and m. The values with a hat are the ones we checked to be sharp.

One more time, the symmetric degrees in $\mathbf{z}$ and $\mathbf{w}$ is not the only property shared by the polynomials of the system $\nabla_{\mathbf{z},\mathbf{w}} D_{\mathbf{u},\mathbf{v}}(\mathbf{w},\mathbf{z}) = \mathbf{0}$. In fact, this system is invariant under the action of the map $*$ and we do not know if the true value of vHDdeg(X) could be lower. Moreover, as already said, the number max HDdeg(X) could be lower than vHDdeg, see the next example.

Example 4.4.3. (Parabola). The parabola

$$X = V\left(z_2 - z_1^2\right) \subseteq \mathbb{C}^2$$

satisfies vHDdeg$(X) = 5$ as predicted by Proposition 4.4.1 and HDdeg$(X) = \{1,3\}$. The value of vHDdeg(X) can be seen by computing the degree of the critical ideal of X for general parameters $(\mathbf{u},\mathbf{v})$. This ideal is generated by the polynomials of the system

$$\begin{cases} p_1(\mathbf{z},\mathbf{w}) = z_2 - z_1^2 = 0 \\ \bar{p}_1(\mathbf{w},\mathbf{z}) = w_2 - w_1^2 = 0 \\ p_2(\mathbf{z},\mathbf{w}) = z_1 - u_1 + 2w_1(z_2 - u_2) = 0 \\ \bar{p}_2(\mathbf{w},\mathbf{z}) = w_1 - v_1 + 2z_1(w_2 - v_2) = 0 \end{cases}$$

where the upper two polynomials are given by the definition of X and the bottom two polynomials are given by the determinant of the 2×2 matrix obtained from the perpendicularity condition. In particular, directly follows the containment HDdeg$(X) \subseteq \{1,3,5\}$. We now compute the true values of HDdeg(X).

Set $\mathbf{v} = \bar{\mathbf{u}}$. Consider the bijective parametrization of the parabola

$$\begin{aligned} \psi\colon \mathbb{C} &\to X = V(z_2 - z_1^2) \subseteq \mathbb{C}^2 \\ z &\mapsto \quad (z, z^2) \end{aligned}$$

so that critical points of the Hermitian distance satisfy the equation

$$\begin{aligned} \partial_z \|\psi(z) - \mathbf{u}\|_{\mathbb{C}}^2 &= \partial_z \left[(z-u_1)(\bar{z}-\bar{u}_1) + (z^2-u_2)(\bar{z}^2-\bar{u}_2)\right] \\ &= (\bar{z}-\bar{u}_1) + 2z(\bar{z}^2-\bar{u}_2) = 0. \end{aligned}$$

The system we obtain introducing $\mathbf{w}$ is

$$\begin{cases} p(z,w) = (z-u_1) + 2w(z^2-u_2) = 0 \\ \bar{p}(w,z) = (w-\bar{u}_1) + 2z(w^2-\bar{u}_2) = 0 \end{cases}$$

and it is equivalent to applying substitution to the Hermitian critical ideal and setting $\mathbf{v} = \bar{\mathbf{u}}$. The matrix representing the Hermitian Killing form, see Subsection 3.1, with respect to the basis

$$\{[1],[z],[w],[zw],[w^2]\}$$

of the quotient $\mathbb{C}[z,w]/\langle p, p^*\rangle$ is

$$\mathcal{K}^1_{\mathbb{C}} = \begin{bmatrix} 5 & -\frac{u_1}{2u_2} & -\frac{\bar{u}_1}{2\bar{u}_2} & -2 & \frac{16|u_2|^2\bar{u}_2+\bar{u}_1^2}{4\bar{u}_2^2} \\ * & -2 & \frac{16|u_2|^2\bar{u}_2+\bar{u}_1^2}{4\bar{u}_2^2} & \frac{8u_1\bar{u}_2+\bar{u}_1}{4\bar{u}_2} & \frac{-\bar{u}_1^3-6u_1\bar{u}_2^2}{8\bar{u}_2^2} \\ * & * & -2 & \frac{8\bar{u}_1u_2+u_1}{4u_2} & \frac{8u_1\bar{u}_2+\bar{u}_1}{4\bar{u}_2} \\ * & * & * & 4|u_2|^2+1 & \frac{-2|u_1|^2\bar{u}_2-32|u_2|^2\bar{u}_2-\bar{u}_1^2}{8\bar{u}_2^2} \\ * & * & * & * & 4|u_2|^2+1 \end{bmatrix}$$

where we assumed without loss of generality $u_2 \neq 0$. In particular, the second diagonal entry is negative, thus this matrix possesses at least one negative eigenvalue and from Corollary 3.1.11 we obtain $\max \mathrm{HDdeg}(X) \leq 4-1 = 3$ and since by testing points $\mathbf{u}$ on the real plane we get both one or three critical points it follows $\mathrm{HDdeg}(X) = \{1,3\}$.

We note that, when $u_2 = 0$ and $u_1 \neq 0$ then the number of solutions of the Hermitian critical ideal drops from 5 to 3, while for the origin $\mathbf{u} = \mathbf{0}$, the system simplifies the solutions are the origin of $\mathbb{C}^4$ and the set of infinite cardinality $\{(z, z^2, -1/2z, (-1/2z)^2) \in \mathbb{C}^4 \mid z \in \mathbb{C}\}$.

We will continue to study this problem in Example 4.6.5.

4.5 HDdeg of conics

In this subsection we focus on the case of conics. We have already seen in Subsection 4.3 that the vHDdeg for generic conics is 8. We try to completely address this problem here by also computing the HDdeg of projective and affine conics.

We start by considering projective conics, so that we set

$$X = V(az_1^2 + 2\cdot bz_1z_2 + cz_2^2 + 2\cdot dz_1z_3 + 2\cdot ez_2z_3 + fz_3^2) \subseteq \mathbb{P}^2$$

where $a,b,c,d,e,f \in \mathbb{C}$. In particular, X is given by the zero set of the symmetric bilinear form

$$\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \mathbf{z}^T A \mathbf{z}$$

that is uniquely determined by the complex symmetric matrix A.

Let us recall a well known fact on symmetric complex matrices that can be found in [HJ13, Corollary 4.4.4 (c)].

Proposition 4.5.1. *Let A be a complex symmetric matrix, then there exists a unitary matrix U such that the product UAU^T is a nonnegative diagonal matrix whose diagonal entries are the singular values of A, in any desired order.*

We will use this result to limit the number of cases we have to address.

Theorem 4.5.2. *Let X be a projective conic and let A be its representative complex symmetric matrix.*

If A is singular then

i) $\mathrm{vHDdeg}(X) = 1$ *and* $\mathrm{HDdeg}(X) = \{1\}$ *if* $\mathrm{rk}(A) = 1$,

ii) $\mathrm{vHDdeg}(X) = 2$ *and* $\mathrm{HDdeg}(X) = \{2\}$ *if* $\mathrm{rk}(A) = 2$.

If A is non singular then

i) $\mathrm{vHDdeg}(X) = 2$ *and* $\mathrm{HDdeg}(X) = \{2\}$ *if A has only one singular value,*

ii) $\mathrm{vHDdeg}(X) = 6$ *if A has two different singular values,*

iii) $\mathrm{vHDdeg}(X) = 8$ *if A has three different singular values.*

Proof. By Lemma 4.2.15, since the Hermitian inner product is invariant under the action of unitary matrices on the complex vector space, we can study of the Hermitian distance problem for X on the variety $U \cdot X$ where U is a 3×3 unitary matrix. We combine this fact with Proposition 4.5.1 and choose a 3×3 unitary matrix U such that relabeling the parameters and applying a change of variables, the quadratic form representing $U \cdot X$ is

$$\mathbf{z}^T A \mathbf{z} = \mathbf{z}^T U^T \bar{U} A U^H U \mathbf{z} = (U\mathbf{z})^T \begin{bmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & f \end{bmatrix} (U\mathbf{z})$$

where $0 \leq f \leq c \leq a \in \mathbb{R}$ are the singular values of the starting matrix A. Moreover, having avoided the trivial case in which the singular values are all zero, without loss of generality using Lemma 4.2.10 and applying the symmetric matrix $\sqrt{a} I_3$, we replace X with the variety $\sqrt{a} U \cdot X$, so that we can also set $a = 1$.

Suppose A is singular. If two singular values are equal to zero, say $c = f = 0$ without loss of generality, then $X = V(z_1^2)$ is an hyperplane and point i) follows since we already discussed this type of problems in Subsection 4.2. If only one singular values is equal to zero, say $f = 0$, then $z_1^2 + c z_2^2 = (z_1 + i\sqrt{c} z_2)(z_1 - i\sqrt{c} z_2)$ so that X is the union of two lines and point ii) follows similarly to point i).

Suppose now that A is non singular. The values of $\mathrm{vHDdeg}(X)$ for all points i), ii) and iii) follow by computing the degree of the Hermitian critical ideal of X for general parameters $(\mathbf{u}, \mathbf{v})$ on Macaulay2. The value of $\mathrm{HDdeg}(X)$ for point i) follows from the properties of the HDdeg. □

Note that, applying this transformation of Proposition 4.5.1, and assuming all three parameters a, c, f to be positive, the variety $U \cdot X$ can be two-to-one parametrized by the map

$$\begin{aligned} \psi \colon \mathbb{C}^2 &\longrightarrow U \cdot X \subseteq \mathbb{C}^3 \\ (z_1, z_2) &\mapsto \left(\frac{z_1^2 + z_2^2}{2\sqrt{a}}, \frac{i(z_1^2 - z_2^2)}{2\sqrt{c}}, \frac{iz_1 z_2}{\sqrt{f}} \right) \end{aligned}$$

and thus, using Proposition 4.4.1, we get the bound $\mathrm{vHDdeg}(X) \leq 33$. However, we have seen in the last result that even for general parameters the true value is much lower. With this result we have almost totally characterized the Hermitian distance problem on projective conics.

Example 4.5.3. The variety

$$X = V(z_1^2 + z_2^2 + f z_3^2) \subseteq \mathbb{P}^2$$

where $f \in \mathbb{R}$ is positive satisfies $\mathrm{vHDdeg}(X) = 6$. The value of $\mathrm{vHDdeg}(X)$ can be seen by computing the degree of the Hermitian critical ideal of X for general parameters $(\mathbf{u}, \mathbf{v})$ that is

$$\left\langle \begin{matrix} z_1^2+z_2^2+fz_3^2, & \text{2-minors of} \begin{bmatrix} z_1 - u_1 & z_2 - u_2 & z_3 - u_3 \\ w_1 & w_2 & fw_3 \end{bmatrix} \\ w_1^2+w_2^2+fw_3^2, & \text{2-minors of} \begin{bmatrix} w_1 - v_1 & w_2 - v_2 & w_3 - v_3 \\ z_1 & z_2 & fz_3 \end{bmatrix} \end{matrix} \right\rangle : (\langle z_1, z_2, z_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle)^\infty$$

and in particular directly follows the containment $\mathrm{HDdeg}(X) \subseteq \{2, 4, 6\}$.

Using the parametrization described above, if we relabel $1/\sqrt{f}$ as simply f, critical points satisfy the equations

$$\begin{aligned} \partial_{z_1} \|\psi(z_1, z_2) - \mathbf{u}\|_{\mathbb{C}}^2 &= 2z_1(\bar{z}_1^2 + \bar{z}_2^2 - \bar{u}_1) + 2iz_1(-i(\bar{z}_1^2 - \bar{z}_2^2) - \bar{u}_2) + ifz_2(-if\bar{z}_1\bar{z}_2 - \bar{u}_3) \\ &= 4z_1\bar{z}_1^2 + f^2\bar{z}_1 z_2 \bar{z}_2 - 2(\bar{u}_1 + i\bar{u}_2)z_1 - if\bar{u}_3 z_2 = 0 \end{aligned}$$

and

$$\begin{aligned} \partial_{z_2} \|\psi(z_1, z_2) - \mathbf{u}\|_{\mathbb{C}}^2 &= 2z_2(\bar{z}_1^2 + \bar{z}_2^2 - \bar{u}_1) - 2iz_2(-i(\bar{z}_1^2 - \bar{z}_2^2) - \bar{u}_2) + ifz_1(-if\bar{z}_1\bar{z}_2 - \bar{u}_3) \\ &= 4z_2\bar{z}_2^2 + f^2 z_1\bar{z}_1\bar{z}_2 - 2(\bar{u}_1 - i\bar{u}_2)z_2 - if\bar{u}_3 z_1 = 0. \end{aligned}$$

We can apply a change of variables on the point $\mathbf{u}$ and relabel the parameter f^2 in order to also collect the scalar 4 so that solving the two equations above is equivalent to solving the system

$$\begin{cases} p_1(\mathbf{z}, \bar{\mathbf{z}}) = z_1\bar{z}_1^2 + f^2\bar{z}_1 z_2\bar{z}_2 - \bar{u}_1 z_1 - \bar{u}_3 z_2 = 0 \\ p_2(\mathbf{z}, \bar{\mathbf{z}}) = z_2\bar{z}_2^2 + f^2 z_1\bar{z}_1\bar{z}_2 + \bar{u}_2 z_2 - \bar{u}_3 z_1 = 0 \end{cases}$$

We now consider affine conics, so that we set

$$X = V(az_1^2 + 2 \cdot bz_1z_2 + cz_2^2 + 2 \cdot dz_1 + 2 \cdot ez_2 + f) \subseteq \mathbb{C}^2$$

where $a, b, c, d, e, f \in \mathbb{C}$. Similarly to the projective case, the variety X is uniquely determined by the complex symmetric matrix A just studied.

Theorem 4.5.4. *Let X be an affine conic and let A be its representative complex symmetric matrix and let A_1 be its 2×2 leading principal submatrix.*

If A is singular

i) $\mathrm{vHDdeg}(X) = 1$ *and* $\mathrm{HDdeg}(X) = \{1\}$ *if* $rk(A) = 1$,

ii) $\mathrm{vHDdeg}(X) = 2$ *and* $\mathrm{HDdeg}(X) = \{2\}$ *if* $rk(A) = 2$.

If A is non singular then

i) $\mathrm{vHDdeg}(X) = 5$ *and* $\mathrm{HDdeg}(X) = \{1, 3\}$ *if A_1 is singular,*

ii) $\mathrm{vHDdeg}(X) = 6$ *and* $\mathrm{HDdeg}(X) = \{2, 4\}$ *if A_1 is non singular and has two equal singular values,*

iii) $\mathrm{vHDdeg}(X) = 8$ *and* $\{2, 4\} \subseteq \mathrm{HDdeg}(X) \subseteq \{2, 4, 6\}$ *if A_1 is non singular and has two different singular values.*

Proof. Let $X = V(az_1^2 + 2bz_1z_2 + cz_2^2 + 2dz_1 + 2ez_2 + f)$ with $a, b, c, d, e, f \in \mathbb{C}$. If A is singular then

$$az_1^2 + 2bz_1z_2 + cz_2^2 + 2dz_1 + 2ez_2 + f = (\alpha_1 z_1 + \beta_1 z_2 + \gamma_1)(\alpha_2 z_1 + \beta_2 z_2 + \gamma_2)$$

for some $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}$ for $k = 1, 2$ so that X is the union of two lines. Moreover, the two line coincide exactly when $\mathrm{rk}(A) = 1$. We already discussed this kind of problems in Subsection 4.2, thus we get points i) and ii).

In general, similarly to the start of the proof of Theorem 4.5.2, by relabeling the parameters and applying a change of variables, we replace X with the variety $U \cdot X$, where

$$U = \begin{bmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$$

with U_1 a 2×2 unitary matrix such that

$$\begin{bmatrix} z_1 & z_2 & 1 \end{bmatrix} A \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix}^T U^T \bar{U} A U^H U \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix} = \left(U \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix} \right)^T \begin{bmatrix} a & 0 & d \\ 0 & c & e \\ d & e & f \end{bmatrix} \left(U \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix} \right)$$

where $0 \le c \le a \in \mathbb{R}$ are the singular value of the starting submatrix A_1 and $d, e, f \in \mathbb{C}$. With these choices it holds

$$\det(A) = \det \begin{bmatrix} a & 0 & d \\ 0 & c & e \\ d & e & f \end{bmatrix} = acf - ae^2 - cd^2,$$

thus if A is non singular at least one parameter among a and c must be non zero. Without loss of generality say $a \neq 0$. Using Lemma 4.2.10 we can translate by the vector $(d/a, 0) \in \mathbb{C}^2$ to set $d = 0$ and scale by the factor $\sqrt{a} \in \mathbb{C}$, that is the same of applying the symmetric matrix

$$\begin{bmatrix} \sqrt{a} I_2 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{to the vector} \quad \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix},$$

to set $a = 1$. In the end, we are left with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & e \\ 0 & e & f \end{bmatrix}$$

where $0 \leq c \leq 1$ and $e, f \in \mathbb{C}$. Since translations and scaling do not affect the ratio of the singular values of A_1 and in particular if either $\det(A_1) = c$ vanishes or not, we consider two mutually exclusive cases depending on the parameter c, case (1) $c = 0$ and case (2) $c \neq 0$.

In case (1), since we are in the hypothesis $\det(A) = -e^2 \neq 0$ then it holds $e \neq 0$. Thus, we translate by the vector $\mathbf{b} = (0, f/2e) \in \mathbb{C}^2$ to set $f = 0$ and scale by the factor $-1/2e \in \mathbb{C}$, that is the same of applying the symmetric matrix

$$\begin{bmatrix} -\frac{1}{2e} I_2 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{to the vector} \quad \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix},$$

so that we can multiply the equation obtained by $1/4e^2 \in \mathbb{C}$ to set $e = -1/2$ and replace our variety with $X = V(z_1^2 - z_2)$. In the end, point i) follows from Example 4.4.3.

In case (2) we apply the translation by the vector $(0, e/c) \in \mathbb{C}^2$ to set $e = 0$. Moreover, since we are in the hypothesis $\det(A) = cf \neq 0$ then it holds $f \neq 0$. Thus, we scale by the factor $i\sqrt{c/f} \in \mathbb{C}$, that is the same of applying the symmetric matrix

$$\begin{bmatrix} i\sqrt{c/f} I_2 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{to the vector} \quad \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix},$$

so that we can multiply the equation obtained by $-c/f \in \mathbb{C}$ to set $f = -c$ and replace our variety with $X = V(z_1^2 + cz_2^2 - c)$ where $0 < c \leq 1$. If $c = 1$ that means the starting submatrix A_1 has two equal singular values and point ii) follows from Example 4.2.16. If $c < 1$ that means the starting submatrix A_1 has two different singular values and point iii) follows from Example 4.3.2. □

The case iii) of A non singular remains still incomplete.

4.6 HD correspondence and duality

Let $X \subseteq \mathbb{V}$ be an algebraic variety, we define the *virtual Hermitian Distance correspondence* (vHD correspondence) of X denoted $\mathrm{v}\mathcal{H}_X$ as the subvariety of $\mathbb{V}^2 \times \mathbb{V}^2$ defined by the Hermitian critical ideal of expression (4.2.1) in the polynomial ring $\mathbb{C}[\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}]$. In particular, it holds the containment $\mathrm{v}\mathcal{H}_X \subseteq X \times \overline{X} \times \mathbb{V}^2$. The following theorem is the generalization of Lemma 4.2.1.

Theorem 4.6.1. *Let $X \subseteq \mathbb{V}$ be an algebraic variety of codimension c, then the vHD correspondence $\mathrm{v}\mathcal{H}_X$ is a $2n$ dimensional irreducible variety. The projection $\pi_{X \times \overline{X}}$ on the first $2n$ components is a vector bundle of rank $2c$. The projection $\pi_{\mathbb{V}^2}$ on the second $2n$ components has generic fibers of cardinality equal to* $\mathrm{vHDdeg}(X)$. *Moreover,* $\min \mathrm{HDdeg}(X)$ *is positive.*

Proof. The first part follows as [Dra+16, Theorem 4.1]. The only remaining part is the fact that the projection into the second component is a dominant map.

Firstly note that the double diagonal $\Delta(X) \coloneqq \{(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{z}, \bar{\mathbf{z}}) \in \mathbb{V}^2 \times \mathbb{V}^2 \mid \mathbf{z} \in X\}$ is contained in $\mathrm{v}\mathcal{H}_X$. Fix a point $\mathbf{z} \in X_{\mathrm{reg}}$ so that $\mathrm{v}\mathcal{H}_X$ is smooth at the point $(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{z}, \bar{\mathbf{z}})$. The tangent space $T_{(\mathbf{z},\bar{\mathbf{z}},\mathbf{z},\bar{\mathbf{z}})}\mathrm{v}\mathcal{H}_X$ contains both the tangent space $T_{(\mathbf{z},\bar{\mathbf{z}},\mathbf{z},\bar{\mathbf{z}})}\Delta(X) = \Delta(T_{\mathbf{z}}X)$ and $\{\mathbf{0}\} \times (T_{\bar{\mathbf{z}}}\overline{X})^{\perp_{\mathbb{R}}} \times (T_{\mathbf{z}}X)^{\perp_{\mathbb{R}}}$. The image of the derivative of the composition of $\pi_{\mathbb{V}^2}$ with the projection $\pi_{\mathbb{V}} \colon \mathbb{V}^2 \to \mathbb{V}$ onto the first n components contains both $T_{\mathbf{z}}X$ and $(T_{\bar{\mathbf{z}}}\overline{X})^{\perp_{\mathbb{R}}}$. Since $(T_{\bar{\mathbf{z}}}\overline{X})^{\perp_{\mathbb{R}}} = (T_{\mathbf{z}}X)^{\perp_{\mathbb{C}}}$ these spaces intersect trivially and span all of $\mathbb{V}$, thus the derivative of $\pi_{\mathbb{V}} \circ \pi_{\mathbb{V}^2}$ at $(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{z}, \bar{\mathbf{z}})$ is surjective and $\pi_{\mathbb{V}} \circ \pi_{\mathbb{V}^2}$ is dominant. □

Remark 4.6.2. The *ED correspondence* is defined in a similar way in [Dra+16] by considering the zero locus of the critical ideal (4.2.2) of the EDdegree in the polynomial ring $\mathbb{C}[\mathbf{z}, \mathbf{u}]$.

The next corollary provide a parametrization of $\mathrm{v}\mathcal{H}_X$ when the variety X is parametrized itself.

Corollary 4.6.3. *If X is (uni-)rational then so is the vHD correspondence $\mathrm{v}\mathcal{H}_X$.*

Proof. Follows as [Dra+16, Corollary 4.2]. However, we provide the proof which contains the formula for the parametrization.

Let $\psi \colon \mathbb{C}^m \to \mathbb{C}^n$ be a rational map that parametrizes X, where $m = \dim(X) = n - c$. Its Jacobian $J(\psi)$ is an $n \times m$ matrix of rational functions in the standard coordinates $t_1, \dots, t_m$ on $\mathbb{C}^m$. The columns of $J(\psi)$ span the tangent space of X at the point $\psi(\mathbf{z})$ for generic $\mathbf{z} \in \mathbb{C}^m$. The left kernel of $J(\psi)$ is a linear space of dimension c. We can write down a basis $\{\beta_1(\mathbf{z}), \dots, \beta_c(\mathbf{z})\}$ of that kernel by applying Cramer's rule to the matrix $J(\psi)$. In particular, β_k for $k = 1, \dots, c$ will also be rational functions in the z_j for $j = 1, \dots, m$. In the end, the map

$$\begin{aligned} &\mathbb{C}^{2m} \times \mathbb{C}^{2c} \to \mathrm{v}\mathcal{H}_X \\ &(\mathbf{z}, \mathbf{w}, \mathbf{t}, \mathbf{s}) \mapsto \left(\psi(\mathbf{z}), \bar{\psi}(\mathbf{w}), \psi(\mathbf{z}) + \sum_{k=1}^{c} t_k \bar{\beta}_k(\mathbf{w}), \bar{\psi}(\mathbf{w}) + \sum_{k=1}^{c} s_k \beta_k(\mathbf{z}) \right) \end{aligned}$$

is a parametrization of $\mathrm{v}\mathcal{H}_X$. In the end, it is esay to see that this parametrization is birational if and only if ψ is birational, in particular if ψ admits a rational inverse, by solving two linear system we can find a rational inverse for the parametrization. □

If X is an affine cone, we consider the closure of the image

$$\begin{aligned} \mathrm{v}\mathcal{H}_X \cap ((\mathbb{V} \setminus \{\mathbf{0}\})^2 \times \mathbb{V}^2) &\longrightarrow (\mathbb{P}\mathbb{V})^2 \times \mathbb{V}^2 \\ ((\mathbf{z}, \mathbf{w}), (\mathbf{u}, \mathbf{v})) &\longmapsto (([\mathbf{z}], [\mathbf{w}]), (\mathbf{u}, \mathbf{v})) \end{aligned}$$

This closure is called the *projective virtual Hermitian Distance correspondence* (projective vHD correspondence) of X, and it is denoted $\mathrm{v}\mathcal{PH}_X$. It possesses the properties resumed in the following theorem.

Theorem 4.6.4. *Let $X \subseteq \mathbb{V}$ be an irreducible affine cone, then the projective vHD correspondence $\mathrm{v}\mathcal{PH}_X$ is a $2n$ dimensional irreducible variety inside $(\mathbb{P}\mathbb{V})^2 \times \mathbb{V}^2$. It is the zero set of the Hermitian critical ideal in (4.2.4). The projection $\pi_{X \times \overline{X}}$ is a vector bundle of rank $2c + 2$. The projection $\pi_{\mathbb{V}^2}$ has generic fibers of cardinality equal to* vHDdeg(X).

Proof. Follows as [Dra+16, Theorem 4.4]. The only non trivial point is the fact that $\mathrm{v}\mathcal{PH}_X$ is the zero set of the Hermitian critical ideal of expression (4.2.4).

First, if $(\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}) \in \mathrm{v}\mathcal{H}_X$, then $([\mathbf{z}], [\mathbf{w}], \mathbf{u}, \mathbf{v})$ lies in the zero locus of the Hermitian critical ideal in (4.2.4). Conversely, if $([\mathbf{z}], [\mathbf{w}], \mathbf{u}, \mathbf{v})$ lies in the variety of that ideal, then there exist unique $\mu_{\mathbf{z}}, \mu_{\mathbf{w}} \in \mathbb{C}$ such that $(\mu_{\mathbf{z}}\mathbf{z}, \mu_{\mathbf{w}}\mathbf{w}, \mathbf{u}, \mathbf{v}) \in \mathrm{v}\mathcal{H}_X$. If both $\mu_{\mathbf{z}}$ and $\mu_{\mathbf{w}}$ are non zero, then this means that $([\mathbf{z}], [\mathbf{w}], \mathbf{u}, \mathbf{v})$ lies in $\mathrm{v}\mathcal{PH}_X$. Suppose $\mu_{\mathbf{z}} = 0$ and $\mu_{\mathbf{w}} \neq 0$, then $\mathbf{u} \perp_{\mathbb{R}} T_{\mathbf{w}}\overline{X}$ and $\mathbf{v} - \mathbf{w} \perp_{\mathbb{R}} T_{\mathbf{z}}X$, hence $([\varepsilon\mathbf{z}], [\mathbf{w}], \mathbf{u} + \varepsilon\mathbf{z}, \mathbf{v}) \in \mathrm{v}\mathcal{PH}_X$ for any non zero $\varepsilon \in \mathbb{C}$, where we used the fact $T_{\mathbf{z}}X = T_{\varepsilon\mathbf{z}}X$. The limit of $([\varepsilon\mathbf{z}], [\mathbf{w}], \mathbf{u} + \varepsilon\mathbf{z}, \mathbf{v})$ for $\varepsilon \to 0$ equals $([\mathbf{z}], [\mathbf{w}], \mathbf{u}, \mathbf{v})$, so the latter point still lies in the projective ED correspondence $\mathrm{v}\mathcal{PH}_X$. Similarly arguments apply in general if $\mu_{\mathbf{z}}\mu_{\mathbf{w}} = 0$. □

Since we are interested in critical points, we define the *Hermitian Distance correspondence* (HD correspondence) $\mathcal{H}_X$ as the image of the projection

$$\begin{aligned} \mathrm{v}\mathcal{H}_X \cap \{(\mathbf{z}, \bar{\mathbf{z}}, \mathbf{u}, \bar{\mathbf{u}}) \mid (\mathbf{z}, \mathbf{u}) \in \mathbb{V}^2\} &\to \mathcal{H}_X \subseteq \mathbb{V} \times \mathbb{V} \\ (\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}) &\mapsto (\mathbf{z}, \mathbf{u}) \end{aligned}$$

and similarly the *projective Hermitian Distance correspondence* (projective HD correspondence) $\mathcal{PH}_X$ if X is a projective variety.

In particular, X is parametrized by ψ, from Corollary 4.6.3 a parametrization of the HD correspondence is given by the map

$$\begin{aligned} \mathbb{C}^m \times \mathbb{C}^c &\to \mathcal{H}_X \subseteq \mathbb{V} \times \mathbb{V} \\ (\mathbf{z}, \mathbf{t}) &\mapsto \left(\psi(\mathbf{z}), \psi(\mathbf{z}) + \sum_{k=1}^{c} t_k \bar{\beta}_k(\bar{\mathbf{z}})\right) \end{aligned}$$

We apply these concepts on a simple example.

Example 4.6.5. (Parabola) The parametrization for the vHD correspondence of the parabola $X = V(z_2 - z_1^2) \subseteq \mathbb{C}^2$ given by Corollary 4.6.3 is

$$\begin{aligned} &\mathbb{C}^2 \times \mathbb{C}^2 \to \mathcal{H}_X \subseteq \mathbb{C}^4 \times \mathbb{C}^4 \\ &(z, w, s, t) \mapsto \big((z, z^2, w, w^2), (z + 2sw, z^2 - s, w + 2tz, w^2 - t)\big) \end{aligned}$$

Thus the HD correspondence can be parametrized by

$$\begin{aligned} &\mathbb{C} \times \mathbb{C} \to \mathcal{H}_X \subseteq \mathbb{C}^2 \times \mathbb{C}^2 \\ &(z, s) \mapsto ((z, z^2), (z + 2s\bar{z}, z^2 - s)) \end{aligned}$$

We will continue to study this problem in Example 5.2.4.

We now briefly discuss the notion of dual variety. For a more general introduction on dual varieties that does not need the presence of the inner product see [GKZ94, Section 1].

Consider an affine cone $X \subseteq \mathbb{V}$, or equivalently the corresponding projective variety $X \subseteq \mathbb{P}\mathbb{V}$ and let $\mathbb{V}^\vee$ be the dual complex vector space of $\mathbb{V}$. Such a variety possesses a dual variety $X^\vee \subseteq \mathbb{V}^\vee$, which is defined as the following Zariski closure

$$X^\vee := \overline{\{\mathbf{s} \in \mathbb{V}^\vee \mid \exists\, \mathbf{z} \in X_{\mathrm{reg}} \colon \mathbf{s} \perp_{\mathbb{R}} T_{\mathbf{z}}X\}}^{Z} \subseteq \mathbb{V}^\vee.$$

Clearly the conjugate operation and the dual commute in the sense that it holds $\overline{X^\vee} = \overline{X}^\vee$. Moreover note that, even if X is an affine subspace, then the perpendicular subspace $X^{\perp_{\mathbb{C}}}$ is not contained in the dual space $X^\vee$. However, it holds the containment $X^{\perp_{\mathbb{C}}} \subseteq \overline{X}^\vee$. We also recall that by the Biduality Theorem it holds $(X^\vee)^\vee = X$, see [GKZ94, Theorem 1.1].

A classical known fact is that in the presence of a bilinear form, that is $q(\ ,\ ^-)$, there exists a canonical isomorphism between $\mathbb{V}$ and $\mathbb{V}^\vee$. In particular we can think of $X^\vee$ as a variety in $\mathbb{V}$.

We recall the *conormal variety* is the Zariski closure

$$\mathcal{N}_X := \overline{\{(\mathbf{z}, \mathbf{s}) \in \mathbb{V}^2 \mid \mathbf{z} \in X_{\mathrm{reg}},\ \mathbf{s} \perp_{\mathbb{R}} T_{\mathbf{z}}X\}}^{Z} \subseteq \mathbb{V}^2$$

which is the zero set of the bi-homogeneous saturation ideal

$$N_X := \left(I_X + \left\langle (c+1)\text{-minors of } \begin{bmatrix} \mathbf{s} \\ J(f) \end{bmatrix} \right\rangle\right) : \left(I_{X_{\mathrm{sing}}}\right)^\infty \subseteq \mathbb{C}[\mathbf{z}, \mathbf{s}],$$

where c is the codimension of X. Recall from Subsection 4.2 that we denote $X_\circ = X_{\mathrm{reg}} \times \overline{X}_{\mathrm{reg}}$. The following theorem characterizes the action of the dual operator on the Hermitian distance problem.

Theorem 4.6.6. *Let $X \subseteq \mathbb{V}$ be an irreducible affine cone and $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ be a point. The map $(\mathbf{z}, \mathbf{w}) \mapsto (\mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{z})$ is a bijection from the set of solutions of the Hermitian critical ideal of X of $(\mathbf{u}, \mathbf{v})$ to the set of solutions of the Hermitian critical ideal of $X^\vee$ of $(\mathbf{v}, \mathbf{u})$ that sends critical points to critical points. In particular, there hold*

$$\mathrm{vHDdeg}(X) = \mathrm{vHDdeg}(X^\vee) \qquad \text{and} \qquad \mathrm{HDdeg}(X) = \mathrm{HDdeg}(X^\vee).$$

Moreover, the map is proximity reversing for critical points.

Proof. If $(\mathbf{z}, \mathbf{w}) \in X_\circ$ is in the zero set of the Hermitian critical ideal of X, of $(\mathbf{u}, \mathbf{v})$ then $\mathbf{v} - \mathbf{w} \perp_{\mathbb{R}} T_{\mathbf{z}}X$ and hence $\mathbf{v} - \mathbf{w} \in X^\vee$ or more generally $(\mathbf{z}, \mathbf{v} - \mathbf{w}) \in \mathcal{N}_X$. Similarly, we can say $(\mathbf{w}, \mathbf{u} - \mathbf{z}) \in \mathcal{N}_{\overline{X}}$. Since $(\mathbf{u}, \mathbf{v})$ is generic, we can assume all points $(\mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{z})$ thus obtained are in $(X^\vee)_\circ$. By the Biduality Theorem, we have $\mathbf{u} - (\mathbf{u} - \mathbf{z}) = \mathbf{z} \perp_{\mathbb{R}} T_{\mathbf{v}-\mathbf{w}}X^\vee$ and similarly $\mathbf{v} - (\mathbf{v} - \mathbf{w}) \perp_{\mathbb{R}} T_{\mathbf{u}-\mathbf{z}}\overline{X}^\vee$. Thus, $(\mathbf{v} - \mathbf{w}, \mathbf{u} - \mathbf{z}) \in (X^\vee)_\circ$ is a solution of the Hermitian critical ideal of $X^\vee$ of $(\mathbf{v}, \mathbf{u})$.

Considering the case $\mathbf{v} = \bar{\mathbf{u}}$ we get the part about critical points.

Applying the same argument to $X^\vee$, since $(X^\vee)^\vee = X$, we prove bijectivity.

For the last statement, we restrict our attention to points such that $\mathbf{v} = \bar{\mathbf{u}}$ and $\mathbf{w} = \bar{\mathbf{z}}$, in particular we compute the distance by considering the first components of the tuples. Observe that it holds $\mathbf{u} - \mathbf{z} \perp_{\mathbb{R}} \mathbf{w} \in T_{\mathbf{w}}\overline{X}$, or in other words $\mathbf{u} - \mathbf{z} \perp_{\mathbb{C}} \bar{\mathbf{w}} = \mathbf{z}$ and thus

$$\|\mathbf{u} - \mathbf{z}\|_{\mathbb{C}}^2 + \|\mathbf{v} - (\mathbf{v} - \mathbf{w})\|_{\mathbb{C}}^2 = \|\mathbf{u} - \mathbf{z}\|_{\mathbb{C}}^2 + \|\mathbf{w}\|_{\mathbb{C}}^2 = \|\mathbf{u} - \mathbf{z}\|_{\mathbb{C}}^2 + \|\mathbf{z}\|_{\mathbb{C}}^2 = \|\mathbf{u}\|_{\mathbb{C}}^2.$$

□

Introduce now the variables $\mathbf{t}$ in the ideal

$$N_{\overline{X}} \coloneqq \left((I_X)^* + \left\langle (c+1)\text{-minors of } \begin{bmatrix} \mathbf{t} \\ J(f)^* \end{bmatrix} \right\rangle \right) : \left((I_{X_{\mathrm{sing}}})^* \right)^\infty \subseteq \mathbb{C}[\mathbf{w}, \mathbf{t}].$$

that is the analogue of the ideal N_X for the variety $\overline{X}$. Again following [Dra+16], duality and the isomorphism above lead us to define the *joint virtual Hermitian Distance correspondence* (joint vHD correspondence) of the cone X in two equivalent ways

$$\begin{aligned} \mathrm{v}\mathcal{H}_X^\vee &\coloneqq \overline{\{(\mathbf{z}, \mathbf{w}, \mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{z}, \mathbf{u}, \mathbf{v}) \in X_\circ \times \mathbb{V}^2 \times \mathbb{V}^2 \mid \mathbf{v}-\mathbf{w} \perp_\mathbb{R} T_\mathbf{z} X,\ \mathbf{u}-\mathbf{z} \perp_\mathbb{R} T_\mathbf{w}\overline{X}\}}^Z \\ &= \overline{\{(\mathbf{u}-\mathbf{t}, \mathbf{v}-\mathbf{s}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}) \in \mathbb{V}^2 \times (X^\vee)_\circ \times \mathbb{V}^2 \mid \mathbf{u}-\mathbf{t} \perp_\mathbb{R} T_\mathbf{s} X^\vee,\ \mathbf{v}-\mathbf{s} \perp_\mathbb{R} T_\mathbf{t} \overline{X}^\vee\}}^Z, \end{aligned}$$

and similarly the *projective joint virtual Hermitian Distance correspondence* (projective joint vHD correspondence) $\mathrm{v}\mathcal{PH}_X^\vee$.

Recall from Subsection 4.2 that the variety $X_\circ$ is never contained in $\tilde{Q} \coloneqq V(\langle \mathbf{z}, \mathbf{w} \rangle_\mathbb{R})$. With the next result we exploit the connection between the vHD correspondence and the dual of an affine variety.

Proposition 4.6.7. *Let $X \subseteq \mathbb{V}$ be an irreducible affine cone, then $\mathrm{v}\mathcal{PH}_X^\vee$ is an irreducible $2n$-dimensional variety in $(\mathbb{PV})^2 \times (\mathbb{PV})^2 \times \mathbb{V}^2$. It is the zero set of the hexa-homogeneous ideal*

$$\left(N_X + N_{\overline{X}} + \left\langle 3\text{-minors of } \begin{bmatrix} \mathbf{u} \\ \mathbf{t} \\ \mathbf{z} \end{bmatrix} \right\rangle + \left\langle 3\text{-minors of } \begin{bmatrix} \mathbf{v} \\ \mathbf{s} \\ \mathbf{w} \end{bmatrix} \right\rangle \right) : \langle \langle \mathbf{z}, \mathbf{w} \rangle_\mathbb{R} \rangle^\infty \tag{4.6.1}$$

in the polynomial ring $\mathbb{C}[\mathbf{z}, \mathbf{w}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}]$.

Proof. To see that $\mathrm{v}\mathcal{PH}_X^\vee$ is defined by the ideal (4.6.1), note first that any point $(\mathbf{z}, \mathbf{w}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v})$ for which $(\mathbf{z}, \mathbf{w}) \in X_\circ$ and $\mathbf{s} \perp_\mathbb{R} T_\mathbf{z} X$, $\mathbf{t} \perp_\mathbb{R} T_\mathbf{w}\overline{X}$ satisfies $(\mathbf{z}, \mathbf{s}) \in \mathcal{N}_X$ and $(\mathbf{w}, \mathbf{t}) \in \mathcal{N}_{\overline{X}}$. Now, since $\mathbf{z}+\mathbf{t} = \mathbf{u}$ and $\mathbf{w}+\mathbf{s} = \mathbf{v}$ then the spaces $\mathrm{span}\{\mathbf{z}, \mathbf{t}, \mathbf{u}\}$ and $\mathrm{span}\{\mathbf{w}, \mathbf{s}, \mathbf{v}\}$ have dimension at most 2.

Conversely, let $([\mathbf{z}], [\mathbf{w}], [\mathbf{s}], [\mathbf{t}], \mathbf{u}, \mathbf{v})$ be in the variety of the ideal (4.6.1). We can assume $(\mathbf{z}, \mathbf{w}) \in X_\circ$ and $(\mathbf{s}, \mathbf{t}) \in (X^\vee)_\circ$. Since $(\mathbf{z}, \mathbf{s}) \in \mathcal{N}_X$ and $(\mathbf{w}, \mathbf{t}) \in \mathcal{N}_{\overline{X}}$ then $\mathbf{s} \perp_\mathbb{R} T_\mathbf{z} X$ and $\mathbf{t} \perp_\mathbb{R} T_\mathbf{w}\overline{X}$ and in particular $\mathbf{s} \perp_\mathbb{R} \mathbf{z}$ and $\mathbf{t} \perp_\mathbb{R} \mathbf{w}$. Applying the hypothesis of saturation must hold that both the pairs $\mathbf{s}, \mathbf{w}$ and $\mathbf{t}, \mathbf{z}$ are linearly independent, thus there hold $\mathbf{u} = \mu_1\mathbf{z} + \lambda_1\mathbf{t}$ and $\mathbf{v} = \mu_2\mathbf{w} + \lambda_2\mathbf{s}$ for unique constants $\mu_1, \mu_2, \lambda_1, \lambda_2 \in \mathbb{C}$. If all those scalars are non zero, then we find that $([\mu_1\mathbf{z}], [\mu_2\mathbf{w}], [\lambda_2\mathbf{s}], [\lambda_1\mathbf{t}], \mathbf{u}, \mathbf{v}) \in \mathrm{v}\mathcal{PH}_X^\vee$. If all are non zero except for $\lambda_2 = 0$, then the same reasoning as in the proof of Theorem 4.6.4 proves $([\mu_1\mathbf{z}], [\mu_2\mathbf{w}], [\varepsilon\mathbf{s}], [\lambda_1\mathbf{t}], \mathbf{u}, \mathbf{v}+\varepsilon\mathbf{s}) \in \mathrm{v}\mathcal{PH}_X^\vee$ for any non zero $\varepsilon \in \mathbb{C}$ and the limit point $([\mu_1\mathbf{z}], [\mu_2\mathbf{w}], [\mathbf{s}], [\lambda_1\mathbf{t}], \mathbf{u}, \mathbf{v})$ for $\varepsilon \to 0$ lies in $\mathrm{v}\mathcal{PH}_X^\vee$.

Similar arguments apply in general when $\mu_1\mu_2\lambda_1\lambda_2 = 0$. ☐

In our context could be more natural to consider another isomorphism we now introduce.

In the presence of a sesquilinear form, that is q, there exists a canonical isomorphism between $\mathbb{V}$ and $\overline{\mathbb{V}^\vee} = \overline{\mathbb{V}}^\vee$ and in particular it holds

$$\overline{X}^\vee = \overline{\{\mathbf{s} \in \mathbb{V} \mid \exists\, \mathbf{z} \in X_{\mathrm{reg}} \colon \mathbf{s} \perp_\mathbb{C} T_\mathbf{z} X\}}^Z \subseteq \mathbb{V}.$$

In this case we should consider

$$\mathrm{v}\tilde{\mathcal{H}}_X^\vee \coloneqq \overline{\{(\mathbf{z},\mathbf{w},\mathbf{v}-\mathbf{w},\mathbf{u}-\mathbf{z},\mathbf{u},\mathbf{v}) \in X_\circ \times \mathbb{V}^2 \times \mathbb{V}^2 \mid \mathbf{u}-\mathbf{z} \perp_{\mathbb{C}} T_{\mathbf{z}}X,\ \mathbf{v}-\mathbf{w} \perp_{\mathbb{C}} T_{\mathbf{w}}\overline{X}\}}^Z$$

instead of $\mathrm{v}\mathcal{H}_X^\vee$.

5 The HD discriminant

For the Euclidean distance problem, the *ED discriminant* $\Sigma_X \subseteq \mathbb{V}$ of a real algebraic variety $X \subseteq \mathbb{V}$ is defined in [Dra+16]. This variety possesses many properties, in particular it contains the points for which a critical point of the Euclidean distance has multiplicity greater than one, moreover the number of critical points changes only on neighborhoods of points in Σ_X.

We apply similar concepts for the Hermitian case.

Let $X \subseteq \mathbb{V}$ be an algebraic variety, by Theorem 4.6.1 the vHDdeg(X) is the cardinality of the generic fiber of the projection $\pi_{\mathbb{V}^2}\colon \mathrm{v}\mathcal{H}_X \subseteq X \times \overline{X} \times \mathbb{V}^2 \to \mathbb{V}^2$ of the vHD correspondence, see Subesction 4.6.

Definition 5.0.1. Let $X \subseteq \mathbb{V}$ be an algebraic variety, the branch locus of $\pi_{\mathbb{V}^2}$ defines a variety $\mathrm{v}\Xi_X \subseteq \mathbb{V}^2$, which we call the *virtual Hermitian Distance discriminant* (vHD discriminant) of X.

The Nagata-Zariski Purity Theorem implies that the last definition is well posed and that the vHD discriminant is typically a hypersurface and we are interested in its defining polynomial, which can be interpreted as a polynomial in $\mathbb{C}[\mathbf{u}, \mathbf{v}]$. In particular, this polynomial $\mathrm{v}\Xi_X$ is invariant under the action of the map $*$ defined in Subsection 4.2.

Moreover, the vHD discriminant is contained in the Zariski closure

$$\overline{\{(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2 \mid \#\pi_{\mathbb{V}^2}^{-1}(\mathbf{u}, \mathbf{v}) < \mathrm{vHDdeg}(X)\}}^Z$$

where $\#$ denotes the set theoretical cardinality. In particular, $\mathrm{v}\Xi_X$ contains the points in which a solution of the Hermitian critical ideal has multiplicity greater than one.

If $X \subseteq \mathbb{P}\mathbb{V}$ is projective, since the variety $\mathrm{v}\mathcal{H}_X$ is defined by quadri-homogeneous equations in $\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}$, also the branch locus is defined by homogeneous equations.

Because we are interested in critical points we specialize our definition.

Definition 5.0.2. Let $X \subseteq \mathbb{V}$ be an algebraic variety, the *Hermitian Distance discriminant* (HD discriminant) of X is the set $\Xi_X \coloneqq \{\mathbf{u} \in \mathbb{V} \mid (\mathbf{u}, \bar{\mathbf{u}}) \in \mathrm{v}\Xi_X\} \subseteq \mathbb{V}$.

Note that the HD discriminant Ξ_X is defined via the generalized polynomial obtained by setting $\mathbf{v} = \bar{\mathbf{u}}$ in the equation defining the vHD discriminant $\mathrm{v}\Xi_X$ and thus in general is not a complex algebraic variety, although it is a real algebraic variety.

On the other hand, the real locus of Ξ_X coincides with the projection onto the first n components of the real locus of $\mathrm{v}\Xi_X$. In other words, the defining polynomial of the real locus of Ξ_X is obtained setting $\mathbf{v} = \mathbf{u}$ in the polynomial defining $\mathrm{v}\Xi_X$ and thus is a real algebraic variety contained in Ξ_X.

Proposition 5.0.3. *Let $X \subseteq \mathbb{V}$ be a real algebraic variety then the real locus of $\Sigma_X \subseteq \mathbb{V}$ is contained in the real locus of $\Xi_X \subseteq \mathbb{V}$.*

Proof. Follows setting $\mathbf{v} = \mathbf{u}$ real and $\mathbf{w} = \mathbf{z}$, and noting that with these choices the two ideals defining the HD correspondence and the ED correspondence coincide, see Remark 4.6.2. □

For the sake of simplicity, we will use the symbol $\mathrm{v}\Xi_X$ to also intend the equation defining the vHD discriminant and similarly for Ξ_X.

We can consider two cases for the points in the vHD discriminant:

i) There is exactly one solution of the critical ideal with multiplicity equal to 2.

ii) There are two or more solutions of the critical ideal with multiplicity equal to 2.

iii) Any other possibility.

In case i) when setting $\mathbf{v} = \bar{\mathbf{u}}$ the number of critical points in a neighborhood could change by 2. On the other hand, in cases ii) and iii) when setting $\mathbf{v} = \bar{\mathbf{u}}$ the change in the number of critical points could be higher. In particular, the multiplicity of at least a solution of the critical ideal is greater than 2 for the points of case iii), moreover these points are contained in the singular locus of $\mathrm{v}\Xi_X$. With a slight abuse of notation we call the singular locus of the HD discriminant Ξ_X the set obtained by setting $\mathbf{v} = \bar{\mathbf{u}}$ in the singular locus of $\mathrm{v}\Xi_X$.

We have already seen in Example 4.2.13 a variety for which there are jumps by more than 2 between the numbers of HDdeg. More generally, let $X \subseteq \mathbb{V}$ and $Y \subseteq \mathbb{W}$ be algebraic varieties and consider $X \times Y$. The critical ideal of $X \times Y$ is the sum of the critical ideals of X and Y. The HD discriminant $\Xi_{X\times Y}$ is $(\Xi_X \times \mathbb{W}) \cup (\mathbb{V} \times \Xi_Y) \subseteq \mathbb{V} \times \mathbb{W}$ and if $\min\{\mathrm{vHDdeg}(X), \mathrm{vHDdeg}(Y)\} > 1$ any of its points satisfy conditions ii) or iii) above. In fact, if two solutions coincide, then there must coincide in the compontents of X and Y simoultaneously, but that should happen for any solution of the critical ideal with that components of X or Y and by our hypothesis such other solutions exist.

From the results on the ED discriminant Σ_X, we obtain the following.

Proposition 5.0.4. *Let $X \subseteq \mathbb{V}$ be a real algebraic variety. Then the set $\Xi_X \setminus \Sigma_X \subseteq \mathbb{V}$ contains a dense subset of the set of points that admit only non real critical points of the Hermitian distance with multiplicity greater than one, while the real locus of the variety $\Sigma_X \subseteq \mathbb{V}$ contains almost all points that admit only real critical point of the Hermitian distance with multiplicity greater than one.*

From the work in Section 3 we get the following result.

Corollary 5.0.5. *Let X be an algebraic variety, Ξ_X is contained in the zero locus of the determinant of the Hermitian Killing form of the system given by the Hermitian critical ideal* (4.2.1). *Moreover, the number of critical points of the Hermitian distance from X changes on a path crossing Ξ_X if the sign of the evaluation of Ξ_X changes.*

Proof. The zero locus of the determinant of the Hermitian Killing form of the system given by the Hermitian critical ideal determines exactly the points for which a solution of the Hermitian critical ideal has multiplicity greater than one that is the definition of the vHD discriminant restricted to points with $\mathbf{v} = \bar{\mathbf{u}}$. □

5.1 Complex evolute

Here we present known properties of the ED discriminant of a real algebraic curve. Then we apply similar ideas to the case of the HD discriminant of an algebraic curve.

Let us recall that an *osculating circle* of a curve $X \subseteq \mathbb{R}^2$ is a circle that is tangent to the curve at a regular point and has the same curvature in the same direction as the curve at the point of tangency. Two simple examples of osculating circles are shown in Figure 5.1.

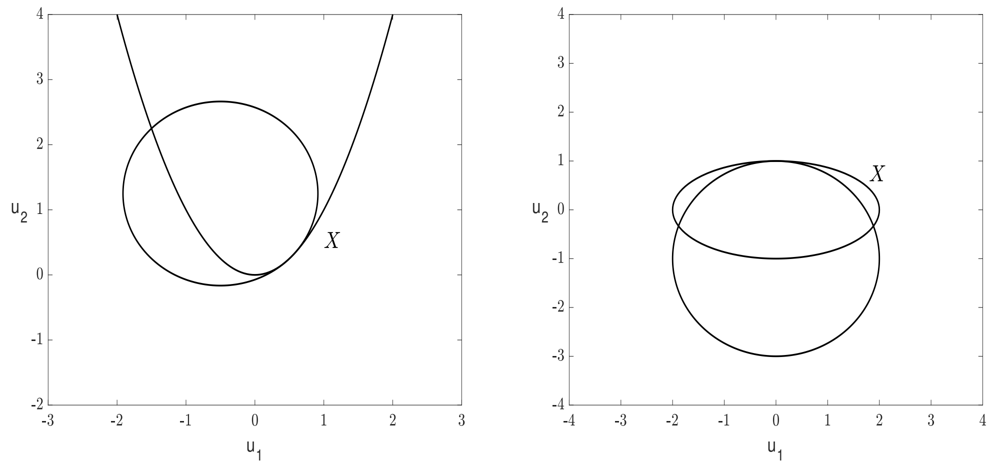

Figure 5.1: Osculating circles of the parabola $X = V(z_2 - z_1^2)$ at the point $\mathbf{u} = (1/2, 1/4)$ (Left) and of the ellipse $X = V(z_1^2 + 4z_2^2 - 4)$ at the point $\mathbf{u} = (0, 1)$ (Right).

We are ready to state the definition of the evolute of a real algebraic curve, a rigorous introduction can be found in [BG92], other works about the more general concept of *Focal loci* are [Tri98; CT07].

Definition 5.1.1. Let $X = V(f) \subseteq \mathbb{R}^2$ with $f \in \mathbb{R}[\mathbf{z}]$ be a plane curve, the *evolute* of X is the union of the centers of the osculating circles of X. Its defining polynomial is the generator of the ideal obtained by eliminating the variables $\mathbf{z}$ from the ideal

$$I_X^E := \langle f, g, \varphi\rangle : \langle \partial_{z_1} f, \partial_{z_2} f\rangle^\infty \subseteq \mathbb{C}[\mathbf{z}, \mathbf{u}],$$

where $g := (u_2 - z_2)\partial_{z_1} f - (u_1 - z_1)\partial_{z_2} f$ and $\varphi := (\partial_{z_1} g)(\partial_{z_2} f) - (\partial_{z_2} g)(\partial_{z_1} f)$.

In particular, the evolute coincides with the envelope of the normal lines of X, see [BG92].

Moreover, if $X \subseteq \mathbb{R}^2$ is a smooth curve parametrized by $\psi = (\psi_1, \psi_2)$, the evolute can be obtained using the formula

$$\psi^E := \left(\psi_1 - \psi_2' \frac{(\psi_1')^2 + (\psi_2')^2}{\psi_1'\psi_2'' - \psi_1''\psi_2'},\ \psi_2 + \psi_1' \frac{(\psi_1')^2 + (\psi_2')^2}{\psi_1'\psi_2'' - \psi_1''\psi_2'}\right) \tag{5.1.1}$$

which plots the center of the osculating circle at ψ. It is a matter of computations to check $\langle \psi', (\psi^E)'\rangle_{\mathbb{R}} = 0$.

The following known result completely solves the problem of finding the ED discriminant of a real curve. The equivalence simply follows by noting that the points of the evolute are precisely those points with a critical point $\mathbf{z} \in X$ of multiplicity greater than one, or in other terms those points which admit critical points of g on X. However, we give a rigorous proof that we will generalize in our setting.

Proposition 5.1.2. *Let $X = V(f)$ with $f \in \mathbb{R}[\mathbf{z}]$ be a plane curve, then the ED discriminant and the evolute coincide.*

Proof. We let g and φ be the polynomials of Definition 5.1.1. The critical ideal (4.2.2) of the EDdegree is $\langle f, g\rangle \subseteq \mathbb{C}[\mathbf{z}, \mathbf{u}]$. Since the definitions of the curves involved in the assertion are local, by the implicit function theorem we can assume z_2 is a function of z_1 and eliminate f. Thus, applying the implicit function theorem, the ED discriminant can be defined as the common zero locus of the polynomials g and

$$\partial_{z_1} g = \begin{bmatrix} 1 & \partial_{z_1} z_2 \end{bmatrix} \begin{bmatrix} \partial_{z_1} g \\ \partial_{z_2} g \end{bmatrix} = \partial_{z_1} g + (\partial_{z_1} z_2)(\partial_{z_2} g) = \partial_{z_1} g - \frac{\partial_{z_1} f}{\partial_{z_2} f} \partial_{z_2} g = \frac{\varphi}{\partial_{z_2} f},$$

Resuming, the ED discriminant can be obtained through the same ideal I_X^E defining the evolute. □

The following object is the Hermitian analogue of the evolute introducing the variables $\mathbf{w}$ and $\mathbf{v}$ of Definition 5.1.1.

Definition 5.1.3. Let $X = V(f)$ be an algebraic curve, we call the *virtual complex evolute* of X the zero locus of the generator of the ideal obtained by eliminating the variables $\mathbf{z}$ and $\mathbf{w}$ from the ideal

$$I_X^{\tilde{E}} := \langle f, f^*, \tilde{g}, \tilde{g}^*, \xi\rangle : \langle \partial_{z_1} f, \partial_{z_2} f, \partial_{w_1} f^*, \partial_{w_2} f^*\rangle^\infty \subseteq \mathbb{C}[\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}],$$

where $\tilde{g} \coloneqq (v_2 - w_2)\partial_{z_1} f - (v_1 - w_1)\partial_{z_2} f$ and

$$\xi \coloneqq \tilde{\varphi}\tilde{\varphi}^* - \langle \nabla_{\mathbf{z}} f, \nabla_{\mathbf{w}} f^* \rangle_{\mathbb{R}}^2 \tag{5.1.2}$$

with $\tilde{\varphi} \coloneqq (\partial_{z_2} f)(\partial_{z_1}\tilde{g}) - (\partial_{z_1} f)(\partial_{z_2}\tilde{g})$. We call the *complex evolute* of X the zero locus of the generalized polynomial obtained setting $\mathbf{v} = \bar{\mathbf{u}}$.

Note that, when considering points with $\mathbf{v} = \bar{\mathbf{u}}$ and critical points $\mathbf{w} = \bar{\mathbf{z}}$ the formula simplifies

$$\xi = |\tilde{\varphi}|^2 - \|\nabla_{\mathbf{z}} f\|_{\mathbb{C}}^4 = \left(|\tilde{\varphi}| - \|\nabla_{\mathbf{z}} f\|_{\mathbb{C}}^2\right)\left(|\tilde{\varphi}| + \|\nabla_{\mathbf{z}} f\|_{\mathbb{C}}^2\right)$$

and in particular we can bound from above the term on the left as

$$|\tilde{\varphi}| - \|\nabla_{\mathbf{z}} f\|_{\mathbb{C}}^2 \leq \|\nabla_{\mathbf{z}}\tilde{g}\|_{\mathbb{C}}\|\nabla \mathbf{z} f\|_{\mathbb{C}} - \|\nabla_{\mathbf{z}} f\|_{\mathbb{C}}^2 = \|\nabla_{\mathbf{z}} f\|_{\mathbb{C}}\left(\|\nabla_{\mathbf{z}}\tilde{g}\|_{\mathbb{C}} - \|\nabla_{\mathbf{z}} f\|_{\mathbb{C}}\right).$$

We now exploit the connection between the HD discriminant and the complex evolute of a curve.

Proposition 5.1.4. *Let $X = V(f)$ be an algebraic curve, then the virtual complex evolute of X is the vHD discriminant $\mathrm{v}\Xi_X$. In particular, the complex evolute of X is the HD discriminant Ξ_X and this is exactly the set where the number of critical points changes in a neighborhood.*

Proof. We let $\tilde{g}$ be the polynomial of Definition 5.1.3. The Hermitian critical ideal takes the form $\langle f, f^*, \tilde{g}, \tilde{g}^* \rangle$ and we can assume z_2 and w_2 to be holomorphic functions of z_1 and w_1 respectively and thus eliminate f and f^*. Moreover, we can assume w_1 to be a holomorphic function of z_1 and eliminate $\tilde{g}^*$. Thus, applying the implicit function Theorem to derive $\partial_{z_1} z_2(z_1)$ and $\partial_{w_1} w_2(w_1)$ and the chain rule, the vHD discriminant is the common zero locus of the polynomials $\tilde{g}$ and

$$\begin{aligned}
\partial_{z_1}\tilde{g} &= \begin{bmatrix} 1 & \partial_{z_1} z_2 & \partial_{z_1} w_1 & \partial_{z_1} w_2 \end{bmatrix} \begin{bmatrix} \partial_{z_1}\tilde{g} \\ \partial_{z_2}\tilde{g} \\ \partial_{w_1}\tilde{g} \\ \partial_{w_2}\tilde{g} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\partial_{z_1} f}{\partial_{z_2} f} & \partial_{z_1} w_1 & -\partial_{z_1} w_1 \frac{\partial_{w_1} f^*}{\partial_{w_2} f^*} \end{bmatrix} \begin{bmatrix} \partial_{z_1}\tilde{g} \\ \partial_{z_2}\tilde{g} \\ \partial_{w_1}\tilde{g} \\ \partial_{w_2}\tilde{g} \end{bmatrix} \\
&= \partial_{z_1}\tilde{g} - \frac{\partial_{z_1} f}{\partial_{z_2} f}\partial_{z_2}\tilde{g} + \partial_{z_1} w_1 \left(\partial_{w_1}\tilde{g} - \frac{\partial_{w_1} f^*}{\partial_{w_2} f^*}\partial_{w_2}\tilde{g}\right) \\
&= \frac{(\partial_{z_2} f)(\partial_{z_1}\tilde{g}) - (\partial_{z_1} f)(\partial_{z_2}\tilde{g}) + \frac{\partial_{z_2} f}{\partial_{w_2} f^*}\partial_{z_1} w_1 \left((\partial_{z_2} f)(\partial_{w_2} f^*) + (\partial_{z_1} f)(\partial_{w_1} f^*)\right)}{\partial_{z_2} f}.
\end{aligned}$$

Now, we use again the implicit function Theorem to derive $\partial_{z_1} w_1(z_1)$. Starting from the equalities

$$\begin{aligned}
\partial_{z_1} w_1 = -\frac{\partial_{z_1}\tilde{g}^*}{\partial_{w_1}\tilde{g}^*} &= -\frac{\partial_{z_1}\left((u_2 - z_2)\partial_{w_1} f^* - (u_1 - z_1)\partial_{w_2} f^*\right)}{\partial_{w_1}\left((u_2 - z_2)\partial_{w_1} f^* - (u_1 - z_1)\partial_{w_2} f^*\right)} \\
&= -\frac{1}{\partial_{z_2} f}\frac{(\partial_{z_1} f)(\partial_{w_1} f^*) + (\partial_{z_2} f)(\partial_{w_2} f^*)}{(u_2 - z_2)\partial_{w_1}(\partial_{w_1} f^*) - (u_1 - z_1)\partial_{w_1}(\partial_{w_2} f^*)},
\end{aligned}$$

we apply the equalities

$$\partial_{w_1}(\partial_{w_1} f^*) = \begin{bmatrix} 1 & -\frac{\partial_{w_1} f^*}{\partial_{w_2} f^*} \end{bmatrix} \begin{bmatrix} \partial^2_{w_1} f^* \\ \partial_{w_1}\partial_{w_2} f^* \end{bmatrix} = \partial^2_{w_1} f^* - \frac{\partial_{w_1} f^*}{\partial_{w_2} f^*} \partial_{w_1}\partial_{w_2} f^*$$
$$\partial_{w_1}(\partial_{w_2} f^*) = \begin{bmatrix} 1 & -\frac{\partial_{w_1} f^*}{\partial_{w_2} f^*} \end{bmatrix} \begin{bmatrix} \partial_{w_1}\partial_{w_2} f^* \\ \partial^2_{w_2} f^* \end{bmatrix} = \partial_{w_1}\partial_{w_2} f^* - \frac{\partial_{w_1} f^*}{\partial_{w_2} f^*} \partial^2_{w_2} f^*$$

obtained by the chain rule to write

$$\partial_{z_1} w_1 = -\frac{\partial_{w_2} f^*}{\partial_{z_2} f} \frac{(\partial_{z_2} f)(\partial_{w_2} f^*) + (\partial_{z_1} f)(\partial_{w_1} f^*)}{(\partial_{w_2} f^*)(\partial_{w_1} \tilde{g}^*) - (\partial_{w_1} f^*)(\partial_{w_2} \tilde{g}^*)}.$$

Now applying this substitution in the formula for $\partial_{z_1}\tilde{g}$ results ξ of equation (5.1.2) in the numerator. On the other hand, every point of the virtual complex evolute is a point of the vHD discriminant and thus we get the first part of the statement.

The second statement follows from the definitions of the complex evolute and critical points. □

In the following subsections we consider X to be a curve and we will provide a useful description of the HD discriminant.

5.2 Outward evolute

To our purpose of this subsection it is useful to introduce the notion of an *outward osculating circle* of a curve $X \subseteq \mathbb{R}^2$, that is a circle that is tangent to the curve at a regular point and has the same curvature with opposite direction as the curve at point of tangency. Two simple examples of outward osculating circles are shown in Figure 5.2.

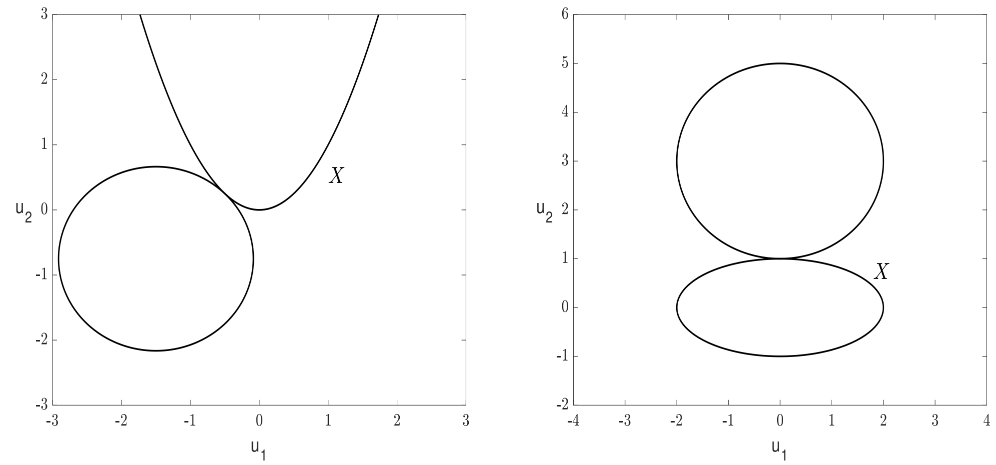

Figure 5.2: Outward osculating circles of the parabola $X = V(z_2 - z_1^2)$ at the point $\mathbf{u} = (-1/2, 1/4)$ (Left) and of the ellipse $X = V(z_1^2 + 4z_2^2 - 4)$ at the point $\mathbf{u} = (0, 1)$ (Right).

Using the outward osculating circles we define our object of interest.

Definition 5.2.1. Let $X = V(f) \subseteq \mathbb{R}^2$ with $f \in \mathbb{R}[\mathbf{z}]$ be a plane curve, the *outward evolute* of X is the union of the centers of the outward osculating circles of X. Its defining polynomial is the generator of the ideal obtained eliminating the variables $\mathbf{z}$ from the ideal

$$I_X^{\exists} := \langle f, g, \varphi - 2\|\nabla_{\mathbf{z}} f\|_{\mathbb{R}}^2 \rangle : \langle \partial_{z_1} f, \partial_{z_2} f \rangle^{\infty} \subseteq \mathbb{C}[\mathbf{z}, \mathbf{u}],$$

where g and φ are the polynomials of Definition 5.1.1.

If ψ is a parametrization of the smooth curve $X \subseteq \mathbb{R}^2$ we can modify formula (5.1.1) of the evolute as

$$\psi^{\exists} := \left(\psi_1 + \psi_2' \frac{(\psi_1')^2 + (\psi_2')^2}{\psi_1'\psi_2'' - \psi_1''\psi_2'},\ \psi_2 - \psi_1' \frac{(\psi_1')^2 + (\psi_2')^2}{\psi_1'\psi_2'' - \psi_1''\psi_2'} \right)$$

to obtain the outward evolute of X. In particular, $\psi^{\exists} = 2\psi - \psi^E$ and $\langle \psi', (\psi^{\exists})' \rangle_{\mathbb{R}} = 2\|\psi'\|_{\mathbb{R}}^2$. Moreover, $\psi - \psi^E = \psi^{\exists} - \psi$ is the radial curve of X.

The following proposition shed some light on the real locus of the complex evolute of a real algebraic curve defined in the previous subsection.

Proposition 5.2.2. *Let $X = V(f)$ be an algebraic curve with $f \in \mathbb{R}[\mathbf{z}]$, then the union of the evolute and the outward evolute is the real locus of the complex evolute. Moreover, for points of the evolute at least two real critical points coincide, while for points of the outward evolute at least two non real critical points coincide.*

Proof. Let $\mathbf{v} = \mathbf{u}$ and write

$$\tilde{\varphi} = (\partial_{z_2} f)(\partial_{z_1} \tilde{g}) - (\partial_{z_1} f)(\partial_{z_2} \tilde{g}) = \varphi + (\partial_{z_2} f)(\partial_{z_1}(\tilde{g} - g)) - (\partial_{z_1} f)(\partial_{z_2}(\tilde{g} - g)).$$

Compute

$$\partial_{z_1}(\tilde{g}-g) = \partial_{z_1}\left((z_2-w_2)\partial_{z_1}f - (z_1-w_1)\partial_{z_2}f\right)$$
$$= (z_2-w_2)\partial^2_{z_1}f - \partial_{z_2}f - (z_1-w_1)\partial_{z_1}\partial_{z_2}f$$

and similarly

$$\partial_{z_2}(\tilde{g}-g) = \partial_{z_1}f + (z_2-w_2)\partial_{z_1}\partial_{z_2}f - (z_1-w_1)\partial^2_{z_2}f.$$

Now, since the variety is real as discussed in Remark 4.2.8 for any critical point $\mathbf{z}$ we also have $\bar{\mathbf{z}}$ and then we can assume $\mathbf{w}=\mathbf{z}$. In the end, substituting these equation we can write $\tilde{\varphi} = \varphi - \|\nabla_{\mathbf{z}} f\|^2_{\mathbb{R}}$ and the claim follows from

$$\xi = \tilde{\varphi}\tilde{\varphi}^* - \langle \nabla_{\mathbf{z}} f, \nabla_{\mathbf{w}} f^*\rangle^2_{\mathbb{R}} = \left(\tilde{\varphi} + \|\nabla_{\mathbf{z}} f\|^2_{\mathbb{R}}\right)\left(\tilde{\varphi} - \|\nabla_{\mathbf{z}} f\|^2_{\mathbb{R}}\right) = \varphi\left(\varphi - 2\|\nabla_{\mathbf{z}} f\|^2_{\mathbb{R}}\right).$$

In fact, from the Euclidean distance problem we know that exaclty for points of the evolute at least two real critical points coincide □

We can confront the last result with the figures of Examples 5.2.4, 5.2.5 and 5.2.6.

Corollary 5.2.3. *Let $X = V(f)$ be an algebraic curve with $f \in \mathbb{R}[\mathbf{z}]$, then the outward evolute is contained in the real singular locus of the HD discriminant.*

Proof. In any open real neighborhood of a point of the outward evolute there exist points that admit a non real solution of the critical ideal that is not a critical point. As discussed in Remark 4.2.8, when $\mathbf{v} = \mathbf{u}$ is real, any non real solution of the critical ideal that is not a critical point yields by conjugation three other different solutions and the four are grouped into two associated pairs. On the outward evolute two points of an associated pair coincide and the coincident point must be real critical point. In particular, the four solutions of the critical ideal coincide thus getting a solution of multiplicity at least 4 of the critical ideal. □

We now present some examples for the three different cases of non degenerate conic.

Example 5.2.4. (Parabola) The complex evolute of the parabola $X = V(z_2 - z_1^2) \subseteq \mathbb{C}^2$ is the zero locus of

$$\begin{aligned}
&65536|u_2|^{12} + 6|u_1|^4(u_1^2\bar{u}_2 + \bar{u}_1^2u_2) - |u_2|^4(49152|u_2|^4 + 10752|u_1|^2)(u_1^2\bar{u}_2 + \bar{u}_1^2u_2 + |u_2|^2)\\
&- 2|u_1|^6 + 6|u_1|^4|u_2|^2 + 384|u_2|^2(|u_1|^4\bar{u}_1^2u_2 + |u_1|^4u_1^2\bar{u}_2 + u_1^4\bar{u}_2^2 + \bar{u}_1^4u_2^2 + |u_2|^2u_1^2\bar{u}_2 + |u_2|^2\bar{u}_1^2u_2)\\
&- 5952|u_1u_2|^4 - 1024(u_1^6\bar{u}_2^3 + \bar{u}_1^6u_2^3 + |u_2|^6) + |u_2|^2(96|u_1|^2 - 44544|u_2|^4)(\bar{u}_1^2u_2 + u_1^2\bar{u}_2 + |u_1|^4)\\
&- 172032|u_1|^2|u_2|^8 - 27(u_1^4\bar{u}_2^2 + \bar{u}_1^4u_2^2 + |u_1|^8) + (12288|u_2|^4 - 384|u_1|^2)(u_1^4\bar{u}_2^2 + \bar{u}_1^4u_2^2 + |u_2|^4).
\end{aligned}$$

If we restrict our attention to real points it simplifies

$$(2(2u_2-1)^3 - 27u_1^2)(8u_2^2(2u_2+1) + u_1^2)^3$$

and we can spot the equation defining the evolute as the first factor and the outward evolute as the second factor. The real zero locus of this polynomial is plotted in Figure 5.3.

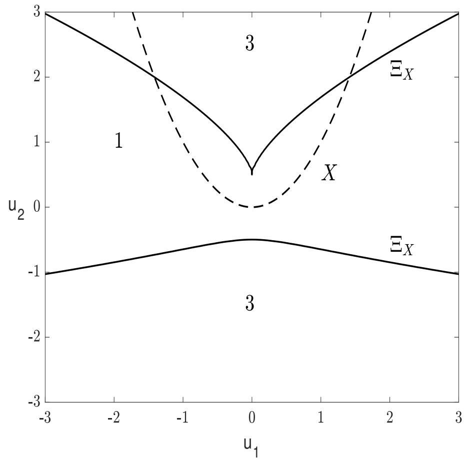

Figure 5.3: Example 4.6.5 - Evolute and outward evolute of X (dashed line) on the real plane. The values are the numbers of critical points in each area: from top to bottom we have 3 real critical points, 1 real critical point, 1 real and 2 non real critical points.

Note that the exponent in which appears the outward evolute is 3, this is related to Corollary 5.2.3.

We will continue to study this problem in Example 6.0.13.

Example 5.2.5. (Circle) The real locus of the complex evolute of $X = V(z_1^2 + z_2^2 - 1) \subseteq \mathbb{C}^2$ when $\mathbf{v} = \mathbf{u}$ coincides with the union of the evolute and the outward evolute and it is the real zero locus of

$$(u_1^2 + u_2^2)(u_1^2 + u_2^2 - 4)^3,$$

this locus is plotted in Figure 5.4.

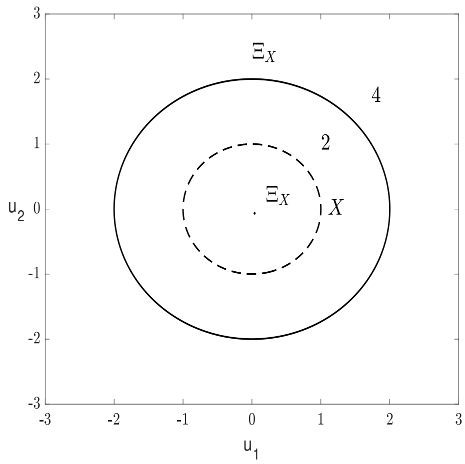

Figure 5.4: Example 5.2.5 - Evolute and outward evolute of X (dashed line) on the real plane. The values are the numbers of critical points in each area: outside the outward evolute we have 2 real and 2 non real critical points, inside we have 2 real critical points.

Note that the exponent in which appears the outward evolute is 3, this is related to Corollary 5.2.3.

We will continue to study this problem in Example 6.0.14.

Example 5.2.6. (Ellipse) The real locus of the complex evolute of $X = V(z_1^2 + 4z_2^2 - 4) \subseteq \mathbb{C}^2$ when $\mathbf{v} = \mathbf{u}$ coincides with the union of the evolute and the outward evolute and it is the zero locus of

$$\left((4u_1^2 + u_2^2 - 9)^3 + 972u_1^2u_2^2\right)\left((4u_1^2 + u_2^2)^3 - 880u_1^4 - 236u_1^2u_2^2 + 5u_2^4 + 3900u_1^2 - 525u_2^2 - 5625\right)^3,$$

the real zero locus is plotted in Figure 5.5.

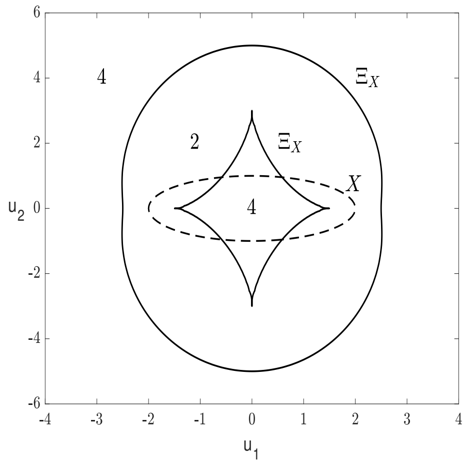

Figure 5.5: Example 5.2.6 - Evolute and outward evolute of X (dashed line) on the real plane. The values are the numbers of critical points in each area: inside the evolute we have 4 real critical points, outside the outward evolute we have 2 real and 2 non real critical points, in between we have 2 real critical points.

Note that the exponent in which appears the outward evolute is 3, this is related to Corollary 5.2.3.

6 The HD polynomial

In this section we introduce a polynomial that encodes all the notions we introduced so far about the Hermitian distance problem, from the exact distances to the HD discriminant of Section 5. The *Euclidean distance polynomial* (ED polynomial), introduced in [OS20] is the equivalent concept in the case of the Euclidean distance problem, a consistent study about it can be found in [HW18], the same concept is called the *offset polynomial* in [BKS24].

Definition 6.0.1. Let X be an algebraic variety. The sum of the Hermitian critical ideal (4.2.1) of X and the ideal $\langle t^2 - \langle \mathbf{u}-\mathbf{z}, \mathbf{v}-\mathbf{w}\rangle_{\mathbb{R}}\rangle$ in the ring $\mathbb{C}[\mathbf{z}, \mathbf{w}, \mathbf{u}, \mathbf{v}, t]$ defines a variety of dimension $2n$.

The *virtual Hermitian Distance polynomial* (vHD polynomial) $\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2)$ of X at $(\mathbf{u}, \mathbf{v})$ is, up to a scalar factor, the generator of the projection of the sum of ideals above in the ring $\mathbb{C}[\mathbf{u}, \mathbf{v}, t]$. When the variety X is defined by polynomials with real (rational) coefficients, the elimination procedure yields $\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}$ with real (rational) coefficients. Moreover, the *Hermitian Distance polynomial* (HD polynomial) is $\mathrm{HDpol}_{X,\mathbf{u}} \coloneqq \mathrm{vHDpol}_{X,\mathbf{u},\bar{\mathbf{u}}}$.

Since the polynomial ring is a UFD, the projection of the sum of the ideals is generated by a single polynomial in t^2 and thus this object is well-defined.

Recall from Subsection 4.2 that we denote $X_\circ = X_{\mathrm{reg}} \times \overline{X}_{\mathrm{reg}}$. From the definition we obtain the following result.

Proposition 6.0.2. *Let X be an algebraic variety and $(\mathbf{u}, \mathbf{v})$ a generic point. The roots of the polynomial $\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2)$ are of the form $t^2 = \langle \mathbf{u}-\mathbf{z}, \mathbf{v}-\mathbf{w}\rangle_{\mathbb{R}}$, where $(\mathbf{z}, \mathbf{w}) \in X_\circ$ is a solution of the Hermitian critical ideal of $(\mathbf{u}, \mathbf{v})$. In particular, the distance of $\mathbf{u}$ from X is a root of $\mathrm{HDpol}_{X,\mathbf{u}}(t^2)$ and the degree of $\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2)$ in t^2 is equal to $\mathrm{vHDdeg}(X)$.*

The next proposition highlights a useful property of the HD polynomial.

Proposition 6.0.3. *Consider the pair-wise disjoint union $X = X_1 \cup \ldots \cup X_r$ for some $r \in \mathbb{N}$, where $X_k \subseteq \mathbb{V}$ is a reduced variety for every $k = 1, \ldots, r$. Then*

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \prod_{k=1}^{r} \mathrm{vHDpol}_{X_k,\mathbf{u},\mathbf{v}}(t^2).$$

Proof. For a general $(\mathbf{u},\mathbf{v}) \in \mathbb{V}^2$ the variety of the Hermitian critical ideal (4.2.1) of X is the union of the varieties of the ideals (4.2.1) of the varieties X_k. The conclusion follows by Proposition 6.0.2. □

The following lemma generalizes Lemma 4.2.10.

Lemma 6.0.4. *Let $X \subseteq \mathbb{V}$ be an algebraic variety, $0 \neq c \in \mathbb{C}$ and $\mathbf{b} \in \mathbb{V}$, then it holds the equality*

$$\mathrm{vHDpol}_{cX+\mathbf{b},c\mathbf{u}+\mathbf{b},\bar{c}\mathbf{v}+\bar{\mathbf{b}}}(|c|^2t^2) = \mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2).$$

Proof. The assertion follows from Lemma 4.2.10 and noting

$$\langle c\mathbf{u}+\mathbf{b}-(c\mathbf{z}+\mathbf{b}), \bar{c}\mathbf{v}+\bar{\mathbf{b}}-(\bar{c}\mathbf{w}+\bar{\mathbf{b}})\rangle_{\mathbb{R}} = |c|^2\langle \mathbf{u}-\mathbf{z},\mathbf{v}-\mathbf{w}\rangle_{\mathbb{R}}.$$

□

We denote the discriminant of the vHD polynomial as $\Delta_X \coloneqq \Delta_{t^2}\,\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2)$. We get the following result connecting the vHD polynomial and the vHD discriminant.

Proposition 6.0.5. *Let X be an algebraic variety, then the polynomial defining the vHD discriminant divides Δ_X. Moreover, if X is symmetric with respect to r affine hyperplanes $L_1,\dots,L_r$ of equations $l_1,\dots,l_r$, then the product $l_1\cdots l_r l_1^*\cdots l_r^*$ divides Δ_X. We are assuming $l_1,\dots,l_r$ to be polynomials in $\mathbf{z}$ and $*$ is the map of Subection 4.2.*

Proof. By definition of $\mathrm{v}\Xi_X$, any point $(\mathbf{u},\mathbf{v}) \in \mathrm{v}\Xi_X$ satisfies the equation of Δ_X, since two roots in t^2 coincide, by Proposition 6.0.2. Now, let $(\mathbf{u},\mathbf{v}) \in L_k \times \overline{L}_j$ be generic and let

$$(\mathbf{z},\mathbf{w}) \in (X \setminus (X_{\mathrm{sing}} \cup L_k)) \times \left(\overline{X} \setminus (\overline{X}_{\mathrm{sing}} \cup \overline{L}_j)\right)$$

for some $k,j \in [r]$ be a solution of the Hermitian critical ideal. Denote by $(\tilde{\mathbf{z}},\tilde{\mathbf{w}})$ the reflection of $(\mathbf{z},\mathbf{w})$ with respect to $L_k \times \overline{L}_j$. Thus $(\tilde{\mathbf{z}},\tilde{\mathbf{w}}) \in X_\circ$ as well and $(\tilde{\mathbf{z}},\tilde{\mathbf{w}})$ is again a solution of the Hermitian critical ideal. Since $\langle \mathbf{u}-\mathbf{z},\mathbf{v}-\mathbf{w}\rangle_{\mathbb{R}} = \langle \mathbf{u}-\tilde{\mathbf{z}},\mathbf{v}-\tilde{\mathbf{w}}\rangle_{\mathbb{R}}$, then $(\mathbf{u},\mathbf{v})$ is a zero of Δ_X. □

Proposition 6.0.6. *Let $X \subseteq \mathbb{V}$ be an algebraic variety given by real polynomials for which it holds $\max\mathrm{HDdeg}(X) < \mathrm{vHDdeg}(X)$, then the set $\{(\mathbf{u},\mathbf{u}) \in \mathbb{V}^2 \mid \mathbf{u} \in \mathbb{V}\} \subseteq \mathbb{V}^2$ is in the zero locus of Δ_X.*

Proof. Similarly as discussed in Remark 4.2.8, if we consider points $(\mathbf{z},\mathbf{v})$ with $\mathbf{v} = \mathbf{u}$, then by the hypothesis there exists at least a solution $(\mathbf{z},\mathbf{w})$ of the Hermitian critical ideal with $\mathbf{z} \neq \mathbf{w}$ which in turn yields another solution $(\mathbf{w},\mathbf{z})$ and those solutions satisfy $\langle \mathbf{u}-\mathbf{z},\mathbf{u}-\mathbf{w}\rangle_{\mathbb{R}} = \langle \mathbf{u}-\mathbf{w},\mathbf{u}-\mathbf{z}\rangle_{\mathbb{R}} = t^2$. In particular, the HD polynomial has a solution with multiplicity greater than 1 and the assertion follows by definition of Δ_X. □

The last proposition highlights the fact that for a real point there exist two critical points at the same distance which are conjugate one to each other.

Proposition 6.0.7. *Let $X \subseteq \mathbb{V}$ be an algebraic variety and let $G \subseteq \mathrm{Iso}(\mathbb{V})$ be a subgroup that leaves X invariant, then for any $g \in G$ it holds $\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}} = \mathrm{vHDpol}_{X,g\cdot\mathbf{u},\bar{g}\cdot\mathbf{v}}$ and the coefficients of $\mathrm{HDpol}_{X,\mathbf{u}}$ are G invariant.*

Proof. Follows from Lemma 4.2.14. □

Proposition 6.0.8. *Let X be an affine subspace, then* $\mathrm{vHDdeg}(X) = 1$ *and* $\mathrm{HDdeg}(X) = \{1\}$. *Moreover, if* $\mathbf{b} \in X$ *then*

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = t^2 - \langle \pi_{(X-\mathbf{b})^{\perp_{\mathbb{C}}}}(\mathbf{u}-\mathbf{b}), \pi_{\overline{(X-\mathbf{b})}^{\perp_{\mathbb{C}}}}(\mathbf{v}-\bar{\mathbf{b}})\rangle_{\mathbb{R}}.$$

Proof. The tangent spaces of X and $\overline{X}$ are everywhere equal to $\mathrm{span}\{e_k\}_{k=1}^m$ and $\mathrm{span}\{\bar{e}_k\}_{k=1}^m$ respectively, where $m \in \mathbb{N}$ is the dimension of X and $e_1, \ldots, e_m \in \mathbb{V}$ are orthonormal generators of the tangent space which is isomorphic to the variety since $X = \mathrm{span}\{e_1, \ldots, e_m\} + \mathbf{b}$. If for $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ the point $(\mathbf{z}, \mathbf{w})$ is a solution of the Hermitian critical ideal, then $\mathbf{v} - \mathbf{w} \perp_{\mathbb{R}} \mathrm{span}\{e_k\}_{k=1}^m$ and $\mathbf{u} - \mathbf{z} \perp_{\mathbb{R}} \mathrm{span}\{\bar{e}_k\}_{k=1}^m$. We complete $\{e_k\}_{k=1}^m$ with vectors $e_{m+1}, \ldots, e_n$ to a complete orthonormal basis of $\mathbb{V}$. By writing $\mathbf{v} = v_1\bar{e}_1 + \ldots + v_n\bar{e}_n$ and $\mathbf{u} = u_1 e_1 + \ldots + u_n e_n$, it is easy to check that the only possible solution is

$$\mathbf{z} = (u_1 - \langle \mathbf{b}, e_1\rangle_{\mathbb{C}})e_1 + \ldots + (u_m - \langle \mathbf{b}, e_m\rangle_{\mathbb{C}})e_m + \mathbf{b} \in X \quad \text{and} \quad \mathbf{w} = \bar{\mathbf{z}} \in \overline{X}.$$

The equation for the HD polynomial follows.

Clearly, the only critical point is $\pi_{(X-\mathbf{b})}(\mathbf{u}-\mathbf{b}) + \mathbf{b}$. □

Corollary 6.0.9. *Let $X = V(a_1 z_1 + \ldots + a_n z_n + b)$ with $a_1, \ldots, a_n \in \mathbb{C}$ not all zero and $b \in \mathbb{C}$ be a hyperplane. Let be $\mathbf{a} = (a_1, \ldots, a_n)$ then*

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = t^2 - \frac{\langle \|\mathbf{a}\|_{\mathbb{C}}^2\mathbf{u} - b\bar{\mathbf{a}}, \bar{\mathbf{a}}\rangle_{\mathbb{C}} \langle \|\mathbf{a}\|_{\mathbb{C}}^2\mathbf{v} - \bar{b}\mathbf{a}, \mathbf{a}\rangle_{\mathbb{C}}}{\|\mathbf{a}\|_{\mathbb{C}}^4}.$$

Example 6.0.10. (Linear regression). Let $a_1, \ldots, a_{n-1}, b \in \mathbb{C}$, we bijectively parametrize the hyperplane

$$\begin{aligned} \psi\colon \mathbb{C}^{n-1} &\longrightarrow X = V\left(z_n - \sum_{k=1}^{n-1} a_k z_k - b\right) \subseteq \mathbb{C}^n \\ (z_1, \ldots, z_{n-1}) &\mapsto \left(z_1, \ldots, z_{n-1}, \sum_{k=1}^{n-1} a_k z_k + b\right) \end{aligned}$$

Critical points of $q_{\mathbf{u}}$ satisfy the following equation for any $k = 1, \ldots, n-1$

$$\partial_{z_k} \|\psi(z_1, \ldots, z_{n-1}) - \mathbf{u}\|_{\mathbb{C}}^2 = (\bar{z}_k - \bar{u}_k) + a_k \left(\sum_{k=1}^{n-1} \bar{a}_k \bar{z}_k + \bar{b} - \bar{u}_n\right) = 0.$$

Denoting the row vectors $\mathbf{a} = (a_1, \ldots, a_{n-1})$, $\tilde{\mathbf{u}} = (u_1, \ldots, u_{n-1})$ and $\tilde{\mathbf{z}} = (z_1, \ldots, z_{n-1})$, in other terms we aim to solve the $(n-1) \times (n-1)$ system

$$\left(I_{n-1} + \mathbf{a}^H \mathbf{a}\right) \tilde{\mathbf{z}}^T = (u_n - b)\mathbf{a}^H + \tilde{\mathbf{u}}^T$$

and we can apply the inverse $I_{n-1} - \frac{1}{1+\|\mathbf{a}\|_{\mathbb{C}}^2}\mathbf{a}^H\mathbf{a} \in \mathcal{M}_{n-1}$ of the matrix on the left hand side to find $\tilde{\mathbf{z}}$. So that by computing the last coordinate $z_n = \langle \mathbf{a}, \tilde{\mathbf{z}} \rangle_{\mathbb{R}} + b$ we obtain the explicit unique critical point

$$(\tilde{\mathbf{z}}, z_n) = \left(\frac{u_n - b - \langle \mathbf{a}, \tilde{\mathbf{u}} \rangle_{\mathbb{R}}}{1 + \|\mathbf{a}\|_{\mathbb{C}}^2} \bar{\mathbf{a}} + \tilde{\mathbf{u}}, \frac{u_n \|\mathbf{a}\|_{\mathbb{C}}^2 + b + \langle \mathbf{a}, \tilde{\mathbf{u}} \rangle_{\mathbb{R}}}{1 + \|\mathbf{a}\|_{\mathbb{C}}^2} \right) \in X.$$

We parametrize the HD correspondence as

$$\begin{aligned} \mathbb{C}^{n-1} \times \mathbb{C} &\longrightarrow \mathcal{H}_X \subseteq \mathbb{C}^n \times \mathbb{C}^n \\ (\tilde{\mathbf{z}}, s) &\mapsto \left((\tilde{\mathbf{z}}, \langle \mathbf{a}, \tilde{\mathbf{z}} \rangle_{\mathbb{R}} + b), (\tilde{\mathbf{z}} + s\bar{\mathbf{a}}, \langle \mathbf{a}, \tilde{\mathbf{z}} \rangle_{\mathbb{R}} + b - s) \right). \end{aligned}$$

If $X = V(z_1 - a_1, \dots, z_{n-1} - a_{n-1}) \subseteq \mathbb{C}^n$, then the critical point is $(\mathbf{a}, u_n) \in X$.

The following result relates the HD polynomial of a variety X to the one of its dual variety $X^\vee$, see Subsection 4.6.

Theorem 6.0.11. *Let X be an affine cone, then*

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \mathrm{HDpol}_{X^\vee,\mathbf{v},\mathbf{u}}(\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}} - t^2).$$

Proof. Consider a point $(\mathbf{u}, \mathbf{v}) \in \mathbb{V}^2$ and let $(\mathbf{z}, \mathbf{w}) \in X_\circ$ be a zero of the Hermitian critical ideal. Then, by Proposition 6.0.2, $\langle \mathbf{u}-\mathbf{z}, \mathbf{v}-\mathbf{w} \rangle_{\mathbb{R}}$ is a root of $\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}$. Applying Theorem 4.6.6, $(\mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{z})$ is a zero of the Hermitian critical ideal of $X^\vee$ of $(\mathbf{v}, \mathbf{u})$. Hence, again by Proposition 6.0.2, $\langle \mathbf{w}, \mathbf{z} \rangle_{\mathbb{R}}$ is a root of $\mathrm{HDpol}_{X^\vee,(\mathbf{v},\mathbf{u})}$. In the end, since X is an affine cone, summing the two roots we get

$$\langle \mathbf{u} - \mathbf{z}, \mathbf{v} - \mathbf{w} \rangle_{\mathbb{R}} + \langle \mathbf{w}, \mathbf{z} \rangle_{\mathbb{R}} = \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}} - \langle \mathbf{z}, \mathbf{v} - \mathbf{w} \rangle_{\mathbb{R}} - \langle \mathbf{u} - \mathbf{z}, \mathbf{w} \rangle_{\mathbb{R}} = \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}},$$

thus we can rewrite the solutions of $\mathrm{HDpol}_{X^\vee,(\mathbf{v},\mathbf{u})}$ as

$$\langle \mathbf{w}, \mathbf{z} \rangle_{\mathbb{R}} = \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}} - \langle \mathbf{u} - \mathbf{z}, \mathbf{v} - \mathbf{w} \rangle_{\mathbb{R}}.$$

□

The next result states the quite obvious fact that the points of the variety are at distance zero from the variety itself.

Proposition 6.0.12. *Let X be an algebraic variety, then X is contained in the zero locus of* $\mathrm{HDpol}_{X,\mathbf{u}}(0)$.

Proof. If we consider generic points such that $\mathbf{v} = \bar{\mathbf{u}} \in X$ clearly holds that $(\mathbf{u}, \bar{\mathbf{u}}) \in X_\circ$ is a solution of the Hermitian critical ideal and in particular is a critical point $(\mathbf{z}, \mathbf{w})$ such that $\langle \mathbf{u} - \mathbf{z}, \mathbf{v} - \mathbf{w} \rangle_{\mathbb{R}} = \langle \mathbf{0}, \mathbf{0} \rangle_{\mathbb{R}} = 0$. □

Example 6.0.13. (Parabola) The HD polynomial of the parabola $X = V(z_2 - z_1^2) \subseteq \mathbb{C}^2$ has degree 5 in t^2 and consist of 131 monomials. For real points the HD polynomial consists in 79 terms and decomposes

$$\mathrm{HDpol}_{X,\mathbf{u}}(t^2) = (4u_2t^2 + (u_2 - u_1^2)(4u_2 + 1))^2(16t^6 + c_2t^4 + c_1t^2 + c_0)$$

where

$$\begin{aligned}
c_2 &= -8(6u_1^2 + 2u_2^2 + 4u_2 - 1),\\
c_1 &= 48u_1^4 + 32u_1^2u_2^2 - 8u_1^2u_2 + 32u_2^3 + 20u_1^2 + 8u_2^2 - 8u_2 + 1,\\
c_0 &= -(u_1^2 - u_2)^2(16u_1^2 + (4u_2 - 1)^2).
\end{aligned}$$

Example 6.0.14. (Circle) The HD polynomial of the unit circle $X = V(z_1^2 + z_2^2 - 1) \subseteq \mathbb{C}^2$ has degree 6 in t^2 and consist in 759 monomials, for real points we get

$$\mathrm{HDpol}_{X,\mathbf{u}}(t^2) = (t^2 - \|\mathbf{u}\|_{\mathbb{C}}^2 + 1)^2(2t^2 - \|\mathbf{u}\|_{\mathbb{C}}^2 + 2)^2(t^2 - 2t - \|\mathbf{u}\|_{\mathbb{C}}^2 + 1)(t^2 + 2t - \|\mathbf{u}\|_{\mathbb{C}}^2 + 1).$$

Example 6.0.15. (Fermat curve of degree 2). The Fermat curve

$$X = V(z_1^2 + z_2^2 - z_3^2) \subseteq \mathbb{P}^2$$

is the projective closure of the variety of Example 4.2.16, moreover it satisfies the equality $X^\vee = X$. From Theorem 4.5.2, we already know $\mathrm{vHDdeg}(X) = 2$ and $\mathrm{HDdeg}(X) = \{2\}$. The polynomial defining the vHD discriminant $\mathrm{v}\Xi_X$ is

$$((v_1u_2 - u_1v_2)^2 - (v_1u_3 + u_1v_3)^2 - (v_2u_3 + u_2v_3)^2)^2$$

and in particular the generalized polynomial defining the HD discriminant Ξ_X is

$$((\bar{u}_1u_2 - u_1\bar{u}_2)^2 - (\bar{u}_1u_3 + u_1\bar{u}_3)^2 - (\bar{u}_2u_3 + u_2\bar{u}_3)^2)^2.$$

The vHD polynomial is

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = 4t^4 - 4(u_1v_1 + u_2v_2 + u_3v_3)t^2 + (u_1^2 + u_2^2 - u_3^2)(v_1^2 + v_2^2 - v_3^2),$$

in particular we obtain

$$\mathrm{HDpol}_{X,\mathbf{u}}(t^2) = 4t^4 - 4\|\mathbf{u}\|_{\mathbb{C}}^2t^2 + |u_1^2 + u_2^2 - u_3^2|^2.$$

7 Determinantal varieties

Let us identify the vector space $\mathbb{V}$ with the space of matrices $\mathcal{M}_n^m := \mathbb{C}^n \otimes \mathbb{C}^m$ or simply $\mathcal{M}_n$ if $n = m$.

We discuss the Hermitian Distance degree of $\mathrm{rk}_r(\mathcal{M}_n^m)$ the subvariety of matrices with rank bounded by some $r \in [\min\{n, m\}]$. Well known facts are $\mathrm{rk}_r(\mathcal{M}_n^m)^\vee = \mathrm{rk}_{\min\{n,m\}-r}(\mathcal{M}_n^m) \subseteq \mathcal{M}_n^m$, see [GKZ94, Proposition 4.11], and $\mathrm{rk}_r(\mathcal{M}_n^m)_{\mathrm{sing}} = \mathrm{rk}_{r-1}(\mathcal{M}_n^m)$, for example see [BV88, Proposition 1.1].

For the Euclidean case see [OSS14; Dra+16].

We can write the Hermitian inner product

$$\langle A, B\rangle_{\mathbb{C}} = \sum_{j_1=1}^{n} \sum_{j_2=1}^{m} A_{j_1 j_2} \bar{B}_{j_1 j_2} = \mathrm{Tr}(AB^H).$$

and $\| \ \|_{\mathbb{C}}$ it is also called the *Frobenius norm* or the *Hilbert–Schmidt norm.*

We state the matrix version of the Terracini Lemma, for a general version see [Lan12, Subsection 5.3].

Theorem 7.0.1 (Terracini Lemma for matrices). *Let $A \in \mathrm{rk}_r(\mathcal{M}_n^m) \subseteq \mathcal{M}_n^m$, then*

$$T_A \, \mathrm{rk}_r(\mathcal{M}_n^m) = \mathrm{span}\{T_{A_1} \mathrm{rk}_1(\mathcal{M}_n^m), \ldots, T_{A_r} \mathrm{rk}_1(\mathcal{M}_n^m)\}$$

where $A_1, \ldots, A_r \in \mathrm{rk}_1(\mathcal{M}_n^m)$ satisfy $A \in \mathrm{span}\{A_1, \ldots, A_r\}$.

We report a classical result about critical points of the Hermitian distance function from determinantal varieties. We firstly prove two useful lemmas and recall the singular value decomposition.

Lemma 7.0.2. *Let $A = a^{(1)} \otimes a^{(2)} = a^{(1)}(a^{(2)})^T \in \mathcal{M}_n^m$ and $B = b^{(1)} \otimes b^{(2)} = b^{(1)}(b^{(2)})^T \in \mathcal{M}_n^m$ for $a^{(1)}, b^{(1)} \in \mathbb{C}^n$ and $a^{(2)}, b^{(2)} \in \mathbb{C}^m$, then*

$$\langle A, B\rangle_{\mathbb{C}} = \langle a^{(1)}, b^{(1)}\rangle_{\mathbb{C}} \langle a^{(2)}, b^{(2)}\rangle_{\mathbb{C}}.$$

Proof. The assertion follows from the equalities

$$\langle A,B\rangle_{\mathbb{C}} = \mathrm{Tr}(AB^H) = \mathrm{Tr}\left(a^{(1)}(a^{(2)})^T\bar{b}^{(2)}(b^{(1)})^H\right) = \sum_{j_2=1}^{m} a^{(2)}_{j_2}\bar{b}^{(2)}_{j_2}\,\mathrm{Tr}(a^{(1)}(b^{(1)})^H)$$

$$= \sum_{j_2=1}^{m} a^{(2)}_{j_2}\bar{b}^{(2)}_{j_2}\sum_{j_1=1}^{n} a^{(1)}_{j_1}\bar{b}^{(1)}_{j_1} = \langle a^{(2)},b^{(2)}\rangle_{\mathbb{C}}\langle a^{(1)},b^{(1)}\rangle_{\mathbb{C}}.$$

□

Lemma 7.0.3. *Let $A \in \mathcal{M}_n^m$ and $a \in \mathbb{C}^n$, $b \in \mathbb{C}^m$, then:*

i) *If $\langle A, a\otimes\mathbb{C}^m\rangle_{\mathbb{C}} = 0$, then $\langle \mathrm{Col}(A), a\rangle_{\mathbb{C}} = 0$.*

ii) *If $\langle A, \mathbb{C}^n\otimes b\rangle_{\mathbb{C}} = 0$, then $\langle \mathrm{Row}(A), b\rangle_{\mathbb{C}} = 0$.*

Proof. i) Let $\{e_1,\dots,e_m\}$ be the canonical basis of $\mathbb{C}^m$, for any $k=1,\dots,m$ we get

$$0 = \langle A, a\otimes e_k\rangle_{\mathbb{C}} = \mathrm{Tr}(Ae_k a^H) = \mathrm{Tr}(A_{\cdot,k}a^H) = \langle A_{\cdot,k}, a\rangle_{\mathbb{C}}$$

and thus it holds $\langle \mathrm{Col}(A), a\rangle_{\mathbb{C}} = 0$.

Similarly for ii). □

Definition 7.0.4. Let $\|\ \|_2$ be the 2-norm on $\mathbb{C}^n$ and $\mathbb{C}^m$, then the *induced matrix norm* of $\mathcal{M}_n^m$ is defined to be

$$\|A\|_2 \coloneqq \sup_{\mathbf{z}\in\mathbb{C}^m\setminus\{0\}} \frac{\|A\mathbf{z}\|_2}{\|\mathbf{z}\|_2} = \max_{\substack{\mathbf{z}\in\mathbb{C}^m\\ \|\mathbf{z}\|_2=1}} \|A\mathbf{z}\|_2 \qquad \text{for } A\in\mathcal{M}_n^m.$$

We present now a well known decomposition of matrices. Let us recall that the *singular values* $\sigma_1,\dots,\sigma_{\mathrm{rk}(A)} \in \mathbb{R}$ of a matrix $A\in\mathcal{M}_n^m$ are uniquely determined as the square roots of the non zero eigenvalues of the matrix $AA^H \in \mathcal{M}_n$.

Theorem (SVD). *For a matrix $A\in\mathcal{M}_n^m$ there exists a* singular value decomposition (SVD) *of the form $A = U\Sigma V^H$ where $U\in\mathcal{M}_n$ and $V\in\mathcal{M}_m$ are unitary and*

$$\Sigma = \mathrm{diag}(\sigma_1,\dots,\sigma_{\mathrm{rk}(A)},0,\dots,0) \in \mathcal{M}_n^m$$

with $\sigma_1 \geq \ldots \geq \sigma_{\mathrm{rk}(A)} > 0$ singular values. *In particular,*

$$A = \sum_{k=1}^{\mathrm{rk}(A)} \sigma_k U_{\cdot,k}V^H_{k,\cdot} = \sum_{k=1}^{\mathrm{rk}(A)} \sigma_k U_{\cdot,k}\otimes \overline{V}_{\cdot,k}$$

Moreover, if A is real then also U and V can be chosen to be real.

Remark 7.0.5. We point out that in $\mathcal{M}_n^m$ the Hermitian norm $\|\ \|_{\mathbb{C}}$ and the induced norm $\|\ \|_2$ do not coincide. In general, if $\sigma_1 \geq \ldots \geq \sigma_{\min\{n,m\}}$ are the singular values of $A\in\mathcal{M}_n^m$ then

$$\|A\|_{\mathbb{C}}^2 = \sum_{k=1}^{\min\{n,m\}} \sigma_k^2 \qquad \text{and} \qquad \|A\|_2^2 = \sigma_1^2.$$

7.1 General matrices

We are ready to prove the Eckart-Young-Mirsky Theorem. This result completely solves the Hermitian distance problem for determinantal varieties.

Theorem (Complex Eckart-Young-Mirsky). *Let $A \in \mathcal{M}_n^m$ with $A = U\Sigma V^H$ a SVD and let $1 \leq r \leq \mathrm{rk}(A)$. All the critical points of the Hermitian distance function from A to the variety $\mathrm{rk}_r(\mathcal{M}_n^m) \setminus \mathrm{rk}_{r-1}(\mathcal{M}_n^m)$ are $U(\Sigma_{k_1} + \ldots + \Sigma_{k_r})V^H$ with $1 \leq k_1 < \ldots < k_r \leq \mathrm{rk}(A)$, where $\Sigma_k \in \mathcal{M}_n^m$ is everywhere zero apart from the (k,k) entry equal to the k-th singular value for $k = 1, \ldots, \mathrm{rk}(A)$.*

Proof. The matrix $B \coloneqq U(\Sigma_{k_1}+\ldots+\Sigma_{k_r})V^H$ is a critical point if and only if $A-B$ is orthogonal to the tangent space

$$T_B \,\mathrm{rk}_r(\mathcal{M}_n^m) = (U_{\cdot,k_1} \otimes \mathbb{C}^m + \mathbb{C}^n \otimes \overline{V}_{\cdot,k_1}) + \ldots + (U_{\cdot,k_r} \otimes \mathbb{C}^m + \mathbb{C}^n \otimes \overline{V}_{\cdot,k_r}),$$

or in other words $A - B \perp_{\mathbb{C}} T_B \,\mathrm{rk}_r(\mathcal{M}_n^m)$. Note that,

$$A - B = U(\Sigma_{j_1} + \ldots + \Sigma_{j_{\mathrm{rk}(A)-r}})V^H = \sum_{\ell=1}^{\mathrm{rk}(A)-r} \sigma_{j_\ell} U_{\cdot,j_\ell} \otimes \overline{V}_{\cdot,j_\ell}$$

where $\{j_1, \ldots, j_{\mathrm{rk}(A)-r}\} = \{1, \ldots, \mathrm{rk}(A)\} \setminus \{k_1, \ldots, k_r\}$.

Let now $\{e_1, \ldots, e_m\}$ be a basis of $\mathbb{C}^m$, by Lemma 7.0.2 we get

$$\langle \sigma_{j_s} U_{\cdot,j_s} \otimes \overline{V}_{\cdot,j_s}, U_{\cdot,k_h} \otimes e_\ell \rangle_{\mathbb{C}} = \sigma_{j_s} \langle U_{\cdot,j_s}, U_{\cdot,k_h} \rangle_{\mathbb{C}} \langle \overline{V}_{\cdot,j_s}, e_\ell \rangle_{\mathbb{C}} = 0$$

since $U_{\cdot,j_s}$ and $U_{\cdot,k_h}$ are two different columns of the unitary matrix U. Similarly for the spaces $\mathbb{C}^n \otimes \overline{V}_{\cdot,k_s}$ and then $A - B$ is orthogonal to $T_B \,\mathrm{rk}_r(\mathcal{M}_n^m)$.

To prove that these are all the critical points, let B of rank k be a critical point and consider the two singular value decompositions

$$B = U'(\Sigma'_1 + \ldots + \Sigma'_r)(V')^H \quad \text{and} \quad A - B = U''(\Sigma''_1 + \ldots + \Sigma''_\ell)(V'')^H$$

so that

$$T_B \,\mathrm{rk}_r(\mathcal{M}_n^m) = (U'_{\cdot,1} \otimes \mathbb{C}^m + \mathbb{C}^n \otimes \overline{V}'_{\cdot,1}) + \ldots + (U'_{\cdot,r} \otimes \mathbb{C}^m + \mathbb{C}^n \otimes \overline{V}'_{\cdot,r}).$$

Since $A - B \perp_{\mathbb{C}} T_B \,\mathrm{rk}_r(\mathcal{M}_n^m)$, by Lemma 7.0.3 we get

$$\langle \mathrm{Col}(A-B), U'_{\cdot,k} \rangle_{\mathbb{C}} = \langle \mathrm{Row}(A-B), \overline{V}'_{\cdot,k} \rangle_{\mathbb{C}} = 0.$$

for $k = 1, \ldots, r$. In particular, $\mathrm{Col}(A-B)$ has dimension at most $n-r$ and $\mathrm{Row}(A-B)$ has dimension at most $m-r$, thus $\mathrm{rk}(A-B) \leq \min\{n, m\} - r$.

By definition, $\mathrm{Col}(A-B) = \mathrm{span}\{U''_{\cdot,1}, \ldots, U''_{\cdot,\ell}\}$ and $\mathrm{Row}(A-B) = \mathrm{span}\{\overline{V}''_{\cdot,1}, \ldots, \overline{V}''_{\cdot,\ell}\}$ and this implies that the orthonormal collection $U'_{\cdot,1}, \ldots, U'_{\cdot,r}, U''_{\cdot,1}, \ldots, U''_{\cdot,\ell}$ can be completed with

$n-r-\ell$ vectors in $\mathbb{C}^n$ to obtain a unitary matrix $U''' \in \mathcal{M}_n$. Similarly, using the first r and ℓ columns of $\overline{V}'$ and $\overline{V}''$ respectively, we complete to a unitary matrix $\overline{V}''' \in \mathcal{M}_m$. Then, we can write

$$B = U''' \begin{bmatrix} \Sigma' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} (V''')^H \quad \text{and} \quad A - B = U''' \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma'' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} (V''')^H$$

where $\Sigma' = \operatorname{diag}(\sigma_1', \dots, \sigma_r') \in \mathcal{M}_r$ and $\Sigma'' = \operatorname{diag}(\sigma_1'', \dots, \sigma_\ell'') \in \mathcal{M}_\ell$. Thus,

$$A = (A - B) + B = U''' \begin{bmatrix} \Sigma' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma'' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} (V''')^H$$

can be reordered to obtain a singular value decomposition and the thesis is proved. ☐

As anticipated this theorem implies the following result.

Corollary 7.1.1. *Let* $X = \operatorname{rk}_r(\mathcal{M}_n^m)$ *then* $\operatorname{HDdeg}(X) = \{\binom{\min\{n,m\}}{r}\}$.

Note that in this particular family of varieties, the HDdeg collapses to a single value.

We now compute the number of solutions of the Hermitian critical ideal.

Lemma 7.1.2. *Let* $\mathbf{u}, \mathbf{v} \in \mathcal{M}_n^m$ *and* $X = \operatorname{rk}_r(\mathcal{M}_n^m) \subseteq \mathcal{M}_n^m$ *with* $r \le \operatorname{rk}(\mathbf{u}\mathbf{v}^T)$, *then the number of solutions of the Hermitian critical ideal of* $(\mathbf{u}, \mathbf{v})$ *is* $\binom{\operatorname{rk}(\mathbf{u}\mathbf{v}^T)}{r}$.

Proof. Consider a point $(\mathbf{z}, \mathbf{w}) \in X \times \overline{X}$ and the formulations

$$\mathbf{z} = \sum_{k=1}^r \lambda_{\mathbf{z}}^{(k)} \mathbf{z}^{(k,1)} \otimes \mathbf{z}^{(k,2)} \qquad \text{and} \qquad \mathbf{w} = \sum_{j=1}^r \lambda_{\mathbf{w}}^{(j)} \mathbf{w}^{(j,1)} \otimes \mathbf{w}^{(j,2)}$$

where $\{\mathbf{z}^{(k,1)}\}_{k=1}^r, \{\mathbf{w}^{(j,1)}\}_{j=1}^r \subseteq \mathbb{C}^n$, $\{\mathbf{z}^{(k,2)}\}_{k=1}^r, \{\mathbf{w}^{(j,2)}\}_{j=1}^r \subseteq \mathbb{C}^m$ and $0 \ne \lambda_{\mathbf{z}}^{(k)}, \lambda_{\mathbf{w}}^{(j)} \in \mathbb{C}$ for any $k, j = 1, \dots, r$ to write

$$\begin{aligned} \langle \mathbf{z} - \mathbf{u}, \mathbf{w} - \mathbf{v} \rangle_{\mathbb{R}} &= \left\langle \sum_{k=1}^r \lambda_{\mathbf{z}}^{(k)} \mathbf{z}^{(k,1)} \otimes \mathbf{z}^{(k,2)} - \mathbf{u}, \sum_{j=1}^r \lambda_{\mathbf{w}}^{(j)} \mathbf{w}^{(j,1)} \otimes \mathbf{w}^{(j,2)} - \mathbf{v} \right\rangle_{\mathbb{R}} \\ &= \sum_{k=1}^r \sum_{j=1}^r \lambda_{\mathbf{z}}^{(k)} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{z}^{(k,1)}, \mathbf{w}^{(j,1)} \rangle_{\mathbb{R}} \langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(j,2)} \rangle_{\mathbb{R}} - \sum_{k=1}^r \lambda_{\mathbf{z}}^{(k)} \langle \mathbf{z}^{(k,1)} \otimes \mathbf{z}^{(k,2)}, \mathbf{v} \rangle_{\mathbb{R}} \\ &\quad - \sum_{j=1}^r \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{u}, \mathbf{w}^{(j,1)} \otimes \mathbf{w}^{(j,2)} \rangle_{\mathbb{R}} + \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}}. \end{aligned}$$

The derivative with respect to $z_\ell^{(k,1)}$ for $\ell = 1, \dots, n$ is

$$\lambda_{\mathbf{z}}^{(k)} \left(\sum_{j=1}^r \lambda_{\mathbf{w}}^{(j)} \mathbf{w}_\ell^{(j,1)} \langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(j,2)} \rangle_{\mathbb{R}} - \langle e_\ell \otimes \mathbf{z}^{(k,2)}, \mathbf{v} \rangle_{\mathbb{R}} \right)$$

and we want to study the zero locus. By considering the derivatives with respect to any $z_\ell^{(k,s)}$ and $w_\ell^{(j,s)}$ for $s = 1, 2$, we get the conditions

$$\mathbf{u}^T\mathbf{w}^{(j,1)} = \sum_{k=1}^{r} \lambda_{\mathbf{z}}^{(k)} \langle \mathbf{z}^{(k,1)}, \mathbf{w}^{(j,1)}\rangle_{\mathbb{R}} \mathbf{z}^{(k,2)}, \qquad \mathbf{u}\mathbf{w}^{(j,2)} = \sum_{k=1}^{r} \lambda_{\mathbf{z}}^{(k)} \langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(j,2)}\rangle_{\mathbb{R}} \mathbf{z}^{(k,1)}, \tag{7.1.1}$$

$$\mathbf{v}^T\mathbf{z}^{(k,1)} = \sum_{j=1}^{r} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{z}^{(k,1)}, \mathbf{w}^{(j,1)}\rangle_{\mathbb{R}} \mathbf{w}^{(j,2)}, \qquad \mathbf{v}\mathbf{z}^{(k,2)} = \sum_{j=1}^{r} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(j,2)}\rangle_{\mathbb{R}} \mathbf{w}^{(j,1)}. \tag{7.1.2}$$

From equations (7.1.2), by knowing $\{\mathbf{z}^{(k,1)}\}_{k=1}^r$, we are able to choose $\{\mathbf{w}^{(j,2)}\}_{j=1}^r$ basis of the subspace given by the images under $\mathbf{v}^T$ and similarly for the basis $\{\mathbf{w}^{(j,1)}\}_{j=1}^r$. One has to convince that there is no essential difference in the choice of these bases. On the other hand, by applying the maps $\mathbf{u}$ and $\mathbf{u}^T$ respectively we get

$$\mathbf{u}\mathbf{v}^T\mathbf{z}^{(\ell,1)} = \sum_{k=1}^{r} \left(\sum_{j=1}^{r} \lambda_{\mathbf{z}}^{(k)} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{z}^{(k,1)}, \mathbf{w}^{(j,1)}\rangle_{\mathbb{R}} \langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(j,2)}\rangle_{\mathbb{R}} \right) \mathbf{z}^{(k,1)},$$

$$\mathbf{u}^T\mathbf{v}\mathbf{z}^{(\ell,2)} = \sum_{k=1}^{r} \left(\sum_{j=1}^{r} \lambda_{\mathbf{z}}^{(k)} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{z}^{(k,1)}, \mathbf{w}^{(j,1)}\rangle_{\mathbb{R}} \langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(j,2)}\rangle_{\mathbb{R}} \right) \mathbf{z}^{(k,2)}$$

for $\ell = 1, \ldots, r$. Thus, by knowing $\{\mathbf{z}^{(k,1)}\}_{k=1}^r$ there is no essentially different choice for the basis $\{\mathbf{z}^{(k,2)}\}_{k=1}^r$. In particular, we need to choose a collection $\{\mathbf{z}^{(k,1)}\}_{k=1}^r$ which generates a r dimensional eigenspace of $\mathbf{u}\mathbf{v}^T$.

Multiplying on the left the left equations of (7.1.1) and (7.1.2) by $(\mathbf{w}^{(j,2)})^T$ and $(\mathbf{z}^{(k,2)})^T$ respectively, we get the equalities

$$\langle \mathbf{z}^{(k,1)} \otimes \mathbf{z}^{(k,2)}, \mathbf{v}\rangle_{\mathbb{R}} = \sum_{j=1}^{r} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{z}^{(k,1)}, \mathbf{w}^{(j,1)}\rangle_{\mathbb{R}} \langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(j,2)}\rangle_{\mathbb{R}},$$

$$\langle \mathbf{u}, \mathbf{w}^{(j,1)} \otimes \mathbf{w}^{(j,2)}\rangle_{\mathbb{R}} = \sum_{k=1}^{r} \lambda_{\mathbf{z}}^{(k)} \langle \mathbf{z}^{(k,1)}, \mathbf{w}^{(k,1)}\rangle_{\mathbb{R}} \langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(k,2)}\rangle_{\mathbb{R}},$$

and we compute the scalars $\lambda_{\mathbf{z}}^{(k)}, \lambda_{\mathbf{w}}^{(j)}$ for $k, j = 1, \ldots, r$ by solving the linear systems provided by these last equations.

At the end, we have $\binom{\mathrm{rk}(\mathbf{u}\mathbf{v}^T)}{r}$ essentially different possible choices for the collection $\{\mathbf{z}^{(k,1)}\}_{k=1}^r$ of eigenvectors of $\mathbf{u}\mathbf{v}^T$ which in turn permits to compute the other vectors and the scalars. □

Proposition 7.1.3. *Let* $X = \mathrm{rk}_r(\mathcal{M}_n^m)$ *then* $\mathrm{vHDdeg}(X) = \binom{\min\{n,m\}}{r}$. *Moreover, the roots in* t^2 *of* $\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}$ *are the sums of* $\min\{n, m\} - r$ *eigenvalues of* $\mathbf{u}\mathbf{v}^T$ *if* $n \le m$ *or of* $\mathbf{u}^T\mathbf{v}$ *if* $n \ge m$.

Proof. The first part follows by Lemma 7.1.2.

To prove the second part, note that from the proof of Lemma 7.1.2 we get that

$$\langle \mathbf{z}, \mathbf{v}\rangle_{\mathbb{R}} = \langle \mathbf{u}, \mathbf{w}\rangle_{\mathbb{R}} = \sum_{k=1}^{r}\sum_{j=1}^{r} \lambda_{\mathbf{z}}^{(k)}\lambda_{\mathbf{w}}^{(j)}\langle \mathbf{z}^{(k,1)}, \mathbf{w}^{(j,1)}\rangle_{\mathbb{R}}\langle \mathbf{z}^{(k,2)}, \mathbf{w}^{(j,2)}\rangle_{\mathbb{R}} = \langle \mathbf{z}, \mathbf{w}\rangle_{\mathbb{R}}.$$

is the sum of r non zero eigenvalues of $\mathbf{u}\mathbf{v}^T$. Let $\lambda_1, \ldots, \lambda_{\min\{n,m\}}$ be the eigenvalues of $\mathbf{u}\mathbf{v}^T$ or $\mathbf{u}^T\mathbf{v}$. Then, if $\mathbf{z}, \mathbf{w}$ is a solution associated to the eigenvalues $\lambda_{k_1}, \ldots, \lambda_{k_r}$ for different $k_1, \ldots, k_r \in [\min\{n,m\}]$, the squared Hermitian distance function takes the value

$$\langle \mathbf{z}, \mathbf{w}\rangle_{\mathbb{R}} - \langle \mathbf{z}, \mathbf{v}\rangle_{\mathbb{R}} - \langle \mathbf{u}, \mathbf{w}\rangle_{\mathbb{R}} + \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}} = \sum_{k\in W}\lambda_k - \sum_{k\in W}\lambda_k - \sum_{k\in W}\lambda_k + \sum_{j=1}^{\min\{n,m\}}\lambda_j = \sum_{k\notin W}\lambda_k$$

where $W = \{k_1, \ldots, k_r\} \subseteq [\min\{n,m\}]$ is a subset of cardinality r. Thus, the claim follows. □

We perform some computations in the simplest non trivial case.

Example 7.1.4. Let $X = \mathrm{rk}_1(\mathcal{M}_2) \subseteq \mathcal{M}_2$, for which it holds $\mathrm{HDdeg}(X) = \{\mathrm{vHDdeg}(X) = 2\}$. We show how to proceed to compute the solution of the Hermitian critical ideal, thus let

$$\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in \mathcal{M}_2 \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \in \mathcal{M}_2.$$

We consider the two matrices

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} \mathbf{u}_{1,\cdot}\mathbf{v}_{1,\cdot}^T & \mathbf{u}_{1,\cdot}\mathbf{v}_{2,\cdot}^T \\ \mathbf{u}_{2,\cdot}\mathbf{v}_{1,\cdot}^T & \mathbf{u}_{2,\cdot}\mathbf{v}_{2,\cdot}^T \end{bmatrix} \in \mathcal{M}_2 \quad \text{and} \quad \mathbf{u}^T\mathbf{v} = \begin{bmatrix} \mathbf{u}_{\cdot,1}^T\mathbf{v}_{\cdot,1} & \mathbf{u}_{\cdot,1}^T\mathbf{v}_{\cdot,2} \\ \mathbf{u}_{\cdot,2}^T\mathbf{v}_{\cdot,1} & \mathbf{u}_{\cdot,2}^T\mathbf{v}_{\cdot,2} \end{bmatrix} \in \mathcal{M}_2.$$

These matrices have two equal eigenvalues

$$\lambda^{\pm} = \frac{\mathrm{Tr}(\mathbf{u}\mathbf{v}^T) \pm \sqrt{(\mathbf{u}_{1,\cdot}\mathbf{v}_{1,\cdot}^T - \mathbf{u}_{2,\cdot}\mathbf{v}_{2,\cdot}^T)^2 + 4\mathbf{u}_{1,\cdot}\mathbf{v}_{2,\cdot}^T\mathbf{u}_{2,\cdot}\mathbf{v}_{1,\cdot}^T}}{2}$$

with associate eigenvectors

$$\mathbf{z}^{(\pm,1)} = \begin{bmatrix} \lambda^{\pm} - 2\mathbf{u}_{2,\cdot}\mathbf{v}_{2,\cdot}^T \\ 2\mathbf{u}_{2,\cdot}\mathbf{v}_{1,\cdot}^T \end{bmatrix} \in \mathbb{C}^2 \quad \text{and} \quad \mathbf{z}^{(\pm,2)} = \begin{bmatrix} \lambda^{\pm} - 2\mathbf{u}_{\cdot,2}^T\mathbf{v}_{\cdot,2} \\ 2\mathbf{u}_{\cdot,2}^T\mathbf{v}_{\cdot,1} \end{bmatrix} \in \mathbb{C}^2$$

respectively. Then we compute the vectors $\mathbf{w}^{(\pm,1)} = \mathbf{v}\mathbf{z}^{(\pm,2)} \in \mathbb{C}^2$ and $\mathbf{w}^{(\pm,2)} = \mathbf{v}^T\mathbf{z}^{(\pm,1)} \in \mathbb{C}^2$ and the scalars

$$\lambda_{\mathbf{z}}^{\pm} = \frac{(\mathbf{w}^{(\pm,2)})^T\mathbf{u}\mathbf{w}^{(1),\pm}}{(\mathbf{z}^{(\pm,1)})^T\mathbf{w}^{(\pm,1)}(\mathbf{z}^{(\pm,2)})^T\mathbf{w}^{(\pm,2)}} \quad \text{and} \quad \lambda_{\mathbf{w}}^{\pm} = \frac{(\mathbf{z}^{(\pm,2)})^T\mathbf{v}\mathbf{z}^{(1),\pm}}{(\mathbf{z}^{(\pm,1)})^T\mathbf{w}^{(\pm,1)}(\mathbf{z}^{(\pm,2)})^T\mathbf{w}^{(\pm,2)}}$$

to get the four matrices $\lambda_{\mathbf{z}}^{\pm}\mathbf{z}^{(\pm,1)} \otimes \mathbf{z}^{(\pm,2)} \in \mathcal{M}_2$ and $\lambda_{\mathbf{w}}^{\pm}\mathbf{w}^{(\pm,2)} \otimes \mathbf{w}^{(\pm,2)} \in \mathcal{M}_2$ which yields the two solutions of the Hermitian critical ideal

$$\left(\lambda_{\mathbf{z}}^{+}\mathbf{z}^{(+,1)} \otimes \mathbf{z}^{(+,2)}, \lambda_{\mathbf{w}}^{+}\mathbf{w}^{(+,1)} \otimes \mathbf{w}^{(+,2)}\right) \quad \text{and} \quad \left(\lambda_{\mathbf{z}}^{-}\mathbf{z}^{(-,1)} \otimes \mathbf{z}^{(-,2)}, \lambda_{\mathbf{w}}^{-}\mathbf{w}^{(-,1)} \otimes \mathbf{w}^{(-,2)}\right).$$

Corollary 7.1.5. *Let $X = \mathrm{rk}_r(\mathcal{M}_n^m)$, then the zero locus of the discriminant $\Delta_{t^2}\,\mathrm{HDpol}_{X,\mathbf{u}}(t^2)$ defines the subset of matrices in $\mathcal{M}_n^m$ that admit two different sums of $\min\{n,m\}-r$ of their singular values to have the same result. If $r = \min\{n,m\}-1$ then*

$$\begin{cases} \mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \det(t^2 I_n - \mathbf{u}\mathbf{v}^T) & \text{when } n \le m, \\ \mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \det(t^2 I_m - \mathbf{v}^T\mathbf{u}) & \text{when } n \ge m, \end{cases}$$

while if $r = 1$ then

$$\begin{cases} \mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \det\left((\mathrm{Tr}(\mathbf{u}\mathbf{v}^T) - t^2)I_n - \mathbf{v}\mathbf{u}^T\right) & \text{when } n \le m, \\ \mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \det\left((\mathrm{Tr}(\mathbf{u}\mathbf{v}^T) - t^2)I_m - \mathbf{u}\mathbf{v}^T\right) & \text{when } n \ge m. \end{cases}$$

In general, let $\lambda_1, \dots, \lambda_{\min\{n,m\}}$ be the eigenvalues of $\mathbf{u}\mathbf{v}^T$ if $n \le m$ or of $\mathbf{v}\mathbf{u}^T$ if $n \ge m$, then

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \prod_{W \subseteq [\min\{n,m\}] \text{ of cardinality } \min\{n,m\}-r} \left(t^2 - \sum_{k \notin W} \lambda_k\right).$$

Example 7.1.6. Let $X = \mathrm{rk}_1(\mathcal{M}_2^m) \subseteq \mathcal{M}_2^m$, then

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \det(t^2 I_2 - \mathbf{u}\mathbf{v}^T) = t^4 - \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}} t^2 + \det(\mathbf{u}\mathbf{v}^T).$$

The HD polynomial then equals

$$\mathrm{HDpol}_{X,\mathbf{u}}(t^2) = t^4 - \|\mathbf{u}\|_{\mathbb{C}}^2 t^2 + \det(\mathbf{u}\mathbf{u}^H)$$

and since $\Delta_{t^2}\,\mathrm{HDpol}_{X,\mathbf{u}}(t^2) = \|\mathbf{u}\|_{\mathbb{C}}^4 - 4\det(\mathbf{u}\mathbf{u}^H) \ge 0$ by the inequality of arithmetic and geometric means, we obtain two critical points if and only if $\det(\mathbf{u}\mathbf{u}^H) \neq 0$ or equivalently $\mathbf{u}$ has two positive singular values.

Let now $m = 2$, the vHD discriminant is defined by the polynomial

$$\Delta_{t^2}\,\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}}^2 - 4|\det(\mathbf{u}\mathbf{v}^T)|.$$

Letting $\mathbf{v} = \bar{\mathbf{u}}$, the inequality $\det(\mathbf{u}\mathbf{u}^H) \neq 0$ is equivalent to $\det(\mathbf{u}) \neq 0$ and the two critical points have same distance if and only if the equation of the HD discriminant

$$\Delta_{t^2}\,\mathrm{HDpol}_{X,\mathbf{u}}(t^2) = \left(\|\mathbf{u}\|_{\mathbb{C}}^2 + 2|\det(\mathbf{u})|\right)\left(\|\mathbf{u}\|_{\mathbb{C}}^2 - 2|\det(\mathbf{u})|\right)$$

vanishes. If $\mathbf{u}$ is real it becomes

$$\Delta_{t^2}\,\mathrm{HDpol}_{X,\mathbf{u}}(t^2) = \left((u_{11}+u_{22})^2 + (u_{12}-u_{21})^2\right)\left((u_{11}-u_{22})^2 + (u_{12}+u_{21})^2\right).$$

Remark 7.1.7. Note that, in Example 7.1.6, the equation defining the HD discriminant is nonnegative. The fact that this polynomial does not change sign could have been anticipated by Proposition 7.1.2 considering Corollary 5.0.5. Also the discriminant of the HD polynomial is nonnegative. We will see this is not a special case in Subsection 7.3.

7.2 Symmetric matrices

We consider here the subspace of symmetric matrices $\mathrm{S}_n \subseteq \mathcal{M}_n$.

We discuss the Hermitian Distance degree of $\mathrm{rk}_r(\mathrm{S}_n)$ the subvariety of symmetric matrices with rank bounded by some $r \in [\min\{n,m\}]$. Similarly to the general case, the equality $\mathrm{rk}_r(\mathrm{S}_n)^\vee = \mathrm{rk}_{n-r}(\mathrm{S}_n) \subseteq \mathrm{S}_n$ can be proved and $\mathrm{rk}_r(\mathrm{S}_n)_{\mathrm{sing}} = \mathrm{rk}_{r-1}(\mathrm{S}_n)$ follows by group actions, for example see Subsection 7.4.

As in the general case it is straightforward to see the following result.

Corollary 7.2.1. *Let* $X = \mathrm{rk}_r(\mathrm{S}_n)$ *then* $\mathrm{HDdeg}(X) = \{\binom{n}{r}\}$.

We now compute the number of solutions of the Hermitian critical ideal.

Lemma 7.2.2. *Let* $\mathbf{u}, \mathbf{v} \in \mathcal{M}_n$ *and* $X = \mathrm{rk}_r(\mathrm{S}_n) \subseteq \mathcal{M}_n$ *with* $r \leq \mathrm{rk}((\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T))$, *then the number of solutions of the Hermitian critical ideal of* $(\mathbf{u}, \mathbf{v})$ *is* $\binom{\mathrm{rk}((\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T))}{r}$.

Proof. Consider a point $(\mathbf{z}, \mathbf{w}) \in X \times \overline{X}$ and the formulations

$$\mathbf{z} = \sum_{k=1}^{r} \lambda_{\mathbf{z}}^{(k)} (\mathbf{z}^{(k)})^{\otimes 2} \qquad \text{and} \qquad \mathbf{w} = \sum_{j=1}^{r} \lambda_{\mathbf{w}}^{(j)} (\mathbf{w}^{(j)})^{\otimes 2}$$

where $\{\mathbf{z}^{(k)}\}_{k=1}^r, \{\mathbf{w}^{(j)}\}_{j=1}^r \subseteq \mathbb{C}^n$ and $0 \neq \lambda_{\mathbf{z}}^{(k)}, \lambda_{\mathbf{w}}^{(j)} \in \mathbb{C}$ for $k, j = 1, \ldots, r$ to write

$$\begin{aligned}\langle \mathbf{z}-\mathbf{u}, \mathbf{w}-\mathbf{v}\rangle_{\mathbb{R}} = &\sum_{k=1}^{r}\sum_{j=1}^{r} \lambda_{\mathbf{z}}^{(k)} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{z}^{(k)}, \mathbf{w}^{(j)}\rangle_{\mathbb{R}}^2 - \sum_{k=1}^{r} \lambda_{\mathbf{z}}^{(k)} \langle (\mathbf{z}^{(k)})^{\otimes 2}, \mathbf{v}\rangle_{\mathbb{R}} \\ &- \sum_{j=1}^{r} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{u}, (\mathbf{w}^{(k)})^{\otimes 2}\rangle_{\mathbb{R}} + \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}}.\end{aligned}$$

The derivative with respect to $z_\ell^{(k)}$ for $\ell = 1, \ldots, n$ is

$$\lambda_{\mathbf{z}}^{(k)} \left(2 \sum_{j=1}^{r} \lambda_{\mathbf{w}}^{(j)} \mathbf{w}_\ell^{(j)} \langle \mathbf{z}^{(k)}, \mathbf{w}^{(j)}\rangle_{\mathbb{R}} - \langle e_\ell \otimes \mathbf{z}^{(k)}, \mathbf{v}\rangle_{\mathbb{R}} - \langle \mathbf{z}^{(k)} \otimes e_\ell, \mathbf{v}\rangle_{\mathbb{R}} \right)$$

and we want to study the zero locus. By considering the derivatives with respect to any $z_\ell^{(k)}$ and $w_\ell^{(j)}$, we get the conditions

$$\begin{aligned}(\mathbf{u}+\mathbf{u}^T)\mathbf{w}^{(j)} &= 2\sum_{k=1}^{r} \lambda_{\mathbf{z}}^{(k)} \langle \mathbf{z}^{(k)}, \mathbf{w}^{(j)}\rangle_{\mathbb{R}} \mathbf{z}^{(k)}, \\ (\mathbf{v}+\mathbf{v}^T)\mathbf{z}^{(k)} &= 2\sum_{j=1}^{r} \lambda_{\mathbf{w}}^{(j)} \langle \mathbf{z}^{(k)}, \mathbf{w}^{(j)}\rangle_{\mathbb{R}} \mathbf{w}^{(j)}.\end{aligned} \tag{7.2.1}$$

From equations (7.2.1), by knowing $\{\mathbf{z}^{(k)}\}_{k=1}^r$, we are able to choose $\{\mathbf{w}^{(j)}\}_{j=1}^r$ basis of the subspace given by the images under $\mathbf{v}+\mathbf{v}^T$. As in the case of rank r matrices, there is no essential difference in the choice of this basis. On the other hand, by applying the map $\mathbf{u}+\mathbf{u}^T$ to equations (7.2.1), we get

$$(\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T)\mathbf{z}^{(\ell)} = 4\sum_{k=1}^r \left(\sum_{j=1}^r \lambda_{\mathbf{z}}^{(k)}\lambda_{\mathbf{w}}^{(j)}\langle \mathbf{z}^{(k)}, \mathbf{w}^{(j)}\rangle_{\mathbb{R}}^2\right)\mathbf{z}^{(k)},$$

for $\ell = 1,\dots,r$. Thus, we need to choose a collection $\{\mathbf{z}^{(k)}\}_{k=1}^r$ which generates a r dimensional eigenspace of $(\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T)/4$.

Multiplying on the left the equations (7.1.1) and (7.1.2) by $(\mathbf{w}^{(j)})^T$ and $(\mathbf{z}^{(k)})^T$ respectively, we get the equalities

$$\langle (\mathbf{z}^{(k)})^{\otimes 2}, \mathbf{v}+\mathbf{v}^T\rangle_{\mathbb{R}} = 2\sum_{j=1}^r \lambda_{\mathbf{w}}^{(j)}\langle \mathbf{z}^{(k)}, \mathbf{w}^{(j)}\rangle_{\mathbb{R}}^2$$
$$\langle \mathbf{u}+\mathbf{u}^T, (\mathbf{w}^{(j)})^{\otimes 2}\rangle_{\mathbb{R}} = 2\sum_{k=1}^r \lambda_{\mathbf{z}}^{(k)}\langle \mathbf{z}^{(k)}, \mathbf{w}^{(k)}\rangle_{\mathbb{R}}^2,$$

and we compute the scalars $\lambda_{\mathbf{z}}^{(k)}, \lambda_{\mathbf{w}}^{(j)}$ for $k, j = 1,\dots,r$ by solving the linear systems provided by these last equations.

At the end, we have $\binom{\mathrm{rk}((\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T))}{r}$ essentially different possible choices for the collection $\{\mathbf{z}^{(k)}\}_{k=1}^r$ of eigenvectors of $(\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T)/4$ which in turn permits to compute the other vectors and the scalars. □

We list a collection of results that are similar to the general case.

Proposition 7.2.3. *Let* $X = \mathrm{rk}_r(\mathrm{S}_n) \subseteq \mathcal{M}_n$ *then* $\mathrm{vHDdeg}(X) = \binom{n}{r}$. *Moreover, the roots in* t^2 *of* $\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}$ *are the sums of* $n-r$ *non zero eigenvalues of* $(\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T)/4$.

Corollary 7.2.4. *Let* $X = \mathrm{rk}_r(\mathrm{S}_n)$, *then the zero locus of the discriminant* $\Delta_{t^2}\,\mathrm{HDpol}_{X,\mathbf{u}}(t^2)$ *defines the subset of matrices in* $\mathcal{M}_n$ *such that a subset of the singular values of* $\mathbf{u}+\mathbf{u}^T$ *can be summed to zero by additions and subtractions. If* $r = n-1$ *then*

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \det\left(t^2 I_n - \frac{(\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T)}{4}\right)$$

while if $r = 1$ *then*

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \det\left(\left(\mathrm{Tr}\left(\frac{(\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T)}{4}\right) - t^2\right) I_n - \frac{(\mathbf{v}+\mathbf{v}^T)(\mathbf{u}+\mathbf{u}^T)}{4}\right)$$

In general, let $\lambda_1,\dots,\lambda_{\min\{n,m\}}$ *be the eigenvalues of* $(\mathbf{u}+\mathbf{u}^T)(\mathbf{v}+\mathbf{v}^T)/4$, *then*

$$\mathrm{vHDpol}_{X,\mathbf{u},\mathbf{v}}(t^2) = \prod_{W\subseteq[n] \text{ of cardinality } n-r}\left(t^2 - \sum_{k\notin W}\lambda_k\right).$$

Corollary 7.2.5. *Let* $\mathbf{u}, \mathbf{v} \in \mathrm{S}_n$ *then the zero locus of the Hermitian critical ideal of* $\mathrm{rk}_r(\mathcal{M}_n) \subseteq \mathcal{M}_n$ *of* $(\mathbf{u}, \mathbf{v})$ *coincides with the zero locus of the Hermitian critical ideal of* $\mathrm{rk}_r(\mathrm{S}_n) \subseteq \mathcal{M}_n$ *of* $(\mathbf{u}, \mathbf{v})$. *In particular, the critical points of the distance function of* $\mathbf{u}$ *from* $\mathrm{rk}_r(\mathcal{M}_n)$ *are all in* $\mathrm{rk}_r(\mathrm{S}_n)$, *which means that the critical points of a symmetric matrix are all symmetric. Moreover, it holds the equality* $\mathrm{vHDpol}_{\mathrm{rk}_r(\mathrm{S}_n),\mathbf{u},\mathbf{v}}(t^2) = \mathrm{vHDpol}_{\mathrm{rk}_r(\mathcal{M}_n),\mathbf{u},\mathbf{v}}(t^2)$.

Proof. The claim follows from the proof of Lemma 7.2.2 by noting that if $\mathbf{u}$ and $\mathbf{v}$ are symmetric then $(\mathbf{u} + \mathbf{u}^T)(\mathbf{v} + \mathbf{v}^T)/4 = \mathbf{u}\mathbf{v}^T$. □

Example 7.2.6. Let $X = \mathrm{rk}_1(\mathrm{S}_2) = V(z_1z_4 - z_2z_3, z_2 - z_3) \subseteq \mathcal{M}_2$, alternatively to this definition, we can simplify our problem and consider the variety $\hat{X}_2 \subseteq \mathbb{C}^3 \simeq \mathrm{S}_2$ we will study in Example 8.3.5 and the Hermitian form $q(\mathbf{z}, \mathbf{w}) = z_1\bar{w}_1 + 2z_2\bar{w}_2 + z_3\bar{w}_3$. With this choice, the HD polynomial is

$$\mathrm{HDpol}_{\hat{X}_2,\mathbf{u},\mathbf{v}}(t^2) = t^4 - \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}} t^2 + u_2^2 v_2^2 + u_1v_1 + u_3v_3 - u_1v_2^2u_3 - v_1u_2^2v_3$$

and if we consider the point $\mathbf{u}$ as the symmetric matrix $\begin{bmatrix} u_1 & u_2 \\ u_2 & u_3 \end{bmatrix}$ and similarly for $\mathbf{v}$ we obtain the known formula

$$\mathrm{HDpol}_{\hat{X}_2,\mathbf{u},\mathbf{v}}(t^2) = \det(t^2 I_2 - \mathbf{u}\mathbf{v}^T) = t^4 - \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}} t^2 + \det(\mathbf{u}\mathbf{v}^T)$$

from Example 7.1.6.

7.3 Discriminant of a Hermitian matrix

Definition 7.3.1. Consider a matrix $A \in \mathcal{M}_n$ with possibly repeated eigenvalues $\lambda_1, \ldots, \lambda_n$. Its *discriminant* is the product

$$\mathrm{disc}(A) := \Delta_t \det(tI_n - A) = \prod_{k \neq j} (\lambda_k - \lambda_j) = (-1)^{\frac{n(n-1)}{2}} \prod_{1 \leq k < j \leq n} (\lambda_k - \lambda_j)^2.$$

In [Par02, Section 2] Parlett obtains the following formula for the discriminant. Let us indicate for $A \in \mathcal{M}_n^m$ and $\varphi = \{\varphi_1, \ldots, \varphi_k\} \subseteq [n]$ the matrix and vector

$$A(\varphi, :) := \begin{bmatrix} A_{\varphi_1,\cdot} \\ \vdots \\ A_{\varphi_k,\cdot} \end{bmatrix} \in \mathcal{M}_k^m \qquad \text{and} \qquad \mathrm{vec}(A) := \begin{bmatrix} A_{1,\cdot}^T \\ \vdots \\ A_{n,\cdot}^T \end{bmatrix} \in \mathbb{C}^{nm}.$$

Proposition 7.3.2 (Parlett '02). *Let* $A \in \mathcal{M}_n$ *then*

$$\mathrm{disc}(A) = \det(\mathcal{L}_A^T \mathcal{L}_A) = \sum_{\varphi \subseteq [n^2] \text{ of cardinality } n} \det(\mathcal{L}_{A^T}(\varphi, :)) \det(\mathcal{L}_A(\varphi, :))$$

where $\mathcal{L}_A \in \mathcal{M}^n_{n^2}$ is the matrix $\mathcal{L}_A = \begin{bmatrix}\mathrm{vec}(I_n) & \mathrm{vec}(A) & \mathrm{vec}(A^2) & \cdots & \mathrm{vec}(A^{n-1})\end{bmatrix}$. In particular, if A is symmetric its discriminant can be written as a sum of squares.

Using this last proposition we obtain another direct consequence in the case of Hermitian matrices.

Corollary 7.3.3. *The discriminant of a Hermitian matrix $A \in \mathcal{M}_n$ can be written as a sum of squared modules.*

Proof. The assertion follows from Proposition 7.3.2 and the equalities $\mathcal{L}_{A^T} = \mathcal{L}_{\bar{A}} = \overline{\mathcal{L}_A}$ which imply $\det(\mathcal{L}_{A^T}(\varphi, :)) = \overline{\det(\mathcal{L}_A(\varphi, :))}$. □

Corollary 7.3.4. *Let $r \in [\min\{n, m\}]$, the discriminants of the HD polynomials of $\mathrm{rk}_r(\mathcal{M}^m_n)$ and $\mathrm{rk}_r(\mathrm{S}_n)$ can be written as sums of squared modules. Moreover, the real codimension of the HD discriminant of both varieties is greater than or equal to 2.*

Proof. Let $X = \mathrm{rk}_r(\mathcal{M}^m_n)$ and assume without loss of generality $n \le m$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathbf{u}\mathbf{u}^H$, the statement follows from the equalities

$$\begin{aligned}\mathrm{HDpol}_{X,\mathbf{u}}(t^2) &= \prod_{W \subseteq [n] \text{ of cardinality } n-r} \left(t^2 - \sum_{k \notin W} \lambda_k\right)\\ &= \det\left(t^2 I_{\binom{n}{r}} - \begin{bmatrix} \sum_{k\notin W_1} \lambda_k & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \sum_{k \notin W_{\binom{n}{r}}} \lambda_k\end{bmatrix}\right)\end{aligned}$$

exploited in Corollary 7.1.5 where W_j for $j = 1, \dots, \binom{n}{r}$ are the different subsets of $[n]$ of cardinality $n - r$ and Corollary 7.3.3.

Similarly for $\mathrm{rk}_r(\mathrm{S}_n)$ considering the Hermitian matrix $(\mathbf{u} + \mathbf{u}^T)(\mathbf{u}^H + \bar{\mathbf{u}})/4$.

The last part follows from the fact that the sum is non trivial in the coefficients of $\mathbf{u}$. □

In particular, this last results confirms that the set HDdeg contains only one value for both the determinantal varieties and as a consequence we get the complex Eckart-Young-Mirsky Theorem recalled in Subsection 7.1.

We can consider the subset φ, introduced in the previous page, as choosing n entries of a $n \times n$ matrix whose diagonal position are $\{1, n+2, 2n+3, \dots, n^2\}$. If φ contains no diagonal positions then $\mathcal{L}_A(\varphi, :)_{\cdot,1}$ is the zero vector and then $\det(\mathcal{L}_A(\varphi, :)) = 0$. Thus, the number of non zero terms in the sum of the formula given by the Proposition 7.3.2 is bounded by

$$\binom{n^2}{n} - \binom{n(n-1)}{n}. \tag{7.3.1}$$

Moreover, if $\hat{\varphi}$ is the subset containing the symmetric entries to the subset φ, then

$$\det(\mathcal{L}_A(\varphi, :)) = \det(\mathcal{L}_{A^T}(\hat{\varphi}, :)) = \det(\mathcal{L}_A(\hat{\varphi}, :))$$

if A is symmetric and

$$\det(\mathcal{L}_A(\varphi, :)) = \det(\mathcal{L}_{A^T}(\hat{\varphi}, :)) = \det(\mathcal{L}_{\bar{A}}(\hat{\varphi}, :)) = \overline{\det(\mathcal{L}_A(\hat{\varphi}, :))}$$

if A is Hermitian, thus in both cases the bound of formula (7.3.1) can be refined to

$$\frac{\binom{n^2}{n} - \binom{n(n-1)}{n} + 1}{2}. \tag{7.3.2}$$

Example 7.3.5. Let $X = \mathrm{rk}_1(\mathcal{M}_2^m) \subseteq \mathcal{M}_2^m$, we compute the discriminant of the Hermitian matrix $\mathbf{u}\mathbf{u}^H \in \mathcal{M}_2$ where $\mathbf{u} \in \mathcal{M}_2^m$. Thus,

$$\mathcal{L}_{\mathbf{u}\mathbf{u}^H} = \begin{bmatrix}\mathrm{vec}(I_2) & \mathrm{vec}(\mathbf{u}\mathbf{u}^H)\end{bmatrix} = \begin{bmatrix} 1 & \mathbf{u}_{1,\cdot}\mathbf{u}_{1,\cdot}^H \\ 0 & \mathbf{u}_{1,\cdot}\mathbf{u}_{2,\cdot}^H \\ 0 & \mathbf{u}_{2,\cdot}\mathbf{u}_{1,\cdot}^H \\ 1 & \mathbf{u}_{2,\cdot}\mathbf{u}_{2,\cdot}^H \end{bmatrix} \in \mathcal{M}_4^2$$

and a quick computation shows

$$\Delta_{t^2} \operatorname{HDpol}_{X,\mathbf{u}}(t^2) = \operatorname{disc}(\mathbf{u}\mathbf{u}^H) = \left|\|\mathbf{u}_{2,\cdot}\|_{\mathbb{C}}^2 - \|\mathbf{u}_{1,\cdot}\|_{\mathbb{C}}^2\right|^2 + 4|\mathbf{u}_{1,\cdot}\mathbf{u}_{2,\cdot}^H|^2.$$

Remark 7.3.6. Since a Hermitian matrix $A \in \mathcal{M}_2$ can be written $A = \mathbf{u}\mathbf{u}^H$ for some $\mathbf{u} \in \mathcal{M}_2^m$, from Example 7.3.5 follows that $\operatorname{disc}(A)$ can be written as the sum of 5 non zero terms as stated in formula (7.3.1) with $n = 2$. The bound of formula (7.3.2) sharpen the result to 3. We have seen that the final formula basically contains only 2 terms.

Note that, for A symmetric, if $\hat{\varphi}$ differs from φ from an element and the two indicates symmetric entries to each other, then $\det(\mathcal{L}_A(\varphi, :)) = \pm\det(\mathcal{L}_A(\hat{\varphi}, :))$, while if φ contains two symmetric entries, then $\det(\mathcal{L}_A(\varphi, :)) = 0$. Thus, in this case the bound of formula (7.3.2) can be sharpened.

In the same work [Par02], Parlett refines this bound. For a matrix subspace $U \subseteq \mathcal{M}_n^m$ and $\mathcal{B}$ basis of U, we denote with $\mathrm{vec}^{\mathcal{B}}(A)$ the vector of the coefficients of $A \in U$ with respect to the basis $\mathcal{B}$.

Consider the space of symmetric matrices $\mathrm{S}_n \subseteq \mathcal{M}_n$ and let $\mathcal{B}$ the basis of cardinality $\binom{n+1}{2}$ consisting in the matrices

$$e_{k,j} := \begin{cases} e_k e_k^T & \text{if } k = j, \\ \frac{1}{\sqrt{2}}(e_k e_j^T + e_j e_k^T) & \text{if } k \neq j, \end{cases}$$

for $k \leq j$, where $\{e_k\}_{k=1}^n$ is the canonical basis of $\mathbb{C}^n$.

Corollary 7.3.7 (Parlett '02). *Consider a matrix $A \in \mathrm{S}_n$, then*

$$\operatorname{disc}(A) = (\mathcal{L}_A^{\mathcal{B}})^T \mathcal{L}_A = \sum_{\varphi \subseteq [\binom{n+1}{2}] \text{ of cardinality } n} \det(\mathcal{L}_A^{\mathcal{B}}(\varphi, :))^2$$

where $\mathcal{L}_A^{\mathcal{B}} \in \mathcal{M}^n_{\binom{n+1}{2}}$ is the matrix $\mathcal{L}_A = \begin{bmatrix}\mathrm{vec}^{\mathcal{B}}(I) & \mathrm{vec}^{\mathcal{B}}(A) & \mathrm{vec}^{\mathcal{B}}(A^2) & \cdots & \mathrm{vec}^{\mathcal{B}}(A^{n-1})\end{bmatrix}$.

The number of non zero terms in the sum of the formula given by the last proposition is bounded by $\binom{\binom{n+1}{2}}{n} - \binom{\binom{n}{2}}{n}$.

Conjecture 7.3.8. *It holds the bound $\binom{\binom{n+1}{2}}{n} - \binom{\binom{n}{2}}{n}$ for the number of non zero terms in the discriminant of a $n \times n$ Hermitian matrix.*

This last conjecture can not be obtained using the same techiniques of Parlett since the set of Hermitian matrices is not a complex subspace of $\mathcal{M}_n$.

7.4 Orbits in matrix spaces

In this subsection we study the dimension of various determinantal varieties. The following results are well known but not very present in the literature. We start by considering the case of symmetric matrices.

Let $\mathrm{GL}_n \subseteq \mathcal{M}_n$ be the general linear group for which it holds $\dim_{\mathbb{R}} \mathrm{GL}_n = 2n^2$ and let $\mathrm{O}_n \subseteq \mathrm{GL}_n$ be the subgroup of orthogonal matrices for which it holds $\dim_{\mathbb{R}} \mathrm{O}_n = n(n-1)$.

Let $\mathrm{S}_n \subseteq \mathcal{M}_n$ be the subspace of symmetric matrices for which it holds $\dim_{\mathbb{R}} \mathrm{S}_n = n(n+1)$. Let $0 \le r \le n$ and denote $\mathrm{rk}_r(\mathrm{S}_n) := \mathrm{S}_n \cap \mathrm{rk}_r(\mathcal{M}_n)$. From the isomorphism

$$\mathrm{rk}_1(\mathrm{S}_n) = \{\mathbf{u}\mathbf{u}^T \mid \mathbf{u} \in \mathbb{C}^n\} \simeq \mathbb{C}^n/\{\pm 1\}$$

we get $\dim_{\mathbb{R}} \mathrm{rk}_1(\mathrm{S}_n) = 2n$.

Proposition 7.4.1. *It holds* $\dim_{\mathbb{R}} \mathrm{rk}_r(\mathrm{S}_n) = r(2n - r + 1)$.

Proof. Consider the action

$$\begin{aligned} \mathrm{GL}_n \times \mathrm{S}_n &\to \mathrm{S}_n \\ (g, A) &\mapsto g^T A g \end{aligned}$$

The subsets $\mathrm{rk}_r(\mathrm{S}_n) \setminus \mathrm{rk}_{r-1}(\mathrm{S}_n)$ are GL_n orbits. In the case of rank r, for suitable blocks of g the stabilizers are given by the equations

$$\begin{bmatrix} g_{11}^T & g_{21}^T \\ g_{12}^T & g_{22}^T \end{bmatrix} \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11}^T g_{11} & g_{11}^T g_{12} \\ g_{12}^T g_{11} & g_{12}^T g_{12} \end{bmatrix} = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathcal{M}_n$$

thus, $g_{11} \in \mathrm{O}_r$ is orthogonal and from $g_{11}^T g_{12} = \mathbf{0} \in \mathcal{M}_r^{n-r}$ follows $g_{12} = \mathbf{0} \in \mathcal{M}_r^{n-r}$. On the other hand $\begin{bmatrix} g_{21} & g_{22} \end{bmatrix} \in \mathcal{M}_{n-r}^n$. This isotropy group is of real dimension $r(r-1) + 2n(n-r)$ and so is the general isotropy group for rank r, so that the general orbit dimension is

$$\dim_{\mathbb{R}} \mathrm{GL}_n - r(r-1) - 2n(n-r) = r(2n - r + 1).$$

□

We consider now the case of Hermitian matrices.

Let $\mathrm{U}_n \subseteq \mathrm{GL}_n$ be the subgroup of unitary matrices for which it holds $\dim_{\mathbb{R}} \mathrm{U}_n = n^2$ and let $\mathrm{H}_n \subseteq \mathcal{M}_n$ be real subspace of Hermitian matrices for which it holds $\dim_{\mathbb{R}} \mathrm{H}_n = n^2$. Let $0 \leq r \leq n$ and denote $\mathrm{rk}_r(\mathrm{H}_n) := \mathrm{H}_n \cap \mathrm{rk}_r(\mathcal{M}_n)$. From the isomorphism

$$\mathrm{rk}_1(\mathrm{H}_n) = \{\mathbf{u}\mathbf{u}^H \mid \mathbf{u} \in \mathbb{C}^n\} \simeq \mathbb{C}^n/\mathbb{S}^1$$

we get $\dim_{\mathbb{R}} \mathrm{rk}_1(\mathrm{H}_n) = 2n - 1$.

Proposition 7.4.2. *It holds* $\dim_{\mathbb{R}} \mathrm{rk}_r(\mathrm{H}_n) = r(2n - r)$.

Proof. Consider the action

$$\begin{aligned} \mathrm{GL}_n \times \mathrm{H}_n &\to \mathrm{H}_n \\ (g, A) &\mapsto g^H A g \end{aligned}$$

The subsets $\mathrm{rk}_r(\mathrm{H}_n) \setminus \mathrm{rk}_{r-1}(\mathrm{H}_n)$ are GL_n orbits. In the case of rank r, for suitable blocks of g the stabilizers are given by the equations

$$\begin{bmatrix} g_{11}^H & g_{21}^H \\ g_{12}^H & g_{22}^H \end{bmatrix} \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11}^H g_{11} & g_{11}^H g_{12} \\ g_{12}^H g_{11} & g_{12}^H g_{12} \end{bmatrix} = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathcal{M}_n$$

thus, $g_{11} \in \mathrm{U}_r$ is unitary and from $g_{11}^H g_{12} = \mathbf{0} \in \mathcal{M}_r^{n-r}$ follows $g_{12} = \mathbf{0} \in \mathcal{M}_r^{n-r}$. On the other hand $\begin{bmatrix} g_{21} & g_{22} \end{bmatrix} \in \mathcal{M}_{n-r}^n$. This isotropy group is of real dimension $r^2 + 2n(n - r)$ and so is the general isotropy group for rank r, so that the general orbit dimension is

$$\dim_{\mathbb{R}} \mathrm{GL}_n - r^2 - 2n(n - r) = r(2n - r).$$

□

In this case, the subset of Hermitian matrices of signature (r, s) is also a GL_n orbit. Let $\mathrm{U}_{n,r,s} := \{g \in \mathcal{M}_n \mid g^H I_{r,s} g = I_{r,s}\}$ where

$$I_{r,s} = \begin{bmatrix} I_r & & \\ & -I_s & \\ & & \mathbf{0} \end{bmatrix} \in \mathcal{M}_n,$$

then $\dim_{\mathbb{R}} \mathrm{U}_{n,r,s} = (r + s)^2$. Consider

$$\mathfrak{u}_n(r, s) = \mathrm{Lie}\, \mathrm{U}_{n,r,s} = \{g \in \mathcal{M}_n \mid g^H I_{r,s} + I_{r,s} g = 0\},$$

in this case for suitable blocks of g we have

$$g^H I_{r,s} + I_{r,s} g = \begin{bmatrix} g_{11}^H + g_{11} & -g_{21}^H + g_{12} & g_{13} \\ g_{12}^H - g_{21} & -g_{22}^H - g_{22} & -g_{23} \\ g_{13}^H & -g_{23}^H & \mathbf{0} \end{bmatrix} = \mathbf{0} \in \mathcal{M}_n$$

thus, $g_{11} \in \mathcal{M}_r$ and $g_{22} \in \mathcal{M}_s$ are skew-Hermitian, $g_{13} = \mathbf{0} \in \mathcal{M}_r^{n-r-s}$, $g_{23} = \mathbf{0} \in \mathcal{M}_s^{n-r-s}$ and $g_{12} = g_{21}^H \in \mathcal{M}_r^s$. On the other hand $\begin{bmatrix} g_{31} & g_{32} & g_{33} \end{bmatrix} \in \mathcal{M}_{n-r-s}^n$. Thus,

$$\dim_{\mathbb{R}} \mathfrak{u}_n(r, s) = r^2 + s^2 + 2n(n - r - s) + 2rs = (r + s)^2 + 2n(n - r - s)$$

and this is the real dimension of the general isotropy group for signature (r, s), so that, as we already know, the general orbit dimension is $\dim_{\mathbb{R}} \mathrm{GL}_n - \dim_{\mathbb{R}} \mathfrak{u}_n(r, s) = (r + s)(2n - r - s)$.

8 Tensors spaces

We turn our attention to the varieties of tensors, the Euclidean distance case is treated in [Dra+16; OF14; DH16]. In particular, in [OF14] it is shown a useful way to compute the EDdegree of the Segre-Veronese variety, while in [DH16] the authors provide a formula to compute the average number of critical points for real tensors, where the average is taken considering a probability measure, and some of these values are computed.

Let $d \in \mathbb{N}$, $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ be a vector of indices and let $\mathbb{V}_1, \ldots, \mathbb{V}_d$ be complex vector spaces of dimension n_k endowed with a Hermitian form q_k respectively for $k = 1, \ldots, d$. In this section we consider the vector space $\mathbb{V}$ to be the tensor product $\mathbb{V}_1 \otimes \ldots \otimes \mathbb{V}_d$.

It is well known that there exists a unique Hermitian form q on $\mathbb{V} = \bigotimes_{k=1}^d \mathbb{V}_k$ such that

$$q(\mathbf{z}^{(1)} \otimes \ldots \otimes \mathbf{z}^{(d)}, \mathbf{w}^{(1)} \otimes \ldots \otimes \mathbf{w}^{(d)}) = \prod_{k=1}^{d} q_k(\mathbf{z}^{(k)}, \mathbf{w}^{(k)})$$

where $\mathbf{z}^{(k)}, \mathbf{w}^{(k)} \in \mathbb{V}_k$ for $k \in [d]$, which is called the *Bombieri-Weyl* or *Frobenius* form. In particular, the Bombieri-Weyl form is unitary invariant in the sense that for any $\mathbf{z}, \mathbf{w} \in \mathbb{V}$ and unitary $g \in \mathrm{U}(\mathbb{V}_1) \times \ldots \times \mathrm{U}(\mathbb{V}_d)$ it holds $q(g \cdot \mathbf{z}, g \cdot \mathbf{w}) = q(\mathbf{z}, \mathbf{w})$.

If we set a basis on each $\mathbb{V}_k$, we denote the space $\mathcal{T}^{\mathbf{n}} = \bigotimes_{k=1}^d \mathbb{C}^{n_k}$ or simply $\mathcal{T}_d^n$ when $n_1 = \ldots = n_d = n \in \mathbb{N}$, thus we identify $\mathcal{T}^{(n,m)} = \mathcal{M}_n^m$.

Assume we have chosen orthonormal bases, for which any q_k is the canonical Hermitian product. Let $\mathcal{A} = (a_{j_1 \ldots j_d}), \mathcal{B} = (b_{j_1 \ldots j_d}) \in \mathcal{T}^{\mathbf{n}}$ be tensors, then the Bombieri-Weyl form is the Hermitian inner product

$$\langle \mathcal{A}, \mathcal{B} \rangle_{\mathbb{C}} = \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} a_{j_1 \ldots j_d} \bar{b}_{j_1 \ldots j_d}.$$

Definition 8.0.1. Let $\mathcal{A} = (a_{j_1 \ldots j_d}) \in \mathcal{T}^{\mathbf{n}}$ and $\mathcal{B} = (b_{j_1 \ldots j_e}) \in \mathcal{T}^{\mathbf{m}}$ be tensors such that $\mathbf{m} = (n_{k_1}, \ldots, n_{k_e})$ for $1 \le k_1 \le \ldots \le k_e \le d$. Then, the *contraction* $\mathcal{A} \times \mathcal{B} \in \mathcal{T}^{\mathbf{n} \setminus \mathbf{m}}$ where

$$\mathbf{n} \setminus \mathbf{m} \coloneqq (n_1, \ldots, n_{k_1-1}, n_{k_1+1}, \ldots, n_{k_2-1}, n_{k_2+1}, \ldots, n_d)$$

of $\mathcal{A}$ and $\mathcal{B}$ is such that

$$(\mathcal{A} \times \mathcal{B})_{j_1 \dots j_{k_1-1} j_{k_1+1} \dots j_{k_2-1} j_{k_2+1} \dots j_d} = \sum_{j_{k_1}=1}^{n_{k_1}} \cdots \sum_{j_{k_e}=1}^{n_{k_e}} a_{j_1 \dots j_d} b_{j_{k_1} \dots j_{k_e}}.$$

Remark 8.0.2. Note that, if $\mathcal{A}, \mathcal{B} \in \mathcal{T}^{\mathbf{n}}$, then we have the equivalence $\mathcal{A} \times \mathcal{B} = \langle \mathcal{A}, \bar{\mathcal{B}} \rangle_{\mathbb{C}}$.

Definition 8.0.3. The *spectral norm* in $\mathcal{T}^{\mathbf{n}}$ is defined to be

$$\|\mathcal{A}\|_S := \sup_{\mathbf{z}^{(k)} \in \mathbb{C}^{n_k} \setminus \{0\}} \frac{|\langle \mathcal{A}, \otimes_{k=1}^d \mathbf{z}^{(k)} \rangle_{\mathbb{C}}|}{\prod_{k=1}^d \|\mathbf{z}^{(k)}\|_2} = \max_{\substack{\mathbf{z}^{(k)} \in \mathbb{C}^{n_k} \\ \|\mathbf{z}^{(k)}\|_2 = 1}} |\langle \mathcal{A}, \otimes_{k=1}^d \mathbf{z}^{(k)} \rangle_{\mathbb{C}}| \qquad \text{for } \mathcal{A} \in \mathcal{T}^{\mathbf{n}}.$$

The following notion is the generalization of Definition 7.0.4 in the case of tensors.

Definition 8.0.4. Let $\| \ \|_2$ be the 2-norm on $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_d}$, then the *induced tensor norm* on $\mathcal{T}^{\mathbf{n}}$ is defined to be

$$\|\mathcal{A}\|_2 := \sup_{\substack{j \in [d] \\ \mathbf{z}^{(k)} \in \mathbb{C}^{n_k} \setminus \{0\}}} \frac{\left\| \mathcal{A} \times \otimes_{\substack{k=1 \\ k \neq j}}^d \mathbf{z}^{(k)} \right\|_2}{\prod_{\substack{k=1 \\ k \neq j}}^d \|\mathbf{z}^{(k)}\|_2} \qquad \text{for } \mathcal{A} \in \mathcal{T}^{\mathbf{n}}.$$

Proposition 8.0.5. *In $\mathcal{T}^{\mathbf{n}}$ there holds the equality of norms $\| \ \|_S = \| \ \|_2$.*

Proof. In the case $d = 2$ we have $\mathcal{T}^{\mathbf{n}} = \mathcal{M}_{n_1}^{n_2}$ and we have already studied this case in Section 7 and the assertion follows from the equalities

$$\|A\|_2 = \sup_{\mathbf{z} \in \mathbb{C}^m \setminus \{0\}} \frac{\|A\mathbf{z}\|_2}{\|\mathbf{z}\|_2} = \sup_{\mathbf{z}^{(k)} \in \mathbb{C}^{n_k} \setminus \{0\}} \frac{|(\mathbf{z}^{(1)})^T A \mathbf{z}^{(2)}|}{\|\mathbf{z}^{(1)}\|_2 \|\mathbf{z}^{(2)}\|_2} = \max_{\substack{\mathbf{z}^{(k)} \in \mathbb{C}^{n_k} \\ \|\mathbf{z}^{(k)}\|_2 = 1}} |(\mathbf{z}^{(1)})^T A \mathbf{z}^{(2)}|$$

where we used the known fact

$$\|A\mathbf{z}\|_2 = \sup_{\mathbf{w} \in \mathbb{C}^n \setminus \{0\}} \frac{|\mathbf{w}^T A \mathbf{z}|}{\|\mathbf{w}\|_2} = \max_{\substack{\mathbf{w} \in \mathbb{C}^n \\ \|\mathbf{w}\|_2 = 1}} |\mathbf{w}^T A \mathbf{z}| \qquad \text{for } A \in \mathcal{M}_n^m \text{ and } \mathbf{z} \in \mathbb{C}^m,$$

Iterating this procedure the assertion follows. □

Remark 8.0.6. Note that in general, as we have seen in the case of matrices in Section 7, the Hermitian norm $\| \ \|_{\mathbb{C}}$ and the spectral norm $\| \ \|_S$ do not coincide, see Corollary 8.1.6.

Given a tensor, we search for its best approximation up to certain rank with respect to the Hermitian distance.

8.1 Segre variety

Consider $\operatorname{Seg}\mathbb{V} \subseteq \mathbb{V}$ the Segre variety given by the Segre embedding

$$\begin{gathered}\mathbb{P}\mathbb{V}_1 \times \ldots \times \mathbb{P}\mathbb{V}_d \to \operatorname{Seg}\mathbb{V} \subseteq \mathbb{P}\mathbb{V} \\ \left([\mathbf{z}^{(1)}], \ldots, [\mathbf{z}^{(d)}]\right) \mapsto [\mathbf{z}^{(1)} \otimes \ldots \otimes \mathbf{z}^{(d)}]\end{gathered}$$

The Segre variety coincides with the variety of rank one, or equivalently decomposable, tensors.

If we set bases on each $\mathbb{V}_k$, we denote $\operatorname{Seg}\mathcal{T}^{\mathbf{n}} \subseteq \mathcal{T}^{\mathbf{n}}$ the Segre variety and the Segre embedding reads

$$\begin{gathered}\mathbb{P}^{n_1-1} \times \ldots \times \mathbb{P}^{n_d-1} \to \operatorname{Seg}\mathcal{T}^{\mathbf{n}} \subseteq \mathbb{P}^{n_1\cdots n_d-1} \\ \left([\mathbf{z}^{(1)}], \ldots, [\mathbf{z}^{(d)}]\right) \mapsto [z_1^{(1)} \cdots z_1^{(d)}, \ldots, z_{j_1}^{(1)} \cdots z_{j_d}^{(d)}, \ldots, z_{n_1}^{(1)} \cdots z_{n_d}^{(d)}]\end{gathered}$$

Let $X = \operatorname{Seg}\mathcal{T}^{\mathbf{n}}$, for a tensor $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$ we aim to solve

$$\min_{Z \in X} \|\mathcal{U} - Z\|_{\mathbb{C}}^2.$$

In order to solve the Euclidean version of this question

$$\min_{Z \in X} \|\mathcal{U} - Z\|_{\mathbb{R}}^2,$$

in [Lim05] Lim proposes a variational approach on real tensors that links this problem to the problem of maximizing the function $\mathcal{U} \times Z = \langle \mathcal{U}, Z \rangle_{\mathbb{R}}$ and which has led to various results. In particular, in the same work Lim introduces the notion of a *singular vector tuple* of $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$.

The same approach is then used again in [OF14] where, more generally, the notion of singular vector tuple is linked to the solutions of the Euclidean critical ideal of a complex tensor. Here, we will consider this last terminology for which a singular vector tuple is a d-tuple

$$(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(d)}) \in (\mathbb{C}^{n_1} \setminus \{\mathbf{0}\}) \times \ldots \times (\mathbb{C}^{n_d} \setminus \{\mathbf{0}\})$$

of unit vectors such that for $k = 1, \ldots, d$ there hold the equalities

$$\mathcal{U} \times (\otimes_{j \in [d] \setminus \{k\}} \mathbf{z}^{(j)}) = \langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)} \rangle_{\mathbb{R}} \mathbf{z}^{(k)} \tag{8.1.1}$$

where the scalar $\langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)} \rangle_{\mathbb{R}}$ is possibly non real and it is the associated *singular value*. This way the singular vector tuples exactly characterize the solutions of the critical ideal (4.2.2) of the EDdegree using the formula $\langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)} \rangle_{\mathbb{R}} \otimes_{k=1}^d \mathbf{z}^{(k)} \in \operatorname{Seg}\mathcal{T}^{\mathbf{n}}$.

What follows is the Hermitian version of this argument and can be found in literature for example in the work of Hiling and Sudbery [HS10].

A rank one tensor $Z \in X$ can be written as $Z = \mu\, \mathbf{z}^{(1)} \otimes \ldots \otimes \mathbf{z}^{(d)}$ where $\mu \in \mathbb{C}$ and the vectors $\mathbf{z}^{(k)} \in \mathbb{C}^{n_k}$ are unit vectors or in other terms satisfy $\|\mathbf{z}^{(k)}\|_{\mathbb{C}}^2 = 1$. Thus, we obtain

$$\begin{aligned}
\min_{Z\in X} \|\mathcal{U} - Z\|_{\mathbb{C}}^2 &= \min_{Z\in X} \left(\|\mathcal{U}\|_{\mathbb{C}}^2 - \langle \mathcal{U}, Z\rangle_{\mathbb{C}} - \overline{\langle \mathcal{U}, Z\rangle_{\mathbb{C}}} + \|Z\|_{\mathbb{C}}^2\right)\\
&= \min_{\substack{\|\mathbf{z}^{(k)}\|_{\mathbb{C}}^2=1\\ \mu\in\mathbb{C}}} \left(\|\mathcal{U}\|_{\mathbb{C}}^2 - \mu\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)}\rangle_{\mathbb{C}} - \bar{\mu}\overline{\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)}\rangle_{\mathbb{C}}} + |\mu|^2\right)\\
&= \max_{\|\mathbf{z}^{(k)}\|_{\mathbb{C}}^2=1} \frac{\left(\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)}\rangle_{\mathbb{C}} + \overline{\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)}\rangle_{\mathbb{C}}}\right)^2}{4} - \|\mathcal{U}\|_{\mathbb{C}}^2\\
&= \max_{\|\mathbf{z}^{(k)}\|_{\mathbb{C}}^2=1} \left|\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)}\rangle_{\mathbb{C}}\right|^2 - \|\mathcal{U}\|_{\mathbb{C}}^2
\end{aligned}$$

where we have chosen without loss of generality $\mu = \langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)}\rangle_{\mathbb{C}}$ to be real and nonnegative, see also below. Using the Lagrange multipliers we define the real-valued function

$$L\colon \mathbb{C}^{n_1} \times \ldots \times \mathbb{C}^{n_d} \times \mathbb{R}^k \to \mathbb{R}$$

such that

$$\begin{aligned}
L(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(d)}, \lambda_1, \ldots, \lambda_k) &\coloneqq \langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)}\rangle_{\mathbb{C}} + \overline{\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)}\rangle_{\mathbb{C}}} - \sum_{k=1}^d \lambda_k(\|\mathbf{z}^{(k)}\|_{\mathbb{C}}^2 - 1)\\
&= \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} (u_{j_1\ldots j_n} \bar{z}_{j_1}^{(1)} \cdots \bar{z}_{j_d}^{(d)} + \bar{u}_{j_1\ldots j_n} z_{j_1}^{(1)} \cdots z_{j_d}^{(d)}) - \sum_{k=1}^d \lambda_k \left(\sum_{j=1}^{n_k} |z_j^{(k)}|^2 - 1\right)
\end{aligned}$$

and its derivatives vanish if and only if for any $k \in [d]$ there hold the equalities

$$\mathcal{U} \times (\otimes_{j\in[d]\setminus\{k\}} \bar{\mathbf{z}}^{(j)}) = \lambda_k \mathbf{z}^{(k)} \qquad \text{and} \qquad \|\mathbf{z}^{(k)}\|_{\mathbb{C}}^2 = 1. \tag{8.1.2}$$

Moreover, from the equalities

$$\lambda_k = \langle \lambda_k \mathbf{z}^{(k)}, \bar{\mathbf{z}}^{(k)}\rangle_{\mathbb{R}} = \langle \mathcal{U} \times \otimes_{j\in[d]\setminus\{k\}} \bar{\mathbf{z}}^{(j)}, \bar{\mathbf{z}}^{(k)}\rangle_{\mathbb{R}} = \langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)}\rangle_{\mathbb{C}}$$

we deduce the independence of λ_k from k.

Thus, let $\lambda_k = \langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)}\rangle_{\mathbb{C}}$ for any k. If this value along with the vectors $\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(d)}$ satisfies the qualities (8.1.2), then for any $\xi \in \mathbb{S}^1 \subseteq \mathbb{C}$ the value $\xi^d \langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)}\rangle_{\mathbb{C}}$ along with the vectors $\bar{\xi}\mathbf{z}^{(1)}, \ldots, \bar{\xi}\mathbf{z}^{(d)}$ satisfies the equalities (8.1.2). Thus, again we can assume that $\langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)}\rangle_{\mathbb{C}}$ is real and nonnegative.

Definition 8.1.1. A *Hermitian singular vector tuple* of $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$ is a d-tuple

$$(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(d)}) \in (\mathbb{C}^{n_1} \setminus \{\mathbf{0}\}) \times \ldots \times (\mathbb{C}^{n_d} \setminus \{\mathbf{0}\})$$

of unit vectors such that for any $k = 1, \ldots, d$ it holds the equality

$$\mathcal{U} \times (\otimes_{j\in[d]\setminus\{k\}} \bar{\mathbf{z}}^{(j)}) = \langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)}\rangle_{\mathbb{C}} \mathbf{z}^{(k)} \tag{8.1.3}$$

and the scalar $\langle \mathcal{U}, \otimes_{j=1}^d \mathbf{z}^{(j)}\rangle_{\mathbb{C}}$ is real and nonnegative and it is the associated *Hermitian singular value.*

We discuss now the notion of Hermitian singular vector tuples when $d = 2$, in other words we set $X = \mathrm{rk}_1(\mathcal{M}_n^m)$.

Example 8.1.2. For a matrix $A \in \mathcal{M}_n^m$, the notion of a Hermitian singular vector tuple translates in the existence of two unitary vectors $\mathbf{z}^{(1)} \in \mathbb{C}^n$ and $\mathbf{z}^{(2)} \in \mathbb{C}^m$ such that they satisfy a condition equivalent to the one of a singular pair $(\mathbf{z}^{(1)}, \bar{\mathbf{z}}^{(2)})$ of A since there hold the equalities

$$A\bar{\mathbf{z}}^{(2)} = A \times \bar{\mathbf{z}}^{(2)} = \langle A, \mathbf{z}^{(1)} \otimes \mathbf{z}^{(2)} \rangle_{\mathbb{C}} \mathbf{z}^{(1)} = \left((\mathbf{z}^{(1)})^H A \bar{\mathbf{z}}^{(2)} \right) \mathbf{z}^{(1)}$$
$$A^H \mathbf{z}^{(1)} = \bar{A} \times \mathbf{z}^{(1)} = \langle A, \mathbf{z}^{(1)} \otimes \mathbf{z}^{(2)} \rangle_{\mathbb{C}} \bar{\mathbf{z}}^{(2)} = \left((\mathbf{z}^{(1)})^H A \bar{\mathbf{z}}^{(2)} \right) \bar{\mathbf{z}}^{(2)}$$

and the critical points of the Hermitian distance function are of the form

$$\langle A, \mathbf{z}^{(1)} \otimes \mathbf{z}^{(2)} \rangle_{\mathbb{C}} \; \mathbf{z}^{(1)} \otimes \mathbf{z}^{(2)} = \left((\mathbf{z}^{(1)})^H A \bar{\mathbf{z}}^{(2)} \right) \mathbf{z}^{(1)} (\bar{\mathbf{z}}^{(2)})^H.$$

In other words, $(\mathbf{z}^{(1)}, \mathbf{z}^{(2)})$ is a Hermitian singular vector tuples of A if and only if $(\mathbf{z}^{(1)}, \bar{\mathbf{z}}^{(2)})$ is a singular pair of A and the value $(\mathbf{z}^{(1)})^H A \bar{\mathbf{z}}^{(2)}$ is both a Hermitian singular value and singular value of A in the classical sense.

On the other hand, singular values in the sense of equations (8.1.1) coincide with classical singular values only for real matrices and real singular vectors. These consideration suggests that the Hermitian singular value could be the more general and thus natural concept to look at.

Remark 8.1.3. Note that, differently from the Euclidean case for which the singular values could be complex numbers, in this case the Hermitian singular values are defined to be nonnegative real numbers. In this sense the Hermitian singular values are a more natural generalization of the notion of singular values of matrices to tensors.

Moreover, if a Hermitian singular vector tuples of a tensor $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$ is real-valued, then that is also a singular vector tuple of $\mathcal{U}$ and the relative Hermitian singular value is also a singular value.

In the end, for a given Hermitian singular value $0 \le \mu \in \mathbb{R}$ with Hermitian singular vector tuples $(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(d)})$ of $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$, then there exist at least d Hermitian singular vector tuples of μ given by $(\xi\mathbf{z}^{(1)}, \ldots, \xi\mathbf{z}^{(d)})$ where $\xi^d = 1$. However, these all yield the same critical point.

We obtain a direct consequence from the definition of Hermitian singular vector tuple.

Corollary 8.1.4. *Let $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$, then $(\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(d)})$ is a Hermitian singular vector tuple of $\mathcal{U}$ iff*

$$\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)} \rangle_{\mathbb{C}} \otimes_{j=1}^d \mathbf{z}^{(j)} \in \mathrm{Seg}\, \mathcal{T}^{\mathbf{n}}$$

is a critical point of the Hermitian distance from $\mathrm{Seg}\, \mathcal{T}^{\mathbf{n}}$.

The next proposition follows from the observation that one of the singular vector tuple is the maximizer of $|\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)} \rangle_{\mathbb{C}}|$.

Proposition 8.1.5. *The maximum between the modules of the Hermitian singular values of a tensor $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$ is $\|\mathcal{U}\|_S = \|\mathcal{U}\|_2$.*

Proof. The statement follows considering the maximization problem and any of the definitions of the norms. □

Corollary 8.1.6. *In $\mathcal{T}^{\mathbf{n}}$ it holds the inequality of norms $\| \ \|_S \leq \| \ \|_{\mathbb{C}}$.*

Proof. Let $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$ and take $Z = \|\mathcal{U}\|_S \otimes_{k=1}^d \mathbf{z}^{(k)} \in \operatorname{Seg} \mathcal{T}^{\mathbf{n}}$ the critical point associated to the largest Hermitian singular value of $\mathcal{U}$, then the assertion follows from the evaluation

$$0 \leq \|\mathcal{U} - Z\|_{\mathbb{C}}^2 = \|\mathcal{U}\|_{\mathbb{C}}^2 - \langle \mathcal{U}, Z \rangle_{\mathbb{C}} - \overline{\langle \mathcal{U}, Z \rangle_{\mathbb{C}}} + \|Z\|_{\mathbb{C}}^2 = \|\mathcal{U}\|_{\mathbb{C}}^2 - \|\mathcal{U}\|_S^2.$$

□

There is an established general theory about complex singular vector d-tuple with singular value equals to zero. Such tuples are solutions of the system

$$\langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)} \rangle_{\mathbb{C}} = 0 \quad \text{and} \quad \nabla_{\bar{\mathbf{z}}^{(k)}} \langle \mathcal{U}, \otimes_{k=1}^d \mathbf{z}^{(k)} \rangle_{\mathbb{C}} = 0 \quad \text{for } k = 1, \ldots, d. \tag{8.1.4}$$

The number of independent equations of (8.1.4) is one more than the number of variables, so the variables can be eliminated to get a polynomial equation in the coefficients of $\mathcal{U}$.

Theorem 8.1.7 (Cayley). *There exists a polynomial function* $\operatorname{hdet}(\mathcal{U})$ *in the coefficients of* $\mathcal{U}$*, such that the equations* (8.1.4) *have a solution with all* $\mathbf{z}^{(k)}$ *nonzero for* $k = 1, \ldots, d$ *if and only if* $\operatorname{hdet}(\mathcal{U}) = 0$.

The function hdet is called the *hyperdeterminant.* It is a special case of the discriminant of a function of several variables, see [GKZ94]. The hyperdeterminant has appeared in the theory of multipartite entanglement, for example in [Miy03].

Remark 8.1.8. The hyperderminant is important also in the Euclidean case since the set of tensors with zero as a Hermitian singular value is defined by the same equations (8.1.4) of the set of tensors with zero as a singular value.

It is known that the dual variety of $\operatorname{Seg} \mathcal{T}^{\mathbf{n}}$ is a hypersurface if and only if it holds the inequality

$$2 \max\{n_1, \ldots, n_d\} \leq n_1 + n_2 + \ldots + n_d - d + 2, \tag{8.1.5}$$

see [GKZ94, Chapter XIV]. In this case, the dual variety is defined by the hyperdeterminant.

In particular, from Theorem 4.6.6 and the fact that the varieties are all defined by real polynomials we get the following result.

Corollary 8.1.9. *Let $\mathcal{U}$ be a tensor and Z be a critical point of the Hermitian distance, if inequality* (8.1.5) *holds then* $\operatorname{hdet}(\mathcal{U} - Z) = 0$.

Set now $d = 3$ and $n_1 = n_2 = n_3 = 2$ so that $X = \operatorname{Seg}\mathcal{T}_3^2 \subseteq \mathcal{T}_3^2$. In [HS10] the authors investigated the Hermitian distance problem on X, they proved that the square of a singular value of a tensors must be the zero of a polynomial of degree 12, thus providing a bound for the number of critical points which in turn yields $\max \operatorname{HDdeg}(X) \leq 12$. In the same work, for a subset of tensors in $\subseteq \mathcal{T}_3^2$ the bound is refined to 5 and the formulas to compute the Hermitian singular values along with the Hermitian singular vector tuples is provided. In particular, from that work, and the fact that $\operatorname{vHDdeg}(X) = 8$ that we will see below, follows the inequality $\max \operatorname{HDdeg}(X) \geq 6$.

The Hermitian distance problem is widely investigated on X, see also [WG03].

Proposition 8.1.10. *For* $\operatorname{Seg}\mathcal{T}_3^2 \subseteq \mathcal{T}_3^2$ *there hold*

$$\operatorname{vHDdeg}(\operatorname{Seg}\mathcal{T}_3^2) = 8 \qquad \textit{and} \qquad \{4,6,8\} \subseteq \operatorname{HDdeg}(\operatorname{Seg}\mathcal{T}_3^2) \subseteq \{2,4,6,8\}.$$

Proof. We will compute in Example 8.1.11 the value of the vHDdeg.

Now, let $\mathcal{U} \in \mathcal{T}_3^2$ be symmetric with 5 Hermitian eigenvector (see Subsection 8.2) which we know exists from Proposition 8.3.2. By Proposition 8.2.4, a Hermitian eigenvector $\mathbf{z} \in \mathbb{C}^2$ generates the Hermitian singular vector tuple $(\mathbf{z},\mathbf{z},\mathbf{z}) \in (\mathbb{C}^2)^3$ with the same Hermitian singular value and then this tensor possesses at least 5 critical points. Since the number of solutions of the Hermitian critical ideal must be equal to 8 there are 3 more solutions. Moreover, since $\mathcal{U}$ is symmetric, any permutation of a Hermitian singular vector tuple yields another one. Thus, the only possible outcome is to have 3 complex singular vectors that are of the form $(\mathbf{z}^{(1)},\mathbf{z}^{(1)},\mathbf{z}^{(2)})$, $(\mathbf{z}^{(1)},\mathbf{z}^{(2)},\mathbf{z}^{(1)})$, $(\mathbf{z}^{(2)},\mathbf{z}^{(1)},\mathbf{z}^{(1)})$ for some $\mathbf{z}^{(1)} \neq \mathbf{z}^{(2)} \in \mathbb{C}^2$ and then this tensor possesses 3 other critical points. In particular, it holds $\max \operatorname{HDdeg}(\operatorname{Seg}\mathcal{T}_3^2) = 8$. A similar reasoning provides tensors with 4 and 6 critical points, such examples can also be easily found by testing random points.

For example the tensor

$$z_{000} = 1,\ z_{010} = \frac{2001}{1000},\ z_{100} = 2,\ z_{110} = 3,\ z_{001} = 2,\ z_{011} = 3,\ z_{101} = 3,\ z_{111} = 5$$

admits 8 critical points, the tensor

$$z_{000} = \frac{8}{5},\ z_{010} = 2,\ z_{100} = \frac{1}{4},\ z_{110} = \frac{8}{9},\ z_{001} = \frac{4}{5},\ z_{011} = \frac{3}{2},\ z_{101} = \frac{5}{9},\ z_{111} = \frac{9}{7}$$

admits 6 critical points, and the tensor

$$z_{000} = \frac{5}{2},\ z_{010} = \frac{8}{7},\ z_{100} = \frac{6}{7},\ z_{110} = \frac{2}{9},\ z_{001} = \frac{3}{8},\ z_{011} = \frac{7}{9},\ z_{101} = \frac{1}{2},\ z_{111} = \frac{9}{7}$$

admits 4 critical points. □

Example 8.1.11. Let $X = \operatorname{Seg}\mathcal{T}_3^2$ defined by the ideal

$$\begin{aligned} I_X = \langle & z_{111}z_{221} - z_{121}z_{211}, && z_{112}z_{222} - z_{122}z_{212}, && z_{111}z_{212} - z_{112}z_{211} \\ & z_{121}z_{222} - z_{122}z_{221}, && z_{111}z_{122} - z_{112}z_{121}, && z_{211}z_{222} - z_{212}z_{221} \\ & z_{121}z_{212} - z_{112}z_{221}, && z_{111}z_{222} - z_{121}z_{212}, && z_{111}z_{222} - z_{211}z_{122}\rangle. \end{aligned}$$

We can bijectively parametrize a dense subvariety of X as

$$\begin{aligned}\psi\colon \mathbb{C}^4 &\longrightarrow X \subseteq \mathcal{T}_3^2\\ (z_1,z_2,z_3,z_4) &\mapsto \begin{bmatrix} z_1\\ 1\end{bmatrix} \otimes \begin{bmatrix} z_2\\ 1\end{bmatrix} \otimes \begin{bmatrix} z_3\\ z_4\end{bmatrix}\end{aligned}$$

and the singular locus of this parametrization is $V(z_3, z_4) \subseteq \mathbb{C}^4$. Critical points satisfy the equations

$$\frac{\partial}{\partial z_k}\|\psi(\mathbf{z}) - \mathbf{u}\|_{\mathbb{C}}^2 = 0$$

for $k = 1, \dots, 4$. Introducing $\mathbf{w}$ and $\mathbf{v}$ we compute that the Hermitian critical ideal is of degree 8 and thus $\mathrm{vHDdeg}(X) = 8$.

8.2 Veronese variety

Let $n, d \in \mathbb{N}$ and let $\mathbb{V}_1$ be a complex vector space of dimension n endowed with a Hermitian form q_1, in this section we consider $\mathbb{V}$ to be the $\binom{n-1+d}{d}$ dimensional vector subspace of symmetric d tensors $\mathbb{V} := \mathrm{Sym}^d \mathbb{V}_1 \subseteq \bigotimes_{k=1}^d \mathbb{V}_1$.

Similarly to the incipit of Section 8, there exists a unique Hermitian form q on $\mathbb{V}$ such that

$$q(\mathbf{z}^d, \mathbf{w}^d) = q_1(\mathbf{z}, \mathbf{w})^d$$

where $\mathbf{z}, \mathbf{w} \in \mathbb{V}_1$ which is exactly the restriction on $\mathbb{V}$ of the Bombieri-Weyl form on $\bigotimes_{k=1}^d \mathbb{V}_1$. If we set a basis on $\mathbb{V}_1$, we denote $\mathcal{S}_d^n \subseteq \mathcal{T}_d^n$ the subspace of symmetric tensors that is the space of tensors $\mathcal{U}$ such that for any σ permutation of d elements it holds $\mathcal{U}_{j_1 j_2 \dots j_d} = \mathcal{U}_{\sigma(j_1)\sigma(j_2)\dots\sigma(j_d)}$.

There exists a bijection

$$\begin{aligned}\mathcal{S}_d^n &\to \mathbb{C}[\mathbf{z}]_d \subseteq \mathbb{C}[\mathbf{z}]\\ \mathcal{U} &\mapsto \mathcal{U} \times \mathbf{z}^{\otimes d} = \sum_{\|\alpha\|_1 = d} \binom{d}{\alpha} u_\alpha \mathbf{z}^\alpha\end{aligned}$$

from $\mathcal{S}_d^n$ to the space of homogeneus polynomial of degree d in n variables $\mathbb{C}[\mathbf{z}]_d$.

Consider $v_d\mathbb{V}_1 \subseteq \mathbb{V}$ the Veronese variety given by the Veronese embedding

$$\begin{aligned}v_d\colon \mathbb{P}\mathbb{V}_1 &\to v_d\mathbb{V}_1 \subseteq \mathbb{P}\mathbb{V}\\ [\mathbf{z}] &\mapsto [\mathbf{z}^d]\end{aligned}$$

The Veronese variety coincides with the variety of rank one symmetric d tensors.

If we set a basis, we denote $v_d\mathbb{C}^n \subseteq \mathcal{S}_d^n$ the Veronese variety and the Veronese embedding reads

$$\begin{aligned} \mathbb{P}^{n-1} &\to v_d\mathbb{C}^n \subseteq \mathbb{P}^{\binom{n-1+d}{d}-1} \\ [\mathbf{z}] &\mapsto [z_1^d,\ z_1^{d-1}z_2, \dots, z_n^{d-1}z_{n-1},\ z_n^d] \end{aligned}$$

The tensors in the Veronese variety are of the form $\mathcal{U} = (a_1z_1 + a_2z_2 + \ldots + a_nz_n)^d$ for some $a_1, \dots, a_n \in \mathbb{C}$ not simultaneously vanishing.

Similarly to what happens for the Segre variety, in the Euclidean distance case the notion of *eigenvector* and relative *eigenvalue* of a tensor $\mathcal{U} \in \mathcal{T}_d^n$ is introduced. We can now generalize this notion. For a general $\mathcal{U} \in \mathcal{T}_d^n$ using the same reasoning of Subsection 8.1 the conditions on a Hermitian singular vector tuple becomes the one contained in the following definition.

Definition 8.2.1. A *Hermitian eigenvector* of $\mathcal{U} \in \mathcal{T}_d^n$ is a unit vector $\mathbf{z} \in \mathbb{C}^n$ such that it holds the equality

$$\begin{bmatrix} \sum_{k=1}^d \mathcal{U} \times \left(\bar{\mathbf{z}}^{\otimes(k-1)} \otimes e_1 \otimes \bar{\mathbf{z}}^{\otimes d-k}\right) \\ \vdots \\ \sum_{k=1}^d \mathcal{U} \times \left(\bar{\mathbf{z}}^{\otimes(k-1)} \otimes e_n \otimes \bar{\mathbf{z}}^{\otimes d-k}\right) \end{bmatrix} = \langle \mathcal{U}, \mathbf{z}^{\otimes d} \rangle_{\mathbb{C}} \mathbf{z} \tag{8.2.1}$$

and the scalar $\langle \mathcal{U}, \mathbf{z}^{\otimes d} \rangle_{\mathbb{C}}$ is real and nonnegative and it is the associated *Hermitian eigenvalue*. Here e_j for $j = 1, \dots, n$ is the j-th vector of the canonical basis of $\mathbb{C}^n$.

In particular, if $\mathcal{U} \in \mathcal{S}_d^n$ by symmetry the notation for equation (8.2.1) simplifies to

$$\mathcal{U} \times \bar{\mathbf{z}}^{\otimes d-1} = \langle \mathcal{U}, \mathbf{z}^{\otimes d} \rangle_{\mathbb{C}} \mathbf{z}. \tag{8.2.2}$$

Remark 8.2.2. If a Hermitian eigenvector of a tensor $\mathcal{U} \in \mathcal{T}^{\mathbf{n}}$ is real-valued, then that is also an eigenvector of $\mathcal{U}$ and the relative eigenvalue it is also a Hermitian eigenvalue.

Since the definition above descends, in the same way of the one of Hermitian singular vector tuple, from the solutions of a variational problem, we get the following direct result.

Corollary 8.2.3. *Let $\mathcal{U} \in \mathcal{S}_d^n$, then $\mathbf{z}$ is a Hermitian eigenvector of $\mathcal{U}$ iff*

$$\langle \mathcal{U}, \mathbf{z}^{\otimes d} \rangle_{\mathbb{C}} \mathbf{z}^d \in v_d\mathbb{C}^n$$

is a critical point of the Hermitian distance from $v_d\mathbb{C}^n$.

Generalizing the two definitions of Hermitian singular vector tuple and Hermitian eigenvector by introducing the variables $\mathbf{w}$ we get the following result.

Proposition 8.2.4. *Let $\mathcal{U}, \mathcal{V} \in \mathcal{S}_d^n$, then the zero locus of the Hermitian critical ideal of $v_d\mathbb{C}^n \subseteq \mathcal{T}_d^n$ of $(\mathcal{U}, \mathcal{V})$ is contained in the zero locus of the Hermitian critical ideal of $\operatorname{Seg} \mathcal{T}_d^n \subseteq \mathcal{T}_d^n$ of $(\mathcal{U}, \mathcal{V})$. In particular, a Hermitian eigenvector $\mathbf{z} \in \mathbb{C}^n$ of $\mathcal{U}$ generates a Hermitian singular vector tuple $(\mathbf{z}, \dots, \mathbf{z}) \in (\mathbb{C}^n)^d$ of $\mathcal{U}$. Moreover, the polynomial $\mathrm{vHDpol}_{v_d\mathbb{C}^n, \mathcal{U}, \mathcal{V}}(t^2)$ divides the polynomial $\mathrm{vHDpol}_{\operatorname{Seg} \mathcal{T}_d^n, \mathcal{U}, \mathcal{V}}(t^2)$. In particular there hold $\mathrm{vHDdeg}(\operatorname{Seg} \mathcal{T}_d^n) \geq \mathrm{vHDdeg}(v_d\mathbb{C}^n)$ and $\max \mathrm{HDdeg}(\operatorname{Seg} \mathcal{T}_d^n) \geq \max \mathrm{HDdeg}(v_d\mathbb{C}^n)$.*

8.3 Binary forms

In this subsection we consider the case of binary forms, or equivalently the Veronese variety of Subsection 8.2 setting $n = 2$. In [OF14] it is shown $\mathrm{EDdegree}(v_d\mathbb{C}^2) = d$, more precisely in [Mac18, Theorem 1] Maccioni shows that the number of real critical points of the Euclidean distance of a symmetric tensor is bounded from below by the number of real zeros of the polynomial representing it. From this we get that $\max \mathrm{HDdeg}(v_d\mathbb{C}^2) \geq d$.

We recall that we identify a tensor $\mathcal{U} \in \mathcal{S}_d^2$ with the homogeneous polynomial

$$\mathcal{U} = \sum_{k=0}^{d} \binom{d}{k} u_k z_1^{d-k} z_2^k \in \mathbb{C}[z_1, z_2].$$

In this context condition (8.2.2) can be expressed in the nice form of the bivariate generalized polynomial equation

$$\begin{aligned} 0 = \det \begin{bmatrix} \partial_{z_1}\overline{\mathcal{U}} & \bar{z}_1 \\ \partial_{z_2}\overline{\mathcal{U}} & \bar{z}_2 \end{bmatrix} &= \begin{bmatrix} \sum_{k=0}^{d-1}(d-k)\binom{d}{k}\bar{u}_k z_1^{d-k-1} z_2^k & \bar{z}_1 \\ \sum_{k=1}^{d} k\binom{d}{k}\bar{u}_k z_1^{d-k} z_2^{k-1} & \bar{z}_2 \end{bmatrix} \\ &= \left(\sum_{k=0}^{d-1}(d-k)\binom{d}{k}\bar{u}_k z_1^{d-k-1} z_2^k\right)\bar{z}_2 - \left(\sum_{k=1}^{d} k\binom{d}{k}\bar{u}_k z_1^{d-k} z_2^{k-1}\right)\bar{z}_1. \end{aligned}$$

Since we consider an Hermitian eigenvector to be normalized, without loss of generality we divide this equation by $z_1^{d-1}\bar{z}_1$ and setting $z = \frac{z_2}{z_1}$ we obtain the univariate generalized polynomial equation

$$\left(\sum_{k=0}^{d-1}(d-k)\binom{d}{k}\bar{u}_k z^k\right)\bar{z} - \sum_{k=1}^{d} k\binom{d}{k}\bar{u}_k z^{k-1} = 0. \tag{8.3.1}$$

The same reasoning could be applied to a tensor $\mathcal{U} \in \mathcal{T}_d^2$ and condition (8.2.1) in order to get a similar polynomial.

Corollary 8.3.1. *Let $\mathcal{U} \in \mathcal{T}_d^2$ be a generic tensor, then there exist at least $d-2$ Hermitian eigenvalues of $\mathcal{U}$. In particular,* $\min \mathrm{HDdeg}(v_d\mathbb{C}^2) \geq d-2$.

Proof. Consider a binary form $\mathcal{U}$ such that $u_{d-1} \neq 0$. Then the generalized polynomial of equation (8.3.1) is of the form of Proposition 2.2.5 where the term with the highest degree is $d\bar{u}_{d-1}z^{d-1}\bar{z}$ and there must exist at least $(d-1)-1 = d-2$ solutions. In a similar way the result can be derived for a generic tensor $\mathcal{U} \in \mathcal{T}_d^2$ considering the coefficient obtained using the same steps from condition (8.2.1). ☐

Proposition 8.3.2. *Let $\mathcal{U} \in \mathcal{S}_d^2$ be a generic binary form, then there exist at most $d^2 - 2d + 2$ Hermitian eigenvalues of $\mathcal{U}$. More specifically,* $\mathrm{vHDdeg}(v_d\mathbb{C}^2) = d^2 - 2d + 2$.

Proof. Let p be the generalized polynomial of equation (8.3.1). Introducing the variable w and a binary form $\mathcal{V}$, using the Bernstein–Kushnirenko Theorem the number of solutions of the system $p(z,w) = \bar{p}(w,z) = 0$ is bounded by $\mathrm{MV}_2(\triangle_1, \triangle_2)$ where the arguments are the convex hulls

$$\triangle_1 = \mathrm{conv}(\{(k,j) \in \mathbb{N}^2 \mid k \le d-1,\ j \le 1\}),$$
$$\triangle_2 = \mathrm{conv}(\{(k,j) \in \mathbb{N}^2 \mid k \le 1,\ j \le d-1\}).$$

Now, the use of formula (4.3.1) yields

$$\begin{aligned}\mathrm{MV}_2(\triangle_1, \triangle_2) &= -\mathrm{Vol}_2(\triangle_1) - \mathrm{Vol}_2(\triangle_2) + \mathrm{Vol}_2(\triangle_1 + \triangle_2)\\ &- (d-1) - (d-1) + d^2 = d^2 - 2d + 2\end{aligned}$$

and hence the bound. The equality with the hypothesis that the polynomial $p(z,w)$ is generic enough. □

Proposition 8.3.3. *For the variety* $v_3\mathbb{C}^2 \subseteq \mathcal{T}_3^2$ *there hold*

$$\mathrm{vHDdeg}(v_3\mathbb{C}^2) = 5 \qquad \textit{and} \qquad \mathrm{HDdeg}(v_3\mathbb{C}^2) = \{1,3,5\}.$$

Proof. Let $X = v_3\mathbb{C}^2$ that is defined by the sum of the ideal defining the variety $\mathrm{Seg}\,\mathcal{T}_3^2 \subseteq \mathcal{T}_3^2$ (see Example 8.1.11) and the ideal

$$\langle z_{100} - z_{010},\ z_{100} - z_{001},\ z_{110} - z_{011},\ z_{110} - z_{101}\rangle.$$

Considering Proposition 4.2.9, we can simplify our problem and study the variety $\hat{X}_3 \subseteq \mathbb{C}^4 \simeq \mathcal{S}_3^2$ we will see in Example 8.3.6 and the Hermitian form $q(\mathbf{z},\mathbf{w}) = z_1\bar{w}_1 + 3z_2\bar{w}_2 + 3z_3\bar{w}_3 + z_4\bar{w}_4$. Then, it is easy to see $\mathrm{vHDdeg}(X) = 5$ as predicted by Proposition 8.3.2. We used Macaulay2 to compute the following results. The tensor

$$\mathcal{U} = z_1^3 + 3\cdot 2z_1^2z_2 + 3\cdot 3z_1z_2^2 + 5z_2^3 \in \mathrm{S}_3^2$$

possesses 3 real (Hermitian) eigenvectors with relative (Hermitian) eigenvalues given by

$$\mathbf{z}^{(1)} = \begin{bmatrix} -\frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{bmatrix} \qquad \langle \mathcal{U}, (\mathbf{z}^{(1)})^{\otimes 3}\rangle_{\mathbb{C}} = \frac{1}{\sqrt{13}},$$

$$\mathbf{z}^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{\varphi^2+1}} \\ \frac{\varphi}{\sqrt{\varphi^2+1}} \end{bmatrix} \qquad \langle \mathcal{U}, (\mathbf{z}^{(2)})^{\otimes 3}\rangle_{\mathbb{C}} = \frac{3\varphi^2+4\varphi+1}{\sqrt{\varphi^2+1}} = \frac{15+7\sqrt{5}}{\sqrt{10+2\sqrt{5}}},$$

$$\mathbf{z}^{(3)} = \begin{bmatrix} -\frac{1}{\sqrt{\varphi^2-2\varphi+2}} \\ \frac{\varphi-1}{\sqrt{\varphi^2-2\varphi+2}} \end{bmatrix} \qquad \langle \mathcal{U}, (\mathbf{z}^{(3)})^{\otimes 3}\rangle_{\mathbb{C}} = \frac{-3\varphi^2+10\varphi-8}{\sqrt{\varphi^2-2\varphi+2}} = \frac{-15+7\sqrt{5}}{\sqrt{10-2\sqrt{5}}},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. These eigenvalues yield the critical points

$$\frac{1}{\sqrt{13}}(\mathbf{z}^{(1)})^{\otimes 3}, \qquad \frac{15+7\sqrt{5}}{\sqrt{10+2\sqrt{5}}}(\mathbf{z}^{(2)})^{\otimes 3} \qquad \text{and} \qquad \frac{-15+7\sqrt{5}}{\sqrt{10-2\sqrt{5}}}(\mathbf{z}^{(3)})^{\otimes 3}$$

respectively. Moreover, for $\mathcal{U}$ we find 2 Hermitian eigenvectors with relative Hermitian eigenvalues that are approximated by

$$\mathbf{z}^{(4)} \approx \begin{bmatrix} \frac{1}{0.690474-1.01910i} \\ \frac{-0.59375-0.0403436i}{0.690474-1.01910i} \end{bmatrix} \qquad \langle \mathcal{U}, (\mathbf{z}^{(4)})^{\otimes 3}\rangle_{\mathbb{C}} \approx 0.670012,$$

$$\mathbf{z}^{(5)} = \bar{\mathbf{z}}^{(4)} \approx \begin{bmatrix} \frac{1}{0.690474+1.01910i} \\ \frac{-0.59375+0.0403436i}{0.690474+1.01910i} \end{bmatrix} \qquad \langle \mathcal{U}, (\mathbf{z}^{(5)})^{\otimes 3}\rangle_{\mathbb{C}} \approx 0.670012,$$

which yields the approximate relative critical points

$$0.670012(\mathbf{z}^{(4)})^{\otimes 3} \qquad \text{and} \qquad 0.670012(\bar{\mathbf{z}}^{(4)})^{\otimes 3}.$$

Other tensors with 5 critical points are

$$25z_1^3 + 3 \cdot 200z_1^2z_2 + 3 \cdot 10z_1z_2^2 + 36z_2^3 \quad \text{and} \quad 35z_1^3 + 3 \cdot 12 + z_1^2z_2 + 3 \cdot 160z_1z_2^2 + 20z_2^3.$$

On the other hand is easy to find tensors with 1 and 3 critical points, for example take

$$2z_1^3 + 3 \cdot 2z_1^2z_2 + 3 \cdot 10z_1z_2^2 + 2z_2^3 \quad \text{and} \quad 50z_1^3 + 3 \cdot 105z_1^2z_2 + 3 \cdot 42z_1z_2^2 + 63z_2^3$$

respectively. □

Table 8.1 shows some values of the variety of rank one binary forms.

d	EDdegree(X)	rEDdegree(X)	vHDdeg(X)	HDdeg(X)
2	2	2	2	2
3	3	1,3	5	1,3,5
4	4	2,4	10	$\subseteq \{2, 4\ldots, 10\}$
5	5	1,3,5	17	$\subseteq \{3, 5\ldots, 17\}$

Table 8.1: Values for $X = v_d\mathbb{C}^2$ for different d.

The variety $v_d\mathbb{C}^2$ can be isometrically identified to the rational normal curve of degree d we present below. We treat now this variety in presence of the canonical Hermitian inner product.

Let X_d be the moment curve of degree d given by the parametrization

$$\begin{aligned} \psi_d \colon \mathbb{C} &\to X_d \subseteq \mathbb{C}^d \\ z &\mapsto (z, z^2, \ldots, z^d) \end{aligned}$$

and $\hat{X}_d \simeq v_d\mathbb{C}^2$ be the rational normal curve of degree d given by the embedding

$$\begin{aligned} v_d \colon \mathbb{P}^1 &\to \hat{X}_d \subseteq \mathbb{P}^d \\ [\mathbf{z}] &\mapsto [z_1^d, z_1^{d-1}z_2, \ldots, z_2^d] \end{aligned}$$

The latter is the projective closure of the former. In [Dra+16, Example 5.12] it is proved that when the ambient space $\mathbb{C}^{d+1}$ is endowed with a generic symmetric bilinear forms it holds $\mathrm{EDdegree}(\hat{X}_d) = 3d-2$, while in the case of the Euclidean inner product we can have $\mathrm{EDdegree}(\hat{X}_d) = d$ as stated above in the beginning of this subsection. In particular, since X_d is a parametrized variety of degree d it holds the equality

$$\mathrm{EDdegree}(X_d)^2 = (2d-1)^2 = d\,\mathrm{EDdegree}(\hat{X}_d) + (d-1)^2.$$

We can also see the equality of the numbers at the extremes by noting that $\hat{X}_d = v_d([1, z_2]) \cup v_d([z_1, 1])$ and the intersection of this two subsets consist only in the point $[1, 1]$. Thus, since the cones over $v_d([1, z_2])$ and $v_d([z_1, 1])$ are both equal to X_d, for generic points we obtain $\mathrm{EDdegree}(X_d)$ solutions of the critical ideal (4.2.2) from points $[1, z_2]$ and $\mathrm{EDdegree}(X_d)$ solutions of the critical ideal (4.2.2) from points $[z_1, 1]$ which in sum yields $\mathrm{EDdegree}(X_d)^2$ solutions (z_1, z_2) in the cone over $\hat{X}_d$. In the end, since the origin is a solution of degree $(d-1)^2$ of the critical ideal (4.2.2) and the parametrization v_d is d-to-one the result follows.

We have seen in Proposition 4.4.1 that for generic Hermitian forms it holds $\mathrm{vHDdeg}(X_d) = 2d^2 - 2d + 1$, we show some values in Table 8.2.

d	1	2	3	4	5	6	7	8	9	10
$\mathrm{vHDdeg}(X_d)$	1	5	13	25	41	61	85	113	145	181
$\max \mathrm{HDdeg}(X_d)$	1	5	≤ 11	≤ 25	≤ 41	≤ 61	≤ 85	≤ 113	≤ 145	≤ 181

Table 8.2: Values of the vHDdeg and bound for max HDdeg for the moment curve of degree d.

A similar argument as above applies in the Hermitian case and we obtain the following result.

Proposition 8.3.4. *Let $X_d \subseteq \mathbb{C}^d$ be the moment curve of degree d and $\hat{X}_d \subseteq \mathbb{P}^d$ be the rational normal curve of degree d, then*

$$\mathrm{vHDdeg}(X_d)^2 = d^2\,\mathrm{vHDdeg}(\hat{X}_d) + (d-1)^4$$

and in particular $\mathrm{vHDdeg}(\hat{X}_d) = 3d^2 - 4d + 2$.

The varieties of degree $d = 1$ are trivial. We perform some computations for the lowest non trivial cases.

Example 8.3.5. (Moment curve of degree 2). This variety is a parabola which we already studied in Example 4.4.3.

(Rational normal curve of degree 2). The affine cone of the rational normal curve is two-to-one parametrized by

$$\begin{aligned} \hat{\psi}_2 \colon \mathbb{C}^2 &\longrightarrow \hat{X}_2 = V(z_1 z_3 - z_2^2) \subseteq \mathbb{C}^3 \\ (z_1, z_2) &\mapsto \quad (z_1^2,\ z_1 z_2,\ z_2^2) \end{aligned}$$

Critical points satisfy the equations

$$\partial_{z_1} \|\hat{\psi}_2(z_1, z_2) - \mathbf{u}\|_{\mathbb{C}}^2 = 2z_1(\bar{z}_1^2 - \bar{u}_1) + z_2(\bar{z}_1 \bar{z}_2 - \bar{u}_2) = 0$$

and

$$\partial_{z_2}\|\hat{\psi}_2(z_1,z_2)-\mathbf{u}\|^2_{\mathbb{C}} = z_1(\bar{z}_1\bar{z}_2-\bar{u}_2)+2z_2(\bar{z}_2^2-\bar{u}_3)=0.$$

Introducing $\mathbf{w},\mathbf{v}$ we get a zero-dimensional system of degree 5^2. Since the origin is a solution of degree 1^2 and the parametrization is two-to-one we get $\mathrm{vHDdeg}(\hat{X}_2)=(5^2-1^2)/2^2=6$ in accordance to Proposition 8.3.4.

The polynomial defining the vHD discriminant $\mathrm{v}\Xi_{\hat{X}_2}$ when $\mathbf{v}=\mathbf{u}$ is

$$\begin{aligned}&(u_1^4u_2^2+8u_1^4u_3^2-2u_1^2u_2^2u_3^2+20u_1^3u_3^3+8u_1^2u_3^4+u_2^2u_3^4)^3\\&\quad(32u_1^6+435u_1^4u_2^2+384u_1^2u_2^4+256u_2^6-240u_1^5u_3-960u_1^3u_2^2u_3-960u_1u_2^4u_3+696u_1^4u_3^2\\&\quad+1098u_1^2u_2^2u_3^2+384u_2^4u_3^2-980u_1^3u_3^3-960u_1u_2^2u_3^3+696u_1^2u_3^4+435u_2^2u_3^4-240u_1u_3^5+32u_3^6).\end{aligned}$$

Note that $(\hat{X}_2)^\vee = V(4z_1z_3-z_2^2)\subseteq\mathbb{C}^3$.

Example 8.3.6. (Moment curve of degree 3). The moment curve is bijectively parametrized by

$$\begin{aligned}\psi_3\colon \mathbb{C}&\to X_3=V(z_3-z_1^3,\ z_2-z_1^2)\subseteq\mathbb{C}^3\\ z&\mapsto\quad (z,\ z^2,\ z^3)\end{aligned}$$

Critical points satisfy the equation

$$\partial_z\|\psi_3(z)-\mathbf{u}\|^2_{\mathbb{C}}=(\bar{z}-\bar{u}_1)+2z(\bar{z}^2-\bar{u}_2)+3z^2(\bar{z}^3-\bar{u}_3)=0.$$

Introducing $\mathbf{w},\mathbf{v}$ we compute $\mathrm{vHDdeg}(X_3)=13$. The diagonal of the matrix representing the Hermitian Killing form (see Section 3) of the system

$$\begin{cases}p(z,w)=(z-u_1)+2w(z^2-u_2)+3w^2(z^3-u_3)=0\\ \bar{p}(w,z)=(w-\bar{u}_1)+2z(w^2-\bar{u}_2)+3z^2(w^3-\bar{u}_3)=0\end{cases}$$

which generates the Hermitian critical ideal when $\mathbf{v}=\bar{\mathbf{u}}$, with respect to the basis

$$\{[1],[z],[w],[z^2],[zw],[w^2],[z^2w],[zw^2],[w^3],[z^2w^2],[zw^3],[w^4],[zw^4]\}$$

is $\begin{bmatrix}13 & -4 & -4 & -\frac{4}{3} & -\frac{4}{3} & -\frac{4}{3} & a & a & a & b & b & b & c\end{bmatrix}$ where

$$a=\frac{81|u_3|^2+20}{9},\qquad b=\frac{96|u_2|^2-432|u_3|^2-28}{27},\qquad c=\frac{45|u_1|^2-480|u_2|^2+810|u_3|^2-4}{81}.$$

In particular the Rayleigh quotient of this matrix possesses a negative minimum and from Corollary 3.1.11 we obtain $\max\mathrm{HDdeg}(X_3)\le 13-2=11$.

The parametrization for the vHD correspondence $\mathrm{v}\mathcal{H}_X$, maps (z,w,s_1,s_2,t_1,t_2) to

$$(z,z^2,z^3,w,w^2,w^3),(z-3s_1w^2-2s_2w,\ z^2+s_2,\ z^3+s_1,\ w-3t_1z^2-2t_2z,\ w^2+t_2,\ w^3+t_1).$$

(Rational normal curve of degree 3 / Twisted cubic). The affine cone of the twisted cubic is three-to-one parametrized by

$$\begin{aligned}\hat{\psi}_3\colon \mathbb{C}^2 &\longrightarrow \hat{X}_3 = V(z_1z_3 - z_2^2,\ z_2z_4 - z_3^2,\ z_1z_4 - z_2z_3) \subseteq \mathbb{C}^4\\ (z_1, z_2) &\mapsto \quad (z_1^3,\ z_1^2z_2,\ z_1z_2^2,\ z_2^3)\end{aligned}$$

Critical points satisfy the equations

$$\partial_{z_1}\|\hat{\psi}_3(z_1,z_2) - \mathbf{u}\|_{\mathbb{C}}^2 = 3z_1^2(\bar{z}_1^3 - \bar{u}_1) + 2z_1z_2(\bar{z}_1^2\bar{z}_2 - \bar{u}_2) + z_2^2(\bar{z}_1\bar{z}_2^2 - \bar{u}_3) = 0$$

and

$$\partial_{z_2}\|\hat{\psi}_3(z_1,z_2) - \mathbf{u}\|_{\mathbb{C}}^2 = z_1^2(\bar{z}_1^2\bar{z}_2 - \bar{u}_2) + 2z_1z_2(\bar{z}_1\bar{z}_2^2 - \bar{u}_3) + 3z_2^2(\bar{z}_2^3 - \bar{u}_4) = 0.$$

Introducing $\mathbf{w}, \mathbf{v}$ we get a zero-dimensional system of degree 13^2. Since the origin is a solution of degree 4^2 and the parametrization is three-to-one we get $\mathrm{vHDdeg}(\hat{X}_3) = (13^2 - 4^2)/3^2 = 17$ in accordance to Proposition 8.3.4.

Example 8.3.7. (Moment curve of degree 4). The moment curve is bijectively parametrized by

$$\begin{aligned}\psi_4\colon \mathbb{C} &\to X_4 = V(z_4 - z_1^4,\ z_3 - z_1^3,\ z_2 - z_1^2) \subseteq \mathbb{C}^4\\ z &\mapsto \quad (z,\ z^2,\ z^3,\ z^4)\end{aligned}$$

Critical points satisfy the equation

$$\partial_z\|\psi(z) - \mathbf{u}\|_{\mathbb{C}}^2 = (\bar{z} - \bar{u}_1) + 2z(\bar{z}^2 - \bar{u}_2) + 3z^2(\bar{z}^3 - \bar{u}_3) + 4z^3(\bar{z}^4 - \bar{u}_4) = 0.$$

Introducing $\mathbf{w}, \mathbf{v}$ we compute $\mathrm{vHDdeg}(X_4) = 25$.

(Rational normal curve of degree 4). The affine cone of the twisted cubic is four-to-one parametrized by

$$\begin{aligned}\hat{\psi}_4\colon \mathbb{C}^2 &\longrightarrow \hat{X}_4 = V(z_1z_3 - z_2^2,\ z_2z_4 - z_3^2,\ z_3z_5 - z_4^2,\ z_1z_5 - z_2z_4) \subseteq \mathbb{C}^5\\ (z_1, z_2) &\mapsto \quad (z_1^4,\ z_1^3z_2,\ z_1^2z_2^2,\ z_1z_2^3,\ z_2^4)\end{aligned}$$

Critical points satisfy the equations $\partial_{z_1}\|\hat{\psi}_4(z_1,z_2) - \mathbf{u}\|_{\mathbb{C}}^2 = \partial_{z_2}\|\hat{\psi}_4(z_1,z_2) - \mathbf{u}\|_{\mathbb{C}}^2 = 0$.

Introducing $\mathbf{w}, \mathbf{v}$ we get a zero-dimensional system of degree 25^2. Since the origin is a solution of degree 9^2 and the parametrization is four-to-one we get $\mathrm{vHDdeg}(\hat{X}_4) = (25^2 - 9^2)/4^2 = 34$ in accordance to Proposition 8.3.4.

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