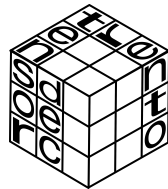




**UNIVERSITÀ
DI TRENTO**

**Dipartimento di
Matematica**

UNIVERSITÀ DI TRENTO
DIPARTIMENTO DI MATEMATICA
Doctoral Programme in Mathematics - cycle XXXVI



Thesis for the Degree of Philosophiae Doctor

**Secant varieties of Spinor varieties
and of other generalized Grassmannians**

CANDIDATE :
Vincenzo Galgano

ADVISOR :
Prof. Alessandra Bernardi

CO-ADVISOR :
Prof. Giorgio Ottaviani

Acknowledgement

The author of this thesis has been partially supported by the italian “National Group for Algebraic and Geometric Structures, and their Applications” (GNSAGA-INdAM).

The three years of Ph.D. have been quite intense and challenging, but also exciting thanks to all the people I have dealt with, even if only for a while. I have great regard for the mathematical community I became part of, among which I have also found several friends.

First of all, I am sincerely and deeply grateful to Alessandra Bernardi and Giorgio Ottaviani for being more than supervisors for me. Every moment shared together has been and is very dear to me. Besides proposing this beautiful project and guiding me through it, you have shared with me not only (part of) your knowledge but also your experience in the academic life and your way of conceiving research. Our stimulating discussions and your always effective advice made me more mature both as a researcher and as a person. Thank you for having always supported me, even at times when I lacked self-confidence. For always being available to patiently answer my questions and doubts, despite your many obligations with the teaching load and other academic duties. For having pushed me to put myself at stake and having opened my mind to international experiences. I recently found out that in Germany doctoral advisors are called “Doktorvater” and “Doktormutter” (literally, doctoral father and doctoral mother), which is exactly what you have been for me.

I would very much like to thank Mateusz Michałek for the five months I spent within his group in Konstanz. Your contagious enthusiasm for maths influenced my research and made my time working with you remarkably inspiring.

I am very grateful to Frédéric Holweck for our collaboration, for his always kind advice and his interest in my research path.

Special thanks to Laurent Manivel for welcoming me in Toulouse, for sharing his curiosity for maths with me. Your enlightening suggestions allowed me to improve the quality of this thesis and more in general of my research.

I thank my colleagues of the *Tensor Dec Lab* in Trento, in particular Alessandra Bernardi, Alessandro Oneto, Mima Stanojkovski, Elisa Postinghel, Edoardo Ballico, for creating a pleas-

ant and stimulating atmosphere in which working.

I really enjoyed every discussion with my academic brothers and sisters Pierpaola Santarsiero, Reynaldo Staffolani, Claudia De Lazzari, Ettore Teixeira Turatti and Cosimo Flavi.

I am thankful to everyone who has supported me, being there for either mathematical discussions or personal advice (or even both). Among those, I am particularly grateful to Fulvio Gesmundo for wisely and friendly advising me over these last months.

I care to thank the secretariat wonderful team Cinzia, Orietta, Michela and Francesca, for their so kind help and for patiently listening to all my complaints about bureaucracy over the years.

Special thanks to my friend, colleague and flatmate Giacomo Vianello. Thanks for all the moments spent together, from the coffee breaks in the department to our dinners at our apartment, for the several discussions on our academic and private life, and for bringing me an authentic *baccalà mantecato* from Venice several times.

Ringrazio di cuore la mia famiglia per avermi sempre supportato in ogni modo possibile, e per ispirarmi ad essere una persona migliore giorno dopo giorno. Le nostre brevi telefonate quotidiane ed i nostri abbracci (molto meno frequenti a causa della distanza) mi hanno sempre dato la forza di andare avanti. La parte migliore delle vacanze è sempre stata poter tornare a casa da voi.

My sweetest thanks are for my partner Jessica. I couldn't have done it without you. Thank you for always believing in me, for always being on my side, for your understanding in times of work overload, for your patience in reminding me not to neglect my private life for work, for the lovely time we spend together. During these years, you have been for me the lighthouse to follow in the fog of my insecurities, and the port in which to find myself again.

Preface

The topics in this thesis are the main part of my study and research during the three-year Doctoral Programme at University of Trento, under the supervision of Alessandra Bernardi and Giorgio Ottaviani. Some of these results have been achieved in collaboration with Reynaldo Staffolani and are collected in the accepted paper [GS23] and in the preprint [Gal23]. Besides this, I have been collaborating on other problems too.

In the published paper [GH23], Frédéric Holweck and I have been investigating a correspondence between orbits of graph states and orbits in the variety of principal minors of symmetric matrices with coefficients in the binary field \mathbb{F}_2 .

During a five-month-long secondment period at Universität Konstanz I have been working with Mateusz Michałek and Hanieh Keneshlou, computing equivariant Euler characteristics of classes in the equivariant Grothendieck ring $K_T^0(X_n)$ on a permutohedral variety X_n . The results appear in the accepted paper [GKM23].

The unifying spirit of all of my research has been exploiting symmetries in geometric contexts which are enriched by the action of a group. I have been enjoying in mixing tools from Algebraic Geometry with tools from Representation Theory, and I have been trying to make this a trademark of my research. I hope to have managed to convey this spirit throughout this manuscript.

Contents

Introduction	vii
Notations	xv
List of Figures	xvii
List of Tables	xvii
1 Preliminaries from Representation Theory	1
1.1 Toolkit on Lie groups and Lie algebras	1
1.2 Irreducible representations of semisimple Lie algebras	4
1.3 Rational homogeneous varieties	6
1.4 Cominuscule varieties	11
2 The geometry of tensors	13
2.1 Secant varieties and identifiability	13
2.2 Apolarity Theory: from classical to nonabelian	16
2.3 Hamming distance on projective varieties	19
3 Diving into spinors	21
3.1 Spin groups from Clifford algebras	21
3.2 Spin representations	25
3.3 Spinor varieties	28
3.4 The diameter of Spinor varieties	35
3.5 Some homogeneous bundles on Spinor varieties	37
3.6 Clifford apolarity	39
4 Identifiability and singular locus of $\sigma_2(\text{Gr}(k, N))$	45
4.1 The poset of SL_N -orbits in $\sigma_2(\text{Gr}(k, N))$	46
4.1.1 The secant branch	48
4.1.2 The tangent branch	49
4.1.3 Inclusions among closures of SL_N -orbits	51
4.2 Identifiability in $\sigma_2(\text{Gr}(k, N))$	53

4.3	Tangential-identifiability in $\tau(\mathrm{Gr}(k, N))$	55
4.4	Dimensions of SL_N -orbits in $\sigma_2(\mathrm{Gr}(k, N))$	56
4.5	The 2-nd Terracini locus of $\mathrm{Gr}(k, N)$	57
4.6	The singular locus of $\sigma_2(\mathrm{Gr}(k, N))$	58
5	Identifiability and singular locus of $\sigma_2(\mathbb{S}_N^\pm)$	67
5.1	The poset of Spin_{2N} -orbits in $\sigma_2(\mathbb{S}_N^\pm)$	68
5.1.1	Secant orbits in $\sigma_2^\circ(\mathbb{S}_N^+)$	70
5.1.2	Tangent orbits in $\tau(\mathbb{S}_N^+)$	72
5.1.3	Inclusions among closures of Spin_{2N} -orbits	73
5.2	Identifiability in $\sigma_2(\mathbb{S}_N^+)$	75
5.3	Tangential-identifiability in $\tau(\mathbb{S}_N^+)$	78
5.4	Dimensions of Spin_{2N} -orbits in $\sigma_2(\mathbb{S}_N^+)$	81
5.5	The 2-nd Terracini locus of \mathbb{S}_N^+	84
5.6	Results on the singular locus of $\sigma_2(\mathbb{S}_N^+)$	85
6	What about other generalized Grassmannians?	89
6.1	G -orbits in $\sigma_2(G/P)$ for G/P cominuscule	90
6.2	A non-cominuscule example: the isotropic Grassmannian $\mathrm{IG}(k, 2N)$	94
6.2.1	P_k -orbits in $\mathfrak{sp}_{2N}/\mathfrak{p}_k$	96
6.2.2	Inclusions and dimensions of the orbits	99
7	Appendix	105
7.1	The degree of Spinor varieties	105
7.2	Spin_M -orbits in $\mathbb{Q}^{M-2} \times \mathbb{Q}^{M-2}$	107
7.3	Secant bundle on Spinor varieties	109
	Bibliography	112

Introduction

The aim of this thesis is to give results on long-standing problems in the theory of Tensor Decomposition, namely on the problems of identifiability and singularity of points in the secant varieties of lines of both Grassmannians and Spinor varieties, and possible generalizations to other homogeneous spaces. The main spirit of this thesis is to investigate such topics from a representation-theoretical perspective, considering the action of groups on such varieties.

Tensors are everywhere in our world: they can be thought as collections of data, and as such they appear in several scientific areas, eg. Statistics, Data Science, Quantum Information, Phylogenetics and so on. In some sense, the Wachowski sisters got very close in their film “The Matrix”, a better name would have been “The Tensor”: nowadays tensorists would say that they just got the *format* wrong.

But what are tensors? Matrices are the simplest example (besides the trivial one, eg. vectors): a matrix $A \in \text{Mat}_{n \times m}(\mathbb{k})$ with coefficients in a field \mathbb{k} is a tensor in the vector space $\mathbb{k}^n \otimes \mathbb{k}^m$ of order 2 and format $n \times m$. More in general, given d \mathbb{k} -vector spaces V_1, \dots, V_d of dimensions $\dim V_i = n_i$, an element of the tensor \mathbb{k} -vector space $V_1 \otimes \dots \otimes V_d$ is a tensor of order d and format (n_1, \dots, n_d) .

Certain tensors have the nice property that they are the “building blocks” for constructing any other tensor: here the intuition should be the same as in certain toys like LEGO and GEOMAG. In the case of matrices, these are the rank-1 matrices, as any matrix $A \in \text{Mat}_{n \times m}(\mathbb{k})$ of rank r is sum of r matrices $A_i = v_i \cdot ({}^t w_i) \in \text{Mat}_{n \times m}(\mathbb{k})$ of rank 1 (here $v_i \in \mathbb{k}^n$ and $w_i \in \mathbb{k}^m$ are vectors). Similarly, any tensor $T \in V_1 \otimes \dots \otimes V_d$ is sum of tensors of the form $v_i^{(1)} \otimes \dots \otimes v_i^{(d)} \in V_1 \otimes \dots \otimes V_d$, called *decomposable* or *simple* or *rank-1*.

This setting can be extended to any irreducible non-degenerate projective variety $X \subset \mathbb{P}_{\mathbb{k}}^M$. Given a point $f \in \mathbb{P}_{\mathbb{k}}^M$, one defines the X -rank of f with respect to X as the minimum number of points of X whose linear span contains f . In particular, point of X have X -rank 1. Then, starting from X , for any $r \in \mathbb{Z}_{>0}$ one can define the r -th secant variety $\sigma_r(X) \subset \mathbb{P}_{\mathbb{k}}^M$ as the Zariski closure of points of $\mathbb{P}_{\mathbb{k}}^M$ of X -rank at most r .

The theory of *Tensor Decomposition* studies tensors, their decompositions and the varieties related to them in an algebro-geometric perspective. Its widely-recognized importance relies in the fact that decomposing tensors into simpler ones is equivalent to extracting simpler (more

feasible, hence more efficient) information from arbitrarily confusing collections of data: correction of noises, image resolutions, entanglements of quantum states can all be restated in terms of tensors and their decompositions. We refer to [Lan12; Ber+18; OR20] for a general overview on secant varieties and tensors, and [BC12; Com14; Lim21] for applications of the study of secant varieties and tensors. Motivated by most of the applications in the sciences, the theory of Tensor Decomposition is considered over fields of zero characteristics, mainly either the real field \mathbb{R} or the complex field \mathbb{C} . However, even when the applied setting is over the real numbers, almost all results are obtained by working in the complexified setting. In this respect, in the following as well as all along the thesis, we assume $k = \mathbb{C}$.

Secant varieties have been studied for decades, but several aspects of their geometry are still mysterious and difficult to handle with. Even computing their dimensions is a hard task and a current topic of research. A state of the art on the dimension of such varieties can be found in [Ber+18], and more detailed results are in [GO22; BDD07; LP13]. Two crucial aspects of secant varieties which are fundamental for applications and are still unknown in general are the *identifiability* and the *singularity* of their points.

Identifiability means uniqueness. Namely, a given point $f \in \mathbb{P}^M$ of X -rank r is identifiable if it admits a unique decomposition as sum of r elements of X . From an applicative point of view, this corresponds to uniqueness in recovering data. On the other hand, singularity means unfeasibility in computations. Formally, singular points in $\sigma_r(X)$ are those such that the dimension of the tangent spaces at these points overcome the dimension of the secant variety. The bigger the tangent space at a point, the more unfeasible the computations in a neighbourhood of that point. Of course, singularities are of great impact to theory too: results having the smoothness (ie. non-singularity) of the variety among the hypotheses are a dense in Geometry. For instance, in [COV14, Prop. 5.1] the authors give a criterion for identifiability of specific tensors in a secant variety under the assumption that these tensors are smooth. In fact, identifiability and smoothness are quite related each other and often one notion suggests the other (as in this work), although in general both implications admit counterexamples.

Owing to the above, determining the *singular locus* $\text{Sing}(\sigma_r(X))$ of a secant variety $\sigma_r(X)$ is a central problem in both theory and applications. Classically, if $\sigma_r(X)$ is not a linear space, it is known that the singular locus $\text{Sing}(\sigma_r(X))$ contains the secant variety $\sigma_{r-1}(X)$ but only in few cases $\text{Sing}(\sigma_r(X))$ is actually determined. For instance, the case $r = 2$ for *Segre varieties* is solved by M. Michałek, L. Oeding and P. Zwiernik in [MOZ15] via tools from toric geometry; for *Veronese varieties* V. Kanev [Kan99] and K. Han [Han18] solve the cases $r = 2$ and $r = 3$ respectively, while partial results for higher cases $r \geq 4$ are obtained by K. Furukawa and K. Han in [FH21]; L. Manivel and M. Michałek in [MM15] obtain partial results for the 2-nd secant variety of *Grassmannians* and other *cominuscule varieties*. The latter case is exactly the one we focus on in this thesis.

Segre varieties, Veronese varieties and Grassmannians are examples of homogeneous vari-

eties (ie. varieties on which a group acts transitively), and more precisely of projective *rational homogeneous varieties* (RHV). Also known as generalized flag varieties, which we assume to be projective, they are a class of varieties described as quotients G/P of a *semisimple complex Lie group* G by a *parabolic subgroup* P , or equivalently as unique closed orbits (of the highest weight vectors) into projectivized representations of such groups. The Representation Theory behind these varieties allows to derive several geometric properties: in this respect, the geometry of RHVs has been largely studied by J.M. Landsberg and L. Manivel [LM03; LM04]. In particular, a wide literature has been devoted to the secant variety of lines $\sigma_2(G/P)$ (ie. Zariski closures of the union of all secant lines) and the tangential variety $\tau(G/P)$ (ie. unions of all tangent lines) to a RHV G/P , starting with Zak’s key work [Zak93] and continuing with [Kaj99; LW07; LW09; MM15; Rus16].

A RHV G/P which is defined by a maximal parabolic subgroup P (ie. for which the Picard group $\text{Pic}(G/P) \simeq \mathbb{Z}$ is monogenic) is called *generalized Grassmannian*: besides the classical Grassmannians (of Dynkin–type A), other well-known examples are the *isotropic Grassmannians* (of type C) and the *orthogonal Grassmannians* (of type B or D). However, some of the generalized Grassmannians have nicer properties than the other ones: these are the *cominuscule varieties* and are characterised by the property that their tangent space $\mathfrak{g}/\mathfrak{p}$ is an irreducible P –module (or equivalently, their tangent bundle $\tau(G/P)$ is irreducible as homogeneous bundle). All classical Grassmannians are cominuscule, but the only isotropic and orthogonal Grassmannians to be cominuscule are the ones whose subspaces have maximum dimension.

The main part of this thesis is devoted to the study of the 2–nd secant varieties (aka. *secant varieties of lines*) to Grassmannians and to *Spinor varieties* (ie. maximal orthogonal Grassmannians). The Grassmannian $\text{Gr}(k, N) \subset \mathbb{P}(\wedge^k \mathbb{C}^N)$ is the projective varieties of k –dimensional linear subspaces of \mathbb{C}^N , and it is homogeneous with respect to the action of the special linear group SL_N .

On the other hand, given a non-degenerate quadratic form $q \in \text{Sym}^2(\mathbb{C}^{2N})^\vee$, the set of q –isotropic subspaces of \mathbb{C}^{2N} of maximal dimension N splits into two connected components, namely the Spinor varieties \mathbb{S}_N^\pm , or also the maximal orthogonal Grassmannians $\text{OG}^\pm(N, 2N)$. These varieties live in the so-called *half-spin representations* $\wedge^{ev} \mathbb{C}^N$ and $\wedge^{od} \mathbb{C}^N$ and are homogeneous under the action of the *Spin group* Spin_{2N} . Roughly known as the universal double cover of the special orthogonal group SO_{2N} , the spin group has an elegant and formal description via *Clifford algebras*. Despite their description as maximal orthogonal Grassmannians, already known to Cartan [Car67], Spinor varieties are not well understood as well as the classical Grassmannians. However, their intrinsic relation with Clifford algebras makes them an interesting and rich topic of research.

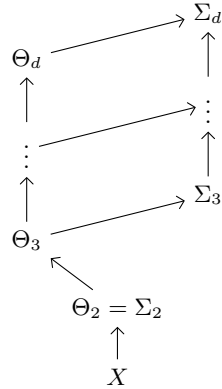
However, the beauty of such varieties rests not only in the very elegant combinatorial and algebraic description of them, but also in their versatility in several applied sciences: they are sets of separable states in bosonic and fermionic spaces respectively in Quantum Information [ST13; LH15; LH18]; tropical Grassmannians appear as spaces of phylogenetics trees [SS04]; nonlinear eigenvalue problems on Grassmannians are of interest in Quantum Chemistry [BSS23]; positive

Grassmannians have been introduced for studying scattering amplitudes in Quantum Physics [Wil21].

The key idea giving the kick-off to the whole thesis is that the action of a group G (SL_N for the Grassmannian $\mathrm{Gr}(k, N)$, and Spin_{2N} for the Spinor varieties \mathbb{S}_N^\pm) leaves the secant variety of lines invariant, which then splits into G -orbits, actually finitely many ones. But there's more: the properties of identifiability and singularity are invariant under such an action, and in particular the singular locus of the secant variety of lines is union of orbits. This implies that it is enough to check such properties for only one representative of each orbit.

Let's denote by X either the Grassmannian $\mathrm{Gr}(k, N)$ or the Spinor variety \mathbb{S}_N^\pm . We treat separately orbits of points lying on bisecant lines to X , and orbits of points lying on tangent lines to X : we refer to the former as *secant orbits*, to the latter as *tangent orbits*. We parametrize the secant orbits via the notion of *Hamming distance* between points of X : namely, the Hamming distance between two points of X is the minimum number of lines lying in X and connecting the two points. The tangent orbits are instead parametrized by the notion of rank in the tangent space $\mathfrak{g}/\mathfrak{p}$. We describe the poset (partially ordered set) of G -orbits in the secant variety of lines together with their dimensions. Our first result is the following: we collect at once the results obtained separately fro Grassmannians and Spinor varieties. By duality of Grassmannians we also assume $k \leq \lfloor \frac{N}{2} \rfloor$.

Theorem (Theorem 4.1.12, Theorem 5.1.12). *Let (X, G, d) be either $(\mathrm{Gr}(k, N), \mathrm{SL}_N, k)$ or $(\mathbb{S}_N^+, \mathrm{Spin}_{2N}, \lfloor \frac{N}{2} \rfloor)$. Then the poset of G -orbits of the secant variety of lines $\sigma_2(X)$ is*



where arrows denote the inclusion of an orbit into the closure of another orbit, the orbits Θ_l are made of tangent points to X , the orbits Σ_l are made of points lying on bisecant lines to X . Moreover, the orbits Θ_d and Σ_d are respectively dense in the tangential variety $\tau(X)$ and in the secant variety of lines $\sigma_2(X)$.

We completely determine such orbits, by exhibiting representatives and inclusions of their closures (cf. Sec. 4.1 and Sec. 5.1), and computing their dimensions (cf. Sec. 4.4 and Sec. 5.4).

Next we consider the problem of identifiability and *tangential-identifiability*: we say that a

tangent point in $\tau(X)$ is tangential-identifiable if it lies on a unique tangent line to X (cf. Def. 2.1.1). The results for Grassmannians are obtained by considering suitable wedge-multiplication maps between different fundamental SL_N -representations. The case of Spinor varieties is a little bit trickier and we use the theory of *nonabelian apolarity*, introduced by J.M. Landsberg and G. Ottaviani in [LO13] and extending the classical apolarity theory: when applied to Spinor varieties, such theory allows to define what we call *Clifford apolarity* (cf. Sec. 3.6). The solutions to the problems of (tangential-)identifiability for both Grassmannians and Spinor varieties are collected in the following theorem.

Theorem (cf. Sec. 4.2, Sec. 5.2). *In the notation of the previous Theorem, the following facts hold:*

- *the points in the orbits Σ_l for $3 \leq l \leq d$ are identifiable;*
- *the points in the orbits Θ_l for $3 \leq l \leq d$ are tangential-identifiable;*
- *for $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$, each point in the orbit $\Sigma_2 \subset \sigma_2(\mathrm{Gr}(k, N))$ is unidentifiable and its set of equivalent decompositions is 4-dimensional;*
- *for $N \geq 4$, each point in the orbit $\Sigma_2 \subset \sigma_2(\mathbb{S}_N^+)$ is unidentifiable and its set of equivalent decompositions is 6-dimensional.*

The unidentifiability of the orbit Σ_2 in both cases follows inductively from the fact the Grassmannian $\mathrm{Gr}(2, 4) \subset \mathbb{P}^5$ and the Spinor variety $\mathbb{S}_4^+ \subset \mathbb{P}^7$ are just quadrics.

We go one step further by also determining the 2-nd *Terracini locus* $\mathrm{Terr}_2(X)$ of X , namely the locus of pairs of points of X whose tangent spaces have non-trivial intersection.

Theorem (Theorem 4.5.2, Theorem 5.5.1). *The 2-nd Terracini locus $\mathrm{Terr}_2(X)$ corresponds to the orbit closure $\overline{\Sigma}_2 = X \sqcup \Sigma_2$.*

The Terracini locus tells us where the differential of the projection from the abstract secant variety onto the secant variety drops rank. More importantly, combined with the identifiability, it is a very useful tool for deducing smoothness of points lying on bisecant lines. However, for tangent points other arguments are needed. Our main results on the singular loci of secant varieties of lines to Grassmannians and to Spinor varieties are the following.

Theorem (Theorem 4.6.10). *For any $N \geq 7$ and any $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$, the singular locus of the secant variety of lines $\sigma_2(\mathrm{Gr}(k, N))$ is*

$$\mathrm{Sing}(\sigma_2(\mathrm{Gr}(k, N))) = \overline{\Sigma}_2$$

of dimension $k(N - k) + 2(N - 2) - 3$.

The above result corrects a previous statement in [AOP12, before Figure 1] in which the authors stated that $\mathrm{Sing}(\sigma_2(\mathrm{Gr}(3, 7))) = \mathrm{Gr}(3, 7)$.

Theorem (cf. Sec. 5.6). *For any $N \geq 7$, the singular locus of the secant variety of lines $\sigma_2(\mathbb{S}_N^+)$ is bounded as follows:*

$$\overline{\Sigma_2} \subset \text{Sing}(\sigma_2(\mathbb{S}_N^+)) \subset \tau(\mathbb{S}_N^+) .$$

We conjecture that the first inclusion on the left actually is an equality, as for the Grassmannian case. This is suggested by some arguments with Hilbert schemes of 2 points and secant bundles (cf. Sec. 7.3).

Quite remarkably, for other cominuscule varieties we *partially* deduce a poset of G -orbits in the secant variety of lines which is similar to the one obtained for Grassmannians and Spinor varieties (cf. Sec. 6.1): we say *partially* because the only inclusions that we haven't proved yet are the ones $\Theta_l \hookrightarrow \Sigma_l$.

We conclude our study by exhibiting an example of a non-cominuscule variety for which such a poset graph fails. Our example is the (non-maximal) isotropic Grassmannian (cf. Sec. 6.2): we show that the orbits in the tangential variety are not totally ordered.

The thesis is organized as follows.

- Chapter 1 is thought as a crash course in the theory of Lie algebras, Lie groups and their representations. We start by recalling the notions of Lie algebras, root systems, Lie Groups and their parabolic subgroups, as well as the theory of representations. Then we introduce the rational homogeneous varieties, study homogeneous bundles on them and get our focus on the cominuscule varieties.
- Chapter 2 collects basic notation and results from Tensor Decomposition. We introduce secant varieties, abstract secant varieties and the problem of identifiability, and we give an overview on the apolarity theory, from the classical one for symmetric tensors to the skew-apolarity for skew-symmetric tensors, ending with the nonabelian apolarity. A small section on the notion of Hamming distance concludes the chapter.
- Chapter 3 is devoted to the world of spinors. First we define algebraically the spin groups as multiplicative subgroups of Clifford algebras, then we give the definition of the spin representations. In the second part of the chapter we introduce Spinor varieties and we determine their diameter. Finally, we describe some homogeneous bundles on them playing a central role in the Clifford apolarity.
- In Chapter 4 we determine the poset of SL_N -orbits in the secant variety of lines to a Grassmannian $\text{Gr}(k, N)$, we solve the problem of identifiability of its points, and as a consequence we compute the orbit dimensions. Moreover, we determine both the second Terracini locus to the Grassmannian and the singular locus of the secant variety of lines. The results in this chapter appear in the work [GS23] joint with R. Staffolani.
- In Chapter 5 we move our focus to the secant varieties of lines to Spinor varieties. We determine their poset of spin-orbits and we use Cliffors apolarity for solving the problem

of identifiability of their points. After computing the orbit dimensions, we also determine the second Terracini locus to a Spinor variety. The last section of the chapter contains partial results on the singular locus of the secant variety of lines to a Spinor variety and a conjecture for the complete result. All results of this chapter are collected in the preprint [\[Gal23\]](#).

- In Chapter [6](#) we investigate generalizations of results obtained on the poset of orbits in the previous chapters. We give a partial description of the poset in the case of cominuscule varieties. Finally, we show that such a poset does not hold for non-cominuscule variety, by determining the poset of parabolic orbits in the tangent space to an isotropic Grassmannian. The results in the last section have been obtained during a visit at Institut de Mathématiques de Toulouse under the supervision of L. Manivel.
- The first two sections in the appendix in Chapter [7](#) collects well-known results for which we haven't been able to find a proper reference, thus we propose them as solved exercises. The last section on secant bundles is a motivation to the conjecture at the end of Chapter [5](#).

Notations

INDICES AND INDEX SUBSETS

$$\begin{array}{lll}
 \ell = a : b & \ell \in \{a, a+1, \dots, b-1, b\} & \forall a \leq b \in \mathbb{Z}_{>0} \\
 [n] & \{1, \dots, n\} & \forall n \in \mathbb{Z}_{>0} \\
 2^{[n]} & \{I \subset [n]\} & \\
 \binom{[n]}{k} & \{\{i_1 < \dots < i_k\} \subset [n]\} & \forall k \leq n \in \mathbb{Z}_{>0} \\
 \mathbf{e}_I & e_{i_1} \wedge \dots \wedge e_{i_k} & \forall I = \{i_1, \dots, i_k\}
 \end{array}$$

ROOTS, GROUPS AND REPRESENTATIONS

$$\begin{array}{ll}
 G & \text{semisimple (simply conn.) complex Lie group} \\
 \mathfrak{g} & \text{complex Lie algebra} \\
 \Delta := \{\alpha_1, \dots, \alpha_N\} & \text{simple roots} \\
 \omega_1, \dots, \omega_N & \text{fundamental weights} \\
 \Lambda^{++} \subset \Lambda^+ \subset \Lambda & \text{regular} \subset \text{dominant} \subset \text{weight lattice} \\
 w_0 \in \mathcal{W}_G & \text{longest element in the Weyl group of } G \\
 P_I & \text{parabolic subgr. def.}^{ed} \text{ by } I \subset \Delta \\
 V_\lambda^{\mathcal{D}} = V_\lambda^{\mathfrak{g}} = V_\lambda^G & \text{irred. repres. of } \mathfrak{g} \text{ of Dynkin type } \mathcal{D} \\
 v_\lambda, \ell_\lambda & \text{highest and lowest weight vectors in } V_\lambda^{\mathcal{D}} \\
 S_N = \bigwedge^\bullet \mathbb{C}^N & \text{spin repres. of type } B_N \text{ (cf. Sec. 3.2)} \\
 S_N^+ = \bigwedge^{ev} \mathbb{C}^N, S_N^- = \bigwedge^{od} \mathbb{C}^N & \text{half-spin repres. of type } D_N \text{ (cf. Sec. 3.2)}
 \end{array}$$

(SEMI*)SIMPLE COMPLEX LIE ALGEBRAS OF CLASSICAL TYPE

$$\begin{array}{ll}
 A_N^* & \mathfrak{gl}_{N+1} = \text{Mat}_{N+1} \\
 A_N & \mathfrak{sl}_{N+1} = \{X \in \text{Mat}_{N+1} \mid \text{tr}(X) = 0\} \\
 B_N & \mathfrak{so}_{2N+1}^Q = \{A \in \text{Mat}_{2N+1} \mid {}^t A Q = -Q A\} \quad Q \in \text{Sym}^2 \mathbb{C}^{2N+1} \\
 C_N & \mathfrak{sp}_{2N}^\Omega = \{A \in \text{Mat}_{2N} \mid {}^t A \Omega = -\Omega A\} \quad \Omega \in \bigwedge^2 \mathbb{C}^{2N} \\
 D_N & \mathfrak{so}_{2N}^Q = \{A \in \text{Mat}_{2N} \mid {}^t A Q = -Q A\} \quad Q \in \text{Sym}^2 \mathbb{C}^{2N}
 \end{array}$$

SIMPLE COMPLEX LIE GROUPS OF CLASSICAL TYPE

$$\begin{array}{ll}
A_N & \mathrm{SL}_{N+1} = \{A \in \mathrm{GL}(N+1) \mid \det A = 1\} \\
B_N & \mathrm{SO}_{2N+1}^Q = \{A \in \mathrm{SL}_{2N+1} \mid {}^tAQA = Q\} \quad Q \in \mathrm{Sym}^2 \mathbb{C}^{2N+1} \\
B_N & \mathrm{Spin}_{2N+1} = \text{cf. Sec 3.1} \\
C_N & \mathrm{Sp}_{2N}^\Omega = \{A \in \mathrm{SL}_{2N} \mid {}^tA\Omega A = \Omega\} \quad \Omega \in \bigwedge^2 \mathbb{C}^{2N} \\
D_N & \mathrm{SO}_{2N}^Q = \{A \in \mathrm{SL}_{2N} \mid {}^tAQA = Q\} \quad Q \in \mathrm{Sym}^2 \mathbb{C}^{2N} \\
D_N & \mathrm{Spin}_{2N} = \text{cf. Sec 3.1}
\end{array}$$

GENERALIZED GRASSMANNIANS OF CLASSICAL TYPE

$$\begin{array}{lll}
\mathbb{Q}^m & m - \text{dim. smooth quadric in } \mathbb{P}^{m+1} & \\
A_N/P_k = \mathrm{Gr}(k, N+1) & \text{Grassmannian} & \\
C_N/P_k = \mathrm{IG}(k, 2N) & \text{isotropic Grassmannian} & \forall k \in [N-1] \\
C_N/P_N = \mathrm{LG}(N, 2N) & \text{Lagrangian Grassmannian} & \\
D_N/P_k = \mathrm{OG}(k, 2N) & \text{orthogonal Grassmannian} & \forall k \in [N-2] \\
D_N/P_{N-1} \simeq D_N/P_N = \mathbb{S}_N & \text{Spinor variety (cf. Sec. 3.3)} &
\end{array}$$

List of Figures

4.1	Poset graph of SL_N -orbits in $\sigma_2(\text{Gr}(k, N))$	53
5.1	Poset graph of $Spin_{2N}$ -orbits in $\sigma_2(\mathbb{S}_N^+)$	75
5.2	Poset graph of the $Spin_{12}$ -orbits in $\sigma_2(\mathbb{S}_6^+)$, and their dimensions.	83
5.3	Poset graph of the $Spin_{16}$ -orbits in $\sigma_2(\mathbb{S}_8^+)$, and their dimensions.	84
6.1	G -orbits in $\sigma_2(G/P_k)$ for G/P_k cominuscule, and some inclusions.	93
6.2	Poset graph of P_3 -orbits in $\mathfrak{sp}_8/\mathfrak{p}_3$, and their dimensions.	103

List of Tables

1.1	Dynkin diagrams of irreducible root systems.	2
1.2	Coefficients of simple roots in the longest root.	4
1.3	Cominuscule weights, cominuscule varieties and their tangent bundles.	12
1.4	Freudenthal's magic square.	12
3.1	Half-spin representations of $Spin_{2N}$ for $N \equiv 0 \pmod{2}$	26
3.2	Half-spin representations of $Spin_{2N}$ for $N \equiv 1 \pmod{2}$	27
5.1	Spinor varieties in low dimensions, and their secants of lines.	68
6.1	Secant varieties of lines to cominuscule varieties: dimensions and $d_{G,k}$	91
7.1	Degrees of small Spinor varieties \mathbb{S}_N for $N = 2 : 8$	107

Chapter 1

Preliminaries from Representation Theory

1.1 Toolkit on Lie groups and Lie algebras

We refer to [FH91; Pro07; Man13] for the theory about Lie Groups and Lie algebras appearing in this section. We restrict to consider only Lie groups and Lie algebras over the complex field \mathbb{C} which are semisimple.

Let G be a semisimple complex Lie group, and let $T \subset B \subset G$ be a maximal torus and a Borel subgroup respectively. We denote the corresponding complex Lie algebras by gothic letters \mathfrak{g} , \mathfrak{t} and \mathfrak{b} : from now on we adopt the gothic notation for any other Lie algebra. Given $\Phi \subset \mathfrak{t}^\vee \setminus \{0\}$ an irreducible root system for G , let $\Phi^+ = \Phi(B)$ be the set of positive roots with respect to the Borel subgroup B , and $\Phi^- = \Phi \setminus \Phi^+$ its complement. We set $\Delta = \{\alpha_1, \dots, \alpha_N\} \subset \Phi^+$ to be the set of (positive) simple roots.

For any $\alpha \in \Phi$ we denote by $\mathfrak{g}_\alpha := \{v \in \mathfrak{g} \mid [h, v] = \alpha(h)v \ \forall h \in \mathfrak{t}\}$ the corresponding weight space (or eigenspace). In particular, one gets the Cartan decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \underbrace{\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha}_{=\mathfrak{b}} \oplus \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha .$$

The algebraic properties of (isomorphism classes of) irreducible root systems are encoded in the Dynkin diagrams. We list them in Table 1.1 following the Bourbaki notation.

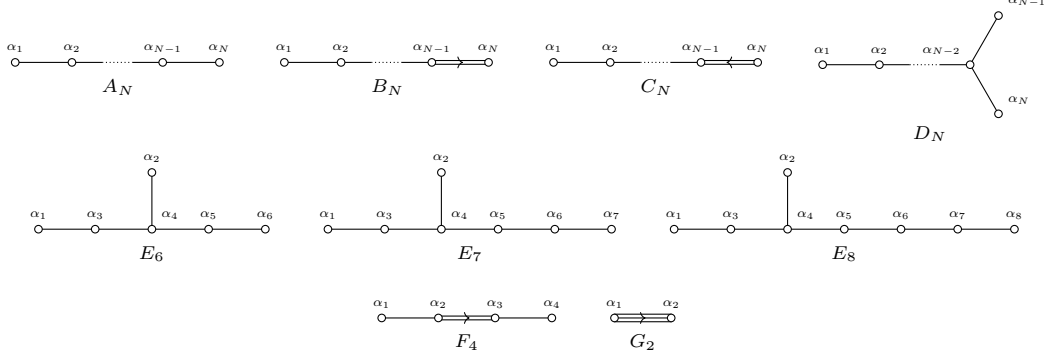


Table 1.1: Dynkin diagrams of irreducible root systems.

Weights. Let (\cdot, \cdot) be the standard scalar product in $\langle \Phi \rangle_{\mathbb{R}} = \mathbb{R}^N$. For any two roots $\alpha, \beta \in \Phi$, the Cartan integer $\langle \beta | \alpha \rangle = 2 \frac{(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$ is linear only in β , not in α . Then one defines the lattices in $\mathbb{R}^N = \langle \Phi \rangle_{\mathbb{R}}$:

- (weights) $\Lambda := \{\lambda \in \mathbb{R}^N \mid \langle \lambda | \alpha \rangle \in \mathbb{Z} \forall \alpha \in \Phi\}$;
- (dominant weights) $\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda | \alpha_i \rangle \geq 0 \forall \alpha_i \in \Delta\}$;
- (regular dominant weights) $\Lambda^{++} := \{\lambda \in \Lambda \mid \langle \lambda | \alpha_i \rangle \geq 1 \forall \alpha_i \in \Delta\}$;

The *fundamental weights* $\omega_1, \dots, \omega_N \in \Lambda$ are those such that $\langle \omega_i | \alpha_j \rangle = \delta_{ij}$ for any $\alpha_j \in \Delta$. In particular, for any $\lambda \in \Lambda$ it holds $\lambda = \sum_i \langle \lambda | \alpha_i \rangle \omega_i$, and $\Lambda = \sum_i \mathbb{Z} \omega_i$. Notice that the root lattice $\sum_i \mathbb{Z} \alpha_i$ is a subgroup of the weight lattice Λ of index $\det(\langle \alpha_i | \alpha_j \rangle)$. On the weight lattice Λ one considers the partial *dominance order*: for any two weights $\lambda, \mu \in \Lambda$

$$\lambda \succ \mu \iff \lambda - \mu \in \sum_{\beta \in \Phi^+} \mathbb{R}_{\geq 0} \beta.$$

The Weyl group. For any $\alpha \in \Phi$, let $s_\alpha \in \text{Orth}(\langle \Phi \rangle_{\mathbb{Q}})$ be the reflection $s_\alpha(v) = v - \frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha$. We denote by $s_i = s_{\alpha_i}$ the simple reflection for $\alpha_i \in \Delta$. Then the *Weyl group* of G is the finite group

$$\mathcal{W}_G := N_G(T)/T = \langle s_\alpha \mid \alpha \in \Phi \rangle = \langle s_i \mid i = 1 : N \rangle.$$

Since \mathcal{W}_G preserves Φ , it preserves the weight lattice Λ too.

We recall that any Weyl element $w \in \mathcal{W}_G$ admits a reduced expression in terms of the simple reflections $w = s_{i_1} \cdots s_{i_{\ell(w)}}$, where $\ell(w)$ is the length of w . We denote by $w_0 \in \mathcal{W}_G$ the (unique) *longest* element of \mathcal{W}_G such that $\ell(w_0) = |\Phi^+|$ and $w_0(\Phi^+) = \Phi^-$.

REMARK 1.1.1. The involution $-w_0$ defines an involution on the Dynkin diagram \mathcal{D} of G . Thus for groups $\mathcal{D} = B_N, C_N, E_7, E_8, F_4, G_2$ whose Dynkin diagram does not admit an involution different from the identity, it holds $-w_0 = id$. For the other types $-w_0$ acts as follows:

$$A_N : \omega_k \leftrightarrow \omega_{N+1-k} \quad \forall k \in [N] \quad , \quad D_N : \begin{cases} \omega_{N-1} \leftrightarrow \omega_N & \text{if } N \equiv 1 \pmod{2} \\ id & \text{if } N \equiv 0 \pmod{2} \end{cases} \quad , \quad E_6 : \omega_1 \leftrightarrow \omega_6 \quad , \quad \omega_3 \leftrightarrow \omega_5 .$$

Bruhat decomposition. To any root $\alpha \in \Phi$ one can associate the unipotent subgroup U_α such that $Lie(U_\alpha) = \mathfrak{g}_\alpha$, called *root subgroup*. Such subgroups generate G and allow to describe the Bruhat cells. More precisely, given the Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$, any Bruhat cell BwB is isomorphic (as an algebraic variety) to $B \times \left(\prod_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-} U_\alpha \right)$. The Bruhat cells stratify G/B and their closures \overline{BwB} inside G/B (which is a projective variety, see Sec. 1.3) are called *Schubert varieties*.

Parabolic subgroups and Levi decomposition. For any subset of simple roots $I \subset \Delta$, one defines the *parabolic subgroup* $P_I := \langle B, U_{-\alpha} \mid \alpha \notin I \rangle$ (any parabolic group arises as such). In particular, any simple root $\alpha_i \in \Delta$ defines a maximal parabolic subgroup $P_i := P_{\{\alpha_i\}}$.

REMARK. We stress out that we fix the notation such that the parabolic subgroup P_I defined by the subset $I \subset \Delta$ is generated by the root subgroups whose roots do *not* lie in I .

Set $\Phi(I) := \{\alpha \in \Phi \mid \langle \alpha, \omega_i \rangle \geq 0 \quad \forall i \in I\}$, $\Phi(I)^0 := \{\alpha \in \Phi \mid \langle \alpha, \omega_i \rangle = 0 \quad \forall i \in I\}$ and $\Phi(I)^+ := \Phi(I) \setminus \Phi(I)^0$. At the Lie algebra level, one has

$$\mathfrak{p}_I = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(I)} \mathfrak{g}_\alpha = \mathfrak{t} \oplus \underbrace{\bigoplus_{\alpha \in \Phi(I)^0} \mathfrak{g}_\alpha}_{\mathfrak{l}_I} \oplus \underbrace{\bigoplus_{\alpha \in \Phi(I)^+} \mathfrak{g}_\alpha}_{\mathfrak{p}_I^u} \quad (1.1.1)$$

where \mathfrak{l}_I and \mathfrak{p}_I^u are respectively reductive and nilpotent (with respect to the adjoint action) Lie algebras. Notice that \mathfrak{p}_I actually contains the Borel subalgebra \mathfrak{b} : indeed, for any positive root $\beta \in \Phi^+$ such that $\beta = \sum_{j=1}^N m_{\alpha_j}(\beta) \alpha_j$ for $m_{\alpha_j}(\beta) \in \mathbb{Z}_{\geq 0}$, it holds for any $i \in I$

$$\langle \beta, \omega_i \rangle = \sum_{j=1}^N m_{\alpha_j}(\beta) \langle \alpha_j, \omega_i \rangle = \sum_{j=1}^N m_{\alpha_j}(\beta) \frac{(\alpha_j, \alpha_j)}{(\omega_i, \omega_i)} \delta_{ij} \geq 0 ,$$

hence $\Phi^+ \subset \Phi(I)$. As a reductive Lie algebra, \mathfrak{l}_I splits as direct sum of a semisimple component \mathfrak{s}_I and its centre \mathfrak{z}_I : more precisely, given t_{α_i} the semisimple element in the \mathfrak{sl}_2 -triplet for $\alpha_i \in \Delta$, it holds

$$\mathfrak{s}_I = \underbrace{\bigoplus_{\alpha_i \notin I} \mathbb{C} t_{\alpha_i}}_{\mathfrak{t}_I} \oplus \bigoplus_{\alpha \in \Phi(I)^0} \mathfrak{g}_\alpha \quad , \quad \mathfrak{z}_I = \mathfrak{t} \setminus \mathfrak{t}_I .$$

In particular, $\Phi(I)^0$ is a linear section Φ , hence it is a root system itself whose Dynkin diagram is obtained by keeping the I -indexed nodes from the one of \mathfrak{g} . Moreover, $\Phi(I)^0$ is the root system of the semisimple Lie algebra \mathfrak{s}_I : in this respect, we say that $\Phi(I)^0$ are the roots of the parabolic algebra \mathfrak{p}_I . Similarly, we can define the weights of \mathfrak{p}_I (strictly formally, of \mathfrak{s}_I) as

$$\Lambda_I := \sum_{i \in I} \mathbb{Z} \omega_i \subset \Lambda ,$$

and denote by Λ_I^+ and Λ_I^{++} its dominant weights and regular dominant weights respectively.

Going back to the group level, the splitting $\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{p}_I^u$ corresponds to the *Levi decomposition* $P_I = L_I P_I^u$, where L_I and P_I^u are respectively reductive and unipotent subgroups. Finally, the Weyl group of P_I is

$$\mathcal{W}_{P_I} := N_{P_I}(T)/T = \langle s_i \mid \alpha_i \notin I \rangle .$$

\mathbb{Z} -gradings on Lie algebras. Fix a simple root $\alpha_k \in \Delta$ and consider $P = P_k$ its corresponding maximal parabolic subgroup. For any root $\beta \in \Phi$ we denote by $m_k(\beta)$ the coefficient of α_k in β . Then we consider on \mathfrak{g} the \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{\mathbb{Z}} \mathfrak{g}_i$ defined by

$$\mathfrak{g}_0 = \mathfrak{t} \oplus \bigoplus_{\beta \in \langle \Delta \setminus \{\alpha_k\} \rangle_{\mathbb{Z}}} \mathfrak{g}_{\beta} \quad , \quad \mathfrak{g}_i = \bigoplus_{m_k(\beta)=i} \mathfrak{g}_{\beta} . \quad (1.1.2)$$

In particular, $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ and $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \dots \simeq \mathfrak{p}^u$, where \mathfrak{p}^u is a P -module after adjoint action and the last identification of P -modules is given by the Killing form on \mathfrak{g} . Notice that only the summand \mathfrak{g}_{-1} is a \mathfrak{p} -submodule of $\mathfrak{g}/\mathfrak{p}$. Moreover, given ρ the longest root in Φ , one has $m_k(\beta) \leq m_k(\rho)$. Thus, $\mathfrak{g}/\mathfrak{p}$ splits in as many summands as $m_k(\rho)$. We list the Dynkin diagrams labeled with the coefficients in ρ of the corresponding simple roots.

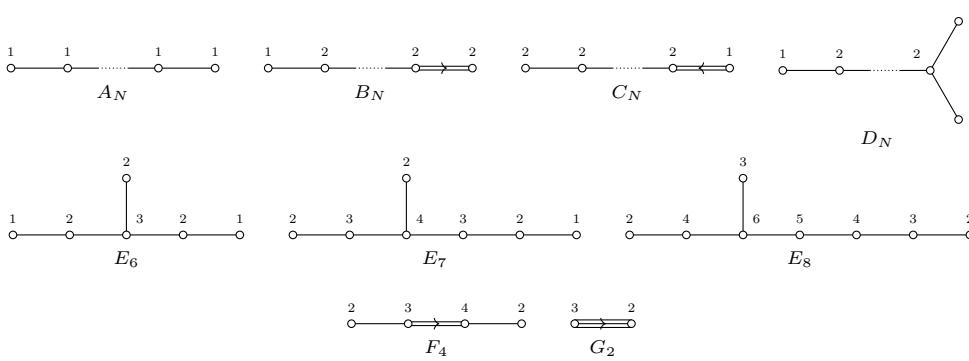


Table 1.2: Coefficients of simple roots in the longest root.

1.2 Irreducible representations of semisimple Lie algebras

In this section we recall the main results of irreducible representations of semisimple Lie algebras and semisimple (simply connected) Lie groups.

Our main references are [FH91; Pro07].

We use the terms “ G -representation” and “ G -module” indistinctly. We recall that a G -equivariant map is a map between G -structures (eg. G -modules, G -varieties) commuting with the group action. By Weyl’s theorem, every finite-dimensional representation of a semisimple complex Lie group G splits as a direct sum of irreducible G -modules. A key result in the theory of representations is the following.

Schur's Lemma ([FH91], Sec. 1.2, Lemma 1.7). *Let V, W be irreducible G -representations and let $f, g : V \rightarrow W$ be G -equivariant maps. Then the following facts hold:*

- (i) *either f is an isomorphism (of G -modules) or $f = 0$;*
- (ii) *if $V = W$, then $f = \lambda \cdot Id_V$ for some $\lambda \in \mathbb{C}^\times$;*
- (iii) *$f = \lambda g$ for some $\lambda \in \mathbb{C}^\times$.*

From the Lie group-Lie algebras correspondence one knows that every G -module is a \mathfrak{g} -module, and that, under the (necessary) assumption that G is simply connected, every \mathfrak{g} -module comes from a G -module.

In this respect, from now on G is a simply connected semisimple complex Lie group.

Let V be a finite-dimensional irreducible \mathfrak{g} -module (hence an irreducible G -module). The Cartan subalgebra \mathfrak{t} acts diagonally on V , which decomposes as

$$V = \bigoplus_{\alpha \in \mathfrak{t}^*} V_\alpha \quad , \quad V_\alpha := \{v \in V \mid h \cdot v = \alpha(h)v \ \forall h \in \mathfrak{t}\} \ ,$$

where α and V_α are said *weights* and *weight spaces* of V respectively. Actually, the weights appearing above are exactly the weights in Λ defined from the root system.

Finite-dimensional irreducible G -representations are in one-to-one correspondence with the dominant weights Λ^+ of \mathfrak{g} . In particular, given $\mathfrak{u} := \mathfrak{b} \setminus \mathfrak{t}$, every (finite-dimensional) irreducible \mathfrak{g} -representation V admits a unique dominant weight $\lambda_V \in \Lambda^+$ such that the weight space V_{λ_V} coincides with the 1-dimensional subspace $V^+ := \{v \in V \mid \mathfrak{u} \cdot v = 0\}$. The weight $\lambda_V \in \Lambda^+$ is said *highest weight* of V , while a non-zero vector $v_{\lambda_V} \in V^+$ is called a *highest weight vector* of V .

Given G simply connected semisimple of Dynkin type \mathcal{D} , we denote by V_λ^G or $V_\lambda^{\mathcal{D}}$ the irreducible \mathfrak{g} -representation with highest weight λ , and by $v_\lambda \in V_\lambda^G$ its highest weight vector.

REMARK. We distinguish the weight space $V_\lambda = V^+$ from the irreducible representation V_λ^G by writing G (or its Dynkin type) at exponent.

Recall that the dominant weight lattice $\Lambda^+ = \sum_i \mathbb{Z}_{\geq 0} \omega_i$ is spanned by the fundamental weights $\omega_1, \dots, \omega_N$: then for any dominant weight $\lambda = a_1 \omega_1 + \dots + a_N \omega_N \in \Lambda^+$ it holds

$$V_\lambda^G \subset \text{Sym}^{a_1}(V_{\omega_1}^G) \otimes \dots \otimes \text{Sym}^{a_N}(V_{\omega_N}^G) \ .$$

We refer to an irreducible representation $V_{\omega_k}^G$, for ω_k a fundamental weight, as to *fundamental representation*.

EXAMPLE. For Dynkin type A_N the simply connected simple Lie group is $G = \text{SL}_{N+1}$. Then for any $k \in [N]$ the k -th fundamental G -representation is $V_{\omega_k}^G = V_{\omega_k}^{A_N} = \bigwedge^k \mathbb{C}^{N+1}$. For any $k \in [N]$ and any $a \in \mathbb{Z}_{>0}$, the dominant weight $a\omega_k \in \Lambda^+$ defines the irreducible G -representation $V_{a\omega_k}^{A_N} \subset \text{Sym}^a(\bigwedge^k \mathbb{C}^{N+1})$. Notice that the latter inclusion is strict for any $k \neq 1$.

The theory of characters provides a formula for the dimension of irreducible representations. In the following κ denotes the Killing form, while

$$\zeta := \sum_{i=1}^N \omega_i$$

is the sum of all the fundamental weights.

Weyl's dimension formula ([Pro07], Sec. 11.10.5). *The irreducible G -representation V_λ^G of highest weight $\lambda \in \Lambda^+$ has dimension*

$$\dim V_\lambda^G = \prod_{\alpha \in \Phi^+} \frac{\kappa(\lambda + \zeta, \alpha)}{\kappa(\zeta, \alpha)}. \quad (1.2.1)$$

Dual representations. Let V^G be a G -representation and let $(V^G)^\vee$ be its dual space. The action of G on $(V^G)^\vee$ is defined by $(g \cdot f)(v) = f(g^{-1} \cdot v)$ for any $g \in G$, $v \in V^G$ and $f \in (V^G)^\vee$. If $\Lambda(V^G)$ denotes the weights of V^G , then $\Lambda((V^G)^\vee) = -\Lambda(V^G)$. Now assume $V^G = V_\lambda^G$ to be irreducible of highest weight $\lambda \in \Lambda^+$ and consider w_0 the longest element in the Weyl group \mathcal{W}_G . Then $-\lambda$ is the lowest weight of $(V_\lambda^G)^\vee$ and $-w_0\lambda$ is the highest weight of $(V_\lambda^G)^\vee$, implying

$$(V_\lambda^G)^\vee = V_{-w_0(\lambda)}^G.$$

Parabolic representations. Let P_I be a parabolic subgroup with Levi decomposition $P_I = L_I P_I^u$, where L_I and P_I^u are reductive and unipotent respectively. Let \mathfrak{p}_I , \mathfrak{l}_I and \mathfrak{p}_I^u be the corresponding Lie algebras. Recall that \mathfrak{p}_I^u is nilpotent, and that \mathfrak{l}_I is reductive with semisimple component \mathfrak{s}_I . The irreducible root system associated to \mathfrak{s}_I is $\Phi(I)^0$ as in (1.1.1) and irreducible \mathfrak{s}_I -representations are in bijection with the dominant weights Λ_I^+ . Moreover, for any $\lambda \in \Lambda_I^+$, the irreducible \mathfrak{s}_I -representation $V_\lambda^{\mathfrak{s}_I}$ is also an irreducible \mathfrak{l}_I -representation $V_\lambda^{\mathfrak{l}_I}$, which extends to an irreducible \mathfrak{p}_I -representation $V_\lambda^{\mathfrak{p}_I}$ by letting the nilpotent component \mathfrak{p}_I^u act trivially.

1.3 Rational homogeneous varieties

Rational homogeneous varieties are the main characters of the Geometric Representation Theory, studying the geometry arising from representations. However they are of great interest in several areas of Algebraic Geometry as they offer promising driving examples for the behaviour of more general varieties. All notation and assumptions from the previous section are assumed. For references we suggest [Ott95], [Sno89, Sec. 5-6] and [FH91, Sec. 23.3].

Rational homogeneous varieties. Let G be a simply connected simple complex Lie group and let V_λ^G an irreducible G -representation with highest weight $\lambda \in \Lambda^+$ and highest weight vector $v_\lambda \in V_\lambda^G$. Then G acts on the projective space $\mathbb{P}(V_\lambda^G)$: in particular, the point $[v_\lambda] \in \mathbb{P}(V_\lambda^G)$ corresponds to the weight space $V^+ = \mathbb{C} \cdot v_\lambda$. The orbit $G \cdot [v_\lambda]$ is the unique closed

G -orbit in $\mathbb{P}(V_\lambda^G)$, hence it is a projective homogeneous variety. Moreover, the stabilizer of $[v_\lambda]$ is a parabolic subgroup (as it contains the Borel subgroup): more precisely,

$$\text{stab}_G([v_\lambda]) = P_{I_\lambda} \quad , \quad I_\lambda := \{\alpha_i \in \Delta \mid \langle \lambda, \alpha_i \rangle \neq 0\} .$$

As parabolic subgroups are closed, $G/P_{I_\lambda} = G \cdot [v_\lambda] \subset \mathbb{P}(V_\lambda^G)$ is compact, and one can prove it is rational too. On the other hand, every projective rational homogeneous variety is isomorphic to a product of finitely many compact rational homogeneous varieties G_i/P_i for certain G_i simple and P_i parabolic subgroups. The compact rational homogeneous varieties G/P are also called *generalized flag varieties*, as they generalize the complete flag varieties G/B . When G is simply connected of Dynkin type \mathcal{D} , we use indistinctly the notation $G/P = \mathcal{D}/P$.

The following result highlights a very important geometric property of rational homogeneous varieties: the statement appears in [Lan12, Theorem 6.10.6.5] and a proof is sketched in [Lan12, Sec. 16.2.2].

Kostant's Theorem. *Let V_λ^G be an irreducible G -module with highest weight $\lambda \in \Lambda^+$, and let $G/P \subset \mathbb{P}(V_\lambda^G)$ be the orbit of a highest weight line $[v_\lambda]$. Then the ideal $\mathcal{I}(G/P) \subset \text{Sym}^\bullet(V_\lambda^G)^\vee$ is generated in degree 2 by $(V_{2\lambda}^G)^\perp \subset \text{Sym}^2(V_\lambda^G)^\vee$. In particular, rational homogeneous varieties are intersections of quadrics.*

Generalized Grassmannians. We are interested in the varieties G/P_k where G is simply connected simple and $P_k = P_{\alpha_k}$ is a maximal parabolic subgroup defined by a simple root $\alpha_k \in \Delta$. Such G/P_k are called *generalized Grassmannians* as they are generalization of Grassmannians (length-1 flags). They are the unique closed orbits in the projective spaces of the irreducible representations $V_{d\omega_k}^G$ defined by the highest weights $d\omega_k$ as $d \in \mathbb{Z}_{>0}$ varies: indeed, given λ the highest weight such that $G/P_k \subset \mathbb{P}(V_\lambda^G)$, one has $\langle \lambda, \alpha_i \rangle = 0$ for any $i \neq k$, that is $\lambda = d\omega_k$ for some $d \in \mathbb{Z}_{>0}$.

EXAMPLE 1.3.1 (Grassmannians). For $G = \text{SL}_{N+1}$ and $k \in [N]$, the unique closed orbit $G/P_k = A_N/P_k$ in the fundamental representation $\mathbb{P}(V_{\omega_k}^G) = \mathbb{P}(\bigwedge^k \mathbb{C}^{N+1})$ is the *Grassmannian*

$$\text{Gr}(k, N+1) := \left\{ [v_1 \wedge \dots \wedge v_k] \in \mathbb{P} \left(\bigwedge^k \mathbb{C}^{N+1} \right) \right\} \simeq \{ W \subset \mathbb{C}^{N+1} \mid W \simeq \mathbb{C}^k \} .$$

We denote points in $\text{Gr}(k, N+1)$ by both $[W]$ and $[w_1 \wedge \dots \wedge w_k]$ where $W = \langle w_1, \dots, w_k \rangle_{\mathbb{C}}$. After fixing a basis (e_1, \dots, e_{N+1}) of \mathbb{C}^{N+1} , the point corresponding to the highest weight vector is $[E] = [e_1 \wedge \dots \wedge e_k] = [v_{\omega_k}]$. Then, as stabilizer of $[E]$, the parabolic subgroup P_k is

$$P_k = \left\{ \begin{bmatrix} A & B \\ & C \end{bmatrix} \in \text{GL}_{N+1} \mid A \in \text{GL}_k, B \in \text{Mat}_{(k, N+1-k)}, C \in \text{GL}_{N+1-k} \right\} .$$

In particular, P_k has Levi decomposition given by $L_k \simeq \text{GL}_k \times \text{GL}_{N+1-k}$ and $P_k^u \simeq \text{Mat}_{(k, N+1-k)}$. Notice that the Levi Lie algebra \mathfrak{l}_k is semisimple of Dynkin type A_{k-1} and it acts on E with highest weight ω_1 . As any irreducible \mathfrak{l}_k -module with a given highest weight extends to an irreducible \mathfrak{p}_k -module with the same highest weight, we get that E is the irreducible P_k -representation with highest weight ω_1 .

EXAMPLE 1.3.2 (Isotropic Grassmannians). Let $\Omega \in \bigwedge^2(\mathbb{C}^{2N})^\vee$ be a non-degenerate symplectic form on \mathbb{C}^{2N} . For any $W \subset \mathbb{C}^{2N}$, let W^\perp be its orthogonal with respect to Ω . Recall that W is Ω -isotropic if $W \subset W^\perp$. For $G = \mathrm{Sp}_{2N}^\Omega$ and $k \in [N-1]$, the k -th fundamental representation is $V_{\omega_k}^G = \bigwedge^k \mathbb{C}^{2N}$ and the unique closed orbit $G/P_k = C_N/P_k$ is the *isotropic Grassmannian*

$$\begin{aligned} \mathrm{IG}_\Omega(k, 2N) &:= \left\{ [v_1 \wedge \dots \wedge v_k] \in \mathbb{P} \left(\bigwedge^k \mathbb{C}^{2N} \right) \mid \Omega(v_i, v_j) = 0 \ \forall i, j \in [k] \right\} \\ &\simeq \{ W \subset \mathbb{C}^{2N} \mid W \simeq \mathbb{C}^k, W \subset W^\perp \} . \end{aligned}$$

EXAMPLE 1.3.3 (Lagrangian Grassmannians). In the same notation as Example 1.3.2, for $k = N$ the variety $G/P_N = C_N/P_N$ is defined in the same way. It is called *Lagrangian Grassmannian* and denoted by $\mathrm{LG}_\Omega(N, 2N)$.

EXAMPLE 1.3.4 (Orthogonal Grassmannians). Let $q \in \mathrm{Sym}^2(\mathbb{C}^{2N})^\vee$ be a non-degenerate quadratic form on \mathbb{C}^{2N} (similarly on \mathbb{C}^{2N+1}). For any $W \subset \mathbb{C}^{2N}$ (resp. \mathbb{C}^{2N+1}), let W^\perp be its orthogonal with respect to q : then W is q -isotropic if $W \subset W^\perp$. The simply connected simple group G of Dynkin type D_N (resp. B_N) is the spin group Spin_{2N}^q (resp. Spin_{2N+1}^q), which will be defined in Sec. 3.1. However, the fundamental representations $V_{\omega_k}^G$ for $k \in [N-2]$ (resp. $[N-1]$) are also representations for the group SO_{2N}^q (resp. SO_{2N+1}^q) and they are $V_{\omega_k}^G = \bigwedge^k \mathbb{C}^{2N}$. For such a k , the unique closed orbit $G/P_k = D_N/P_k$ (resp. B_N/P_k) is the *orthogonal Grassmannian*

$$\begin{aligned} \mathrm{OG}_q(k, 2N) &:= \left\{ [v_1 \wedge \dots \wedge v_k] \in \mathbb{P} \left(\bigwedge^k \mathbb{C}^{2N} \right) \mid q(v_i, v_j) = 0 \ \forall i, j \in [k] \right\} \\ &\simeq \{ W \subset \mathbb{C}^{2N} \mid W \simeq \mathbb{C}^k, W \subset W^\perp \} . \end{aligned}$$

(similarly for type B_N). However, for $k = N-1, N$ (resp. N) the fundamental representations of the spin group Spin_{2N} (resp. Spin_{2N+1}) are not representations of SO_{2N}^q (resp. SO_{2N+1}^q) and the above description fails: we describe them in Sec. 3.2. The varieties G/P arising in this way are the *Spinor varieties*, which will be the topic of Sec. 3.3.

EXAMPLE 1.3.5 (Veronese varieties). Consider $G = \mathrm{SL}_{N+1}$ and $d \in \mathbb{Z}_{>0}$. The irreducible representation of highest weight $d\omega_1$ is $V_{d\omega_1}^{A_N} = \mathrm{Sym}^d(V_{\omega_1}^{A_N}) = \mathrm{Sym}^d \mathbb{C}^{N+1}$: this is not true for the other fundamental weights, as for $k, d \geq 2$ the representation $\mathrm{Sym}^d(V_{\omega_k}^{A_N})$ is not irreducible anymore. After fixing a suitable basis (e_1, \dots, e_{N+1}) of \mathbb{C}^{N+1} , the highest weight vector in $\mathrm{Sym}^d \mathbb{C}^{N+1}$ is the monomial $v_{d\omega_1} = e_1^d$ and the unique closed orbit $A_N/P_1 = \mathrm{SL}_{N+1} \cdot [e_1^d]$ in $\mathbb{P}(V_{d\omega_1}^{A_N})$ is the *degree- d Veronese variety*

$$\nu_d(\mathbb{P}^N) := \left\{ [v^d] \in \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^{N+1}) \mid [v] \in \mathbb{P}^N \right\} .$$

In the language of homogeneous bundles (see next paragraph), one says that the degree- d Veronese variety $\nu_d(\mathbb{P}^N)$ is the rational homogeneous variety A_N/P_1 projectively embedded via the (homogeneous) line bundle $\mathcal{O}(d)$.

Homogeneous bundles. Let G be a simply connected semisimple complex Lie Group and let P be a parabolic subgroup. A rank- r vector bundle \mathcal{E} on G/P is G -homogeneous if there exists a G -action on \mathcal{E} commuting with the fibration $\mathcal{E} \rightarrow G/P$. The action of G permutes the fibers of \mathcal{E} : given $[gP] \in G/P$ and $h \in G$, one has $h \cdot \mathcal{E}_{[gP]} = \mathcal{E}_{[hgP]}$. Moreover, given $[P] \in G/P$ the base point, one has $P \cdot \mathcal{E}_{[P]} = \mathcal{E}_{[P]}$, hence $\mathcal{E}_{[P]}$ is a P -representation. On the other hand, starting from a representation $\rho : P \rightarrow \mathrm{GL}(V^P)$ such that $V^P \simeq \mathbb{C}^r$, one can construct a rank- r G -homogeneous bundle \mathcal{E}_ρ (cf. [Sno89, Sec. 5]): one defines \mathcal{E}_ρ as the quotient $G \times_\rho V^P = (G \times V^P) / \sim_P$ where $(g, v) \sim_P (h, w)$ if and only if $(h, w) = (gp^{-1}, \rho(p)v)$ for some $p \in P$. If $V^P = V_\lambda^P$ is an irreducible P -representation of highest weight λ , then \mathcal{E}_ρ is said *irreducible* G -homogeneous bundle, and we denote it by \mathcal{E}_λ .

The above construction allows to describe the global sections of \mathcal{E}_ρ :

$$H^0(G/P, \mathcal{E}_\rho) \simeq \{s : G \rightarrow V^P \mid s(gp^{-1}) = p \cdot s(g) \ \forall g \in G, p \in P\} .$$

In particular, the vector space $H^0(G/P, \mathcal{E}_\rho)$ is endowed with the natural action of g

$$(g \cdot s)(h) = s(g^{-1}h) \quad \forall g, h \in G, s \in H^0(G/P, \mathcal{E}_\rho) ,$$

hence it is a G -representation.

All line bundles on G/P are G -homogeneous and, if P_I is the parabolic subgroup defined by the subset of simple roots $I \subset \Delta$, then [Sno89, Theorem 6.4]

$$\mathrm{Pic}(G/P_I) \simeq \mathbb{Z}^I \simeq \Lambda_I = \sum_{i \in I} \mathbb{Z}\omega_i .$$

In particular, if P_k is maximal defined by the simple root $\alpha_k \in \Delta$, then the Picard group of G/P_k is $\mathrm{Pic}(G/P_k) = \mathbb{Z}\mathcal{O}_{G/P_k}(1) \simeq \mathbb{Z}\omega_k$.

Recall that Λ_I , Λ_I^+ and Λ_I^{++} denote the lattices of weights, dominant weights and regular dominant weights respectively. Recall that Λ_I is in one-to-one correspondence with the line bundles on G/P_I , while the dominant weight lattice Λ^+ is in one-to-one correspondence with the irreducible G -representations.

Borel-Weil's Theorem ([Sno89], Theorem 6.5). *Let G be a semisimple simply connected complex Lie group and let P_I be a parabolic subgroup defined by the subset of simple roots $I \subset \Delta$. Let $\mathcal{L}_\lambda \in \mathrm{Pic}(G/P_I)$ be a line bundle with highest weight $\lambda \in \Lambda_I$. Then:*

- (i) \mathcal{L}_λ is spanned at one point of G/P_I iff \mathcal{L} is spanned at every point of G/P_I , iff $\lambda \in \Lambda_I^+$;
- (ii) \mathcal{L}_λ is ample iff \mathcal{L} is very ample, iff $\lambda \in \Lambda_I^{++}$;
- (iii) $H^0(G/P_I, \mathcal{L}_\lambda) = 0$ if $\lambda \in \Lambda_I \setminus \Lambda_I^+$;
- (iv) $H^0(G/P_I, \mathcal{L}_\lambda) \simeq (V_\lambda^G)^\vee$ if $\lambda \in \Lambda_I^+$.

The above result generalizes to G -homogeneous bundles of higher rank too. It is a consequence of the fact that any irreducible G -homogeneous bundle on a rational homogeneous variety G/P is isomorphic to the pull-back $\pi^*\mathcal{L}$ of a line bundle \mathcal{L} on G/B via the projection $\pi : G/B \rightarrow G/B$: a proof is sketched in [Ott95, Sec. 10].

Borel-Weil's Theorem (Generalized). *Let G be a semisimple simply connected complex Lie group and let P_I be a parabolic subgroup defined by the subset $I \subset \Delta$ of simple roots. Let \mathcal{E}_λ be an irreducible G -homogeneous bundle with highest weight $\lambda \in \Lambda_+^+$. Then*

$$H^0(G/P_I, \mathcal{E}_\lambda) \simeq (V_\lambda^G)^\vee = V_{-w_0(\lambda)}^G .$$

EXAMPLE 1.3.6 (Minimal homogeneous embedding). Let G/P_k be the generalized flag defined by the maximal parabolic subgroup P_k . The highest weight defining the line bundle $\mathcal{O}(1)$ is the fundamental weight ω_k , which is regular dominant. Then by Borel-Weil's Theorem we get the minimal homogeneous embedding

$$G/P_k \hookrightarrow \mathbb{P}(H^0(G/P_k, \mathcal{O}(1))^\vee) \simeq \mathbb{P}(V_{\omega_k}^G) .$$

EXAMPLE 1.3.7 (Universal bundles). In the notation of Example 1.3.1, let \mathcal{U}^\vee be the rank- k bundle (dual to the universal bundle \mathcal{U}) on the Grassmannian $A_N/P_k = \text{Gr}(k, N+1)$ with fibers $\mathcal{U}_{[W]}^\vee \simeq W$ for any $[W] \in \text{Gr}(k, N+1)$. Since $P_k = \text{stab}_{A_N}([E])$, the fiber $\mathcal{U}_{[E]}^\vee \simeq E$ is a P_k -representation with highest weight ω_1 . Then

$$H^0(\text{Gr}(k, N+1), \mathcal{U}^\vee) \simeq (V_{\omega_1}^{A_N})^\vee \simeq (\mathbb{C}^{N+1})^\vee .$$

EXAMPLE 1.3.8 (The bundle $\mathcal{U}(1)$ on Grassmannians). In the notation of the above examples, consider the rank- k vector bundle $\mathcal{U}(1) = \mathcal{U} \otimes \mathcal{O}(1)$ on the Grassmannian $\text{Gr}(k, N+1)$. The rank- k universal bundle \mathcal{U} has no global sections, but its twisting $\mathcal{U}(1)$ does. Indeed, the *determinant bundle* of \mathcal{U} is $\det(\mathcal{U}) := \bigwedge^k \mathcal{U} \simeq \mathcal{O}(1)$ and it holds

$$\mathcal{U} \otimes \mathcal{O}(1) \simeq \mathcal{U} \otimes \det(\mathcal{U}) \simeq \bigwedge^{k-1} \mathcal{U}^\vee .$$

In particular, the fiber at $[W] \in \text{Gr}(k, N+1)$ of the bundle $\mathcal{U}(1)$ is $(\mathcal{U}(1))_{[W]} \simeq \bigwedge^{k-1} W^\vee$, and its global sections are

$$H^0(\text{Gr}(k, N+1), \mathcal{U}(1)) \simeq \bigwedge^{k-1} (\mathbb{C}^{N+1})^\vee .$$

We stress out that the above description does not always apply: for instance, it fails for Spinor varieties (cf. Sec. 3.5).

EXAMPLE 1.3.9. Let $\mathcal{T}_{G/P}$ be the tangent bundle on G/P . The fiber at $[P] \in G/P$ is $(\mathcal{T}_{G/P})_{[P]} \simeq \mathfrak{g}/\mathfrak{p} \simeq \mathfrak{p}^u$, which is a P -module under the adjoint action. We know that in general $\mathfrak{g}/\mathfrak{p}$ is not irreducible as P -module (neither if P is maximal, cf. Sec. 6.2), hence the tangent bundle is not always an irreducible homogeneous bundle. The generalized flags G/P for which it is irreducible are the *cominuscule varieties* (see Sec. 1.4).

1.4 Cominuscule varieties

The “cominuscule” varieties appears in different corners of the literature: in the theory of algebraic groups and their representations (eg. parabolic subgroups with abelian unipotent radical [RRS92]), in differential geometry (eg. compact hermitian symmetric spaces [Kos61, Sec. 8]), in Quantum Information (eg. varieties parametrizing certain simple quantum states [ST13]).

DEFINITION. A fundamental weight ω_i is *cominuscule* if the longest root $\rho \in \Phi$ has coefficient 1 on the simple root α_i . Given P_i a maximal parabolic subgroup defined by a fundamental weight ω_i , the generalized Grassmannian G/P_i is *cominuscule* if ω_i is so.

Table 1.2 shows that cominuscule (fundamental) weights (and cominuscule varieties) only appear in Dynkin types $ABCD$ and E_6, E_7 .

Proposition 1.4.1. *Let G a simple simply connected complex Lie group. Let ω be a cominuscule weight and let P be the corresponding maximal parabolic subgroup. The following facts are equivalent:*

1. *The weight ω is cominuscule;*
2. *The \mathbb{Z} -grading of \mathfrak{g} with respect to P is $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$;*
3. *The tangent space $\mathfrak{g}/\mathfrak{p}$ is an irreducible \mathfrak{p} -module;*
4. *The unipotent radical P^u is an abelian group.*

Proof. Let α be the simple root corresponding to the fundamental weight ω . Consider the \mathbb{Z} -grading on \mathfrak{g} induced by α as in (1.1.2). The non-zero component of highest grade corresponds to the coefficient $m_\alpha(\rho)$ of α in the longest root ρ . Moreover, the only \mathfrak{p} -submodule of $\mathfrak{g}/\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \dots \oplus \mathfrak{g}_{-m_\alpha(\rho)}$ is \mathfrak{g}_{-1} . Thus the equivalence among 1, 2 and 3 is straightforward. The equivalence between 1 and 3 is proved in [RRS92, Lemma 2.2]. \square

The above proposition and Example 1.3.9 imply the following characterization of cominuscule varieties.

Corollary. *Let G/P be a projective rational homogeneous variety defined by a maximal parabolic subgroup P . The following facts are equivalent:*

1. *The variety G/P is cominuscule;*
2. *The tangent bundle $\mathcal{T}(G/P)$ is an irreducible P -homogeneous bundle.*

In Table 1.3 we list all the cominuscule varieties together with their tangent bundles. We denote by \mathcal{U} both the rank- k universal bundle on the Grassmannian $\text{Gr}(k, N+1)$ and, when $N+1 = 2M$, its pullbacks on the Lagrangian Grassmannian $\text{LG}(M, 2M)$ (cf. Example 1.3.3) and on the Spinor varieties \mathbb{S}_M^\pm (cf. Sec. 3.3); \mathcal{Q} (italic style) is the rank- $(N+1-k)$ quotient

bundle on the Grassmannian $\text{Gr}(k, N + 1)$; \mathbb{Q}^m (non-italic style) is the m -dimensional quadric in \mathbb{P}^{m+1} ; the variety $\mathbb{O}\mathbb{P}^2 := E_6/P_1$ is called *Cayley plane*; $V_{\omega_5}^{D_5}$ is (one of) the half-spin representation of type D_5 (cf. Sec. 3.2).

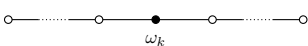
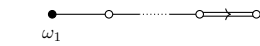
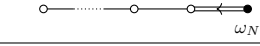
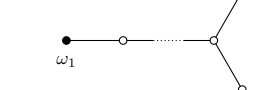
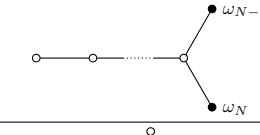
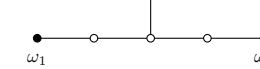
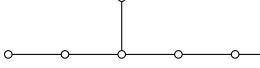
Dynkin	G	cominuscule weights	G/P	$\mathcal{T}(G/P)$
A_N	SL_{N+1}		$\text{Gr}(k, N + 1)$	$\mathcal{U}^\vee \otimes \mathcal{Q}$
B_N	Spin_{2N+1}		\mathbb{Q}^{2N-1}	$\mathcal{U}^\vee \otimes (\mathcal{U}^\perp/\mathcal{U})$
C_N	Sp_{2N}		$\text{LG}(N, 2N)$	$\text{Sym}^2 \mathcal{U}^\vee$
D_N	Spin_{2N}		\mathbb{Q}^{2N-2}	$\mathcal{U}^\vee \otimes (\mathcal{U}^\perp/\mathcal{U})$
D_N	Spin_{2N}		$\mathbb{S}_N^+, \mathbb{S}_N^-$	$\wedge^2 \mathcal{U}^\vee$
E_6	E_6		$\mathbb{O}\mathbb{P}^2$	$V_{\omega_5}^{D_5} = \wedge^{od} \mathbb{C}^5$
E_7	E_7		E_7/P_7	$V_{\omega_1}^{E_6}$

Table 1.3: Cominuscule weights, cominuscule varieties and their tangent bundles.

REMARK 1.4.2. The varieties $\text{Gr}(3, 6)$, $\text{LG}(3, 6)$, \mathbb{S}_6^\pm and E_7/P_7 are known as *Legendrian varieties*, while $\mathbb{O}\mathbb{P}^2$ is a Severi variety. They fit in the third and second row respectively of the *Freudenthal's magic square* (Table 1.4), which has been studied in detail by J.M. Landsberg and L. Manivel in [LM01; LM07].

$\nu_2(\mathbb{Q}^1)$	$\mathbb{P}(T\mathbb{P}^2)$	$\text{LG}(2, 6)$	$\mathbb{O}\mathbb{P}_0^2$
$\nu_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$\text{Gr}(2, 6)$	$\mathbb{O}\mathbb{P}^2$
$\text{LG}(3, 6)$	$\text{Gr}(3, 6)$	\mathbb{S}_6^\pm	E_7/P_7
F_4^{ad}	E_6^{ad}	E_7^{ad}	E_8^{ad}

Table 1.4: Freudenthal's magic square.

In the above table, G^{ad} denotes the *adjoint variety* of G , that is the unique closed orbit in $\mathbb{P}(\mathfrak{g})$ under the adjoint action. The fourth row contains some adjoint varieties, while the third row (the Legendrian one) is given by the varieties of lines through a point of the adjoint variety in the same column. The second row contains the *Severi varieties*, which have the property that a generic hyperplane section is still homogeneous: such hyperplane sections give the first row.

Chapter 2

The geometry of tensors

2.1 Secant varieties and identifiability

Our main references about the geometry of tensors are [Lan12; Ber+18; OR20]. In this section we recall some notions and results from the theory of tensor decomposition.

We work over the complex field \mathbb{C} .

Secant varieties. Given $X \subset \mathbb{P}^M$ an irreducible non-degenerate projective variety, the X -rank of a point $p \in \mathbb{P}^M$, denoted by $r_X(p)$, is the minimum number of points of X whose span contains p . The r -th secant variety $\sigma_r(X)$ of X in \mathbb{P}^M is defined as the Zariski closure of the set $\sigma_r^\circ(X) := \{p \in \mathbb{P}^M : r_X(p) \leq r\} \subset \mathbb{P}^M$ of points of rank at most r , i.e.

$$\sigma_r(X) := \overline{\sigma_r^\circ(X)} = \overline{\{p \in \mathbb{P}^M : r_X(p) \leq r\}} \subset \mathbb{P}^M .$$

The *border X -rank* of a point $p \in \mathbb{P}^M$, denoted by $br_X(p)$, is the minimum integer r such that $p \in \sigma_r(X)$. Although computing the dimension of secant varieties is a hard problem in general, the following inequality always holds:

$$\dim \sigma_r(X) \leq \text{expdim } \sigma_r(X) := \min\{r(\dim X + 1) - 1, M\} ,$$

where the right-hand side is called *expected dimension* of the secant variety. This is a straightforward consequence of the celebrated Terracini's Lemma.

Terracini's Lemma. *In the above notation, let $p_1, \dots, p_r \in X$ be r distinct general points and let $q \in \langle p_1, \dots, p_r \rangle \subset \sigma_r^\circ(X)$. Then*

$$T_q \sigma_r(X) = \langle T_{p_1} X, \dots, T_{p_r} X \rangle .$$

If $\dim \sigma_r(X) \leq \text{expdim } \sigma_r(X)$, then the secant variety $\sigma_r(X)$ is said to be *defective*. We say that $\sigma_r(X)$ *overfills* the ambient space if $\dim \sigma_r(X) = M \leq r(\dim X + 1) - 1$, while it *perfectly fills* the ambient space if $\dim \sigma_r(X) = M = r(\dim X + 1) - 1$. The latter is said *perfect case*.

In the above notation, quite often one refers to points in \mathbb{P}^M as to “tensors” and to the points of X -rank 1 (i.e. points in X) as to “simple (or decomposable) tensors”. As the chain of secant varieties to X always stabilizes at \mathbb{P}^M , it is quite common to look at points in X as to “building blocks” for the whole space \mathbb{P}^M .

This general theory comes from a more concrete setting, motivating the choice of the word “tensor”. Indeed, the most basic example is the one of matrices, for which the ambient space $\mathbb{P}^M = \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^m)$ is given by all $n \times m$ matrices, the variety $X = \text{Seg}(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1})$ is given by rank-1 matrices and the secant variety $\sigma_r(X)$ is given by the matrices of rank less or equal than r . Although basic, this is a “degenerate” case in which rank and border rank coincides.

However, the situation gets more complicated (and more interesting) when considering tensor spaces of higher order $d \geq 3$ (i.e. with d tensor entries) $\mathbb{C}^{(n_1, \dots, n_d)} := \mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_d}$. An element $f \in \mathbb{C}^{(n_1, \dots, n_d)}$ is said to be a *tensor* of format (n_1, \dots, n_d) (and order d). Tensors of the form $f = v_1 \otimes \dots \otimes v_d \in \mathbb{C}^{(n_1, \dots, n_d)}$ for some $v_i \in \mathbb{C}^{n_i}$ are said *decomposable* or *simple* or *rank-1*: the set of projective classes of such elements is the Segre variety $\text{Seg}(n_1, \dots, n_d) := \text{Seg}(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_d-1}) \subset \mathbb{P}(\mathbb{C}^{(n_1, \dots, n_d)})$. Clearly, any element $f \in \mathbb{C}^{(n_1, \dots, n_d)}$ can be written as sum of decomposable elements, and the notion of *rank* is quite immediate as previous X -rank where $X = \text{Seg}(n_1, \dots, n_d)$.

Other examples of tensors which are ubiquitous in the literature are the space of symmetric tensors $\text{Sym}^d \mathbb{C}^n$ whose “symmetric rank”-1 elements define the degree- d Veronese variety $\nu_d(\mathbb{P}^{n-1})$ (cf. Example 1.3.5), and the space of skewsymmetric tensors $\wedge^k \mathbb{C}^n$ whose “skewsymmetric rank”-1 elements give the Grassmannian $\text{Gr}(k, n)$ (cf. Example 1.3.1). More in general, from Sec. 1.3 we know that any (projective) rational homogeneous variety G/P embedded in an irreducible representation V_λ^G is a rank-1 variety.

Identifiability. For any $p \in \mathbb{P}^M$ with $r_X(p) = r$, the *decomposition locus* of p is the set of all r -tuples of points of X giving a minimal (i.e. length- r) decomposition of p

$$\text{Dec}_X(p) := \{(p_1, \dots, p_r) \mid p_i \in X, p \in \langle p_1, \dots, p_r \rangle, r = r_X(p)\} \subset X_{/\mathfrak{S}_r}^r,$$

where $X_{/\mathfrak{S}_r}^r$ denotes the symmetric quotient of X^r by the symmetric group \mathfrak{S}_r acting on r elements. An element $(p_1, \dots, p_r) \in \text{Dec}_X(p)$ is called a *decomposition* of p . A point p is *identifiable* if there exists a unique decomposition of p , i.e. $\text{Dec}_X(p)$ is a singleton. Otherwise one says that p is *unidentifiable*.

For any subset $Y \subset \mathbb{P}^M$, we say that Y is (un)identifiable if any point of Y is so. In particular, an orbit is (un)identifiable if and only if a representative of it is so.

Tangential-identifiability The *tangential variety* of X in \mathbb{P}^M is the union of all lines in \mathbb{P}^M which are tangent to X , i.e. it is the set of all tangent points to X

$$\tau(X) := \bigcup_{p \in X} T_p X \subset \mathbb{P}^M.$$

In particular, $X \subset \tau(X) \subset \sigma_2(X)$. Moreover, from Zak's key result [Zak93, Theorem 1.4] it holds that either $\dim \tau(X) = 2 \dim X$ and $\dim \sigma_2(X) = 2 \dim X + 1$, or $\tau(X) = \sigma_2(X)$.

DEFINITION 2.1.1. A tangent point $q \in \tau(X)$ is *tangential-identifiable* if it lies on a unique tangent line to X , or equivalently if there exists a unique $p \in X$ such that $q \in T_p X$. Otherwise it is *tangential-unidentifiable*.

We say that an orbit is tangential-(un)identifiable if all of its elements are so.

DEFINITION 2.1.2. Given a tangent point $q \in \tau(X)$, its *tangential-locus* is the set of points $p \in X$ such that $q \in T_p X$.

Clearly, if q is tangential-identifiable, then its tangential-locus is given by a single point at X .

Hilbert scheme of 2 points. Given X a smooth projective variety of dimension n , the *Hilbert scheme of 2 points* $Hilb_2(X)$ on X is the scheme of 0-dimensional subschemes of X of length 2: since X is smooth, $Hilb_2(X)$ is smooth too (this is false for Hilbert schemes of higher lengths). The subschemes in $Hilb_2(X)$ are of two types: the *reduced* subschemes, corresponding to subsets of distinct points $\{p, q\} \subset X$, and the *non-reduced* subschemes supported at only one point, lying on the boundary and parametrized by points of X along with their tangent directions (a.k.a. *2-jets*), that is $\{p, v\}$ such that $p \in X$ and $v \in T_p X$. The Hilbert scheme $Hilb_2(X)$ admits the *universal subscheme*

$$\Phi := \{(x, \mathcal{Z}) \in X \times Hilb_2(X) \mid x \in \mathcal{Z}\}$$

$$\begin{array}{ccc} & & \\ & \swarrow \pi_X & \searrow \pi_H \\ X & & Hilb_2(X) \end{array}$$

Notice that $\pi_H^{-1}(\mathcal{Z}) \simeq \mathcal{Z}$. The universal subscheme Φ is isomorphic to the blow-up $Bl_{\Delta}(X \times X)$ of the product $X \times X$ along the diagonal Δ_X (see [Ver01, Section 3], [Ull16, Section 1]).

REMARK 2.1.3. Let $X^2_{/\mathfrak{S}_2} := (X \times X)/\mathfrak{S}_2$ be the symmetric square of X obtained by quotienting $X \times X$ by the action of the symmetric group acting on two elements and switching the entries in every pair: it is singular along the diagonal Δ_X (which by abuse of notation we denote in the same way as the diagonal in $X \times X$). The isomorphism $\Phi \simeq Bl_{\Delta}(X \times X)$ leads to the commutative diagram

$$\begin{array}{ccc} Bl_{\Delta}(X \times X) & \xrightarrow{Bl} & X \times X \\ / \mathfrak{S}_2 \downarrow & & \downarrow / \mathfrak{S}_2 \\ Hilb_2(X) & \longrightarrow & X^2_{/\mathfrak{S}_2} \end{array}$$

in which the projection $\pi_H : \Phi \rightarrow Hilb_2(X)$ corresponds to the quotient by the \mathfrak{S}_2 -action $Bl_{\Delta}(X \times X) \rightarrow Hilb_2(X)$. In particular, the Hilbert scheme of 2 points can be realized as the blow-up of the symmetric square $X^2_{/\mathfrak{S}_2}$ along the diagonal, i.e.

$$Hilb_2(X) \simeq Bl_{\Delta_X}(X^2_{/\mathfrak{S}_2}).$$

Abstract secant varieties of lines. Motivated by the problem of identifiability and tangential-identifiability of points in the secant variety of lines, one can consider an incidence variety which “solves” the unidentifiabilities by distinguishing among the different bisecant or tangent lines on which each point lies. Such a variety is the 2-nd *abstract secant variety*. In the literature there are several (and *non-equivalent*) definitions of abstract secant varieties. Due to our purposes, we define the 2-nd abstract secant variety as the smooth variety

$$Ab\sigma_2(X) := \{(\mathcal{Z}, p) \in Hilb_2(X) \times \mathbb{P}^M \mid p \in \langle \mathcal{Z} \rangle\} ,$$

where $Hilb_2(X)$ is the Hilbert scheme of 2 points on X , and $\langle \mathcal{Z} \rangle \subset \mathbb{P}^M$ denotes the linear span of \mathcal{Z} . The 2-nd abstract secant variety comes with the natural projections onto the two factors $\pi_1 : Ab\sigma_2(X) \rightarrow Hilb_2(X)$ and $\pi_2 : Ab\sigma_2(X) \rightarrow \mathbb{P}^M$: in particular,

$$\pi_2 (Ab\sigma_2(X)) = \sigma_2(X) .$$

The fiber of π_2 at an identifiable point $[a + b] \in \sigma_2^o(X)$ is given by just the reduced subscheme $\{[a], [b]\} \subset X$, while the fiber at a tangential-identifiable point $[q] \in \tau(X)$ such that $[q] \in T_{[x]}X$ is just the 2-jet $\{[x], [q]\} \in Hilb_2(X)$. More in general, the fiber at a given point $p \in \sigma_2^o(X)$ coincides with the decomposition locus of p . On the other hand, the projection π_1 onto the first factor is a \mathbb{P}^1 -fibration, i.e. fibers are isomorphic to projective lines.

REMARK 2.1.4. With this definition, $Ab\sigma_2(X)$ is smooth (as $Hilb_2(X)$ is so) and closed. However, different definitions of the 2-nd abstract secant variety may require to take the closure, and may even be singular. For instance, if one replaces the Hilbert scheme by the product $X \times X$, then one needs to take the closure and gets a smooth variety. On the other hand, one can also replace $Hilb_2(X)$ by the symmetric square $X^2_{\mathfrak{S}_2} := (X \times X)/\mathfrak{S}_2$: in this case, again the closure is needed but the variety is singular along the preimage of the diagonal via the projection onto the first factor.

2.2 Apolarity Theory: from classical to nonabelian

Apolarity Theory is a very rich toolbox for studying several properties of tensors: historically it was born in the setting of symmetric tensors [IK99] and then spread to wider classes of varieties [Arr+21; LO13; Sta23]. We assume notation and arguments from previous sections.

Classical apolarity. Given a degree- d homogeneous polynomial (i.e. symmetric tensor) $f \in \text{Sym}^d \mathbb{C}^{N+1}$, the *Waring decomposition problem* consists in finding a minimal decomposition (i.e. as many summands as the symmetric rank) of f as sum of d -powers of linear forms (i.e. rank-1 symmetric tensors):

$$f = \sum_{i=1}^r \ell_i^d \quad , \quad \ell_i \in \mathbb{C}^{N+1} \quad , \quad .$$

In the language from Sec. 2.1, this is equivalent to asking for which $r \in \mathbb{N}$ the point $[f] \in \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^{N+1})$ lies in the set $\sigma_r^\circ(\nu_d(\mathbb{P}^N)) \setminus \sigma_{r-1}^\circ(\nu_d(\mathbb{P}^N))$. A problem of great interest, both in pure and applied mathematical areas, is whether a minimal Waring decomposition of f is either unique (i.e. f is identifiable) or there are either finitely or infinitely many of them.

Consider the dual space $\mathrm{Sym}^d(\mathbb{C}^{N+1})^\vee$, whose elements can be seen as degree- d homogeneous derivations. Given $f \in \mathrm{Sym}^d \mathbb{C}^{N+1}$, for any $h \leq d$ the pairing

$$\mathcal{C}^{h,d-h} : \mathrm{Sym}^d \mathbb{C}^{N+1} \otimes \mathrm{Sym}^h(\mathbb{C}^{N+1})^\vee \rightarrow \mathrm{Sym}^{d-h} \mathbb{C}^{N+1}$$

induces the h -th catalecticant map of f

$$\begin{array}{ccc} \mathcal{C}_f^{h,d-h} : \mathrm{Sym}^h(\mathbb{C}^{N+1})^\vee & \longrightarrow & \mathrm{Sym}^{d-h} \mathbb{C}^{N+1} \\ \partial & \mapsto & \partial(f) \end{array} .$$

The apolar ideal of f is the ideal of the graded algebra $\mathrm{Sym}^\bullet(\mathbb{C}^{N+1})^\vee$

$$(f)^\perp := \{g \in \mathrm{Sym}^\bullet(\mathbb{C}^{N+1})^\vee \mid g(f) = 0\} = \bigoplus_{h \geq 0} \ker(\mathcal{C}_f^{h,d-h}) ,$$

where we also admit the values $h \geq d$ for which $\ker(\mathcal{C}_f^{h,d-h}) = \mathrm{Sym}^h(\mathbb{C}^{N+1})^\vee$. The apolar ideal plays a central role in the following key result in Apolarity Theory [IK99].

Apolarity Lemma. *Let $f \in \mathrm{Sym}^d \mathbb{C}^{N+1}$ be a degree- d homogeneous polynomial and let $\ell_1, \dots, \ell_r \in \mathbb{C}^{N+1}$ be r distinct linear forms. Then*

$$f = \sum_{i=1}^r \ell_i^d \iff \mathcal{I}([\ell_1], \dots, [\ell_r]) \subset (f)^\perp .$$

Roughly, Apolarity Lemma allows to look for decompositions of f among the saturated 0-dimensional ideals inside $(f)^\perp$.

REMARK 2.2.1. As the pairing $\mathcal{C}^{h,d-h}$ is SL_{N+1} -equivariant, one can read it in terms of representations as $V_{d\omega_1}^{AN} \otimes (V_{h\omega_1}^{AN})^\vee \simeq V_{d\omega_1}^{AN} \otimes V_{h\omega_N}^{AN} \rightarrow V_{(d-h)\omega_1}^{AN}$. As there is only one copy of the irreducible module $V_{(d-h)\omega_1}^{AN}$ as summand in the tensor module on the left-hand side, by Schur's Lemma it follows that the map $\mathcal{C}^{h,d-h}$ is the unique (up to scalars) SL_{N+1} -equivariant map between the two above modules. Since $\mathrm{Sym}^d \mathbb{C}^{N+1} = H^0(\mathbb{P}^N, \mathcal{O}(d))^\vee$, another interpretation of the map $\mathcal{C}^{h,d-h}$ comes from the natural contraction map between spaces of global sections of line bundles on \mathbb{P}^N : indeed, the classical apolarity is a special case of the nonabelian apolarity which we introduce later in this section.

Skewsymmetric apolarity. An analog of the h -th catalecticant map from the classical apolarity for skewsymmetric tensors is the following [Arr+21, Def. 4]. We denote decomposable elements by $\mathbf{v}_I = \mathbf{v}_{\{i_1, \dots, i_k\}} = v_{i_1} \wedge \dots \wedge v_{i_k} \in \bigwedge^k \mathbb{C}^{N+1}$ and their dual elements by $\mathbf{w}^J = \mathbf{w}^{\{j_1, \dots, j_k\}} = w^1 \wedge \dots \wedge w^k \in \bigwedge^k (\mathbb{C}^{N+1})^\vee$. For any $h \leq k$ and any rank-1 skewsymmetric tensor $\mathbf{v}_I \in \bigwedge^k \mathbb{C}^{N+1}$, one considers

$$\begin{array}{ccc} \mathcal{C}_{\mathbf{v}_I}^{h,k-h} : \bigwedge^h (\mathbb{C}^{N+1})^\vee & \rightarrow & \bigwedge^{k-h} \mathbb{C}^{N+1} \\ \mathbf{w}^J & \mapsto & (\mathbf{w}^J) \cdot (\mathbf{v}_I) \end{array}$$

where

$$(\mathbf{w}^J) \cdot (\mathbf{v}_I) = \sum_{S \in \binom{I}{h}} \text{sign}(S) \det(w^{j_p}(v_{s_q})) \mathbf{v}_{I \setminus S}$$

for S varying among the (ordered) subsets of h elements in I , $\text{sign}(S)$ being the sign of the permutation that sends the sequence $\{i_1, \dots, i_k\}$ to the sequence $\{s_1, \dots, s_h, I \setminus S\}$, and $(w^{j_p}(v_{s_q}))$ being the $h \times h$ matrix with J -indexed rows and S -indexed columns. Then the *skewapolarity action* is defined by extending by linearity the above map to the bilinear map

$$\mathcal{C}^{h, k-h} : \bigwedge^k \mathbb{C}^{N+1} \otimes \bigwedge^h (\mathbb{C}^{N+1})^\vee \longrightarrow \bigwedge^{k-h} \mathbb{C}^{N+1}. \quad (2.2.1)$$

The restriction $\mathcal{C}_f^{h, k-h} \in \text{Hom}(\bigwedge^h (\mathbb{C}^{N+1})^\vee, \bigwedge^{k-h} \mathbb{C}^{N+1})$ are called *skew-catalecticants*. Given $f \in \bigwedge^k \mathbb{C}^{N+1}$ one defines its orthogonal with respect to the skew-apolar action

$$f^\perp := \left\{ g \in \bigwedge_{h \leq N+1} (\mathbb{C}^{N+1})^\vee \mid g \cdot f = 0 \right\}.$$

Moreover, given r decomposable elements $\mathbf{v}_i = v_{i,1} \wedge \dots \wedge v_{i,k} \in \text{Gr}(k, N+1)$ for $i \in [r]$ corresponding to the k -dimensional subspaces $H_{\mathbf{v}_i} \subset \mathbb{C}^{N+1}$, one defines $\mathcal{I}(\mathbf{v}_1, \dots, \mathbf{v}_r) := \bigcap_{i \in [r]} H_{\mathbf{v}_i}^\perp$ and by $\mathcal{I}(\mathbf{v}_1, \dots, \mathbf{v}_r)_k := \mathcal{I}(\mathbf{v}_1, \dots, \mathbf{v}_r) \cap \bigwedge^k \mathbb{C}^{N+1}$ its k -degree component. There is a result similar to the Apolarity Lemma holds [Arr+21, Lemma 12].

Skew-apolarity Lemma (ARRONDO-BERNARDI-MARQUES-MOURRAIN). *In the above notation, the following are equivalent:*

- (i) $f = \sum_{i \in [r]} \mathbf{v}_i$;
- (ii) $\mathcal{I}(\mathbf{v}_1, \dots, \mathbf{v}_r) \subset f^\perp$;
- (iii) $(\mathcal{I}(\mathbf{v}_1, \dots, \mathbf{v}_r))_k \subset (f^\perp)_k$.

REMARK 2.2.2. Similarly to Remark 2.2.1, the skew-apolarity action can be interpreted in terms of SL_{N+1} -representations as the SL_{N+1} -equivariant map $V_{\omega_k}^{A_N} \otimes V_{\omega_{N+1-h}}^{A_N} \rightarrow V_{\omega_{k-h}}^{A_N}$. Again, by Schur's Lemma one can prove that this map is uniquely determined (up to scalars). Moreover, also in this case one can translate everything in the language of vector bundles, since $\bigwedge^k \mathbb{C}^{N+1} = V_{\omega_k}^{A_N} \simeq H^0(\text{Gr}(k, N+1), \mathcal{O}(1))^\vee$.

Nonabelian apolarity The classical apolarity action and the skew-apolarity action are particular cases of a more general apolarity, namely the *nonabelian apolarity*, introduced at first in [LO13, Section 1.3]. Let $X \subset \mathbb{P}^n$ be a projective variety and let \mathcal{L} be a very ample line bundle on X giving the embedding $X \subset \mathbb{P}^n = \mathbb{P}(H^0(X, \mathcal{L})^\vee)$. Let \mathcal{E} be a rank- e vector bundle on X such that $H^0(X, \mathcal{E}^\vee \otimes \mathcal{L})$ is not trivial, where \mathcal{E}^\vee denotes the bundle dual to \mathcal{E} . The natural contraction map

$$H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}^\vee \otimes \mathcal{L}) \longrightarrow H^0(X, \mathcal{L})$$

leads to the morphism $A : H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{L})^\vee \longrightarrow H^0(X, \mathcal{E}^\vee \otimes \mathcal{L})^\vee$ and, after fixing a given $f \in H^0(X, \mathcal{L})^\vee$, one gets the linear map

$$A_f : H^0(X, \mathcal{E}) \longrightarrow H^0(X, \mathcal{E}^\vee \otimes \mathcal{L})^\vee$$

defined as $A_f(s) := A(s \otimes f)$ for any $s \in H^0(X, \mathcal{E})$, and called *nonabelian apolarity action*. Let $H^0(X, \mathcal{I}_Z \otimes \mathcal{E})$ and $H^0(X, \mathcal{I}_Z \otimes \mathcal{E}^\vee \otimes \mathcal{L})$ be the spaces of global sections vanishing on a 0-dimensional subscheme $Z \subset X$, and let $\langle Z \rangle \subset H^0(X, \mathcal{L})^\vee$ be the linear span of Z . For a proof of the following result we refer to [LO13, Proposition 5.4.1].

Proposition 2.2.3 (LANDSBERG-OTTAVIANI). *Let $f \in H^0(X, \mathcal{L})^\vee$ and let $Z \subset X$ be a 0-dimensional subscheme such that $f \in \langle Z \rangle$. Then it holds $H^0(X, \mathcal{I}_Z \otimes \mathcal{E}) \subseteq \ker A_f$ and $H^0(X, \mathcal{I}_Z \otimes \mathcal{E}^\vee \otimes \mathcal{L}) \subseteq \operatorname{Im} A_f^\perp$.*

It is worth remarking that the above result holds for non-reduced subschemes $Z \subset X$ too. The schematic non-reduced version of the nonabelian apolarity has already been considered in [OR20, Theorem 6.10] for Veronese varieties, and more generally in [Gal17, Proposition 7]. In the classical apolarity, the case of minimal subschemes is considered with respect to cactus varieties and cactus rank [BB14; BR13], while for non-minimal subschemes see [BT20]. Finally, for reduced subschemes we recall the following result [OO13, Proposition 4.3].

Proposition 2.2.4 (OEDING-OTTAVIANI). *Let $f \in H^0(X, \mathcal{L})^\vee$ and let $Z \subset X$ be a 0-dimensional reduced subscheme of length r , minimal with respect to the property $f \in \langle Z \rangle$. Assume that $\operatorname{Rk} A_f = r \cdot \operatorname{Rk} \mathcal{E}$. Then it holds $H^0(X, \mathcal{I}_Z \otimes \mathcal{E}) = \ker A_f$ and $H^0(X, \mathcal{I}_Z \otimes \mathcal{E}^\vee \otimes \mathcal{L}) = \operatorname{Im} A_f^\perp$. In particular, Z is contained in the common zero locus of $\ker A_f$ and $\operatorname{Im} A_f^\perp$.*

2.3 Hamming distance on projective varieties

We introduce a notion of distance between points of a projective variety with respect to its embedding. The name ‘‘Hamming’’ comes from the similarity with the Hamming distance in Code Theory, which measures how much two code arrays differ. This notion has already appeared in [CGG05; BDD07; AOP12].

DEFINITION 2.3.1. Let $X \subset \mathbb{P}^M$ be an irreducible projective variety. The *Hamming distance* $d(p, q)$ between two points $p, q \in X$ is the minimum number of lines in \mathbb{P}^M which are fully contained in X and connect p to q , that is

$$d(p, q) := \min \{r \mid \exists p_1, \dots, p_{r-1} \in X \text{ s.t. } L(p_i, p_{i+1}) \subset X, i = 0 : r - 1\}$$

where $p_0 = p$ and $p_r = q$ and $L(p_i, p_{i+1})$ is the line passing through p_i and p_{i+1} .

In particular, $d(p, q) = 0$ if and only if $p = q$, while $d(p, q) = 1$ if and only if $L(p, q) \subset X$ and $p \neq q$. The maximum Hamming distance possible between points of X is called the *diameter* of X :

$$\operatorname{diam}(X) := \max \{d(p, q) \mid p, q \in X\} .$$

REMARK 2.3.2. Given G a group, W a G -representation and $X \subset \mathbb{P}(W)$ a variety which is invariant under the G -action induced on $\mathbb{P}(W)$, then the G -action preserves the Hamming distance between points in X , that is $d(g \cdot p, g \cdot q) = d(p, q)$.

REMARK. As any generalized flag variety G/P is intersection of quadrics, for any two distinct points $p, q \in G/P$ by Bèzout one gets $d(p, q) \geq 2$ if and only if $L(p, q) \cap (G/P) = \{p, q\}$.

REMARK. For cominuscule varieties the Hamming distance coincides with the minimum possible degree of rational curves passing through the two points [Buc+13, Lemma 4.2]. For non-cominuscule varieties this is not true in general (a counterexample is given in Sec. 6.2).

Chapter 3

Diving into spinors

3.1 Spin groups from Clifford algebras

Spin groups are known as the universal double coverings of the special orthogonal groups. This section is devoted to construct algebraically spin groups starting from Clifford algebras.

We refer to [Pro07, Secc. 5.4, 5.5].

Clifford algebras. Let V be an N -dimensional complex euclidean space endowed with a non-degenerate quadratic form $q \in \text{Sym}^2 V^\vee$. We denote the norm and the inner product defined by q as $q(v) = \|v\|^2$ and $q(v, w)$ respectively. We recall the polarization formula $2q(v, w) = q(v + w) - q(v) - q(w)$. Given $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ the graded tensor algebra over V , one defines the *Clifford algebra* of q over V as the quotient algebra

$$\text{Cl}_q(V) := \frac{T(V)}{(v^{\otimes 2} - q(v))} .$$

We denote by $v \cdot w$ (or simply, vw) the ring product in $\text{Cl}_q(V)$: for any $v, w \in V$,

$$v \cdot w + w \cdot v = (v + w)^2 - v^2 - w^2 = q(v + w) - q(v) - q(w) = 2q(v, w) .$$

In particular, orthogonal elements in V anticommute in $\text{Cl}_q(V)$: $v \perp w \implies v \cdot w = -w \cdot v$. Although the ideal $I = (v^{\otimes 2} - q(v))$ is not homogeneous, it is of even degree and it decomposes as $I_0 \oplus I_1$, where I_0 is the even-degree component and I_1 the odd-degree one. It follows that the quotient algebra $\text{Cl}_q(V)$ is $\mathbb{Z}/2\mathbb{Z}$ -graded with graded decomposition

$$\text{Cl}_q(V) = \frac{T^{\text{ev}}(V)}{I_0} \oplus \frac{T^{\text{od}}(V)}{I_1} = \text{Cl}_q^+(V) \oplus \text{Cl}_q^-(V) .$$

The $[0]_2$ -degree component $\text{Cl}_q^+(V)$ is called *even Clifford algebra* of q over V .

It is useful to give a functorial definition for Clifford algebras.

UNIVERSAL PROPERTY. The *Clifford algebra* $\text{Cl}_q(V)$ of q over V is the algebra with the following universal property: for any algebra R such that $V \subset R$ and for any linear map $\phi : V \rightarrow R$ such that $\phi(v)^2 = q(v) \cdot \mathbf{1}_R$, there exists a unique algebra homomorphism $\psi : \text{Cl}_q(V) \rightarrow R$ making the following diagram commuting

$$\begin{array}{ccc} V & \xrightarrow{\phi} & R \\ & \searrow i & \nearrow \psi \\ & \text{Cl}_q(V) & \end{array}$$

REMARK. By functoriality, for any orthogonal transformation of vector spaces $f : (V, q) \rightarrow (V', q')$ there is an algebra homomorphism $f_* : \text{Cl}_q(V) \rightarrow \text{Cl}_{q'}(V')$. This also leads to the identification of the orthogonal group as subgroup of the automorphism group of the Clifford algebra: $O(V, q) < \text{Aut}(\text{Cl}_q(V))$.

Given (e_1, \dots, e_N) a q -orthogonal basis of V , then $(e_{i_1} \cdots e_{i_k} \mid k \geq 0, i_1 < \dots < i_k)$ is a basis of $\text{Cl}_q(V)$ [Pro07, Sec. 5.4.1, Lemma 1]: in particular, $\dim \text{Cl}_q(V) = 2^N$. As dimensions may suggest, Clifford algebras are strictly related to exterior algebras. Indeed, there is a canonical isomorphism of vector spaces [LM89, Sec. 1.1, Proposition 1.3]

$$\begin{aligned} \bigwedge V & \xrightarrow{\cong} & \text{Cl}_q(V) \\ v_1 \wedge \dots \wedge v_k & \mapsto & \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma v_{\sigma(1)} \cdots v_{\sigma(k)} \end{aligned} \quad (3.1.1)$$

REMARK. One can always reduce to study Clifford algebras over even-dimensional vector spaces. Indeed, given $\dim V = 2N + 1$ and (e_1, \dots, e_{2N+1}) a q -orthonormal basis of V , the element $c = e_1 \cdots e_{2N+1}$ is central in $\text{Cl}_q(V)$ and $c^2 = (-1)^N$ [Pro07, Sec. 5.4.1, Lemma 2]. Since $e_{2N+1} = e_{2N} e_{2N-1} \cdots e_2 e_1 c$, one has $\text{Cl}_q(V) = \text{Cl}_q(\langle e_1, \dots, e_{2N} \rangle_{\mathbb{C}}) + \text{Cl}_q(\langle e_1, \dots, e_{2N} \rangle_{\mathbb{C}}) \cdot c$. In particular, as $(-1)^N$ has a square root in \mathbb{C} , the following isomorphism holds

$$\text{Cl}_q(V) \simeq \text{Cl}_q(\langle e_1, \dots, e_{2N} \rangle_{\mathbb{C}}) \oplus \text{Cl}_q(\langle e_1, \dots, e_{2N} \rangle_{\mathbb{C}}) \cdot c. \quad (3.1.2)$$

Clifford multiplication. In respect of the previous remark, we assume $\dim V = 2N$. Consider a hyperbolic standard basis $(e_1, \dots, e_N, f_1, \dots, f_N)$ of V such that q is described by the symmetric matrix

$$Q = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix} \in \text{Sym}^2 \mathbb{C}^{2N}.$$

Then V decomposes as $V = E \oplus F$, where $E = \langle e_1, \dots, e_N \rangle_{\mathbb{C}}$ and $F = \langle f_1, \dots, f_N \rangle_{\mathbb{C}}$. In particular, E and F are two fully isotropic subspaces of V (i.e. $E = E^\perp$ and $F = F^\perp$) of maximal dimension $\lfloor \frac{2N}{2} \rfloor = N$. Up to isomorphism, we identify $F = E^\vee$ and $V = E \oplus E^\vee$: via this identification, it holds $q((v, \varphi)) = \varphi(v)$ for any $v \in E$ and $\varphi \in E^\vee$.

Consider the linear maps

$$\begin{aligned} \mu : E & \longrightarrow \text{End}(\bigwedge E) \\ v & \mapsto v \wedge \bullet \end{aligned}$$

$$\begin{aligned} \lambda : E^\vee &\longrightarrow \text{End}(\bigwedge E) \\ \varphi &\mapsto \left(v_1 \wedge \dots \wedge v_k \mapsto \sum_{l=1}^k (-1)^{l-1} \varphi(v_l) v_1 \wedge \dots \wedge \widehat{v}_l \wedge \dots \wedge v_k \right) \end{aligned} \quad (3.1.3)$$

A straightforward count shows that $\mu(v)^2 = \lambda(\varphi)^2 = 0$ and $\mu(v)\lambda(\varphi) + \lambda(\varphi)\mu(v) = \varphi(v) \cdot id$. Hence the linear map $\mu + \lambda : E \oplus E^\vee \rightarrow \text{End}(\bigwedge E)$ is such that

$$(\mu + \lambda)(v, \varphi)^2 = (\mu(v) + \lambda(\varphi))^2 = \varphi(v) = q((v, \varphi)) ,$$

that is, by the universal property of Clifford algebras, there exists an algebra homomorphism (actually, an isomorphism [Pro07, Sec. 5.4.1, Theorem 2])

$$\theta : \text{Cl}_q(V) = \text{Cl}_q(E \oplus E^\vee) \xrightarrow{\simeq} \text{End}(\bigwedge E) . \quad (3.1.4)$$

From (3.1.2) and (3.1.4), it follows that

$$\text{Cl}_q(\mathbb{C}^{2N+1}) \simeq \text{Mat}_{2^N}(\mathbb{C}) \oplus \text{Mat}_{2^N}(\mathbb{C}) \quad , \quad \text{Cl}_q(\mathbb{C}^{2N}) \simeq \text{Mat}_{2^N}(\mathbb{C}) . \quad (3.1.5)$$

In particular, $\text{Cl}_q(V)$ has centre

$$Z(\text{Cl}_q(V)) = \begin{cases} \mathbb{C} & \text{if } \dim V \text{ even} \\ \mathbb{C} + \mathbb{C} \cdot c \simeq \mathbb{C} \oplus \mathbb{C} & \text{if } \dim V \text{ odd} \end{cases} .$$

REMARK. The isomorphisms above hold since we are over an algebraically closed field. We refer to [Pro07, Sec. 5.4.2] for details over more general fields.

The contraction map λ (3.1.3) and the isomorphism (3.1.1) lead to a nice description of the multiplication in the Clifford algebra, which we call *Clifford multiplication*. As $E = E^\perp$, the map λ extends to $V = E \oplus E^\vee$ trivially, and for any $v \in V$ and $x \in \text{Cl}_q(V)$, via the vector space isomorphism $\text{Cl}_q(V) \simeq \bigwedge V$, it holds

$$v \cdot x \simeq v \wedge x + \lambda(v)(x) . \quad (3.1.6)$$

Even Clifford algebras. Now assume V of dimension $\dim V = N + 1$, endowed with a non-degenerate quadratic form $q \in \text{Sym}^2 V^\vee$. Let (c, e_1, \dots, e_N) be a q -orthogonal basis of V and set $V' = \langle e_1, \dots, e_N \rangle_{\mathbb{C}}$: then $V = \mathbb{C} \cdot c \oplus V'$. Let $q' \in \text{Sym}^2(V')^\vee$ be the restriction of q to V' . Given $\gamma := c^2 = q(c) \in \mathbb{C}^\times$, one can construct an algebra isomorphism between the Clifford algebra of $-\gamma q'$ on V' and the even Clifford algebra of q on V [Pro07, Sec. 4.4, Proposition 1]

$$\tilde{f} : \text{Cl}_{-\gamma q'}(V') \rightarrow \text{Cl}_q^+(V) . \quad (3.1.7)$$

Spin groups. Let V be a complex vector space and let $q \in \text{Sym}^2 V^\vee$ be a quadratic form on it. Let $\text{Cl}_q(V)$ be the Clifford algebra of q over V and let $\text{Cl}_q(V)^\times$ be the multiplicative subgroup given by the invertible elements in the Clifford algebras. Clearly, $\text{Cl}_q(V)^\times$ acts by

conjugacy on $\text{Cl}_q(V)$. One defines the *Clifford group* $\Gamma(V, q)$ to be the normalizer in $\text{Cl}_q(V)^\times$ of V

$$\Gamma(V, q) := \{x \in \text{Cl}_q(V)^\times \mid xVx^{-1} = V\} .$$

Let $\Gamma^+(V, q) = \Gamma(V, q) \cap \text{Cl}_q^+(V)$ be the intersection between the Clifford group and the even Clifford algebra. Given $x \in \Gamma(V, q)$, the conjugacy endomorphism $\text{conj}_x : v \mapsto xvx^{-1}$ on V is orthogonal: indeed, since $v^2 = q(v)$, it holds $q(xvx^{-1}) = (xvx^{-1})^2 = xv^2x^{-1} = q(v)$. Thus one has the group homomorphism

$$\begin{aligned} \text{conj} : \Gamma(V, q) &\longrightarrow O^q(V) \\ x &\mapsto (\text{conj}_x : v \mapsto xvx^{-1}) . \end{aligned}$$

Notice that, for any $v \in V$ such that $q(v) \neq 0$, v is invertible and it holds $v^{-1} = \frac{v}{q(v)}$. Moreover, given $v, w \in V$ with v invertible, one has

$$\text{conj}_v(w) = v w v^{-1} = (2q(v, w) - wv)v^{-1} = \frac{2q(v, w)}{q(v)}v - w = -\rho_v(w) ,$$

where ρ_v is the orthogonal reflection with respect to the hyperplane v^\perp . This implies that any $x = v_1 \cdots v_k \in \Gamma(V, q)$ induces a product of k reflections. In particular, any $x = v_1 \cdots v_{2k} \in \Gamma^+(V, q)$ (in the even Clifford algebra) induces an even product of reflections, hence it induces a special orthogonal transformation, that is

$$\text{conj} : \Gamma^+(V, q) \longrightarrow \text{SO}^q(V) .$$

REMARK. By the Cartan-Dieudonné theorem, any special orthogonal transformation is an even product of reflections, thus the conjugacy map above is surjective. Moreover, if $\dim V$ is even, then $\text{conj}_v = -\rho_v$ is in $O^q(V) \setminus \text{SO}^q(V)$ and the map $\Gamma(V, q) \rightarrow O^q(V)$ is surjective too, while for $\dim V$ odd the image is $\text{SO}^q(V)$ [Pro07, Sec. 5.5, Proposition 1].

Let $x \in \ker(\text{conj}) \subset \Gamma(V, q)$: then $xvx^{-1} = v$ for any $v \in V$ and, since V generates the Clifford algebra, it means that $x \in Z(\text{Cl}_q(V))$. Thus we have the exact sequence

$$1 \longrightarrow Z(\text{Cl}_q(V))^\times \longrightarrow \Gamma(V, q) \longrightarrow O^q(V) .$$

REMARK. For $\dim V$ even, we know that $Z(\text{Cl}_q(V))^\times = \mathbb{C}^\times$. On the other hand, for $\dim V$ odd the map $\Gamma(V, q) \rightarrow O^q(V)$ has image $\text{SO}^q(V)$, thus any element $x \in \Gamma(V, q)$ is of the form $x = av_1 \cdots v_{2h}$ for $a \in Z(\text{Cl}_q(V))^\times$ and v_1, \dots, v_{2h} reflections. In particular, if $x \in \Gamma^+(V, q)$ then $a \in \mathbb{C}^\times$.

The above remark implies that the restriction of the latter exact sequence to the even Clifford group $\Gamma^+(V, q)$ leads to the short exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \Gamma^+(V, q) \longrightarrow \text{SO}^q(V) \longrightarrow 1 . \quad (3.1.8)$$

independently from the parity of $\dim V$.

For any $x = v_1 \cdots v_k \in \Gamma(V, q)$, one defines its *spinor norm* as

$$N(x) = xx^* = q(v_1) \cdots q(v_k) \in \mathbb{C}^\times ,$$

where $*$ is the *involution* on $\text{Cl}_q(V)$ given by $(v_1 \cdots v_k)^* = v_k \cdots v_1$. When restricted to the even Clifford group, the spinor norm induces a group homomorphism $N : \Gamma^+(V, q) \rightarrow \mathbb{C}^\times$, whose kernel is the *spin group*

$$\text{Spin}^q(V) := \{x \in \Gamma^+(V, q) \mid N(x) = 1\} .$$

If $V = \mathbb{C}^N$, we write Spin_N^q for the spin group $\text{Spin}^q(\mathbb{C}^N)$.

REMARK. The spin group $\text{Spin}^q(V)$ is a double covering of $\text{SO}^q(V)$: in particular, they share the same Lie algebras

$$\text{Lie}(\text{Spin}^q(V)) \simeq \mathfrak{so}^q(V) \simeq \bigwedge^2 V .$$

Indeed, for any $x = v_1 \cdots v_{2k} \in \Gamma^+(V, q)$, there exists $a \in \mathbb{C}$ such that $N(ax) = a^2 N(x) = 1$, hence $ax \in \text{Spin}(V)$. It follows that the map conj in the short exact sequence (3.1.8) restricted to $\text{Spin}^q(V)$ is still surjective onto $\text{SO}^q(V)$. Moreover, for any $a \in \mathbb{C}^\times$, it holds $N(a) = a^2 = 1$ if and only if $a = \pm 1$. Thus one gets the short exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \text{Spin}(V) \xrightarrow{2:1} \text{SO}^q(V) \longrightarrow 1 .$$

REMARK. The (complex) spin group $\text{Spin}^q(V)$ is simply connected. This is a consequence (actually, equivalent to) the fact that the fundamental group of $\text{SO}^q(V)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$: for details we refer to [FH91, Sec. 23.1].

3.2 Spin representations

Following [Pro07, Sec. 11.7] we introduce the *spin representations*, that is the fundamental representations of the spin group which are not representations for the special orthogonal group. In the even dimensional case, the two half-spin representations are known to physicists as “chiral spin representations”, whose sum gives the “fermionic Fock space” [LH15].

Let V be a M -dimensional complex vector space endowed with a non-degenerate quadratic form $q \in \text{Sym}^2 V^\vee$. Depending on the parity of $\dim V$, say either $M = 2N$ or $M = 2N + 1$, we assume q to be described by the symmetric matrix

$$\text{either } Q = \frac{1}{2} \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix} \quad \text{or} \quad Q = \frac{1}{2} \begin{bmatrix} 1 & & & \\ & 0 & I_N & \\ & I_N & 0 & \\ & & & 1 \end{bmatrix} ,$$

where the rescaling by $\frac{1}{2}$ is in order to simplify counts. We fix either $(e_1, \dots, e_N, f_1, \dots, f_N)$ or $(u, e_1, \dots, e_N, f_1, \dots, f_N)$ respectively for the standard hyperbolic basis so that the quadratic form q is described by the matrix Q with respect to this basis.

Consider the linear subspace of the even Clifford algebra $L := \langle v \cdot w \mid v, w \in V \rangle_{\mathbb{C}} \subset \text{Cl}_q^+(V)$ and the linear map

$$\begin{aligned} \epsilon : \bigwedge^2 V &\longrightarrow \text{Cl}_q^+(V) \\ v \wedge w &\mapsto \frac{1}{2}[v, w] = \frac{1}{2}(vw - wv) . \end{aligned} \tag{3.2.1}$$

Given (e_1, \dots, e_M) a q -orthogonal basis of V , we have

$$\epsilon(e_i \wedge e_j) = \frac{1}{2}(e_i e_j - e_j e_i) = \frac{1}{2} \cdot 2e_i e_j = e_i e_j ,$$

thus ϵ is injective and we can identify $\bigwedge^2 V$ as subset of $\text{Cl}_q^+(V)$. After this identification, the subset L is precisely $\mathbb{C} \oplus \bigwedge^2 V$ and it is a subalgebra of $\text{Cl}_q^+(V)$. Moreover, $[L, L] = \bigwedge^2 V$ and, under the adjoint action, L acts as $\mathfrak{so}^q(V)$ on V [Pro07, Sec. 5.4.4, Proposition 2].

As algebra, $\text{Cl}_q^+(V)$ is generated by $\bigwedge^2 V$. It follows that any irreducible representation of $\text{Cl}_q^+(V)$ is an irreducible representation of $\text{Spin}^q(V)$: indeed, given an irreducible representation of $\text{Cl}_q^+(V)$, if it was reducible as $\text{Spin}^q(V)$ -module, then it would be so as $\mathfrak{so}^q(V)$ -module, that is as $\bigwedge^2 V$ -module, hence as $\mathbb{C}[\bigwedge^2 V]$ -module; but $\mathbb{C}[\bigwedge^2 V] = \text{Cl}_q^+(V)$ would leads to a contradiction.

Thus we study the irreducible representations of $\text{Cl}_q^+(V)$ in order to find the ones of $\text{Spin}^q(V)$. As the Lie algebra of the spin group is $\mathfrak{so}(V)$, we talk about “type- B_N case” if $\dim V = 2N + 1$ and about “type- D_N case” if $\dim V = 2N$.

Type- D_N case. Set $V = E \oplus E^\vee$ with $\dim V = 2N$. Then from the isomorphisms (3.1.5) and (3.1.7) one gets

$$\text{Cl}_q^+(V) \simeq \text{Cl}_{q'}(\mathbb{C}^{2N-1}) \simeq \text{Mat}_{2^{N-1}}(\mathbb{C}) \oplus \text{Mat}_{2^{N-1}}(\mathbb{C}) ,$$

hence $\text{Cl}_q^+(V)$ has only two non-isomorphic irreducible representations of dimension 2^{N-1} . Via the isomorphism (3.1.4) $\text{Cl}_q(V) \simeq \text{End}(\bigwedge E)$, the even Clifford algebra $\text{Cl}_q^+(V)$ embeds in $\text{End}(\bigwedge E)$ and the two irreducible representations correspond to $\bigwedge^{ev} E$ and $\bigwedge^{od} E$.

DEFINITION. The two irreducible Spin_{2N} -modules

$$S_N^+ := \bigwedge^{ev} E \quad , \quad S_N^- := \bigwedge^{od} E .$$

are said *half-spin representations*. Given $(\epsilon_1, \dots, \epsilon_N)$ a suitable orthogonal basis of $\langle \Phi^{D_N} \rangle_{\mathbb{R}}$, the half-spin representations are defined by the fundamental weights of Dynkin type D_N

$$\omega_{N-1} = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{N-1} - \epsilon_N) \quad , \quad \omega_N = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{N-1} + \epsilon_N) .$$

More precisely, given $(e_1, \dots, e_N, f_1, \dots, f_N)$ a hyperbolic basis of $V = E \oplus E^\vee$, the highest weights, the highest weight vectors and the lowest weight vectors (we denote them by ℓ_{ω_i}) of S_N^+ and S_N^- depend on the parity of $\dim E$ (hence on the value of $\dim V \pmod{4}$) as follows: we denote by $\mathbb{1}$ the scalar $1 \in \mathbb{C}$ in order to remark its property of being a lowest weight vector.

$N \equiv 0 \pmod{2}$			
	$V_{\omega_i}^{D_N}$	v_{ω_i}	ℓ_{ω_i}
ω_{N-1}	$S_N^- = \bigwedge^{od} E$	$\mathbf{e}_{[N-1]} = e_1 \wedge \dots \wedge e_{N-1}$	e_1
ω_N	$S_N^+ = \bigwedge^{ev} E$	$\mathbf{e}_{[N]} = e_1 \wedge \dots \wedge e_N$	$\mathbb{1}$

Table 3.1: Half-spin representations of Spin_{2N} for $N \equiv 0 \pmod{2}$.

$N \equiv 1 \pmod{2}$			
	$V_{\omega_i}^{D_N}$	v_{ω_i}	ℓ_{ω_i}
ω_{N-1}	$S_N^+ = \bigwedge^{ev} E$	$\mathbf{e}_{[N-1]} = e_1 \wedge \dots \wedge e_{N-1}$	$\mathbb{1}$
ω_N	$S_N^- = \bigwedge^{od} E$	$\mathbf{e}_{[N]} = e_1 \wedge \dots \wedge e_N$	e_1

Table 3.2: Half-spin representations of Spin_{2N} for $N \equiv 1 \pmod{2}$.

Type- B_N case. Let V be of dimension $2N + 1$ and let $(u, e_1, \dots, e_N, f_1, \dots, f_N)$ be a hyperbolic basis of it. By setting $a_i = e_i u$ and $b_i = u f_i$, one defines $E = \langle a_1, \dots, a_N \rangle_{\mathbb{C}}$ and $q' \in \text{Sym}^2(E \oplus E^\vee)^\vee$ (induced by q). From (3.1.4) one gets the isomorphism

$$\text{Cl}_q^+(V) = \text{Cl}_{q'}(E \oplus E^\vee) \simeq \text{End}(\bigwedge E).$$

In particular, $\bigwedge E$ is the only irreducible representation of $\text{Cl}_q^+(V)$ of dimension 2^N , hence of $\text{Spin}(V)$.

DEFINITION. The irreducible Spin_{2N+1} -module

$$S_N = \bigwedge E.$$

is said the *spin representation* of type B_N . Given $(\epsilon_1, \dots, \epsilon_N)$ a suitable orthogonal basis of $\langle \Phi^{B_N} \rangle_{\mathbb{R}}$, the highest weight of the spin representation is the fundamental weight of Dynkin type B_N

$$\omega_N = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_N)$$

and its highest weight vector is $v_{\omega_N} = a_1 \wedge \dots \wedge a_N = e_1 u \wedge \dots \wedge e_N u$.

REMARK. We stress out the fact that in the odd case the even Clifford algebra is *isomorphic* to $\text{End}(\bigwedge E)$, while in the even case it just *embeds* in the latter. Moreover, in the odd case the subspaces $\bigwedge^{ev} E$ and $\bigwedge^{od} E$ are not $\text{Cl}_q^+(V)$ -invariant: indeed, in the notations above, for any $\varphi \in \bigwedge E$ it holds

$$u \cdot \varphi = \begin{cases} \varphi & \text{if } \varphi \in \bigwedge^{ev} E \\ -\varphi & \text{if } \varphi \in \bigwedge^{od} E \end{cases},$$

thus for any $w \in V \setminus \langle u \rangle_{\mathbb{C}}$ we get by (3.2.1)

$$\begin{aligned} (u \wedge w) \cdot \varphi &= \frac{1}{2}[u, w]\varphi = \frac{1}{2}uw\varphi - \frac{1}{2}w \underbrace{u\varphi}_{\pm\varphi} \\ &= \frac{1}{2}(-wu + q(u, w))\varphi \mp \frac{1}{2}w\varphi \\ &= \mp \frac{1}{2}w\varphi + \frac{1}{2} \underbrace{q(u, w)}_0 \varphi \mp \frac{1}{2}w\varphi = \mp w\varphi \end{aligned}$$

that is $(u \wedge w) \cdot \varphi = \mp w\varphi$ is in $\bigwedge^{ev} E$ (resp. $\bigwedge^{od} E$) if φ is in $\bigwedge^{od} E$ (resp. $\bigwedge^{ev} E$).

REMARK 3.2.1 (Duality). From Remark 1.1.1 and the duality of irreducible representations, one gets the following duality between the two half-spin representations

$$\left(V_{\omega_{N-1}}^{D_N}\right)^\vee = V_{-w_0(\omega_{N-1})}^{D_N} = \begin{cases} V_{\omega_N}^{D_N} & \text{if } N \equiv 1 \pmod{2} \\ V_{\omega_{N-1}}^{D_N} & \text{if } N \equiv 0 \pmod{2} \end{cases}$$

and similarly for $V_{\omega_N}^{D_N}$. On the other hand, the spin representation of type B_N is always self-dual.

REMARK (Triality). The case of Dynkin type D_4 is very special. Indeed, there exists a subgroup of automorphism of the Dynkin diagram of type D_4 isomorphic to \mathfrak{S}_3 which acts on the three extremal nodes corresponding to the fundamental weights $\omega_1, \omega_3, \omega_4$. In particular, it induces isomorphisms among the three 8-dimensional irreducible Spin_8 -representations

$$V_{\omega_1}^{D_4} = \mathbb{C}^8 \quad , \quad V_{\omega_3}^{D_4} = \mathbb{C}^4 \oplus \bigwedge^3 \mathbb{C}^4 \quad , \quad V_{\omega_4}^{D_4} = \mathbb{C} \oplus \bigwedge^2 \mathbb{C}^4 \oplus \bigwedge^4 \mathbb{C}^4 \quad .$$

3.3 Spinor varieties

Spinor varieties are the generalized Grassmannians lying in the spin representations. Their elements are called “pure spinors” and they have been studied since E. Cartan [Car67] and C. Chevalley [Che54]. Spinor varieties are of interest to physicists as Spin-orbits of the vacuum state in the fermionic Fock spaces [LH15]. We keep notation from previous sections.

From the theory of rational homogeneous varieties in Sec. 1.3 one can abstractly define the *Spinor varieties* to be the Generalized Grassmannians

$$\begin{aligned} \mathbb{S}_N &:= B_N/P_{\omega_N} = \text{Spin}_{2N+1} \cdot [v_{\omega_N}] \subset \mathbb{P}(V_{\omega_N}^{B_N}) \quad , \\ \mathbb{S}_N^+ &:= D_N/P_{\omega_+} = \text{Spin}_{2N} \cdot [v_{\omega_+}] \subset \mathbb{P}(S_N^+) \quad , \\ \mathbb{S}_N^- &:= D_N/P_{\omega_-} = \text{Spin}_{2N} \cdot [v_{\omega_-}] \subset \mathbb{P}(S_N^-) \quad , \end{aligned}$$

where ω_+, ω_- are the fundamental weights ω_{N-1}, ω_N of Dynkin type D_N depending on the parity of N , accordingly to Table 3.1 and 3.2: namely, for N even it holds $\mathbb{S}_N^+ = D_N/\omega_N$ and $\mathbb{S}_N^- = D_N/\omega_{N-1}$, while for N odd one gets $\mathbb{S}_N^+ = D_N/P_{\omega_{N-1}}$ and $\mathbb{S}_N^- = D_N/P_{\omega_N}$. In the literature, the elements of the above orbits are said *pure spinors* while the element of the spin representations are said *spinors*.

REMARK. From Sec. 1.4 we know that the D_N -type Spinor varieties \mathbb{S}_N^\pm are cominuscule varieties, while the B_N -type Spinor varieties \mathbb{S}_N are not. However, in Theorem 3.3.1 we are going to see that the Spinor varieties of type B_N and D_{N+1} are projectively equivalent, and that one can always consider an action of Spin_{2N+2} on the Spinor variety \mathbb{S}_N . In particular, \mathbb{S}_N and \mathbb{S}_{N+1}^\pm share the same geometry inside their minimal homogeneous embeddings.

It is useful to enlight the geometry of the Spinor varieties, by giving a description of them analogous to the one of the Grassmannian $\text{Gr}(k, N)$ as variety of k -planes in \mathbb{C}^N . More precisely, we are going to exhibit a correspondence between Spinor varieties and the maximal orthogonal Grassmannians.

Spinors and isotropic subspaces. Given q a non-degenerate quadratic form, let H^\perp be the q -orthogonal space to a subspace H : recall that a subspace H is q -isotropic if and only if $H \subset H^\perp$, and for any q -isotropic subspace $H \subset V$ it holds $\dim H \leq \lfloor \frac{\dim V}{2} \rfloor$. We denote by

$$\text{OG}(N, 2N+1) := \{H \in \text{Gr}(N, 2N+1) \mid H = H^\perp\}$$

the connected *maximal orthogonal Grassmannian* of maximal (i.e. N -dimensional) q -isotropic subspaces of \mathbb{C}^{2N+1} , and by

$$\text{OG}_q^+(N, 2N) := \{H \in \text{Gr}(N, 2N) \mid H = H^\perp, \dim(H \cap E) \equiv 0 \pmod{2}\}$$

$$\text{OG}_q^-(N, 2N) := \{H \in \text{Gr}(N, 2N) \mid H = H^\perp, \dim(H \cap E) \equiv 1 \pmod{2}\}$$

the two connected components of the *maximal orthogonal Grassmannian* of q -isotropic N -planes in \mathbb{C}^{2N} .

Let V be a complex vector space of dimension $2N$ (resp. $2N+1$) endowed with $q \in \text{Sym}^2 V^\vee$. Let $V = E \oplus E^\vee$ (resp. $V = E \oplus E^\vee \oplus \mathbb{C}u$) a hyperbolic decomposition of V with respect to q , where $E = \langle e_1, \dots, e_N \rangle_{\mathbb{C}}$ and $E^\vee = \langle f_1, \dots, f_N \rangle_{\mathbb{C}}$ are such that $q(e_i, e_j) = q(f_i, f_j) = 0$ and $q(e_i, f_j) = \delta_{ij}$: in particular, $E = E^\perp$ and $E^\vee = (E^\vee)^\perp$.

For any spinor $a \in \bigwedge E$ (not necessarily pure), the Clifford multiplication (3.1.6) defines a map

$$\begin{aligned} \psi_a : V &\longrightarrow \bigwedge E \\ v &\longmapsto v \cdot a \end{aligned} \quad (3.3.1)$$

We denote by

$$H_a := \ker(\psi_a) = \{v \in V \mid v \cdot a = 0\} \quad (3.3.2)$$

REMARK. For any non-zero spinor $a \in \bigwedge E$ (not necessarily pure) the subspace H_a is always q -isotropic (hence $\dim H_a \leq \lfloor \frac{\dim V}{2} \rfloor = N$): indeed, for any $v \in H_a$ one has

$$0 = v \cdot (v \cdot a) = q(v)a \stackrel{a \neq 0}{\implies} q(v) = 0.$$

On the other hand, given $H \subset V$ a q -isotropic subspace, one can consider the subspace of $\bigwedge E$

$$S_H := \left\{ b \in \bigwedge E \mid H \cdot b = 0 \right\}$$

of spinors annihilated by H . For pure spinors the subspaces $H_a \subset V$ and $S_H \subset \bigwedge E$ gives a correspondence between Spinor varieties and maximal orthogonal Grassmannian.

Lemma. *In the above setting and notation:*

1. If a is a pure spinor, then H_a has maximal dimension N .
2. For any q -isotropic subspace $H \subset V$ it holds $\dim S_H = 2^{N-\dim H}$. In particular, if H is maximal q -isotropic, then $\dim S_H = 1$.
3. If H is maximal q -isotropic, then $S_H = \mathbb{C}a_H$ for a certain pure spinor a_H .

Proof.

1. [Pro07, Sec. 11.7.2, Theorem 1] It is useful to work with the lowest weight vectors as they do not depend on the parity of N : indeed, $\mathbb{1}$ is always the lowest weight vector of S_N^+ and $V_{\omega_N}^{BN}$, while e_1 is always the lowest weight vector of S_N^- . From the Clifford multiplication (3.1.6), elements of E acts via wedge-multiplication while elements of E^\vee acts as derivations. Thus for any $v \in V$ it holds $v \cdot \mathbb{1} = 0$ if and only if $v \in E^\vee$, and $v \cdot e_1 = 0$ if and only if $v \in \langle e_1, f_2, \dots, f_N \rangle_{\mathbb{C}}$. In particular, $H_{\mathbb{1}} = E^\vee$ and $H_{e_1} = \langle e_1, f_2, \dots, f_N \rangle_{\mathbb{C}}$ are of maximal dimension N .

Now we work with $\mathbb{1} \in S_N^+$ but the same argument apply to the other orbits S_N^- and S_N . For any pure spinor $a \in S_N^+$ there exist $g \in G = \text{Spin}(V)$ and $\lambda \in \mathbb{C}^\times$ such that $a = g \cdot \lambda \mathbb{1}$. Pick $v \in H_a$. Since by definition $\text{Spin}(V)$ normalizes V , we can consider $w \in V$ such that $v = gwg^{-1}$ and

$$0 = v \cdot a = (gwg^{-1})(g \cdot \lambda \mathbb{1}) = g \cdot (w \cdot \lambda \mathbb{1}) \iff w \cdot \mathbb{1} = 0$$

that is $H_a = gH_{\mathbb{1}}g^{-1} = gE^\vee g^{-1}$. As conjugation of subspaces preserve dimensions, one gets $\dim H_a = N$.

2. [Bat14, Sec. 3, Theorem 1] Given H q -isotropic, one can complete a basis of it to obtain a maximal q -isotropic subspace \hat{H} giving an equivalent decomposition $V = E \oplus E^\vee = \hat{H} \oplus \hat{H}^\vee$. Thus we may assume $H = \langle f_1, \dots, f_s \rangle_{\mathbb{C}} \subset E^\vee$ for $s = \dim H$. Then a basis of S_H is

$$(\mathbb{1}, e_{i_1} \wedge \dots \wedge e_{i_t} \mid t = 1 : N - s, \{i_1, \dots, i_t\} \subset \{s + 1, \dots, N\}) ,$$

hence $\dim S_H = \sum_{j=0}^{N-s} \binom{N-s}{j} = 2^{N-s}$.

3. For any $g \in \text{Spin}(V)$ and any q -isotropic subspace $H \subset V$ it holds

$$\begin{aligned} S_{gHg^{-1}} &= \left\{ b \in \bigwedge E \mid gH \underbrace{g^{-1} \cdot b}_{=: a} = 0 \right\} = \left\{ g \cdot a \in \bigwedge E \mid gH \cdot a = 0 \right\} \\ &\stackrel{\clubsuit}{=} \left\{ g \cdot a \in \bigwedge E \mid H \cdot a = 0 \right\} \\ &= g \cdot S_H , \end{aligned}$$

where the equality (\clubsuit) holds as $g \in \text{Cl}_q(V)^\times$ is an automorphism of $\bigwedge E$. In particular, if H is maximal q -isotropic, then it is $\text{Spin}(V)$ -conjugated to either $H_{\mathbb{1}}$ or H_{e_1} (depending on the parity of $\dim V$, on the parity of N and on the connected component in which H lies). For instance, if $H = gH_{\mathbb{1}}g^{-1}$, then $a_H = \lambda g \cdot \mathbb{1}$ for some $\lambda \in \mathbb{C}^\times$, that is a_H is a pure spinor.

□

REMARK. We stress out the action of $\text{Spin}(V)$ on H_a and S_H (cf. the above proof):

$$H_{g \cdot a} = gH_a g^{-1} \quad , \quad S_{gH_a g^{-1}} = g \cdot S_H \ .$$

Theorem. *The Spinor varieties \mathbb{S}_N^+ , \mathbb{S}_N^- and \mathbb{S}_N are in correspondence with the connected maximal orthogonal Grassmannians $\text{OG}^+(N, 2N)$, $\text{OG}^-(N, 2N)$ and $\text{OG}(N, 2N + 1)$ respectively. More precisely, the one-to-one correspondence is given by*

$$\begin{array}{ccc} \mathbb{S}_N^+ & \longleftrightarrow & \text{OG}^+(N, 2N) \\ [a] & \longrightarrow & H_a \\ [a_H] := \mathbb{P}(S_H) & \longleftarrow & H \end{array} \quad (3.3.3)$$

and similarly for the other Spinor varieties.

Proof. The previous lemma shows that pure spinors and maximal q -isotropic subspaces are in one-to-one correspondence. Moreover, the fact that this correspondence preserves the connected components (in the sense that pure spinors in \mathbb{S}_N^+ correspond to maximal q -isotropic subspaces in $\text{OG}^+(N, V)$) follows from [Car67, Sec. VI.I.124]. □

REMARK. The map ψ_a (3.3.1) allows to give equations for Spinor varieties. Indeed, a spinor $a \in \bigwedge E$ is pure if and only if $H_a = \ker(\psi_a)$ has maximum dimension N , hence if and only if $\text{Rk}(\psi_a) \leq N$. Thus pure spinors are determined by the vanishing of the $(N + 1) \times (N + 1)$ minors of ψ_a .

Parametrization. Being generalized Grassmannians, the Spinor varieties \mathbb{S}_N^\pm and \mathbb{S}_N are rational varieties. In particular, they are parametrized by skew-symmetric matrices $\bigwedge^2 \mathbb{C}^N$ and they have dimension

$$\dim \mathbb{S}_N^\pm = \binom{N}{2} \ .$$

Indeed, given a pure spinor a , up to choosing a Plücker basis, the subspace H_a is defined by the $2N \times N$ matrix $X_a = \begin{bmatrix} I_N \\ Y_a \end{bmatrix}$. Given $Q = \begin{bmatrix} & I_N \\ I_N & \end{bmatrix}$ the symmetric matrix defining the quadratic form q , by isotropicity of H_a it holds $0 = {}^t X_a \cdot Q \cdot X_a = Y_a + {}^t Y_a$, that is ${}^t Y_a = -Y_a$.

$D_{N+1} - B_N$ projective equivalence. The spin representations $S_{N+1}^\pm \in \{V_{\omega_N}^{D_{N+1}}, V_{\omega_{N+1}}^{D_{N+1}}\}$ and $S_N = V_{\omega_N}^{B_N}$ have both dimension 2^N , hence they are isomorphic as vector spaces. Actually, a stronger result holds.

Theorem 3.3.1. *The Spinor varieties $\mathbb{S}_{N+1}^\pm \subset \mathbb{P}(V_{\omega_{N+1}}^{D_{N+1}})$ and $\mathbb{S}_N \subset \mathbb{P}(V_{\omega_N}^{B_N})$ are projectively equivalent.*

Proof. We describe an isomorphism between \mathbb{S}_{N+1}^\pm and \mathbb{S}_N in terms of maximal orthogonal Grassmannians as follows. Any q' -isotropic subspace $W \subset \mathbb{C}^{2N+1}$ has dimension at most N , and the set of maximal (i.e. of dimension N) q' -isotropic subspaces is connected. On one hand,

any q -isotropic subspace $W \subset \mathbb{C}^{2N+2}$ of maximal dimension $N+1$ restricts to a q' -isotropic subspace $W' = W \cap H$ of dimension at least N , hence of dimension exactly N . On the other hand, any maximal q' -isotropic subspace $W' \subset \mathbb{C}^{2N+1}$ lifts to two possible maximal q -isotropic subspaces of \mathbb{C}^{2N+2} , namely $W_1 = W' \oplus \mathbb{C}e_{N+1}$ and $W_2 = W' \oplus \mathbb{C}f_{N+1}$. One concludes that the maximal orthogonal Grassmannians $\text{OG}_{q'}(N, 2N+1)$ and $\text{OG}_q^\pm(N+1, 2N+2)$ are isomorphic. This isomorphism among pure spinors maps basis elements of $\bigwedge^\bullet \mathbb{C}^{2N+1}$ to basis elements of $\bigwedge^{ev} \mathbb{C}^{2N+2}$, inducing the aimed projective equivalence. \square

In respect of Theorem 3.3.1, from now on we only work with the D_N -type spin group $\text{Spin}(2N)$ for N even. In particular, we focus on the Spinor variety

$$\mathbb{S}_N^+ = D_N/P_{\omega_N} \subset \mathbb{P}(V_{\omega_N}^{D_N}) = \mathbb{P}\left(\bigwedge^{ev} E\right).$$

The setting in Table 3.1 is assumed.

Pfaffian coordinates. Let N be even (cf. Table 3.1). Consider the Spinor variety $\mathbb{S}_N^+ = \text{Spin}_{2N} \cdot [v_{\omega_N}] \subset \mathbb{P}(\bigwedge^{ev} \mathbb{C}^N)$ where $v_{\omega_N} = \mathbf{e}_{[N]} = e_1 \wedge \dots \wedge e_N$ is the highest weight vector. A neighbourhood of $[v_{\omega_N}]$ is given by pure spinors $[a]$ such that the subspace H_a is described by the matrix in Plücker form $\begin{bmatrix} I_N \\ Y_a \end{bmatrix}$ where $Y_a \in \bigwedge^2 \mathbb{C}^N$. On the other hand, any skew-symmetric matrix $Y = (y_{ij}) \in \bigwedge^2 \mathbb{C}^N$ defines in a neighbourhood of $[v_{\omega_N}]$ the pure spinor (cf. [Man09, Sec. 2.3])

$$a_Y = \left(e_1 + \sum_{i=2}^N y_{1,i} f_i \right) \wedge \dots \wedge \left(e_N + \sum_{i=1}^{N-1} y_{N,i} f_i \right) = \sum_{\#I \text{ even}} \text{Pf}_I(Y) \mathbf{e}_{[N] \setminus I}, \quad (3.3.4)$$

where $\text{Pf}_I(Y)$ is the pfaffian of the principal submatrix of Y whose rows and columns are indexed by I , and $\mathbf{e}_{[N] \setminus I} = e_{j_1} \wedge \dots \wedge e_{j_{N-2r}}$ indexed by the set $[N] \setminus I = \{j_1, \dots, j_{N-2r}\}$.

REMARK. The middle-term in the above chain of equations describes the subspace H_{a_Y} but it is not an element of the half-spin representation as the vectors f_j 's appear; on the other hands, the termn on the right-hand side is an element of $\bigwedge^{ev} \mathbb{C}^N$ and one can prove the second equality by multiplying on the right both terms by $f_1 \cdots f_N \in \text{Cl}_q^+(\mathbb{C}^{2N})$.

In particular, after the isomorphism $\mathbb{P}(\mathbb{S}_N^+) \simeq \mathbb{P}^{2^{N-1}-1}$, one gets the following coordinate description for pure spinors in a neighbourhood of $[v_{\omega_N}]$

$$\begin{aligned} \mathbb{S}_{n+1}^+ &\hookrightarrow \mathbb{P}^{2^n-1} \\ [a_Y] &\mapsto [1 : \text{Pf}_{\{i,j\}}(Y) : \dots : \text{Pf}_I(Y) : \dots] \end{aligned}$$

From $\text{OG}^+(N, 2N)$ to \mathbb{S}_N^+ . We assume the notation for $V = E \oplus E^\vee$ as in the previous paragraphs, for $\dim V = 2N$ and N even. Recall that, under these assumptions, the highest

weight vector $[v_{\omega_N}]$ corresponds to the maximal isotropic subspace $H_{v_{\omega_N}} = E$.

Given $H \in \text{OG}^+(N, V) \cup \text{OG}^-(N, V)$, we are interested in expliciting the coordinates (in the standard basis of $\bigwedge E$) of the corresponding pure spinor $[a_H] \in \mathbb{S}_N^+ \cup \mathbb{S}_N^-$ with respect to the coordinates of a given basis of H . Given (h_1, \dots, h_N) a basis of H , via the correspondence (3.3.3) we know that the pure spinor $[a_H]$ is given by the (unique up to scalars) spinor

$$a_H = \sum_I \beta_I \mathbf{e}_I \in \bigwedge E$$

such that $h_j \cdot a = 0$ for all $j = 1 : N$.

Assume $p := \dim(H \cap E) \equiv 0 \pmod{2}$, so that $H \in \text{OG}^+(N, V)$ and $[a_H] \in \mathbb{S}_N^+$. Up to rotations we may assume $H = \langle e_1, \dots, e_p, g_{p+1}, \dots, g_N \rangle_{\mathbb{C}}$ where for all $j = p+1 : N$

$$g_j = f_j + \sum_{k=p+1}^N \alpha_{kj} e_k \quad , \quad \alpha_{kj} = -\alpha_{jk} \quad , \quad (3.3.5)$$

where the coefficients α_{jk} define a skew-symmetric matrix $A = (\alpha_{jk}) \in \bigwedge^2 \mathbb{C}^{N-p}$. Since $[a_H] \in \mathbb{S}_N^+ \subset \mathbb{P}(\bigwedge^{\text{ev}} E)$, one gets $\beta_I = 0$ for any $I \in 2^{[N]}$ such that $\#I$ is odd. Moreover, as basis elements of H , the vectors e_1, \dots, e_p annihilates a_H , hence $\beta_I = 0$ also for any I not containing $[p]$. Thus we restrict to consider spinors of the form

$$a_H = \sum_{I \subset [N] \setminus [p], \#I \text{ even}} \beta_I \mathbf{e}_{[p]} \wedge \mathbf{e}_I \quad .$$

Proposition 3.3.2. *Let $p \equiv N \equiv 0 \pmod{2}$. Let $H = \langle e_1, \dots, e_p, g_{p+1}, \dots, g_N \rangle_{\mathbb{C}} \in \text{OG}^+(N, 2N)$, where g_j are as in (3.3.5) and define the matrix $A = (\alpha_{kj}) \in \bigwedge^2 \mathbb{C}^{N-p}$. Then H corresponds to the pure spinor in \mathbb{S}_N^+*

$$a_H = \sum_{I \subset [N] \setminus [p], \#I \text{ even}} \text{Pf}(A_I) \mathbf{e}_{[p]} \wedge \mathbf{e}_I \quad . \quad (3.3.6)$$

Proof. It is enough to prove that each generator of H annihilates (via Clifford multiplication (3.1.6)) the above spinor a_H . This clearly holds for e_1, \dots, e_p . Consider the generator g_j for $j \in [N] \setminus [p]$ fixed. Then

$$\begin{aligned} g_j \cdot a &= \left(\sum_{h=p+1}^N \alpha_{hj} e_h \right) \wedge a_H + \lambda(f_j) a_H \\ &= \sum_{h=p+1}^N \sum_{I \not\ni h} \alpha_{hj} \text{Pf}(A_I) e_h \wedge \mathbf{e}_{[p]} \wedge \mathbf{e}_I + \sum_{I \ni j} \text{Pf}(A_I) (-1)^{\text{pos}(I,j)+1} \mathbf{e}_{[p]} \wedge \mathbf{e}_{I \setminus \{j\}} \end{aligned}$$

where: in the first summand one restricts only to subsets I not containing h , since for $h \in I$ it holds $e_h \wedge \mathbf{e}_I = 0$; in the second summand one restricts only to subsets I containing j since for $j \notin I$ it holds $\lambda(f_j)(\mathbf{e}_{[p]} \wedge \mathbf{e}_I) = 0$; $\text{pos}(I, j)$ denotes the position of the index j in the ordered subset I . Notice that, for $h \in [N] \setminus [p]$ and $t = 0 : \frac{N-p}{2} - 1$ fixed, it holds

$$\sum_{I \in \binom{[N] \setminus [p]}{2t} : I \not\ni h} \alpha_{hj} \text{Pf}(A_I) e_h \wedge \mathbf{e}_{[p]} \wedge \mathbf{e}_I = \sum_{J \in \binom{[N] \setminus [p]}{2t+1}} \alpha_{hj} \text{Pf}(A_{J \setminus \{h\}}) (-1)^{\text{pos}(J,h)+1} \mathbf{e}_{[p]} \wedge \mathbf{e}_J$$

since for the J 's not containing h the submatrix $A_{J \setminus \{h\}} = A_J$ is of odd size, hence its pfaffian is zero. By a similar argument, after fixing $j \in [N] \setminus [p]$ and $c = 1 : \frac{N-p}{2}$, one gets

$$\sum_{I \in \binom{[N] \setminus [p]}{2c} : I \ni j} \text{Pf}(A_I) (-1)^{\text{pos}(I,j)+1} \mathbf{e}_{[p]} \wedge \mathbf{e}_{I \setminus \{j\}} = \sum_{J \in \binom{[N] \setminus [p]}{2c-1}} \text{Pf}(A_{J \cup \{j\}}) (-1)^{\text{pos}(J \cup \{j\}, j)+1} \mathbf{e}_{[p]} \wedge \mathbf{e}_J.$$

Thus one has

$$\begin{aligned} g_j \cdot a &= \sum_{h=p+1}^N \sum_{t=0}^{\frac{N-p}{2}-1} \sum_{J \in \binom{[N] \setminus [p]}{2t+1}} \alpha_{hj} \text{Pf}(A_{J \setminus \{h\}}) (-1)^{\text{pos}(J,h)+1} \mathbf{e}_{[p]} \wedge \mathbf{e}_J + \\ &+ \sum_{c=0}^{\frac{N-p}{2}-1} \sum_{J \in \binom{[N] \setminus [p]}{2c+1}} \text{Pf}(A_{J \cup \{j\}}) (-1)^{\text{pos}(J \cup \{j\}, j)+1} \mathbf{e}_{[p]} \wedge \mathbf{e}_J \\ &= \sum_{t=0}^{\frac{N-p}{2}-1} \sum_{J \in \binom{[N] \setminus [p]}{2t+1}} \underbrace{\left[\sum_{h=p+1}^N \alpha_{hj} \text{Pf}(A_{J \setminus \{h\}}) (-1)^{\text{pos}(J,h)+1} + \text{Pf}(A_{J \cup \{j\}}) (-1)^{\text{pos}(J \cup \{j\}, j)+1} \right]}_{=: \Gamma_J} \mathbf{e}_{[p]} \wedge \mathbf{e}_J. \end{aligned}$$

Consider the *Heaviside step function*

$$\mathbb{H}(x) = \mathbf{1}_{x>0} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $\mathbb{H}(x) = 1 - \mathbb{H}(-x)$. Then, for any $t = 0 : \frac{N-p}{2} - 1$ and for any subset $J \in \binom{[N] \setminus [p]}{2t+1}$, one has

$$\begin{aligned} \sum_{h=p+1}^N \alpha_{hj} \text{Pf}(A_{J \setminus \{h\}}) (-1)^{\text{pos}(J,h)+1} &= \sum_{h=p+1}^N (-\alpha_{jh}) \text{Pf}(A_{(J \cup \{j\}) \setminus \{h,j\}}) (-1)^{\text{pos}(J,h)+1} \\ &= \sum_{h=p+1}^N \alpha_{jh} \text{Pf}(A_{(J \cup \{j\}) \setminus \{h,j\}}) (-1)^{\text{pos}(J \cup \{j\}, h) + \mathbb{H}(h-j)} \\ &= (-1)^{\text{pos}(J \cup \{j\}, j)} \text{Pf}(A_{J \cup \{j\}}) \end{aligned}$$

where the last equality follows by the recursive equation for the pfaffian (for an index i fixed)

$$\text{Pf}(B) = \sum_{j \neq i} (-1)^{i+j+1+\mathbb{H}(i-j)} b_{ij} \text{Pf}(B_{i,j}).$$

It follows that for all index subsets J it holds

$$\Gamma_J = (-1)^{\text{pos}(J \cup \{j\}, j)} \text{Pf}(A_{J \cup \{j\}}) + (-1)^{\text{pos}(J \cup \{j\}, j)+1} \text{Pf}(A_{J \cup \{j\}}) = 0,$$

hence $g_j \cdot a = 0$ for any $j \in [N] \setminus [p]$. This concludes the proof. \square

3.4 The diameter of Spinor varieties

In this section we describe the Hamming distance between points on Spinor varieties as codimension of the intersection of their corresponding maximal isotropic subspaces, hence we determine the diameter of Spinor varieties. These computations have been made before finding out that the distance on Spinor varieties (as well as on any cominuscule variety) was already known in [Buc+13, Sec. 4].

In light of Theorem 3.3.1 we assume N to be even and we consider the $2N$ -dimensional complex vector space $V = E \oplus E^\vee$ with basis $(e_1, \dots, e_N, f_1, \dots, f_N)$ which is hyperbolic with respect to the quadratic form $q = \sum_{i=1}^N x_i y_i$. Consider the Spinor variety $\mathbb{S}_N^+ \subset \mathbb{P}(\bigwedge^{ev} E)$ being the closed Spin_{2N} -orbit of the highest weight vector $v_{\omega_N} = \mathbf{e}_{[N]} = e_1 \wedge \dots \wedge e_N$, as well as of the lowest weight vector $\ell_{\omega_N} = \mathbb{1}$.

Let $[a], [b] \in \mathbb{S}_N^+$ be two pure spinors and let $H_a, H_b \in \text{OG}^+(N, V)$ be their corresponding maximal q -isotropic subspaces. Consider the line $L([a], [b]) = \{[\lambda a + \mu b] \mid [\lambda : \mu] \in \mathbb{P}^1\} \subset \mathbb{P}(\bigwedge^{ev} E)$. Any point in $L([a], [b])$ defines via the map (3.3.1) a q -isotropic subspace $H_{[\mu a + \lambda b]}$ of dimension $\dim H_{[\mu a + \lambda b]} \leq N$. Moreover, Spinor varieties being intersections of quadrics, by Bèzout it holds either $L([a], [b]) \subset \mathbb{S}_N^+$ or $L([a], [b]) \cap \mathbb{S}_N^+ = \{[a], [b]\}$. In particular, in the former case $[a], [b]$ have Hamming distance $d([a], [b]) = 1$ and

$$L([a], [b]) \subset \mathbb{S}_N^+ \iff [a + b] \in \mathbb{S}_N^+ \iff \dim H_{a+b} = N .$$

If $d([a], [b]) = k \geq 1$, then $L([a], [b]) \not\subset \mathbb{S}_N^+$ and we look for distinct pure spinors $[c_1], \dots, [c_{k-1}] \in \mathbb{S}_N^+$ such that $L([c_j], [c_{j+1}]) \subset \mathbb{S}_N^+$ (or equivalently $H_{c_j + c_{j+1}} \in \text{OG}^+(N, V)$) for all $j = 0 : k-1$, where $[c_0] = [a]$ and $[c_k] = [b]$. Moreover, since each H_{c_i} has to lie in the same connected component $\text{OG}^+(N, V)$, it holds $\text{codim}_{H_{c_i}}(H_{c_i} \cap H_{c_{i+1}}) \equiv 0 \pmod{2}$ for any $i = 0 : k-1$. For simplicity we write H_i for H_{c_i} .

Lemma 3.4.1. *Let $[a], [b] \in \mathbb{S}_N^+$ be two pure spinors corresponding to the subspaces $H_a, H_b \in \text{OG}^+(N, V)$. Then*

$$d([a], [b]) = 1 \iff \dim(H_a \cap H_b) = N - 2 .$$

Proof. From Remark 2.3.2, the action of Spin_{2N} preserves the Hamming distance between pure spinors in \mathbb{S}_N^+ : thus we may assume $a = v_{\omega_N} = \mathbf{e}_{[N]}$ with $H_a = H_{\mathbf{e}_{[N]}} = E$.

Let $p := \dim(E \cap H_b) \equiv 0 \pmod{2}$ and consider a basis $(e_1, \dots, e_p, g_{p+1}, \dots, g_N)$ of H_b for g_j as in (3.3.5). From Proposition 3.3.2 we know that

$$b = \sum_{I \subset \{p+1, \dots, N\}} \text{Pf}(A_I) \mathbf{e}_{[N]} \wedge \mathbf{e}_I ,$$

where $A = (\alpha_{kj}) \in \bigwedge^2 \mathbb{C}^{N-p}$ is defined by the coefficients in the g_j 's, and we set $\mathbf{e}_\emptyset := \mathbb{1}$.

If $p = \dim(E \cap H_b) = N - 2$, then we can write $b = \mathbf{e}_{[N-2]} \wedge (\mathbb{1} + \alpha_{N-1, N}(e_{N-1} \wedge e_N))$ and $\mathbf{e}_{[N]} + b = (1 + \alpha_{N-1, N})\mathbf{e}_{[N]} + \mathbf{e}_{[N-2]}$, hence

$$H_{\mathbf{e}_{[N]} + b} = \langle e_1, \dots, e_{N-2}, f_{N-1} - (1 - \alpha_{N-1, N})e_N, f_N + (1 - \alpha_{N-1, N})e_{N-1} \rangle_{\mathbb{C}} \in \text{OG}^+(N, V) ,$$

implying $d([\mathbf{e}_{[N]}], [b]) = 1$.

Conversely, if $d([\mathbf{e}_{[N]}], [b]) = 1$, then $E \cap H_b \subsetneq H_{\mathbf{e}_{[N]}+b} \in \text{OG}^+(N, V)$, hence $H_{\mathbf{e}_{[N]}+b} = \langle e_1, \dots, e_p, e_{p+1}, \dots, e_q, h_{q+1}, \dots, h_N \rangle_{\mathbb{C}}$ for some $q \geq p$: actually, it has to be $q = p$, otherwise $H_{\mathbf{e}_{[N]}+b} \cap E = H_{\mathbf{e}_{[N]}+b} \cap H_{-\mathbf{e}_{[N]}} \subset H_{\mathbf{e}_{[N]}+b-\mathbf{e}_{[N]}} = H_b$ would lead to contradiction. Then one has $H_{\mathbf{e}_{[N]}+b} = \langle e_1, \dots, e_p, h_{p+1}, \dots, h_N \rangle_{\mathbb{C}}$ for some vectors h_j as in (3.3.5) defined by a certain $B \in \bigwedge^2 \mathbb{C}^{N-p}$. By Proposition (3.3.2) one gets

$$\mathbf{e}_{[N]} + b = \sum_{I \subset \{p+1, \dots, N\}} \text{Pf}(B_I) \mathbf{e}_{[p]} \wedge \mathbf{e}_I .$$

Notice that both in b and $\mathbf{e}_{[N]} + b$ the summand $\mathbf{e}_{[p]} \wedge \mathbf{e}_{\emptyset} = \mathbf{e}_{[p]}$ appears with coefficient 1, thus it cancels out in the above equation leading to

$$\mathbf{e}_{[N]} = \sum_{\emptyset \neq I \subset \{p+1, \dots, N\}} (\text{Pf}(B_I) - \text{Pf}(A_I)) \mathbf{e}_{[p]} \wedge \mathbf{e}_I .$$

If it was $p \leq N-2$, then for any $I = \{i, j\} \subset \{p+1, \dots, N\}$ the coefficient $\text{Pf}(B_I) - \text{Pf}(A_I)$ has to be zero, hence all the 2×2 pfaffians of A and B would coincide: in particular, by recursiveness of pfaffians, also the maximum pfaffians $\text{Pf}(A_{\{p+1, \dots, N\}})$ and $\text{Pf}(B_{\{p+1, \dots, N\}})$ should coincide, leading to the contradiction $\mathbf{e}_{[N]} = 0$. Thus it has to be $p = N-2$.

This completes the proof. \square

Proposition 3.4.2. *Let $[a], [b] \in \mathbb{S}_N^+$ be two pure spinors with corresponding subspaces $H_a, H_b \in \text{OG}^+(N, V)$. Then*

$$d([a], [b]) = \frac{N - \dim(H_a \cap H_b)}{2} .$$

Proof. As in the previous proof, we may assume $a = \mathbf{e}_{[N]}$ and $b = \sum_{I \subset \{p+1, \dots, N\}} \text{Pf}(A_I) \mathbf{e}_{[p]} \wedge \mathbf{e}_I$, where $A = (\alpha_{kj}) \in \bigwedge^2 \mathbb{C}^{n+1-p}$. Then $H_b = \langle e_1, \dots, e_p, g_{p+1}, \dots, g_N \rangle_{\mathbb{C}}$ where $p = \dim(E \cap H_b) \equiv 0 \pmod{2}$. Let $d([\mathbf{e}_{[N]}], [b]) \geq 2$.

For any $j = 1 : \frac{N-p}{2}$ consider the pure spinor $[c_j] \in \mathbb{S}_N^+$ corresponding to the maximal q -isotropic subspace

$$H_j = \langle e_1, \dots, e_p, e_{p+1}, \dots, e_{N-2j}, g_{N-2j+1}, \dots, g_N \rangle_{\mathbb{C}} \in \text{OG}^+(N, V) .$$

Notice that $H_{\frac{N-p}{2}} = H_b$ so that $[c_{\frac{N-p}{2}}] = [b]$. Moreover, for any $j = 1 : \frac{N-p}{2}$ it holds

$$\dim(E \cap H_j) = \dim(H_j \cap H_{j+1}) = N-2$$

which by Lemma 3.4.1 is equivalent to $d([\mathbf{e}_{[N]}], [c_1]) = d([c_j], [c_{j+1}]) = 1$: in particular, we get

$$d([\mathbf{e}_{[N]}], [b]) \leq \frac{N-p}{2} = \frac{N - \dim(E \cap H_b)}{2} .$$

Let $d := d([\mathbf{e}_{[N]}], [b])$ and assume the thesis holds for $d-1$. Consider a minimal sequence of pure spinors $[p_1], \dots, [p_{d-1}] \in \mathbb{S}_N^+$ such that for any $j = 0 : d-1$ it holds $d([p_j], [p_{j+1}]) = 1$ where

we set $p_0 = \mathbf{e}_{[N]}$ and $p_d = b$, that is $\dim(H_{p_j} \cap H_{p_{j+1}}) = 2$. By minimality of the sequence and by inductive hypothesis, it holds

$$d - 1 = d([\mathbf{e}_{[N]}, [p_{d-1}]) = \frac{N - \dim(E \cap H_{p_{d-1}})}{2},$$

hence $\dim(E \cap H_{p_{d-1}}) = N - 2d + 2$. From the Grassmann dimension formula we get

$$\begin{aligned} \dim(E \cap H_{p_{d-1}} \cap H_b) &= \\ &= \dim(E \cap H_{p_{d-1}}) + \dim(H_{p_{d-1}} \cap H_b) - \dim((E \cap H_{p_{d-1}}) + (H_{p_{d-1}} \cap H_b)) \\ &= (N - 2d + 2) + (N - 2) - \dim((E \cap H_{p_{d-1}}) + (H_{p_{d-1}} \cap H_b)) \\ &= 2N - 2d - \dim((E \cap H_{p_{d-1}}) + (H_{p_{d-1}} \cap H_b)). \end{aligned}$$

The latter dimension can be a value in $\{N - 2, N - 1, N\}$, then one has

$$N - 2d \leq \dim(E \cap H_{p_{d-1}} \cap H_b) \leq N - 2d + 2.$$

If it was $N - 2d + 1 \leq \dim(E \cap H_{p_{d-1}} \cap H_b) \leq N - 2d + 2$, from the inequality $d \leq \frac{N-p}{2}$ already proven, one would get

$$N - 2d + 1 \leq \dim(E \cap H_{p_{d-1}} \cap H_b) \leq \dim(E \cap H_b) = p \leq N - 2d,$$

leading to contradiction. It follows that $\dim(E \cap H_{p_{d-1}} \cap H_b) = N - 2d$ must hold and, from a similar chain of inequalities as above, one conclude that $2d = N - p$, that is the thesis. \square

It follows straightforward from Proposition 3.4.2 that the maximum possible distance between two pure spinors $[a], [b] \in \mathbb{S}_N^+$ is realized when $H_a \cap H_b = \{0\}$. Moreover, similar argument apply to the case of N odd. The following result is now just a corollary.

Theorem 3.4.3. *The Spinor variety $\mathbb{S}_N^\pm \subset \mathbb{P}(V_{\omega_N}^{D_N})$ has diameter*

$$\text{diam}(\mathbb{S}_N^\pm) = \left\lfloor \frac{N}{2} \right\rfloor.$$

REMARK 3.4.4. The Spinor varieties $\mathbb{S}_2^\pm \simeq \mathbb{P}^1$ and $\mathbb{S}_3^\pm \simeq \mathbb{P}^3$ are linear spaces, thus their diameter is trivially 1. The Spinor variety \mathbb{S}_4^\pm is isomorphic to the 6-dimensional quadric $Q^6 \subset \mathbb{P}^7$ and it has diameter 2: this holds for any quadric $Q^N \subset \mathbb{P}^{N+1}$. The first Spinor variety having diameter at least 3 is $\mathbb{S}_6^\pm \subset \mathbb{P}^{31}$.

3.5 Some homogeneous bundles on Spinor varieties

In this section we describe some homogeneous bundles on the Spinor varieties which will be useful for defining the Clifford apolarity in Sec. 3.6.

We keep the notation from previous sections. We recall that we may assume N to be even in light of Theorem 3.3.1 and of the duality between the half-spin representations. In particular,

$$D_N/P_N = \mathbb{S}_N^+ \subset \mathbb{P}(V_{\omega_N}^{D_N}) = \mathbb{P}\left(\bigwedge^{ev} E\right).$$

Consider the half-spin representations $V_{\omega_{N-1}}^{D_N} = \bigwedge^{od} E$ and $V_{\omega_N}^{D_N} = \bigwedge^{ev} E$ with highest weight vectors $v_{\omega_{N-1}} = \mathbf{e}_{[N-1]} = e_1 \wedge \dots \wedge e_{N-1}$ and $v_{\omega_N} = \mathbf{e}_{[N]} = e_1 \wedge \dots \wedge e_{N-1}$ respectively: since N is even, by Remark 3.2.1 both half-spin representations are self-dual. Let $V_{\omega_1}^{D_N} = V = E \oplus E^\vee$ be the standard representation with highest weight vector $v_{\omega_1} = e_1$.

Let $(\epsilon_1, \dots, \epsilon_N)$ be an orthonormal basis of the vector space $\langle \Phi^{D_N} \rangle_{\mathbb{R}} \simeq \mathbb{R}^N$ spanned by the D_N -type root system, so that the simple roots of type D_N are of the form

$$\alpha_1 = \epsilon_1 - \epsilon_2 \quad , \quad \dots \quad , \quad \alpha_{N-1} = \epsilon_{N-1} - \epsilon_N \quad , \quad \alpha_N = \epsilon_{N-1} + \epsilon_N \quad ,$$

while the fundamental weights are

$$\begin{aligned} \omega_1 &= \epsilon_1 \quad , \quad \omega_2 = \epsilon_1 + \epsilon_2 \quad , \quad \dots \quad , \quad \omega_{N-2} = \epsilon_1 + \epsilon_2 + \dots + \epsilon_{N-2} \quad , \\ \omega_{N-1} &= \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{N-1} - \epsilon_N) \quad , \quad \omega_N = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{N-1} + \epsilon_N) \quad . \end{aligned}$$

The line bundle $\mathcal{O}_{\mathbb{S}}(1)$. Being a generalized Grassmannian, the Picard group of the Spinor variety \mathbb{S}_N^+ is monogenic, generated by the very ample line bundle

$$\mathcal{O}_{\mathbb{S}}(1) = \mathcal{E}_{\omega_N}$$

defined by the irreducible P_N -representation with highest weight ω_N . Moreover, by Borel–Weil Theorem such line bundle gives the minimal homogeneous embedding (cf. Example 1.3.6)

$$\mathbb{S}_N^+ \hookrightarrow \mathbb{P}(H^0(\mathbb{S}_N^+, \mathcal{O}_{\mathbb{S}}(1))^\vee) \simeq \mathbb{P}(V_{\omega_N}^{D_N}) \quad .$$

After the identification between the Spinor variety \mathbb{S}_N^+ and the maximal orthogonal Grassmannian $\text{OG}^+(N, 2N)$, one gets the inclusion

$$\iota : \mathbb{S}_N^+ \rightarrow \text{Gr}(N, 2N) \quad .$$

The pull-back via ι of the line bundle $\mathcal{O}_{\text{Gr}}(1)$ on the Grassmannian is related to the line bundle $\mathcal{O}_{\mathbb{S}}(1)$ on the Spinor variety as follows:

$$\iota^* \mathcal{O}_{\text{Gr}}(1) = \mathcal{O}_{\mathbb{S}}(1)^{\otimes 2} = \mathcal{O}_{\mathbb{S}}(2) \quad .$$

REMARK. This relation follows from the fact that the maximal q -isotropic subspaces in $\text{Gr}(N, 2N)$ splits in two isomorphic connected components, and it is classically stated as “the line bundle on the Spinor variety is the square root of the one on the Spinor variety” or also, by abuse of notation $\mathcal{O}_{\mathbb{S}}(1) = \mathcal{O}_{\text{Gr}}(\frac{1}{2})$.

The dual bundle $\mathcal{U}_{\mathbb{S}}^{\vee}$. The pull-back via ι of the bundle $\mathcal{U}_{\text{Gr}}^{\vee}$ dual to the universal bundle on the Grassmannian $\text{Gr}(N, 2N)$ (see Example 1.3.7) gives the dual bundle

$$\mathcal{U}_{\mathbb{S}}^{\vee} = \iota^* \mathcal{U}_{\text{Gr}}^{\vee}$$

on \mathbb{S}_N^+ , defined by the P_N -representation of highest weight ω_1 , hence by Borel–Weil Theorem it holds

$$H^0(\mathbb{S}_N^+, \mathcal{U}^{\vee}) \simeq (V_{\omega_1}^{D_N})^{\vee} = V_{\omega_1}^{D_N},$$

where the last equality follows from $-w_0(\omega_1) = \omega_1$ (see Remark 1.1.1).

The twisted bundle $\mathcal{U}_{\mathbb{S}}(1)$. Consider the rank- N bundle

$$\mathcal{U}_{\mathbb{S}}(1) = \mathcal{U}_{\mathbb{S}} \otimes \mathcal{O}_{\mathbb{S}}(1)$$

on the Spinor variety \mathbb{S}_N^+ . Notice that the description in Example 1.3.8 fails in this case: indeed,

$$\det(\mathcal{U}_{\mathbb{S}}) = \det(\iota^* \mathcal{U}_{\text{Gr}}) = \iota^* \det(\mathcal{U}_{\text{Gr}}) = \iota^* \mathcal{O}_{\text{Gr}}(1) = \mathcal{O}_{\mathbb{S}}(2).$$

However, as a homogeneous bundle, $\mathcal{U}_{\mathbb{S}}(1)$ is defined by a P_N -representation whose weights are of the form $\lambda + \mu$ as λ and μ vary among the weights defining $\mathcal{U}_{\mathbb{S}}$ and $\mathcal{O}_{\mathbb{S}}(1)$ respectively. The line bundle $\mathcal{O}_{\mathbb{S}}(1)$ has unique weight ω_N , while the rank- N bundle $\mathcal{U}_{\mathbb{S}}$ has weights $-\epsilon_1, \dots, -\epsilon_N$: then the bundle $\mathcal{U}_{\mathbb{S}}(1)$ has weights $\frac{1}{2}(c_1\epsilon_1 + \dots + c_N\epsilon_N)$ for $c_i \in \{\pm 1\}$ such that $\sum_{j=1}^N c_j = N-1$. In particular, the highest weight of the P_N -representation defining $\mathcal{U}_{\mathbb{S}}(1)$ is $\frac{1}{2}(\epsilon_1 + \dots + \epsilon_{N-1} - \epsilon_N) = \omega_{N-1}$, and by Borel–Weil’s Theorem we get the global sections

$$H^0(\mathbb{S}_N^+, \mathcal{U}_{\mathbb{S}}(1)) = (V_{\omega_{N-1}}^{D_N})^{\vee} = V_{\omega_{N-1}}^{D_N},$$

where the last equality follows from Remark 1.1.1 under the assumption that N is even.

3.6 Clifford apolarity

In this section we analyze the nonabelian apolarity in the case of Spinor varieties. For certain vector bundles, this leads to what we call “Clifford apolarity”. We also exhibit the vanishing condition of some global sections on Spinor varieties.

We assume the same setting, notation and results from Sec. 3.5. We recall that we may assume N to be even in light of Theorem 3.3.1 and of the duality between the half-spin representations.

In light of the global section spaces exhibited in Sec. 3.5, the contraction map $H^0(\mathbb{S}_N^+, \mathcal{U}^{\vee}) \otimes H^0(\mathbb{S}_N^+, \mathcal{U}(1)) \rightarrow H^0(\mathbb{S}_N^+, \mathcal{O}(1))$ is equivalent to

$$(V_{\omega_1}^{D_N})^{\vee} \otimes (V_{\omega_{N-1}}^{D_N})^{\vee} \longrightarrow (V_{\omega_N}^{D_N})^{\vee},$$

which under the assumption of N even and after the duality of half-spin representations (Remark 3.2.1) coincides with

$$(E \oplus E^\vee) \otimes \bigwedge^{od} E \longrightarrow \bigwedge^{ev} E . \quad (3.6.1)$$

Such map is uniquely determined as $\text{Spin}(V)$ -equivariant morphism (up to scalars): this is a consequence of Schur's lemma applied to the following result.

Theorem 3.6.1. *The irreducible $\text{Spin}(V)$ -module $\bigwedge^{ev} E$ (resp. $\bigwedge^{od} E$) appears with multiplicity 1 in the $\text{Spin}(V)$ -module $V \otimes \bigwedge^{od} E$ (resp. $V \otimes \bigwedge^{ev} E$).*

Proof. In light of the natural inclusion $\text{SL}(E) \subset \text{Spin}(V)$, for any $\text{Spin}(V)$ -module W among the above ones we can consider its restriction as $\text{SL}(E)$ -module $\text{Res}_{\text{SL}(E)}^{\text{Spin}(V)}(W)$: we lighten up the notation by simply writing W and specifying its module structure. Since N is even, one can rewrite the $\text{SL}(E)$ -module $(E \oplus E^\vee) \otimes \bigwedge^{od} E$ as

$$(E \oplus E^\vee) \otimes \bigwedge^{od} E = \bigoplus_{k=0}^{\frac{N-2}{2}} \left(E \otimes \bigwedge^{2k+1} E \right) \oplus \left(E^\vee \otimes \bigwedge^{2k+1} E \right) .$$

For each $k = 0 : \frac{N-2}{2}$, from Pieri's formula [Pro07, Sec. 9.10.2], one gets the following decompositions into irreducible $\text{SL}(E)$ -modules:

$$E \otimes \bigwedge^{2k+1} E = S^{\overbrace{(2, 1, \dots, 1)}^{2k+1}} \oplus \bigwedge^{2k+2} E \quad , \quad E^\vee \otimes \bigwedge^{2k+1} E = S^{\overbrace{(2, \dots, 2, 1, \dots, 1)}^{2k+1} \overbrace{N-2-2k}^{N-2-2k}} \oplus \bigwedge^{2k} E .$$

where we used that $E^\vee \simeq \bigwedge^{N-1} E$, $\mathbb{C}^\vee \simeq \bigwedge^N E$ and $\bigwedge^h E = 0$ for any $h \geq N+1$. Thus in the decomposition of $(E \oplus E^\vee) \otimes \bigwedge^{od} E$ into $\text{SL}(E)$ -modules

$$V \otimes \bigwedge^{od} E = \mathbb{C} \oplus 2 \left(\bigoplus_{k=1}^{\frac{N-4}{2}} \bigwedge^{2k} E \right) \oplus \bigwedge^N E \oplus \left(\bigoplus_{k=0}^{\frac{N-2}{2}} S^{\overbrace{(2, 1, \dots, 1)}^{2k+1}} \oplus S^{\overbrace{(2, \dots, 2, 1, \dots, 1)}^{2k+1} \overbrace{N-2-2k}^{N-2-2k}} \right)$$

there is only one copy of $\bigwedge^{ev} E$, hence there has to be only one copy in the $\text{Spin}(V)$ -module decomposition as well. \square

By fixing $q \in \bigwedge^{ev} E$, from the map (3.6.1) we get a $\text{Spin}(V)$ -equivariant map $\psi_q : (E \oplus E^\vee) \rightarrow \bigwedge^{od} E$ which is unique (up to scalars) by Theorem 3.6.1, thus it coincides with the map $\psi_q : V \rightarrow \bigwedge E$ defined in (3.3.1). Moreover, the natural contraction (3.6.1) is equivalent to a map

$$\Phi : \bigwedge^{ev} E \otimes \bigwedge^{od} E^\vee \longrightarrow E \oplus E^\vee$$

which again by Theorem 3.6.1 is uniquely determined as $\text{Spin}(V)$ -equivariant morphism: more precisely, it is the projection onto the unique copy of the irreducible $\text{Spin}(V)$ -submodule $E \oplus E^\vee$.

REMARK 3.6.2. By uniqueness, the map Φ is intrinsically related to the skew-apolarity map (2.2.1). Consider the splitting in $\mathrm{SL}(E)$ -modules

$$\bigwedge^{ev} E \otimes \bigwedge^{od} E^\vee = \bigoplus_{r=0}^{\frac{N}{2}} \bigoplus_{s=0}^{\frac{N-2}{2}} \left(\bigwedge^{2r} E \otimes \bigwedge^{2s+1} E^\vee \right).$$

The map Φ is $\mathrm{Spin}(V)$ -equivariant, hence $\mathrm{SL}(E)$ -equivariant, so are its restrictions

$$\Phi_{2r,2s+1} : \bigwedge^{2r} E \otimes \bigwedge^{2s+1} E^\vee \longrightarrow E \oplus E^\vee.$$

By Schur's lemma, each restriction $\Phi_{2r,2s+1}$ is non-zero if and only if either E or E^\vee appears as irreducible $\mathrm{SL}(E)$ -summand in the tensor product on the left-hand side: this happens if and only if $|2r - (2s + 1)| = 1$. In particular, it holds

$$\mathrm{Im}(\Phi_{2r,2s+1}) = \begin{cases} E & \text{if } 2r = (2s + 1) + 1 \\ E^\vee & \text{if } 2r = (2s + 1) - 1 \\ 0 & \text{otherwise} \end{cases}.$$

For $2r = (2s + 1) + 1$ (resp. $2r = (2s + 1) - 1$), the module $\bigwedge^{2r} E \otimes \bigwedge^{2s+1} E^\vee$ has a unique copy of E (resp. E^\vee) as irreducible $\mathrm{SL}(E)$ -submodule, thus (up to scalars) there exists a unique $\mathrm{SL}(E)$ -equivariant morphism from the tensor product onto E (resp. E^\vee). By uniqueness, the restrictions $\Phi_{2r,2s+1}$ are given (up to composing with a projection $\pi_{E \oplus E^\vee}$ onto $E \oplus E^\vee$) by generalizing the skew-catalecticant maps $\mathcal{C}_t^{s,d-s}$ in (2.2.1) as follows:

$$\begin{aligned} \Phi_{2r,2s+1} : \bigwedge^{2r} E \otimes \bigwedge^{2s+1} E^\vee &\rightarrow \bigwedge^{2r-(2s+1)} E \oplus \bigwedge^{2s+1-2r} E^\vee \xrightarrow{\pi_{E \oplus E^\vee}} E \oplus E^\vee \\ e \otimes f &\mapsto \mathcal{C}_e^{2s+1,2r-(2s+1)}(f) + \mathcal{C}_f^{2r,2s+1-2r}(e) \end{aligned}$$

where $\mathcal{C}_t^{s,d-s} = 0$ for $s > d$: roughly, depending on the sign of $2r - (2s + 1)$, one looks at vectors in E^\vee as derivations on E , or viceversa.

DEFINITION 3.6.3. The *Clifford apolarity action* is the $\mathrm{Spin}(E \oplus E^\vee)$ -equivariant map

$$\begin{aligned} \Phi : \bigwedge^{ev} E \otimes \bigwedge^{od} E^\vee &\longrightarrow E \oplus E^\vee \\ e \otimes f &\mapsto C_e(f)|_E + C_f(e)|_{E^\vee} \end{aligned} \quad (3.6.2)$$

where $C_e(f)|_E$ (resp. $C_f(e)|_{E^\vee}$) is the projection onto E (resp. E^\vee) of the contraction $f \cdot e \in \bigwedge E$ (resp. $e \cdot f \in \bigwedge E^\vee$) obtained via Clifford multiplication.

Vanishing of sections in $H^0(\mathbb{S}_N^+, \mathcal{U}(1))$. Via the isomorphism

$$H^0(\mathbb{S}_N^+, \mathcal{U}(1)) \simeq \left(V_{\omega_{N-1}}^{D_N} \right)^\vee = \bigwedge^{od} E^\vee,$$

any section $j_f \in H^0(\mathbb{S}_N^+, \mathcal{U}(1))$ corresponds to a spinor $f \in \bigwedge^{od} E^\vee$: we describe the zero locus of a section j_f by generalizing an argument from [Man21, proof of Prop. 2].

For any $[a] \in \mathbb{S}_N^+$, one has the fiber $(\mathcal{U}(1))_{[a]} \simeq \text{Hom}((\mathcal{O}(1))_{[a]}^\vee, \mathcal{U}_{[a]})$. In particular, it holds $(\mathcal{O}(1))_{[a]}^\vee = \langle a \rangle_{\mathbb{C}} \subset \bigwedge^{ev} E$ and $\mathcal{U}_{[a]} = H_a := \{v \in E \oplus E^\vee \mid v \cdot a = 0\}$. Thus we can identify $j_f([a]) \in (\mathcal{U}(1))_{[a]}$ with a homomorphism $j_f([a]) : \langle a \rangle_{\mathbb{C}} \rightarrow H_a$. We want to determine $j_f([a])(\lambda a) \in H_a$ as the scalar $\lambda \in \mathbb{C}$ varies.

REMARK 3.6.4. The scalar product $q(\cdot, \cdot)$ on $V = E \oplus E^\vee$ extends to a scalar product $\langle \cdot, \cdot \rangle$ on $\bigwedge V$ defined on decomposable elements $x = x_1 \wedge \dots \wedge x_k \in \bigwedge^k V$ and $y = y_1 \wedge \dots \wedge y_h \in \bigwedge^h V$ as

$$\langle x, y \rangle = \begin{cases} \det(q(x_i, y_j)) & \text{if } k = h \\ 0 & \text{otherwise} \end{cases}$$

and extended by linearity. The isomorphism of vector spaces $\bigwedge V \simeq \text{Cl}(V, q)$ defines such scalar product on the Clifford algebra too. The well-known adjointness between exterior product and contraction $\langle v \wedge x, y \rangle = \langle x, v \lrcorner y \rangle$ for $v \in V$, $x \in \bigwedge^{h-1} V$ and $y \in \bigwedge^h V$, together with the Clifford multiplication (3.1.6), implies $\langle x, v \cdot y \rangle = \langle v \cdot x, y \rangle$ for any $x, y \in \bigwedge V$ and any $v \in V$: in the Clifford algebra, this property extends to any element $z \in \text{Cl}(V, q)$ as $\langle x, zy \rangle = \langle z^* x, y \rangle$, where z^* is the reverse involution

$$z = x_1 \cdots x_{2k} \implies z^* = x_{2k} \cdots x_1 .$$

The function $\psi_{f,a}(v) = \langle f, v \cdot a \rangle$ is linear in $v \in V$ and vanishes on H_a , thus it belongs to $H_a^\perp = \{\psi : V \rightarrow \mathbb{C} \mid \psi(H_a) = 0\}$. But $H_a = H_a^\perp$ (since a is a pure spinor), thus $\psi_{f,\bullet} \in \text{Hom}(\langle a \rangle_{\mathbb{C}}, H_a)$. It follows that we have two maps

$$\begin{array}{ccc} \bigwedge^{od} E^\vee & \longrightarrow & H^0(\mathbb{S}_N^+, \mathcal{U}(1)) \\ j : f & \mapsto & j_f \\ \Psi : f & \mapsto & \psi_{f,\bullet} \end{array}$$

where j is the $\text{Spin}(V)$ -equivariant isomorphism given by Borel–Weil Theorem. Next remark shows that the maps j and Ψ coincide up to scalars.

REMARK 3.6.5. The map Ψ is $\text{Spin}(V)$ -equivariant too. Indeed, for any G -homogeneous bundle on a variety G/P , the G -action on a global section s is given by $(g \cdot s)(a) = s(g^{-1} \cdot a)$. Since for any $x \in \text{Spin}(V)$ it holds $x^{-1} = x^*$, one gets $(x \cdot \psi_{f,\bullet})(a) = \psi_{f,\bullet}(x^* a) = \psi_{f,x^* a} \in \text{Hom}(\langle x^* a \rangle, H_{x^* a})$ where $\psi_{f,x^* a}(\lambda x^* a) \in H_{x^* a}^\perp = H_{x^* a}$ is the functional on V given by $\psi_{f,x^* a}(\lambda x^* a)(v) = \langle f, v \cdot \lambda x^* a \rangle$. But $x^* \in \text{Spin}(V)$ acts by conjugacy on V , thus $V = x^* V x$ and by adjointness one concludes $\langle f, v \cdot \lambda x^* a \rangle = \langle f, (x^* w x) \cdot \lambda x^* a \rangle = \langle f, x^* w \cdot \lambda a \rangle \stackrel{adj.}{=} \langle x \cdot f, w \cdot \lambda a \rangle$, that is $x \cdot \psi_{f,\bullet} = \psi_{x \cdot f, \bullet}$.

Lemma 3.6.6. *In the previous notation, the zero locus $\mathcal{Z}(j_f)$ of a global section $j_f \in H^0(\mathbb{S}_N^+, \mathcal{U}(1))$ corresponding to a spinor $f \in \bigwedge^{od} E^\vee$ is*

$$\mathcal{Z}(j_f) = \mathbb{S}_N^+ \cap (V \cdot f)^\perp . \quad (3.6.3)$$

Proof. By Schur's lemma, the equivariant maps j and Ψ coincide up to scalar, hence the section $j_f \in H^0(\mathbb{S}_N^+, \mathcal{U}(1))$ is such that $j_f([a])(\lambda a) = \langle f, \bullet \cdot (\lambda a) \rangle \in H_a$ for any $\lambda \in \mathbb{C}$. We conclude that the zero locus $\mathcal{Z}(j_f)$ of the global section j_f is given by the pure spinors $[a] \in \mathbb{S}_N^+$ such that $0 = \langle f, v \cdot a \rangle = \langle v \cdot f, a \rangle$ for any $v \in V$, that is the thesis. \square

Chapter 4

Identifiability and singular locus of $\sigma_2(\mathrm{Gr}(k, N))$

Despite Grassmannians are ubiquitous objects in theoretical and applied areas of Mathematics, there are several aspects of their geometry still mysterious. Among these, only partial results have been obtained about their secant varieties (eg. see [MM15]). This chapter is devoted to solve the identifiability problem and to determine the singular locus of the secant varieties of lines to Grassmannians. The results appearing in this chapter come from the work [GS23] joint with Dr. Reynaldo Staffolani.

Let V be an N -dimensional complex vector space. After the identification with \mathbb{C}^N , we fix the standard basis (e_1, \dots, e_N) . For $k \leq N$ we consider the Grassmannian (cf. Example 1.3.1)

$$\mathrm{Gr}(k, N) = A_{N-1}/P_k \subset \mathbb{P}(V_{\omega_k}^{A_{N-1}}) = \mathbb{P}\left(\bigwedge^k \mathbb{C}^N\right).$$

We recall that the highest weight vector and the lowest weight vector in $V_{\omega_k}^{A_{N-1}}$ are respectively $v_{\omega_k} = \mathbf{e}_{[k]} = e_1 \wedge \dots \wedge e_k$ and $\ell_{\omega_k} = \mathbf{e}_{[N] \setminus [N-k]} = e_{N-k+1} \wedge \dots \wedge e_N$. For any $[v_1 \wedge \dots \wedge v_k] \in \mathrm{Gr}(k, N)$ we denote the corresponding k -dimensional linear subspace by $H_p = \langle v_1, \dots, v_k \rangle_{\mathbb{C}}$.

Secant variety of lines to $\mathrm{Gr}(k, N)$. Consider the secant variety of lines

$$\sigma_2(\mathrm{Gr}(k, N)) = \overline{\sigma_2^\circ(\mathrm{Gr}(k, N))} = \overline{\{[a + b] \in \mathbb{P}(V_{\omega_k}^{A_{N-1}}) \mid [a], [b] \in \mathrm{Gr}(k, N)\}} \subset \mathbb{P}(V_{\omega_k}^{A_{N-1}}).$$

The dense subset $\sigma_2^\circ(\mathrm{Gr}(k, N))$ is given by the union of bisecant lines, while the union of tangent lines defines the tangential variety $\tau(\mathrm{Gr}(k, N))$. It is known that the secant variety of lines is quasi-homogeneous, in the sense that it admits a dense orbit: more precisely, $\sigma_2(\mathrm{Gr}(k, N)) = \overline{\mathrm{SL}_N \cdot [v_{\omega_k} + \ell_{\omega_k}]}$ [Zak93, Theorem 1.4]. Moreover, it is *non-defective* for any $3 \leq k \leq \frac{N}{2}$

[CGG05, Theorem 2.1], thus its dimension is the expected one

$$\begin{aligned} \dim \sigma_2(\text{Gr}(k, N)) &= \min \{ 2 \dim \text{Gr}(k, N) + 1, \dim \mathbb{P}(V_{\omega_k}^{A_{N-1}}) \} \\ &= \min \left\{ 2k(N-k) + 1, \binom{N}{k} - 1 \right\} . \end{aligned}$$

On the other hand, the secant variety of lines $\sigma_2(\text{Gr}(2, N))$ to the Grassmannians of (affine) planes is *defective* for any $N \geq 5$ with defect equal to 4 [CGG05, Theorem 2.1].

REMARK 4.0.1. In same terminology from Sec. 2.1, the secant variety $\sigma_2(\text{Gr}(3, 6))$ is a *perfect* case. The Grassmannian $\text{Gr}(3, 6)$ is a Legendrian variety and, as such, its secant variety of lines has been studied by J.M. Landsberg and L. Manivel (cf. Remark 1.4.2).

The duality of irreducible representations $\bigwedge^k(\mathbb{C}^N)^\vee = (V_{\omega_k}^{A_{N-1}})^\vee = V_{\omega_{N-k}}^{A_{N-1}} = \bigwedge^{N-k} \mathbb{C}^N$ induces a duality of Grassmannians

$$\text{Gr}(k, N) \simeq \text{Gr}(N-k, N) .$$

Toy-case $\text{Gr}(2, N)$. For $k = 2$, the SL_N -module $V_{\omega_2}^{A_{N-1}} = \bigwedge^2 \mathbb{C}^N$ parameterizes the space of $N \times N$ skew-symmetric matrices, while the secant variety of lines $\sigma_2(\text{Gr}(2, N))$ parameterizes the $N \times N$ skew-symmetric matrices of rank at most 4. Since $\sigma_2(\text{Gr}(2, 4)) = \mathbb{P}^5$ and $\sigma_2(\text{Gr}(2, 5)) = \mathbb{P}^9$, we assume $N \geq 6$.

By defectivity we know that $\dim \sigma_2(\text{Gr}(2, N)) = 4(N-2) - 3$. Moreover, from the key result [Zak93, Theorem 1.4] one has the identity $\tau(\text{Gr}(2, N)) = \sigma_2(\text{Gr}(2, N))$.

Theorem. For $N \geq 6$ the singular locus of the secant $\text{Sing}(\sigma_2(\text{Gr}(2, N)))$ is exactly $\text{Gr}(2, N)$.

Proof. Since $\text{Gr}(2, N)$ parametrizes the $N \times N$ skew-symmetric matrices of rank 2 (i.e. skew-symmetric rank 1), the variety $\sigma_2(\text{Gr}(2, N))$ is given by the union of only two orbits: $\text{Gr}(2, N)$ and $\sigma_2(\text{Gr}(2, N)) \setminus \text{Gr}(2, N)$. Since in general $\text{Sing}(\sigma_r(\text{Gr}(k, N))) \supset \sigma_{r-1}(\text{Gr}(k, N))$ holds, the thesis follows. \square

*In respect of the duality of Grassmannians and of the above theorem,
through out all this chapter we assume*

$$3 \leq k \leq \left\lfloor \frac{N}{2} \right\rfloor .$$

4.1 The poset of SL_N -orbits in $\sigma_2(\text{Gr}(k, N))$

The secant variety of lines to the Grassmannian $\text{Gr}(k, N)$ is invariant under SL_N -action, and its subsets $\sigma_2^o(\text{Gr}(k, N))$ and $\tau(\text{Gr}(k, N))$ too. We refer to the orbits in $\sigma_2(\text{Gr}(k, N))$ as *secant orbits* and to the ones in $\tau(\text{Gr}(k, N))$ as *tangent orbits*: we already stress out that there could be points being both secant and tangent (see Remark 4.1.9). The spirit is at first determining

such orbits separately, and then analyzing how they interact. We fix the notation for any $l \in [k]$

$$\mathfrak{e}_l := \mathfrak{e}_{[k-l]} \wedge \mathfrak{e}_{[k+l] \setminus [k]} = e_1 \wedge \dots \wedge e_{k-l} \wedge e_{k+1} \wedge \dots \wedge e_{k+l} .$$

In particular, $\mathfrak{e}_k = e_{k+1} \wedge \dots \wedge e_{2k}$. Moreover, for any $l \in [k]$ we fix the notation for the intersection

$$E_{k-l} := \langle e_1, \dots, e_{k-l} \rangle_{\mathbb{C}} = H_{\mathfrak{e}_{[k]}} \cap H_{\mathfrak{e}_l} .$$

We are going to use a more linear-algebraic description of the Hamming distance between two points in $\text{Gr}(k, N)$ (cf. Sec. 2.3).

Lemma 4.1.1. *Let $[p], [q] \in \text{Gr}(k, N)$ and let $H_p, H_q \subset \mathbb{C}^N$ be their corresponding k -dimensional subspaces. Then $[p], [q]$ are joined by a line in the Grassmannian if and only if the corresponding subspaces H_p, H_q meet along a common hyperplane. In particular,*

$$d([p], [q]) = k - \dim(H_p \cap H_q) .$$

Proof. The trivial case $[p] = [q]$ (i.e. $H_p = H_q$) implies $d([p], [q]) = 0 = k - \dim(H_p \cap H_q)$. If $d([p], [q]) = 1$, then $\langle [p], [q] \rangle \subset \text{Gr}(k, N)$ and $[p + q] \in \text{Gr}(k, N)$, which is equivalent to $\dim(H_p \cap H_q) = k - 1$.

For the general case, assume $d([p], [q]) = \ell \geq 2$ and let $\dim(H_p \cap H_q) = s$. First, we prove that $\ell \leq k - s$. We may assume $p = x_1 \wedge \dots \wedge x_s \wedge \dots \wedge x_k$ and $q = x_1 \wedge \dots \wedge x_s \wedge y_{s+1} \wedge \dots \wedge y_k$. Consider the points $[p_1], \dots, [p_{k-s-1}] \in \text{Gr}(k, N)$ corresponding to the k -dimensional subspaces

$$H_{p_j} = \langle x_1, \dots, x_s, \dots, x_{k-j}, y_{k-j+1}, \dots, y_k \rangle \quad , \quad \forall j = 1 : k - s - 1 .$$

Since $\dim(H_{p_j} \cap H_{p_{j+1}}) = k - 1$ for any $j = 0 : k - s$ (where $p_0 = p$ and $p_{k-s} = q$), it follows that $d([p_j], [p_{j+1}]) = 1$ for any $j = 0 : k - s$, that is $d([p], [q]) = \ell \leq k - s$.

On the other hand, from Definition 2.3.1 there exists $[c_1], \dots, [c_{\ell-1}] \in \text{Gr}(k, N)$ such that $\langle [c_i], [c_{i+1}] \rangle \subset \text{Gr}(k, N)$ for any $i = 0 : \ell - 1$, where $c_0 = p$ and $c_{\ell} = q$. Clearly it holds $d([p], [c_{\ell-1}]) = \ell - 1$, hence by induction one has $\ell - 1 = d([p], [c_{\ell-1}]) = k - \dim(H_p \cap H_{c_{\ell-1}})$, that is $\dim(H_p \cap H_{c_{\ell-1}}) = k - \ell + 1$. Now consider the intersection $H_p \cap H_{c_{\ell-1}} \cap H_q$: using Grassmann Formula its dimension is

$$\begin{aligned} \dim(H_p \cap H_{c_{\ell-1}} \cap H_q) &= \\ &= \dim(H_p \cap H_{c_{\ell-1}}) + \dim(H_{c_{\ell-1}} \cap H_q) - \dim((H_p \cap H_{c_{\ell-1}}) + (H_{c_{\ell-1}} \cap H_q)) \\ &= 2k - \ell - \dim((H_p \cap H_{c_{\ell-1}}) + (H_{c_{\ell-1}} \cap H_q)) . \end{aligned}$$

Since the latter dimension can be either $k - 1$ or k , one has

$$k - \ell \leq \dim(H_p \cap H_{c_{\ell-1}} \cap H_q) \leq k - \ell + 1 .$$

If it was $\dim(H_p \cap H_{c_{\ell-1}} \cap H_q) = k - \ell + 1$, from the inequality $\ell \leq k - s$ one would get

$$k - \ell + 1 = \dim(H_p \cap H_{c_{\ell-1}} \cap H_q) \leq \dim(H_p \cap H_q) = s \leq k - \ell ,$$

leading to contradiction. It follows that $\dim(H_p \cap H_{p_{\ell-1}} \cap H_q) = k - \ell$ must hold and, from a similar chain of inequalities as above, we conclude that $\ell = k - s$, that is the thesis. \square

Corollary. *The Grassmannian $\text{Gr}(k, N)$ has diameter $\text{diam Gr}(k, N) = k$.*

Notice that for any $l \in [k]$ it holds $d([\mathbf{e}_{[k]}], [\mathbf{e}_l]) = l$ as the intersection $H_{\mathbf{e}_{[k]}} \cap H_{\mathbf{e}_l} = E_{k-l}$ has dimension $k - l$.

4.1.1 The secant branch

We show that the secant SL_N -orbits in $\sigma_2(\text{Gr}(k, N))$ are as many as the diameter of $\text{Gr}(k, N)$, and the points $[\mathbf{e}_{[k]} + \mathbf{e}_l]$ are their representatives. For any $l \in [k]$, we denote

$$\Sigma_l^{k,N} := \{[p + q] \in \sigma_2(\text{Gr}(k, N)) \mid d([p], [q]) = l\} . \quad (4.1.1)$$

Notice that $\Sigma_1^{k,N} = \text{Gr}(k, N)$. Since the SL_N -action preserves the Hamming distance between points in $\text{Gr}(k, N)$ (cf. Remark 2.3.2), the action of SL_N preserves $\Sigma_l^{k,N}$.

Proposition 4.1.2. *For any $l \in [k]$, the set $\Sigma_l^{k,N}$ is an SL_N -orbit. More precisely,:*

$$\Sigma_l^{k,N} = \text{SL}_N \cdot [\mathbf{e}_{[k]} + \mathbf{e}_l] .$$

In particular, the SL_N -orbit partition of the dense subset $\sigma_2^\circ(\text{Gr}(k, N))$ is

$$\sigma_2^\circ(\text{Gr}(k, N)) = \text{Gr}(k, N) \sqcup \bigsqcup_{l=2}^k \Sigma_l^{k,N} .$$

Proof. Clearly, the orbit $\text{SL}_N \cdot [\mathbf{e}_{[k]} + \mathbf{e}_l]$ is contained in $\Sigma_l^{k,N}$ but actually equality holds: given $[p + q] \in \Sigma_l^{k,N}$, we can write it as

$$p + q = v_1 \wedge \cdots \wedge v_{k-l} \wedge (v_{k-l+1} \wedge \cdots \wedge v_k + v_{k+1} \wedge \cdots \wedge v_{k+l})$$

and one can always find a $g \in \text{SL}_N$ such that $g(v_i) = e_i$ for any $i \in [k + l]$, that is $g \cdot [p + q] = [\mathbf{e}_{[N]} + \mathbf{e}_l]$. \square

REMARK 4.1.3. The orbit $\Sigma_k^{k,N}$ is dense: indeed, another representative is given by $[\mathbf{e}_{[k]} + \mathbf{e}_{[N] \setminus [N-k]}] = [v_{\omega_k} + \ell_{\omega_k}]$. Moreover, the closures $\overline{\Sigma_l^{k,N}}$ are already known in the literature as *restricted chordal varieties* [FH91, Hard exercise 15.44].

In the following we reinterpret the vector subspaces corresponding to Grassmannian points as kernels and we associate certain vector subspaces to secant points too: we underline that the latter is *not* a 1:1 correspondence, in the sense that to any vector subspace could correspond more secant points.

For any point $q \in \bigwedge^k \mathbb{C}^N$, we consider the multiplication map

$$\begin{aligned} \psi_q : \mathbb{C}^N &\longrightarrow \bigwedge^{k+1} \mathbb{C}^N \\ x &\longmapsto x \wedge q \end{aligned} \quad (4.1.2)$$

and we associate to the point q the subspace

$$H_q := \ker(\psi_q) .$$

For instance, for $[q] = [v_1 \wedge \dots \wedge v_k] \in \mathrm{Gr}(k, N)$ one recovers the corresponding subspace $\ker(\psi_p) = H_p$. Notice that at the moment we know the dimension and a basis of H_q only for $[q] \in \mathrm{Gr}(k, N)$.

Lemma 4.1.4. *Let $[p+q] \in \Sigma_k^{k,N}$ be a generic secant point. Then $H_{p+q} = \{0\}$.*

Proof. By homogeneity of the dense orbit $\Sigma_k^{k,N}$, we may assume $p+q = \mathbf{e}_{[k]} + \mathbf{e}_k$. Notice that $\{0\} = H_p \cap H_q \subset H_{p+q} = \ker(\psi_{p+q})$. Let $y \in H_{p+q} \subset \mathbb{C}^N$ with $y = \sum_{i=1}^N \beta_i e_i$: then

$$\begin{aligned} 0 &= y \wedge (\mathbf{e}_{[k]} + \mathbf{e}_k) \\ &= \sum_{i=k+1}^N (-1)^k \beta_i \mathbf{e}_{[k]} \wedge e_i + \sum_{i=1}^k \beta_i e_i \wedge \mathbf{e}_k + \sum_{i=2k+1}^N (-1)^k \beta_i \mathbf{e}_k \wedge e_i . \end{aligned}$$

From the linear independence of the summands above in $\bigwedge^{k+1} \mathbb{C}^N$, it follows $\beta_i = 0$ for any $i \in [N]$, that is $y = 0$ and the thesis follows. \square

Proposition 4.1.5. *Let $2 \leq l \leq k$ and let $[p+q] \in \Sigma_l^{k,N}$. Then*

$$H_{p+q} = H_p \cap H_q .$$

Proof. By Lemma 4.1.4 we know that the thesis holds for $l = k$. Fix $2 \leq l \leq k-1$ and a point $[p+q] \in \Sigma_l^{k,N}$. Let $p = v_1 \wedge \dots \wedge v_k$ and $q = v_1 \wedge \dots \wedge v_{k-l} \wedge w_{k-l+1} \wedge \dots \wedge w_k$, so that $H_p \cap H_q = \langle v_1, \dots, v_{k-l} \rangle_{\mathbb{C}}$.

Consider the multiplication map ψ_{p+q} as in (4.1.2): then it clearly holds $H_p \cap H_q \subset H_{p+q} := \ker(\psi_{p+q})$. Take $y \in H_{p+q}$ being linearly independent from v_1, \dots, v_{k-l} : in particular, if we complete $\{v_1, \dots, v_{k-l}\}$ to a basis of \mathbb{C}^N , we may assume that y does not depend on v_1, \dots, v_{k-l} . Then we get

$$\begin{aligned} 0 &= y \wedge (p+q) = y \wedge v_1 \wedge \dots \wedge v_{k-l} \wedge (v_{k-l+1} \wedge \dots \wedge v_k + w_{k-l+1} \wedge \dots \wedge w_k) \\ &= (-1)^{k-l} v_1 \wedge \dots \wedge v_{k-l} \wedge \left(y \wedge \underbrace{v_{k-l+1} \wedge \dots \wedge v_k}_{=: a} + y \wedge \underbrace{w_{k-l+1} \wedge \dots \wedge w_k}_{=: b} \right) . \end{aligned}$$

Since $y \wedge (a+b)$ is linearly independent on v_1, \dots, v_{k-l} , it follows that $y \wedge (a+b) = 0$. Notice that, if we denote $V' := \mathbb{C}^N / H_p \cap H_q \simeq \mathbb{C}^{N-k+l}$, we have $[a], [b] \in \mathrm{Gr}(l, V')$ with $d([a], [b]) = l$, thus $[a+b]$ lies in the dense orbit $\Sigma_l^{l, V'}$. Moreover, $y \in \ker(\psi_{a+b} : V' \rightarrow \bigwedge^{l+1} V') =: H_{a+b}$ and by Lemma 4.1.4 it holds $H_{a+b} = \{0\}$. It follows that $y = 0$, thus $H_{p+q} = H_p \cap H_q$. \square

4.1.2 The tangential branch

Now we focus on orbits in the tangential variety $\tau(\mathrm{Gr}(k, N))$. By homogeneity of the Grassmannian, it is enough to study only one tangent space. Indeed, it holds

$$g \cdot T_x \mathrm{Gr}(k, N) = T_{g \cdot x} \mathrm{Gr}(k, N) \quad , \forall g \in \mathrm{SL}_N, \forall x \in \mathrm{Gr}(k, N) ,$$

hence the SL_N -orbits in $\tau(\mathrm{Gr}(k, N))$ are in bijection with the orbits in $T_{[\mathbf{e}_{[k]}]} \mathrm{Gr}(k, N)$ under the action of the parabolic subgroup $P_k = \mathrm{stab}(\mathbf{e}_{[k]})$.

It is known that the (affine) tangent space at $[\mathbf{e}_{[k]}]$ is

$$\begin{aligned} T_{[\mathbf{e}_{[k]}]} \mathrm{Gr}(k, N) &= \sum_{j=1}^k e_1 \wedge \dots \wedge \underbrace{\mathbb{C}^N}_{j^{\mathrm{th}}} \wedge \dots \wedge e_k \\ &= \langle [\mathbf{e}_{[k]}], e_1 \wedge \dots \wedge \widehat{e}_j \wedge \dots \wedge e_k \wedge e_r \mid j \in [k], r \in [N] \setminus [k] \rangle \end{aligned} \quad (4.1.3)$$

where \widehat{e}_i denotes that the vector e_i has been removed.

REMARK 4.1.6. For a given $[p] = [v_1 \wedge \dots \wedge v_k] \in \mathrm{Gr}(k, N)$ one has the following isomorphism [Har13, Example 16.1] which is compatible with the action of $\mathrm{stab}_{\mathrm{SL}_N}(p)$:

$$T_{[p]} \mathrm{Gr}(k, N) = \bigwedge^{k-1} H_p \wedge V \simeq \mathrm{Hom}(H_p, \mathbb{C}^N / H_p) .$$

REMARK 4.1.7. One can describe the tangent space to the Grassmannian at some point $[p] \in \mathrm{Gr}(k, N)$ using the notion of Hamming distance. If $p = v_1 \wedge \dots \wedge v_k$, then from (4.1.3) one gets that

$$T_{[p]} \mathrm{Gr}(k, N) = \langle \{ [q] \in \mathrm{Gr}(k, N) \mid d([p], [q]) \leq 1 \} \rangle ,$$

that is the tangent space at a point is generated by lines passing through that point.

Consider the k elements in $T_{[\mathbf{e}_{[k]}]} \mathrm{Gr}(k, N)$

$$\theta_l := \sum_{j=1}^l e_1 \wedge \dots \wedge e_{j-1} \wedge e_{k+j} \wedge e_{j+1} \wedge \dots \wedge e_k \quad , \quad \forall l \in [k] . \quad (4.1.4)$$

From Remark 4.1.6, any element of $T_{[\mathbf{e}_{[k]}]} \mathrm{Gr}(k, N)$ corresponds to an $(N - k) \times k$ matrix in $\mathbb{C}^{N-k} \otimes \mathbb{C}^k$: in particular, any θ_l corresponds to an $(N - k) \times k$ matrix of rank l . The only invariant in $\mathbb{C}^{N-k} \otimes \mathbb{C}^k$ is the rank and, since the isomorphism is compatible with the group action, so is for $T_{[\mathbf{e}_{[k]}]} \mathrm{Gr}(k, N)$. In particular, all points in $T_{[\mathbf{e}_{[k]}]} \mathrm{Gr}(k, N)$ of rank l are conjugated to θ_l . Finally, by homogeneity of $\mathrm{Gr}(k, N)$, the action of SL_N conjugates all tangent spaces, and for any $l \in [k]$ the unions of all the rank- l orbits (as the tangent space varies) gives an SL_N -orbit in the tangential variety, namely

$$\Theta_l^{k, N} := \{ t \in \tau(\mathrm{Gr}(k, N)) \mid \mathrm{Rk}(t) = l \} . \quad (4.1.5)$$

From the arguments above we conclude the following result.

Proposition 4.1.8. *For any $l \in [k]$, the set $\Theta_l^{k, N}$ coincides with the SL_N -orbit*

$$\Theta_l^{k, N} = \mathrm{SL}_N \cdot [\theta_l] .$$

In particular, the SL_N -orbit partition of the tangential variety is

$$\tau(\mathrm{Gr}(k, N)) = \bigsqcup_{l=1}^k \Theta_l^{k, N} .$$

REMARK 4.1.9. From (4.1.4) we have $[\theta_1] = [e_2 \wedge \dots \wedge e_k \wedge e_{k+1}] \in \text{Gr}(k, N)$, thus

$$\Theta_1^{k,N} = \text{Gr}(k, N) = \Sigma_1^{k,N}.$$

Moreover, $\theta_2 = e_2 \wedge \dots \wedge e_k \wedge e_{k+1} + e_1 \wedge e_3 \wedge \dots \wedge e_k \wedge e_{k+2}$ is sum of two points in $\text{Gr}(k, N)$ having Hamming distance 2, thus $[\theta_2] \in \Sigma_2^{k,N}$ and we get

$$\Sigma_2^{k,N} = \Theta_2^{k,N}.$$

Finally, the orbit $\Theta_k^{k,N}$ is given by points corresponding to $(N - k) \times k$ matrices of maximum rank, which are a dense subset in $\mathbb{C}^{N-k} \otimes \mathbb{C}^k$. Thus the orbit $\Theta_k^{k,N}$ is *dense* in $\tau(\text{Gr}(k, N))$.

4.1.3 Inclusions among closures of SL_N -orbits

From the previous subsections we get the SL_N -orbit partition

$$\sigma_2(\text{Gr}(k, N)) = \text{Gr}(k, N) \sqcup \left(\bigsqcup_{l=2}^k \Sigma_l^{k,N} \right) \cup \left(\bigsqcup_{l=3}^k \Theta_l^{k,N} \right).$$

In the following we determine the closures of the orbits and we prove that the above “weak union” actually is a “disjoin union”.

Proposition 4.1.10.

1. For any $l \in [k - 1]$ it holds $\Sigma_l^{k,N} \subset \overline{\Sigma_{l+1}^{k,N}}$.
2. For any $l \in [k - 1]$ it holds $\Theta_l^{k,N} \subset \overline{\Theta_{l+1}^{k,N}}$.
3. For any $l = 3 : k$ it holds $\Theta_l^{k,N} \subset \overline{\Sigma_l^{k,N}}$.

Proof. 1. Fix $l \in [k - 1]$. By homogeneity, it is enough to show that a representative of the distance- l orbit lies in the closure of the distance- $(l + 1)$ orbit. The representative $[\mathbf{e}_{[k]} + \mathbf{e}_l] \in \Sigma_l^{k,N}$ is limit for $\epsilon \rightarrow \infty$ of the sequence

$$\mathbf{e}_{[k]} + \mathbf{e}_{[k-l-1]} \wedge \left(e_{k-l} + \frac{1}{\epsilon} e_{k+l+1} \right) \wedge e_{k+1} \wedge \dots \wedge e_{k+l} \in \Sigma_{l+1}^{k,N}.$$

2. The tangent points in $\Theta_l^{k,N}$ correspond to $(N - k) \times k$ matrices of rank l , while points $\Theta_{l+1}^{k,N}$ to $(N - k) \times k$ matrices of rank $l + 1$. The thesis follows from the fact that the former matrices lie in the closure of the latter ones.
3. Given $[\mathbf{e}_{[k]} + \mathbf{e}_l]$ and $[\theta_l]$ the representatives of the orbits $\Sigma_l^{k,N}$ and $\Theta_l^{k,N}$, respectively, it is enough to find elements $g_\epsilon \in GL_N$ such that $\lim_{\epsilon \rightarrow 0} (g_\epsilon \cdot (\mathbf{e}_{[k]} + \mathbf{e}_l)) = \theta_l$. For any $\epsilon > 0$ consider the element $g_\epsilon \in GL_N$ acting as

$$e_i \mapsto e_i + \epsilon \cdot e_{k+i}, \quad e_{k+j} \mapsto e_{k+j-l} + \epsilon^2 \cdot e_{k+j}, \quad e_{k+l} \mapsto -e_k + \epsilon^2 \cdot e_{k+l}$$

for any $i \in [l]$ and $j \in [l-1]$, and as the identity on the other basis vectors: since the images of the basis vectors are all linearly independent, the linear map g_ϵ actually belongs to GL_N . From a straightforward count one gets $g_\epsilon \cdot (\mathbf{e}_{[k]} + \mathbf{e}_l) = \epsilon \cdot \theta_l + \epsilon^2 \cdot \bar{t}$ for a suitable $\bar{t} \in \bigwedge^k \mathbb{C}^N$, hence $\lim_{\epsilon \rightarrow 0} (\epsilon^{-1} g_\epsilon \cdot (\mathbf{e}_{[k]} + \mathbf{e}_l)) = \theta_l$, that is the thesis. \square

We complete Remark 4.1.9 by showing that for any $l = 3 : k$ it holds $\Theta_l^{k,N} \neq \Sigma_l^{k,N}$. For the dense case $l = k$ the equality does not hold since for $k \geq 3$ the secant variety is always non-defective and $\tau(\text{Gr}(k, N)) \subsetneq \sigma_2(\text{Gr}(k, N))$ [Zak93, Theorem 1.4].

Proposition 4.1.11. *For any $l = 3 : k-1$ it holds $\Sigma_l^{k,N} \neq \Theta_l^{k,N}$.*

Proof. By contradiction, we assume that there exists $3 \leq l \leq k-1$ such that $\Sigma_l^{k,N} = \Theta_l^{k,N}$. Then from the inclusions in Proposition 4.1.10 we easily get that $\Sigma_i^{k,N} = \Theta_i^{k,N}$ for any $2 \leq i \leq l$: in particular, $\Sigma_3^{k,N} = \Theta_3^{k,N}$. Thus it is enough to assume by contradiction that $\Sigma_3^{k,N} = \Theta_3^{k,N}$. Since we don't want the dense orbit, we consider $k \geq 4$. Consider the representative of $\Theta_3^{k,N}$

$$\begin{aligned} \theta_3 &= e_2 \wedge \dots \wedge e_k \wedge e_{k+1} + e_1 \wedge e_3 \wedge \dots \wedge e_k \wedge e_{k+2} + e_1 \wedge e_2 \wedge e_4 \wedge \dots \wedge e_k \wedge e_{k+3} \\ &= e_4 \wedge \dots \wedge e_k \wedge \underbrace{\left(e_2 \wedge e_3 \wedge e_{k+1} + e_1 \wedge e_3 \wedge e_{k+2} + e_1 \wedge e_2 \wedge e_{k+3} \right)}_{=: \eta} \end{aligned}$$

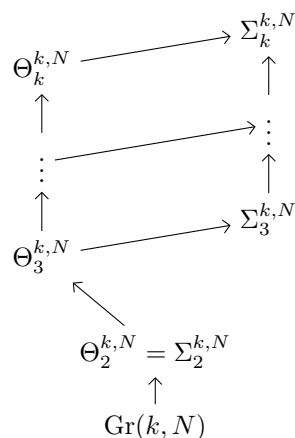
From the multiplication map ψ_{θ_3} as in (4.1.2) we get the subspace $H_{\theta_3} = \ker(\psi_{\theta_3}) = \langle e_4, \dots, e_k \rangle_{\mathbb{C}}$ having dimension $\dim H_{\theta_3} = k-3$. Define $W := \langle e_1, e_2, e_3, e_{k+1}, \dots, e_N \rangle_{\mathbb{C}}$.

Since $[\theta_3] \in \Theta_3^{k,N} = \Sigma_3^{k,N}$, there exist $[p], [q] \in \text{Gr}(k, N)$ such that $[\theta_3] = [p+q]$ and $d([p], [q]) = 3$. Then, by definition as kernels, one gets $\langle e_4, \dots, e_k \rangle_{\mathbb{C}} = H_{\theta_3} = H_{p+q} = H_p \cap H_q$, where the last equality follows from Proposition 4.1.5. This implies that we can write $p+q = e_4 \wedge \dots \wedge e_k \wedge (a+b)$ for a certain $[a+b] \in \Sigma_3^{3,W}$. Given the multiplication map

$$\begin{aligned} \mu : \bigwedge^3 W &\longrightarrow \bigwedge^k \mathbb{C}^N \\ t &\mapsto e_4 \wedge \dots \wedge e_k \wedge t \end{aligned}$$

it holds $\mu(\eta) = \theta_3 = p+q = \mu(a+b)$, and by injectivity of μ we get $\eta = a+b$. But $[\eta]$ is exactly the representative of the orbit $\Theta_3^{3,W}$, while $[a+b]$ is in the orbit $\Sigma_3^{3,W}$: in particular, it follows that $\tau(\text{Gr}(3, W)) = \sigma_2(\text{Gr}(3, W))$ which is a contradiction. \square

Theorem 4.1.12. *For any $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$, the poset of SL_N -orbits in the secant variety of lines $\sigma_2(\text{Gr}(k, N))$ is described by the graph in Figure 4.1, where the arrows denote the inclusion of an orbit into the closure of the other orbit. In particular, the orbits $\Theta_k^{k,N}$ and $\Sigma_k^{k,N}$ are the dense orbits of the tangential and secant variety respectively.*

Figure 4.1: Poset graph of SL_N -orbits in $\sigma_2(\text{Gr}(k, N))$.

4.2 Identifiability in $\sigma_2(\text{Gr}(k, N))$

In this section we prove that the secant orbit $\Sigma_2^{k, N}$ is unidentifiable while the secant orbits $\Sigma_l^{k, N}$ for $l = 3 : k$ are identifiable. We refer to [BV18, Theorem 1.1] for the most updated results about generic identifiability for skew-symmetric tensors when $N \leq 14$. We refer to Sec. 2.1 for the notions of identifiability and decomposition locus.

Unidentifiability of $\Sigma_2^{k, N}$. The unidentifiability of the distance-2 orbit $\Sigma_2^{k, N}$ is a consequence of the fact that the Grassmannian $\text{Gr}(2, 4)$ is just a quadric in \mathbb{P}^5 . This is clear if one considers the representative

$$[\mathbf{e}_{[k]} + \mathfrak{e}_2] = [\mathbf{e}_{[k-2]} \wedge (e_{k-1} \wedge e_k + e_{k+1} \wedge e_{k+2})] \in \Sigma_2^{k, N},$$

and notices that the sum in the round brackets actually is a sum of two points lying on a quadric in $\mathbb{P}(\wedge^2 \langle e_{k-1}, e_k, e_{k+1}, e_{k+2} \rangle_{\mathbb{C}})$, which is unidentifiable: for instance, another decomposition is

$$e_{k-1} \wedge e_k + e_{k+1} \wedge e_{k+2} = e_{k-1} \wedge (e_k + e_{k+1}) + e_{k+1} \wedge (e_{k+2} + e_{k-1}).$$

Thus the point $[\mathbf{e}_{[k]} + \mathfrak{e}_2]$ is unidentifiable too, hence the orbit $\Sigma_2^{k, N}$ is so. Moreover, given $W := \langle e_{k-1}, \dots, e_N \rangle_{\mathbb{C}}$, the dimension of the decomposition locus of $[e_{k-1} \wedge e_k + e_{k+1} \wedge e_{k+2}]$ in $\Sigma_2^{2, W} \subset \sigma_2^\circ(\text{Gr}(2, W))$ is equal to the defect of $\sigma_2(\text{Gr}(2, W))$ which is 4, as recalled at the beginning of this chapter.

Corollary 4.2.1. *The distance-2 orbit $\Sigma_2^{k, N}$ is unidentifiable. Moreover, the decomposition locus of any point in $\Sigma_2^{k, N}$ has dimension 4.*

Identifiability of $\Sigma_l^{k, N}$ for $l \geq 3$. First we prove that the dense $\Sigma_k^{k, N}$ is identifiable for $k \geq 3$, and then we conclude the identifiability of any orbit $\Sigma_l^{k, N}$ for $l \geq 3$.

Lemma 4.2.2. *For any $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$, the dense orbit $\Sigma_k^{k, N}$ is identifiable.*

Proof. Consider the multiplication map $\bigwedge^k V \otimes \bigwedge^2 V \rightarrow \bigwedge^{k+2} V$ defined on decomposable elements as $x \otimes y \mapsto x \wedge y$ and extended by linearity. Given a secant point $[p + q] \in \Sigma_k^{k, N}$, the induced multiplication map $(p + q) \wedge \bullet : \bigwedge^2 V \rightarrow \bigwedge^{k+2} V$ has kernel

$$\ker((p + q) \wedge \bullet) = \left\{ \sum_i v_i \wedge w_i \mid \sum_i p \wedge v_i \wedge w_i = - \sum_i q \wedge v_i \wedge w_i \right\} = H_p \wedge H_q$$

where the last equality follows from the fact that $d([p], [q]) = k$, that is $H_p \cap H_q = 0$. Then, given the point $[p + q]$, the subspaces H_p and H_q (hence $[p]$ and $[q]$) can be recovered in a unique way as follows. First, one recovers $H_p \oplus H_q$ as kernel of $V \wedge \ker((p + q) \wedge \bullet) \rightarrow \bigwedge^3 V$.

Let $[p' + q'] = [p + q]$ be another decomposition: since $H_{p'} \cap H_{q'} \subset H_{p'+q'} = H_{p+q} = H_p \cap H_q = \{0\}$, it holds $d([p'], [q']) = k$. Clearly, as kernel of a multiplication map with respect to $[p + q] = [p' + q']$, it holds

$$H_p \wedge H_q = H_{p'} \wedge H_{q'} \quad , \quad H_p \oplus H_q = H_{p'} \oplus H_{q'} .$$

In particular, for any $v \in H_p$ it holds either $v \in H_{p'}$ or $v \in H_{q'}$ (similar for $w \in H_q$): indeed, if $v = v_{p'} + v_{q'}$ and $w = w_{p'} + w_{q'}$ in $H_{p'} \oplus H_{q'}$ for some $v_{p'}, v_{q'}, w_{p'}, w_{q'} \neq 0$, then $0 = (v \wedge w) \wedge (p + w) = (v_{p'} \wedge w_{p'} + v_{p'} \wedge w_{q'} + v_{q'} \wedge w_{p'} + v_{q'} \wedge w_{q'}) \wedge (p' + q') = v_{p'} \wedge w_{p'} \wedge q' + v_{q'} \wedge w_{q'} \wedge p'$, leading to a contradiction since $\dim H_{p'} = \dim H_{q'} = k \geq 3$ (together with $H_{p'} \cap H_{q'} = \{0\}$) implies $\langle H_{p'}, v_{q'}, w_{q'} \rangle \neq \langle H_{q'}, v_{p'}, w_{p'} \rangle$.

Now, assume by contradiction that $\{H_p, H_q\} \neq \{H_{p'}, H_{q'}\}$, that is there exist $v_1, v_2 \in H_p$ such that $v_1 \in H_{p'}$ and $v_2 \in H_{q'}$, and similarly $w_1, w_2 \in H_q$ such that $w_1 \in H_{p'}$ and $w_2 \in H_{q'}$. Then one gets

$$\begin{aligned} 0 &= (v_1 \wedge w_1 + v_2 \wedge w_2) \wedge (p + q) = (v_1 \wedge w_1 + v_2 \wedge w_2) \wedge (p' + q') \\ &= v_1 \wedge w_1 \wedge p' + v_2 \wedge w_2 \wedge q' \neq 0 , \end{aligned}$$

hence a contradiction. We conclude that $\{H_p, H_q\} = \{H_{p'}, H_{q'}\}$, hence $\{[p], [q]\} = \{[p'], [q']\}$. \square

Theorem 4.2.3. *For any $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$ and any $3 \leq l \leq k$, the secant orbit $\Sigma_l^{k, N}$ is identifiable.*

Proof. From Lemma 4.2.2 we already know that the dense orbit $\Sigma_k^{k, N}$ is identifiable, thus we fix $3 \leq l \leq k - 1$. By homogeneity, it is enough to prove the thesis for the representative

$$[\mathbf{e}_{[k]} + \mathbf{e}_l] = [\mathbf{e}_{[k-l]} \wedge (e_{k-l+1} \wedge \dots \wedge e_k + e_{k+1} \wedge \dots \wedge e_{k+l})] \in \Sigma_l^{k, N} .$$

Let $[p], [q] \in \text{Gr}(k, N)$ be such that $[p + q] = [\mathbf{e}_{[k]} + \mathbf{e}_l]$: in particular, $H_p \cap H_q = H_{p+q} = H_{\mathbf{e}_{[k]} + \mathbf{e}_l} = E_{k-l}$. Given $W := \langle e_{k-l+1}, \dots, e_N \rangle_{\mathbb{C}} \simeq \mathbb{C}^{N-k+l}$, we can write

$$p = \mathbf{e}_{[k-l]} \wedge \underbrace{v_{k-l+1} \wedge \dots \wedge v_k}_{=:a} \quad , \quad q = \mathbf{e}_{[k-l]} \wedge \underbrace{w_{k-l+1} \wedge \dots \wedge w_k}_{=:b} ,$$

and it holds $H_a \cap H_b = \{0\}$, that is $[a + b] \in \Sigma_l^{l, W}$. Now, the multiplication map

$$\begin{aligned} \mu : \bigwedge^l W &\longrightarrow \bigwedge^k \mathbb{C}^N \\ t &\mapsto \mathbf{e}_{[k-l]} \wedge t \end{aligned}$$

restricts to the map $\Sigma_l^{l, W} \rightarrow \Sigma_l^{k, N}$. Since

$$\mu(a + b) = p + q = \mathbf{e}_{[k]} + \mathbf{e}_l = \mu(e_{k-l+1} \wedge \dots \wedge e_k + e_{k+1} \wedge \dots \wedge e_{k+l}) ,$$

by injectivity we get $a + b = e_{k-l+1} \wedge \dots \wedge e_k + e_{k+1} \wedge \dots \wedge e_{k+l} \in \Sigma_l^{l, W}$. But from Lemma 4.2.2 the orbit $\Sigma_l^{l, W} \subset \sigma_2(\text{Gr}(l, W))$ is identifiable (as dense orbit), thus $\{a, b\} = \{e_{k-l+1} \wedge \dots \wedge e_k, e_{k+1} \wedge \dots \wedge e_{k+l}\}$ and $\{p, q\} = \{\mathbf{e}_{[k]}, \mathbf{e}_l\}$, that is $[\mathbf{e}_{[k]} + \mathbf{e}_l]$ is identifiable. \square

4.3 Tangential-identifiability in $\tau(\text{Gr}(k, N))$

In this section we focus on the tangent orbits $\Theta_l^{k, N} \subset \tau(\text{Gr}(k, N))$. We point out that each tangent orbit $\Theta_l^{k, N}$ for $l \geq 2$ is *unidentifiable*: indeed, any representative $[\theta_l] \in \Theta_l^{k, N}$ in (4.1.4) admits the equivalent decomposition

$$(e_1 \wedge (e_{k+2} + e_{k+1}) + e_{k+1} \wedge (e_1 + e_2)) \wedge e_3 \wedge \dots \wedge e_k + \sum_{j=3}^l e_1 \wedge \dots \wedge e_{j-1} \wedge e_{k+j} \wedge e_{j+1} \wedge \dots \wedge e_k .$$

REMARK 4.3.1. The distance-2 orbit $\Sigma_2^{k, N} = \Theta_2^{k, N}$ is *not* tangential-identifiable: indeed, the representative $[\theta_2] = [e_{k+1} \wedge e_2 \wedge \dots \wedge e_k + e_1 \wedge e_{k+2} \wedge e_3 \wedge \dots \wedge e_k]$ lies on both the tangent spaces

$$[\theta_2] \in T_{[\mathbf{e}_{[k]}]} \text{Gr}(k, N) \cap T_{[\mathbf{e}_{[k+2] \setminus [2]}]} \text{Gr}(k, N) .$$

However, for any equivalent decomposition of $[\theta_2]$ one can exhibit two tangent spaces on which that decomposition lies. Thus we conclude that *the tangential-locus of $[\theta_2]$ has the same dimension of the decomposition locus of $[\theta_2]$* , which by Corollary 4.2.1 is 4.

Theorem 4.3.2. *For any $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$ and any $3 \leq l \leq k$, the tangent orbit $\Theta_l^{k, N}$ is tangential-identifiable.*

Proof. Fix $l \geq 3$. By homogeneity, it is enough to prove the thesis for the representative $[\theta_l] = [\sum_{j=1}^l \theta_{l,j}] \in \Theta_l^{k, N}$ where

$$\theta_{l,j} := e_1 \wedge \dots \wedge e_{j-1} \wedge e_{k+j} \wedge e_{j+1} \wedge \dots \wedge e_k$$

are the summands appearing in (4.1.4). We already know that $[\theta_l] \in T_{[\mathbf{e}_{[k]}]} \text{Gr}(k, N)$. We want to prove that, if $[q] \in \text{Gr}(k, N)$ is such that $[\theta_l] \in T_{[q]} \text{Gr}(k, N)$, then $[q] = [\mathbf{e}_{[k]}]$.

Assume $[\theta_l] \in T_{[q]} \text{Gr}(k, N)$ for some $[q] \in \text{Gr}(k, N)$, hence $[\theta_l] \in T_{[\mathbf{e}_{[k]}]} \text{Gr}(k, N) \cap T_{[q]} \text{Gr}(k, N)$.

Notice that

$$\begin{aligned} T_{[\mathbf{e}_{[k]}]} \mathrm{Gr}(k, N) \cap T_{[q]} \mathrm{Gr}(k, N) &= \left(\bigwedge^{k-1} H_{\mathbf{e}_{[k]}} \wedge V \right) \cap \left(\bigwedge^{k-1} H_q \wedge V \right) \\ &= \bigwedge^{k-1} (H_{\mathbf{e}_{[k]}} \cap H_q) \wedge V + \bigwedge^{k-2} (H_{\mathbf{e}_{[k]}} \cap H_q) \wedge H_{\mathbf{e}_{[k]}} \wedge H_q . \end{aligned} \quad (4.3.1)$$

If $\dim(H_{\mathbf{e}_{[k]}} \cap H_q) \leq k-2$ (i.e. $d(\mathbf{e}_{[k]}, [q]) \geq 3$), then $T_{[\mathbf{e}_{[k]}]} \mathrm{Gr}(k, N) \cap T_{[q]} \mathrm{Gr}(k, N) = \{0\}$, leading to a contradiction. If $k-2 \leq \dim(H_{\mathbf{e}_{[k]}} \cap H_q) \leq k-1$ (i.e. $1 \leq d([\mathbf{e}_{[k]}], [q]) \leq 2$), as each summand $\theta_{l,j}$ for $j \in [l]$ is a simple element in the space (4.3.1), it follows that in each $\theta_{l,j}$ there are at least $k-2$ wedge-entries lying in $H_{\mathbf{e}_{[k]}} \cap H_q$, that is

$$H_{\mathbf{e}_{[k]}} \cap H_q \subset \langle e_1, \dots, \hat{e}_j, e_{k+j}, \dots, e_k \rangle_{\mathbb{C}} \quad , \quad \forall j \in [l] .$$

Since $l \geq 3$, one deduces $H_q = H_{\mathbf{e}_{[k]}}$, which is in contradiction with the condition $1 \leq d([\mathbf{e}_{[k]}], [q]) \leq 2$. We conclude that $\dim(H_{\mathbf{e}_{[k]}} \cap H_q) = k$ must hold, that is $[q] = [\mathbf{e}_{[k]}]$ and the point $[\theta_i]$ is tangential-identifiable. \square

4.4 Dimensions of SL_N -orbits in $\sigma_2(\mathrm{Gr}(k, N))$

This section is devoted to the computation of the dimensions of the SL_N -orbits in the secant variety $\sigma_2(\mathrm{Gr}(k, N))$. Since $k \geq 3$, the secant variety is non-defective and the tangential variety has codimension 1, thus the corresponding dense orbits have dimensions

$$\begin{aligned} \dim \Sigma_k^{k,N} &= \dim \sigma_2(\mathrm{Gr}(k, N)) = 2k(N-k) + 1 , \\ \dim \Theta_k^{k,N} &= \dim \tau(\mathrm{Gr}(k, N)) = 2k(N-k) . \end{aligned}$$

First, we determine the dimensions of the secant orbits: unlike the case of the tangent orbits, the computation for secant orbits does not require information about identifiability.

Proposition 4.4.1. *For $l = 2 : k-1$, the distance- l secant orbit $\Sigma_l^{k,N}$ has dimension*

$$\dim \Sigma_l^{k,N} = \begin{cases} k(N-k) + 2(N-2) - 3 & \text{for } l = 2 \\ k(N-k) + l(N-l) + 1 & \text{for } l \geq 3 . \end{cases}$$

Proof. Fix $l = 2 : k-1$. Consider the fibration

$$\begin{aligned} \xi : \quad \Sigma_l^{k,N} &\longrightarrow \mathrm{Gr}(k-l, N) \\ [p+q] &\mapsto H_p \cap H_q \end{aligned}$$

which is well-defined by Proposition 4.1.5. Define $W := \langle e_{k-l+1}, \dots, e_N \rangle_{\mathbb{C}} \simeq \mathbb{C}^{N-k+l}$. Then, following the same arguments in the proof of Proposition 4.1.11, one gets the fibre

$$\begin{aligned} \xi^{-1}(E_{k-l}) &= \left\{ [p+q] \in \Sigma_l^{k,N} \mid H_p \cap H_q = E_{k-l} \right\} \\ &= \left\{ [\mathbf{e}_{[k-l]} \wedge (a+b)] \mid [a], [b] \in \mathrm{Gr}(l, W), d([a], [b]) = l \right\} \\ &\simeq \Sigma_l^{l,W} \end{aligned}$$

where the last isomorphism is given by restriction of the multiplication map

$$e_{[k-l]} \wedge \bullet : \bigwedge^l W \longrightarrow \bigwedge^k \mathbb{C}^N .$$

In particular, $\Sigma_l^{l,W}$ is the dense orbit in the secant variety $\sigma_2(\text{Gr}(l, W))$, thus

$$\dim \xi^{-1}(E_{k-l}) = \begin{cases} 2 \dim \text{Gr}(2, W) - 3 = 4(N - k) - 3 & \text{for } l = 2 \\ 2 \dim \text{Gr}(l, W) + 1 = 2l(N - k) + 1 & \text{for } l \geq 3 . \end{cases}$$

From the fibre dimension theorem we conclude that

$$\dim \Sigma_l^{k,N} = \dim \text{Gr}(k - l, N) + \dim \xi^{-1}(E_{k-l}) = \begin{cases} k(N - k) + 2(N - 2) - 3 & \text{for } l = 2 \\ k(N - k) + l(N - l) + 1 & \text{for } l \geq 3 . \end{cases}$$

□

Next we determine the dimensions of tangent orbits by using the results from Sec. 4.3. From the equality $\Sigma_2^{k,N} = \Theta_2^{k,N}$ we only consider $l \geq 3$.

Proposition 4.4.2. *For any $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$ and any $3 \leq l \leq k$, the distance- l tangent orbit $\Theta_l^{k,N}$ has dimension*

$$\dim \Theta_l^{k,N} = k(N - k) + l(N - l) .$$

In particular, for any $l \geq 3$ the closure $\overline{\Theta}_l^{k,N}$ has codimension 1 in the closure $\overline{\Sigma}_l^{k,N}$.

Proof. By definition (4.1.5), the orbit $\Theta_l^{k,N}$ exactly corresponds to the set of all rank- l matrices in $T_{[p]} \text{Gr}(k, N) \simeq H_p \otimes (\mathbb{C}^N / H_p) \simeq \mathbb{C}^k \otimes \mathbb{C}^{N-k}$ as $[p] \in \text{Gr}(k, N)$ varies. We recall that the subset of rank- l matrices $[\mathbb{C}^k \otimes \mathbb{C}^{N-k}]_l$ has dimension $l(N - k + k - l) = l(N - l)$.

Finally, since $l \geq 3$, the tangential-identifiability (cf. Proposition 4.3.2) implies that any tangent point lies on a unique tangent space to the Grassmannian, hence $\Theta_l^{k,N} = \text{Gr}(k, N) \times [\mathbb{C}^k \otimes \mathbb{C}^{N-k}]_l$. It follows that

$$\dim \Theta_l^{k,N} = \dim \text{Gr}(k, N) + \dim [\mathbb{C}^k \otimes \mathbb{C}^{N-k}]_l = k(N - k) + l(N - l) .$$

□

4.5 The 2-nd Terracini locus of $\text{Gr}(k, N)$

In the following we determine the *Terracini locus*, introduced in [BC21; BBS20]. Its importance relies in the fact that it gives information on the singularities of points lying on bisecant lines.

DEFINITION 4.5.1. The r -th *Terracini locus* of a variety X is

$$\text{Terr}_r(X) := \overline{\{ \{p_1, \dots, p_r\} \in \text{Hilb}_r(X) \mid \dim \langle T_{p_1} X, \dots, T_{p_r} X \rangle < \dim \sigma_r(X) \}} ,$$

where $\text{Hilb}_r(X)$ is the Hilbert scheme of 0-dimensional subschemes of X of length r (see Sec. 7.3 for details).

From Sec. 2.1 we also recall the definition of the abstract secant variety

$$\text{Ab}\sigma_2(X) := \{(\mathcal{Z}, p) \in \text{Hilb}_2(X) \times \mathbb{P}^m \mid p \in \langle \mathcal{Z} \rangle\} ,$$

together with the two natural projections $\pi_1 : \text{Ab}\sigma_2(X) \rightarrow \text{Hilb}_2(X)$ and $\pi_2 : \text{Ab}\sigma_2(X) \rightarrow \sigma_2(X) \subset \mathbb{P}^m$.

Theorem 4.5.2. *For any $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$, the second Terracini locus $\text{Terr}_2(\text{Gr}(k, N))$ of the Grassmannian $\text{Gr}(k, N)$ corresponds to the distance-2 orbit closure $\overline{\Sigma_2^{k, N}}$. More precisely, in the above notation, it holds*

$$\text{Terr}_2(\text{Gr}(k, N)) = (\pi_1 \circ \pi_2^{-1}) \left(\overline{\Sigma_2^{k, N}} \right) .$$

Proof. Consider a point $[p + q] \in \sigma_2(\text{Gr}(k, N))$ for certain $[p], [q] \in \text{Gr}(k, N)$. We show that $\dim \langle T_{[p]} \text{Gr}(k, N), T_{[q]} \text{Gr}(k, N) \rangle$ drops only for $[p + q] \in \overline{\Sigma_2^{k, N}} = \text{Gr}(k, N) \sqcup \Sigma_2^{k, N}$. As in (4.3.1), it holds

$$T_{[p]} \text{Gr}(k, N) \cap T_{[q]} \text{Gr}(k, N) = \bigwedge^{k-1} (H_p \cap H_q) \wedge V + \bigwedge^{k-2} (H_p \cap H_q) \wedge H_p \wedge H_q .$$

In particular, for $d([p], [q]) \geq 3$ one gets $T_{[p]} \text{Gr}(k, N) \cap T_{[q]} \text{Gr}(k, N) = \{0\}$, hence the dimension of the span does not drop. On the other hand, for $d([p], [q]) \leq 2$ the above intersection has positive dimension and the dimension of the span drops. \square

4.6 The singular locus of $\sigma_2(\text{Gr}(k, N))$

We are now ready to determine the singular locus of the secant variety $\sigma_2(\text{Gr}(k, N))$, proving that it exactly coincides with the distance-2 orbit closure $\overline{\Sigma_2^{k, N}}$. From the previous sections, we recall that we assume $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$.

REMARK. From Remark 4.0.1, we recall that the Grassmannian $\text{Gr}(3, 6)$ has been studied in [LM07] as one of the Legendrian varieties, and it has been proven that the tangential variety $\tau(\text{Gr}(3, 6))$ has singular locus coinciding with $\overline{\Sigma_2^{3, 6}}$ (corresponding to σ_+ in [LM07]).

In the respect of the above remark, from now on we consider $k \geq 3$ and $N \geq 7$: for any $N \geq 7$, the secant variety $\sigma_2(\text{Gr}(k, N))$ does not fill up the ambient space $\mathbb{P}^{\binom{N}{k}-1}$.

In the following we deduce the singularity of the distance-2 orbit $\Sigma_2^{k, N}$ in the secant variety from its tangential-*un*identifiability. First, we recall two general lemmas which are well-known to experts. The first result is a weaker version of classical Terracini's Lemma (cf. [Rus16, Theorem 1.3.1]).

Lemma 4.6.1. *Let $X \subset \mathbb{P}^M$ be an irreducible smooth projective variety. Given any $p, q \in X$ and $p + q \in \sigma_2^\circ(X)$, the following inclusion holds*

$$\langle T_p X, T_q X \rangle \subset T_{p+q} \sigma_2(X) .$$

Proof. By symmetry, it is enough to prove the inclusion $T_p X \subset T_{p+q} \sigma_2(X)$. For any curve $\gamma(t) \subset X$ such that $\gamma(0) = p$ and $\gamma'(0) = v \in T_p X$, the curve $\delta(t) := q + \gamma(t) \subset \sigma_2(X)$ is such that $\delta(0) = q + \gamma(0) = p + q$ and $\delta'(0) = \gamma'(0) = v$, that is $v \in T_{p+q} \sigma_2(X)$. \square

Lemma 4.6.2. *Let $X \subset \mathbb{P}^M$ be an irreducible smooth projective variety. For any $q \in \tau(X)$ and any $p \in X$ such that $q \in T_p X$, it holds*

$$T_p X \subset T_q \sigma_2(X) .$$

Proof. Let $q \in \sigma_2(X)$ and $p \in X$ such that $q \in T_p X$. Given a tangent vector $v \in T_p X$, the line $\gamma(t) := q + t \cdot v$ lies in the tangent space $T_p X \subset \sigma_2(X)$, where the latter inclusion holds by definition of secant variety. Since $\gamma(0) = q$ and $\gamma'(0) = v$, we conclude that $v \in T_q \sigma_2(X)$. \square

Lemma 4.6.3. *For any $3 \leq k \leq \lfloor \frac{N}{2} \rfloor$, the distance-2 orbit is singular in the secant variety, i.e.*

$$\Sigma_2^{k, N} \subset \text{Sing}(\sigma_2(\text{Gr}(k, N))) .$$

Proof. By homogeneity, it is enough to prove the singularity of the representative

$$\mathbf{e}_{[k]} + \mathbb{e}_2 = e_1 \wedge \cdots \wedge e_{k-2} \wedge (e_{k-1} \wedge e_k + e_{k+1} \wedge e_{k+2}) .$$

From Lemma 4.6.1 we know that $\langle T_{\mathbf{e}_{[k]}} \text{Gr}(k, N), T_{\mathbb{e}_2} \text{Gr}(k, N) \rangle \subset T_{\mathbf{e}_{[k]} + \mathbb{e}_2} \sigma_2(\text{Gr}(k, N))$. On the other hand, from Remark 4.3.1 we deduce that the tangential-locus of $\mathbf{e}_{[k]} + \mathbb{e}_2$ contains the points

$$\begin{aligned} p_1 &= e_1 \wedge \cdots \wedge e_{k-2} \wedge e_{k-1} \wedge e_{k+1} , & p_2 &= e_1 \wedge \cdots \wedge e_{k-2} \wedge e_{k-1} \wedge e_{k+2} \\ p_3 &= e_1 \wedge \cdots \wedge e_{k-2} \wedge e_k \wedge e_{k+1} , & p_4 &= e_1 \wedge \cdots \wedge e_{k-2} \wedge e_k \wedge e_{k+2} , \end{aligned}$$

that is $\mathbf{e}_{[k]} + \mathbb{e}_2 \in T_{p_i} \text{Gr}(k, N)$ for any $i \in [4]$, hence from Lemma 4.6.2 we get the inclusions $T_{p_i} \text{Gr}(k, N) \subset T_{\mathbf{e}_{[k]} + \mathbb{e}_2} \sigma_2(\text{Gr}(k, N))$ for any $i \in [4]$. In particular, since $T_{\mathbf{e}_{[k]} + \mathbb{e}_2} \sigma_2(\text{Gr}(k, N))$ is a linear space, it must contain the sum

$$\begin{aligned} & T_{\mathbf{e}_{[k]}} \text{Gr}(k, N) + T_{\mathbb{e}_2} \text{Gr}(k, N) + T_{p_1} \text{Gr}(k, N) + T_{p_2} \text{Gr}(k, N) = \\ &= \left(\bigwedge^{k-1} H_{\mathbf{e}_{[k]}} \wedge V \right) + \left(\bigwedge^{k-1} H_{\mathbb{e}_2} \wedge V \right) + \left(\bigwedge^{k-1} H_{p_1} \wedge V \right) + \left(\bigwedge^{k-1} H_{p_2} \wedge V \right) . \end{aligned} \quad (4.6.1)$$

Given $E_{k-2} := \langle e_1, \dots, e_{k-2} \rangle_{\mathbb{C}}$, for any $p \in \{\mathbf{e}_{[k]}, \mathbb{e}_2, p_1, p_2\}$ one has $E_{k-2} \subset H_p$ and

$$\bigwedge^{k-1} H_p \wedge V = \bigwedge^{k-2} E_{k-2} \wedge \frac{H_p}{E_{k-2}} \wedge \frac{V}{H_p} \oplus \bigwedge^{k-2} E_{k-2} \wedge \bigwedge^2 \frac{H_p}{E_{k-2}} \oplus \bigwedge^{k-3} E_{k-2} \wedge \bigwedge^2 \frac{H_p}{E_{k-2}} \wedge \frac{V}{H_p} ,$$

and an easy computation of the generators (and their repetitions among the four tangent spaces) shows that the sum in (4.6.1) has dimension $(N - k)(4k - 4) - 4k + 6$. If we prove that

$$(N - k)(4k - 4) - 4k + 6 \geq 2(N - k)k + 2 = \dim \sigma_2(\text{Gr}(k, N)) + 1$$

we are done. Notice that the above strict inequality is equivalent to $(N-k)(2k-4)-4(k-1) \geq 0$. Moreover, since $N \geq 2k$, it holds $(N-k)(2k-4)-4(k-1) \geq 2k^2-8k+4$.

Now, for $k \geq 4 \geq 2 + \sqrt{2}$ one gets $2k^2-8k+4 \geq 0$, while for $k=3$ and $N \geq 8$ one has $(N-k)(2k-4)-4(k-1) = 2(N-3)-8 \geq 0$. Finally, for $k=3$ and $N=7$, Lemma 4.6.2 implies that the tangent space $T_{\mathbf{e}_{[k]}+\mathbf{e}_2}\sigma_2(\text{Gr}(3,7))$ must contain the sum

$$T_{\mathbf{e}_{[k]}}\text{Gr}(3,7) + T_{\mathbf{e}_2}\text{Gr}(3,7) + T_{p_1}\text{Gr}(3,7) + T_{p_2}\text{Gr}(3,7) + T_{p_3}\text{Gr}(3,7) + T_{p_4}\text{Gr}(3,7)$$

which, by similar computations as above, has dimension $30 \geq 26 = \dim \sigma_2(\text{Gr}(3,7)) + 1$. In each one of the above cases we conclude that $\mathbf{e}_{[k]} + \mathbf{e}_2$ is singular in $\sigma_2(\text{Gr}(k, N))$, hence the orbit $\Sigma_2^{k,N}$ is so. \square

We are left with proving the smoothness of all the secant and tangent orbits of distance greater or equal than 3.

REMARK 4.6.4. In order to prove the inclusion $\text{Sing}(\sigma_2(\text{Gr}(k, N))) \subset \overline{\Sigma_2^{k,N}}$, it is enough to prove the smoothness for the distance-3 tangent orbit $\Theta_3^{k,N}$, as it is contained in the closure of all the orbits (both secant and tangent) of distance greater or equal than 3.

REMARK 4.6.5. We point out that one can deduce the smoothness of the secant orbits $\Sigma_l^{k,N}$ for $l \geq 3$ also from the information about both the second Terracini locus (Theorem 4.5.2) and identifiability (Theorem 4.2.3). In order to see this, it's more convenient to consider the alternative definition of abstract secant variety (cf. Remark 2.1.4)

$$Ab\sigma_2(\text{Gr}(k, N)) := \overline{\left\{ ([a], [b], [q]) \in \text{Gr}(k, N)_{/\mathbb{C}^2}^2 \times \mathbb{P} \left(\bigwedge^k \mathbb{C}^N \right) \mid [q] \in \langle [a], [b] \rangle \right\}}.$$

Indeed, in light of the identifiability of the orbit $\Sigma_l^{k,N}$, the projection from such abstract secant variety onto the second factor restricts to a bijection $\pi' : \pi_2^{-1}(\Sigma_l^{k,N}) \rightarrow \Sigma_l^{k,N}$. Moreover, the differential of this restriction at a point $([p], [q], [p+q]) \in \pi_2^{-1}(\Sigma_l^{k,N})$ is injective: since $\text{Terr}_2(\text{Gr}(k, N))$ corresponds to $\overline{\Sigma_2^{k,N}}$, the tangent spaces at $[p]$ and $[q]$ do not intersect and the differential maps $T_{[p]}\text{Gr}(k, N) \times T_{[q]}\text{Gr}(k, N)$ to $T_{[p]}\text{Gr}(k, N) \oplus T_{[q]}\text{Gr}(k, N)$, hence it is injective. It follows that the projection π_2 is locally an isomorphism onto $\Sigma_l^{k,N}$, implying the smoothness of the latter orbit.

In the following we prove the smoothness in $\sigma_2(\text{Gr}(k, N))$ of the tangent point

$$q_3 := e_2 \wedge \dots \wedge e_k \wedge e_{k+1} - e_1 \wedge e_3 \wedge \dots \wedge e_k \wedge e_{k+2} + e_1 \wedge e_2 \wedge e_4 \wedge \dots \wedge e_k \wedge e_{k+3} \in T_{\mathbf{e}_{[k]}}\text{Gr}(k, N)$$

lying in the orbit $\Theta_3^{k,N}$ (it differs from the representative θ_3 by a sign). Consider the dual space $(\mathbb{C}^N)^\vee$ with coordinates (x_1, \dots, x_N) such that $x_i(e_j) = \delta_{ij}$. We denote by $\mathcal{I}(q) \subset \bigwedge^\bullet (\mathbb{C}^N)^\vee$ the ideal of a point $q \in \sigma_2(\text{Gr}(k, N))$, by $(\mathcal{I}(q)^2)_k$ the k -th graded component of its squared ideal and by

$$(\mathcal{I}(q)^2)_k^\perp := \left\{ v \in \bigwedge^k \mathbb{C}^N \mid \alpha(v) = 0 \forall \alpha \in (\mathcal{I}(q)^2)_k \right\}$$

the subspace of $\bigwedge^k \mathbb{C}^N$ orthogonal to $(\mathcal{I}(q)^2)_k$.

Lemma 4.6.6. *For any $q \in \sigma_2(\text{Gr}(k, N))$, it holds*

$$T_q \sigma_2(\text{Gr}(k, N)) \subset (\mathcal{I}(q)^2)_k^\perp.$$

Proof. Consider $v \in T_q \sigma_2(\text{Gr}(k, N))$ being the direction of a curve $\gamma(t) \subset \sigma_2(\text{Gr}(k, N))$ passing through $\gamma(0) = q$. As any $\alpha \in (\mathcal{I}(q)^2)_k$ is continuous and linear, being a derivation, one has

$$\alpha(v) = \alpha \left(\lim_{t \rightarrow 0} \frac{\gamma(t) - q}{t} \right) \stackrel{\text{cont}}{=} \lim_{t \rightarrow 0} \frac{\alpha(\gamma(t) - q)}{t} \stackrel{\text{lin}}{=} \frac{d}{dt} \alpha(\gamma(t)) \Big|_{t=0} = \frac{d\alpha}{d\gamma(t)} \Big|_{\gamma(0)=q} \cdot \underbrace{\frac{d\gamma(t)}{dt} \Big|_{t=0}}_{=v}.$$

Since $\alpha \in (\mathcal{I}(q)^2)_k$, we can write $\alpha = \sum_j f_j \wedge g_j$ for some $f_j, g_j \in \mathcal{I}(q)$. Hence

$$\alpha(v) = \sum_j \left[\frac{df_j}{d\gamma(t)} \wedge g_j + f_j \wedge \frac{dg_j}{d\gamma(t)} \right] \Big|_{\gamma(0)=q} \cdot v = \sum_j \left[\underbrace{f_j'}_{=0} \wedge \underbrace{g_j(q)}_{=0} - \underbrace{g_j'}_{=0} \wedge \underbrace{f_j(q)}_{=0} \right] \cdot v = 0.$$

□

The next step is to compute the dimension of $(\mathcal{I}(q_3)^2)_k^\perp$. As it is clear that the multiplication we consider is the wedge product, in the following we lighten up the notation by omitting the wedges: for instance, $x_i x_j$ means $x_i \wedge x_j$. The ideal of q_3 is generated by

$$\begin{aligned} \mathcal{I}(q_3) = & \left(\underbrace{x_{k+4}, \dots, x_N}_{(1)}, \underbrace{x_1 x_{k+1}, x_2 x_{k+2}, x_3, x_{k+3}}_{(2)}, \right. \\ & \underbrace{x_{k+1} x_{k+2}, x_{k+1} x_{k+3}, x_{k+2} x_{k+3}}_{(3)}, \\ & \left. \underbrace{x_2 x_{k+1} + x_1 x_{k+2}, x_3 x_{k+1} + x_1 x_{k+3}, x_2 x_{k+3} + x_3 x_{k+2}}_{(4)} \right) \subset \bigwedge^{\bullet} (\mathbb{C}^N)^\vee, \end{aligned}$$

and a direct computation shows that the generators of the squared ideal $\mathcal{I}(q_3)^2$ are

$$\begin{aligned} (1)^2 & \quad x_i x_j \quad \forall i, j \in \{k+4, \dots, N\} \\ (1)(2) & \quad x_i x_j x_{k+j} \quad \forall i \in \{k+4, \dots, N\}, \quad \forall j \in \{1, 2, 3\} \\ (1)(3) & \quad x_i x_{k+j} x_{k+s} \quad \forall i \in \{k+4, \dots, N\}, \quad \forall j, s \in \{1, 2, 3\} \\ (2)^2 & \quad x_j x_s x_{k+j} x_{k+s} \quad \forall j, s \in \{1, 2, 3\} \\ (2)(3) & \quad x_j x_{k+1} x_{k+2} x_{k+3} \quad \forall j \in \{1, 2, 3\} \\ (1)(4) & \quad x_i (x_2 x_{k+1} + x_1 x_{k+2}), \quad x_i (x_3 x_{k+1} + x_1 x_{k+3}), \quad x_i (x_2 x_{k+3} + x_3 x_{k+2}) \quad \forall i \in \{k+4, \dots, N\} \\ (2)(4) & \quad x_3 x_{k+3} (x_2 x_{k+1} + x_1 x_{k+2}), \quad x_2 x_{k+2} (x_3 x_{k+1} + x_1 x_{k+3}), \quad x_1 x_{k+1} (x_2 x_{k+3} + x_3 x_{k+2}). \end{aligned}$$

Consider the sets of generators $A := \{(1)^2, (1)(2), (1)(3)\}$ and $B := \{A, (2)^2, (2)(3)\}$. In particular, it holds

$$(\mathcal{I}(q_3)^2)_k^\perp = (B)_k^\perp \cap ((1)(4), (2)(4))_k^\perp.$$

First we compute the dimension of $(B)_k^\perp$ and then we compute the linearly independent relations imposed by the generators (1)(4) and (2)(4). Given $[r] := \{1, \dots, r\}$ for any $r \in \mathbb{N}$ and given $s \leq r$, we denote by $\binom{[r]}{s}$ the set of all subsets of $[r]$ having s distinct elements. In the following we distinguish the cases $k = 3$ and $k \geq 4$: the arguments are the same although the computations are slightly different as $(\mathcal{I}(q_3)^2)_3 = (A, (1)(4))_3$.

Proposition 4.6.7. *For any $N \geq 7$, the subspace $(\mathcal{I}(q_3)^2)_3^\perp$ has dimension $6(N - 3) + 2$, i.e.*

$$\dim(\mathcal{I}(q_3)^2)_3^\perp = \dim \sigma_2(\text{Gr}(3, N)) + 1 .$$

Proof. Since $(\mathcal{I}(q_3)^2)_3^\perp = (A, (1)(4))_3^\perp$, first we study $(A)_3^\perp$ and then we cut it by the relations obtained from (1)(4).

Since A is given by monomials, also $(A)_3^\perp$ has to be spanned by monomials of the form $e_{i_1} \wedge e_{i_2} \wedge e_{i_3}$. Let $\mathbf{e}_{[3]} = e_1 \wedge e_2 \wedge e_3$ be the (unique by tangential-identifiability) point of tangency of q_3 . Set $e_i \wedge e_j \wedge e_\ell$ a possible generator of $(B)_3^\perp$, and $d := d(\mathbf{e}_{[3]}, e_i \wedge e_j \wedge e_\ell)$.

- If $d = 3$, then $\{i, j, \ell\} \subset \{4, \dots, N\}$. Since $e_i \wedge e_j \wedge e_\ell$ has to vanish on A , the conditions from (1)² impose that at least two indices lie in $\{4, 5, 6\}$. However, the conditions from (1)(3) imply that there cannot be an index lying in $\{7, \dots, N\}$. Thus the only possibility is that $\{i, j, \ell\} = \{4, 5, 6\}$, leading to a unique generator of $(A)_3^\perp$ having distance 3 from $\mathbf{e}_{[3]}$.
- If $d = 2$, then we may assume $i \in \{1, 2, 3\}$ and $\{j, \ell\} \subset \{4, \dots, N\}$. The relations from (1)² imply that $\{j, \ell\} \not\subset \{7, \dots, N\}$. If $j \in \{4, 5, 6\}$ and $\ell \in \{7, \dots, N\}$, the conditions (1)(2) impose that $j \neq 3 + i$, thus one gets $3 \cdot 2 \cdot (N - 6)$ generators. On the other hand, the case $\{j, \ell\} \subset \{4, 5, 6\}$ leads to other $3 \cdot 3$ generators.
- The case $d = 1$ leads to $3(N - 3)$ generators as from Remark 4.1.7 these elements span the tangent space at $\mathbf{e}_{[3]}$.
- The case $d = 0$ trivially leads to the generator $\mathbf{e}_{[3]}$ itself.

As the above generators are all linearly independent, we get $\dim(A)_3^\perp = 9(N - 3) - 7$.

Next we impose on $(A)_3^\perp$ the equations from (1)(4): since the $3(N - 6)$ elements in (1)(4) impose linearly independent relations on $(A)_3^\perp$ we get

$$\dim(\mathcal{I}(q_3)^2)_3^\perp = \dim(B)_3^\perp - \dim\langle(1)(4)\rangle = 9(N - 3) - 7 - 3(N - 6) = 6(N - 3) + 2 .$$

□

Proposition 4.6.8. *In the above notation, for $k \geq 4$ the dimension of $(B)_k^\perp$ is*

$$\dim(B)_k^\perp = 5 + 3(N - k - 3)(k - 1) + 6(k - 2) + k(N - k) .$$

Proof. Since $(B)_k \subset \bigwedge^k (\mathbb{C}^N)^\vee$ is spanned by monomials of the form $x_{i_1} \wedge \dots \wedge x_{i_k}$, then $(B)_k^\perp$ has to be spanned by monomials of the form $e_{j_1} \wedge \dots \wedge e_{j_k}$. More precisely, if $(B)_k =$

$\langle x_{i_1} \wedge \dots \wedge x_{i_k} \mid (i_1 \leq \dots \leq i_k) \in I \rangle$ for a certain subset of ordered k -tuples $I \subset \binom{[N]}{k}$, then $(B)_k^\perp = \langle e_{j_1} \wedge \dots \wedge e_{j_k} \mid (j_1 \leq \dots \leq j_k) \notin I \rangle$. We recall that $q_3 \in T_{\mathbf{e}_{[k]}} \text{Gr}(k, N) \times_3^{k, N}$ where $\mathbf{e}_{[k]} = e_1 \wedge \dots \wedge e_k$, and that $\mathbf{e}_{[k]}$ is the only point of tangency for q_3 , by tangential-identifiability. We analyze the possible generators $e_{i_1} \wedge \dots \wedge e_{i_k}$ of $(B)_k^\perp$ as the Hamming distance with respect to $\mathbf{e}_{[k]}$ varies. We set $d := d(\mathbf{e}_{[k]}, e_{i_1} \wedge \dots \wedge e_{i_k})$.

- (i) If $d \geq 4$, we may assume $i_1, \dots, i_{k-d} \in [k]$ and $i_{k-d+1}, \dots, i_k \in \{k+1, \dots, N\}$. The conditions imposed by the generators (1)² imply that there cannot be two or more indices in $\{i_{k-d+1}, \dots, i_k\} \cap \{k+4, \dots, N\}$. Moreover, if it was $i_k \in \{k+4, \dots, N\}$ and $i_{k-d+1}, \dots, i_{k-1} \in \{k+1, k+2, k+3\}$, then it would be $d = 4$ leading to contradiction with the conditions imposed by (1)(3). Finally, it cannot be $i_{k-d+1}, \dots, i_k \in \{k+1, k+2, k+3\}$ as it would imply $d = 3$. Thus there are no generators in $(B)_k^\perp$ having distance at least 4 from $\mathbf{e}_{[k]}$.
- (ii) If $d = 3$, we may assume $i_1, \dots, i_{k-3} \in [k]$ and $i_{k-2}, i_{k-1}, i_k \in \{k+1, \dots, N\}$. Again, it has to be $\{i_{k-2}, i_{k-1}, i_k\} \cap \{k+4, \dots, N\} = \emptyset$ because of the conditions from (1)² and (1)(3). On the other hand, for $\{i_{k-2}, i_{k-1}, i_k\} = \{k+1, k+2, k+3\}$ the conditions from (2)² and (2)(3) impose $\{i_1, \dots, i_{k-3}\} = [k] \setminus \{3\}$. Thus there is only one generator in $(B)_k^\perp$ having distance 3 from $\mathbf{e}_{[k]}$, namely $e_4 \wedge \dots \wedge e_k \wedge e_{k+1} \wedge e_{k+2} \wedge e_{k+3}$.
- (iii) If $d = 2$, we may assume $i_1, \dots, i_{k-2} \in [k]$ and $i_{k-1}, i_k \in \{k+1, \dots, N\}$. Similarly to the above cases, the conditions from (1)² impose that $\{i_{k-1}, i_k\} \not\subseteq \{k+4, \dots, N\}$. If $i_k \in \{k+4, \dots, N\}$ and $i_{k-1} \in \{k+1, k+2, k+3\}$, then the conditions from (1)(2) imply $\{i_1, \dots, i_{k-2}\} \subset [k] \setminus \{i_{k-1} - k\}$, leading to $(N - k - 3) \cdot 3 \cdot (k - 1)$ generators. Finally, if $\{i_{k-1}, i_k\} \subset \{k+1, k+2, k+3\}$, then (2)² implies that $\{i_{k-1} - k, i_k - k\} \not\subseteq \{i_1, \dots, i_{k-2}\}$: in particular, the case $\{i_1, \dots, i_{k-2}\} = [k] \setminus \{i_{k-1} - k, i_k - k\}$ leads to $3 \cdot 1$ generators, while the case $|\{i_{k-1} - k, i_k - k\} \cap \{i_1, \dots, i_{k-2}\}| = 1$ leads to $2 \cdot 3 \cdot (k - 2)$ generators.
- (iv) All of the $k(N - k)$ monomials $e_{i_1} \wedge \dots \wedge e_{i_k}$ of distance $d = 1$ from $\mathbf{e}_{[k]}$ are generators, as by Remark 4.1.7 they span $T_{\mathbf{e}_{[k]}} \text{Gr}(k, N)$.
- (v) For $d = 0$ there trivially is the generator $\mathbf{e}_{[k]}$ itself.

Clearly, all of the above generators are linearly independent, hence the thesis follows. \square

Proposition 4.6.9. *For $k \geq 4$, the subspace $(\mathcal{I}(q_3)^2)_k^\perp$ has dimension $2k(N - k) + 2$, that is*

$$\dim(\mathcal{I}(q_3)^2)_k^\perp = \dim \sigma_2(\text{Gr}(k, N)) + 1 .$$

Proof. In order to compute the dimension of $(\mathcal{I}(q_3)^2)_k^\perp = ((1)(4), (2)(4))_k^\perp \cap (B)_k^\perp$, we cut $(B)_k^\perp$ by the relations in $((1)(4), (2)(4))_k$. Thus we determine the generators from $((1)(4), (2)(4))_k$ which are linearly independent from $(B)_k^\perp$. Notice that any generator in (2)(4) multiplied by x_i for $i = k+4, \dots, N$ lies in the ideal generated by (1)(4).

Let us start from the relations in $((2)(4))_k$. By symmetry, we may consider the element

$x_1x_{k+1}(x_2x_{k+3} + x_3x_{k+2}) \in (2)(4)$. The generators in degree k coming from the above element are obtained by multiplying it with monomials $x_{i_1} \wedge \dots \wedge x_{i_{k-4}} \in \bigwedge^{k-4}(\mathbb{C}^N)^\vee$. Clearly, $\{i_1, \dots, i_{k-4}\}$ cannot contain $1, k+1$, otherwise the multiplication goes to zero. On the other hand, it cannot contain $2, 3, k+2, \dots, N$, otherwise the multiplication would give linear combinations of elements of B . Thus the generators in degree k coming from $x_1x_{k+1}(x_2x_{k+3} + x_3x_{k+2})$ are obtained from the $k-3$ monomials with indices $\{i_1, \dots, i_{k-4}\} \in \binom{[k] \setminus [3]}{k-4}$. By symmetry, the same holds for the other two generators in $(2)(4)$. Moreover, a direct computation shows that the generators of $((2)(4))_k$ obtained as above

$$x_1x_{k+1}(x_2x_{k+3} + x_3x_{k+2}) \wedge x_{i_1} \cdots x_{i_{k-4}} \quad \forall \{i_1, \dots, i_{k-4}\} \in \binom{[k] \setminus [3]}{k-4}$$

$$x_2x_{k+2}(x_1x_{k+3} + x_3x_{k+1}) \wedge x_{j_1} \cdots x_{j_{k-4}} \quad \forall \{j_1, \dots, j_{k-4}\} \in \binom{[k] \setminus [3]}{k-4}$$

$$x_3x_{k+3}(x_2x_{k+1} + x_1x_{k+2}) \wedge x_{s_1} \cdots x_{s_{k-4}} \quad \forall \{s_1, \dots, s_{k-4}\} \in \binom{[k] \setminus [3]}{k-4}$$

are linearly independent. It follows that $((2)(4))_k$ imposes $3(k-3)$ conditions on $(B)_k^\perp$.

Finally, we focus on the relations from $((1)(4))_k$. By symmetry, we consider the set of elements $x_i(x_1x_{k+2} + x_2x_{k+1}) \in (1)(4)$ for $i \in \{k+4, \dots, N\}$ and we multiply it with a monomial $x_{i_1} \wedge \dots \wedge x_{i_{k-3}}$. Similarly to the previous argument, the set of indices $\{i_1, \dots, i_{k-3}\}$ cannot contain $1, 2, k+1, \dots, N$ otherwise we would get linear combinations of elements of B . However, in this case one can have $3 \in \{i_1, \dots, i_{k-3}\}$, thus we get $(N-k-3)(k-2)$ generators of $((1)(4))_k$ from $\{x_i(x_1x_{k+2} + x_2x_{k+1}) \mid i = k+4 : N\}$ by multiplying it with the $k-2$ monomials indexed by $\{i_1, \dots, i_{k-3}\} \subset \{3, \dots, k\}$. Analogously, the remaining sets of elements $\{x_j(x_1x_{k+3} + x_3x_{k+1}) \mid j = k+4 : N\}$ and $\{x_s(x_3x_{k+2} + x_2x_{k+3}) \mid s = k+4 : N\}$ in $(1)(4)$ give $(N-k-3)(k-2)$ generators each, after multiplying with monomials indexed by $\{j_1, \dots, j_{k-3}\} \subset \{2, 4, \dots, k\}$ and $\{s_1, \dots, s_{k-3}\} \subset \{1, 4, \dots, k\}$ respectively. Again, one has to check possible linear combinations among the above $3(N-k-3)(k-2)$ generators: a direct computation shows that the only possible linear combinations are of the form

$$\begin{aligned} x_i(x_1x_{k+2} + x_2x_{k+1}) \wedge x_3x_{i_2} \cdots x_{i_{k-3}} + x_i(x_1x_{k+3} + x_3x_{k+1}) \wedge x_2x_{i_2} \cdots x_{i_{k-3}} + \\ + x_i(x_3x_{k+2} + x_2x_{k+3}) \wedge x_1x_{i_2} \cdots x_{i_{k-3}} = 0 \end{aligned}$$

as $i \in \{k+4, \dots, N\}$ and $\{i_2, \dots, i_{k-3}\} \in \binom{[k] \setminus [3]}{k-4}$ vary. It follows that $((1)(4))_k$ imposes $3(N-k-3)(k-2) - (N-k-3)(k-3)$ conditions on $(B)_k^\perp$.

We conclude that the dimension of $(\mathcal{I}(q_3)^2)_k^\perp$ is

$$\begin{aligned} \dim(\mathcal{I}(q_3)^2)_k^\perp &= \dim\left((B)_k^\perp \cap ((1)(4), (2)(4))_k^\perp\right) \\ &= [5 + 3(N - k - 3)(k - 1) + 6(k - 2) + k(N - k)] - 3(k - 3) \\ &\quad - [3(N - k - 3)(k - 2) - (N - k - 3)(k - 3)] \\ &= 2k(N - k) + 2. \end{aligned}$$

□

Theorem 4.6.10. *For any $3 \leq k \leq \frac{N}{2}$ and $N \geq 7$, the singular locus of the secant variety of lines to the Grassmannian $\text{Gr}(k, N)$ coincides with the closure of the distance-2 orbit, i.e.*

$$\text{Sing}(\sigma_2(\text{Gr}(k, N))) = \overline{\Sigma_2^{k, N}}.$$

Proof. From Lemma 4.6.3 we already know that the inclusion $\overline{\Sigma_2^{k, N}} \subset \text{Sing}(\sigma_2(\text{Gr}(k, N)))$ holds. From Remark 4.6.4 it is enough to prove the smoothness of $\Theta_3^{k, N}$ for deducing the smoothness of all the remaining orbits. Moreover, by homogeneity it is enough to check the smoothness of a representative of $\Theta_3^{k, N}$, say q_3 . Finally, from Lemma 4.6.6, Proposition 4.6.7 (for $k = 3$) and Proposition 4.6.9 (for $k \geq 4$) we get the chain of inequalities

$$2k(N - k) + 2 \leq \dim T_{q_3} \sigma_2(\text{Gr}(k, N)) \leq \dim(\mathcal{I}(q_3)^2)_k^\perp = 2k(N - k) + 2,$$

leading to $\dim T_{q_3} \sigma_2(\text{Gr}(k, N)) = \dim \sigma_2(\text{Gr}(k, N)) + 1$, hence the point q_3 is smooth in the secant variety. □

REMARK. Theorem 4.6.10 corrects a previous statement in [AOP12, before Figure 1] in which the authors states that $\text{Sing}(\sigma_2(\text{Gr}(3, 7))) = \text{Gr}(3, 7)$.

Chapter 5

Identifiability and singular locus of $\sigma_2(\mathbb{S}_N^\pm)$

Similarly to the Grassmannian case, only partial results appear on the secant variety of lines to Spinor varieties (eg. see [MM15; Man09]). In this chapter we solve the identifiability problem and we determine upper and lower bounds for the singular locus of the secant varieties of lines to Grassmannians. The results appearing in this chapter are collected in the preprint [Gal23].

In light of Theorem 3.3.1 we work in the D_N -type setting for N even. In particular, we assume notation in Table 3.1. We consider a $2N$ -dimensional complex vector space $V = E \oplus E^\vee$ endowed with the quadratic form $q = \sum_{i=1}^N x_i x_{N+i}$ and we fix the q -hyperbolic basis $(e_1, \dots, e_N, f_1, \dots, f_N)$. We focus on the Spinor variety

$$\mathbb{S}_N^+ = D_N/P_{\omega_N} \subset \mathbb{P}(V_{\omega_N}^{D_N}) = \mathbb{P}\left(\bigwedge^{ev} E\right).$$

We recall that, under these assumptions, the highest weight vector and the lowest weight vector are respectively $v_{\omega_N} = \mathbf{e}_{[N]} = e_1 \wedge \dots \wedge e_N$ and $\ell_{\omega_N} = \mathbf{1}$, with corresponding maximal q -isotropic subspaces $E = H_{\mathbf{e}_{[N]}}$ and $E^\vee = H_{\mathbf{1}}$.

Secant variety of lines to \mathbb{S}_N^+ . Consider the secant variety of lines of \mathbb{S}_N^+

$$\sigma_2(\mathbb{S}_N^+) = \overline{\sigma_2^{\circ}(\mathbb{S}_N^+)} = \overline{\{[a+b] \in \mathbb{P}(V_{\omega_N}^{D_N}) \mid [a], [b] \in \mathbb{S}_N^+\}} \subset \mathbb{P}(V_{\omega_N}^{D_N}).$$

A dense subset is $\sigma_2^{\circ}(\mathbb{S}_N^+)$, given by the union of bisecant lines, while the union of tangent lines defines the tangential variety $\tau(\mathbb{S}_N^+)$. It is known that the secant variety of lines is quasi-homogeneous, in the sense that it admits a dense orbit: more precisely, $\sigma_2(\mathbb{S}_N^+) = \overline{\text{Spin}_{2N} \cdot [v_{\omega_N} + \ell_{\omega_N}]}$ [Zak93, Theorem 1.4]. Moreover, it is *non-defective* for any N [Kaj99;

Ang11], thus its dimension is the expected one:

$$\dim \sigma_2(\mathbb{S}_N^\pm) = \min \{2 \dim \mathbb{S}_N^\pm + 1, \dim \mathbb{P}(V_{\omega_N}^{D_N})\} = \begin{cases} 2^{N-1} - 1 & \text{for } N \leq 6 \\ N(N-1) + 1 & \text{for } N \geq 6 \end{cases}.$$

We list the dimensions of Spinor varieties and their secant varieties for $N \leq 8$.

Spin_{2N}	\mathbb{S}_N^\pm	$\dim \mathbb{S}_N^\pm$	$\dim \sigma_2(\mathbb{S}_N^\pm)$	$\mathbb{P}(V_{\omega_N}^{D_N})$
Spin_4	$\mathbb{S}_2^+ \simeq \mathbb{P}^1$	1	1	$\mathbb{P}(\wedge^{ev} \mathbb{C}^2) \simeq \mathbb{P}^1$
Spin_6	$\mathbb{S}_3^- \simeq \mathbb{P}^3$	3	3	$\mathbb{P}(\wedge^{od} \mathbb{C}^3) \simeq \mathbb{P}^3$
Spin_8	$\mathbb{S}_4^+ \simeq \mathbb{Q}^6$	6	7	$\mathbb{P}(\wedge^{ev} \mathbb{C}^4) \simeq \mathbb{P}^7$
Spin_{10}	\mathbb{S}_5^-	10	15	$\mathbb{P}(\wedge^{od} \mathbb{C}^5) \simeq \mathbb{P}^{15}$
Spin_{12}	\mathbb{S}_6^+	15	31	$\mathbb{P}(\wedge^{ev} \mathbb{C}^6) \simeq \mathbb{P}^{31}$
Spin_{14}	\mathbb{S}_7^-	21	43	$\mathbb{P}(\wedge^{od} \mathbb{C}^7) \simeq \mathbb{P}^{63}$
Spin_{16}	\mathbb{S}_8^+	28	57	$\mathbb{P}(\wedge^{ev} \mathbb{C}^8) \simeq \mathbb{P}^{127}$

Table 5.1: Spinor varieties in low dimensions, and their secants of lines.

REMARK. In same terminology from Sec. 2.1, for $N \leq 5$ the secant variety $\sigma_2(\mathbb{S}_N^\pm)$ *overfills* the ambient space $\mathbb{P}^{2^{N-1}-1}$, for $N \geq 7$ it is strictly contained, while for $N = 6$ we are in the *perfect* case. We recall that the Spinor variety \mathbb{S}_6^\pm is a Legendrian variety and, as such, its secant variety of lines has been studied by J.M. Landsberg and L. Manivel (cf. Remark 1.4.2).

5.1 The poset of Spin_{2N} -orbits in $\sigma_2(\mathbb{S}_N^\pm)$

The secant variety of lines $\sigma_2(\mathbb{S}_N^\pm)$ is invariant under Spin_{2N} -action, as well as its subsets $\sigma_2^\circ(\mathbb{S}_N^\pm)$ and $\tau(\mathbb{S}_N^\pm)$. We refer to the orbits lying in $\sigma_2^\circ(\mathbb{S}_N^\pm)$ as *secant orbits*, and to the orbits lying in $\tau(\mathbb{S}_N^\pm)$ as *tangent orbits*. Recall that the Spinor variety has diameter $\text{diam}(\mathbb{S}_N^\pm) = \frac{N}{2}$ (cf. Theorem 3.4.3). Moreover, for any $l = 1 : (\frac{N}{2} - 1)$ we consider the following pure spinors and their corresponding maximal q -isotropic subspaces

$$\mathbf{e}_{[N-2l]} = e_1 \wedge \dots \wedge e_{N-2l} \in \mathbb{S}_N^+, \quad (5.1.1)$$

$$E_{N-2l} := H_{\mathbf{e}_{[N-2l]}} = \langle e_1, \dots, e_{N-2l}, f_{N-2l+1}, \dots, f_N \rangle_{\mathbb{C}} \in \text{OG}^+(N, V).$$

By convention, for $l = \frac{N}{2}$ we set $\mathbf{e}_{[0]} = \ell_{\omega_N} = \mathbb{1}$. On the other hand, for $l = 0$ one gets $\mathbf{e}_{[N]} = v_{\omega_N}$.

Spin $_{2N}$ -orbits in $\mathbb{S}_N^+ \times \mathbb{S}_N^+$. The action of Spin_{2N} on \mathbb{S}_N^+ naturally induces the action on the direct product $\mathbb{S}_N^+ \times \mathbb{S}_N^+$ given by $g \cdot ([a], [b]) = ([g \cdot a], [g \cdot b])$. We recall that the notation $g \cdot a$ stands for the action of $\text{Spin}^q(V) \subset (\text{Cl}_q(V))^\times$ on $\wedge E$ via Clifford multiplication (3.1.6).

REMARK 5.1.1. Via the bijection (3.3.3), the action on $\mathbb{S}_N^+ \times \mathbb{S}_N^+$ is equivalent to the action of $\text{Spin}^q(V)$ on $\text{OG}^+(N, V)^{\times 2}$ given by $g \cdot (H_a, H_b) = (g \cdot H_a, g \cdot H_b)$, where $g \cdot H_a = gH_ag^{-1}$ is the *conjugacy* action: this follows from the inclusion $\text{Spin}^q(V) \subset N_{\text{Cl}_q(V) \times (V)}$.

From Remark 2.3.2, the Hamming distance in \mathbb{S}_N^+ is invariant under $\text{Spin}^q(V)$ -action, as well as the dimensions of subspaces in V are preserved under conjugacy. In particular, the actions of $\text{Spin}^q(V)$ on $\mathbb{S}_N^+ \times \mathbb{S}_N^+$ and on $\text{OG}^+(N, V)^{\times 2}$ restrict to actions on the subsets

$$\mathcal{O}_{l,N} := \{([a], [b]) \in \mathbb{S}_N^+ \times \mathbb{S}_N^+ \mid d([a], [b]) = l\} \quad (5.1.2)$$

$$\widehat{\mathcal{O}}_{l,N} := \{(H_a, H_b) \in \text{OG}^+(N, V)^{\times 2} \mid \dim(H_a \cap H_b) = N - 2l\} \quad (5.1.3)$$

respectively, for any $l = 0 : \frac{N}{2}$ where $\frac{N}{2} = \text{diam}(\mathbb{S}_N^+)$. Since for $l = 0$ one gets the diagonals $\Delta_{\mathbb{S}_N^+}$ and $\Delta_{\text{OG}^+(N, V)}$ we consider $l \geq 1$. From Proposition 3.4.2, for any distance $l \geq 1$, the subsets $\mathcal{O}_{l,N}$ and $\widehat{\mathcal{O}}_{l,N}$ are equivalent one to the other, and they give partitions of the corresponding direct products (as sets). Since $\dim(H_{\mathbf{e}_{[N]}} \cap H_{\mathbf{e}_l}) = \dim(E \cap E_l) = N - 2l$, for any $l = 1 : \frac{N}{2}$ one gets the inclusion $\text{Spin}^q(V) \cdot ([\mathbf{e}_{[N]}], [\mathbf{e}_{[N-2l]}) \subset \mathcal{O}_{l,N}$.

REMARK. It is likely well-known that the Spin_{2N} -orbit partition of $\mathbb{Q}^{2N-2} \times \mathbb{Q}^{2N-2}$ is uniquely determined by the Hamming distance: in particular,

$$\mathbb{Q}^{2N-2} \times \mathbb{Q}^{2N-2} = \Delta_{\mathbb{Q}^{2N-2}} \sqcup \text{Spin}_{2N} \cdot ([e_1], [e_2]) \sqcup \text{Spin}_{2N} \cdot ([e_1], [f_1]) .$$

However, we haven't been able to find a proper citation, hence we prove this result in Sec. 7.2.

Proposition 5.1.2. *For any $l = 1 : \text{diam}(\mathbb{S}_N^+)$, the spin group $\text{Spin}^q(V)$ acts transitively on $\mathcal{O}_{l,N} \subset \mathbb{S}_N^+ \times \mathbb{S}_N^+$. In particular, it holds*

$$\mathcal{O}_{l,N} = \text{Spin}^q(V) \cdot ([\mathbf{e}_{[N]}], [\mathbf{e}_{[N-2l]}) .$$

Proof. Given $([a], [b]) \in \mathcal{O}_{l,N}$, by homogeneity of \mathbb{S}_N^+ we may assume $a = \mathbf{e}_{[N]}$.

First we prove the result for $l = \frac{N}{2}$. Since $d([\mathbf{e}_{[N]}], [b]) = \frac{N}{2}$, it holds $E \cap H_b = \{0\}$ and we may assume $H_b = \langle g_1, \dots, g_N \rangle_{\mathbb{C}}$ for generators g_j as in (3.3.5). In light of Theorem 7.2.1 $\text{Spin}^q(V)$ conjugates the q -isotropic vector $g_1 = f_1 + \sum_{i=2}^N \alpha_{i1} e_i$ (having Hamming distance 2 from e_1) to f_1 by leaving e_1 fixed. Now, consider $V' = \langle e_2, \dots, e_N, f_2, \dots, f_N \rangle_{\mathbb{C}}$ and the subspaces $E' = E \cap V'$ and $H'_b = H_b / \langle e_1, f_1 \rangle_{\mathbb{C}}$: again $\text{Spin}(V')$ conjugates $g_2 - \alpha_{12} e_1 \in H'_b$ to f_2 by leaving e_2 fixed. In particular, $\text{Spin}^q(V)$ conjugates $g_2 \in H_b$ to f_2 by leaving e_1, f_1, e_2 fixed. By iterating, $\text{Spin}^q(V)$ conjugates H_b to E^\vee , hence $([\mathbf{e}_{[N]}], [b])$ to $(\mathbf{e}_{[N]}, [\mathbb{1}])$.

On the other hand, for $l \leq \frac{N}{2}$ one has $H_b = \langle h_1, \dots, h_{N-2l}, g_{N-2l+1}, \dots, g_N \rangle_{\mathbb{C}}$ for g_j 's as in (3.3.5) and $E \cap H_b = \langle h_1, \dots, h_{N-2l} \rangle_{\mathbb{C}}$. Up to reordering h_1, \dots, h_{N-2l} , one gets that h_j has Hamming distance 1 from e_j for any $j = 1 : N - 2l$, hence by applying Theorem 7.2.1 and similar arguments as above one can conjugates $E \cap H_b$ to E_{N-2l} . Finally, by working in $W := V/E_{N-2l}$ one can conjugates E/E_{N-2l} to H_b/E_{N-2l} via $\text{Spin}(W)$ and lifting this to a conjugation under $\text{Spin}^q(V)$ leaving E_{N-2l} fixed. The thesis follows. \square

Corollary 5.1.3. *The product $\mathbb{S}_N^+ \times \mathbb{S}_N^+$ splits in the Spin_{2N} -orbits*

$$\mathbb{S}_N^+ \times \mathbb{S}_N^+ = \Delta_{\mathbb{S}_N^+} \sqcup \bigsqcup_{l=1}^{\frac{N}{2}} \text{Spin}_{2N} \cdot ([\mathbf{e}_{[N]}], [\mathbf{e}_{[N-2l]}]) .$$

5.1.1 Secant orbits in $\sigma_2^\circ(\mathbb{S}_N^+)$

We deduce the orbit partition of the dense subset $\sigma_2^\circ(\mathbb{S}_N^+)$ from Proposition 5.1.2. We recall that each spinor $[a+b] \in \mathbb{P}(\bigwedge^{ev} E)$ defines, via the map (3.3.1), a q -isotropic subspace $H_{a+b} = \ker(\psi_{a+b}) \subset \mathbb{Q}^{2N-2}$, which has maximal dimension N if and only if $[a+b]$ is pure. By definition of these subspaces as annihilators, it clearly holds

$$H_a \cap H_b \subseteq H_{a+b} .$$

For any two distinct pure spinors $[a], [b] \in \mathbb{S}_N^+$ such that $d([a], [b]) = 1$, the spinor $[a+b]$ is pure too, since the line $L([a], [b])$ fully lies in the Spinor variety: in particular, in this case one has $\dim(H_a \cap H_b) \stackrel{(3.4.2)}{=} N-2$ while $\dim H_{a+b} = N$, thus the strict inclusion $H_a \cap H_b \subsetneq H_{a+b}$ holds. However, for higher Hamming distances the equality holds.

Lemma 5.1.4. *In the above notation, the following holds:*

$$H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}} = E \cap E_{N-2l} \iff l \neq 1 .$$

In particular, for any $l \geq 2$, it holds $\dim H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}} = N - 2l$.

Proof. We already know that $\langle e_1, \dots, e_{N-2l} \rangle_{\mathbb{C}} = E \cap E_{N-2l} \subset H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}}$ holds. Assume that there exists $v \in H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}}$ such that $v \notin (E \cap E_{N-2l}) \setminus \{0\}$. Thus, since $H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}}$ is q -isotropic, up to linear combinations we can consider a decomposition of v with respect to the standard hyperbolic basis $(e_1, \dots, e_N, f_1, \dots, f_N)$ of the form $v = \alpha_1 e_{N-2l+1} + \dots + \alpha_{2l} e_N + \beta_1 f_{N-2l+1} + \dots + \beta_{2l} f_N$ for some $\alpha_i, \beta_j \in \mathbb{C}$. Since $H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}} = \ker(\psi_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}})$, it holds

$$\begin{aligned} 0 &= v \cdot (\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}) \\ &\stackrel{(3.1.6)}{=} \sum_{i=1}^{2l} \alpha_i (e_1 \wedge \dots \wedge e_{N-2l} \wedge e_{N-2l+i}) + \sum_{j=1}^{2l} (-1)^{j+1} \beta_j (e_1 \wedge \dots \wedge \hat{e}_{N-2l+j} \wedge \dots \wedge e_N) , \end{aligned}$$

where \hat{e}_{N-2l+j} denotes that the vector is missing in the wedge product.

Now, the summands above may simplify one to each other if and only if $l = 1$, otherwise they do not since they are all independent vectors in the standard basis of $\bigwedge E$. It follows that, for $l \geq 2$, it holds $v \cdot (\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}) = 0$ if and only if $\alpha_i = \beta_j = 0$ for all $i, j \in [2l]$, that is $H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}} = E \cap E_l$. On the other hand, for $l = 1$, the conditions $\alpha_1 = \beta_2$ and $\alpha_2 = -\beta_1$ give a non-zero vector $0 \neq v \in H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}} \setminus (E \cap E_l)$, thus $E \cap E_1 \subsetneq H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}} = \langle e_1, \dots, e_{N-2}, e_{N-1} + f_N, e_N - f_{N-1} \rangle_{\mathbb{C}}$. \square

Corollary 5.1.5. *For any $[a+b] \in \sigma_2^\circ(\mathbb{S}_N^+) \setminus \mathbb{S}_N^+$, the equality $H_{a+b} = H_a \cap H_b$ holds.*

Proof. Let $[a + b] = [c + d] \in \sigma^\circ(\mathbb{S}_N^+) \setminus \mathbb{S}_N^+$ be such that $d([a], [b]) = l$ and $d([c], [d]) = m$ for certain $2 \leq l, m \leq \frac{N}{2}$. By Proposition 5.1.2 there exists $g \in \text{Spin}^q(V)$ such that $g \cdot ([a], [b]) = ([\mathbf{e}_{[N]}], [\mathbf{e}_{[N-2l]}])$ and $g \cdot ([c], [d]) = ([c'], [d'])$. In particular, $[\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}] = g \cdot [a + b] = g \cdot [c + d] = [c' + d']$, and by Proposition 5.1.4 we get $H_{c'} \cap H_{d'} \subset H_{c'+d'} = H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}} = E \cap E_{N-2l}$, where the last equality follows from Lemma 5.1.4 since $l \geq 2$. Dimensionally we have $N - 2m = \dim(H_{c'} \cap H_{d'}) \leq \dim(E \cap E_{N-2l}) = N - 2l$. But, by symmetry, one also gets $N - 2l = \dim(H_{a'} \cap H_{b'}) \leq \dim(E \cap E_{N-2m}) = N - 2m$, thus $l = m$ and the thesis follows. \square

Corollary 5.1.5 allows to define for any $l = 2 : \frac{N}{2}$ the subset

$$\begin{aligned} \Sigma_{l,N} &:= \{[a + b] \in \sigma_2^\circ(\mathbb{S}_N^+) \mid \dim H_{a+b} = N - 2l\} \\ &= \{[a + b] \in \sigma_2^\circ(\mathbb{S}_N^+) \mid d([a], [b]) = l\} . \end{aligned} \quad (5.1.4)$$

Moreover, we set $\Sigma_{1,N} := \mathbb{S}_N^+$. Our claim is that the subsets $\Sigma_{l,N}$ are exactly the $\text{Spin}^q(V)$ -orbits in $\sigma_2^\circ(\mathbb{S}_N^+)$.

The action of $\text{Spin}^q(V)$ on $\sigma_2(\mathbb{S}_N^+)$ preserves the subsets $\Sigma_{l,N}$, as by Remark 5.1.1 the spin group acts on V and its subspaces by conjugacy. Moreover, by Proposition 5.1.2 any two pairs $([a], [b])$ and $([c], [d])$ of Hamming distance l are conjugated, hence their lines $L([a], [b])$ and $L([c], [d])$ are so. Finally, the following result proves that $\text{Spin}^q(V)$ acts transitively on points on a same line $L([a], [b]) \setminus \{[a], [b]\}$ too.

Lemma 5.1.6. *For any two distinct pure spinors $[a], [b] \in \mathbb{S}_N^+$, the spin group $\text{Spin}^q(V)$ acts transitively on $L([a], [b]) \setminus \{[a], [b]\}$.*

Proof. Since the lines defined by pairs of pure spinors having the same Hamming distance are all conjugated, it is enough to prove the transitivity on the line $L([\mathbf{e}_{[N]}], [\mathbf{e}_{[N-2l]}])$. Moreover, given a point $[\lambda \mathbf{e}_{[N]} + \mu \mathbf{e}_{[N-2l]}] = [\mathbf{e}_{[N]} + z \mathbf{e}_{[N-2l]}] \in L([\mathbf{e}_{[N]}], [\mathbf{e}_{[N-2l]}])$, we can rewrite it as $\mathbf{e}_{[N]} + z \mathbf{e}_{[N-2l]} = e_1 \wedge \dots \wedge e_{N-2l} \wedge (e_{N-2l+1} \wedge \dots \wedge e_N + z \mathbb{1})$. Since $[e_{N-2l+1} \wedge \dots \wedge e_N]$ and $[z \mathbb{1}]$ are pure spinors in \mathbb{S}_{2l}^+ , we can restrict to consider the line $L([\mathbf{e}_{[N]}], [\mathbb{1}])$.

Given $\mathbf{e}_{[N]} + z \mathbb{1}$, we look for a spin element $g \in \text{Spin}^q(V)$ such that $\mathbf{e}_{[N]} + z \mathbb{1} = k(\mathbf{e}_{[N]} + \mathbb{1})$ for some $k \in \mathbb{C}^\times$. We consider the element $\tilde{g} = (a_1 e_1 + b_1 f_1) \cdots (a_N e_N + b_N f_N) \in \text{Cl}_q^+(V)$ for certain $a_i, b_i \in \mathbb{C}^\times$: it is product of an even number of vectors in V by the assumption as N is even. Then $\tilde{g} \in \text{Spin}^q(V)$ if and only if it is invertible and it has unitary spinor norm, that is

$$\tilde{g} \in \text{Spin}^q(V) \iff \prod_{i=1}^N a_i b_i \stackrel{\clubsuit}{=} 1 .$$

Via Clifford multiplication (3.1.6) it holds $(a_1 e_1 + b_1 f_1) \cdot (\mathbf{e}_{[N]} + z \mathbb{1}) = a_1 z e_1 + b_1 e_2 \wedge \dots \wedge e_N$, and by iterating for $i = 1 : N$ one gets $\tilde{g} \cdot (\mathbf{e}_{[N]} + z \mathbb{1}) = a_1 \cdots a_N z \mathbf{e}_{[N]} + b_1 \cdots b_N \mathbb{1}$. In particular, the second required condition is

$$\tilde{g} \cdot (\mathbf{e}_{[N]} + z \mathbb{1}) = k(\mathbf{e}_{[N]} + \mathbb{1}) \iff z \prod_{i=1}^N a_i \stackrel{\spadesuit}{=} \prod_{i=1}^N b_i .$$

By putting together the conditions (\clubsuit) and (\spadesuit), it is straightforward that for the choice $a_1 = \dots = a_N = \sqrt[2N]{z^{-1}}$ and $b_1 = \dots = b_N = \sqrt[2N]{z}$ one gets $\tilde{g} \cdot (\mathbf{e}_{[N]} + z \mathbb{1}) = \sqrt{z}(\mathbf{e}_{[N]} + \mathbb{1})$. \square

It follows that for any $l = 1 : \frac{N}{2}$ the subset $\Sigma_{l,N}$ is a $\text{Spin}^q(V)$ -orbit in $\sigma_2^\circ(\mathbb{S}_N^+)$. Moreover, by Proposition 5.1.2 we deduce that for any l it holds $\Sigma_{l,N} = \text{Spin}(V) \cdot [\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}]$.

In conclusion, we have proved the following theorem.

Theorem 5.1.7. *The dense set $\sigma_2^\circ(\mathbb{S}_N^+)$ splits under the action of Spin_{2N} in the orbits*

$$\sigma_2^\circ(\mathbb{S}_N^+) = \bigsqcup_{l=1}^{\frac{N}{2}} \text{Spin}_{2N} \cdot [\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}] .$$

5.1.2 Tangent orbits in $\tau(\mathbb{S}_N^+)$

From the non-defectivity of $\sigma_2(\mathbb{S}_N^+)$ and the dicotomy between tangential and secant varieties [FH79, Corollary 4], we know that $\tau(\mathbb{S}_4^+) = \sigma_2(\mathbb{S}_4^+) = \mathbb{P}^7$ and $\tau(\mathbb{S}_5^+) = \sigma_2(\mathbb{S}_5^+) = \mathbb{P}^{15}$, while for $N \geq 6$ the tangential variety $\tau(\mathbb{S}_N^+)$ is a divisor in $\sigma_2(\mathbb{S}_N^+) \subset \mathbb{P}^{2^N-2-1}$. We deduce the orbit partition of $\tau(\mathbb{S}_N^+)$ from the tangent bundle on the Spinor variety.

Tangent bundle. Let $\mathcal{T}_{\mathbb{S}_N^+}$ be the tangent bundle on the Spinor variety \mathbb{S}_N^+ . Under the identification $\mathbb{S}_N^+ = \text{Spin}_{2N}/P$, the base point $[P] \in D_N/P$ corresponds to the pure spinor $[v_{\omega_N}] = [\mathbf{e}_{[N]}]$, hence to the maximal q -isotropic subspace $E \in \text{OG}^+(N, V)$, and

$$P = \text{stab}(E) \simeq \left\{ \begin{bmatrix} A & B \\ & A^{-1} \end{bmatrix} \mid A \in \text{SL}(E), B \in \bigwedge^2 \mathbb{C}^N \right\} .$$

It follows that the fiber of the tangent bundle at the base point $[P]$ is $(\mathcal{T}_{\mathbb{S}_N^+})_{[P]} \simeq \mathfrak{p}^u \simeq \bigwedge^2 \mathbb{C}^N$. From the parametrization of Spinor varieties, one can describe the fiber at any pure spinor $[a] \in \mathbb{S}_N^+$ as

$$(\mathcal{T}_{\mathbb{S}_N^+})_{[a]} = T_{[a]}\mathbb{S}_N^+ \simeq \bigwedge^2 H_a .$$

In particular, this leads to the isomorphism of homogeneous bundles

$$\mathcal{T}_{\mathbb{S}_N^+} \simeq \bigwedge^2 \mathcal{U}^\vee , \quad (5.1.5)$$

where \mathcal{U} is the rank- N universal bundle on \mathbb{S}_N^+ obtained by pulling back the universal bundle on the Grassmannian $\text{Gr}(N, V)$ (cf. Example 1.3.7). Notice that the fiber at the point $[v_{\omega_N}] \simeq E$ is $T_{[v_{\omega_N}]} \mathbb{S}_N^+ \simeq \bigwedge^2 E$ and it is an irreducible $\text{SL}(E)$ -module with highest weight ω_2 : as such the P -orbits in the tangent space are uniquely determined by the rank of skew-symmetric matrices.

Tangent orbits. By homogeneity of \mathbb{S}_N^+ , all tangent spaces are conjugated one to each other by transformations in $\text{Spin}_{2N} \setminus P$. Thus the Spin_{2N} -orbits of points in the tangential variety $\tau(\mathbb{S}_N^+)$ are in bijection with the P -orbits in the tangent space $T_{[v_{\omega_N}]} \mathbb{S}_N^+$, which are parametrized by the possible ranks in $\bigwedge^2 \mathbb{C}^N$.

Let $[T_{[a]}\mathbb{S}_N^+]_{2l}$ be the set of tangent points to \mathbb{S}_N^+ at the pure spinor $[a]$ corresponding to

skew-symmetric matrices of size N and rank $2l$. Then for any $l = 1 : \frac{N}{2}$ we denote the set of all tangent points corresponding to rank- $2l$ skew-symmetric matrices by

$$\Theta_{l,N} := \{[q] \in \tau(\mathbb{S}_N^+) \mid \exists [a] \in \mathbb{S}_N^+ : [q] \in [T_{[a]}\mathbb{S}_N^+]_{2l}\} . \quad (5.1.6)$$

The above arguments ensure that each subset $\Theta_{l,N}$ is indeed a $\text{Spin}^q(V)$ -orbit, and all together they give the $\text{Spin}^q(V)$ -orbit partition of $\tau(\mathbb{S}_N^+)$.

Finally, for any $l = 1 : \frac{N}{2}$ the tangent orbit $\Theta_{l,N}$ admits as representative the spinor

$$[q_l] := \left[\sum_{i=1}^l e_{2i-1} \wedge e_{2i} \right] . \quad (5.1.7)$$

Indeed, the curve of rank- $2l$ skew-symmetric matrices of size N

$$tC_l = t \begin{bmatrix} P_1 & & & \\ & \ddots & & \\ & & P_l & \\ & & & 0_{N-2l} \end{bmatrix} \quad \text{where} \quad P_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (5.1.8)$$

defines the curve of maximal q -isotropic subspaces

$$H_{c(t)} = \left\langle f_1 + t \sum_{k=1}^N c_{k1} e_k, \dots, f_N + t \sum_{k=1}^N c_{kN} e_k \right\rangle_{\mathbb{C}} \in \text{OG}^+(N, V) ,$$

which by Proposition 3.3.2 corresponds to the curve of pure spinors $c(t) = \sum_{I \in \mathcal{I}_{2[N]}} \text{Pf}(C(t)_I) \mathbf{e}_I$ passing at $c(0) = \mathbb{1}$ with direction $c'(0) = q_l$, thus $[q_l] \in T_{[\mathbb{1}]} \mathbb{S}_N^+$.

Theorem 5.1.8. *The tangential variety $\tau(\mathbb{S}_N^+)$ splits in the Spin_{2N} -orbits*

$$\tau(\mathbb{S}_N^+) = \bigsqcup_{l=1}^{\frac{N}{2}} \text{Spin}_{2N} \cdot \left[\sum_{i=1}^l e_{2i-1} \wedge e_{2i} \right] .$$

5.1.3 Inclusions among closures of Spin_{2N} -orbits

We have treated the secant orbits and the tangent orbits separately, now we analyze the behaviour of their inclusions. First of all, notice that the tangent representative $[q_1] = [e_1 \wedge e_2] \in \Theta_{1,N}$ is a pure spinor, hence

$$\Theta_{1,N} = \mathbb{S}_N^+ = \Sigma_{1,N} .$$

REMARK. The orbit $\Theta_{1,N}$ is given by skew-symmetric matrices having rank (as matrices) 2, hence it is described by the Grassmannian of planes $\text{Gr}(2, N)$. This agrees with the more theoretical result [LM03, Prop. 2.5 + Subsec. 3.1] $T_x \mathbb{S}_N^+ \cap \mathbb{S}_N^+ \simeq \text{Gr}(2, N)$ for any $x \in \mathbb{S}_N^+$.

Moreover, for $l = 2$, a representative of $\Theta_{2,N}$ is $[q_2] = [e_1 \wedge e_2 + e_3 \wedge e_4]$. But $[e_1 \wedge e_2]$ and $[e_3 \wedge e_4]$ are pure spinors with corresponding subspaces $H_{e_1 \wedge e_2} = E_{[2]}$ and $H_{e_3 \wedge e_4} = \langle f_1, f_2, e_3, e_4, f_5, \dots, f_N \rangle_{\mathbb{C}}$, hence they have Hamming distance 2 and $[q_2] \in \Sigma_{2,N}$. Thus

$$\Theta_{2,N} = \Sigma_{2,N} .$$

REMARK 5.1.9. Via the map ψ_q in (3.3.1) every $[q] \in \Theta_{l,N}$ defines a subspace $H_q \in \text{OG}(N - 2l, V)$: indeed, the representative $[q_l]$ (5.1.7) defines the subspace $H_{q_l} = \langle f_{2l+1}, \dots, f_N \rangle_{\mathbb{C}}$ of dimension $N - 2l$ and $\text{Spin}^q(V)$ acts by conjugacy on the subspaces, preserving their dimensions. Moreover, by definition, a tangent point $[q] \in T_{[a]}\mathbb{S}_N^+ \cap \Theta_{l,N}$ is the direction of a curve $\gamma(t) = \{[a(t)] \mid t \in (-\epsilon, \epsilon)\} \subset \mathbb{S}_N^+$ passing at $\gamma(0) = [a]$ and, up to considering a smaller neighbourhood, one can assume $d([a], [a(t)]) = m$ for any $t \in (0, \epsilon)$. Then for any $t > 0$ the spinor $\frac{a-a(0)}{t}$ defines a subspace $H_t \in \text{OG}(N - 2l, V)$ and one gets the equality $H_q = \lim_{t \rightarrow 0} H_t \in \text{OG}(N - 2l, V)$.

From the previous arguments we deduce the following description of the secant variety of lines

$$\sigma_2(\mathbb{S}_N^+) = \mathbb{S}_N^+ \sqcup \left(\bigsqcup_{l=2}^{\lfloor \frac{N}{2} \rfloor} \Sigma_{l,N} \right) \cup \left(\bigcup_{l=3}^{\lfloor \frac{N}{2} \rfloor} \Theta_{l,N} \right),$$

where non-disjoint unions appear since we haven't proved $\Sigma_{l,N} \neq \Theta_{l,N}$ for $l \geq 3$ yet. Since we are interested in considering $l \geq 3$, we assume $N \geq 6$.

Lemma 5.1.10.

1. For any $l = 2 : \frac{N}{2}$ it holds $\Sigma_{l-1,N} \subset \overline{\Sigma_{l,N}}$.
2. For any $l = 2 : \frac{N}{2}$ it holds $\Theta_{l-1,N} \subset \overline{\Theta_{l,N}}$.
3. For any $l = 3 : \frac{N}{2}$ it holds $\Theta_{l,N} \subset \overline{\Sigma_{l,N}}$.

Proof. 1. For $\epsilon > 0$ consider the sequence $[\mathbf{e}_{[N]} + a_\epsilon] \in \Sigma_{l,N}$ for the pure spinors $[a_\epsilon]$ defined by the maximal q -isotropic subspaces

$$H_{a_\epsilon} = \langle e_1, \dots, e_{N-2l}, g_{N-2l+1}(\epsilon), g_{N-2l+2}(\epsilon), g_{N-2l+3}, \dots, g_N \rangle_{\mathbb{C}}$$

where

$$g_{N-2l+1}(\epsilon) = \frac{1}{\epsilon} f_{N-2l+1} + e_{N-2l+2} \quad , \quad g_{N-2l+2}(\epsilon) = \frac{1}{\epsilon} f_{N-2l+2} - e_{N-2l+1}$$

and $g_h = f_h + \sum_{k=N-2l+3}^N \alpha_{kj} e_k$ as in (3.3.5). Then the sequence $[\mathbf{e}_{[N]} + a_\epsilon]$ has limit $[\mathbf{e}_{[N]} + a]$ where the pure spinor $[a] \in \mathbb{S}_N^+$ corresponds to the maximal q -isotropic subspace $H_a = \langle e_1, \dots, e_{N-2l+2}, g_{N-2l+3}, \dots, g_N \rangle_{\mathbb{C}}$: in particular, $[\mathbf{e}_{[N]} + a] \in \Sigma_{l-1,N}$. By reversing this argument, one can always look at a point in $\Sigma_{l-1,N}$ as limit of a sequence in $\Sigma_{l,N}$.

2. The tangent points in $\Theta_{l-1,N}$ correspond to skew-symmetric matrices of size N and rank $2l - 2$, while points in $\Theta_{l,N}$ to rank- $2l$ skew-symmetric matrices of size N . As the former matrices lie in the closure of the latter ones, the thesis follows.
3. Consider $[q] \in \Theta_{l,N} \subset \tau(\mathbb{S}_N^+)$: then by Remark 5.1.9 $\dim H_q = N - 2l$ and there exists a curve of pure spinors $\gamma(t) = \{[a(t)] \mid t \in (-\epsilon, \epsilon)\} \subset \mathbb{S}_N^+$ with direction $\gamma'(0) = [q]$ such that (up to a smaller ϵ) $d([a(t)], [a(0)]) = l$ for any $t \in (0, \epsilon)$. In particular, for any $t > 0$ it holds $\left[\frac{a(t)+a(0)}{t} \right] \in \Sigma_{l,N}$ and by definition $[q] = \left[\lim_{t \rightarrow 0} \frac{a(t)-a(0)}{t} \right] = \lim_{t \rightarrow 0} \left[\frac{a(t)+a(0)}{t} \right] \in \overline{\Sigma_{l,N}}$. \square

Lemma 5.1.11. *For any $N \geq 6$ and $l = 3 : \frac{N}{2}$, it holds $\Sigma_{l,N} \neq \Theta_{l,N}$.*

Proof. For $l = \frac{N}{2}$ the equality does not hold since for $N \geq 6$ one has $\tau(\mathbb{S}_N^+) \subsetneq \sigma_2(\mathbb{S}_N^+)$. By contradiction, assume that there exists $3 \leq l \leq \frac{N}{2} - 1$ such that the equality holds. Then Lemma 5.1.10 implies $\Sigma_{i,N} = \Theta_{i,N}$ for any $3 \leq i \leq l$: in particular, $\Sigma_{3,N} = \Theta_{3,N}$. Since we want to deal with $3 \lesseqgtr \frac{N}{2}$, we assume $N \geq 7$.

Consider the tangent representative $[q_3] = [e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6] \in \Theta_{3,N}$. It defines the subspace $H_{q_3} = \langle f_7, \dots, f_N \rangle_{\mathbb{C}}$ of dimension $\dim H_{q_3} = N - 6$. Set $E \cap E_{[6]} = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle_{\mathbb{C}}$. Since $[q_3] \in \Sigma_{3,N}$, there exist pure spinors $[a], [b] \in \mathbb{S}_N^+$ such that $[q_3] = [a + b]$ and $d([a], [b]) = 3$. Then, being kernels, one gets $\langle f_7, \dots, f_N \rangle_{\mathbb{C}} = H_{q_3} = H_{a+b} = H_a \cap H_b$, where the last equality follows from Corollary 5.1.5. This implies that $a + b \in \bigwedge^{ev}(E \cap E_{[6]})$ and $[a], [b] \in \mathbb{S}_6^+$: in particular, the equality $[q_3] = [a + b]$ holds in $\Sigma_{3,6}$ for the representative $[q_3] \in \Theta_{3,6}$. But this means that $\Theta_{3,6} = \Sigma_{3,6}$, that is $\tau(\mathbb{S}_6^+) = \sigma_2(\mathbb{S}_6^+)$ which is a contradiction. \square

Theorem 5.1.12. *For any $N \geq 6$, the poset of Spin_{2N} -orbits in the secant variety of lines $\sigma_2(\mathbb{S}_N^+)$ is described by the graph in Figure (5.1), where arrows denote the inclusion of an orbit into the closure of the other orbit. In particular, the orbits $\Theta_{\frac{N}{2},N}$ and $\Sigma_{\frac{N}{2},N}$ are the dense orbits of the tangential and secant variety respectively.*

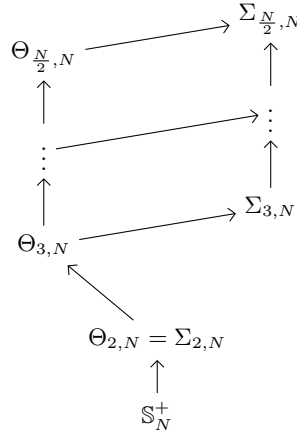


Figure 5.1: Poset graph of Spin_{2N} -orbits in $\sigma_2(\mathbb{S}_N^+)$.

5.2 Identifiability in $\sigma_2(\mathbb{S}_N^+)$

In Sec. 2.1 we have defined a point $[q] \in \sigma_2^\circ(\mathbb{S}_N^+)$ to be *identifiable* if there exists a unique pair of pure spinors $([a], [b]) \in (\mathbb{S}_N^+)_{\mathbb{C}_2}^{\times 2}$ such that $[q] = [a + b]$, which geometrically means that $[q]$ lies on a unique bisecant line to \mathbb{S}_N^+ . On the contrary, $[q]$ is *unidentifiable* (or non-identifiable) if it lies on at least two distinct bisecant lines to \mathbb{S}_N^+ . Recall that we say that a subset $Y \subset \sigma_2^\circ(\mathbb{S}_N^+)$

(eg. an orbit) is (un)identifiable if any of its points is so. Finally, the *decomposition locus* of $[q] \in \sigma_2^o(\mathbb{S}_N^+)$ is the set $Dec([q]) = \left\{ ([x], [y]) \in (\mathbb{S}_N^+)^{\times 2} /_{\mathfrak{S}_2} \mid [x + y] = [q] \right\}$.

Unidentifiability and decomposition loci for $\Sigma_{2,N}$ The 6-dimensional Spinor variety $\mathbb{S}_4^+ \subset \mathbb{P}^7$ coincides with the 6-dimensional quadric $\mathbb{Q}^6 \subset \mathbb{P}^7$. In this case, the secant orbit $\Sigma_{2,4}$ is dense in $\sigma_2(\mathbb{S}_4^+) = \mathbb{P}^7$. Given $[q] \in \mathbb{P}^7 \setminus \mathbb{Q}^6$, for any pair $([x], [y]) \in Dec([q])$ in the decomposition locus it holds $d([x], [y]) = 2$, since $L([x], [y]) \not\subset \mathbb{Q}^6$.

Proposition 5.2.1. *For any $[q] \in \mathbb{P}^7 \setminus \mathbb{Q}^6$, the decomposition locus $Dec([q])$ is parametrized by the lines $\mathbb{P}^1 \subset \mathbb{P}^7$ passing at $[q]$ and intersecting \mathbb{Q}^6 in two distinct points, i.e.*

$$Dec([q]) \simeq \mathbb{P}^6 \setminus \mathbb{Q}^5 .$$

In particular, the distance-2 orbit $\Sigma_{2,4}$ is unidentifiable.

Proof. The set of lines $\mathbb{P}^1 \subset \mathbb{P}^7$ passing through $[q]$ and intersecting \mathbb{Q}^6 in two distinct points is isomorphic to $\mathbb{P}^6 \setminus \mathbb{Q}^5$: indeed, points in $\mathbb{Q}^5 \subset \mathbb{P}^6$ correspond to lines $\mathbb{P}^1 \subset \mathbb{P}^7$ which are tangent to \mathbb{Q}^6 . A pair $([x], [y]) \in (\mathbb{Q}^6)^{\times 2}$ gives a decomposition for $[q] = [x + y]$ if and only if the line $L([x], [y]) \subset \mathbb{P}^7$ contains $[q]$ and intersects \mathbb{Q}^6 exactly in the distinct points $[x], [y]$. Thus the thesis follows. \square

EXAMPLE 5.2.2. For \mathbb{S}_4^+ , consider the representative $[\mathbf{e}_{[4]} + \mathbb{1}]$ of the dense orbit $\Sigma_{2,4}$: an alternative decomposition is

$$[\mathbf{e}_{[4]} + \mathbb{1}] = [(\mathbf{e}_{[4]} - \mathbf{e}_{[2]}) + (\mathbf{e}_{[2]} + \mathbb{1})]$$

where both summands are pure spinors since $d([\mathbf{e}_{[4]}], [\mathbf{e}_{[2]})] = d([\mathbf{e}_{[2]}], [\mathbb{1}]) = 1$.

Proposition 5.2.3. *For any $N \geq 6$, let $[q] \in \Sigma_{2,N}$ defining the $(N - 4)$ -dimensional isotropic subspace $H_q \subset \mathbb{C}^{2N}$ via ψ_q in (3.3.1). Let $\mathbb{Q}^6 \subset \mathbb{P}^7 = \mathbb{P}(H_q^\perp/H_q)$ be the quadric in the (projectivization of the) orthogonal quotient of H_q . For any $[\tilde{a}] \in \mathbb{Q}^6$, we denote by $[a] = [\tilde{a} \wedge H_q]$ the pure spinor in \mathbb{S}_N^+ . Then the decomposition locus of $[q]$ is 6-dimensional and isomorphic to the open set*

$$Dec([q]) \simeq \mathbb{Q}^6 \setminus \{ [\tilde{a}] \in \mathbb{Q}^6 \mid L([a], [q]) \subset \mathbb{S}_N^+ \} .$$

In particular, the distance-2 orbit $\Sigma_{2,N}$ is unidentifiable.

Proof. Fix $[q] \in \Sigma_{2,N}$. For any $([a], [b]) \in Dec([q])$, the pure spinors $[a], [b]$ have Hamming distance 2 and define two maximal q -isotropic subspaces H_a, H_b such that $H_a \cap H_b = H_q$. In particular, $\dim(H_a \cap H_b) = N - 4$ and $\dim\left(\frac{(H_a \cap H_b)^\perp}{H_a \cap H_b}\right) = 8$. Thus, in the orthogonal quotient space $W := H_q^\perp/H_q \simeq \mathbb{C}^8$, they give the 4-dimensional (maximal) isotropic subspaces $H_a/H_q, H_b/H_q$ which intersect trivially, hence they correspond to two pure spinors $[\tilde{a}], [\tilde{b}] \in \mathbb{S}_4^+ \simeq \mathbb{Q}^6$ such that $d([\tilde{a}], [\tilde{b}]) = 2$: in particular, the line $L([\tilde{a}], [\tilde{b}]) \not\subset \mathbb{Q}^6$ as well as $L([a], [b]) = L([a], [q]) \not\subset \mathbb{S}_N^+$.

On the other hand, start from a pure spinor $[\tilde{a}] \in \mathbb{Q}^6$ (look at the quadric as the Spinor

variety \mathbb{S}_4^+ constructed from W) and consider the lifting of its corresponding subspace $\tilde{H}_{\tilde{a}} \in \text{OG}^+(4, W)$ to the subspace $H := \langle \tilde{H}_{\tilde{a}}, H_q \rangle_{\mathbb{C}} \in \text{OG}^+(N, V)$, which corresponds to the pure spinor $[a] = [\tilde{a} \wedge H_q] \in \mathbb{S}_N^+$. If $[\tilde{a}] \in \mathbb{Q}^6$ is such that $L([a], [q]) \not\subseteq \mathbb{S}_N^+$, then by Bézout (\mathbb{S}_N^+ is intersection of quadrics) there exists a unique $[b] \in \mathbb{S}_N^+$ such that $L([a], [q]) \cap \mathbb{S}_N^+ = \{[a], [b]\}$. We conclude that each $[\tilde{a}] \in \mathbb{Q}^6$ such that $L([a], [q]) \not\subseteq \mathbb{S}_N^+$ corresponds to a unique pair $([a], [b]) \in \text{Dec}([q])$ in the decomposition locus of $[q]$, hence the thesis. \square

Identifiability of orbits $\Sigma_{l,N}$ for $l \geq 3$. We show the identifiability of the secant orbits $\Sigma_{l,N}$ for $l \geq 3$ via an inductive argument, based on the injectivity of a wedge-multiplication map. The *base case* of the induction is given by the identifiability of the dense orbit $\Sigma_{\frac{N}{2}, N}$, which we prove via Clifford apolarity introduced in Sec. 3.6.

REMARK. The case of the 15-dimensional spinor variety $\mathbb{S}_6^+ \subset \mathbb{P}^{31}$ was already known [AR03, Example 5] as an example of *variety with one apparent double point* (OADP variety), namely a n -dimensional variety $X \subset \mathbb{P}^{2n+1}$ such that through a general point of \mathbb{P}^{2n+1} there passes a unique secant line to X [AR03, Definition 3].

Lemma 5.2.4. *For any $N \geq 6$, the dense secant orbit $\Sigma_{\frac{N}{2}, N}$ is identifiable.*

Proof. We prove that the representative $[\mathbf{e}_{[N]} + \mathbb{1}] \in \Sigma_{\frac{N}{2}, N}$ is identifiable. In the notation of Sec. 3.6, consider the map $\Phi_{\mathbf{e}_{[N]} + \mathbb{1}} : H^0(\mathbb{S}_N^+, \mathcal{U}(1)) \rightarrow H^0(\mathbb{S}_N^+, \mathcal{U}^\vee)^\vee$ corresponding to the Clifford apolarity (3.6.2)

$$\begin{aligned} \Phi_{\mathbf{e}_{[N]} + \mathbb{1}} : \bigwedge^{od} E^\vee &\rightarrow E \oplus E^\vee \\ f &\mapsto C_{\mathbf{e}_{[N]} + \mathbb{1}}(f)|_E + C_f(\mathbf{e}_{[N]} + \mathbb{1})|_{E^\vee} \end{aligned}$$

Since $\text{Rk } \Phi_{\mathbf{e}_{[N]} + \mathbb{1}} = 2N = 2 \text{Rk } \mathcal{U}(1)$, from Proposition 2.2.4 it is enough to prove that the common zero locus of $\ker(\Phi_{\mathbf{e}_{[N]} + \mathbb{1}})$ is $\mathcal{Z}(\ker(\Phi_{\mathbf{e}_{[N]} + \mathbb{1}})) = \{[\mathbf{e}_{[N]}], [\mathbb{1}]\}$.

Given $f = \sum_{k=0}^{\frac{N-2}{2}} \sum_{I \in \binom{[N]}{2k+1}} c_I \mathbf{f}_I \in \bigwedge^{od} E^\vee$, one gets

$$\Phi_{\mathbf{e}_{[N]} + \mathbb{1}}(f) = (f \cdot \mathbf{e}_{[N]} + f \cdot \mathbb{1})|_E + (\mathbf{e}_{[N]} \cdot f + \mathbb{1} \cdot f)|_{E^\vee} = (f \cdot \mathbf{e}_{[N]})|_E + f|_{E^\vee},$$

hence

$$\ker(\Phi_{\mathbf{e}_{[N]} + \mathbb{1}}) = \bigwedge^3 E^\vee \oplus \dots \oplus \bigwedge^{N-3} E^\vee.$$

Consider $a = \sum_{s=0}^{\frac{N}{2}} \sum_{J \in \binom{[N]}{2s}} \alpha_J \mathbf{e}_J \in \bigwedge^{ev} E$ such that $[a] \in \mathcal{Z}(\ker(\Phi_{\mathbf{e}_{[N]} + \mathbb{1}}))$. From the vanishing of global sections in Lemma 3.6.6, we know that $\langle V \cdot f, a \rangle = 0$ for any $f \in \ker(\Phi_{\mathbf{e}_{[N]} + \mathbb{1}})$. In particular, for any index subset $I \subset [N]$ of odd cardinality from 3 to $N-3$ (so that \mathbf{f}_I lies in the kernel), and for any basis vector $e_\lambda \in E$, we get

$$0 = \langle e_\lambda \cdot \mathbf{f}_I, a \rangle = \langle \mathbf{f}_I, e_\lambda \wedge a \rangle = \sum_{s=0}^{\frac{N}{2}} \sum_{J \in \binom{[N]}{2s}} \alpha_J \langle \mathbf{f}_I, e_\lambda \wedge \mathbf{e}_J \rangle \quad (5.2.1)$$

$$= \sum_{s=0}^{\frac{N}{2}} \sum_{J \in \binom{[N]}{2s} : J \not\supset \lambda} (-1)^{\text{pos}(\lambda, J \cup \{\lambda\})+1} \alpha_J \langle \mathbf{f}_I, \mathbf{e}_{J \cup \{\lambda\}} \rangle = (-1)^{\text{pos}(\lambda, I)+1} \alpha_{I \setminus \{\lambda\}}, \quad (5.2.2)$$

where $\text{pos}(\lambda, I)$ denotes the position of λ in the ordered subset I , and the last equality follows from Remark 3.6.4. Thus in a there is no summand indexed by a subset J such that $J \subset I$ and $|J| = |I| - 1$ for any subset I with $|I| = 3, 5, \dots, N - 3$. On the other hand, for any subset $I \subset [N]$ of odd cardinality from 3 to $N - 3$ (i.e. $\mathbf{f}_I \in \ker(\Phi_{\mathbf{e}_{[N]+1}})$) and for any basis vector $f_\nu \in E^\vee$ such that $\nu \notin I$, it holds

$$0 = \langle f_\nu \cdot \mathbf{f}_I, a \rangle = \langle f_\nu \wedge \mathbf{f}_I, a \rangle = (-1)^{\text{pos}(\nu, I \cup \{\nu\})+1} \sum_{s=0}^{\frac{n+1}{2}} \sum_{J \in \binom{[N]}{2s}} \alpha_J \langle \mathbf{f}_{I \cup \{\nu\}}, \mathbf{e}_J \rangle \quad (5.2.3)$$

$$= (-1)^{\text{pos}(\nu, I \cup \{\nu\})+1} \alpha_{I \cup \{\nu\}}, \quad (5.2.4)$$

implying that a has no summand indexed by a subset J such that $4 \leq |J| \leq N - 2$. We deduce that every $[a] \in \mathcal{Z}(\ker(\Phi_{\mathbf{e}_{[N]+1}}))$ is such that $a = \alpha + \beta \mathbf{e}_{[N]} \in \mathbb{C} \oplus \bigwedge^N E$. Since $d(\mathbb{1}, \mathbf{e}_{[N]}) = \frac{N}{2}$, the line $L([\mathbf{e}_{[N]}], [\mathbb{1}])$ does not lie in \mathbb{S}_N^+ : as Spinor varieties are intersections of quadrics, by Bézout the only points of the form $[\alpha \mathbb{1} + \beta \mathbf{e}_{[N]}]$ being pure spinors are for either $\alpha = 0$ or $\beta = 0$, that is $\mathcal{Z}(\ker(\Phi_{\mathbf{e}_{[N]+1}})) = \{[\mathbf{e}_{[N]}], [\mathbb{1}]\}$. \square

Theorem 5.2.5. *For any $N \geq 6$ and $l \geq 3$, the secant orbit $\Sigma_{l,N} \subset \sigma_2(\mathbb{S}_N^+)$ is identifiable.*

Proof. Fix $l \geq 3$ and consider the orbit $\Sigma_{l,N} = \text{Spin}_{2N} \cdot [\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}]$. By homogeneity, it is enough to show that the spinor $[\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}]$ is identifiable.

Assume *ad absurdum* that there exist two pure spinors $[a], [b] \in \mathbb{S}_N^+ \setminus \{[\mathbf{e}_{[N]}], [\mathbf{e}_{[N-2l]}\}]$ such that $[a + b] = [\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}]$. In particular, since $l \neq 1$, Corollary 5.1.5 implies that $H_a \cap H_b = H_{a+b} = H_{\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}} = E \cap E_{N-2l}$, where we recall that $E_{N-2l} = \langle e_1, \dots, e_{N-2l}, f_{N-2l+1}, \dots, f_N \rangle_{\mathbb{C}}$. Since $\langle e_1, \dots, e_{N-2l} \rangle_{\mathbb{C}} = E \cap E_{N-2l} \subset H_a, H_b$, by Proposition 3.3.2 it follows that there exist two spinors $a', b' \in \bigwedge^{ev}(\langle e_{N-2l+1}, \dots, e_N \rangle_{\mathbb{C}})$ such that $a = \mathbf{e}_{[N-2l]} \wedge a'$ and $b = \mathbf{e}_{[N-2l]} \wedge b'$: by maximality of H_a and H_b , the spinors $[a'], [b'] \in \mathbb{S}_{2l}^+$ are pure in a smaller Spinor variety where they have maximum Hamming distance l . Moreover, since $[a], [b] \in \mathbb{S}_N^+ \setminus \{[\mathbf{e}_{[N]}], [\mathbf{e}_{[N-2l]}\}]$, we also know that $[a'], [b'] \in \mathbb{S}_{2l}^+ \setminus \{[e_{N-2l+1} \wedge \dots \wedge e_N], [\mathbb{1}]\}$.

The spinor $[a + b] = [\mathbf{e}_{[N-2l]} \wedge (a' + b')] \in \Sigma_{l,N}$ is the image of $[a' + b'] \in \Sigma_{l,2l}$ via the wedge multiplication map

$$\mathbf{e}_{[N-2l]} \wedge \bullet : \bigwedge^{ev} \langle e_{N-2l+1}, \dots, e_N \rangle_{\mathbb{C}} \longrightarrow \bigwedge^{ev} E \quad (5.2.5)$$

restricting to $(\mathbf{e}_{[N-2l]} \wedge \bullet) : \Sigma_{l,2l} \rightarrow \Sigma_{l,N}$. Since the above linear map is injective, the equality $(\mathbf{e}_{[N-2l]} \wedge \bullet)([a' + b']) = (\mathbf{e}_{[N-2l]} \wedge \bullet)([e_{N-2l+1} \wedge \dots \wedge e_N + \mathbb{1}])$ implies that $[a' + b'] = [e_{N-2l+1} \wedge \dots \wedge e_N + \mathbb{1}]$. But the spinor $[e_{N-2l+1} \wedge \dots \wedge e_N + \mathbb{1}]$ is a representative for the dense orbit $\Sigma_{l,2l} \subset \sigma_2(\mathbb{S}_{2l}^+)$, which is identifiable by Lemma 5.2.4 (since $l \geq 3$). Thus it holds $\{[a'], [b']\} = \{[e_{N-2l+1} \wedge \dots \wedge e_N], [\mathbb{1}]\}$, in contradiction to $[a'], [b'] \in \mathbb{S}_{2l}^+ \setminus \{[e_{N-2l+1} \wedge \dots \wedge e_N], [\mathbb{1}]\}$. \square

5.3 Tangential-identifiability in $\tau(\mathbb{S}_N^+)$

In this section we prove that any point of a tangent orbit $\Theta_{l,N}$ for $l \geq 3$ is tangential-identifiable (cf. Definition 2.1.1), and we do so via Clifford apolarity (cf. Section 3.6). The

setting at the beginning of the chapter is assumed. Since for $N \leq 5$ it holds $\overline{\Theta_{2,N}} = \tau(\mathbb{S}_N^+) = \sigma_2(\mathbb{S}_N^+)$, in the following we assume $N \geq 6$.

REMARK 5.3.1. The 15-dimensional Spinor variety $\mathbb{S}_6^+ \subset \mathbb{P}^{31}$ has been deeply studied in the context of the *Freudenthal's magic square*, (cf. Table 1.4). Indeed, \mathbb{S}_6^+ is the third of the four *Legendrian varieties* lying in the third row of the square (associated to the group Spin_{12} - see [LM01]), while $\mathbb{P}^{31} = \sigma_2(\mathbb{S}_6)^+$ is the third prehomogenous space in the fourth row (associated to the exceptional group E_7 - see [Cle03]). In both the above references the generic tangential-identifiability of $\tau(\mathbb{S}_6^+)$ is obtained: we refer to [LM01, Propp. 5.8-5.12] and [Cle03, Propp. 8.4, 9.8] for details.

In the notation of Section 3.6, for a tangent spinor $[q] \in \tau(\mathbb{S}_N^+) \setminus \mathbb{S}_N^+$ we consider the map

$$\Phi_q : H^0(\mathbb{S}_N^+, \mathcal{U}(1)) \longrightarrow H^0(\mathbb{S}_N^+, \mathcal{U}^\vee)^\vee$$

corresponding to the Clifford apolarity in (3.6.2). Similarly to the proof of Lemma 5.2.4, we apply the nonabelian apolarity, but in this case we deal with non-reduced subschemes. Let $Y \subset \mathbb{S}_N^+$ be a non-reduced subscheme of \mathbb{S}_N^+ of length 2 such that $[q] \in \langle Y \rangle$, and let Y_{supp} be its support: in particular, such a Y corresponds to $\{[p], [q]\}$ for $[p] \in \mathbb{S}_N^+$ such that $[q] \in T_{[p]}\mathbb{S}_N^+$, and $Y_{supp} = \{[p]\}$. Then by Proposition 2.2.3 we get $H^0(\mathbb{S}_N^+, I_Y \otimes \mathcal{U}(1)) \subset \ker(\Phi_q)$ and in particular $\mathcal{Z}(\ker(\Phi_q)) \subset Y_{supp}$, where $\mathcal{Z}(\ker(\Phi_q))$ is the common zero locus of global sections in $\ker(\Phi_q)$. If such common zero locus is given by only one pure spinor $[p] \in \mathbb{S}_N^+$, then for any 0-dimensional subscheme $Y \subset X$ of length 2 such that $[q] \in \langle Y \rangle$ it holds $Y_{supp} = \{[p]\}$, that is $[q]$ lies on only one tangent space, namely $T_{[p]}\mathbb{S}_N^+$, and $[q]$ is tangential-identifiable.

Theorem 5.3.2. *For any $N \geq 6$ and $l \geq 3$, the tangent orbit $\Theta_{l,N} \subset \tau(\mathbb{S}_N^+)$ is tangential-identifiable.*

Proof. Fix $l = 3 : \frac{N}{2}$. From the above argument it is enough to prove that $\mathcal{Z}(\ker(\Phi_{q_l})) = \{[\mathbb{1}]\}$ for $[q_l] \in T_{[\mathbb{1}]\mathbb{S}_N^+} \setminus \mathbb{S}_{n+1}^+$ being the representative of $\Theta_{l,N}$ as in (5.1.7):

$$q_l = \sum_{i=1}^l e_{2i-1} \wedge e_{2i} \in \bigwedge^2 E.$$

From the Clifford apolarity, for any $f \in \bigwedge^{od} E^\vee$ we have

$$\Phi_{q_l}(f) = C_{q_l}(f)|_E + C_f(q_l)|_{E^\vee}.$$

First, notice that for any $f = \sum_{r=1}^N \beta_r f_r \in E^\vee$ one gets

$$C_{q_l}(f) = \sum_{r=1}^N \sum_{i=1}^l \beta_r [f_r \cdot (e_{2i-1} \wedge e_{2i})] = \sum_{i=1}^l (\beta_{2i-1} e_{2i} - \beta_{2i} e_{2i-1}),$$

implying that $\ker(\Phi_{q_l}) \cap E^\vee = \{0\}$. Moreover, since $q_l \in \bigwedge^2 E$, it is straightforward that

$$\bigwedge_{od \geq 5} E^\vee := \bigoplus_{k \geq 2} \bigwedge^{2k+1} E^\vee \subset \ker(\Phi_{q_l}).$$

As the images $\Phi_{q_l}(\bigwedge^{2k+1} E^\vee)$ are linearly independent one to each other, we get

$$\ker(\Phi_{q_l}) = \bigwedge^{od \geq 5} E^\vee \oplus \left(\ker(\Phi_{q_l}) \cap \bigwedge^3 E^\vee \right),$$

thus $\mathcal{Z}(\ker(\Phi_{q_l}))$ is the intersection of the common zero loci of the two summands above.

Consider $[a] \in \mathcal{Z}(\ker(\Phi_{q_l}))$ such that $a = \sum_{s=0}^{\frac{N}{2}} \sum_{J \in \binom{[N]}{2s}} \alpha_J \mathbf{e}_J$. From Lemma 3.6.6 it holds

$$\langle V \cdot f, a \rangle = 0, \forall f \in \bigwedge^{od \geq 5} E^\vee \oplus \left(\ker(\Phi_{q_l}) \cap \bigwedge^3 E^\vee \right).$$

The same computation in (5.2.1) shows that for any index subset $I \subset [N]$ such that $|I| = 2k+1 \geq 5$ and for any basis vector $e_\lambda \in E$ it holds $0 = \langle e_\lambda \cdot \mathbf{f}_I, a \rangle = (-1)^{pos(\lambda, I)+1} \alpha_{I \setminus \{\lambda\}}$ where again $pos(\lambda, I)$ denotes the position of λ in the ordered index subset I . We deduce that in a there is no summand indexed by a subset J such that $J \subset I$ and $|J| = |I| - 1$ for any $|I| \geq 5$, hence $a \in \mathbb{C} \oplus \bigwedge^2 E \oplus \bigwedge^{n+1} E$. On the other hand, the same computation as in (5.2.3) shows that for any $I \subset [N]$ such that $|I| = 3$ and any basis vector $f_\nu \in E^\vee$ such that $\nu \notin I$, it holds $0 = \langle f_\nu \cdot \mathbf{f}_I, a \rangle = (-1)^{pos(\nu, I \cup \{\nu\})+1} \alpha_{I \cup \{\nu\}}$, hence a has no summand in $\bigwedge^N E$ either. It follows

$$\mathcal{Z}(\ker(\Phi_{q_l})) = \mathbb{P} \left(\mathbb{C} \oplus \bigwedge^2 E \right) \cap \mathcal{Z} \left(\bigwedge^3 E^\vee \cap \ker(\Phi_{q_l}) \right).$$

Clearly, $[1] \in \mathcal{Z}(\ker(\Phi_{q_l}))$ since for any $\lambda \in \mathbb{C}$ and any $f \in \bigwedge^3 E^\vee$ one has $\langle f, v \cdot \lambda \rangle = \lambda \langle f, v \rangle = 0$. Moreover, for any $b \in \bigwedge^2 E$ it holds that $b \in \mathcal{Z}(\ker(\Phi_{q_l}))$ if and only if $\lambda + b \in \mathcal{Z}(\ker(\Phi_{q_l}))$ for any $\lambda \in \mathbb{C}$: in particular, it is enough to prove that $\mathcal{Z}(\ker(\Phi_{q_l})) \cap \mathbb{P}(\bigwedge^2 E) = \emptyset$ in order to conclude.

Let $[a] \in \mathcal{Z}(\ker(\Phi_{q_l}))$ be such that $a = \sum_{\{s \leq t\} \subset [N]} \alpha_{st} e_s \wedge e_t$. First, consider an index subset $I \subset [N]$ such that $|I| = 3$ and $\{2k-1, 2k\} \not\subset I$ for any $k \in [l]$: the condition $C_{\mathbf{f}_I}(q_l)|_{E^\vee} = 0$ implies $\mathbf{f}_I \in \ker(\Phi_{q_l})$. Then, for any $e_\lambda \in E$ one gets

$$0 = \langle e_\lambda \cdot \mathbf{f}_I, a \rangle = \delta_{\lambda, i_1} \alpha_{i_2, i_3} - \delta_{\lambda, i_2} \alpha_{i_1, i_3} + \delta_{\lambda, i_3} \alpha_{i_1, i_2},$$

for δ_{xy} being the Kronecker symbol. Since $N \geq 6$, given any two distinct indices $\{i \leq j\} \subset [N]$ such that $\{i, j\} \neq \{2k-1, 2k\}$ for any $k \in [l]$, one can always find a third index $r \in [N] \setminus \{i, j\}$ such that $\{2k-1, 2k\} \not\subset I = \{i, j, r\}$ for any $k \in [l]$. Thus for any $\{i, j\} \neq \{2k-1, 2k\}$ it holds $\alpha_{ij} = 0$ and

$$\mathcal{Z}(\ker(\Phi_{q_l})) \cap \mathbb{P} \left(\bigwedge^2 E \right) \subset \left\{ [a] = \left[\sum_{k=1}^l \alpha_{2k-1, 2k} e_{2k-1} \wedge e_{2k} \right] \right\}.$$

Now, for any $\{k \leq h\} \subset [l]$ and any $r \in [N] \setminus \{2k-1, 2k, 2h-1, 2h\}$ consider $\mathbf{f}_{2k-1, 2k, r} - \mathbf{f}_{2h-1, 2h, r} \in \bigwedge^3 E^\vee$: it is a straightforward count that $\mathbf{f}_{2k-1, 2k, r} - \mathbf{f}_{2h-1, 2h, r} \in \ker(\Phi_{q_l})$. In particular, $0 = \langle e_r \cdot (\mathbf{f}_{2k-1, 2k, r} - \mathbf{f}_{2h-1, 2h, r}), e_r \cdot a \rangle = \alpha_{2k-1, 2k} - \alpha_{2h-1, 2h}$, hence it holds

$$\alpha_{2k-1, 2k} = \alpha_{2h-1, 2h}, \forall \{h \leq k\} \subset [l].$$

It follows that $a = \alpha \cdot q_l$ for some $\alpha \in \mathbb{C}$. But $[q_l] \notin \mathbb{S}_N^+$ (since $l \geq 3$), thus it has to be $\alpha = 0$ and $\mathcal{Z}(\ker(\Phi_{q_l})) \cap \mathbb{P}(\bigwedge^2 E) = \emptyset$. We conclude that $\mathcal{Z}(\ker(\Phi_{q_l})) = \{[1]\}$. \square

5.4 Dimensions of Spin_{2N} -orbits in $\sigma_2(\mathbb{S}_N^+)$

In this section we compute the dimensions of each orbit in the secant variety of lines to a Spinor variety. We have postponed this computation as we apply tangential-identifiability for computing dimensions in the tangent branch.

From [Zak93, Theorem 1.4] the secant orbit $\Sigma_{\frac{N}{2}, N}$ is dense of dimension

$$\dim \Sigma_{\frac{N}{2}, N} = \dim \sigma_2(\mathbb{S}_N^+) = \begin{cases} 2^{N-1} - 1 & \text{for } N \leq 6 \\ N(N-1) + 1 & \text{for } N \geq 6 \end{cases}. \quad (5.4.1)$$

Moreover, for $N \geq 6$ the tangential variety has codimension 1, hence its dense orbit $\Theta_{\frac{N}{2}, N}$ has dimension

$$\dim \Theta_{\frac{N}{2}, N} = \dim \tau(\mathbb{S}_N^+) = N(N-1), \quad \forall N \geq 6.$$

Recall that for $N \leq 5$ the Spinor variety \mathbb{S}_N^+ has diameter 2, and the distance-2 orbit $\Sigma_{2, N}$ either does not exist (for $N \leq 3$) or it is the dense one. In this respect, in the following we assume $N \geq 6$.

Lemma 5.4.1. *For any $N \geq 6$ and any $l = 2 : \frac{N}{2} - 1$, the fibration*

$$\begin{array}{ccc} \xi : \Sigma_{l, N} & \longrightarrow & \text{OG}(N-2l, V) \\ [a+b] & \mapsto & H_{a+b} \stackrel{\text{Cor. 5.1.5}}{=} H_a \cap H_b \end{array}$$

has fibers isomorphic to the dense orbit on the smaller Spinor variety \mathbb{S}_{2l}^+ , namely $\xi^{-1}(H) \simeq \Sigma_{l, 2l} \subset \sigma_2(\mathbb{S}_{2l}^+)$. In particular,

$$\overline{\xi^{-1}(H)} \simeq \sigma_2(\mathbb{S}_{2l}^+).$$

Proof. Consider the distance- l orbit $\Sigma_{l, N} = \text{Spin}_{2N} \cdot [\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}]$ and the above fibration ξ . By homogeneity, it is enough to determine the fiber at the subspace $\xi([\mathbf{e}_{[N]} + \mathbf{e}_{[N-2l]}) = E \cap E_{N-2l} = \langle e_1, \dots, e_{N-2l} \rangle_{\mathbb{C}}$. We set $V' = \langle e_{N-2l+1}, \dots, e_N, f_{N-2l+1}, \dots, f_N \rangle_{\mathbb{C}}$ and $E' = V' \cap E$.

Let $[a+b] \in \xi^{-1}(E \cap E_{N-2l})$: then $E \cap E_l \subset H_a$ (resp. $E \cap E_l \subset H_b$) and, by definition of the maximal q -isotropic subspaces as kernels of (3.3.1), we can write $[a] = [\mathbf{e}_{[N-2l]} \wedge w_a]$ (resp. $[b] = [\mathbf{e}_{[N-2l]} \wedge w_b]$) for some $w_a \in \wedge^{ev} E'$ (resp. $w_b \in \wedge^{ev} E'$). In particular, since w_a and w_b are defined by $2l$ linearly independent columns in the matrices describing H_a and H_b , they corresponds to maximal isotropic subspaces $H'_a, H'_b \subset \text{OG}^+(2l, V')$, hence $[w_a], [w_b] \in \mathbb{S}_{2l}^+$. Finally, the condition $H_a \cap H_b = E \cap E_{N-2l}$ implies $H'_a \cap H'_b = (0)$, hence $[w_a]$ and $[w_b]$ have maximum Hamming distance $d([w_a], [w_b]) = l$ in \mathbb{S}_{2l}^+ . Therefore the injective wedge-multiplication map $(\mathbf{e}_{[N-2l]} \wedge \bullet) : \Sigma_{l, 2l} \rightarrow \Sigma_{l, N}$ in (5.2.5) gives the biregular isomorphism

$$\begin{aligned} \xi^{-1}(E \cap E_{N-2l}) &= \{[a+b] \in \Sigma_{l, N} \mid H_a \cap H_b = E \cap E_{N-2l}\} \\ &= \{[\mathbf{e}_{[N-2l]} \wedge (w_a + w_b)] \in \Sigma_{l, N} \mid [w_a], [w_b] \in \mathbb{S}_{2l}^+, d([w_a], [w_b]) = l\} \\ &\simeq \{[w_a + w_b] \in \sigma_2^{\circ}(\mathbb{S}_{2l}^+) \mid d([w_a], [w_b]) = l\} \\ &= \Sigma_{l, 2l}. \end{aligned}$$

□

Proposition 5.4.2. *For any $N \geq 6$ and $l = 2 : \frac{N}{2} - 1$, the secant orbit $\Sigma_{l,N} \subset \sigma_2^\circ(\mathbb{S}_N^\pm)$ has dimension*

$$\dim \Sigma_{l,N} = \begin{cases} \frac{N(N-1)}{2} + 4N - 15 & \text{if } l = 2 \\ \frac{N(N-1)}{2} + l(2N-1) - 2l^2 + 1 & \text{if } l \geq 3. \end{cases}$$

Proof. Consider the fibration ξ in Lemma 5.4.1. From the fiber dimension theorem we get

$$\dim \Sigma_{l,N} = \dim \text{OG}(N-2l, V) + \dim \sigma_2(\mathbb{S}_{2l}^+).$$

In general, the orthogonal Grassmannian $\text{OG}(r, M)$ coincides with the kernel of the global section $s_q \in H^0(\text{Gr}(r, M), \text{Sym}^2 \mathcal{U})$ induced by the quadratic form q on \mathbb{C}^M , where \mathcal{U} is the rank- r universal bundle on $\text{Gr}(r, M)$. Thus

$$\dim \text{OG}(r, M) = \dim \text{Gr}(r, M) - \text{Rk}(\text{Sym}^2(\mathcal{U})) = r(M-r) - \binom{r+1}{2}.$$

The thesis follows by substituting $r = N - 2l$ and $M = 2N$, and recovering $\dim \sigma_2(\mathbb{S}_{2l}^+)$ from (5.4.1). \square

The above computation puts on light a particular feature of the second-to-last secant orbit. In the following result we need $N \geq 8$ in order to get an intermediate proper secant orbit between $\Sigma_{2,N}$ and $\Sigma_{\frac{N}{2},N}$.

Corollary 5.4.3. *For any $N \geq 8$, the closure of the second-to-last orbit $\Sigma_{\frac{N-2}{2},N}$ is a divisor in $\sigma_2(\mathbb{S}_N^+)$ parametrized by the vanishing of a pfaffian. Indeed, up to chart-changing, all pure spinors $[a] \in \mathbb{S}_N^+$ such that $d([\mathbf{e}_{[N]}], [a]) = \frac{N-2}{2}$ correspond to maximal q -isotropic subspaces H_a described by matrices $\begin{bmatrix} I_N \\ A \end{bmatrix}$ where $\text{Rk}(A) = N - \dim(E \cap H_a) = N - 2$.*

Proof. The thesis “being a divisor” is a straightforward count from Proposition 5.4.2. We show that the closure of $\Sigma_{\frac{N-2}{2},N}$ is parametrized by the vanishing of a pfaffian.

The pure spinors $[a] \in \mathbb{S}_N^+$ having Hamming distance l from $[\mathbf{e}_{[N]}]$ correspond to subspaces $H_a \in \text{OG}^+(N, V)$ such that $\dim(E \cap H_a) = N - 2l$. Up to chart-changing, we may assume that H_a is described by the matrix $\begin{bmatrix} I_N \\ A \end{bmatrix}$ for a certain $A \in \wedge^2 \mathbb{C}^N$: in particular, $\text{Rk}(A) = N - \dim(E \cap H_a) = 2l$. It follows that the pure spinors having Hamming distance $l = \frac{N-2}{2}$ from $[\mathbf{e}_{[N]}]$ are described (up to chart-changing) by skew-symmetric matrices of rank $N - 2$, that is they are parametrized by the vanishing of the pfaffian of such matrices. Now, given the pure spinor $[\mathbf{e}_{[N]}]$, the subvarieties

$$\mathcal{F}_{\mathbf{e}_{[N]}, \frac{N-2}{2}} = \left\{ [\mathbf{e}_{[N]} + a] \mid [a] \in \mathbb{S}_N^+, d([\mathbf{e}_{[N]}], [a]) \leq \frac{N-2}{2} \right\} \subset \overline{\Sigma_{\frac{N-2}{2},N}},$$

$$\mathcal{F}_{\mathbf{e}_{[N]}} = \{ [\mathbf{e}_{[N]} + a] \mid [a] \in \mathbb{S}_N^+ \} \subset \sigma_2^\circ(\mathbb{S}_N^+)$$

are such that $\mathcal{F}_{\mathbf{e}_{[N]}, \frac{N-2}{2}} = \mathcal{F}_{\mathbf{e}_{[N]}} \cap V(\text{Pf}(A))$: more in general, for any pure spinor $[b] \in \mathbb{S}_N^+$ it holds $\mathcal{F}_{b, \frac{N-2}{2}} = \mathcal{F}_b \cap V(\text{Pf}(A))$, giving the *rational* isomorphism in the thesis. \square

We are left with computing dimensions in the tangent branch for $N \geq 6$. From Proposition 5.4.2 we know that $\Theta_{2,N} = \Sigma_{2,N}$ has dimension $\frac{N(N-1)}{2} + 4N - 15$. Thus in the following we assume $N \geq 6$ and $l = 3 : \frac{N-1}{2} - 1$.

REMARK. In the following proof we use that the tangent orbits $\Theta_{l,N}$ for $l \geq 3$ are tangentially-identifiable (cf. Theorem 5.3.2): each point $[q] \in \Theta_{l,N}$ lies on a unique tangent space $T_{[q]}\mathbb{S}_N^+$.

Proposition 5.4.4. *For any $N \geq 6$ and any $l = 3 : \frac{N}{2} - 1$, the tangent orbit $\Theta_{l,N} \subset \tau(\mathbb{S}_N^+)$ has dimension*

$$\dim \Theta_{l,N} = \frac{N(N-1)}{2} + l(2N-1) - 2l^2.$$

In particular, for such N, l , the closure $\overline{\Theta_{l,N}}$ is a divisor in $\overline{\Sigma_{l,N}}$.

Proof. Given $[\bigwedge^2 \mathbb{C}^N]_{2l}$ the space of rank- $2l$ skew-symmetric matrices of size N , from Sec. 5.1.2 we know that for any $l = 2 : \frac{N}{2}$ it holds

$$\Theta_{l,N} = \bigcup_{[a] \in \mathbb{S}_N^+} [T_{[a]}\mathbb{S}_N^+]_{2l} \simeq \bigcup_{[a] \in \mathbb{S}_N^+} \left[\bigwedge^2 \mathbb{C}^N \right]_{2l}.$$

For $l \geq 3$, any tangent $[q] \in \Theta_{l,N}$ belongs to a unique tangent space $T_{[q]}\mathbb{S}_N^+$, hence

$$\dim \Theta_{l,N} = \dim \mathbb{S}_N^+ + \dim \left[\bigwedge^2 \mathbb{C}^N \right]_{2l}, \quad \forall l \geq 3.$$

The space $[\bigwedge^2 \mathbb{C}^N]_{2l}$ is the $\text{GL}(N)$ -orbit in $\mathbb{C}^N \otimes \mathbb{C}^N$ of the skew-symmetric matrix C_l in (5.1.8) having stabilizer isomorphic to $\text{Sp}(2l) \times \text{GL}(N-2l) \times (\mathbb{C}^{2l} \otimes \mathbb{C}^{N-2l})$. Therefore one gets

$$\dim \left[\bigwedge^2 \mathbb{C}^N \right]_{2l} = N^2 - \left[\binom{2l+1}{2} + 2l(N-2l) + (N-2l)^2 \right] = l(2N-1) - 2l^2$$

and the thesis straightforwardly follows. \square

EXAMPLE 5.4.5. The Spinor variety $\mathbb{S}_6^+ \subset \mathbb{P}(\bigwedge^{ev} \mathbb{C}^6) \simeq \mathbb{P}^{31}$ has diameter 3. Set $G = \text{Spin}(12)$ and $v_{\omega_6} = e_1 \wedge \dots \wedge e_6$. The secant variety $\sigma_2(\mathbb{S}_6^+)$ stratifies in the $\text{Spin}(12)$ -orbits

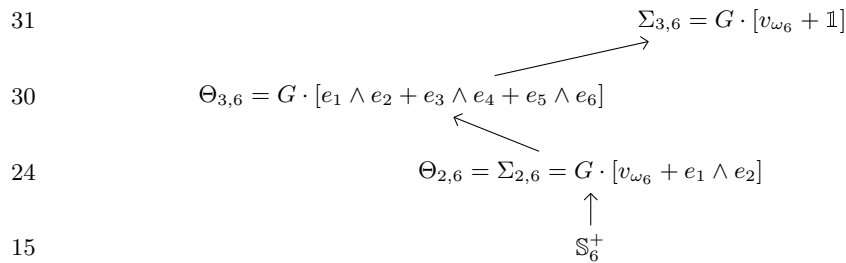


Figure 5.2: Poset graph of the Spin_{12} -orbits in $\sigma_2(\mathbb{S}_6^+)$, and their dimensions.

This poset and the dimensions were already known to J.M. Landsberg and L. Manivel [LM01] in the context of Legendrian varieties. Actually, as confirmed by the authors (which we thank for the confrontation), our arguments allow to recognize a misprint in [LM01, Proposition 5.10], where the dimension of the orbit σ_+ (in the authors' notation, corresponding to our $\Sigma_{2,6}$ for $m = 4$) is $5m + 4$ instead of $5m + 3$.

EXAMPLE 5.4.6. The Spinor variety $\mathbb{S}_8^+ \subset \mathbb{P}(\wedge^{ev} \mathbb{C}^8) \simeq \mathbb{P}^{127}$ has diameter 4: this is the only case in which the inclusion $\overline{\Theta_{3,8}} \subset \overline{\Theta_{4,8}} \cap \overline{\Sigma_{3,8}}$ actually is an equality.

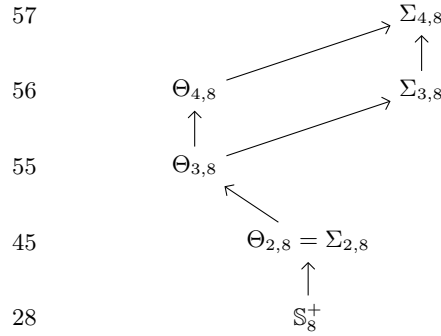


Figure 5.3: Poset graph of the Spin_{16} -orbits in $\sigma_2(\mathbb{S}_8^+)$, and their dimensions.

5.5 The 2–nd Terracini locus of \mathbb{S}_N^+

We recall the definition of the second Terracini locus from Sec. 4.5:

$$\text{Terr}_2(\mathbb{S}_N^+) := \overline{\{ \{p_1, p_2\} \in \text{Hilb}_2(\mathbb{S}_N^+) \mid \dim \langle T_{p_1} \mathbb{S}_N^+, T_{p_2} \mathbb{S}_N^+ \rangle \leq \dim \sigma_2(\mathbb{S}_N^+) \}}.$$

We consider the abstract secant variety (cf. Sec. 2.1)

$$\text{Ab}\sigma_2(\mathbb{S}_N^+) := \left\{ (\mathcal{Z}, [q]) \in \text{Hilb}_2(\mathbb{S}_N^+) \times \mathbb{P} \left(\wedge^{ev} E \right) \mid [q] \in \langle \mathcal{Z} \rangle \right\}$$

and the diagram

$$\begin{array}{ccc} & \text{Ab}\sigma_2(\mathbb{S}_N^+) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \text{Hilb}_2(\mathbb{S}_N^+) & & \sigma_2(\mathbb{S}_N^+) \end{array}$$

where π_1 and π_2 are the natural projection from the abstract secant variety onto the first and second factor respectively.

As already pointed out in Remark 4.6.5 for the case of Grassmannians, the second Terracini locus tells us where the differential of the projection from the abstract secant variety onto the secant variety drops rank: this information, combined with the identifiability, allows to deduce

the smoothness on the locus of points which are both identifiable and outside the Terracini locus. In the following we consider $N \geq 7$ as for $N \leq 6$ the second secant variety coincides with the ambient space.

Theorem 5.5.1. *For any $N \geq 7$, the second Terracini locus $\text{Terr}_2(\mathbb{S}_N^+)$ of the Spinor variety \mathbb{S}_N^+ corresponds to the distance-2 orbit closure $\overline{\Sigma_{2,N}}$. More precisely, in the above notation, it holds*

$$\text{Terr}_2(\mathbb{S}_N^+) = (\pi_1 \circ \pi_2^{-1}) (\overline{\Sigma_{2,N}}) .$$

Proof. Given a point $[a+b] \in \sigma_2(\mathbb{S}_N^+)$ for certain $[a], [b] \in \mathbb{S}_N^+$, we show that $\dim \langle T_{[a]}\mathbb{S}_N^+, T_{[b]}\mathbb{S}_N^+ \rangle$ drops if and only if $[a+b] \in \overline{\Sigma_{2,N}} = \mathbb{S}_N^+ \sqcup \Sigma_{2,N}$, that is if and only if $d([a], [b]) \leq 2$.

If $d([a], [b]) = 1$, then any point on the line $L([a], [b]) \subset \mathbb{S}_N^+$ is direction of the curve defined by such line, that is $L([a], [b]) \subset T_{[a]}\mathbb{S}_N^+ \cap T_{[b]}\mathbb{S}_N^+$.

If $d([a], [b]) = 2$, then there exists $[c] \in \mathbb{S}_N^+$ such that $L([a], [c]), L([c], [b]) \subset \mathbb{S}_N^+$. In particular, from the previous case we deduce $[c] \in T_{[a]}\mathbb{S}_N^+ \cap T_{[b]}\mathbb{S}_N^+$.

On the other hand, if the dimension drops, then there exists a common non-zero tangent point $[x] \in T_{[a]}\mathbb{S}_N^+ \cap T_{[b]}\mathbb{S}_N^+$. In particular, $[x]$ is not tangential-identifiable, hence $[x] \in \Theta_{2,N}$. From the definition (5.1.6) of $\Theta_{2,N} \cap T_{[a]}\mathbb{S}_N^+$ as the set of rank-4 skew-symmetric matrices in $T_{[a]}\mathbb{S}_N^+$, we know that $[x] = [y+z]$ for $[y], [z] \in \Theta_{1,N} \cap T_{[a]}\mathbb{S}_N^+ = \mathbb{S}_N^+ \cap T_{[a]}\mathbb{S}_N^+$ skew-symmetric matrices of rank 2. In particular, $[a+y]$ and $[a+z]$ lie in \mathbb{S}_N^+ , that is $d([a], [y]) = d([a], [z]) = 1$. The same argument shows that also $d([b], [y]) = d([b], [z]) = 1$. We conclude that $d([a], [b]) \leq 2$. \square

5.6 Results on the singular locus of $\sigma_2(\mathbb{S}_N^+)$

This section is devoted to study the singular locus of the secant variety of lines $\sigma_2(\mathbb{S}_N^+)$ to a Spinor variety \mathbb{S}_N^+ . We use results on identifiability from previous sections.

REMARK. As defective cases, the secant varieties $\sigma_2(\mathbb{S}_4^+) = \tau(\mathbb{S}_4^+) = \mathbb{P}^7$ and $\sigma_2(\mathbb{S}_5^+) = \tau(\mathbb{S}_5^+) = \mathbb{P}^{15}$ overfill the ambient space. On the other hand, the secant variety $\sigma_2(\mathbb{S}_6^+) = \mathbb{P}^{31}$ perfectly fills the ambient space (hence it is smooth), but the tangential variety $\tau(\mathbb{S}_6^+)$ is a quartic hypersurface in it. Accordingly to Remark 5.3.1, \mathbb{S}_6^+ has been widely studied in [LM01; LM07] and it has been proven that $\text{Sing}(\tau(\mathbb{S}_6^+)) = \overline{\Sigma_{2,6}}$ (corresponding to σ_+ in [LM07]).

According to the above remark, we assume $N \geq 7$, so that $\sigma_2(\mathbb{S}_N^+) \subsetneq \mathbb{P}^{2^{N-1}-1}$. In the same spirit of Remark 4.6.5, in the following we consider the alternative definition of abstract secant variety (cf. Remark 2.1.4)

$$Ab\sigma_2(\mathbb{S}_N^+) := \overline{\left\{ ([a], [b], [q]) \in (\mathbb{S}_N^+)_{/\mathfrak{S}_2}^2 \times \mathbb{P} \left(\bigwedge^{ev} E \right) \mid [q] \in \langle [a], [b] \rangle \right\}},$$

which is smooth outside the preimage of the diagonal $\Delta_X \subset (\mathbb{S}_N^+)_{/\mathfrak{S}_2}^2$ via the projection onto the first factor. In particular, given π the projection onto the second factor, the preimage $\pi^{-1}(\sigma_2(\mathbb{S}_N^+) \setminus \overline{\Sigma_{2,N}})$ is smooth in $Ab\sigma_2(\mathbb{S}_N^+)$.

Lemma 5.6.1. *For any $N \geq 7$, the distance-2 orbit $\Sigma_{2,N}$ lies in the singular locus $\text{Sing}(\sigma_2(\mathbb{S}_N^+))$ of the secant variety of lines to the Spinor variety \mathbb{S}_N^+ . In particular,*

$$\overline{\Sigma_{2,N}} \subset \text{Sing}(\sigma_2(\mathbb{S}_N^+)) .$$

Proof. Clearly, the Spinor variety is singular in its secant variety of lines, hence we just have to prove the singularity of the distance-2 orbit $\Sigma_{2,N}$. We assume by contradiction that $\Sigma_{2,N}$ is smooth: then the poset in Theorem 5.1.12 implies that all the open subset $\sigma_2(\mathbb{S}_N^+) \setminus \mathbb{S}_N^+$ is smooth. Consider the projection from the abstract secant variety onto the second factor

$$\begin{aligned} \pi : \quad Ab\sigma_2(\mathbb{S}_N^+) &\longrightarrow \sigma_2(\mathbb{S}_N^+) \\ ([a], [b], [q]) &\mapsto [q] . \end{aligned}$$

The (tangential-)identifiability of the orbits $\Sigma_{l,N}$ and $\Theta_{l,N}$ for $l \geq 3$ (cf. Theorem 5.2.5 and Theorem 5.3.2) implies that the restriction of π to the open subset $\pi^{-1}(\sigma_2(\mathbb{S}_N^+) \setminus \overline{\Sigma_{2,N}})$ is a bijection of smooth open subsets, hence it is an isomorphism. On the other hand, the orbit $\Sigma_{2,N}$ is unidentifiable and any of its points has 6-dimensional decomposition locus (cf. Proposition 5.2.3), thus the differential $d(\pi)_{([a],[b],[q])}$ drops rank exactly at the points in the preimage $\pi^{-1}(\overline{\Sigma_{2,N}})$.

It follows that the restriction

$$\pi|_1 : Ab\sigma_2(\mathbb{S}_N^+) \setminus \pi^{-1}(\mathbb{S}_N^+) \longrightarrow \sigma_2(\mathbb{S}_N^+) \setminus \mathbb{S}_N^+$$

is a morphism of smooth varieties of the same dimension: in particular, the locus of points where the rank of the differential drops is a (determinantal) divisor. We show that the preimage

$$\{([a], [b], [q]) \in Ab\sigma_2(\mathbb{S}_N^+) \setminus \pi^{-1}(\mathbb{S}_N^+) \mid \text{Rk } d(\pi)_{([a],[b],[q])} < N(N-1)\} = \pi|_1^{-1}(\overline{\Sigma_{2,N}})$$

cannot be a divisor (leading to a contradiction). Indeed, from Proposition 5.4.2 we know that $\dim \Sigma_{2,N} = \frac{N(N-1)}{2} + 4N - 15$, and from Proposition 5.2.3 that the decomposition locus of any point in $\Sigma_{2,N}$ is 6-dimensional, thus from the fiber dimension theorem we get $\dim \pi|_1^{-1}(\overline{\Sigma_{2,N}}) = \dim \Sigma_{2,N} + 6 = \frac{N(N-1)}{2} + 4N - 9$ and

$$\dim Ab\sigma_2(\mathbb{S}_N^+) - \dim \pi|_1^{-1}(\overline{\Sigma_{2,N}}) = \frac{N(N-1)}{2} - 4N + 10 \underset{N \geq 7}{\neq} 1 .$$

Thus $\pi|_1^{-1}(\overline{\Sigma_{2,N}})$ cannot be a divisor for $N \geq 7$, giving the contradiction. We deduce the inclusion $\Sigma_{2,N} \subset \text{Sing}(\sigma_2(\mathbb{S}_N^+))$. \square

Lemma 5.6.2. *For any $N \geq 7$ and any $l = 3 : \frac{N}{2}$, the secant orbit $\Sigma_{l,N}$ is smooth in the secant variety of lines $\sigma_2(\mathbb{S}_N^+)$. In particular, the singular locus of the secant variety of lines to the Spinor variety \mathbb{S}_N^+ lies in the tangential variety:*

$$\text{Sing}(\sigma_2(\mathbb{S}_N^+)) \subset \tau(\mathbb{S}_N^+) .$$

Proof. From Proposition 5.2.5 we know that the secant orbits $\Sigma_{l,N}$ for $l = 3 : N$ are identifiable. Moreover, from the poset in Figure 5.1 it holds

$$\Sigma_{3,N} \sqcup \dots \sqcup \Sigma_{\frac{N}{2},N} = \sigma_2(\mathbb{S}_N^+) \setminus \tau(\mathbb{S}_N^+) .$$

Then the projection $\pi : Ab\sigma_2(\mathbb{S}_N^+) \rightarrow \sigma_2(\mathbb{S}_N^+)$ from the abstract secant variety onto the second factor restricts to a bijection

$$\pi' : \pi^{-1} \left(\sigma_2(\mathbb{S}_N^+) \setminus \tau(\mathbb{S}_N^+) \right) \rightarrow \sigma_2(\mathbb{S}_N^+) \setminus \tau(\mathbb{S}_N^+) .$$

Finally, the points in $\sigma_2(\mathbb{S}_N^+) \setminus \tau(\mathbb{S}_N^+)$ are outside the Terracini locus $\text{Terr}_2(\mathbb{S}_N^+)$ (cf. Theorem 5.5.1), thus for any point $[p+q] \in \sigma_2(\mathbb{S}_N^+) \setminus \tau(\mathbb{S}_N^+)$ it holds $T_p\mathbb{S}_N^+ \cap T_q\mathbb{S}_N^+ = \{0\}$. In particular, for such points the differential $d\pi'_{([p],[q],[p+q])}$ maps $T_p\mathbb{S}_N^+ \times T_q\mathbb{S}_N^+$ to $T_p\mathbb{S}_N^+ \oplus T_q\mathbb{S}_N^+$, hence it is injective. It follows that π' is an isomorphism and the open subset $\sigma_2(\mathbb{S}_N^+) \setminus \tau(\mathbb{S}_N^+)$ is smooth in the secant variety of lines. \square

Collecting Lemma 5.6.1 and Lemma 5.6.2 we get the bound on the singular locus

$$\overline{\Sigma_{2,N}} \subset \text{Sing}(\sigma_2(\mathbb{S}_N^+)) \subset \tau(\mathbb{S}_N^+) .$$

However, at this point we haven't been able to get more information on the singular locus of $\sigma_2(\mathbb{S}_N^+)$, but we have a conjecture whose proof is left for future work.

Conjecture 5.6.3. *For any $N \geq 7$, the singular locus of the secant variety of lines to the Spinor variety \mathbb{S}_N^+ is the closure of the distance-2 orbit, i.e.*

$$\text{Sing}(\sigma_2(\mathbb{S}_N^+)) = \overline{\Sigma_{2,N}} = \mathbb{S}_N^+ \sqcup \Sigma_{2,N} .$$

Although we don't have a proof yet, in Sec. 7.3 we propose an argument suggesting that the conjecture is actually true.

Chapter 6

What about other generalized Grassmannians?

The identity between the poset graphs of G -orbits in the secant varieties of lines to Grassmannians (cf. Figure 4.1) and to Spinor varieties (cf. 5.1) suggests that such poset graph may hold for other generalized Grassmannians too. This chapter is addressed to investigate so. In Sec. 6.1 we show that cominuscule varieties are very good candidates: a complete proof is left for future work. In Sec. 6.2 we show that such graph does not hold for every generalized Grassmannian, by exhibiting an example: this has been obtained during a visit at Institut de Mathématiques de Toulouse with Prof. Laurent Manivel, whom the candidate thanks for the inspiring suggestions.

We assume notation and results from Chap. 1. Let G be a semisimple simply connected complex Lie group and let P_k be a maximal parabolic subgroup corresponding to the simple root $\alpha_k \in \Delta$, or equivalently to the fundamental weight ω_k . Let \mathcal{W} and \mathcal{W}_{P_k} be the Weyl groups of G and P_k respectively. Given $V_{\omega_k}^G$ the irreducible G -representation with highest weight ω_k , we denote by $v_{\omega_k} \in V_{\omega_k}^G$ a highest weight vector, so that $P_k = \text{stab}_G(v_{\omega_k})$. Let $G/P_k = G \cdot [v_{\omega_k}] \subset \mathbb{P}(V_{\omega_k}^G)$ be the projective generalized Grassmannian minimally embedded in $\mathbb{P}(H^0(X, \mathcal{O}_{G/P_k}(1))^\vee) = \mathbb{P}(V_{\omega_k}^G)$.

In the notation of Sec. 2.1, let $\sigma_2(G/P_k) \subset \mathbb{P}(V_{\omega_k}^G)$ be the secant variety of lines to G/P_k in its minimal embedding, obtained as union of the dense subset $\sigma_2^\circ(G/P_k)$ of points lying on bisecant lines to G/P_k , and of the tangential variety $\tau(G/P_k)$, whose points lie on tangent lines to G/P_k .

G -orbits in $\sigma_2^\circ(G/P_k)$. The G -orbits in the dense subset $\sigma_2^\circ(G/P_k)$ are in bijection with the G -orbits in $(G/P_k) \times (G/P_k)$, on which G acts diagonally:

$$g \cdot ([x], [y]) = (g \cdot [x], g \cdot [y]) \quad , \quad \forall g \in G, \quad \forall [x], [y] \in G/P_k .$$

By homogeneity of G/P_k , any pair $([x], [y]) \in (G/P_k) \times (G/P_k)$ is G -conjugated to a pair of the form $([v_{\omega_k}], [y'])$, whose first entry is left fixed by $P_k = \text{stab}_G(v_{\omega_k})$. Since P_k moves only the second entry, the G -orbits in $(G/P_k) \times (G/P_k)$ are in bijection with P_k -orbits in G/P_k : in particular, we get the bijection between the cosets

$$\frac{\sigma_2^\circ(G/P_k)}{G} \longleftrightarrow \frac{G/P_k}{P_k} = P_k \backslash G/P_k .$$

From the Bruhat decomposition of G (cf. Sec. 1.1) one recovers the Bruhat decomposition of $G/P_k = \bigsqcup_{w \in \mathcal{W}/\mathcal{W}_{P_k}} BwP_k$, so that

$$P_k \backslash G/P_k = \bigsqcup_{w \in \mathcal{W}_{P_k} \backslash \mathcal{W}/\mathcal{W}_{P_k}} P_k w P_k .$$

REMARK 6.0.1. The P_k -orbits in G/P_k (hence the G -orbits in $\sigma_2^\circ(G/P_k)$) are always finitely many.

G -orbits in $\tau(G/P_k)$. As already pointed out in the previous chapters, by homogeneity of G/P_k , the action of G on the tangential variety $\tau(G/P_k)$ conjugates all tangent spaces $T_{[x]}(G/P_k)$ as $[x] \in G/P_k$ varies, since $g \cdot T_{[x]}(G/P_k) = T_{[g \cdot x]}(G/P_k)$. It follows that any G -orbit \mathcal{O} in $\tau(X)$ is of the form

$$\mathcal{O} = \bigcup_{[x] \in G/P_k} (\mathcal{O} \cap T_{[x]}(G/P_k))$$

hence it is enough to determine $\mathcal{O} \cap T_{[v_{\omega_k}]}(G/P_k)$ and then move it by G -action. But $\mathcal{O} \cap T_{[v_{\omega_k}]}(G/P_k)$ is a P_k -orbit, hence determining the G -orbits in $\tau(G/P_k)$ is equivalent to determining the P_k -orbits in $T_{[v_{\omega_k}]}(G/P_k) \simeq \mathfrak{g}/\mathfrak{p} \otimes \mathbb{C}v_{\omega_k} \simeq \mathfrak{p}_k^u$:

$$\frac{\tau(G/P_k)}{G} \longleftrightarrow \frac{T_{[v_{\omega_k}]}(G/P_k)}{P_k} \longleftrightarrow \frac{\mathfrak{g}/\mathfrak{p}_k}{P_k} .$$

REMARK 6.0.2. Unlike the orbits in $\sigma_2^\circ(G/P_k)$ (cf. Remark 6.0.1), the tangent space $\mathfrak{g}/\mathfrak{p}_k \simeq \mathfrak{p}_k^u$ may contain infinitely many P_k -orbits. The cases in which the nilpotent algebra \mathfrak{p}_k^u has finitely many P -orbits have been classified by L. Hille and G. Röhrle [HR99, Theorem 1.1].

REMARK. It is surprising that the algebra $\mathfrak{g}/\mathfrak{p}$ (hence the tangential variety $\tau(G/P)$) may have infinitely many orbits although the ones in G/P (hence in the dense subset $\sigma_2^\circ(G/P)$) are always finitely many. This means that a secant orbit may degenerate to infinitely many tangent orbits.

6.1 G -orbits in $\sigma_2(G/P)$ for G/P cominuscule

In this section we restrict to consider only fundamental weights ω_k (hence generalized Grassmannians G/P_k) which are cominuscule. We assume notation and results from Sec. 1.4.

From the first part of this chapter we know that the dense subset $\sigma_2(G/P_k)$ has always finitely

many G -orbits parametrized by double cosets $\mathcal{W}_{P_k} \backslash \mathcal{W} / \mathcal{W}_{P_k}$ of the Weyl group. Moreover, from Proposition 1.4.1 we know that the tangent space $\mathfrak{g}/\mathfrak{p}_k = \mathfrak{g}_{-1}$ is an irreducible P_k -module.

In the following we collect already-known results which allow to prove that the P_k -orbits in $\mathfrak{g}/\mathfrak{p}_k$ are finitely many too, and they are as many as the P_k -orbits in G/P_k : we denote by $d_{G,k}$ such value.

Proposition 6.1.1. *In the above notation, let G/P_k be a cominuscule variety. Then:*

1. There are d_{G/P_k} -many P_k -orbits in G/P_k for $d_{G,k}$ as in Table 6.1:

$$G/P_k = \bigsqcup_{j=1}^{d_{G,k}} P_k w_j P_k ;$$

2. For any $j = 2 : d_{G,k}$ the points $w_{j-1}P_k, w_j P_k \in G/P_k$ have Hamming distance 1 and $d(idP_k, w_j P_k) = j$. In particular, the variety G/P_k has diameter $d_{G,k}$;

Moreover, the P_k -orbits in G/P_k are totally ordered:

$$\overline{P_k w_j P_k} = \bigsqcup_{i=1}^j P_k w_i P_k \quad , \quad \forall j \in [d_{G,k}] .$$

Proof. (2) is proved in [Buc+13, Sec. 4, Lemma 4.2]. (1) is proved in [RRS92, Sec. 2, Table 1], while the total order of the orbits is proved in [RRS92, Corollary 3.7(a)]: here one uses the characterization of cominuscule varieties as the generalized Grassmannians having abelian unipotent radical (cf. Proposition 1.4.1). \square

In the following table we list all the cominuscule varieties together with dimensions and values $d_{G,k}$ of their secant varieties. For the Grassmannian $\text{Gr}(k, N)$, the symbol (\star) is to remark that for $k \in \{2, N-2\}$ and $N \geq 6$ the secant variety of lines is defective of dimension $\min\{2k(N-k) + 1, \binom{N}{k} - 1\} - 4$ (cf. Sec. 4). We have computed the values $d_{k,G}$ for $\text{Gr}(k, N)$, \mathbb{Q}^{2N-2} and \mathbb{S}_N^\pm in Sec. 4.1.1, Sec. 7.2 and Sec. 3.4 respectively, while for the other values we refer to [RRS92, Table 1] or also [Buc+13, Sec. 4].

G/P_k	$\dim G/P_k$	$\mathbb{P}(V_{\omega_k}^G)$	$\dim \sigma_2(G/P_k)$	$d_{G,k}$
$\text{Gr}(k, \mathbb{C}^N) (\star)$	$k(N-k)$	$\mathbb{P}(\bigwedge^k \mathbb{C}^N)$	$\min\{2k(N-k) + 1, \binom{N}{k} - 1\}$	$\min\{k, N-k\}$
\mathbb{Q}^{2N-2}	$2N-2$	$\mathbb{P}(\mathbb{C}^{2N})$	$2N-1$ (whole space)	2
$\text{LG}(N, 2N)$	$\binom{N+1}{2}$	$\mathbb{P}(\bigwedge^N \mathbb{C}^{2N})$	$\min\{N(N+1) + 1, \binom{2N}{N} - 1\}$	N
\mathbb{S}_N^\pm	$\binom{N}{2}$	$\mathbb{P}(\bigwedge^{ev/od} \mathbb{C}^N)$	$\min\{N(N-1) + 1, 2^{N-1} - 1\}$	$\lfloor \frac{N}{2} \rfloor$
$\mathbb{O}\mathbb{P}^2$	16	\mathbb{P}^{26}	25 (hypers.)	2
E_7/P_7	27	\mathbb{P}^{55}	55 (whole space)	3

Table 6.1: Secant varieties of lines to cominuscule varieties: dimensions and $d_{G,k}$.

We are left with analyzing the tangent orbits in $\sigma_2(G/P_k)$, or equivalently the P_k -orbits in $\mathfrak{g}/\mathfrak{p}_k \simeq \mathfrak{p}_k^u$. We recall some arguments due to L. Hille and G. Röhrle [HR99] implying that for

cominuscule varieties such orbits are finitely many.

Given a parabolic subgroup P_I defined by the subset $I \subset \Delta$ of simple roots, the *nilpotency class* of the radical unipotent P^u (or equivalently of the nilpotent algebra \mathfrak{p}^u) is

$$\ell(P^u) := \sum_{\alpha \in \Delta \setminus \Phi(I)^0} m_\alpha(\rho) , \quad (6.1.1)$$

where ρ is the longest root in Φ , $m_\alpha(\rho)$ the coefficient of α in ρ , and $\Phi(I)^0$ are the roots of the parabolic algebra \mathfrak{p}_I (cf. (1.1.1)). For $I = \{\alpha_k\}$ and ω_k cominuscule, one gets

$$\ell(P_k^u) = m_{\alpha_k}(\rho) = 1 .$$

Then by [HR99, Theorem 1.1] we get that, for any cominuscule variety G/P_k , the tangent space $\mathfrak{g}/\mathfrak{p}_k$ has finitely many P_k -orbits. Actually, there is more than this.

Proposition 6.1.2. *In the above notation, let G/P_k be a cominuscule variety. Then the tangent space $T_{[v_{\omega_k}]}(G/P_k) \simeq \mathfrak{g}/\mathfrak{p}_k$ has finitely many P_k -orbits. More precisely, there are $d_{G,k}$ -many orbits for $d_{G,k}$ as in Table 6.1 and they are totally ordered:*

$$T_{[v_{\omega_k}]}(G/P_k) = \bigsqcup_{j=1}^{d_{G,k}} \mathcal{R}_j \quad , \quad \overline{\mathcal{R}_i} = \bigsqcup_{s=1}^i \mathcal{R}_s \quad , \quad \forall i \in [d_{G,k}] .$$

Proof. We already know that there are finitely many orbits. Their exact number and the inclusions among their closures can be deduced from the fact that cominuscule varieties are *compact hermitian symmetric spaces*, and as such their tangent spaces stratify accordingly to a notion of rank. However, although unelegant, we avoid to formalize the latter description, and we prove the thesis by analyzing the tangent space case by case.

The cases of the Grassmannians and Spinor varieties have been settled in Sec. 4.1.2 and Sec. 5.1.2 respectively: their tangent spaces are spaces of matrices and the stratification by rank is immediate. Moreover, the maximum ranks are respectively $d_{A_N,k} = \min\{k, N\}$ and $d_{D_N,N} = \lfloor \frac{N}{2} \rfloor$.

The case of Lagrangian Grassmannians is analogous. Accordingly to Table 1.3, the tangent space at $[v_{\omega_N}]$ to the Lagrangian Grassmannian $\text{LG}(N, 2N)$ is isomorphic to the space of $N \times N$ symmetric matrices

$$T_{[v_{\omega_N}]} \text{LG}(N, 2N) \simeq \text{Sym}^2 \mathbb{C}^N ,$$

and the unique invariant is the rank of matrices: the maximum rank is $d_{C_N,N} = N$.

The case E_7/P_7 is already known among the Legendrian varieties [LM01, Sec. 5]: the $d_{E_7,7} = 3$ orbits in $T_{[v_{\omega_7}]}(E_7/P_7)$ are obtained as intersections with the orbits (in the authors' notation) E_7/P_7 , $\sigma_+ \setminus (E_7/P_7)$ and $\tau(E_7/P_7) \setminus \sigma_+$.

Finally, the case of the quadric Q^{2N-2} is trivial, while the secant variety to the Cayley plane $\mathbb{O}\mathbb{P}^2$ is defective and $\sigma_2(\mathbb{O}\mathbb{P}^2) = \tau(\mathbb{O}\mathbb{P}^2)$: in particular, there are just $d_{E_6,1} = 2$ orbits. \square

The results in Proposition 6.1.1 and Proposition 6.1.2 allows to give a partial (actually, almost complete) description of the poset graph of G -orbits in the secant variety of lines to a

cominuscule variety.

For any P_k -orbit \mathcal{R}_j in the tangent space $T_{[v_{\omega_k}]}(G/P_k)$ from Proposition 6.1.2, we denote by

$$\Theta_j := G \cdot [v_{\omega_k} + \mathcal{R}_j] \subset \tau(G/P_k) \quad , \quad \forall j \in [d_{G,k}]$$

the corresponding G -orbit in the tangential variety. Moreover, any P_k -orbit $P_k w_j P_k$ in G/P_k from Proposition 6.1.1 defines the G -orbit

$$\Sigma_j := G \cdot [idP_k + w_j P_k] \subset \sigma_2^\circ(G/P_k) \quad , \quad \forall j \in [d_{G,k}]$$

in the dense subset of points lying on bisecant lines to G/P_k .

REMARK. Since $d(idP_k, w_1 P_k) = 1$ (cf. Proposition 6.1.1), the line $L(idP_k, w_1 P_k)$ lies in G/P_k , hence the representative $[idP_k + w_1 P_k]$ is a point on G/P_k . Moreover, from the proof of Proposition 6.1.2 any $\theta \in \mathcal{R}_1$ corresponds to a rank-1 matrix, hence it defines a point $[v_{\omega_k} + \theta]$ of G/P_k as well. In particular, it holds

$$\Theta_1 = G/P_k = \Sigma_1 .$$

Finally, any point in $\theta \in \mathcal{R}_2$ corresponds to a rank-2 matrix, hence it is of the form $\theta = \theta_1 + \theta_2$ for θ_1, θ_2 rank-1 matrices. Since both $[v_{\omega_k} + \theta_1]$ and $[v_{\omega_k} + \theta_2]$ lies in G/P_k , we conclude that the point $[v_{\omega_k} + \theta_1 + \theta_2]$ lies on the bisecant line $L([\frac{1}{2}v_{\omega_k} + \theta_1], [\frac{1}{2}v_{\omega_k} + \theta_2])$ such that $d([\frac{1}{2}v_{\omega_k} + \theta_1], [\frac{1}{2}v_{\omega_k} + \theta_2]) = 2$, that is

$$\Sigma_2 = \Theta_2 .$$

The above remark leads to the following graph, where the arrows denote the inclusion of an orbit into the closure of the other orbit:

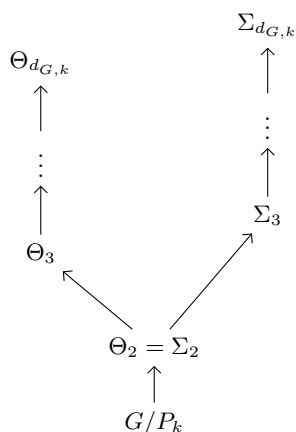


Figure 6.1: G -orbits in $\sigma_2(G/P_k)$ for G/P_k cominuscule, and some inclusions.

Conjecture 6.1.3. *The poset graph of the G -orbit in the secant variety of lines $\sigma_2(G/P_k)$ to a cominuscule variety G/P_k is obtained by adding to the graph in Figure 6.1 the inclusions $\Theta_j \subset \overline{\Sigma_j}$ for any $j \in [d_{G,k}]$.*

REMARK. Conjecture 6.1.3 is very close to be a theorem. Indeed, it has been proved for Grassmannians and Spinor varieties in Theorem 4.1.12 and Theorem 5.1.12 respectively. Moreover, as pointed out in the proof of Proposition 6.1.2, the cases of the quadrics Q^{2N-2} and of the Cayley plane $\mathbb{O}P^2$ are trivial as their secant varieties of lines coincide with the tangential varieties, and split in just two orbits. Finally, the cases $LG(3,6)$ and E_7/P_7 have been proved among the Legendrian varieties. Thus the only case for which the conjecture hasn't be proved yet is for any Lagrangian Grassmannian $LG(N,2N)$.

6.2 A non-cominuscule example: the isotropic Grassmannian $IG(k,2N)$

In this section we show that the graph in Figure 6.1 does not hold for generalized Grassmannians which are not cominuscule. As a counterexample, for $k \lesssim N$, we consider the isotropic Grassmannian $IG(k,2N)$ (cf. Example 1.3.2). Notice that the condition $k \lesssim N$ is necessary, as for $k = N$ one gets the Lagrangian Grassmannian which is cominuscule. Moreover, for $k = 1$ one gets the trivial representation. Thus in this section we assume $2 \leq k \leq N - 1$.

Setting. Let $V \simeq \mathbb{C}^{2N}$ be a complex vector space endowed with a non-degenerate symplectic form $\omega \in \wedge^2 V^\vee$. For any $m \in \mathbb{Z}_{>0}$ we fix the notation

$$\mathbb{J}_m := \begin{bmatrix} & & & 1 \\ & & \cdot & \\ & & & \\ 1 & & & \end{bmatrix} \in \text{Mat}_{m \times m} \quad , \quad \Omega_m := \begin{bmatrix} 0 & \mathbb{J}_m \\ -\mathbb{J}_m & 0 \end{bmatrix} \in \bigwedge^2 \mathbb{C}^{2m} .$$

Let $(e_1, \dots, e_N, e_{-N}, \dots, e_{-1})$ be a basis of V such that the symplectic form ω is represented by the skew-symmetric matrix $\Omega_N \in \wedge^2 \mathbb{C}^{2N}$: in particular, for any $i, j \in [N]$ it holds

$$\omega(e_i, e_j) = \omega(e_{-i}, e_{-j}) = 0 \quad , \quad \omega(e_i, e_{-j}) = \delta_{ij} .$$

Then we consider the symplectic group

$$\text{Sp}_{2N}^\omega = \text{Sp}^\omega(V) = \{A \in \text{SL}(V) \mid {}^t A \Omega_N A = \Omega_N\}$$

and, for any $k \lesssim N$, the isotropic Grassmannian

$$\text{IG}_\omega(k, 2N) = \text{IG}_\omega(k, V) = \{W \subset V \mid W \simeq \mathbb{C}^k, W \subset W^\perp\}$$

of k -dimensional linear subspaces of V which are ω -isotropic. For simplicity we write $\text{Sp}_{2N} = \text{Sp}(V)$ and $\text{IG}(k, 2N) = \text{IG}(k, V)$ by omitting the symplectic form ω . We recall that

$$\text{IG}(k, 2N) = \text{Sp}_{2N}/P_k = C_N/P_k \subset \mathbb{P}(V_{\omega_k}^{C_N}) = \mathbb{P}\left(\bigwedge^k \mathbb{C}^{2N}\right) ,$$

where P_k is the parabolic subgroup associated to the fundamental weight ω_k of Dynkin type C_N . The highest weight vector and its corresponding ω -isotropic subspace are

$$v_{\omega_k} = \mathbf{e}_{[k]} = e_1 \wedge \dots \wedge e_k \quad , \quad E_k = \langle e_1, \dots, e_k \rangle_{\mathbb{C}} \subset V .$$

Moreover, the ω -orthogonal and the dual subspaces to E_k are respectively

$$E_k^{\perp} = \langle e_1, \dots, e_N, e_{-N}, \dots, e_{-k-1} \rangle_{\mathbb{C}} \quad , \quad E_k^{\vee} = \langle e_{-k}, \dots, e_{-1} \rangle_{\mathbb{C}} .$$

Levi decomposition of P_k . We lighten up the notation by fixing $P := P_k$. The parabolic subgroup $P = \text{stab}(v_{\omega_k}) = \text{stab}(E_k)$ stabilizes the subspace E_k . In particular, it admits the Levi decomposition $P = LP^u$ where

$$L = \left\{ \left[\begin{array}{c|c} G & \\ \hline S & \mathbb{J}_k({}^t R^{-1}) \mathbb{J}_k \end{array} \right] \mid \begin{array}{l} G \in \text{GL}(E_k) \\ S \in \text{Sp}(E_k^{\perp}/E_k) \end{array} \right\} \simeq \text{GL}(E_k) \times \text{Sp}(E_k^{\perp}/E_k) \quad (6.2.1)$$

is the Levi factor of P stabilizing the consecutive quotients $E_k, E_k^{\perp}/E_k$ and $E_k^{\vee} \simeq V/E_k^{\perp}$ in the flag $E_k \subset E_k^{\perp} \subset V$, and

$$P^u = \left\{ \left[\begin{array}{ccc|c} I_k & A & B & \\ \hline & I_{2N-2k} & \Omega_{N-k}({}^t A) \mathbb{J}_k & \\ & & I_k & \\ \hline & & & B \mathbb{J}_k - \mathbb{J}_k({}^t B) = A \Omega_{N-k}({}^t A) \end{array} \right] \mid \begin{array}{l} A \in E_k \otimes E_k^{\perp}/E_k \\ B \in E_k \otimes E_k \\ B \mathbb{J}_k - \mathbb{J}_k({}^t B) = A \Omega_{N-k}({}^t A) \end{array} \right\} \quad (6.2.2)$$

is the unipotent radical of P acting trivially on the consecutive quotients of $E_k \subset E_k^{\perp} \subset V$.

The tangent bundle. The inclusion $\iota : \text{IG}(k, 2N) \hookrightarrow \text{Gr}(k, 2N)$ defines the universal bundle \mathcal{U}_{IG} on the isotropic Grassmannian $\text{IG}(k, 2N)$ as pull-back of the universal bundle \mathcal{U}_{Gr} on the Grassmannian $\text{Gr}(k, 2N)$. Moreover, the dual bundle

$$\mathcal{U}_{\text{IG}}^{\vee} = \mathcal{E}_{\omega_1}$$

is defined by the irreducible P_k -representation of highest weight ω_1 .

One way to describing the tangent bundle $\mathcal{T}_{\text{IG}(k, V)}$ is through the short exact sequence

$$0 \longrightarrow \mathcal{U}^{\vee} \otimes (\mathcal{U}^{\perp}/\mathcal{U}) \longrightarrow \mathcal{T}_{\text{IG}(k, V)} \longrightarrow \text{Sym}^2 \mathcal{U}^{\vee} \longrightarrow 0 ,$$

fitting in the commutative diagram

$$\begin{array}{ccccccc} & & & 0 & & & 0 \\ & & & \downarrow & & & \downarrow \\ 0 & \longrightarrow & \mathcal{U}_{\text{IG}}^{\vee} \otimes (\mathcal{U}_{\text{IG}}^{\perp}/\mathcal{U}_{\text{IG}}) & \longrightarrow & \mathcal{U}_{\text{Gr}}^{\vee} \otimes (\mathcal{U}_{\text{Gr}}^{\perp}/\mathcal{U}_{\text{Gr}}) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}_{\text{IG}} & \longrightarrow & \mathcal{T}_{\text{Gr}} = \mathcal{U}_{\text{Gr}}^{\vee} \otimes \mathcal{Q} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Sym}^2 \mathcal{U}_{\text{IG}}^{\vee} & \longrightarrow & \mathcal{U}_{\text{Gr}}^{\vee} \otimes \mathcal{U}_{\text{Gr}}^{\vee} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Another way to describe $\mathcal{T}_{\mathrm{IG}(k,V)}$ is by determining its fiber at $[v_{\omega_k}]$. From the theory in Sec. 1.1 we know that the simple root α_k defines a \mathbb{Z} -grading on the Lie algebra \mathfrak{sp}_{2N} whose degrees are bounded by the coefficient $m_{\alpha_k}(\rho)$ of α_k in the longest root ρ . From Table 1.2 we know that $m_{\alpha_k}(\rho) = 2$, hence

$$T_{[v_{\omega_k}]} \mathrm{IG}(k, 2N) \simeq \mathfrak{p}^u \simeq \mathfrak{sp}_{2N}/\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} .$$

Recall that, as we are not in the cominuscule case, the \mathfrak{p}^u is not irreducible as P -module, and its only submodule is \mathfrak{g}_{-1} . Deriving the description of the unipotent radical P^u in (6.2.2) we get the nilpotent algebra

$$\mathfrak{p}^u = (E_k^\vee \otimes E_k^\perp/E_k) \oplus \mathrm{Sym}^2 E_k^\vee ,$$

whose only P -invariant (for the adjoint action of P) summand is $\mathfrak{g}_{-1} = E_k^\vee \otimes E_k^\perp/E_k$. Notice that such summand is the fiber of $\mathcal{U}^\vee \otimes (\mathcal{U}^\perp/\mathcal{U})$, which is a P -homogeneous sub-bundle of $\mathcal{T}_{\mathrm{IG}(k,V)}$ being the kernel in the above short exact sequence.

6.2.1 P_k -orbits in $\mathfrak{sp}_{2N}/\mathfrak{p}_k$

We keep notation from the beginning of the section. We want to determine the poset of P -orbits in the tangent space $T_{[v_{\omega_k}]} \mathrm{IG}(k, 2N) \simeq \mathrm{Sym}^2 E_k^\vee \oplus (E_k^\vee \otimes E_k^\perp/E_k)$.

REMARK. The parabolic subgroup $P_k \subset \mathrm{Sp}_{2N}$ has nilpotency class $\ell(P_k) = 2$ (cf. (6.1.1)), hence from [HR99, Theorem 1.1] we know that \mathfrak{p}^u has finitely many P -orbits.

An element of the tangent space is of the form

$$\sigma + H \in \mathrm{Sym}^2 E_k^\vee \oplus (E_k^\vee \otimes E_k^\perp/E_k) .$$

Notice that the action of $\mathrm{GL}(E_k) \subset L$ conjugates the above element to one of the form $(e_{-k}^2 + \dots + e_{-k+r-1}^2) + \hat{H}$, where r is the rank of $\sigma \in \mathrm{Sym}^2 E_k^\vee$. In this respect, we may assume

$$\sigma = \sigma_r := e_{-k}^2 + \dots + e_{-k+r-1}^2 .$$

In the following we describe separately how the Levi factor L and the unipotent radical P^u act on an element of the form $\sigma_r + H$ for a certain $r \leq k$.

The action of P^u . We start from the action of the unipotent radical $P^u \subset P$ in (6.2.2), which depends only on the entries $A \in E_k \otimes E_k^\perp/E_k$ and $B \in E_k \otimes E_k$.

REMARK. The action of B on $\mathrm{Sym}^2 E_k^\vee$ is identically zero since there is no non-zero projection from $(E_k \otimes E_k) \otimes \mathrm{Sym}^2 E_k^\vee$ onto $\mathrm{Sym}^2 E_k^\vee \oplus (E_k^\vee \otimes E_k^\perp/E_k)$. Similarly, also the actions of A and B on $E_k^\vee \otimes E_k^\perp/E_k$ are identically zero.

The only non-trivial action is the one of A on $\mathrm{Sym}^2 E_k^\vee$, which by Schur's theorem coincides with the contraction map

$$\begin{aligned} (E_k \otimes E_k^\perp/E_k) \otimes \mathrm{Sym}^2 E_k^\vee &\longrightarrow E_k^\vee \otimes E_k^\perp/E_k \\ \left(\sum_{i,j} a_{ij} e_i \otimes e_j \right) \otimes f &\mapsto \sum_{i,j} a_{ij} \frac{\partial f}{\partial e_i} \otimes e_j . \end{aligned}$$

In particular, given $A = \sum_{i=1}^k \sum_{j=k+1}^{-k-1} a_{ij} e_i \otimes e_j \in E_k \otimes E_k^\perp / E_k$, one gets

$$A \cdot \sigma_r = \sum_j \sum_{i=k-r+1}^k 2a_{ij} e_{-i} \otimes e_j .$$

It follows that, for any unipotent element $g_A \in P^u$ depending on $A \in E_k \otimes E_k^\perp / E_k$, and for any $H = \sum_{i,j} h_{ij} e_{-i} \otimes e_j \in E_k^\vee \otimes E_k^\perp / E_k$, it holds

$$\begin{aligned} g_A \cdot (\sigma_r + H) &= (I_k \cdot \sigma_r) + (I_{2N-2k} \cdot H) + (A \cdot \sigma_r) = \sigma_r + H + A \cdot \sigma_r \\ &= \sigma_r + \sum_j \left[\sum_{i=1}^{k-r} h_{ij} e_{-i} \otimes e_j + \sum_{i=k-r+1}^k (h_{ij} + 2a_{ij}) e_{-i} \otimes e_j \right] . \end{aligned}$$

Thus acting by P^u allows to “truncate” the summand H . We conclude that *there are infinitely many P^u -orbits in $\sigma_r + (E_k^\vee \otimes E_k^\perp / E_k)$* , namely

$$\sigma_r + (E_k^\vee \otimes E_k^\perp / E_k) = \bigsqcup_{Q \in \mathbb{C}^{k-r} \otimes \mathbb{C}^{2N-2k}} P^u \cdot \left(\sigma_r + \sum_{i=1}^{k-r} e_{-i} \otimes Q^i \right) ,$$

where Q^i denotes the i -th column of the matrix $Q \in \mathbb{C}^{k-r} \otimes \mathbb{C}^{2N-2k}$.

REMARK. We point out that to be infinitely-many are the P^u -orbits (not P -orbits) for a fixed $\sigma \in \text{Sym}^2 E_k^\vee$. The additional action of the Levi factor L will reduce the number of orbits in the whole tangent space to finitely many.

The action of L . Now we deal with the action of the Levi factor $L \subset P$ in (6.2.1). In light of the previous arguments, up to acting by both $\text{GL}(E_k) \subset L$ and P^u , we may assume

$$\sigma_r + H = (e_{-k}^2 + \dots + e_{-k+r-1}^2) + \sum_{i=1}^{k-r} e_{-i} \otimes q_i \quad (6.2.3)$$

for certain vectors $q_i \in E_k^\perp / E_k$, and we are left with considering the action of the stabilizer in L of σ_r :

$$\text{stab}_L(\sigma_r) = \left\{ \begin{bmatrix} * & * & & & \\ & * & & & \\ & & N & & \\ & & & O & R \\ & & & & M \end{bmatrix} \in L \left| \begin{array}{l} O \in \text{Orth}(r) \\ M \in \text{GL}(k-r) \\ R \in \mathbb{C}^r \otimes \mathbb{C}^{k-r} \\ N \in \text{Sp}(E_k^\perp / E_k) \end{array} \right. \right\} \simeq \text{stab}_{\text{GL}(E_k)}(\sigma_r) \times \text{Sp}(E_k^\perp / E_k) ,$$

where the first block (the one with asterisks, of size $k \times k$) uniquely depends on the last block $\begin{bmatrix} O & R \\ & M \end{bmatrix}$.

Let $Q := \langle q_1, \dots, q_{k-r} \rangle_{\mathbb{C}} \subset E_k^\perp / E_k$ be the subspace spanned by the second tensor-entries in H (6.2.3), and let

$$h := \dim Q \leq \min\{2N - 2k, k - r\}$$

be its dimension. Acting by a proper permutation matrix in $\text{GL}(k-r) \subset \text{stab}_{\text{GL}(E_k)}(\sigma_r)$, we can reorder the summands in H such that (q_1, \dots, q_h) is a basis of Q . Then for any

$h+1 \leq i \leq k-r$ we can write $q_i = \sum_{j=1}^h \alpha_{ij} q_j$ with respect to this basis, for a suitable matrix $(\alpha_{ij}) \in \mathbb{C}^{k-r-h} \otimes \mathbb{C}^h$, getting

$$\sigma_r + H = \sigma_r + \sum_{i=1}^h \left(e_{-i} + \sum_{\ell=h+1}^{k-r} \alpha_{\ell i} e_{-\ell} \right) \otimes q_i .$$

Moreover, by applying a base change via $\mathrm{GL}(k-r) \subset \mathrm{stab}_{\mathrm{GL}(E_k)} \subset L$ in the above first tensor-entries, we can move such $\sigma_r + H$ to the point $\sigma_r + \sum_{i=1}^h e_{-i} \otimes q_i$. Thus for any $Q = \langle q_1, \dots, q_h \rangle_{\mathbb{C}} \in \mathrm{Gr}(h, E_k^\perp/E_k)$ we set

$$\sigma_r + H_Q := \sigma_r + \sum_{i=1}^h e_{-i} \otimes q_i . \quad (6.2.4)$$

REMARK 6.2.1. The dimension $h = \dim Q$ is an invariant for the P^u -action too (hence for the P -action). Indeed, given $\mathrm{supp}(\sigma_r) = \langle x \in E_k \mid \sigma_r(x) \neq 0 \rangle_{\mathbb{C}} = \langle e_{k-r+1}, \dots, e_k \rangle_{\mathbb{C}} \subset E_k$, one has

$$P^u \cdot (\sigma_r + H_Q) = \{ \sigma_r + H_Q + H' \mid H' \in \mathrm{supp}(\sigma_r)^\vee \otimes E_k^\perp/E_k \} .$$

Then h is well defined as the invariant

$$h := \min\{ \mathrm{Rk}(Q|Q') \mid H_{Q'} \in \mathrm{supp}(\sigma_r)^\vee \otimes E_k^\perp/E_k \} .$$

Up to now, we have used the action of both L and P^u on $\sigma_r + H$ for minimizing the number of summands appearing in H . The next step is to find representatives for such “minimal” summands.

Let $\omega' \in \mathrm{Sym}^2(E_k^\perp/E_k)^\vee$ be the non-degenerate symplectic form obtained as restriction of $\omega \in \mathrm{Sym}^2 V^\vee$. It is represented by the skew-symmetric matrix $\Omega_{N-k} \in \bigwedge^2 \mathbb{C}^{2N-2k}$.

For any subspace $Q \in \mathrm{Gr}(h, E_k^\perp/E_k)$, the restriction of ω' to Q has rank

$$\mathrm{Rk}(\omega'_{|_Q}) = h - \dim(Q \cap Q^\perp) \equiv 0 \pmod{2} .$$

Notice that:

- If $Q \subset Q^\perp$, i.e. $Q \in \mathrm{IG}_{\omega'}(h, E_k^\perp/E_k)$ and $\mathrm{Rk}(\omega'_{|_Q}) = 0$, then $h \leq N-k$ and the symplectic group $\mathrm{Sp}(E_k^\perp/E_k)$ conjugates $\sigma_r + H_Q$ to the point $\sigma_r + \sum_{i=1}^h e_{-i} \otimes e_{k+i}$.
- If $Q \cap Q^\perp = \{0\}$, then $Q \oplus Q^\perp = E_k^\perp/E_k$ and $\mathrm{Rk}(\omega'_{|_Q}) = h$ is even. Thus the symplectic group $\mathrm{Sp}(E_k^\perp/E_k)$ conjugates $\sigma_r + H_Q$ to the point $\sigma_r + \sum_{i=1}^{\frac{h}{2}} e_{-i} \otimes e_{k+i} + \sum_{i=1}^{\frac{h}{2}} e_{-\frac{h}{2}-i} \otimes e_{-k-i}$.

In general, assume that

$$\mathrm{Rk}(\omega'_{|_Q}) = h - t$$

for a certain $1 \leq t \leq h$ such that $h \equiv t \pmod{2}$. Then the intersection $Q \cap Q^\perp \in \mathrm{IG}_{\omega'}(t, E_k^\perp/E_k)$ is ω' -isotropic and $(Q/(Q \cap Q^\perp)) \oplus (Q^\perp/(Q \cap Q^\perp)) = (Q + Q^\perp)/(Q \cap Q^\perp)$, hence the action of

$\mathrm{Sp}(E_k^\perp/E_k)$ conjugates the subspace H_Q in (6.2.4) to the subspace

$$H_{(h,t)} := \sum_{i=1}^t e_{-i} \otimes e_{k+i} + \sum_{i=1}^{\frac{h-t}{2}} e_{-t-i} \otimes e_{k+t+i} + \sum_{i=1}^{\frac{h-t}{2}} e_{-t-\frac{h-t}{2}-i} \otimes e_{-k-t-i}, \quad (6.2.5)$$

that is the point $\sigma_r + H_Q$ to the point $\sigma_r + H_{(h,t)}$.

Theorem 6.2.2. *For any $2 \leq k \leq N-1$, the tangent space $T_{[v_{\omega_k}]} \mathrm{IG}_{\omega}(k, 2N) \simeq \mathfrak{sp}_{2N}/\mathfrak{p}_k$ splits in the finitely many P_k -orbits*

$$\mathcal{O}_{(r,h,t)} := \left\{ \sigma + H \in \mathrm{Sym}^2 E_k^\vee \oplus (E_k^\vee \otimes E_k^\perp/E_k) \left| \begin{array}{l} \mathrm{Rk}(\sigma) = r \\ \ker(H) \subset \mathrm{supp}(\sigma) \\ \dim \mathrm{Im}(H) = h \\ \mathrm{Rk}(\omega|_{\mathrm{Im}(H)}) = h-t \end{array} \right. \right\} = P \cdot (\sigma_r + H_{(h,t)}) \quad (6.2.6)$$

where: $\mathrm{Rk}(\sigma)$ and $\mathrm{supp}(\sigma) := \{x \in E_k \mid \sigma(x) \neq 0\} \subset E_k$ are respectively the rank and the support of the quadratic form $\sigma \in \mathrm{Sym}^2 E_k^\vee$; the matrix $H \in E_k^\vee \otimes E_k^\perp/E_k$ is considered as a linear map in $\mathrm{Hom}(E_k, E_k^\perp/E_k)$; $\mathrm{Rk}(\omega|_{\mathrm{Im}(H)})$ is the rank of the restriction of the symplectic form $\omega \in \wedge^2 V^\vee$ to the subspace $\mathrm{Im}(H) \subset E_k^\perp/E_k$; $\sigma_r + H_{(h,t)}$ is the representative in (6.2.5).

Proof. The thesis has been proved all along the previous arguments. However, we recap the steps remarking how the parabolic subgroup P acts on a given point $\sigma + H \in \mathrm{Sym}^2 E_k^\vee \oplus (E_k^\vee \otimes E_k^\perp/E_k)$:

- given $r := \mathrm{Rk}(\sigma)$, the action of $\mathrm{GL}(E_k) \subset L$ conjugates σ to $\sigma_r = e_{-k}^2 + \dots + e_{-k+r-1}^2$;
- the unipotent radical P^u maps $\sigma + H$ to a certain $\sigma + H'$ such that $\ker(H') \subset \mathrm{supp}(\sigma)$, so that H' is obtained by truncating from H the component in $\mathrm{supp}(\sigma)^\vee \otimes E_k^\perp/E_k$ (cf. (6.2.3));
- the action of $\mathrm{GL}(k-r) \subset \mathrm{stab}_{\mathrm{GL}(E_k)}(\sigma_r) \subset L$ allows to reduce the number of summands in (6.2.3) to as many summands as $h := \dim \mathrm{Im}(H)$, leading to the form (6.2.4);
- the rank $\mathrm{Rk}(\omega|_{\mathrm{Im}(H)})$ is invariant under the action of $\mathrm{Sp}(E_k^\perp/E_k) \subset L$, which conjugates the form (6.2.4) to the representative $\sigma_r + H_{(h,t)}$ in (6.2.5) where $t := \dim \mathrm{Im}(H) - \mathrm{Rk}(\omega|_{\mathrm{Im}(H)})$.

Finally, we notice that, as r, h, t vary, the points $\sigma_r + H_{(h,t)}$ cannot be conjugated with each other via P -action: indeed, even if the unipotent radical P^u moves $\sigma_r + H_{(h,t)}$ to $\sigma_r + H_{(h,t)} + \sum_{i=k-r+1}^k e_{-i} \otimes q_i$ for arbitrary $q_i \in E_k^\perp/E_k$, then the Levi factor L could not mix the vectors $e_{-k}, \dots, e_{-k+r-1}$ with the vectors e_{-k+r}, \dots, e_{-1} without also changing the quadric σ_r . \square

6.2.2 Inclusions and dimensions of the orbits

We know that the invariants of the P -action on the tangent space to $\mathrm{IG}_{\omega}(k, 2N)$ are given by the triplet

$$\left(\underbrace{\mathrm{Rk}(\sigma)}_{=:r}, \underbrace{\dim \mathrm{Im}(H|_{E_k/\mathrm{supp}(\sigma)})}_{=:h}, \underbrace{\dim \mathrm{Im}(H|_{E_k/\mathrm{supp}(\sigma)}) - \mathrm{Rk}(\omega|_{\mathrm{Im}(H|_{E_k/\mathrm{supp}(\sigma)})})}_{=:t} \right).$$

We stress out that such invariants satisfy the inequalities:

$$r \leq k \quad , \quad h \leq \min\{k - r, 2N - 2k\} \quad , \quad t \leq \min\{h, N - k\} \quad , \quad t \equiv h \pmod{2} . \quad (6.2.7)$$

Proposition 6.2.3. *The inclusions among the closures of the P_k -orbits in $\mathfrak{sp}_{2N}/\mathfrak{p}_k$ are ruled by the degeneracies of the ranks in the matrix spaces*

$$\mathrm{Sym}^2 E_k^\vee \quad , \quad \left(\frac{E_k}{\mathrm{supp}(\sigma)} \right)^\vee \otimes (E_k^\perp/E_k) \quad , \quad \bigwedge^2 \left(\frac{(E_k/\mathrm{supp}(\sigma))^\perp}{E_k/\mathrm{supp}(\sigma)} \right)^\vee .$$

In particular, the following inclusions hold (where the constraints on the right-side also respect the conditions (6.2.7)):

- i) $\mathcal{O}_{(r,h,t)} \subset \overline{\mathcal{O}_{(r,h',t')}} \quad \forall h \leq h' \leq \min\{k - r, 2N - 2k\}, \forall t \leq t' \leq h' ;$
- ii) $\mathcal{O}_{(r,h,t)} \subset \overline{\mathcal{O}_{(r',h,t)}} \quad \forall r \leq r' \leq k - h ;$
- iii) $\mathcal{O}_{(r-1,h+1,t-1)} \subset \overline{\mathcal{O}_{(r,h,t)}} \quad \forall h \leq \min\{k - r - 1, 2N - 2k\} ;$
- iv) $\mathcal{O}_{(r,h-1,t+1)} \subset \overline{\mathcal{O}_{(r,h,t)}} \quad \forall t \leq h - 2 ;$
- v) $\mathcal{O}_{(r,h,t)} \subset \overline{\mathcal{O}_{(k-1,1,1)}} \quad \forall (r, h, t) \neq (k, 0, 0) ;$
- vi) $\overline{\mathcal{O}_{(k,0,0)}} = \overline{P \cdot \sigma_k} = \mathfrak{sp}_{2N}/\mathfrak{p} .$

Proof. (i) For r fixed, and $h, t \geq 2$ when necessary, the sequences in $\mathcal{O}_{(r,h,t)}$

$$a(\epsilon) := \sigma_r + H_{(h,t)} + \left(\frac{1}{\epsilon} - 1 \right) e_{-t} \otimes e_{k+t} ,$$

$$b(\epsilon) := \sigma_r + H_{(h,t)} + \left(\frac{1}{\epsilon} - 1 \right) \left(e_{-t-\frac{h-t}{2}} \otimes e_{k+t+\frac{h-t}{2}} + e_{-t-h+t} \otimes e_{-k-t-\frac{h-t}{2}} \right) \quad (\text{if } h - t \geq 2) ,$$

$$c(\epsilon) := \sigma_r + H_{(h,t)} + e_{-t+1} \otimes \frac{1}{\epsilon} e_{-k-t} + e_{-t} \otimes \left[\left(\frac{1}{\epsilon} - 1 \right) e_{k+t} + e_{-k-t+1} \right]$$

have limits for $\epsilon \rightarrow \infty$ respectively

$$a(\epsilon) \rightarrow \sigma_r + H_{(h-1,t-1)} \in \mathcal{O}_{(r,h-1,t-1)} ,$$

$$b(\epsilon) \rightarrow \sigma_r + H_{(h-2,t)} \in \mathcal{O}_{(r,h-2,t)} ,$$

$$c(\epsilon) \rightarrow \sigma_r + H_{(h,t-2)} \in \mathcal{O}_{(r,h,t-2)} .$$

Then $\mathcal{O}_{(r,h,t)} \subset \overline{\mathcal{O}_{(r,h',t')}}$ for $(h', t') \in \{(h+1, t+1), (h+2, t), (h, t+2)\}$ and the thesis follows by an iterative argument.

(ii) Given $t \leq h \lesssim (k - r)$, the sequence

$$\left(e_{-k}^2 + \dots + \frac{1}{\epsilon} e_{-k+(r+1)-1}^2 \right) + H_{(h,t)} \in \mathcal{O}_{(r+1,h,t)}$$

has limit $\sigma_r + H_{(h,t)} \in \mathcal{O}_{(r,h,t)}$ for $\epsilon \rightarrow \infty$, hence the thesis.

(iii) The sequence

$$\sigma_r + \left(\frac{1}{\epsilon} - 1 \right) e_{-k}^2 + H_{(h,t)} + e_{-k} \otimes e_{-k-t}$$

lies in $\mathcal{O}_{(r,h,t)}$ (as it is P^u -conjugated to $\sigma_r + \left(\frac{1}{\epsilon} - 1\right) e_{-k}^2 + H_{(h,t)}$) and for $\epsilon \rightarrow \infty$ has limit in $\mathcal{O}_{(r-1,h+1,t-1)}$.

(iv) The sequence

$$\sigma_r + H_{(h,t)} + \left(\frac{1}{\epsilon} - 1\right) e_{-h} \otimes e_{-k-t-\frac{h-t}{2}}$$

lies in $\mathcal{O}_{(r,h,t)}$ and has limit $\sigma_r + H_{(h-2,t)} + e_{-t-\frac{h-t}{2}} \otimes e_{k+t+\frac{h-t}{2}} \in \mathcal{O}_{(r,h-1,t+1)}$.

(v) If $h \geq 1$, the sequence

$$\sigma_r + \frac{1}{\epsilon} (e_{-k+r}^2 + \dots + e_{-2}^2) + H_{(h,t)}$$

lies in $\mathcal{O}_{(k-1,1,1)}$ (as under P^u -action one can reduce $H_{(h,t)}$ to $e_{-1} \otimes e_{k+1}$) and has limit $\sigma_r + H_{(h,t)} \in \mathcal{O}_{(r,h,t)}$. If $h = 0$, the thesis follows by (i) and (ii).

(vi) In the orbit $\mathcal{O}_{(k,0,0)} = P \cdot \sigma_k$ there are all the points of the form $\sigma_k + \sum_{i=1}^k e_{-i} \otimes q_i$ for any $q_i \in E_k^\perp/E_k$ (via the action of P^u). In particular, for any triplet of invariants (r, h, t) the sequence

$$\sigma_k + \left(\frac{1}{\epsilon} - 1\right) (e_{-k+r}^2 + \dots + e_{-1}^2) + H_{(h,t)}$$

lies in $\mathcal{O}_{(k,0,0)}$ and for $\epsilon \rightarrow \infty$ it has limit $\sigma_r + H_{(h,t)} \in \mathcal{O}_{(r,h,t)}$, hence the thesis. \square

Corollary 6.2.4. *The orbit $\mathcal{O}_{(k,0,0)}$ is dense in $\mathfrak{sp}_{2N}/\mathfrak{p}_k$. Moreover,*

$$\overline{\mathcal{O}_{(k-1,1,1)}} = (\mathfrak{sp}_{2N}/\mathfrak{p}_k) \setminus \mathcal{O}_{(k,0,0)}$$

is the hypersurface defined by the vanishing of the determinant in $\mathrm{Sym}^2 E_k^\vee$.

Dimensions. We conclude the study of the P_k -orbits in $\mathfrak{sp}_{2N}/\mathfrak{p}_k$ by computing their dimensions. From the description of $\mathfrak{p}^u \simeq \mathrm{Sym}^2 E_k^\vee \otimes E_k^\perp/E_k$ we know that

$$\dim \mathrm{IG}_\omega(k, 2N) = \dim \mathfrak{sp}_{2N}/\mathfrak{p}_k = \frac{k(k+1)}{2} + k(2N-2k) = k \left(\frac{4N-3k+1}{2} \right).$$

REMARK. One can recover the above dimension also by noticing that $\mathrm{IG}_\omega(k, 2N)$ is the kernel of the section $s_\omega \in H^0(\mathrm{Gr}(k, 2N), \wedge^2 \mathcal{U})$ corresponding to the symplectic form $\omega \in \wedge^2(\mathbb{C}^{2N})^\vee$ (via Borel–Weil’s Theorem). Thus the fiber dimension theorem implies

$$\dim \mathrm{IG}_\omega(k, 2N) = \dim \mathrm{Gr}(k, 2N) - \mathrm{Rk} \left(\wedge^2 \mathcal{U} \right) = k \left(\frac{4N-3k+1}{2} \right).$$

Proposition 6.2.5. *In the above notation, the orbit $\mathcal{O}_{(r,h,t)}$ has dimension*

$$\dim \mathcal{O}_{(r,h,t)} = \frac{r(r+1)}{2} + (r+h)(2N-k-r) + \frac{t(t+1)}{2} - h^2 - t^2. \quad (6.2.8)$$

Proof. For any h, t satisfying conditions (6.2.7), consider the incidence variety

$$\mathcal{I}_{h,t} := \{(Q', Q) \in \mathrm{IG}_\omega(t, E_k^\perp/E_k) \times \mathrm{Gr}(h, E_k^\perp/E_k) \mid Q' = Q \cap Q^\perp\}$$

whose projection onto the first factor $\pi : \mathcal{I}_{h,t} \rightarrow \text{IG}_{\omega}(t, E_k^{\perp}/E_k)$ has fiber

$$\begin{aligned} \pi^{-1}(W) &\simeq \{Q \in \text{Gr}(h, E_k^{\perp}/E_k) \mid Q \cap Q^{\perp} = W\} \\ &\simeq \{\tilde{Q} \in \text{Gr}(h-t, W^{\perp}/W) \mid \tilde{Q} \cap \tilde{Q}^{\perp} = \{0\}\} . \end{aligned}$$

As the latter set is dense in the Grassmannian $\text{Gr}(h-t, W^{\perp}/W)$, from the fiber dimension theorem we get

$$\begin{aligned} \dim \mathcal{I}_{h,t} &= \dim \text{IG}_{\omega}(t, E_k^{\perp}/E_k) + \dim \text{Gr}(h-t, 2N-2k-2t) \\ &= \frac{t(t+1)}{2} + t(2N-2k-2t) + (h-t)(2N-2k-h-t) \end{aligned}$$

Let $[\text{Sym}^2 \mathbb{C}^k]_r$ be the set of $k \times k$ symmetric matrices of rank r , having dimension $\frac{r(r+1)}{2} + r(k-r)$. For any triplet of invariants (r, h, t) we consider the fibration

$$\begin{aligned} \rho : \mathcal{O}_{(r,h,t)} &\longrightarrow [\text{Sym}^2 E_k^{\vee}]_r \times \mathcal{I}_{h,t} \\ \sigma + H &\mapsto (\sigma, [\text{Im}(H) \cap \text{Im}(H)^{\perp} \subset \text{Im}(H)]) . \end{aligned}$$

Given $U := \langle e_{-k+r}, \dots, e_{-1} \rangle_{\mathbb{C}} \subset E_k^{\vee}$ and $\text{supp}(\sigma_r)^{\vee} = \langle e_{-k}, \dots, e_{-k+r-1} \rangle_{\mathbb{C}} \subset E_k^{\vee}$, the fiber of ρ at the point $x_{\sigma_r, Q} := (\sigma_r, [Q \cap Q^{\perp} \subset Q])$ is

$$\rho^{-1}(x_{\sigma_r, Q}) = \left\{ \sigma_r + \sum_{i=1}^h v_i \otimes q_i + H' \mid v_i \in U, H' \in \text{supp}(\sigma_r)^{\vee} \otimes E_k^{\perp}/E_k \right\}$$

and it has dimension $\dim(U \otimes \mathbb{C}^h) + \dim(\text{supp}(\sigma_r)^{\vee} \otimes E_k^{\perp}/E_k) = h(k-r) + r(2N-2k)$. From the fiber dimension theorem again, we deduce

$$\begin{aligned} \dim \mathcal{O}_{(r,h,t)} &= \dim [\text{Sym}^2 E_k^{\vee}]_r + \dim \mathcal{I}_{h,t} + \dim \rho^{-1}(x_{\sigma_r, Q}) \\ &= \frac{r(r+1)}{2} + (r+h)(2N-k-r) + \frac{t(t+1)}{2} - h^2 - t^2 . \end{aligned}$$

□

EXAMPLE 6.2.6. Consider the isotropic Grassmannian $\text{IG}(3, 8)$. From the constraints in (6.2.7), the invariants (r, h, t) are such that

$$0 \leq r \leq 2 \quad , \quad h \leq \min\{2, 3-r\} \quad , \quad t \leq \min\{h, 1\} .$$

Moreover, the condition $t \equiv h \pmod{2}$ implies that, in such a case, the value t is uniquely determined by the value h : the first non trivial case in which r, h, t are not redundant is $\text{IG}(3, 10)$. Then the tangent space $\mathfrak{sp}_8/\mathfrak{p}_3$ has the following poset of P_3 -orbits, where the arrows denote the inclusion of an orbit into the closure of another.

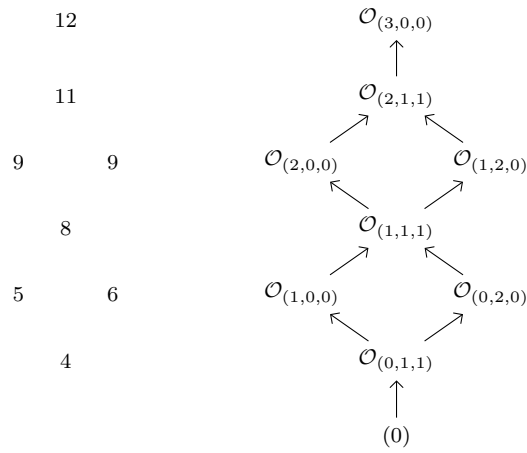


Figure 6.2: Poset graph of P_3 -orbits in $\mathfrak{sp}_8/\mathfrak{p}_3$, and their dimensions.

REMARK. Recall that each P_3 -orbit $\mathcal{O}_{(r,h,t)}$ in the tangent space $\mathfrak{sp}_8/\mathfrak{p}_3$ corresponds to a Sp_8 -orbit, say $\text{Sp}_8 \cdot \mathcal{O}_{(r,h,t)}$, in the tangential variety $\tau(\text{IG}(3, 8))$. Then the graph in Figure 6.2 also describes the poset of Sp_8 -orbits in $\tau(\text{IG}(3, 8))$: of course, the dimensions of the orbits $\text{Sp}_8 \cdot \mathcal{O}_{(r,h,t)}$ depend on the tangential-identifiability of their points.

Chapter 7

Appendix

This chapter collects some results which either are not relevant to the rest of the chapters (but they were made as an exercise) or are well-known but we haven't been able to find a proper reference.

7.1 The degree of Spinor varieties

This section is mainly intended as an exercise and it is independent from the results appearing in other chapters. We have determined a general formula for the degree of Spinor varieties by ourselves. Only after evaluating it in small dimensions (up to $N = 8$), we have found out that the sequence of degrees matches with the series A003121 on the “Online Encyclopedia of Integer Sequences” (OEIS) [\[Slo\]](#).

As usual, we assume notation from previous sections and we treat the case of D_N -type in light of Theorem 3.3.1. Given Φ the root lattice of type D_N with $(\epsilon_1, \dots, \epsilon_N)$ an orthonormal basis of $\langle \Phi \rangle_{\mathbb{R}}$, the simple roots $\Delta = \{\alpha_1, \dots, \alpha_N\}$ are

$$\alpha_1 = \epsilon_1 - \epsilon_2 \quad , \quad \dots \quad , \quad \alpha_{N-1} = \epsilon_{N-1} - \epsilon_N \quad , \quad \alpha_N = \epsilon_{N-1} + \epsilon_N$$

and they define the positive roots $\Phi^+ = \{\epsilon_i - \epsilon_j\}_{i < j} \cup \{\epsilon_i + \epsilon_j\}_{i < j}$. Recall that the D_N -type fundamental weights are

$$\begin{aligned} \omega_1 &= \epsilon_1 \quad , \quad \omega_2 = \epsilon_1 + \epsilon_2 \quad , \quad \dots \quad , \quad \omega_{N-2} = \epsilon_1 + \epsilon_2 + \dots + \epsilon_{N-2} \quad , \\ \omega_{N-1} &= \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{N-1} - \epsilon_N) \quad , \quad \omega_N = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{N-1} + \epsilon_N) \quad . \end{aligned}$$

We denote the sum of all the fundamental weights by

$$\zeta := \sum_{i=1}^N \omega_i = \sum_{l=1}^{N-1} (N-l)\epsilon_l \quad .$$

REMARK. Given a projective variety $X \subset \mathbb{P}^M$ of dimension m , its *degree* $\deg(X)$ is the number of points in which a general $\mathbb{P}^{M-m} \subset \mathbb{P}^M$ intersects X . One can define the *Hilbert polynomial* of X to be the Euler characteristic

$$\text{Hilb}_X(t) := \chi(X, \mathcal{O}(t)) := \sum_{i=0}^m (-1)^i h^i(X, \mathcal{O}(t)) ,$$

where $h^i(X, \mathcal{O}(t)) := \dim H^i(X, \mathcal{O}(t))$. When $\mathcal{O}(t)$ is spanned, all cohomology groups vanish but the one of global sections, thus the above value coincides with

$$\text{Hilb}_X(t) = h^0(X, \mathcal{O}(t)) .$$

A key result about the Hilbert polynomial is [Har13, Sec. 13] is that

$$\text{Hilb}_X(t) = \frac{\deg(X)}{\dim(X)!} t^{\dim X} + O(t^{\dim(X)-1}) .$$

Theorem 7.1.1. *For any N , the spinor variety $\mathbb{S}_N^\pm \subset \mathbb{P}(S_N^\pm)$ has degree*

$$\deg(\mathbb{S}_N^\pm) = \frac{\binom{N}{2}!}{\prod_{1 \leq i < j \leq N} (2N - (i + j))} .$$

Proof. We assume N to be even and we work with $\mathbb{S}_N^+ = D_N/P_N \subset \mathbb{P}(V_{\omega_N}^{D_N})$. We compute the Hilbert polynomial of \mathbb{S}_N^+ , and from its leading coefficient we deduce the degree.

For any $t \geq 1$ the line bundle $\mathcal{O}(t)$ on \mathbb{S}_N^+ is defined by the irreducible P_N -representation with highest weight $t\omega_N$: the latter weight is dominant for any $t \geq 1$, hence by Borel–Weil Theorem the line bundle $\mathcal{O}(t)$ is spanned at any point of \mathbb{S}_N^+ . Thus $\text{Hilb}_{\mathbb{S}_N^+}(t) = h^0(\mathbb{S}_N^+, \mathcal{O}(t))$. Moreover, again by Borel–Weil Theorem, the latter dimension coincides with the dimension of the irreducible representation $V_{t\omega_N}^{D_N}$, which can be computed by the Weyl’s dimension formula (1.2.1):

$$\begin{aligned} \dim V_{t\omega_N}^{D_N} &= \prod_{\alpha \in \Phi^+} \frac{\kappa(t\omega_N + \zeta, \alpha)}{\kappa(\zeta, \alpha)} \\ &= \prod_{1 \leq i < j \leq N} \frac{\kappa(t\omega_N + \zeta, \epsilon_i - \epsilon_j)}{\kappa(\zeta, \epsilon_i - \epsilon_j)} \cdot \frac{\kappa(t\omega_N + \zeta, \epsilon_i + \epsilon_j)}{\kappa(\zeta, \epsilon_i + \epsilon_j)} . \end{aligned}$$

By direct computation one gets

$$\begin{aligned} \kappa(t\omega_N + \zeta, \epsilon_i - \epsilon_j) &= \kappa \left(\sum_{l=1}^N \left(N - l + \frac{t}{2} \right) \epsilon_l, \epsilon_i - \epsilon_j \right) \\ &= \left(N - i + \frac{t}{2} \right) \kappa(\epsilon_i, \epsilon_i) - \left(N - j + \frac{t}{2} \right) \kappa(\epsilon_j, \epsilon_j) \\ &= j - i , \end{aligned}$$

$$\begin{aligned} \kappa(t\omega_N + \zeta, \epsilon_i + \epsilon_j) &= \kappa\left(\sum_{l=1}^N \left(N - l + \frac{t}{2}\right) \epsilon_l, \epsilon_i + \epsilon_j\right) \\ &= \left(N - i + \frac{t}{2}\right) \kappa(\epsilon_i, \epsilon_i) + \left(N - j + \frac{t}{2}\right) \kappa(\epsilon_j, \epsilon_j) \\ &= 2N - (i + j) + t, \end{aligned}$$

$$\kappa(\zeta, \epsilon_i - \epsilon_j) = (N - i) \kappa(\epsilon_i, \epsilon_i) - (N - j) \kappa(\epsilon_j, \epsilon_j) = j - i,$$

$$\kappa(\zeta, \epsilon_i + \epsilon_j) = (N - i) \kappa(\epsilon_i, \epsilon_i) + (N - j) \kappa(\epsilon_j, \epsilon_j) = 2N - (i + j).$$

We conclude that the Hilbert polynomial for \mathbb{S}_N^+ is the degree- $\binom{N}{2}$ polynomial

$$Hilb_{\mathbb{S}_N^+}(t) = \dim V_{t\omega_N}^{D_N} = \prod_{1 \leq i < j \leq N} \frac{2N - (i + j) + t}{2N - (i + j)},$$

having leading coefficient

$$\frac{\deg(\mathbb{S}_N^+)}{\binom{N}{2}!} = \frac{1}{\prod_{i < j} (2N - (i + j))}.$$

□

REMARK. The degrees of the Spinor varieties \mathbb{S}_N^+ for $N = 2 : 8$ are (also cf. integer sequence A003121 on OEIS [Slo])

N	2	3	4	5	6	7	8
$\deg(\mathbb{S}_N)$	1	1	2	12	286	33592	23178480

Table 7.1: Degrees of small Spinor varieties \mathbb{S}_N for $N = 2 : 8$.

From Table 5.1 we recall that $\mathbb{S}_2 \simeq \mathbb{P}^1$ and $\mathbb{S}_3 \simeq \mathbb{P}^3$ are linear (hence of degree 1), while $\mathbb{S}_4 \simeq \mathbb{Q}^6$ is a quadric (hence of degree 2). The first non-trivial case is the ten-fold Spinor variety \mathbb{S}_5 .

7.2 Spin_M-orbits in $\mathbb{Q}^{M-2} \times \mathbb{Q}^{M-2}$

Let V be an M -dimensional complex vector space and let $q \in \text{Sym}^2 V^\vee$ be a non-degenerate quadratic form on it. Let \mathbb{Q}^{M-2} be the projective quadric in $\mathbb{P}^{M-1} = \mathbb{P}(V)$ defined by q . The action of the special orthogonal group SO_M splits \mathbb{P}^{M-1} into the two orbits \mathbb{Q}^{M-2} and $\mathbb{P}^{M-1} \setminus \mathbb{Q}^{M-2}$, depending on the q -isotropy of points. In particular, the SO_M -action on $\mathbb{P}^{M-1} \times \mathbb{P}^{M-1}$ restricts to an action on $\mathbb{Q}^{M-2} \times \mathbb{Q}^{M-2}$.

For points in $\mathbb{Q}^{M-2} \times \mathbb{Q}^{M-2}$ one can consider the notion of Hamming distance (cf. Sec. 2.3). Since quadrics have Hamming distance 2, one gets the set partition

$$\mathbb{Q}^{M-2} \times \mathbb{Q}^{M-2} = \Delta_{\mathbb{Q}^{M-2}} \sqcup \mathcal{O}_1 \sqcup \mathcal{O}_2$$

where $\Delta_{\mathbb{Q}^{M-2}}$ is the diagonal and $\mathcal{O}_\ell := \{([v], [w]) \mid d([v], [w]) = \ell\}$ for $\ell = 1, 2$. By Remark 2.3.2 the Hamming distance is invariant under the action of SO_M , hence SO_M acts on each \mathcal{O}_i for $i = 0, 1, 2$: clearly, the action on the diagonal is transitive.

REMARK. From now on we assume V to be of even dimension $M = 2m$ and equipped with the quadratic form $q(x) = \sum_{i=1}^m x_i x_{m+i} \in \mathrm{Sym}^2 V^\vee$ represented by the matrix

$$Q := \frac{1}{2} \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}.$$

Let $(e_1, \dots, e_m, f_1, \dots, f_m)$ be a standard hyperbolic basis with respect to q . For any $w \in V$ the scalar $q(e_i, w)$ is the coefficient of f_i in the decomposition of w with respect to the above basis (and viceversa, $q(f_i, w)$ is the one of e_i).

Proposition. SO_{2m} acts transitively on \mathcal{O}_1 . In particular, $\mathcal{O}_1 = \mathrm{SO}_{2m} \cdot ([e_1], [e_2])$.

Proof. Given $([v], [w]) \in \mathcal{O}_1$, we may assume $[v] = [e_1]$ up to moving via SO_{2m} . Since $d([e_1], [w]) = 1$, the line $L([e_1], [w])$ lies in \mathbb{Q}^{2m-2} , hence the subspace $\langle e_1, w \rangle_{\mathbb{C}}$ is q -isotropic, that is $\langle e_1, w \rangle_{\mathbb{C}} \in \mathrm{OG}(2, V)$. Since $\mathrm{OG}(2, V)$ is homogeneous under the action of SO_{2m} , it follows that $\langle e_1, w \rangle_{\mathbb{C}}$ is conjugated to the q -isotropic subspace $\langle e_1, e_2 \rangle_{\mathbb{C}}$, hence the thesis. \square

Proposition. SO_{2m} acts transitively on \mathcal{O}_2 . In particular, $\mathcal{O}_2 = \mathrm{SO}_{2m} \cdot ([e_1], [f_1])$.

Proof. Given $([v], [w]) \in \mathcal{O}_2$, we may assume $[v] = [e_1]$ up to moving via SO_{2m} . Since $d([e_1], [w]) = 2$, the line $L([e_1], [w])$ does not lie in \mathbb{Q}^{2m-2} but there exists $[z] \in \mathbb{Q}^{2m-2}$ such that $L([e_1], [z]), L([z], [w]) \subset \mathbb{Q}^{2m-2}$. The subspace $\langle e_1, z, w \rangle_{\mathbb{C}}$ is not q -isotropic since $\langle e_1, w \rangle_{\mathbb{C}}$ is not so, but $\langle e_1, z \rangle_{\mathbb{C}} \in \mathrm{OG}(2, V)$, hence there exists $A \in \mathrm{SO}_{2m}$ conjugating it to $\langle e_1, e_2 \rangle_{\mathbb{C}}$ and we may assume $z = e_2$.

Since $q(e_1, w) \neq 0$ and $q(e_2, w) = 0$ we may assume $q(e_1, w) = 1$ so that $w = q(f_1, w)e_1 + f_1 + q(f_2, w)e_2 + w'$ where $w' \in V' := \langle e_3, \dots, e_m, f_3, \dots, f_m \rangle_{\mathbb{C}}$. Notice that up to acting by $\mathrm{SO}(V') < \mathrm{SO}_{2m}$ we may assume $w' = 0$. Hence we reduce to consider an action of SO_4 on $\langle e_1, e_2, f_1, f_2 \rangle_{\mathbb{C}}$. Finally, the condition $q(w) = 0$ implies $q(f_1, w) = 0$, that is $w = f_1 + q(f_2, w)e_2$, which is SO_4 -conjugated to $f_1 + e_2$. At the end of the day, we have reduced to prove that

$(e_1, f_1 + e_2)$ and (e_1, f_1) are SO_4 -conjugated, and this is done by the matrix $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. \square

Corollary. The group Spin_{2m} acts transitively on \mathcal{O}_1 and on \mathcal{O}_2 .

Proof. The spin group Spin_{2m} is the universal double cover of SO_{2m} and the following SES holds

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_{2m} \longrightarrow \mathrm{SO}_{2m} \longrightarrow 1.$$

Thus any transformation $A \in \mathrm{SO}_{2m}$ lifts to two transformations $\pm A \in \mathrm{Spin}_{2m}$ differing by a sign. But the sign does not affect the action on \mathcal{O}_1 and \mathcal{O}_2 since their points are pairs of projective points. \square

REMARK. The above results still hold for V of odd dimension $M = 2m + 1$. In this case one considers the quadratic form described by the matrix $\begin{bmatrix} Q & \\ & \frac{1}{2} \end{bmatrix}$ with respect to the standard hyperbolic basis $(e_1, \dots, e_m, f_1, \dots, f_m, u)$ for $q(u) = 1$. Then it is enough to repeat the same arguments in the previous proves up to substituting every matrix A with the matrix $\begin{bmatrix} A & \\ & 1 \end{bmatrix}$.

We conclude the following result.

Theorem 7.2.1. *Let G be either SO_M or $Spin_M$. The Hamming distance is the unique G -invariant in $\mathbb{Q}^{M-2} \times \mathbb{Q}^{M-2}$, splitting in the G -orbits*

$$\mathbb{Q}^{M-2} \times \mathbb{Q}^{M-2} = \Delta_{\mathbb{Q}^{M-2}} \sqcup \mathcal{O}_1 \sqcup \mathcal{O}_2 = \left(G \cdot ([e_1], [e_1]) \right) \sqcup \left(G \cdot ([e_1], [e_2]) \right) \sqcup \left(G \cdot ([e_1], [f_1]) \right).$$

7.3 Secant bundle on Spinor varieties

The arguments in this section are inspired from [Ver01; Ver09; Ull16]: we thank L. Manivel for suggesting these references. They have been investigated in the attempt of proving the smoothness of both secant and tangent orbits of distance greater or equal than 3 in the secant variety of lines to a Spinor variety. The idea was to prove that the secant bundle gives a desingularization of the secant variety, but we have just been able to exhibit a bijection (as sets) between dense subsets.

We recall that a line bundle \mathcal{L} on a smooth projective variety X is k -very ample if every 0-dimensional subscheme $\mathcal{Z} \subset X$ of length $k + 1$ imposes independent conditions on \mathcal{L} , or equivalently if the restriction map $H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}})$ is surjective: in particular, \mathcal{L} is 1-very ample if and only if it is very ample, and \mathcal{L} is k -very ample if and only if there does not exist a 0-dimensional subscheme $\mathcal{Z} \subset X$ of length $k + 1$ lying on a linear subspace $\mathbb{P}^{k-1} \subset \mathbb{P}(H^0(X, \mathcal{L})^\vee)$.

REMARK 7.3.1. The line bundle $\mathcal{O}_{\mathbb{S}_N^+}(1)$ on the Spinor variety \mathbb{S}_N^+ is very ample, giving the embedding in $\mathbb{P} \left(H^0(\mathbb{S}_N^+, \mathcal{O}_{\mathbb{S}_N^+}(1))^\vee \right) \simeq \mathbb{P}(\wedge^{ev} E)$, but not 2-very ample. Indeed, it would be 2-very ample if and only if there would not exist a 0-dimensional subscheme of \mathbb{S}_N^+ of length 3 lying on a line $\mathbb{P}^1 \subset \mathbb{P}(\wedge^{ev} E)$. But we know that the Spinor variety contains lines, defined by pairs of pure spinors having Hamming distance 1.

Secant bundle. Let $\mathcal{L} = \mathcal{O}(1)$ be the very ample line bundle on $X := \mathbb{S}_N^+$, and let $Hilb_2(X)$ be the Hilbert scheme of 0-dimensional subschemes of \mathbb{S}_N^+ of length 2. We denote by $\mathcal{E}_{\mathcal{L}}$ the locally free sheaf of rank 2 on $Hilb_2(X)$ having fibers $(\mathcal{E}_{\mathcal{L}})_{\mathcal{Z}} = H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}})$: formally, $\mathcal{E}_{\mathcal{L}} := (\pi_2)_*(\pi_1^*(\mathcal{L}))$ where π_1 and π_2 are the natural projections from the universal family $\Phi := \{(x, \mathcal{Z}) \in X \times Hilb_2(X) \mid x \in \mathcal{Z}\}$ onto X and $Hilb_2(X)$ respectively. From the projection formula, one gets the following global sections of sheaves:

$$H^0(Hilb_2(X), \mathcal{E}_{\mathcal{L}}) = H^0(\Phi, (\pi_X)^*\mathcal{L}) = H^0(X, \mathcal{L} \otimes (\pi_X)_*\mathcal{O}_{\Phi}) = H^0(X, \mathcal{L}),$$

where the last equality follows since π_X is a proper projection and $(\pi_X)_*\mathcal{O}_\Phi = \mathcal{O}_X$.

By pushing forward via π_H the morphism $H^0(X, \mathcal{L}) \otimes \mathcal{O}_\Phi \rightarrow (\pi_X)^*\mathcal{L}$ (of sheaves on Φ) one gets the *evaluation* morphism of sheaves on $Hilb_2(X)$

$$ev : H^0(X, \mathcal{L}) \otimes \mathcal{O}_{Hilb_2(X)} \longrightarrow \mathcal{E}_\mathcal{L} ,$$

defined on the fibre as the restriction: for any $\mathcal{Z} \in Hilb_2(X)$ it holds

$$ev_{\mathcal{Z}} : \begin{array}{ccc} H^0(X, \mathcal{L}) & \longrightarrow & (\mathcal{E}_\mathcal{L})_{\mathcal{Z}} = H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}}) \\ s & \longmapsto & s|_{\mathcal{Z}} \end{array} .$$

As \mathcal{L} is very ample, the evaluation

$$ev_{\mathcal{Z}} : H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}})$$

is a surjection for any $\mathcal{Z} \in Hilb_2(X)$, hence the morphism of sheaves $H^0(X, \mathcal{L}) \otimes \mathcal{O}_{Hilb_2(X)} \rightarrow \mathcal{E}_\mathcal{L}$ is surjective. Moreover, again by very ampleness of \mathcal{L} , one can get the injective morphism

$$\psi : \begin{array}{ccc} Hilb_2(X) & \hookrightarrow & \text{Gr}(2, H^0(X, \mathcal{L})) \\ \mathcal{Z} & \longmapsto & [H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}})] \end{array} .$$

Now, we may think at $\mathbb{P}^M = \mathbb{P}(H^0(X, \mathcal{L})^\vee)$ as to the 1-dimensional quotients of $H^0(X, \mathcal{L})$, i.e.

$$\mathbb{P}^M = \{H^0(X, \mathcal{L}) \rightarrow Q \mid \dim Q = 1\} .$$

We denote by Q^1 a 1-dimensional vector space. Then one defines the (first) *secant bundle* $\mathbb{P}\mathcal{E}_\mathcal{L}$ as the \mathbb{P}^1 -bundle on $Hilb_2(X)$ with fibers $(\mathbb{P}\mathcal{E}_\mathcal{L})_{\mathcal{Z}} = \{(\mathcal{Z}, [H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}}) \rightarrow Q^1])\}$. The following morphism is well-defined:

$$f : \begin{array}{ccc} \mathbb{P}\mathcal{E}_\mathcal{L} & \longrightarrow & \mathbb{P}(H^0(X, \mathcal{L})^\vee) \\ (\mathcal{Z}, [H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}}) \rightarrow Q^1]) & \longmapsto & [H^0(X, \mathcal{L}) \rightarrow Q^1] \end{array} .$$

Notice that $[H^0(X, \mathcal{L}) \rightarrow Q^1] \subset \text{Im}(f)$ if and only if it factors through a certain $H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}})$ for some $\mathcal{Z} \in Hilb_2(X)$: in particular,

$$f(\mathbb{P}\mathcal{E}_\mathcal{L}) = \sigma_2(X) \subset \mathbb{P}(H^0(X, \mathcal{L})^\vee) .$$

For any $\mathcal{Z} \in Hilb_2(X)$, the fibre $(\mathbb{P}\mathcal{E}_\mathcal{L})_{\mathcal{Z}} = \{(\mathcal{Z}, [H^0(X, \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}}) \rightarrow Q^1])\}$ is bijectively mapped via f to the secant line $L(\mathcal{Z}) \subset \sigma_2(X)$ intersecting X in the points of \mathcal{Z} : if $\mathcal{Z} = \{p, q\}$ is a reduced subscheme, then the fibre at \mathcal{Z} gives the bisecant line $(\mathbb{P}\mathcal{E}_\mathcal{L})_{\mathcal{Z}} \xrightarrow{1:1} L(p, q)$, while if \mathcal{Z} is a non-reduced subscheme corresponding to $\{p, v\}$, where $p \in X$ and $v \in T_p X$, then the fibre at \mathcal{Z} gives the tangent line $(\mathbb{P}\mathcal{E}_\mathcal{L})_{\mathcal{Z}} \xrightarrow{1:1} L_v(p) = p + tv$. Finally, in our setting, for any two distinct subschemes $\mathcal{Z}, \mathcal{Y} \in Hilb_2(X)$ the lines $L(\mathcal{Z})$ and $L(\mathcal{Y})$ coincide if and only if they lie in X , as the Spinor variety is intersection of quadrics.

Lemma 7.3.2. *The restriction of f to $f^{-1}(\sigma_2(\mathbb{S}_N^+) \setminus \overline{\Sigma_{2,N}})$ is a bijection.*

Proof. By Theorem 5.2.5, the points in the secant orbits $\Sigma_{l,N}$ for $l \geq 3$ are identifiable, thus any two bisecant lines (given by points in $\Sigma_{l,N}$ for $l \geq 3$, except for the two points in which they intersect \mathbb{S}_N^+) do not intersect away from \mathbb{S}_N^+ . By Theorem 5.3.2, the points in the tangent orbits $\Theta_{l,N}$ for $l \geq 3$ are tangential-identifiable, lying on a unique tangent line, hence any two tangent lines do not intersect away from $\overline{\Sigma_{2,N}}$. Finally, by the orbit partition of $\sigma_2(\mathbb{S}_N^+)$ in Theorem 5.1.12, since $\Theta_{l,N} \neq \Sigma_{l,N}$ for any $l \geq 3$, a tangent line and a bisecant line never intersect away from $\overline{\Sigma_{2,N}}$. It follows that $f : \mathbb{P}\mathcal{E}_{\mathcal{L}} \rightarrow \sigma(\mathbb{S}_N^+)$ is a bijection away from $f^{-1}(\overline{\Sigma_{2,N}})$. \square

REMARK. We point out that Vermeire in [Ver09, Proposition 1.2, Theorem 2.2] and Ullery in [Ull16, Lemma 1.1] prove the smoothness of $\sigma_2(X) \setminus X$ under the assumption of the line bundle giving the embedding to be 3-very ample: this assumption is the one *separating secant and tangent lines* away from X . In our case, the previous hypothesis is not satisfied (see Remark 7.3.1). At the set-level, the identifiability and the tangential-identifiability allow to separate secant and tangent lines (away from $\overline{\Sigma_{2,N}}$) as well. What is missing is that the above restriction has injective differential, so that it would be an isomorphism.

Bibliography

- [AOP12] H. Abo, G. Ottaviani, and C. Peterson. “Non-defectivity of Grassmannians of planes”. *J. Algebraic Geom.* 21.1 (2012), pp. 1–20.
- [AR03] A. Alzati and F. Russo. “Special subhomaloidal systems of quadrics and varieties with one apparent double point”. *Math. Proc. Cambridge Philos. Soc.* 134.1 (2003), pp. 65–82.
- [Ang11] E. Angelini. “Higher secants of spinor varieties”. *Boll. Unione Mat. Ital. (9)* 4.2 (2011), pp. 213–235.
- [Arr+21] E. Arrondo, A. Bernardi, P.M. Marques, and B. Mourrain. “Skew-symmetric tensor decomposition”. *Commun. Contemp. Math.* 23.2 (2021), p. 1950061.
- [BBS20] E. Ballico, A. Bernardi, and P. Santarsiero. “Terracini locus for three points on a Segre variety”. *arXiv preprint arXiv:2012.00574* (2020).
- [BC21] E. Ballico and L. Chiantini. “On the Terracini locus of projective varieties”. *Milan Journal of Mathematics* 89.1 (2021), pp. 1–17.
- [Bat14] C. Batista. “Pure subspaces, generalizing the concept of pure spinors”. *J. Geom. Phys.* 81 (2014), pp. 117–127.
- [BDD07] K. Baur, J. Draisma, and W. De Graaf. “Secant dimensions of minimal orbits: computations and conjectures”. *Experiment. Math.* 16.2 (2007), pp. 239–250.
- [Ber+18] A. Bernardi, E. Carlini, M.V. Catalisano, A. Gimigliano, and A. Oneto. “The hitchhiker guide to: Secant varieties and tensor decomposition”. *Mathematics* 6.12 (2018), p. 314.
- [BC12] A. Bernardi and I. Carusotto. “Algebraic geometry tools for the study of entanglement: an application to spin squeezed states”. *J. Phys. A* 45.10 (2012), pp. 105304, 13. DOI: [10.1088/1751-8113/45/10/105304](https://doi.org/10.1088/1751-8113/45/10/105304).
- [BR13] A. Bernardi and K. Ranestad. “On the cactus rank of cubics forms”. *J. Symbolic Comput.* 50 (2013), pp. 291–297.
- [BT20] A. Bernardi and D. Tauffer. “Waring, tangential and cactus decompositions”. *J. Math. Pures Appl. (9)* 143 (2020), pp. 1–30.

- [BV18] A. Bernardi and D. Vanzo. “A new class of non-identifiable skew-symmetric tensors”. *Annali di Matematica Pura ed Applicata (1923-)* 197.5 (2018), pp. 1499–1510.
- [BSS23] V. Borovik, B. Sturmfels, and S. Sverrisdóttir. “Coupled Cluster Degree of the Grassmannian”. *arXiv preprint, arXiv:2310.15474* (2023).
- [Buc+13] A.S. Buch, P.E. Chaput, L.C. Mihalcea, and N. Perrin. “Finiteness of cominusule quantum K-theory”. *Ann. Sci. Éc. Norm. Supér. (4)* 46.3 (2013), pp. 477–494.
- [BB14] W. Buczyńska and J. Buczyński. “Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes”. *J. Algebraic Geom.* 23.1 (2014), pp. 63–90.
- [Car67] É. Cartan. *The theory of spinors*. The M.I.T. Press, Cambridge, Mass., 1967.
- [CGG05] M. Catalisano, A. Geramita, and A. Gimigliano. “Secant varieties of Grassmann varieties”. *Proc. Amer. Math. Soc.* 133.3 (2005), pp. 633–642.
- [Che54] C. Chevalley. *The algebraic theory of spinors*. Columbia University Press, New York, 1954.
- [COV14] L. Chiantini, G. Ottaviani, and N. Vannieuwenhoven. “An algorithm for generic and low-rank specific identifiability of complex tensors”. *SIAM J. Matrix Anal. Appl.* 35.4 (2014), pp. 1265–1287.
- [Cle03] J.L. Clerc. “Special prehomogeneous vector spaces associated to F4, E6, E7, E8 and simple Jordan algebras of rank 3”. *J. Algebra* 264.1 (2003), pp. 98–128.
- [Com14] P. Comon. “Tensors: a brief introduction”. *IEEE Signal Processing Magazine* 31.3 (2014), pp. 44–53.
- [FH79] W. Fulton and J. Hansen. “A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings”. *Ann. of Math.* 110.1 (1979), pp. 159–166.
- [FH91] W. Fulton and J. Harris. *Representation theory: a first course*. Graduate Texts in Mathematics, 129. Springer-Verlag, New York, 1991.
- [FH21] K. Furukawa and K. Han. “On the singular loci of higher secant varieties of Veronese embeddings”. *preprint arXiv:2111.03254* (2021).
- [Gal17] M. Gałazka. “Vector bundles give equations of cactus varieties”. *Linear Algebra Appl.* 521 (2017), pp. 254–262.
- [Gal23] V. Galgano. “Identifiability and singular locus of secant varieties to spinor varieties”. *preprint arXiv:2302.05295* (2023).
- [GH23] V. Galgano and F. Holweck. “Graph States and the Variety of Principal Minors”. *Annali Matematica Pura ed Applicata (1923-)* (2023). DOI: [10.1007/s10231-023-01361-8](https://doi.org/10.1007/s10231-023-01361-8).

- [GKM23] V. Galgano, H. Keneshlou, and M. Michałek. “Equivariant Euler characteristics on permutohedral varieties”. *Lefschetz Properties - Current and New Directions, Springer INdAM Book Series, to appear, preprint arXiv:2302.04598* (2023).
- [GS23] V. Galgano and R. Staffolani. “Identifiability and singular locus of secant varieties to Grassmannians”. *Collectanea Mathematica, to appear, preprint arXiv:2212.05811* (2023).
- [GO22] F. Galuppi and A. Oneto. “Secant non-defectivity via collisions of fat points”. *Advances in Mathematics* 409 (2022), p. 108657.
- [Han18] K. Han. “On singularities of third secant varieties of Veronese embeddings”. *Linear Algebra and its Applications* 544 (2018), pp. 391–406.
- [Har13] J. Harris. *Algebraic geometry: a first course*. Vol. 133. Springer Science & Business Media, 2013.
- [HR99] L. Hille and G. Röhrle. “A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical”. *Transform. Groups* 4.1 (1999), pp. 35–52. DOI: [10.1007/BF01236661](https://doi.org/10.1007/BF01236661).
- [IK99] A. Iarrobino and V. Kanev. *Power sums, Gorenstein algebras, and determinantal loci*. Lecture Notes in Mathematics, 1721. Springer-Verlag, Berlin, 1999.
- [Kaj99] H. Kaji. “Homogeneous projective varieties with degenerate secants”. *Trans. Amer. Math. Soc.* 351.2 (1999), pp. 533–545.
- [Kan99] V. Kanev. “Chordal varieties of Veronese varieties and catalecticant matrices”. *J. Math. Sc. (New York)* 94.1 (1999), pp. 1114–1125.
- [Kos61] B. Kostant. “Lie algebra cohomology and the generalized Borel-Weil theorem”. *Ann. of Math. (2)* 74 (1961), pp. 329–387. DOI: [10.2307/1970237](https://doi.org/10.2307/1970237).
- [LP13] A. Laface and E. Postingshel. “Secant varieties of Segre-Veronese embeddings of $(\mathbb{P}^1)^r$ ”. *Mathematische Annalen* 356.4 (2013), pp. 1455–1470.
- [Lan12] J.M Landsberg. *Tensors: geometry and applications*. Vol. 128. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012.
- [LM01] J.M Landsberg and L. Manivel. “The projective geometry of Freudenthal’s magic square”. *J. Algebra* 239.2 (2001), pp. 477–512.
- [LM03] J.M Landsberg and L. Manivel. “On the projective geometry of rational homogeneous varieties”. *Comment. Math. Helv.* 78.1 (2003), pp. 65–100.
- [LM04] J.M Landsberg and L. Manivel. “Representation theory and projective geometry”. In: *Algebraic transformation groups and algebraic varieties*. Encyclopedia Math. Sci., 132. Springer, Berlin, 2004, pp. 71–122.
- [LM07] J.M Landsberg and L. Manivel. “Legendrian varieties”. *Asian J. Math.* 11.3 (2007), pp. 341–360.

- [LO13] J.M Landsberg and G. Ottaviani. “Equations for secant varieties of Veronese and other varieties”. *Ann. Mat. Pura Appl.* 192.4 (2013), pp. 569–606.
- [LW09] J.M Landsberg and J. Weyman. “On secant varieties of compact Hermitian symmetric spaces”. *J. Pure Appl. Algebra* 213.11 (2009), pp. 2075–2086.
- [LW07] JM Landsberg and J. Weyman. “On tangential varieties of rational homogeneous varieties”. *J. London Math. Soc.* 76.2 (2007), pp. 513–530.
- [LM89] H.B. Lawson and M.L. Michelsohn. *Spin Geometry*. Princeton Mathematical Series, 38. Princeton University Press, Princeton, 1989.
- [LH15] P. Lévy and F. Holweck. “Embedding qubits into fermionic Fock space: peculiarities of the four-qubit case”. *Phys. Rev. D* 91.12 (2015), pp. 125029, 25. DOI: [10.1103/PhysRevD.91.125029](https://doi.org/10.1103/PhysRevD.91.125029).
- [LH18] P. Lévy and F. Holweck. “A fermionic code related to the exceptional group E8”. *J. Phys. A* 51.32 (2018), p. 325301.
- [Lim21] L.H. Lim. “Tensors in computations”. *Acta Numerica* 30 (2021), pp. 555–764.
- [Man09] L. Manivel. “On spinor varieties and their secants”. *SIGMA Symmetry Integrability Geom. Methods Appl.* 5 (2009).
- [Man13] L. Manivel. “Prehomogeneous spaces and projective geometry”. *Rend. Sem. Mat. Univ. Politec. Torino* 71.1 (2013), pp. 35–118.
- [Man21] L. Manivel. “The double Cayley Grassmannian”. *Math. Research Letters* 28.6 (2021), pp. 1765–1792.
- [MM15] L. Manivel and M. Michałek. “Secants of minuscule and cominuscule minimal orbits”. *Linear Algebra Appl.* 481 (2015), pp. 288–312.
- [MOZ15] M. Michałek, L. Oeding, and P. Zwiernik. “Secant cumulants and toric geometry”. *Int. Math. Res. Not.* 2015.12 (2015), pp. 4019–4063.
- [OO13] L. Oeding and G. Ottaviani. “Eigenvectors of tensors and algorithms for Waring decomposition”. *J. Symbolic Comput.* 54 (2013), pp. 9–35.
- [Ott95] G. Ottaviani. “Rational homogeneous varieties”. *Lecture notes for the summer school in Algebraic Geometry in Cortona* (1995).
- [OR20] G. Ottaviani and P. Reichenbach. “Tensor rank and complexity”. *Series of Congress Reports, EMS Book Series, to appear, arXiv:2004.01492* (2020).
- [Pro07] C. Procesi. *Lie groups. An approach through invariants and representations*. Universitext. Springer, New York, 2007.
- [RRS92] R. Richardson, G. Röhrle, and R. Steinberg. “Parabolic subgroups with abelian unipotent radical”. *Invent. Math.* 110.3 (1992), pp. 649–671. DOI: [10.1007/BF01231348](https://doi.org/10.1007/BF01231348).
- [Rus16] F. Russo. *On the Geometry of Some Special Projective Varieties*. Springer International Publishing, 2016.

- [ST13] A. Sawicki and V.V. Tsanov. “A link between quantum entanglement, secant varieties and sphericity”. *J. Phys. A* 46.26 (2013), pp. 265301, 20. DOI: [10.1088/1751-8113/46/26/265301](https://doi.org/10.1088/1751-8113/46/26/265301).
- [Slo] N.J.A. Sloane. *The On-line Encyclopedia of Integer Sequences (OEIS)*. Available at <https://oeis.org/>.
- [Sno89] D. Snow. “Homogeneous vector bundles”. In: *Group actions and invariant theory (Montreal, PQ, 1988)*. Vol. 10. CMS Conf. Proc. Amer. Math. Soc., Providence, RI, 1989, pp. 193–205. DOI: [10.1007/bf02564684](https://doi.org/10.1007/bf02564684).
- [SS04] D. Speyer and B. Sturmfels. “The tropical grassmannian” (2004).
- [Sta23] R. Staffolani. “Schur apolarity”. *J. Symbolic Comput.* 114 (2023), pp. 37–73. DOI: [10.1016/j.jsc.2022.04.017](https://doi.org/10.1016/j.jsc.2022.04.017).
- [Ull16] B. Ullery. “On the normality of secant varieties”. *Adv. Math.* 288 (2016), pp. 631–647.
- [Ver01] P. Vermeire. “Some results on secant varieties leading to a geometric flip construction”. *Compositio Math.* 125.3 (2001), pp. 263–282.
- [Ver09] P. Vermeire. “Singularities of the secant variety”. *J. Pure Appl. Algebra* 213.6 (2009), pp. 1129–1132.
- [Wil21] L.K. Williams. “The positive Grassmannian, the amplituhedron, and cluster algebras”. *arXiv preprint, arXiv:2110.10856* (2021).
- [Zak93] F.L. Zak. *Tangents and secants of algebraic varieties*. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, 1993.