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DECOMPOSITIONS OF POWERS OF QUADRATIC FORMS

presentata da

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To my family, the only lighthouse in my storms

Abstract

We analyze the Waring decompositions of the powers of any quadratic form over the field of complex numbers. Our main objective is to provide detailed information about their rank and border rank. These forms are of significant importance because of the classical decomposition expressing the space of polynomials of a fixed degree as a direct sum of the spaces of harmonic polynomials multiplied by a power of the quadratic form. Using the fact that the spaces of harmonic polynomials are irreducible representations of the special orthogonal group over the field of complex numbers, we show that the apolar ideal of the s-th power of a non-degenerate quadratic form in n variables is generated by the set of harmonic polynomials of degree $s+1$. We also generalize and improve upon some of the results about real decompositions, provided by B. Reznick in his notes from 1992, focusing on possibly minimal decompositions and providing new ones, both real and complex. We investigate the rank of the second power of a non-degenerate quadratic form in n variables, which is equal to $\left(n^2+n+2\right)/2$ in most cases. We also study the border rank of any power of an arbitrary ternary non-degenerate quadratic form, which we determine explicitly using techniques of apolarity and a specific subscheme contained in its apolar ideal. Based on results about smoothability, we prove that the smoothable rank of the s-th power of such form corresponds exactly to its border rank and to the rank of its middle catalecticant matrix, which is equal to $(s+1)(s+2)/2$.

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Preface

This work represents the research activity carried out during the author's Ph.D. program. The choice of the powers of the quadratic forms as the main object of this study is due to its special properties. Indeed, it represents an important example of invariant form under the action of the orthogonal group. Moreover, of great relevance is the large use of the Laplace operator, appearing in several branches of Mathematics.

Despite describing the problem is quite elegant and simple, establishing the Waring rank of the polynomial q_n^s, with $s, n \in \mathbb{N}$, presents several difficulties and requires many information and notions about several subjects, such as apolarity, representation theory, and secant varieties.

Prerequisites

We will take for granted all the basic notions of linear algebra, like basis of vector spaces, linear applications and dual spaces, for whose details we refer, for instance, to [Lan87] or [SR13]. We will also need some arguments about differential geometry, especially for what concerns the study of Lie groups and Lie algebras. Even if we will redefine some subjects in chapter 1, one can consult, for instance, [AT11], [GQ20], [Spi79], and [Tu11] to get more details about differential geometry in general. For a more accurate view on Lie groups and Lie algebras we refer, instead, to [Kir08] or [Pro07] and, to get more information about linear algebraic groups, also to [MT11]. On the other hand, for what concerns tensor products and multilinear algebra, we suggest to consult [Lan12].

Through the text, there will be also needed a basic knowledge of algebraic structures, like ring, ideals, modules and algebras, for which we refer to [AM69], which is a useful classic resource for an introduction to basic elements of commutative algebra, or also to [ZS58] and [ZS75]. Moreover, we will treat also some results about Gröbner bases, leading ideals and saturation of ideals. In this case, we suggest [Eis95] as a standard reference. Theory of schemes and varieties is necessary as well. Some elementary concepts, like algebraic sets or Zariski topology, can be found in [Per08] or [Har95]. For arguments a bit more demanding, instead, there are many other texts which can be consulted. For a more detailed general overview, we suggest [Har77], [Sha13a] and [Sha13b].

Notation

We include here a list of the main basic notations we use. The other symbols will be introduced and specified in the various chapters. Anyway, all of these will be included in the Glossary of notations at the end of the dissertation.

We denote by $\subseteq$ an inclusion with equality allowed, while we use $\subsetneq$ to consider a proper inclusion.

For every $n, k \in \mathbb{N}$, the binomial coefficient of n choose k is denoted by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and, by convection, it is supposed to be equal to 0 whenever $k > n$.

We denote the sets of natural numbers (including zero), integer numbers, rational numbers, real numbers and complex numbers with the usual notations, given respectively by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. We denote the ring of integers modulo n by $\mathbb{Z}_n$ for every $n \in \mathbb{N}$. The Euler's constant and imaginary unit will be written in roman typefaces respectively as "e" and "i".

Given any field $\mathbb{K}$, we denote the usual groups of classical linear algebra in the traditional way, that is, we use the symbols $\mathrm{GL}_n(\mathbb{K})$, $\mathrm{SL}_n(\mathbb{K})$, $\mathrm{O}_n(\mathbb{K})$ and $\mathrm{SO}_n(\mathbb{K})$ to indicate respectively the linear algebraic group, the special linear group, the orthogonal group and the special orthogonal group over $\mathbb{K}$ in dimension $n \in \mathbb{N}$. The ring of square matrices of order n over $\mathbb{K}$ is instead denoted by $\mathrm{Mat}_n(\mathbb{K})$ and the polynomial ring in n variables $x_1, \dots, x_n$ is denoted by $\mathbb{K}[x_1, \dots, x_n]$.

In dealing with any vector space which is different by the one-dimensional ones, we will denote its elements by boldface writing and its coordinates numbered with subscript. For instance, we use the notation

$$\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$$

with $a_1, a_2 \in \mathbb{R}$. The same holds for multi-indices and monomials, setting, given the vector of coordinates

$$\mathbf{x} = (x_1, \dots, x_n)$$

and a multi-index $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$, every monomial in $\mathbb{K}[x_1, \dots, x_n]$ is written as

$$\mathbf{x}^{\boldsymbol{\delta}} = x_1^{\delta_1} \cdots x_n^{\delta_n}.$$

In particular, the degree of the above monomial is denoted by the quantity

$$|\delta| = \delta_1 + \cdots + \delta_n.$$

Moreover, we denote every linear form in $\mathbb{K}[x_1, \dots, x_n]$, i.e. every polynomial of degree 1, associated to the point $\mathbf{a} \in \mathbb{K}^n$ as

$$l_{\mathbf{a}} = (\mathbf{a} \cdot \mathbf{x}) = a_1 x_1 + \cdots + a_n x_n.$$

For every linear application $f \colon V \longrightarrow W$ between any two vector spaces V and W, we will denote by $\operatorname{Ker} f$ and $\operatorname{Im} f$ respectively the kernel and the image of f. The dual space of V will be denoted by V^*.

In describing summation of powers of linear forms, if not specified, the subscript on the right standing alone will denote the variable varying among all the possible natural values numbering the variables, while the superscript on the right will establish the number of summands of the summation. Moreover, the plus-minus sign $\pm$ will denote that both of the signs must be considered among the summation. For instance, the sum

$$f(x_1, x_2, x_3) = \sum\nolimits_j^4 (x_1 \pm x_j)^2$$

consists of the four summands $(x_1 + x_2)^2$, $(x_1 - x_2)^2$, $(x_1 + x_3)^2$ and $(x_1 - x_3)^2$.

Acknowledgements

As I near the end of my academic journey, I feel a mix of emotions. I am happy to have completed this phase of my life, but also nostalgic for the people and experiences that have shaped me. Nevertheless, I believe that the most important thing is to take what we learn from our path and use it to move forward.

First of all, I would like to express my sincere gratitude to my supervisors, Alessandro Gimigliano and Giorgio Ottaviani. They were much more than just professors; they offered me invaluable guidance and support throughout my research. I owe them a lot, especially for giving me a hand in a difficult moment and continuing to believe in me. Despite all the difficulties that one may encounter, it is important never to give up, never to surrender, because "being stubborn in this profession is not a flaw". I think there is nothing more important for a Ph.D. student at the beginning of their academic career.

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* * *

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Cosimo Flavi

There is hope, even when your brain tells you there isn't.

— John Green

Introduction

The determination of the minimum natural number r such that a homogeneous polynomial of degree $d \in \mathbb{N}$ can be written as a sum of r different d-th powers of linear forms is a classical problem. This value is also known as *Waring rank* of a polynomial, taking its name by E. Waring. He posed in 1770 the problem of determining if, for every natural number k, there exists a positive integer s such that every natural number is the sum of at most s natural numbers raised to the power k (see [War91]). This old and fascinating problem of number theory, after remaining unsolved for over a century, has been completely solved by D. Hilbert in 1909, who provided in [Hil09] a proof of the affirmative answer, thanks to the so called Hilbert-Waring Theorem.

Sum of powers and Waring rank

Given a field $\mathbb{K}$ and a homogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ of degree d, a decomposition of f of size $r \in \mathbb{N}$ is a linear combination of the d-th powers of r different linear forms $l_1, \ldots, l_r \in \mathbb{K}[x_1, \ldots, x_n]_1$, which results to be equal to f, namely,

$$f = \sum_{j=1}^{r} \lambda_j l_j^d$$

for some $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$. If $\mathbb{K}$ is an algebraically closed field, then the definition is limited to the existence of r different linear forms $l_1, \ldots, l_r \in \mathbb{K}[x_1, \ldots, x_n]_1$ such that

$$f = \sum_{j=1}^{r} l_j^d.$$

For example (see [BBT13, Section 2]), a minimal Waring decomposition of the quadratic form $x_1x_2 \in \mathbb{K}[x_1, x_2]$ is given by the formula

$$x_1x_2 = \frac{1}{4}(x_1 + x_2)^2 - \frac{1}{4}(x_1 - x_2)^2.$$

A minimal decomposition of the form $x_1x_2x_3 \in \mathbb{K}[x_1, x_2, x_3]$ is instead given by

$$x_1x_2x_3 = \frac{1}{24}(x_1 + x_2 + x_3)^3 - \frac{1}{24}(x_1 + x_2 - x_3)^3 - \frac{1}{24}(x_1 - x_2 + x_3)^3 + \frac{1}{24}(x_1 - x_2 - x_3)^3.$$

For every homogeneous polynomial f, we denote the Waring rank of f by $\operatorname{rk} f$.

The analogous of the Waring problem for homogeneous polynomials, also known as *Big Waring problem*, concerns the determination of the minimum number r such that a general form of degree d admits a decomposition of size r, also called a *Waring decomposition*. With the term *general*, we mean any form belonging to an open and dense subset in the Zariski topology. As one could expect, the question was not so easy and it was left unanswered up to 1995, when the problem was solved by J. Alexander and A. Hirschowitz in [AH95]. They established in the famous Alexander-Hirschowitz theorem that the rank of a general polynomial in n variables of degree d is given by a precise formula depending on n and d, with the exception of a few cases, regarding some lower values of the degree and the number of variables.

Theorem (J. Alexander, A. Hirschowitz)**.** *The Waring rank of a general form $f \in \mathbb{K}[x_1, \dots, x_n]_d$ is given by the formula*

$$\operatorname{rk} f = \left\lceil \frac{1}{n}\binom{d+n-1}{d} \right\rceil,$$

with the exceptions given by the following cases:

- $d = 2$*;*
- $n = 3,\ d = 4$*;*
- $n = 4,\ d = 4$*;*
- $n = 5,\ d = 3$*;*
- $n = 5,\ d = 4$*.*

More recently, this result has been accurately analyzed by M. C. Brambilla and G. Ottaviani, who provided a shorter version of the proof in [BO08], to which we refer for the details.

However, despite the Big Waring problem has been completely understood, the determination of the rank of a specific polynomial remains in general a hard issue and, currently, there is no general efficient method to solve it, or even to determine some suitable decompositions, independently by the form we consider. Nevertheless, many partial results and methods to attack the Waring problem for a polynomial have been produced among the years. For a more detailed overview about Waring decompositions, there are many texts and papers in the literature. We refer, for instance, to [BCC+18], [BGI11], [CGO14], [Lan12], and [LO13].

For the special case of two variables, the determination of the rank is quite easier to approach and completely analyzed (see [Syl51b] or the more recent work of G. Comas and M. Seiguer in [CS11]). In particular, there are many algorithms leading to explicit decompositions. The first of these is known as Sylvester algorithm, that can be found in [Syl86] or, with a more recent version, also in [CS11], [BGI11] and [BCMT10]. It has further been analyzed among the years, with several other variants (see e.g. [BGI11, Algorithm 2]).

For what concerns instead a higher number of variables, there are still some algorithms which, under highly specific hypothesis, can provide decompositions. One can see, for instance, the generalization of Sylvester algorithms provided by J. Brachat, P. Comon, B. Mourrain, and T. Tsigaridas in [BCMT10, Algorithm 7.1]. However the determination of convenient decompositions for an arbitrary homogeneous polynomial remains in general a quite difficult argument. Despite this, there are several cases related to some specific classes of polynomials, for which the problem has been solved partially with some estimations on lower and upper bound on the rank, or even completely. One of the most important examples has been provided in 2012 by E. Carlini, M. V. Catalisano, and A. V. Geramita in [CCG12, Proposition 3.1]. Thanks to this result, they completely solved the Waring problem for monomials, proving that, for every monomial

$$g = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{K}[x_1, \dots, x_n]_d$$

of degree d, with

$$\alpha_1 + \cdots + \alpha_n = d, \quad 1 \le \alpha_1 \le \cdots \le \alpha_n,$$

its rank is given by the formula

$$\operatorname{rk} g = \frac{1}{\alpha_1 + 1} \prod_{j=1}^{n} (\alpha_j + 1).$$

A very useful subject in view of the Waring problem for polynomials is represented by the apolarity theory. Given a homogeneous polynomial $f \in \mathbb{K}[x_1, \dots, x_n]_d$, it is well known that the ideal of the points appearing in one of its decomposition is contained in another specific ideal. This is the so called *apolar ideal* of a f, denoted by $f^{\perp}$, and we will introduce it in section 1.2. The catalecticant map of f is defined by the derivative action of differential polynomial operators on f. That is, for every monomial $\mathbf{y}^\alpha$ such that $|\alpha| = d$, the map obtained extending by linearity the function defined on monomials by

$$\mathrm{Cat}_f(\mathbf{y}^\alpha) = \frac{\partial^{|\alpha|} f}{\partial \mathbf{x}^\alpha}.$$

The apolar ideal can thus be identified as the kernel of this map and the crucial correspondence described above is essentially the statement of a famous result known as the *apolarity lemma* (cf. Lemma 1.2.17, [IK99, Lemma 1.15]).

Lemma (Apolarity Lemma). *Let $Z = \{[\mathbf{a}_1], \dots, [\mathbf{a}_r]\} \subseteq \mathbb{P}(\mathbb{K}^n)$ and let $f \in \mathbb{K}[x_1, \dots, x_n]_d$. Then the following conditions are equivalent:*
(1) $f = \sum_{j=1}^r \lambda_j (\mathbf{a}_j \cdot \mathbf{x})^d$ for some $\lambda_1, \dots, \lambda_r \in \mathbb{K}$;
(2) $I(Z) \subseteq f^\perp$.

Thanks to this correspondence, the determination of such ideals leads to explicit decompositions, up to solving a linear system. This fact allows, in particular, to get some information on the decompositions by analyzing algebraic invariants, like Hilbert function. Apolarity theory, which represents a quite classical subject, has been mostly formalized in 1999 by A. Iarrobino and V. Kanev in [IK99], to which we refer to get further details.

Another related problem is to determine when a given decomposition of a homogeneous form is unique. Probably the most classical result related to this question is based on generic forms and is due to J. J. Sylvester. It is historically known as the Sylvester's Pentahedral Theorem, appearing for the first time in [Syl51a] in 1851 (see also [Dol12, Theorem 9.4.1] for a more recent reference).

Theorem (J. J. Sylvester). *Every generic cubic form f in four variables can uniquely be written, up to scalars, as*

$$f = l_1^3 + l_2^3 + l_3^3 + l_4^3 + l_5^3$$

where $l_1, \dots, l_5$ are non-proportional linear forms in four variables.

Other classical examples are provided for the case of ternary forms in degree 5 by H. W. Richmond in [Ric04] and by F. Palatini in [Pal03], while the same argument has been treated from a more modern point of view in several recent papers (see e.g. [GM19], [Mel09], and [RS00]). Furthermore, it is quite noticeable the analysis of the uniqueness of decompositions in the case of polynomials which are invariant under the action of a linear group, clearly up to suitable transformations. One of the most clarifying examples in this sense, as we will see later, is represented by the power

$$\left(x_1^2 + x_2^2 + x_3^2\right)^2.$$

This specific form, as proved by B. Reznick in [Rez92, Theorem 9.13], can be represented by a sum of 6 different 4-th powers of linear forms, which must correspond to the vertices of a regular icosahedron.

Apart from the classical concept of Waring rank, several other notions of rank of polynomials have been introduced among the years. A very important one is represented by the *border rank* of a homogeneous polynomial $f \in \mathbb{K}[x_1, \dots, x_n]_d$, denoted by $\operatorname{brk} f$. This is the minimum number $r \in \mathbb{N}$ such that

$$f = \lim_{t \to 0} \sum_{j=1}^r l_j^d(t),$$

for a suitable family $\{l_j(t)\}_{t \in \mathbb{R}}$ for every $j = 1, \dots, r$.

Thanks to the theory of secant varieties (see [Har95, Lecture 8] for a general overview), a strong connection between rank of polynomials and algebraic varieties has been developed, especially for what concerns the theory of secant varieties. We will introduce it briefly in subsection 1.2.2. We can also find many elements in the literature concerning this subject, such as the above mentioned texts [BCC+18], [CGO14], [LO13] or also other papers, such as [BL13], [Bal10a], and [Bal10b].

In these order of ideas, of noticeable relevance is the d-Veronese embedding, defined as

$$\begin{aligned} \nu_d \colon \mathbb{P}\big(\mathbb{K}^n\big) &\longrightarrow \mathbb{P}\Big(\mathbb{K}^{\binom{d+n-1}{d}}\Big) \\ [l] &\longmapsto \big[l^d\big], \end{aligned}$$

and its image $\nu_d\big(\mathbb{K}^n\big)$, known as the d-Veronese variety, to which the quite recent notions of *cactus rank* and *smoothable rank* of a homogeneous form are well connected. These last two concepts will be presented in subsection 1.2.2 and are a little bit beyond our purposes, but they can be compared with rank and border rank and effectively permit to establish upper bounds. The cactus rank corresponds exactly to the notion of *scheme length*, which has been first introduced in 1999 by A. Iarrobino and V. Kanev (see [IK99, Definition 5.1]). It is defined as the natural value

$$\operatorname{crk} f = \min\left\{ r \in \mathbb{N} \,\middle|\, \exists \text{ 0-dim. scheme } Z \subseteq \nu_d\big(\mathbb{P}^{n-1}\big) \colon \deg Z = r,\ [f] \in \langle Z\rangle \right\}.$$

The term *cactus rank* has been used for the first time by K. Ranestad and F.-O. Schreyer in [RS11], inspired by the notion of *cactus variety*, defined by W. Buczyńska and J. Buczyński in [BB14]. It has been analyzed in several other papers, such as [Bal18], [BR13], or [BBG19]. The smoothable rank is defined analogously, with the only difference that the scheme contained in the image of the Veronese embedding must be smoothable, namely, a limit of smooth subschemes. That is the natural value

$$\operatorname{smrk} f = \min\left\{ r \in \mathbb{N} \,\middle|\, \exists \text{ 0-dim. smoothable scheme } Z \subseteq \nu_d\big(\mathbb{P}^{n-1}\big) \colon \deg Z = r,\ [f] \in \langle Z\rangle \right\}.$$

This last definition appeared for the first time in [RS11], motivated by several results appearing in [BGI11], [BB14], and [BGL13]. Our interest in smoothable rank is due to a comparison with border rank (see [BBM14, Remark 2.7]), stating that the inequality

$$\operatorname{brk} f \le \operatorname{smrk} f$$

holds for every homogeneous polynomial f. In particular, we will use it to determine the border rank of any power of a ternary quadratic form.

Tensor decomposition and applications

Given a tensor $T \in V^{\otimes d}$, where V is a finite dimensional vector space over a field of characteristic 0, the rank of T is defined as the minimum number $r \in \mathbb{N}$ such that T can be written as sum of r tensors of the form

$$v_1 \otimes \cdots \otimes v_n,$$

for some $v_1, \dots, v_n \in V$. Such tensors are called *decomposable* or *rank one* tensors. Thus, the rank of T is the value

$$\operatorname{rk} T = \min\left\{ r \in \mathbb{N} \,\middle|\, T = \sum_{j=1}^{r} v_{j,1} \otimes \cdots \otimes v_{j,n} : v_{j,k} \in V,\ k = 1, \dots, n \right\}$$

and, analogously, the *border rank* of T is defined as

$$\operatorname{brk} T = \min\left\{ r \in \mathbb{N} \,\middle|\, T = \lim_{t\to 0} \sum_{j=1}^{r} v_{j,1}(t) \otimes \cdots \otimes v_{j,n}(t) : \{v_{j,k}(t)\}_{t\in\mathbb{C}} \subseteq V,\ k = 1, \dots, n \right\}$$

It is quite easy to show that any d-graded homogeneous polynomial in n variables can be naturally viewed as a symmetric tensor in the d-th symmetric power $S^d\mathbb{K}^n$ (cf. Proposition 1.2.3). This allows to analyze a homogeneous polynomial from the point of view of tensors. In particular, the *symmetric rank* and the *symmetric border rank* of a symmetric tensor $S \in S^d V$ are defined respectively as

$$\operatorname{rk}_s S = \min \left\{ r \in \mathbb{N} \;\middle|\; S = \sum_{j=1}^{r} v_j^{\otimes d} : v_j \in V \right\}$$

and

$$\operatorname{brk}_s S = \min \left\{ r \in \mathbb{N} \;\middle|\; S = \lim_{t \to 0} \sum_{j=1}^{r} v_j^{\otimes d}(t) : \{v_j(t)\}_{t \in \mathbb{C}} \subseteq V \right\}.$$

Especially in recent years, tensor decomposition turned out to have many applications in several scientific branches, both theoretical and applied. In general, for applications, scientific data are collected in multi-dimensional arrays, which are treated as tensors. Some surveys about several uses related to tensor decompositions are provided by J. M. Landsberg in [Lan12, Chapter 1] and by B. W. Bader and T. G. Kolda in [KB09]. Among the more theoretical usages, we can highlight complexity of matrix multiplication (see e.g. [Str83]), the P versus NP complexity problem (see e.g. [Val01]), and the entanglement in quantum physics (see e.g. [BC12]). For what concerns instead applications to more applied sciences, we can consider independent component analysis and blind identifications in signal processing (see respectively [Com94] and [SGB00]), the study of phylogenetic invariants (see e.g. [AR08]), bioinformatics and spectroscopy (see e.g. [CJ10]).

Of great interest, however, are also the applications of symmetric tensor decomposition. In [BGI11, Section 1], A. Bernardi, A. Gimigliano, and M. Idà briefly summarize some uses, such as telecommunications in electrical engineering (see [Che11] and [DLC07]) or cumulant tensors in statistics (see [McC87]).

It is by this reason that the determination of the Waring rank of a polynomial or, equivalently, the symmetric rank of a symmetric tensor, despite its classical origins, still preserves a special role even in more recent days. The concept of border rank of an arbitrary tensor, in particular, was introduced for the first time in 1979 by D. Bini, M. Capovani, G. Lotti, and F. Romani in [BCRL79]. It was then named that way in the following year still by D. Bini, G. Lotti, and F. Romani in [BLR80], where they defined it in a more practical way as the minimum number of decomposable tensors requested to approximate a tensor with an arbitrarily small error. This is the reason why it can be much useful in cases in which specific tensors have a particularly high gap between rank and border rank (see e.g. [Zui17]). Hence, in a more practical way, these could be approximated, in a more convenient way, by tensors of much lower rank.

Powers of quadratic forms

As one could expect, many forms own a special role in both classical and modern subjects, appearing several time in the literature. The central objects we will consider, on the field $\mathbb{C}$ of complex numbers, are represented by the powers of the quadratic forms. It is well known that, by the classical Silvester's law ([Syl52]), every quadratic form with rank equal to $n \in \mathbb{N}$ can be written as

$$q_n = x_1^2 + \cdots + x_n^2.$$

Thus, we can restrict ourselves to the study of the polynomial

$$q_n^s = \left(x_1^2 + \cdots + x_n^2\right)^s$$

for suitable $n, s \in \mathbb{N}$.

Our aim is to gather as much information as possible about the rank and the border rank of any power of the form q_n. While for the case of binary forms the problem is quite easy and completely solved, we cannot say the same for the general case in more variables, about which there is not much information in the literature.

The most complete analysis on this subject up to now is due to B. Reznick, who provides in [Rez92] an accurate survey about both classical and more original results over $\mathbb{R}$. Following the greater relevance which real numbers used to have in applications, with respect to complex ones, B. Reznick focuses in his notes on real Waring decomposition, to which he refers as *representations*. In particular, to get a new vision of this problem, especially considering the recent applications of which tensors are endowed, we would consider also the complex case of Waring decompositions. In relation to the powers of quadratic forms, this represents a new approach, apart from the classical point of view.

Several uses due to decompositions of powers of quadratic forms have been listed by B. Reznick in [Rez92, Section 8], such as in number theory, to study the Waring problem, or even in functional analysis. The quadratic form q_n corresponds exactly, in the language of differential operators, to the well known Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

which has a special role in analysis.

By an algebraic point of view, we can recall the decomposition of the d-th symmetric power $S^d\mathbb{C}^n$ as

$$S^d\mathbb{C}^n = \bigoplus_{j=0}^{\lfloor \frac{d}{2} \rfloor} q_n^j \mathcal{H}_n^{d-2j},$$

where $\mathcal{H}_n^{d-2j}$ represents the space of harmonic polynomials of degree $d-2j$ in n variables for every $j \in \mathbb{N}$. This elegant decomposition, which has been exposed in details by R. Goodman and N. R. Wallach in [GW98, Corollary 5.2.5], takes on much relevance, considering its invariance under the action of the orthogonal complex group $\mathrm{O}_n(\mathbb{C})$.

This last decomposition represents also an important tool we use for the determination of the apolar ideal of q_n^s, representing actually the first result we provide, which will be analyzed in details in chapter 2. B. Reznick already remarks that the catalecticant matrices of q_n^s have all full rank; in particular, denoting by $T_{n,s}$ the size of the middle catalecticant matrix, equal to

$$T_{n,s} = \binom{s+n-1}{s},$$

it easily follows the lower bound (cf. Corollary 2.2.18)

$$\mathrm{rk}\big(q_n^s\big) \geq \mathrm{brk}\big(q_n^s\big) \geq T_{n,s}.$$

We improve this basic result for the polynomial q_n^s by describing exactly the structure of the kernels of the various catalecticant matrices, that is, we prove the following

Theorem. *For every $n, s \in \mathbb{N}$*

$$\big(q_n^s\big)^{\perp} = \big(\mathcal{H}_n^{s+1}\big).$$

Once we know how the apolar ideal is made, our problem turns into the analysis of ideals of points contained in it. This characterization is useful in determining explicit decompositions presenting the same pattern, as we will see in chapter 4. Moreover, for our purposes, it is crucial for determining the border rank for any power in the case of three variables. B. Reznick provides in [Rez92, Chapters 8-9] both classical and new decompositions, giving also proofs of minimality for some of these. In particular, he analyzes, using the language of spherical designs (see [DGS77]), the existence and the uniqueness of *tight representations*, namely, real decompositions having size equal to the rank of the middle catalecticant matrix. He summarizes this in the following

Theorem ([Rez92, Proposition 9.2])**.** *If q_n^s has a real tight decomposition, then one of the following conditions holds:*
(1) $s = 1$ *or* $n = 2$*;*
(2) $s = 2$ *and* $n = 3$*;*
(3) $s = 2$ *and* $n = m^2 - 2$ *for some odd* $m \in \mathbb{N}$*;*
(4) $s = 3$ *and* $n = 3m^2 - 4$ *for some* $m \in \mathbb{N}$*;*
(5) $s = 5$ *and* $n = 24$.

In particular, this theorem guarantees that there is no tight representation for $s \geq 6$. We partially generalize this theorem for complex numbers, proving in chapter 3 for $s = 2$ the following

Theorem. *If q_n^2 has a tight decomposition, then $n = 3$ or $n = m^2 - 2$ for some odd number $m \in \mathbb{N}$.*

In the case of $s = 3$, we get a less powerful result than the statement proposed by B. Reznick.

Theorem. *If q_n^3 has a tight decomposition, then* $n \equiv 2 \bmod 3$.

The proof of this results has been made following exactly the same strategy that B. Reznick uses in [Rez92, pp. 130-132], but the possibility of using it is not so immediate. Indeed, he uses the important fact that every tight representation must be *first caliber*, namely, every point of such a decomposition must have the same norm, that is, if

$$q_n^s = \sum_{j=1}^{r} (\mathbf{a}_j \cdot \mathbf{x})^{2s}$$

is a tight real decomposition with $r \in \mathbb{N}$ and $\mathbf{a}_1, \ldots, \mathbf{a}_r \in \mathbb{R}^n$, then

$$|\mathbf{a}_j| = |\mathbf{a}_k|$$

for every $1 \leq j, k \leq r$. Denoting by $B_{n,s}$ the value of the norm of every point raised to $2s$, for such a tight decomposition, we have (see [Rez92, Corollary 8.18])

$$B_{n,s} = |\mathbf{a}_j|^{2s} = \frac{1}{T_{n,s}} \prod_{j=0}^{s-1} \frac{2j+n}{2j+1} = \binom{s+n-1}{s}^{-1} \prod_{j=0}^{s-1} \frac{2j+n}{2j+1}.$$

Considering this, B. Reznick determines the kernel of the polynomial

$$q_n^2 - B_{n,2} x_1^4,$$

which must be necessarily non-zero and imposes consequent conditions to the remaining point of the decompositions.

The new fact that allows us to generalize the result provided by B. Reznick, extending the notion of first caliber to complex decompositions, is that every tight complex decomposition is first caliber as well. In fact, we prove in section 3.1 the following theorem, which implies in particular that every tight decomposition does not contain any isotropic point.

Theorem. *Every tight decomposition*

$$q_n^s = \sum_{k=1}^{r} (\mathbf{a}_k \cdot \mathbf{x})^{2s}$$

is first caliber. In particular,

$$(\mathbf{a}_k \cdot \mathbf{a}_k)^s = \frac{1}{r} \prod_{j=0}^{s-1} \frac{2j+n}{2j+1}.$$

for every $k = 1, \dots, r$.

In the case of exponent $s = 2$, a quadrature formula presented by A. H. Stroud in [Str67a] is used by B. Reznick to prove that, for $n = 4, 5, 6$, the real rank of q_n^2 is equal to $T_{n,2} + 1$, while in the case of $n = 7$, it gives a tight decomposition of size $T_{7,2}$. By this fact, we prove another result about the rank of q_n^2. This establishes its exact rank for most of the values assumed by $n \geq 9$ and partially solves the determination of the rank in the case of $s = 2$.

Theorem. *Let* $n \geq 3$. *Then the following conditions hold:*
(1) if $n = 3, 7, 23$, *then* $\operatorname{rk}(q_n^2) = T_{n,2}$*;*
(2) if $n > 23$ *and* $n = m^2 - 2$ *for some odd number* $m \in \mathbb{N}$, *then* $T_{n,2} \leq \operatorname{rk}(q_n^2) \leq T_{n,2} + 1$*;*
(3) if $n = 8$, *then* $\operatorname{rk}(q_n^2) \geq T_{n,2} + 1$*;*
(4) otherwise, $\operatorname{rk}(q_n^2) = T_{n,2} + 1$.

Apart this general properties, B. Reznick actually studies the real rank, also called *width*, for several specific cases, including the one of two variables. For this case he concludes that all the points of such minimal decompositions must be the vertices of a regular polygon inscribed in a circle of radius depending only on s. Again, thanks to apolarity, we are able to generalize this fact and also to determine the exact correspondence between polynomials in the apolar ideal and complex decompositions. The latter are found to be equivalent, up to complex orthogonal transformations, to the real ones already known.

The other decompositions analyzed by B. Reznick regard ternary forms. The power q_3^2, in particular, is quite special. Indeed, each of its minimal decompositions, which has size equal to 6 and hence is tight, must be composed by points corresponding to the vertices of a regular icosahedron inscribed in a sphere of radius $5/6$. Again, by the first caliber property, we can extend the uniqueness to the complex field (cf. Theorem 3.3.8). Beside this, denoting by $\operatorname{rk}_{\mathbb{R}} f$ the real rank of a homogeneous polynomial f, he shows also that

$$\operatorname{rk}_{\mathbb{R}}(q_3^3) = T_{3,3} + 1 = 11, \quad \operatorname{rk}_{\mathbb{R}}(q_3^4) = T_{3,4} + 1 = 16.$$

The first case is easily obtained by previous results on exponent $s = 3$, while the second one involves instead a more complicated proceeding. We will see in chapter 5 that the first caliber property imposes strong conditions for the angle between the various points. In particular, by an analysis through the Gram matrices of 4 points in the space $\mathbb{C}^3$, we achieve to prove the following

Theorem. $\operatorname{rk}(q_3^4) = 16$.

The last analysis we make concerns instead the smoothable rank and the border rank, which represented up to now a completely unexplored land for this specific polynomial. Indeed, it could seem a quite surprising fact to state that the border rank of every power a ternary quadratic form is minimal. In particular, we have

Theorem. *For every* $s \in \mathbb{N}$,

$$\operatorname{smrk}(q_3^s) = \operatorname{brk}(q_3^s) = T_{3,s} = \frac{(s+1)(s+2)}{2}.$$

The central instrument used to verify this is the version of apolarity lemma for schemes. Indeed, we determine a smoothable 0-dimensional scheme contained in the apolar ideal and supported over a point. This specific structure can be easily determined for every $s \in \mathbb{N}$, providing an effective proof of the theorem.

Overview

We divide our analysis of the polynomial q_n^s into five parts. We start in chapter 1 by reporting some preliminary material, especially for what concerns Lie groups, Lie algebras and some elements of representations theory. This last one is quite important in relation to the fact that the form q_n^s is invariant under the action of the orthogonal group $\mathrm{O}_n(\mathbb{C})$ and that the space of d-harmonic polynomials $\mathcal{H}_n^d$ is an irreducible $\mathrm{SO}_n(\mathbb{C})$-module ([GW98, Theorem 5.2.4]). After that, we present some notions about both classical and recent apolarity theory, tensor decomposition and secant varieties, such as the apolar ideal, catalecticant map, the various definitions of tensor ranks and also some basic but necessary results, such as the classical apolarity lemma and its alternative version for schemes (cf. Lemma 1.2.29).

In chapter 2, we focus our attention on the apolarity action on the polynomial q_n^s, determining the structure of catalecticant matrices for every $n, s \in \mathbb{N}$ and providing our first main result, proving that the apolar ideal of q_n^s is given by

$$\left(q_n^s\right)^\perp = \left(\mathcal{H}_n^{s+1}\right).$$

Chapter 3, instead, is centered on tight decompositions. We first summarize some known facts and some results exposed by B. Reznick. Then, we extend his results concerning tight decompositions, focusing in particular on the fact that every tight decomposition must be first caliber. After that, we analyze all the decompositions in two variables, proving the uniqueness also for the complex case, and the suitable tight decompositions for the exponents $s = 2, 3$.

In chapter 4, we provide some general decompositions for different values assumed by $n \in \mathbb{N}$. In particular, we first concentrate on the decomposition of q_n^2, establishing its rank for most of the values of $n \in \mathbb{N}$, and then we provide some specific decompositions regarding ternary forms, which seems to follow a quite similar pattern.

Finally, chapter 5 is dedicated to our main result, which is the determination of the border rank of ternary quadratic forms, in which we show that the smoothable rank of q_n^s is equal to the minimal possible value, that is, the rank of the middle catalecticant. Moreover, by the conditions imposed by the first caliber decompositions, we are able to guarantee that, also for complex numbers, the rank of the forms q_3^3 and q_3^4 are given by

$$\mathrm{rk}\left(q_3^3\right) = 11, \quad \mathrm{rk}\left(q_3^4\right) = 16.$$

CHAPTER 1

Preliminaries

Tensor decomposition involves in general several arguments of algebra and geometry. In this chapter we provide some background material we need to achieve our results.

Section 1.1 is focused on some elements of representations theory. We first recall some basic notions about Lie groups and Lie algebras. Then, we introduce some theory related to the notion of representation of these two algebraic structures, focusing on the irreducible ones, for their special role. All the contents we provide about these arguments can be found in [FH91], [Pro07] and [Ser77]. In addition, to get further details about Lie groups and Lie algebras, one can also consult many other textbooks, such as, for instance, [AT11], [GQ20], [GW98], [Kir08], and [MT11]. The latter, in particular, provides a specific overview of linear algebraic groups.

Section 1.2 deals instead with apolarity theory and decompositions of polynomials. In particular, we provide a brief overview about rank and border rank of polynomials. The main reference we use for apolarity is [IK99], but we refer also to [Dol12] to get a more recent point of view. Information about the various notions of rank for tensors, on the other hand, are quite standard and can be found in many texts. For a general overview, we refer, as a textbook, to [Lan12]. However, there are also several papers summarizing the main notions about rank and border rank of symmetric tensors, such as [BCC+18], [LO13] and [LT10].

1.1 Representation theory

Through this section, we will only consider finite-dimensional vector spaces over an algebraically closed field $\mathbb{C}$.

1.1.1 Lie groups and Lie algebras

We start by recalling the basic concept of Lie group, which is very important in differential geometry, as well as in representation theory.

Definition 1.1.1. A differentiable manifold G is called a *Lie group* if it owns a structure of algebraic group such that the product and the inverse maps are differentiable maps. A subset $H \subseteq G$ is called a *closed Lie subgroup* if it is an algebraic subgroup and a submanifold of G.

All the usual classical group of linear algebra represents the most common examples of Lie groups. For instance, we can consider the groups $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}_n(\mathbb{R})$, or $\mathrm{SO}_n(\mathbb{R})$ and also their complex versions. As we will see later, a special class having a special role is that of connected Lie groups. About this, we recall the following property, a proof of which can be found in [Kir08, Corollary 2.10].

Proposition 1.1.2. *Let G be a Lie group. Then every connected open set $U \subseteq G$ containing the identity element $e \in G$ generates the connected component G_0 of G containing e.*

It is possible to define what is a morphism of Lie groups, which is a map preserving both the structures of manifold and of algebraic group.

Definition 1.1.3. A *morphism of Lie groups* is a differentiable map $f: G \to H$ which is also a group homomorphism, that is

$$f(g_1 g_2) = f(g_1)f(g_2)$$

for every $g_1, g_2 \in G$.

We now proceed to introduce Lie algebras.

Definition 1.1.4. A *Lie algebra* $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map

$$[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

called *bracket*, satisfying the *Jacobi identity*, that is,

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$$

for every $X, Y, Z \in \mathfrak{g}$.

Since Lie algebras are vector spaces, it is quite natural to define morphisms of Lie algebras in the following way.

Definition 1.1.5. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A linear map $f: \mathfrak{g} \to \mathfrak{h}$ is a *map of Lie algebras* if it preserves bracket, that is,

$$f([X, Y]) = [f(X), f(Y)]$$

for every $X, Y \in \mathfrak{g}$.

Now, given a differentiable manifold M, we denote by $\mathfrak{X}(M)$ the set of vector fields over M. It is well known that, given a Lie group G, there is a Lie algebra $\mathfrak{g}$ that can be naturally associated to it. This is the tangent space T_eG to G at the identity element $e \in G$, which is called *Lie algebra of* G. This fact is very important for representations of Lie algebras. Anyway, the structure of Lie algebra on T_eG is not immediate and can be obtained by considering it as a subspace of $\mathfrak{X}(G)$.

Given two manifolds M and N and a diffeomorphism $f: M \to N$, if $X \in \mathfrak{X}(M)$, then the *pushforward* of X by f is well-defined (see e.g. [Tu11, Example 14.15]) and corresponds to the vector field $f_*X \in \mathfrak{X}(N)$, given for every $p \in M$ by

$$(f_*X)_{f(p)} = \mathrm{d}f_p(X_p),$$

where $\mathrm{d}f_p: T_pM \to T_{f(p)}N$ is the differential map of f at p. If we consider the left multiplication map

$$\begin{aligned} L_h: G &\longrightarrow G \\ g &\longmapsto hg, \end{aligned}$$

that is clearly a diffeomorphism of Lie groups with inverse ${L_h}^{-1} = L_{h^{-1}}$. We have the following definition.

Definition 1.1.6. A vector field $X \in \mathfrak{X}(G)$ over a Lie group G is *left-invariant* if, for every $h \in G$,

$$(L_h)_*X = X,$$

that is

$$\mathrm{d}(L_h)_g(X_g) = X_{hg}.$$

Denoting by $L(G)$ the vector space of all left-invariant vector fields on G, it turns out that T_eG is naturally isomorphic to $L(G)$ via the isomorphism

$$\begin{aligned} \Phi\colon L(G) &\longrightarrow T_eG \\ X &\longmapsto X_e. \end{aligned}$$

It is clear that Φ represents an isomorphism, since, by definition of left-invariant vector field, we must have

$$X_g = \big((L_g)_*X\big)_e = \mathrm{d}(L_g)_e(X_e)$$

for each $g \in G$. Thus, the vector field X is uniquely defined. Finally, we can define the bracket on T_eG given for every $X_e, Y_e \in T_eG$ by

$$[X_e, Y_e] = [X, Y]_e,$$

which in fact represents a well-defined bracket (see [Tu11, Proposition 16.10]), that provides the structure of Lie algebra on $\mathfrak{g}$ (for further details about the construction of the Lie algebra associated to a Lie group, we refer to [AT11], [GQ20], or [Tu11])

We now provide the construction of the exponential map of a Lie group G, which allows us to determine the values of a morphism of Lie groups by its differential at the identity element $e \in G$. Let us start by recalling the notion of integral curve of a vector field.

Definition 1.1.7. Let $X \in \mathfrak{X}(M)$ be a vector field over a manifold M and let $p \in M$. Let $I \subseteq \mathbb{R}$ be an interval such that $0 \in I$. If $\gamma\colon I \to M$ is a curve such that $\gamma(0) = p$ and

$$\gamma'(t) = X_{\gamma(t)}$$

for every $t \in I$, then γ is said to be an *integral curve* for X passing through p.

It is possible to show that for every $X \in \mathfrak{X}(M)$ and every point $p \in M$, there exists a unique maximal integral curve for X passing through p. The proof of this fact is based on the local existence theorem for differential equations (see e.g. [Wal98, Section III.13.V] and [Lan95, Section IV.1]) and involves also the concept of local flow of a vector field over a manifold. To see this, we refer to [Lan99, Section IV.2].

Definition 1.1.8. Let G be a connected Lie group and let $\theta\colon \mathbb{R} \to G$ a differentiable map. Then θ is said to be a *one-parameter subgroup* of G if it is also a homomorphism of groups, that is

$$\theta(s+t) = \theta(s)\theta(t)$$

for every $s, t \in \mathbb{R}$.

Next proposition is crucial to obtain exponential maps. However, its proof is quite long and we omit it. To see it, one can consult, for instance. [AT11, Lemma 3.6.2] or [Kir08, Proposition 3.1]. It reveals that the integral curve of a Lie group G is in fact the only one-parameter subgroup of G.

Proposition 1.1.9. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. If $v \in \mathfrak{g}$, then there exists a unique one-parameter subgroup $\gamma_v\colon \mathbb{R} \to G$ of G such that*

$$\gamma_v'(0) = v.$$

For every $v \in \mathfrak{g}$, the map γ_v introduced in Proposition 1.1.9 is called the *one-parameter subgroup associated to v* and owns a special role in connecting Lie groups and their associated Lie algebras. The uniqueness of γ_v implies that, given any $\lambda \in \mathbb{R}$, we have

$$\gamma_v(\lambda t) = \gamma_{\lambda v}(t). \tag{1.1.10}$$

This follows immediately from the equalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_v(\lambda t)\bigg|_{t=0} = \lambda\gamma_v'(\lambda t)\big|_{t=0} = \lambda\gamma_v'(0) = \lambda v.$$

Definition 1.1.11. Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. The *exponential map* of G is the map

$$\begin{aligned} \exp\colon \mathfrak{g} &\longrightarrow G \\ X &\mapsto \gamma_X(1), \end{aligned}$$

where $\gamma_X\colon \mathfrak{g} \to G$ is the one-parameter subgroup associated to X.

For every $v \in \mathfrak{g}$, the one-parameter subgroup γ_v is uniquely determined by the curve $\sigma(t) = tv$. Indeed, by relation (1.1.10), we have

$$\gamma_v(t) = \gamma_{tv}(1) = \exp(tv). \tag{1.1.12}$$

It can be easily proved, by the definition of local flow, that the exponential map is differentiable. We refer to [Spi79, Chapter 10] for more explicit details. We focus in particular on the fact that the differential of the exponential map at the identity element of a Lie group is trivial (see e.g. [Kir08, Theorem 3.7]).

Proposition 1.1.13. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Then:*
(1) the differential of the exponential map at 0

$$\mathrm{d}(\exp)_0\colon T_0\mathfrak{g} \cong \mathfrak{g} \to \mathfrak{g}$$

corresponds to the identity map;
(2) the exponential map $\exp\colon \mathfrak{g} \to G$ *is a local diffeomorphism at* $0 \in \mathfrak{g}$*; that is, there exist an open neighborhood* U_0 *of* 0 *such that*

$$\exp\big|_{U_0}\colon U_0 \to \exp(U_0)$$

is a diffeomorphism.

Proof. (1). Given $v \in \mathfrak{g}$, let us consider the curve

$$\begin{aligned} \sigma\colon \mathbb{R} &\longrightarrow \mathfrak{g} \\ t &\longmapsto tv. \end{aligned}$$

Then we have $\sigma'(0) = X$ and

$$\mathrm{d}(\exp)_0(v) = (\exp\circ\sigma)'(0) = \frac{\mathrm{d}}{\mathrm{d}t}\exp(tv)\bigg|_{t=0} = v,$$

where the last equality follows by 1.1.12.

(2). It directly follows by point (1) and by the Inverse Function Theorem (see e.g. [Spi65, Theorem 2-13]), which states that any differentiable map of manifolds $f\colon M \to N$ is a local diffeomorphism at a point $p \in M$ if and only if the differential map $\mathrm{d}f_p$ of f at p is an isomorphism. □

Next proposition represents a crucial fact for exponential maps.

Proposition 1.1.14. *If $f\colon G \to H$ is a homomorphism of Lie groups, then*

$$\exp \circ \, \mathrm{d}f_e = f \circ \exp,$$

that is, the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{d}f_e"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

commutes.

Proof. By Proposition 1.1.9 and formula (1.1.12), it is sufficient to prove that the curve $f\big(\exp(tv)\big)$ is the one-parameter subgroup associated to $\mathrm{d}f_e(v)$. Thus, observing that

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(\exp(tv)\big)\bigg|_{t=0} = \mathrm{d}f_e\left(\frac{\mathrm{d}}{\mathrm{d}t}\exp(tv)\bigg|_{t=0}\right) = \mathrm{d}f_e(v),$$

we get the statement. □

We conclude by the following result, which immediately follows from Proposition 1.1.14.

Corollary 1.1.15. *Let G and H be two Lie groups and let $f\colon G \to H$ a morphism of Lie groups. If G is connected, then f is uniquely determined by the differential map of f at e*

$$\mathrm{d}f_e\colon \mathfrak{g} \to \mathfrak{h}.$$

Proof. By Proposition 1.1.14, we know that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{d}f_e"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

commutes. Moreover, it follows by Proposition 1.1.13 that the exponential map $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism at $0 \in \mathfrak{g}$; in particular, the image of exp contains an open neighborhood of $e \in G$. Therefore, since G is connected, we have by Proposition 1.1.2 that G is generated by $\exp(\mathfrak{g})$. This means that ρ is uniquely determined by its differential $\mathrm{d}\rho_e$. □

For completeness, we provide further notions about Lie algebras, given in next definition. These will be necessary to analyze the Lie algebra of the Lie group $\mathrm{SL}_2\mathbb{C}$.

Definition 1.1.16. A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if

$$[X, Y] \in \mathfrak{h}$$

for every $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$. A Lie algebra is *simple*, if it is non-commutative and does not contain any nonzero proper ideal. A Lie algebra is *semisimple* if it is a direct sum of simple Lie algebras.

1.1.2 Representations and modules

Definition 1.1.17. A group homomorphisms $\rho\colon G \to \mathrm{GL}(V)$ is called a *representation* of G. In this case, V is said to be a *G-module*.

If there is no risk of confusion, given any G-module V, we write $g \cdot v$ to describe the action of g on v, for every $g \in G$ and $v \in V$, without specifying the associated representation.

Definition 1.1.18. Let V be a G-module. A subset $U \subset V$ is said to be *G-invariant* if $g \cdot u \in U$ for every $g \in G$ and $u \in U$. If U is also a linear subspace of V, then U is said to be a *G-submodule* of V.

Clearly, a G-submodule U of V has a natural structure of G-module, given by

$$g \cdot u = \rho(g)|_U(u),$$

for every $u \in U$ and $g \in G$.

Definition 1.1.19. Let V and W be two G-modules. A *G-equivariant map* (or a *G-modules homomorphism*) from V to W is a linear map $\alpha\colon V \to W$ such that

$$\alpha(g \cdot v) = g \cdot \alpha(v),$$

for every $g \in G$ and $v \in V$. If, moreover, α is an isomorphism, then it is said to be a *G-modules isomorphism* and the G-module V and W are said to be *isomorphic*.

The following remark provides some natural structures induced by G-modules. It is straightforward to verify that they are well defined.

Remark 1.1.20. Let V and W be two G-modules.

(1) Given any G-equivariant map $\alpha\colon V \to W$, $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$ are G-submodules respectevely of V and W.

(2) The vector spaces $V \oplus W$ and $V \otimes W$ are G-modules, with action respectevely defined by

$$g \cdot (v + w) = (g \cdot v) + (g \cdot w)$$

and

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$$

for every $g \in G$, $v \in V$ and $w \in W$.

(3) As direct consequence of point (2), for every $d \in \mathbb{N}$, the d-symmetric power S^dV of V is a G-module, with an action defined by

$$g \cdot \left(v_1^{k_1} \cdots v_n^{k_n}\right) = (g \cdot v_1)^{k_1} \cdots (g \cdot v_n)^{k_n},$$

for every $v_1, \dots, v_n \in V$ and $k_1, \dots, k_n \in \mathbb{N}$ such that $k_1 + \cdots + k_n = d$. Moreover, we have that also the symmetric algebra $S(V)$, as direct sum of G-modules and extended by linearity, is a G-module.

(4) If $\rho\colon G \to \mathrm{GL}(V)$ is a representation, then the dual vector space V^* has a natural structure of G-module given by the dual representation $\rho^*\colon G \to \mathrm{GL}(V^*)$. It is defined as

$$\rho^*(g) = {}^{\mathrm{t}}\rho(g)^{-1}$$

for every $g \in G$, so that the dual pairing is preserved, i.e. for every $g \in G$, $\phi \in V^*$ and $v \in V$

$$(g \cdot \phi)(g \cdot v) = \phi(v).$$

G-modules not containing any nontrivial G-module play an important role.

Definition 1.1.21. A G-module V is said to be *irreducible* if it contains no nonzero proper G-submodules, i.e. if W is a G-submodule of V, then $W = 0$ or $W = V$.

We can provide now an elementary but very important statement related to irreducible modules. It is due to I. Schur and appears for the first time in 1905 in [Sch05]. We can find it, for instance, in [Ser77, Proposition 4].

Lemma 1.1.22 (Schur's Lemma)**.** *Let V and W be irreducible G-modules and let $\alpha\colon V \to W$ a G-equivariant map. Then:*
(1) if V and W are not isomorphic, then $\alpha = 0$;
(2) if $V = W$, then $\alpha = \lambda \cdot \mathrm{id}$ for some $\lambda \in \mathbb{K}$.

Proof. Point (1) follows immediately from the fact that $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$ are G-submodules. So, let $V = W$ and let $\lambda \in \mathbb{K}$ be an eigenvalue of α. If we set $\alpha' = \alpha - \lambda \cdot \mathrm{id}$, then we must have $\operatorname{Ker}\alpha' \neq 0$ and hence, by point (1), we get

$$\alpha' = \alpha - \lambda \cdot \mathrm{id} = 0,$$

that is, $\alpha = \lambda \cdot \mathrm{id}$. □

Analogously to algebraic groups, we can extend the concept of representations to Lie algebras, associating a linear application to each element of a Lie algebra. This need not to be an isomorphism.

Definition 1.1.23. Let $\mathfrak{g}$ be a Lie algebra and let V be a vector space. A map of Lie algebras

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V) \cong \mathrm{End}(V)$$

is called a *representation* of $\mathfrak{g}$.

As for G-modules, it is possible to define subrepresentations, focusing on those not containing any proper subrepresentations.

Definition 1.1.24. Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. If W is a vector subspace of V such that

$$\rho(g)(w) \in W,$$

then it is said to be $\mathfrak{g}$-*invariant*. The representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is said *irreducible*, if the vector space V does not contain any proper $\mathfrak{g}$-invariant subspace.

Now, it is natural to ask which are the relations between representations of Lie groups and representations of their Lie algebras. An important fact concerns irreducibility of representations. We summarize it in the next proposition, which can be also found as an exercise in [FH91, Exercise 8.17].

Proposition 1.1.25. *Let G be a connected Lie group and let V be a G-module with associate representation $\rho\colon G \to \mathrm{GL}(V)$. Then a subspace W of V is a G-submodule if and only if it is carried into itself under the action of the Lie algebra $\mathfrak{g}$ of G, that is,*

$$\mathrm{d}\rho_e(v)(W) \subseteq W$$

for every $v \in \mathfrak{g}$.

Proof. By Proposition 1.1.14, we know that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{d}\rho_e} & \mathrm{End}(V) \\ {\scriptstyle\exp}\big\downarrow & & \big\downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\rho} & \mathrm{GL}(V) \end{array} \tag{1.1.26}$$

commutes. Given a subspace $W \subseteq V$, let us suppose that for every $v \in \mathfrak{g}$

$$\mathrm{d}\rho_e(v)(W) \subseteq W.$$

Let us consider the closed Lie subgroup $\mathrm{GL}(W)$ of $\mathrm{GL}(V)$. We have that

$$\mathrm{d}\rho_e(v)\big|_W \in \mathrm{End}(W)$$

and hence, considering the restriction of the exponential map to $\mathrm{End}(W)$, we must have

$$\exp\big|_{\mathrm{End}(W)}\big(\mathrm{d}\rho_e(v)\big|_W\big) \in \mathrm{GL}(W),$$

that is,

$$\exp\big(\mathrm{d}\rho_e(v)\big)\big|_W \in \mathrm{GL}(W).$$

By commutativity of the diagram in (1.1.26), we have

$$\rho\big(\exp(v)\big)\big|_W \in \mathrm{GL}(W)$$

and hence, since G is connected, it follows by Proposition 1.1.13 that G is generated by $\exp(\mathfrak{g})$. Thus, we get

$$\rho(g)\big|_W \in \mathrm{GL}(W),$$

which means that W is a G-submodule of V.

Conversely, if

$$\rho(g)(w) \in W$$

for every $g \in G$ and $w \in W$, then we can define a morphism

$$\rho\colon G \to \mathrm{GL}(W),$$

for which we can consider the identification

$$\rho(g) = \rho(g)\big|_W.$$

In particular, the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{d}\rho_e} & \mathrm{End}(W) \\ {\scriptstyle\exp}\big\downarrow & & \big\downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\rho} & \mathrm{GL}(W) \end{array}$$

is well-defined and commutes. Therefore, we can also restrict the differential $\mathrm{d}\rho_e$, getting

$$\mathrm{d}\rho_e(v)\big|_W \in \mathrm{End}\, W,$$

for every $v \in V$. This implies that W is carried into itself by the action of $\mathfrak{g}$. □

By Proposition 1.1.25, it follows immediately the following corollary.

Corollary 1.1.27. *Let G be a connected Lie group and let V be a G-module. Then, V is irreducible over G if and only if it is irreducible over $\mathfrak{g}$.*

1.1.3 Irreducible representations of $\mathfrak{sl}_2\mathbb{C}$

In this section we focus on a particular example regarding irreducible representations of Lie Algebras. We analyze the Lie algebra of the Lie group $\mathrm{SL}_2(\mathbb{C})$, which is the space

$$\mathfrak{sl}_2\mathbb{C} = \{\, A \in \mathrm{Mat}_2(\mathbb{C}) \mid \operatorname{tr} A = 0 \,\}.$$

All the contents below are taken from [FH91, Section 11.1] and [Kir08, Section 4.8], to which we refer to further details. Let us consider the basis given by the the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying the equations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{1.1.28}$$

Proposition 1.1.29. *The Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is a simple algebra.*

Proof. Let $\mathfrak{a} \subseteq \mathfrak{sl}_2\mathbb{C}$ be an ideal of $\mathfrak{sl}_2\mathbb{C}$. Let us consider any non-zero linear combination

$$aH + bE + cF \in \mathfrak{a}$$

with $a, b, c \in \mathbb{C}$. We observe that if $H \in \mathfrak{a}$, then by equations (1.1.28) we have

$$[H, E] = 2E \in \mathfrak{a}, \quad [H, F] = -2F \in \mathfrak{a}$$

and hence $\mathfrak{a} = \mathfrak{sl}_2\mathbb{C}$. So, if $b = c = 0$, the statement is trivial. Otherwise, in the case of $b = 0$, since

$$[H, aH + bE + cF] = a[H, H] + b[H, E] + c[H, F] = 2bE - 2cF \in \mathfrak{a},$$

we easily get $F \in \mathfrak{a}$ and hence also

$$[E, F] = H \in \mathfrak{a}.$$

Analogously, if $c = 0$, we get $H \in \mathfrak{a}$ as well. Finally, if $b, c \neq 0$, we have

$$(aH + bE + cF) + (bE - cF) = aH + 2bE \in \mathfrak{a}$$

and hence

$$[H, aH + 2bE] = a[H, H] + 2b[H, E] = 4bE \in \mathfrak{a}.$$

This implies, in particular, that

$$[E, F] = H \in \mathfrak{a},$$

proving the statement. □

The classical Jordan decomposition (see e.g. [Bor91, Proposition 4.2]) says that any endomorphism $f: V \to V$ of a complex vector space V can be written uniquely as the sum

$$f = f_s + f_n,$$

where $f_s: V \to V$ and $f_n: V \to V$ are respectively a diagonalizable endomorphism and a nilpotent endomorphism commuting with each other, namely,

$$f_n f_s - f_s f_n = 0.$$

This concept can be extended to Lie algebras in the specific case of semisimple Lie algebras. Indeed, each of these can be decomposed into a sum of two elements, preserving the diagonalizable and the nilpotent parts. The proof of this theorem, which can be found in [FH91, Theorem 9.20, Corollary C.18], is quite technical and we omit it. We suggest to consult [FH91, Appendix C] for more specific details and a complete proof.

Theorem 1.1.30 (Preservation of Jordan decomposition)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra. For every element $X \in \mathfrak{g}$, there exist two element $X_s, X_n \in \mathfrak{g}$ such that for any representation $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ we have*

$$\rho(X_s) = \rho(X)_s, \quad \rho(X_n) = \rho(X)_n.$$

In particular, if ρ is injective and $\mathfrak{g}$ represents a subalgebra of $\mathfrak{gl}(V)$, then the diagonalizable and nilpotent parts of any element X of $\mathfrak{g}$ are again in $\mathfrak{g}$ and are independent of the particular representation ρ.

Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_2\mathbb{C}$. By Theorem 1.1.30, it is clear that the action of H on V is diagonalizable. Therefore, we can write

$$V = \bigoplus_{\lambda \in I} V_\lambda, \tag{1.1.31}$$

where I is a finite set of complex values such that

$$H(v) = \lambda v$$

for every $\lambda \in I$ and $v \in V_\lambda$. Now, given a vector $v \in V_\lambda$, we have

$$\begin{aligned} H\big(E(v)\big) &= E\big(H(v)\big) + [H, E](v) \\ &= E(\lambda v) + 2E(v) \\ &= (\lambda + 2)E(v). \end{aligned}$$

This means that E sends every eigenvector into another eigenspace and, in particular, its restriction to V_λ gives a morphism $E\colon V_\lambda \longrightarrow V_{\lambda+2}$ for every $\lambda \in I$. Analogously, we obtain a morphism $F\colon V_\lambda \longrightarrow V_{\lambda-2}$ such that

$$H\big(F(v)\big) = (\lambda - 2)F(v).$$

The space

$$\bigoplus_{k \in \mathbb{Z}} V_{\lambda_0+2k},$$

is invariant under the action of $\mathfrak{sl}_2\mathbb{C}$ and hence, by irreducibility of V, we must have

$$V = \bigoplus_{k \in \mathbb{Z}} V_{\lambda_0+2k}.$$

This means that the eigenvalues appearing in decomposition (1.1.31) must be congruent modulo 2. Moreover, the spectrum of H must consist of an unbroken string of complex numbers $\lambda_0, \lambda_0+2, \ldots, \lambda_0+2m$ for a certain $m \in \mathbb{N}$. More specifically, if we set $n = \lambda_0 + 2m$, then we can describe the action of $\mathfrak{sl}_2\mathbb{C}$ on V through the diagram

$$\begin{tikzcd} 0 \arrow[r, shift left, "E"] & V_{n-2m} \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] \arrow[loop below, "H"] & V_{n-2m+2} \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] \arrow[loop below, "H"] & \cdots \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] & V_{n-2} \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] \arrow[loop below, "H"] & V_n \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] \arrow[loop below, "H"] & 0 \arrow[l, shift left, "F"] \end{tikzcd} . \tag{1.1.32}$$

Hence, it is sufficient to determine the value of $n \in \mathbb{C}$ to determine all the eigenspaces.

We provide now a series of results, by which we can determine the structure of the space V.

Lemma 1.1.33. *Let $v \in V_n$. Then the set $\{\, F^k(v) \mid k \in \mathbb{N} \,\}$ generates the space V.*

Proof. Let us consider the subspace

$$W = \langle F^k(v) \rangle_{k\in\mathbb{N}} \subseteq V.$$

Then, by irreducibility of V, it is sufficient to show that W is irreducible under the action of $\mathfrak{sl}_2\mathbb{C}$. We clearly have that F preserves the space W. Moreover, since $F^k(v) \in V_{n-2k}$ for every $k \in \mathbb{N}$, it follows that

$$H\big(F^k(v)\big) = (n-2k)F^k(v) \tag{1.1.34}$$

and hence also H preserves the space W. Finally, to prove that

$$E(W) \subseteq W,$$

we can proceed by induction on the power $k \in \mathbb{N}$ to show that

$$E\big(F^k(v)\big) \in W$$

for every $k \in \mathbb{N}$. If $k = 0$, then, since $v \in V_n$, we have

$$E(v) = 0,$$

which clearly belongs to W. Now, let us suppose that $E\big(F^{k-1}(v)\big) \in W$. Then we have

$$\begin{aligned} E\big(F^k(v)\big) &= F\Big(E\big(F^{k-1}(v)\big)\Big) + [E,F]\big(F^{k-1}(v)\big) \\ &= F\Big(E\big(F^{k-1}(v)\big)\Big) + H\big(F^{k-1}(v)\big). \end{aligned}$$

Since both F and H preserves W then we have that also E preserves E as well, which proves that $V = W$. □

Considering formula (1.1.34), we have by Lemma 1.1.33 that

$$V = \langle F^k(v) \rangle_{k\in\mathbb{N}}.$$

We therefore immediately obtain the following corollary.

Corollary 1.1.35. *If V_λ is an eigenspace of H, then* $\dim V_\lambda = 1$.

It is also possible to determine explicitly the image of each power $F^k(v)$ from the operator E.

Lemma 1.1.36. *The equality*

$$E\big(F^k(v)\big) = k(n-k+1)F^{k-1}(v)$$

holds for every $k \in \mathbb{N} \setminus \{0\}$ and $v \in V_n$.

Proof. We proceed by induction on k. If $k = 1$, then

$$E\big(F(v)\big) = F\big(E(v)\big) + [E,F](v) = H(v) = nv,$$

where the second equality follows by $E(v) = 0$. Now, let us suppose the statement true for $k-1$. Then we have by inductive hypothesis and formula (1.1.34) the equalities

$$\begin{aligned} E\big(F^k(v)\big) &= F\Big(E\big(F^{k-1}(v)\big)\Big) + [E,F]\big(F^{k-1}(v)\big) \\ &= F\big((k-1)(n-k+2)F^{k-2}(v)\big) + (n-2k+2)F^{k-1}(v) \\ &= \big((k-1)(n-k+2) + (n-2k+2)\big)\big)F^{k-1}(v) \\ &= k(n-k+1)F^{k-1}(v). \end{aligned}$$

□

Since V has finite dimension, it follows by decomposition (1.1.31) and formula (1.1.34) that there must exists a natural number $m \in \mathbb{N}$, such that

$$F^m(v) = 0.$$

In particular, if we suppose m to be the smallest power of F annihilating v, then we have by Lemma 1.1.36

$$0 = E\big(F^m(v)\big) = m(n - m + 1)F^{m-1}(v).$$

Therefore, since $F^{m-1} \neq 0$, we must have $n = m - 1$. In particular, we conclude that n is a non-negative integer and the set of eigenvalues of H on V is given by a finite succession of integers differing by 2 and it is symmetric with respect to 0 in $\mathbb{Z}$. Namely, we can rewrite the diagram (1.1.32) as

$$\begin{tikzcd} 0 \arrow[r, shift left, "E"] & V_{-n} \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] \arrow[loop below, "H"] & V_{-n+2} \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] \arrow[loop below, "H"] & \cdots \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] & V_{n-2} \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] \arrow[loop below, "H"] & V_n \arrow[l, shift left, "F"] \arrow[r, shift left, "E"] \arrow[loop below, "H"] & 0 \arrow[l, shift left, "F"] \end{tikzcd} \quad (1.1.37)$$

We have thus proved that irreducible representations are unique and depend only on the dimension of V. From the later we can determine the eigenvalues of the action of the operator H, which have a special role.

Definition 1.1.38. Given a representation $V^{(n)}$ of $\mathfrak{sl}_2\mathbb{C}$ of dimension $n + 1$, the eigenvectors of the action of the operator H on $V^{(n)}$, associated to the set of eigenvalues given by

$$\{-n, -n + 2, \dots, n - 2, n\},$$

are called the *weights* of $V^{(n)}$.

Considering the diagram (1.1.37) we conclude that any representation of $\mathfrak{sl}_2\mathbb{C}$ with distinct eigenvalues of multiplicity 1 and having the same parity must be irreducible. In particular, we can see that irreducible representations of $\mathfrak{sl}_2\mathbb{C}$ correspond to the components of the ring of polynomials in two variables and the weights correspond to monomials.

Theorem 1.1.39. *For every $n \in \mathbb{N}$, there exists a unique irreducible representation $V^{(n)}$ of $\mathfrak{sl}_2\mathbb{C}$ such that $\dim V^{(n)} = n + 1$, given by the n-th symmetric power of $\mathbb{C}^2$, that is*

$$V^{(n)} \cong S^n\mathbb{C}^2,$$

and for every $k = 0, \dots, n$, the weight associated to the integer number $n - 2k$ is given by the monomial $x^{n-k}y^k$, that is

$$H\big(x^{n-k}y^k\big) = (n - 2k)x^{n-k}y^k.$$

Proof. The standard representation of $\mathfrak{sl}_2\mathbb{C}$ on $\mathbb{C}^2$ can be defined by the relations on the operator H

$$H(x) = x, \quad H(y) = -y.$$

These can be extended to the n-th symmetric power $S^n\mathbb{C}^2$ by considering its basis

$$\big\{x^n, x^{n-1}y, \dots, xy^{n-1}, y^n\big\}.$$

Indeed, we have

$$H\big(x^{n-k}y^k\big) = (n - k)H(x)x^{n-k-1}y^k + kH(y)x^{n-k}y^{k-1} = (n - 2k)x^{n-k}y^k.$$

Thus, by the considerations we made above, we conclude that $S^n\mathbb{C}^2$ is the unique irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ of dimension $n + 1$. ◻

1.2 Apolarity and sums of powers

Through this section, we will always denote by V a finite-dimensional vector space of dimension $n \in \mathbb{N}$ over a field $\mathbb{K}$ of characteristic 0.

1.2.1 Classical apolarity

Let us denote by

$$\mathcal{R}_n = S(V), \quad \mathcal{D}_n = S(V^*),$$

respectively the symmetric algebra of V and its dual space. We can define in a natural way, for every $d \in \mathbb{N}$, a bilinear map in $S^dV \otimes S^dV^*$, given by

$$\begin{aligned} \circ\colon S^dV^* \times S^dV &\longrightarrow \mathbb{K} \\ (\phi_1 \cdots \phi_d, v_1 \cdots v_d) &\mapsto \sum_{\sigma \in \mathfrak{S}_d} \phi_1(v_{\sigma(1)}) \cdots \phi_d(v_{\sigma(d)}), \end{aligned} \tag{1.2.1}$$

which is also known as *contraction pairing* or *polar pairing*. This one can be also generalized to the *partial polarization map*, defined for $k \leq d$ by

$$\begin{aligned} \circ\colon S^kV^* \times S^dV &\longrightarrow S^{d-k}V \\ (\phi_1 \cdots \phi_k, v_1 \cdots v_d) &\mapsto \sum_{1 \leq i_1 < \cdots < i_k \leq d} (\phi_1 \cdots \phi_k) \circ \left(v_{i_1} \cdots v_{i_k}\right) \prod_{j \neq i_1, \ldots, i_k} v_j. \end{aligned} \tag{1.2.2}$$

It is possible to identify $S(V)$ and $S(V^*)$ with polynomial rings. Let us denote by $\mathbb{K}[V]$ the ring of polynomial functions on V, that is the commutative $\mathbb{K}$-algebra generated by V^*.

Proposition 1.2.3. *For every $d \in \mathbb{N}$, there is a natural isomorphism*

$$S^dV^* \cong \mathbb{K}[V]_d.$$

Proof. Let us consider the $\mathbb{K}$-linear map

$$\begin{aligned} \Phi\colon S^dV^* &\longrightarrow \mathbb{K}[V]_d \\ \phi &\longmapsto f_\phi, \end{aligned}$$

where the function f_ϕ is defined by

$$\begin{aligned} f_\phi\colon V &\longrightarrow \mathbb{K} \\ v &\mapsto \phi(v, \ldots, v). \end{aligned}$$

We should verify that Φ is well defined, that is, f_ϕ is effectively a polynomial function. To see this, we can write ϕ as a linear combination of the basis $\left\{v_1^*, \ldots, v_d^*\right\}$, obtaining

$$\phi(w_1, \ldots, w_d) = \sum_{i_1, \ldots, i_d = 1}^{n} \phi(v_{i_1}, \ldots, v_{i_d}) v_{i_1}^*(w_1) \cdots v_{i_d}^*(w_d). \tag{1.2.4}$$

Therefore, we have

$$f_\phi(v) = \phi(v, \ldots, v) = \sum_{i_1, \ldots, i_d = 1}^{n} \phi(v_{i_1}, \ldots, v_{i_d}) v_{i_1}^*(v) \cdots v_{i_d}^*(v) \tag{1.2.5}$$

for every $v \in V$ and hence f_ϕ is a polynomial function, namely, $f_\phi \in \mathbb{K}[V]$. It remains to verify that Φ is bijective. Let us consider an element $\phi \in S^dV$ and let us suppose that $\Phi(\phi) \equiv 0$. This means that

$$\Phi(\phi)(v) = f_\phi(v) = 0$$

for every $v \in V$. Then, if we consider again formula (1.2.5), we have

$$f_\phi = \sum_{i_1,\dots,i_d=1}^{n} \phi(v_{i_1},\dots,v_{i_d}) v_{i_1}^* \cdots v_{i_d}^* \equiv 0.$$

Moreover, since the set of the polynomial functions of the type

$$v_{i_1}^* \cdots v_{i_d}^* \colon V \longrightarrow \mathbb{K}$$

among the indices $1 \leq i_1,\dots,i_d \leq n$ constitutes a set of linear independent elements, it follows that

$$\phi(v_{i_1},\dots,v_{i_d}) = 0.$$

Thus, as the linear combination describing ϕ in formula (1.2.4) has the same coefficients, we conclude that $\phi \equiv 0$, that is Φ is injective. To prove surjectivity, we consider a polynomial function $f \in \mathbb{K}[V]_d$, written as a linear combination of the arbitrary basis $\{v_1^*,\dots,v_d^*\}$, that is

$$f = \sum_{i_1,\dots,i_d=1}^{n} a_{i_1,\dots,i_d} v_{i_1}^* \cdots v_{i_d}^*$$

for some suitable coefficients $a_{i_1,\dots,i_d} \in \mathbb{K}$. If we consider the symmetric multilinear form $\psi \in S^d V^*$, defined as

$$\psi(w_1,\dots,w_d) = \sum_{i_1,\dots,i_d=1}^{n} a_{i_1,\dots,i_d} v_{i_1}^*(w_1) \cdots v_{i_d}^*(w_d),$$

then it is immediate to verify that $f = f_\psi = \Phi(\psi)$, that is, Φ is surjective. □

Let us consider a basis $\{v_1,\dots,v_n\}$ of V and its dual basis $\{v_1^*,\dots,v_n^*\}$ in V^*. Then, given any

$$v = y_1 v_1 + \cdots + y_n v_n \in V,$$

as each functional v_i^* associates to v its i-th coordinate y_i, we can directly identify $\{v_1^*,\dots,v_n^*\}$ as the sets of coordinates $\{y_1,\dots,y_n\}$. Therefore, by Proposition 1.2.3, we can write

$$\mathcal{R}_n \simeq \mathbb{K}[x_1,\dots,x_n], \quad \mathcal{D}_n \simeq \mathbb{K}[y_1,\dots,y_n],$$

identifying $\mathcal{R}_n$ and $\mathcal{D}_n$ with two polynomial rings.

We can compute the image of each pair of monomials through polarization map, that is

$$\mathbf{y}^{\boldsymbol{\alpha}} \circ \mathbf{x}^{\boldsymbol{\beta}} = \begin{cases} \dfrac{\boldsymbol{\beta}!}{(\boldsymbol{\beta}-\boldsymbol{\alpha})!}\mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\alpha}}, & \text{if } \boldsymbol{\beta}-\boldsymbol{\alpha} \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$, where we use the notation

$$\mathbf{x}^{\boldsymbol{\delta}} = x_1^{\delta_1} \dots x_n^{\delta_n}, \quad \mathbf{y}^{\boldsymbol{\delta}} = y_1^{\delta_1} \dots y_n^{\delta_n}, \quad \boldsymbol{\delta}! = \delta_1! \cdots \delta_n!$$

for every multi-index $\boldsymbol{\delta} = (\delta_1,\dots,\delta_n) \in \mathbb{N}^n$. By these considerations, it follows that we can naturally identify the space $\mathcal{D}_n$ with the space of polynomial differential operators.

Definition 1.2.6. Given any homogeneous polynomial $\phi \in \mathcal{D}_n^k$, with $k \leq d$, the operator

$$\begin{aligned} \mathrm{D}_\phi \colon \mathcal{R}_n^d &\longrightarrow \mathcal{R}_n^{d-k} \\ h &\longmapsto \phi \circ h \end{aligned}$$

is called the *differential operator* associated to ϕ.

The partial polarization map induces in a natural way an action of the space $\mathcal{D}_n$ on the space $\mathcal{R}_n$, considering their elements just as polynomials.

Definition 1.2.7. The *apolarity action* of $\mathcal{D}_n$ on $\mathcal{R}_n$ is defined by naturally extending by linearity the polarization maps for each components of $\mathcal{D}_n$ and $\mathcal{R}_n$, that is

$$\begin{aligned} \circ\colon \mathcal{D}_n \times \mathcal{R}_n &\longrightarrow \mathcal{R}_n \\ (\phi, f) &\longmapsto \mathrm{D}_\phi(f). \end{aligned}$$

Remark 1.2.8. For the case of contraction pairing, defined in (1.2.1), and also in general in dealing with dual spaces $\mathcal{R}_n$ and $\mathcal{D}_n$, we can reverse the roles of variables x_i and y_i, setting an identification between the components $\mathcal{R}_n^j$ and $\mathcal{D}_n^j$. In particular, we can consider the contraction pairing as a symmetric bilinear form.

Given any homogeneous polynomial, we can also provide the following definition, based on the apolarity action of the space $\mathcal{D}_n$ on it.

Definition 1.2.9. For every homogeneous polynomial $f \in \mathcal{R}_n^d$, the *catalecticant map* of f is defined as the linear map

$$\begin{aligned} \mathrm{Cat}_f\colon \mathcal{D}_n &\longrightarrow \mathcal{R}_n \\ \phi &\longmapsto \phi \circ f. \end{aligned}$$

The *apolar ideal* of the polynomial f is defined as the kernel of Cat_f and it is the set

$$f^\perp = \{\, g \in \mathcal{D} \mid g \circ f = 0 \,\}.$$

Now, it is clear that the catalecticant map is graded, that is the map

$$\begin{aligned} \mathrm{Cat}_f^j\colon \mathcal{D}_n^j &\longrightarrow \mathcal{R}_n^{d-j} \\ g &\longmapsto g \circ f, \end{aligned}$$

also called the *j-th catalecticant* of the polynomial f, is well defined. Moreover, fixed any bases of $\mathcal{D}_n^j$ and $\mathcal{R}_n^{d-j}$, the matrix associated to Cat_f^j is called the *j-th catalecticant matrix* of f.

Now, by the identification we have seen in Proposition 1.2.3, we can naturally define the action of the linear group $\mathrm{GL}_n(\mathbb{K})$ on both $\mathcal{R}_n$ and $\mathcal{D}_n$, in the same way of Remark 1.1.20.

Definition 1.2.10. The action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathcal{R}_n$ is defined by naturally extending the action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathcal{R}_n^1$, given by

$$A \cdot x_j = \sum_{k=1}^{n} A_{kj} x_k,$$

for every $A \in \mathrm{GL}_n(\mathbb{K})$ and for every $j = 1, \ldots, n$. The action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathcal{D}_n$ is defined in the same way by the dual action on $\mathcal{D}_n^1$, given by

$$A \cdot y_j = \sum_{k=1}^{n} {}^{\mathrm{t}}A^{-1}_{kj} y_k,$$

for every $A \in \mathrm{GL}_n(\mathbb{K})$ and for every $j = 1, \ldots, n$.

In dealing with apolarity action it can be useful to consider a particular notation for a basis of monomials.

Definition 1.2.11. For every $k \in \mathbb{N}$, the *divided power monomials* of $\mathcal{R}_n^k$ and $\mathcal{D}_n^k$ are the monomials defined as

$$\mathbf{x}^{[\delta]} = x_1^{[\delta_1]} \dots x_n^{[\delta_n]} = \frac{1}{\delta!} x_1^{\delta_1} \dots x_n^{\delta_n}, \quad \mathbf{y}^{[\delta]} = y_1^{[\delta_1]} \dots y_n^{[\delta_n]} = \frac{1}{\delta!} y_1^{\delta_1} \dots y_n^{\delta_n},$$

where

$$\delta! = \delta_1! \cdots \delta_n!$$

for any multi-index $\boldsymbol{\delta} \in \mathbb{N}^n$ such that

$$|\delta| = \delta_1 + \dots + \delta_n = k.$$

The use of divided power monomials can simplify the use of apolarity action in dealing with the coefficients produced by derivations. It is possible to define also a structure of $\mathbb{K}$-algebra over the vector space generated by divided power monomials (see [IK99, Appendix A]). This is given by extending by linearity the operation of multiplication on divided power monomials, which is defined by the equality

$$\mathbf{x}^{[\delta]}\mathbf{x}^{[\gamma]} = \frac{(\delta + \gamma)!}{\delta!\gamma!}\mathbf{x}^{[\delta+\gamma]}$$

for every multi-indices $\boldsymbol{\delta}, \boldsymbol{\gamma} \in \mathbb{N}^n$.

The next property easily follows from the definition of apolarity action and representations on dual spaces.

Proposition 1.2.12. *The apolarity action of $\mathcal{D}_n$ on $\mathcal{R}_n$ is a $\mathrm{GL}_n(\mathbb{K})$-equivariant map.*

Proof. By the action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathcal{R}_n$ and $\mathcal{D}_n$ given in Definition 1.2.10 and Remark 1.1.20, we can define in a natural way the action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathcal{D}_n \times \mathcal{R}_n$ as

$$A \cdot (\phi, f) = (A \cdot \phi, A \cdot f)$$

for every $A \in \mathrm{GL}_n(\mathbb{K})$, $\phi \in \mathcal{D}_n$ and $f \in \mathcal{R}_n$. We first analyze the restriction to the action of $\mathcal{D}_n^1$ on $\mathcal{R}_n^1$. So, let $A \in \mathrm{GL}_n(\mathbb{K})$ and $j, k \in \mathbb{N}$ such that $1 \leq j, k \leq n$. Then we have

$$\begin{aligned}
(A \cdot y_j) \circ (A \cdot x_k) &= \left(\sum_{s=1}^n {}^{\mathrm{t}}A^{-1}_{js} y_s \right) \circ \left(\sum_{t=1}^n A_{tk} x_t \right) \\
&= \sum_{s=1}^n {}^{\mathrm{t}}A^{-1}_{js} \left(\sum_{t=1}^n A_{kt} \frac{\partial x_t}{\partial x_s} \right) \\
&= \sum_{t=1}^n A^{-1}_{tj} A_{kt} = \delta_{jk},
\end{aligned}$$

that is,

$$(A \cdot y_j) \circ (A \cdot x_k) = y_j \circ x_k.$$

This means, in particular, that the set $\{A \cdot y_1, \dots, A \cdot y_n\}$ corresponds to the dual basis of the basis $\{A \cdot x_1, \dots, A \cdot x_n\}$. Therefore, since the action of A is a linear transformation preserving the dual pairing, we have by definition that

$$(A \cdot \phi) \circ (A \cdot f) = A \cdot (\phi \circ f)$$

for every $\phi \in \mathcal{D}_n$ and $f \in \mathcal{R}_n$. □

Linear forms can be view in general as points in $\mathbb{K}^{n*}$. For every point $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{K}^n$, we denote by $l_{\mathbf{a}} \in \mathcal{D}_n^1$ the linear form

$$l_{\mathbf{a}} = \mathbf{a} \cdot \mathbf{x} = a_1 x_1 + \cdots + a_n x_n$$

and we call it the *linear form associated to* $\mathbf{a}$. Sometimes, if there is no risk of confusion, we will refer to it simply by the term *point*.

Definition 1.2.13. Let $\mathbf{a} \in \mathbb{K}^n$ and let $l_{\mathbf{a}}$ be its associated linear form. Then for every $d \in \mathbb{N}$ the form $l_{\mathbf{a}}^{[d]} \in \mathcal{R}_n^d$, defined as

$$l_{\mathbf{a}}^{[d]} = \sum_{\alpha_1 + \cdots + \alpha_n = d} a_1^{\alpha_1} \cdots a_n^{\alpha_n} x_1^{[\alpha_1]} \dots x_n^{[\alpha_n]},$$

is called the *d-th divided power* of $l_{\mathbf{a}}$.

The main properties of divided powers of linear forms are enumerated in the following proposition.

Proposition 1.2.14. *Let be $l_{\mathbf{a}} = a_1 x_1 + \cdots + a_n x_n \in \mathcal{D}_n^1$. Then for every $d, k \in \mathbb{N}$*
(1) $l_{\mathbf{a}}^d = d! l_{\mathbf{a}}^{[d]}$;
(2) $l_{\mathbf{a}}^{[d]}(\phi) = \phi(a_1, \dots, a_n)$ for every $\phi \in \mathcal{R}_n^d$;
(3) $A \cdot l_{\mathbf{a}}^{[d]} = (A \cdot l_{\mathbf{a}})^{[d]}$ for every $A \in \mathrm{GL}_n(\mathbb{K})$;
(4) $d!k! l_{\mathbf{a}}^{[d]} l_{\mathbf{a}}^{[k]} = (d+k)! l_{\mathbf{a}}^{[d+k]}$.

Since the proof is quite simple and consists just of some technical computations, we omit it and we refer to [IK99, Proposition A.9] to check it. Another property to take into consideration is the fact that the image elements through the catalecticant map of the powers of a point are still powers of the same point.

Lemma 1.2.15. *Let $d, k \in \mathbb{N}$ such that $d \geq k$. Then, for every $\mathbf{a} \in \mathbb{K}^n$ and $\phi \in \mathcal{D}_n^k$,*

$$\phi \circ l_{\mathbf{a}}^{[d]} = \phi(\mathbf{a}) l_{\mathbf{a}}^{[d-k]}.$$

Proof. By linearity it is sufficient to prove the formula just for monomials in $\mathcal{D}_n^k$. So, given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ such that $|\boldsymbol{\alpha}| = k$, we have

$$\begin{aligned} y_1^{\alpha_1} \cdots y_n^{\alpha_n} \circ l_{\mathbf{a}}^{[d]} &= y_1^{\alpha_1} \cdots y_n^{\alpha_n} \circ \left(\sum_{|\beta| = d} a_1^{\beta_1} \dots a_n^{\beta_n} x_1^{[\beta_1]} \cdots x_n^{[\beta_n]} \right) \\ &= \sum_{|\beta| = d} a_1^{\beta_1} \cdots a_n^{\beta_n} x_1^{[\beta_1 - \alpha_1]} \cdots x_n^{[\beta_n - \alpha_n]} \\ &= a_1^{\alpha_1} \cdots a_n^{\alpha_n} \left(\sum_{|\gamma| = d-k} a_1^{\gamma_1} \cdots a_n^{\gamma_n} x_1^{[\gamma_1]} \cdots x_n^{[\gamma_n]} \right) \\ &= a_1^{\alpha_1} \cdots a_n^{\alpha_n} l_{\mathbf{a}}^{[d-k]}. \end{aligned}$$

□

Corollary 1.2.16. *For every $d, k \in \mathbb{N}$ and for every $\mathbf{a} \in \mathbb{K}^n$*

$$\mathrm{rk}\left(\mathrm{Cat}^k_{l_{\mathbf{a}}^d} \right) \leq 1.$$

We can finally expose a classical result, crucial for our purposes, which puts homogeneous polynomials in relation with powers of linear forms.

Lemma 1.2.17 (Apolarity lemma). *Let $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{K}^n$, let $l_k = l_{\mathbf{a}_k}$ for every $k = 1, \dots, r$, let*

$$\mathcal{A} = \{[\mathbf{a}_1], \dots, [\mathbf{a}_r]\} \subset \mathbb{P}(\mathbb{K}^n) = \mathbb{P}^{n-1}$$

and let $I_{\mathcal{A}}$ be the homogeneous ideal in $\mathcal{D}_n$ of polynomials vanishing on $\mathcal{A}$. Then:
(1) for every $d, k \in \mathbb{N}$ such that $d \geq k$, if $\phi \in \mathcal{D}_n^k$, then

$$\phi \circ \left(l_1^{[d]} + \cdots + l_r^{[d]}\right) = \phi(\mathbf{a}_1) l_1^{[d-k]} + \cdots + \phi(\mathbf{a}_r) l_r^{[d-k]};$$

(2) given any $k \in \mathbb{N}$, then

$$(I_{\mathcal{A}})_k^{\perp} = \left\langle l_1^{[k]}, \dots, l_r^{[k]} \right\rangle,$$

where $(I_{\mathcal{A}})_k^{\perp}$ represents the orthogonal space to $(I_{\mathcal{A}})_k$ with respect to the contraction pairing

$$\circ \colon \mathcal{D}_n^k \times \mathcal{R}_n^k \to \mathbb{K}.$$

Proof. Point (1) follows directly by Lemma 1.2.15. Now, given any $\phi \in \mathcal{D}_n^k$, we have still by Lemma 1.2.15 that for every $\lambda_1, \dots, \lambda_r \in \mathbb{K}$

$$\phi \circ \left(\sum_{j=1}^{r} \lambda_j l_j^{[k]} \right) = \sum_{j=1}^{r} \lambda_j \phi(\mathbf{a}_j).$$

This means that

$$\left\langle l_1^{[k]}, \dots, l_r^{[k]} \right\rangle^{\perp} = \left\{ \phi \in \mathcal{D}_n^k \;\middle|\; \phi(\mathbf{a}_j) = 0, \forall j = 1, \dots, r \right\} = (I_{\mathcal{A}})_k.$$

Thus, point (2) directly follows from the fact that the contraction pairing is non-degenerate. □

1.2.2 Rank and border rank of a polynomial

The statement of Lemma 1.2.17 provides an efficient method to determine a representation of a polynomial as sum of powers of linear forms. This way of writing polynomials have a special role in both classical and recent mathematics.

Definition 1.2.18. Let $f \in \mathcal{R}_n^d$ be a homogeneous polynomial. For every $r \in \mathbb{N}$, the sum of the d-th powers of r linear form $l_{\mathbf{a}_1}, \dots, l_{\mathbf{a}_r} \in \mathcal{R}_1$ is said to be a *decomposition* of f if

$$f = \sum_{j=1}^{r} l_{\mathbf{a}_j}^d.$$

The elements $\mathbf{a}_1, \dots, \mathbf{a}_r$ are called the *points* of the decomposition.

Now, as already mentioned in the Introduction, the determination of decompositions of polynomials is quite classical and, nowadays, has been approached with the languages of symmetric tensors, as we have seen in Proposition 1.2.3. In dealing with this subject, the main problem consists of finding which is the minimum number $r \in \mathbb{N}$ such that, given a specific $f \in \mathcal{R}_d$, it is possible to represent a decomposition of f of size r.

Definition 1.2.19. Let $f \in \mathcal{R}_d$ be a homogeneous polynomial. The *Waring rank*, or *symmetric tensor rank*, or simply *rank*, of f is the natural number

$$\operatorname{rk} f = \min \left\{ r \in \mathbb{N} \;\middle|\; f = \sum_{j=1}^{r} l_{\mathbf{a}_j}^d : \mathbf{a}_j \in \mathbb{K}^n \right\}.$$

In dealing with decompositions of polynomials from the point of view of algebraic geometry, there is another subject to take into consideration. That is, the minimum number $r \in \mathbb{N}$ such that a polynomial is the limit of polynomials of rank equal to r.

Definition 1.2.20. For every $r \in \mathbb{N}$, let

$$\mathcal{S}^d_{n,r} = \left\{ f \in \mathcal{R}^d_n \,\middle|\, \operatorname{rk} f = r \right\}$$

be the set of polynomials of degree d with rank equal to r. Given a polynomial $f \in \mathcal{R}^d_n$, the *border rank* of f is the natural number

$$\operatorname{brk} f = \min \left\{ r \in \mathbb{N} \,\middle|\, f \in \overline{\mathcal{S}^d_{n,r}} \right\},$$

where the overline represents the closure for the Zariski topology.

In terms of closure, we need to recall that, essentially by a result provided by D. Mumford in [Mum95, Theorem 2.33], setting $\mathbb{K} = \mathbb{C}$, the closure of the set $\mathcal{S}^d_{n,r}$ in Zariski topology equals the closure of the same set in Euclidean topology. Thus, we can think to the border rank of a polynomial $f \in \mathcal{R}^d_n$ as the minimum number $r \in \mathbb{N}$ such that

$$f = \lim_{t \to 0} \sum_{j=1}^{r} l_j^d(t),$$

where $\{l_j(t)\}_{t \in \mathbb{R}}$ is a family of linear forms for every $j = 1, \dots, r$.

If one would like to see the definition of rank and border rank from a more geometric point of view, we should give further definitions (see e.g. [LT10, Section 2]). Given a projective variety $X \subseteq \mathbb{P}^{n-1}$, we can can introduce the sets

$$\sigma_r^0(X) = \bigcup_{\mathbf{a}_1,\dots,\mathbf{a}_r \in X} \langle \mathbf{a}_1, \dots, \mathbf{a}_r \rangle, \quad \sigma_r(X) = \overline{\bigcup_{\mathbf{a}_1,\dots,\mathbf{a}_r \in X} \langle \mathbf{a}_1, \dots, \mathbf{a}_r \rangle}, \tag{1.2.21}$$

where the overline is again meant for the Zariski topology. We thus define the X*-rank* and the X*-border rank* of a point $\mathbf{p} \in \mathbb{P}^{n-1}$ as the values given respectively by

$$\operatorname{rk}_X(\mathbf{p}) = \min \left\{ r \in \mathbb{N} \,\middle|\, \mathbf{p} \in \sigma_r^0(X) \right\}, \quad \operatorname{brk}_X(\mathbf{p}) = \min \{ r \in \mathbb{N} \mid \mathbf{p} \in \sigma_r(X) \}. \tag{1.2.22}$$

In particular, we can relate this generalized vision to the rank and border rank of polynomials.

Definition 1.2.23. For every $d \in \mathbb{N}$ and every vector space V over $\mathbb{K}$, the map

$$\begin{aligned} \nu_d \colon \mathbb{P}V &\longrightarrow \mathbb{P}\big(S^d V\big) \\ [\mathbf{v}] &\longmapsto \big[\mathbf{v}^d\big] \end{aligned}$$

is called the d*-Veronese map* of V.

Now, if we consider the space V as the space of linear polynomials $\mathcal{R}^1_n$, then we can translate the d-Veronese map as

$$\begin{aligned} \nu_d \colon \mathbb{P}\big(\mathcal{R}^1_n\big) &\longrightarrow \mathbb{P}\big(\mathcal{R}^d_n\big) \\ [l] &\longmapsto \big[l^d\big]. \end{aligned}$$

Considering the values defined in formulas (1.2.22) for a polynomial $f \in \mathcal{R}^d_n$, we have that these coincide with Definition 1.2.18 and Definition 1.2.19 in the case of X corresponding to the image of the d-Veronese map. That is

$$\operatorname{rk} f = \operatorname{rk}_{\nu_d(\mathbb{P}(\mathcal{D}^1_n))}(f), \quad \operatorname{brk} f = \operatorname{brk}_{\nu_d(\mathbb{P}(\mathcal{D}^1_n))}(f). \tag{1.2.24}$$

By construction, it is evident that the border rank has a privileged role in algebraic geometry over the classical Waring rank, since the set $\sigma_r(X)$ of formulas (1.2.22) is an algebraic variety and it is indeed called the *r-th secant variety of X*.

Now, by Lemma 1.2.15, we can easily recall a well known lower bound for rank and border rank of a homogeneous polynomial. It is classically attributed to J. J. Sylvester (see [Syl51b]) and also appears many times in the literature (see e.g. [Lan12, Proposition 3.5.1.1]). The proof we provide here is quite simple and can be found, for instance, in [Tei14, pp. 11-12].

Proposition 1.2.25. *Let $d, k \in \mathbb{N}$ be such that $d \geq k$. Then*

$$\operatorname{rk} f \geq \operatorname{brk} f \geq \operatorname{rk}\big(\operatorname{Cat}_f^k\big)$$

for every $f \in \mathcal{R}_n^d$.

Proof. Let us consider the decomposition of size r

$$[f]_r = l_{\mathbf{a}_1}^d + \cdots + l_{\mathbf{a}_r}^d,$$

with $r \in \mathbb{N}$, where $\mathbf{a}_1, \ldots, \mathbf{a}_r \in \mathbb{K}^n$. Then, it follows by Lemma 1.2.15 that for every $j = 1, \ldots, d$, if $\phi \in \mathcal{D}_n^k$, we have

$$\begin{aligned} \operatorname{Cat}_f^k(\phi) = \phi \circ f &= \mathrm{D}_\phi\big(l_{\mathbf{a}_1}^d + \cdots + l_{\mathbf{a}_r}^d\big) \\ &= \mathrm{D}_\phi\big(l_{\mathbf{a}_1}^d\big) + \cdots + \mathrm{D}_\phi\big(l_{\mathbf{a}_r}^d\big) \\ &= \frac{d!}{(d-k)!}\phi(\mathbf{a}_1) l_1^{d-k} + \cdots + \phi(\mathbf{a}_r) l_r^{d-k}, \end{aligned}$$

that is,

$$\operatorname{Im} f \subseteq \big\langle l_1^{d-k}, \ldots, l_r^{d-k} \big\rangle$$

and hence

$$\operatorname{rk}\big(\operatorname{Cat}_f^k\big) \leq r.$$

The second inequality is a direct consequence of the fact that the locus of matrices of rank at most r is closed in Zariski topology for every $r \in \mathbb{N}$. □

The concepts of Waring rank and border rank are not the only notions of tensor rank appearing in the literature. Among the years, several definitions have been provided, generalizing the concept of decompositions, to associate natural values to points of schemes and varieties. An example is given by the following definition.

Definition 1.2.26. Given an algebraic variety $X \subseteq \mathbb{P}^{n-1}$ and a point $[\mathbf{p}] \in \mathbb{P}^{n-1}$, the *$X$-cactus rank* of $\mathbf{p}$ is the natural value

$$\operatorname{crk}_X(\mathbf{p}) = \min\{\, r \in \mathbb{N} \mid \exists \text{ 0-dim. scheme } Z \subseteq X\colon \deg Z = r,\ [\mathbf{p}] \in \langle Z \rangle \,\}.$$

That is, the minimum natural number r such that there exists a scheme Z in X of degree r such that $[\mathbf{p}] \in \langle Z \rangle$.

This particular notion of tensor rank, has been first presented in 1999 by A. Iarrobino and V. Kanev (see [IK99, Definition 5.1]), who called it *scheme length*. The term *cactus rank* has been instead introduced by K. Ranestad and F.-O. Schreyer in [RS11], inspired by the notion of *cactus variety*, defined by W. Buczyńska and J. Buczyński in [BB14]. There are several motivation for its introduction, which have been analyzed in other works, such as [Bal18], [BR13], and [BBG19].

However, for our purposes, another notion of tensor rank becomes relevant. This is the smoothable rank of a tensor, which is quite similar to the cactus rank, but with the difference that the scheme for which we require the existence must be *smoothable*, that is, a limit of smooth subschemes. To be more precise, we recall the classical notion of *Hilbert scheme*, which we denote by $\mathrm{Hilb}_r(\mathbb{P}^n)$. This object, firstly developed by A. Grothendieck (see [Gro61] for a formal definition and further details), basically consists of a scheme parameterizing all the closed 0-dimensional subschemes of $\mathbb{P}^n$ of degree r, which represent the closed points of $\mathrm{Hilb}_r(\mathbb{P}^n)$. Denoting by $\mathrm{Hilb}_r^0(\mathbb{P}^n)$ the open subset of $\mathrm{Hilb}_r(\mathbb{P}^n)$ consisting of the r-tuples of distinct points in $\mathbb{P}^n$, we say that a scheme $Z \subseteq \mathbb{P}^n$ is *smoothable* if it is contained in the closure of $\mathrm{Hilb}_r^0(\mathbb{P}^n)$ (to see more details about smoothability one can consult, for instance, [JKK19]).

Definition 1.2.27. Given an algebraic variety $X \subseteq \mathbb{P}^{n-1}$ and a point $[\mathbf{p}] \in \mathbb{P}^{n-1}$, the *$X$-smoothable rank* of $\mathbf{p}$ is the natural value

$$\mathrm{smrk}_X(\mathbf{p}) = \min\{\, r \in \mathbb{N} \mid \exists \text{ 0-dim. smoothable scheme } Z \subseteq X\colon \deg Z = r,\ [\mathbf{p}] \in \langle Z\rangle \,\}.$$

That is, the minimum natural number r such that there exists a smoothable scheme Z of length r such that $[\mathbf{p}] \in \langle Z\rangle$.

Clearly, it follows by definition that, for every algebraic variety $X \subseteq \mathbb{P}^{n-1}$ and $[\mathbf{p}] \in X$, then

$$\mathrm{crk}_X\, \mathbf{p} \le \mathrm{smrk}_X\, \mathbf{p}.$$

To compare these with many other notions of symmetric tensor ranks, we refer to [BBM14]. Analogously to formulas (1.2.24), we can define the *cactus rank* and the *smoothable rank* of a homogeneous polynomial $f \in \mathcal{R}_n^d$, respectively as

$$\mathrm{crk}\, f = \mathrm{crk}_{\nu_d(\mathbb{P}(\mathcal{R}_n^1))}\, f, \quad \mathrm{smrk}\, f = \mathrm{smrk}_{\nu_d(\mathbb{P}(\mathcal{R}_n^1))}\, f.$$

It is possible to make a comparison also with the border rank of a polynomial. Indeed, as observed by A. Bernardi, J. Brachat and B. Mourrain in [BBM14, Remark 2.7], using [IK99, Lemma 5.17], we have also

$$\mathrm{brk}\, f \le \mathrm{smrk}\, f. \tag{1.2.28}$$

We recall that, for a 0-dimensional scheme Z, its degree is defined as the maximum value assumed by the Hilbert function of the corresponding ideal I, given by

$$\begin{aligned} \mathrm{HF}_I\colon \mathbb{Z} &\longrightarrow \mathbb{Z} \\ k &\longmapsto \dim(\mathcal{D}_k / I_k). \end{aligned}$$

As one could expect, there are many cases in which equality in formula (1.2.28) does not hold. Several examples of polynomials having border rank strictly smaller than smoothable rank are provided by W. Buczyńska and J. Buczyński in [BB15]. Probably the most useful result we need is the version of Lemma 1.2.17 for 0-dimensional schemes. There many ways to enunciate it, but we use the one appearing in [BJMR18, Lemma 1].

Lemma 1.2.29 (Apolarity lemma)**.** *Let $f \in \mathcal{R}_d$ and $Z \subseteq \mathbb{P}^n$ a 0-dimensional scheme and let. Let*

$$\nu_d\colon \mathbb{P}(\mathcal{R}_n^1) \to \mathbb{P}(\mathcal{R}_n^d)$$

be the d-Veronese map. The following conditions are equivalent:
(1) $[f] \in \langle \nu_d(Z)\rangle$;
(2) $I_Z \subseteq f^{\perp}$.

Proof. If $I_Z \subseteq f^\perp$, then we have

$$(I_Z)_d \subseteq f_d^\perp$$

for every $d \in \mathbb{N}$ and hence $[f] \in V\big((I_Z)_d\big) = \langle \nu_d(Z) \rangle$. Conversely, let $[f] \in \langle \nu_d(Z) \rangle$. We have that $(I_Z)_k \subseteq f_k^\perp$ for every $k > d$. If $\varphi \in (I_Z)_k$ for some $k \leq d$, then $\varphi \mathcal{D}_{d-k} \subseteq (I_Z)_d$ and hence, since $[f] \in \langle \nu_d(Z) \rangle$, we have $\varphi \mathcal{D}_{d-k} \subseteq (f^\perp)_d$. That is,

$$\varphi h \circ f = 0$$

for every $h \in \mathcal{D}_n^{d-k}$ and hence we have

$$\varphi \circ f = 0,$$

which proves the second part of the statement. □

Lemma 1.2.29 is particularly useful in dealing with smoothable rank, since the condition of belonging to the space $\langle \nu_d(Z) \rangle$ can be obtained by analyzing the apolar ideal of the considered homogeneous form. We will see a direct application in chapter 5.

CHAPTER 2

Apolarity on powers of quadratic forms

In this chapter, we begin our analysis on the powers of the quadratic forms. As already said previously, our aim is to determine suitable decompositions, trying to establish which are the minimal ones. These elegant polynomials appear many times also in the classical literature. Several authors have provided various decompositions for different exponents and numbers of variables (see e.g. [Dic19, Chapter XXV, pp. 717-724]). A quite recent work is due to B. Reznick, who in [Rez92, Chapters 8-9] analyzes in details many cases, restricting anyway to the case of real decompositions, also called in that context *representations*. From this point on we will consider only vector spaces over the field $\mathbb{C}$ and we will use the same notations of chapter 1 for the polynomial rings. That is,

$$\mathcal{R}_n = \mathbb{C}[x_1, \dots, x_n] \simeq S(\mathbb{C}^n), \quad \mathcal{D}_n = \mathbb{C}[y_1, \dots, y_n] \simeq S(\mathbb{C}^{n*}),$$

for every $n \in \mathbb{N}$, and

$$\mathcal{R}_n^d = \mathbb{C}[x_1, \dots, x_n]_d \simeq S^d\mathbb{C}^n, \quad \mathcal{D}_n^d = \mathbb{C}[y_1, \dots, y_n]_d \simeq S^d\mathbb{C}^{n*},$$

for every $d \in \mathbb{N}$. Up to linear transformations, we can consider every quadratic form equal to the form

$$q_n = x_1^2 + \cdots + x_n^2$$

for some $n \in \mathbb{N}$, which corresponds to the value of its rank. Therefore, if the quadratic form is in $\mathcal{R}_m^2$ for some $m \in \mathbb{N}$, but it has rank equal to $n < m$, we will consider it as a form in $\mathcal{R}_n^2$. An important strategy to obtain decompositions is given by the use of Lemma 1.2.17. By this reason, our first result regards the determination of the apolar ideal of q_n^s for every $s \in \mathbb{N}$.

We start in section 2.1 by an analysis on the space of harmonic polynomials, which is an irreducible representation of the special orthogonal group $\mathrm{SO}_n(\mathbb{C})$. This last fact is a quite classical result (see [GW98, Theorem 5.2.4]). Several facts that are related to quadratic forms involve harmonic polynomials. One of the most interesting ones is that any form can be uniquely decomposed as a sum of harmonic polynomials multiplied by powers of q_n.

Then we proceed in section 2.2 with the analysis on the apolarity action on q_n^s. After a detailed examination on the catalecticant matrices of q_n^s, which result to be full rank (a fact already known from [Rez92]), we move to the analysis of its apolar ideal. Its determination is in fact a quite important result in terms of apolarity, since it provides us with another strategy to attack the problem of decomposing the form q_n^s. The resulting structure is quite elegant: the apolar ideal of the s-th power of a quadratic form of rank n is exactly the ideal generated by harmonic polynomials of degree $s + 1$.

2.1 Harmonic polynomials

To carry on our analysis of the apolar ideal of the form q_n^s, we need to make some consideration on the spaces of harmonic polynomials. These objects appear many times also in other branches of mathematics. Clearly, their presence can be found in analysis, especially in the theory of harmonic functions, for which we refer to [ABR01, Chapter 5] to get further details from an analytic point of view.

2.1.1 Apolarity and Laplace operator

It can be useful to identify the space $\mathcal{D}_n$ with the space of polynomial differential operators, as we have already seen in section 1.2. We first observe that the kernel of a differential operator associated to a homogeneous polynomial $\phi \in \mathcal{D}_n$ is connected with the contraction pairing. Indeed, it can be viewed as the orthogonal complement of the space $\phi\mathcal{D}_n^{d-k}$.

Proposition 2.1.1. *Let $k \leq d$ and let $\phi \in \mathcal{D}_n^k$. Then*

$$\operatorname{Ker}(\mathrm{D}_\phi) = \left(\phi\mathcal{D}_n^{d-k}\right)^\perp \subseteq \mathcal{R}_n^d.$$

Proof. Let $\circ \colon \mathcal{D}_n^d \times \mathcal{R}_n^d \to \mathbb{C}$ be the contraction pairing of degree $d \in \mathbb{N}$. Given any $g \in \mathcal{R}_n^d$, we have by the argument of Remark 1.2.8 that $g \in \left(\phi\mathcal{D}_n^{d-k}\right)^\perp$ if and only if, for every $\psi \in \mathcal{D}_n^{d-k}$,

$$g \circ (\phi\psi) = (\phi\psi) \circ g = 0.$$

That is,

$$\psi \circ (\phi \circ g) = 0,$$

but this means that

$$\mathrm{D}_\phi(g) = \phi \circ g = 0$$

and hence $g \in \operatorname{Ker}(\mathrm{D}_\phi)$. □

Proposition 2.1.1 thus allows us to introduce the space of harmonic polynomials by the contraction pairing. If there is no risk of confusion, we continue to interchange the roles of the variables x_i and y_i passing from one space to its dual, without specifying.

Definition 2.1.2. The space

$$\mathcal{H}_n^d = \left(q_n\mathcal{D}_n^{d-2}\right)^\perp = \operatorname{Ker}(\mathrm{D}_{q_n}) \subseteq \mathcal{R}_n^d$$

is called the space of the *d-harmonic polynomials* or *d-harmonic forms*. The differential operator D_{q_n}, denoted also by Δ, is called the *Laplace operator* and corresponds to the operator $\Delta \colon \mathcal{R}_n^d \to \mathcal{R}_n^{d-2}$, defined by

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

Remark 2.1.3. We can make some considerations on the behavior of the Laplace operator on polynomials. First, given any two forms $g_1, g_2 \in \mathcal{R}_n$, it is straightforward to verify, simply using Leibniz's rule for derivations, that

$$\Delta(g_1g_2) = \Delta(g_1)g_2 + g_1\Delta(g_2) + 2\sum_{j=1}^n \frac{\partial g_1}{\partial x_j}\frac{\partial g_2}{\partial x_j}. \tag{2.1.4}$$

In particular, we need to recall the Euler's formula for a polynomial $f \in \mathcal{R}_n^k$, that is

$$\sum_{j=1}^n x_j \frac{\partial f}{\partial x_j} = kf. \tag{2.1.5}$$

Furthermore, we can observe that

$$
\begin{aligned}
\Delta\big(q_n^s\big) &= \sum_{j=1}^n \frac{\partial^2 q_n^s}{\partial x_j^2} = 2s\sum_{j=1}^n \frac{\partial}{\partial x_j}\big(x_j q_n^{s-1}\big) = 2s\sum_{j=1}^n\left(q_n^{s-1} + x_j\frac{\partial q_n^{s-1}}{\partial x_j}\right)\\
&= 2snq_n^{s-1} + 4s(s-1)\sum_{j=1}^n x_j^2 q_n^{s-2} = 2s\big(n+2(s-1)\big)q_n^{s-1}
\end{aligned}
\tag{2.1.6}
$$

for every $s \geq 2$. Therefore, iterating the process, we get

$$
\Delta^k\big(q_n^s\big) = 2^k\frac{s!}{(s-k)!}\left(\prod_{j=1}^k\big(n+2(s-j)\big)\right)q_n^{s-k} \tag{2.1.7}
$$

for every $k \leq s$. In particular, we can introduce a constant value depending on $n, s \in \mathbb{N}$, given by

$$
C_{n,s} = \Delta^s\big(q_n^s\big) = 2^s s!\prod_{j=0}^{s-1}(n+2j) \neq 0. \tag{2.1.8}
$$

Introducing the notations

$$
q_n^{[k]} = \frac{1}{2^k k!}q_n^k, \quad A_{n,s,k} = \prod_{j=1}^k\big(n+2(s-j)\big)
$$

for every $k \in \mathbb{N}$, formula (2.1.7) assumes a more concise form, since we can write

$$
q_n^k \circ q_n^{[s]} = A_{n,s,k}q_n^{[s-k]}. \tag{2.1.9}
$$

In addition to the apolarity action between powers of the quadratic forms, we can observe another nice property. It regards the apolarity action of powers of q_n on the products of harmonic polynomials by powers of the quadratic form, generalizing formula (2.1.9).

Lemma 2.1.10. *Given a harmonic polynomial $h_m \in \mathcal{H}_n^m$,*

$$
q_n^k \circ \big(q_n^{[d]}h_m\big) = A_{n,d+m,k}q_n^{[d-k]}h_m.
$$

Proof. We begin by the case $k=1$. Now, using formula (2.1.4) and formula (2.1.5), we get

$$
\begin{aligned}
\Delta\big(q_n^{[d]}h_m\big) &= \Delta\big(q_n^d\big)h_m + q_n^{[d]}\Delta(h_m) + 2\sum_{j=1}^n\frac{\partial q_n^{[d]}}{\partial x_j}\frac{\partial h_m}{\partial x_j}\\
&= \Delta\big(q_n^{[d]}\big)h_m + 2q_n^{[d-1]}\sum_{j=1}^n x_j\frac{\partial h_m}{\partial x_j}\\
&= \big(n+2(d-1)\big)q_n^{[d-1]}h_m + 2mq_n^{[d-1]}h_m\\
&= \big(n+2(d+m-1)\big)q_n^{[d-1]}h_m\\
&= A_{n,d+m,1}q_n^{[d-1]}h_m.
\end{aligned}
$$

Thus, iterating the process, we get

$$
\Delta^k\big(q_n^{[d]}h_{s-2d}\big) = A_{n,d+m,k}q_n^{[d-k]}h_{s-2d}.
$$

□

2.1.2 Decompositions and harmonic components

The role of harmonic polynomials is crucial in determining a decomposition of the whole space $\mathcal{R}_n^d$. It is well known indeed, that the space $\mathcal{H}_n^d$ is an irreducible $\mathrm{SO}_n(\mathbb{C})$-module for every $d \in \mathbb{N}$ (see [GW98, Theorem 5.2.4]). Next proposition shows a decomposition of $\mathcal{R}_n^d$ as direct sum of irreducible representations. It is already presented in [GW98, Corollary 5.2.5] and also in [ABR01, Proposition 5.5], but we provide here another proof, for which we use apolarity.

Proposition 2.1.11. *Let $d \in \mathbb{N}$. Then*

$$\mathcal{R}_n^d = q_n \mathcal{R}_n^{d-2} \oplus \mathcal{H}_n^d$$

and, more precisely,

$$\mathcal{R}_n^d = \bigoplus_{j=0}^{\lfloor \frac{d}{2} \rfloor} q_n^j \mathcal{H}_n^{d-2j}. \tag{2.1.12}$$

Proof. Since by duality $q_n \mathcal{R}_n^{d-2} \simeq q_n \mathcal{D}_n^{d-2}$, the aimed decomposition can be obtained simply by proving that

$$q_n \mathcal{R}_n^{d-2} \cap \mathcal{H}_n^d = \{0\}.$$

For every $g \in \mathcal{R}_n^{d-2} \setminus \{0\}$, let $k \in \mathbb{N}$ be the maximum natural number such that $q_n g = q_n^k g_0$ for some $g_0 \in \mathcal{R}_n^{d-2k}$. Then we have that $q_n \nmid g_0$ and, using formulas (2.1.5) and (2.1.7), we get

$$\begin{aligned}
\Delta\big(q_n^k g_0\big) &= \Delta\big(q_n^k\big) g_0 + q_n^k \Delta(h_0) + 2 \sum_{j=1}^n \frac{\partial q_n^k}{\partial x_j} \frac{\partial h_0}{\partial x_j} \\
&= 2k\big(n + 2(k-1)\big) q_n^{k-1} g_0 + q_n^k \Delta(g_0) + 4k q_n^{k-1} \sum_{j=1}^n x_j \frac{\partial g_0}{\partial x_j} \\
&= 2k\big(n + 2(k-1)\big) q_n^{k-1} g_0 + 4k(d-2k) q_n^{k-1} g_0 + q_n^k \Delta(g_0) \\
&= 2k\big(n + 2(d-k-1)\big) q_n^{k-1} g_0 + q_n^k \Delta(g_0) \sum_{j=1}^n \frac{\partial q_n^k}{\partial x_j} \frac{\partial h_0}{\partial x_j}.
\end{aligned}$$

So, if $\Delta\big(q_n^k g_0\big) = 0$, then we have

$$q_n^k \Delta(g_0) = -2k\big(n + 2(d-k-1)\big) q_n^{k-1} g_0$$

and, since $g_0 \neq 0$, it implies that $q_n^k \mid q_n^{k-1} g_0$. Thus we must have $q_n \mid g_0$, but this is absurd by the hypothesis on g_0. We have therefore proved that

$$\mathcal{R}_n^d = q_n \mathcal{R}_n^{d-2} \oplus \mathcal{H}_n^d.$$

Proceeding by induction on d, we easily get the equality

$$\mathcal{R}_n^d = \bigoplus_{j=0}^{\lfloor \frac{d}{2} \rfloor} q_n^j \mathcal{H}_n^{d-2j}. \qquad \square$$

By Proposition 2.1.11, we can determine the dimension of each component of the vector space of harmonic polynomials.

Corollary 2.1.13. *For every $d, n \in \mathbb{N}$*

$$\dim \mathcal{H}_n^d = \dim \mathcal{R}_n^d - \dim \mathcal{R}_n^{d-2} = \binom{d+n-1}{n-1} - \binom{d+n-3}{n-1}.$$

We can also establish a recursive value of the dimension of each component.

Lemma 2.1.14. *For every $d, n \geq 1$,*

$$\dim \mathcal{H}_n^d = \dim \mathcal{R}_{n-1}^{d-1} + \dim \mathcal{R}_{n-1}^d = \dim \mathcal{H}_n^{d-1} + \dim \mathcal{H}_{n-1}^d.$$

Proof. By Corollary 2.1.13, we have

$$\begin{aligned}
\dim \mathcal{H}_n^d &= \binom{d+n-1}{n-1} - \binom{d+n-3}{n-1} \\
&= \frac{(d+n-1)! - d(d-1)(d+n-3)!}{d!(n-1)!} \\
&= \big((d+n-1)(d+n-2) - d(d-1)\big)\frac{(d+n-3)!}{d!(n-1)!} \\
&= \big(2d(n-1) + (n-1)(n-2)\big)\frac{(d+n-3)!}{d!(n-1)!} \\
&= \big(2d+n-2\big)\frac{(d+n-3)!}{d!(n-2)!} \\
&= \binom{(d-1)+(n-1)}{n-1} + \binom{d+n-1}{n-1} = \dim \mathcal{R}_{n-1}^{d-1} + \dim \mathcal{R}_{n-1}^d.
\end{aligned}$$

Furthermore, by this last equality, we get

$$\begin{aligned}
\dim \mathcal{H}_n^d &= \dim \mathcal{R}_{n-1}^{d-1} + \dim \mathcal{R}_{n-1}^d \\
&= \dim \mathcal{R}_{n-1}^{d-1} + \dim \mathcal{R}_{n-1}^{d-2} + \dim \mathcal{H}_{n-1}^d \\
&= \dim \mathcal{H}_n^{d-1} + \dim \mathcal{H}_{n-1}^d.
\end{aligned}$$ □

In terms of the contraction pairing defined in formula (1.2.1), we observe that formula (2.1.12) provides an orthogonal decomposition of the spaces of homogeneous polynomials in each degree. In the following lemma we see how the contraction pairing works between each component of decomposition (2.1.12).

Lemma 2.1.15. *For every $j, k \leq d \in \mathbb{N}$ such that $j \neq k$,*

$$q_n^j h_{d-2j} \circ q_n^k h_{d-2k} = 0$$

for every $h_{d-2j} \in \mathcal{H}_n^{d-2j}$ and $h_{d-2k} \in \mathcal{H}_n^{d-2k}$.

Proof. Recalling Remark 1.2.8, we can suppose by symmetry that $j \geq k$. Thus, we get immediately by Lemma 2.1.10 that

$$q_n^j h_{d-2j} \circ q_n^k h_{d-2k} = 0.$$ □

2.2 The apolar ideal of q_n^s

Now we prove that the apolar ideal of q_n^s is exactly the ideal generated by harmonic polynomials of degree $s+1$. Again, as we have already done above, we will pass from a set of coordinate to its dual without specifying. We start by observing the behavior of the apolarity action of an arbitrary polynomial on q_n^s. By considering monomials of degree 1 first, we get for every $j = 1, \dots, n$

$$y_j \circ q_n^s = \frac{\partial q_n^s}{\partial x_j} = 2s q_n^{s-1} x_j .$$

Therefore, by Leibniz's rule, we can extend it to any polynomial $g \in \mathcal{D}_n^k$, with $k \leq s$, and we get the equation

$$g \circ q_n^s = 2^k \frac{s!}{(s-k)!} q_n^{s-k} g + q_n^{s-k+1} h, \tag{2.2.1}$$

for a suitable $h \in \mathcal{R}_n^{k-2}$. In this case we use the same notation for the polynomial obtained by replacing y_j by x_j for every $j = 1, \dots, n$ in the polynomial g.

2.2.1 Catalecticant matrices of q_n^s

We have already seen that $\mathcal{R}_n$ and $\mathcal{D}_n$ own a natural structure of $\mathrm{GL}_n(\mathbb{C})$-modules. Next result is related to the catalecticant map and it is crucial for our purposes. For every polynomial $g \in \mathcal{R}_n$, we denote by $\big(\mathrm{GL}_n(\mathbb{C})\big)_g$ the stabilizer of g, i.e. the subgroup of $\mathrm{GL}_n(\mathbb{C})$ given by

$$\big(\mathrm{GL}_n(\mathbb{C})\big)_g = \{ A \in \mathrm{GL}_n(\mathbb{C}) \mid A \cdot g = g \} .$$

The stabilizer of $\mathrm{GL}_n(\mathbb{C})$ with respect to the form q_n^s can be described explicitly. In particular, it contains the group of special orthogonal matrices $\mathrm{SO}_n(\mathbb{C})$.

Lemma 2.2.2. *For every $n, s \in \mathbb{N}$ the stabilizer of the action of $\mathrm{GL}_n(\mathbb{C})$ with respect to the form q_n^s is the group*

$$G_{q_n^s} = \mathbb{Z}_s \times \mathrm{O}_n(\mathbb{C}).$$

Proof. Given any $A \in \mathrm{GL}_n(\mathbb{C})$, we have by Definition 1.2.10 that

$$A \cdot q_n^s = A \cdot \left(x_1^2 + \dots + x_n^2\right)^s = \left((A \cdot x_1)^2 + \dots + (A \cdot x_n)^2\right)^s = (A \cdot q_n)^s$$

and hence

$$A \cdot q_n^s = (A \cdot q_n)^s = q_n^s,$$

that is,

$$\left((A \cdot x_1)^2 + \dots + (A \cdot x_n)^2\right)^s = \left(x_1^2 + \dots + x_n^2\right)^s .$$

This means, considering the $(2s)$-th roots of unity, that

$$A \cdot q_n = \mathrm{e}^{\frac{2(j-1)\pi \mathrm{i}}{s}} q_n$$

for some $j \in \mathbb{N}$ such that $1 \leq j \leq s$ and this implies that

$$\mathrm{e}^{-\frac{2(j-1)\pi \mathrm{i}}{s}} A \cdot q_n = q_n .$$

In particular, since the stabilizer of the form q_n corresponds to the orthogonal group $\mathrm{O}_n(\mathbb{C})$, we have

$$A = \mathrm{e}^{\frac{2(j-1)\pi \mathrm{i}}{s}} B$$

for some $B \in \mathrm{O}_n(\mathbb{C})$, which proves the statement. □

We need to notice another basic fact in representation theory, i.e. that, in general, the catalecticant map of a homogeneous polynomial g represents a map of $\mathrm{GL}_n(\mathbb{C})_g$-modules.

Proposition 2.2.3. *Let $g \in \mathcal{R}_n^d$ and let $G = \mathrm{GL}_n(\mathbb{C})$. Then the catalecticant map of g is G_g-equivariant, that is,*

$$\mathrm{Cat}_g(A \cdot f) = A \cdot \mathrm{Cat}_g(f)$$

for every $A \in G_g$.

Proof. By Proposition 1.2.12, we know that the apolarity action of $\mathcal{D}_n$ on $\mathcal{R}_n$ is G-equivariant, that is, given any $\phi \in \mathcal{D}_n$ and $f \in \mathcal{R}_n$,

$$B \cdot (\phi \circ f) = (B \cdot \phi) \circ (B \cdot f)$$

for every $B \in G$. Thus, given any $A \in G_g$, we have

$$\mathrm{Cat}_g(A \cdot \phi) = (A \cdot \phi) \circ g = (A \cdot \phi) \circ (A \cdot g) = A \cdot (\phi \circ g) = A \cdot \mathrm{Cat}_g(\phi)$$

for every $\phi \in \mathcal{D}_n$. □

Before proving that the set of $(s+1)$-harmonic polynomials represents a set of generators of the apolar ideal, we need other considerations. Let us define the polynomials $u, v \in \mathcal{D}_1$, as

$$u = y_1 + \mathrm{i}y_2, \quad v = y_1 - \mathrm{i}y_2. \tag{2.2.4}$$

Lemma 2.2.5. *If $l_{\mathbf{a}} \in \mathcal{D}_n^1$ is a linear form associated to an isotropic point $\mathbf{a} \in \mathbb{C}^n$, namely such that $\mathbf{a} \cdot \mathbf{a} = 0$, then the d-th power $l_{\mathbf{a}}^d$ is harmonic for every $d \in \mathbb{N}$.*

Proof. If $d = 1$ the statement is clear. If $d \geq 2$, we get by Lemma 1.2.15 that

$$\Delta l_{\mathbf{a}}^{[d]} = q_n \circ l_{\mathbf{a}}^{[d]} = (\mathbf{a} \cdot \mathbf{a}) l_{\mathbf{a}}^{[d-2]} = 0.$$ □

As a direct consequence, we have that every power of the polynomials u and v is harmonic.

Lemma 2.2.6. *For every $d \in \mathbb{N}$ such that $d \geq s+1$, $u^d, v^d \in \left(q_n^s\right)^\perp$.*

Proof. We prove the statement only for the linear polynomial u, since the case of v is analogous. We observe that

$$(y_1 + \mathrm{i}y_2) \circ (x_1 + \mathrm{i}x_2) = 1 + \mathrm{i}^2 = 0$$

and

$$u \circ q_n^s = (y_1 + \mathrm{i}y_2) \circ \left(x_1^2 + \cdots + x_n^2\right)^s = 2s(x_1 + \mathrm{i}x_2)\left(x_1^2 + \cdots + x_n^2\right)^{s-1}.$$

Then, if $d \geq s+1$, we obtain by Leibniz's rule

$$u^d \circ q_n^s = (y_1 + \mathrm{i}y_2)^d \circ \left(x_1^2 + \cdots + x_n^2\right)^s = 2^s s!(y_1 + \mathrm{i}y_2)^{d-s} \circ (x_1 + \mathrm{i}x_2)^s = 0,$$

that is, $u^d \in \left(q_n^s\right)^\perp$. □

Remark 2.2.7. It is clear that the product of two powers of the polynomials u and v cannot be harmonic. Indeed, given any $l, m \in \mathbb{N}$ such that $l, m \geq 1$,we have

$$u^l v^m = \left(y_1^2 + y_2^2\right)(y_1 + \mathrm{i}y_2)^{l-1}(y_1 - \mathrm{i}y_2)^{m-1},$$

namely $\left(y_1^2 + y_2^2\right) \mid u^l v^m$.

We can finally analyze each component of the catalecticant map, in order to determine its kernel, which is the apolar ideal of q_n^s. We have seen in Corollary 2.2.11 that the catalecticant matrices of q_n^s are full rank for every $n, s \in \mathbb{N}$. Moreover, we can represent them in their diagonal form by the use of harmonic polynomials. Given any $k \in \mathbb{N}$ such that $k \leq 2s$, let us consider the k-th catalecticant map

$$\mathrm{Cat}^k_{q_n^s} : \mathcal{D}_n^k \to \mathcal{R}_n^{2s-k}.$$

Proposition 2.2.8. *Let $k \in \mathbb{N}$ such that $k \leq 2s$ and $j \leq \lfloor \frac{k}{2} \rfloor$. Then*

$$\mathrm{Cat}^k_{q_n^s}\left(q_n^j h_{k-2j}\right) \in q_n^{s-k+j} \mathcal{H}_n^{k-2j}$$

for every $h_{k-2j} \in \mathcal{H}_n^{k-2j}$. In particular, the restriction

$$\mathrm{Cat}^k_{q_n^s} : q_n^j \mathcal{H}_n^{k-2j} \to q_n^{s-k+j} \mathcal{H}_n^{k-2j}$$

is a well defined isomorphism of $\mathrm{SO}_n(\mathbb{C})$*-modules.*

Proof. We have already recalled that the space of harmonic polynomials $\mathcal{H}_n^s$ is an irreducible $\mathrm{SO}_n(\mathbb{C})$-module ([GW98, Theorem 5.2.4]) and so is $q_n^j \mathcal{H}_n^{k-2j}$. Now, let us consider the linear polynomials u and v defined in (2.2.4) and the restriction

$$\mathrm{Cat}^k_{q_n^s}\Big|_{q_n^j \mathcal{H}_n^{k-2j}} : q_n^j \mathcal{H}_n^{k-2j} \to \mathcal{R}_n^{2s-k}.$$

Then we have, by formula (2.1.7), that

$$\mathrm{Cat}^k_{q_n^s}\left(q_n^j u_1^{k-2j}\right) = \left(q_n^j u_1^{k-2j}\right) \circ q_n^s = u_1^{k-2j} \circ \left(2^k \frac{s!}{(s-k)!} \left(\prod_{l=s-k}^{s-1} (n+2l)\right) q_n^{s-j}\right).$$

We also see that

$$u_1^{k-2j} \circ q_n^{s-j} = (y_1 + \mathrm{i} y_2)^{k-2j} \circ q_n^{s-j} = (y_1 + \mathrm{i} y_2)^{k-2j-1} \circ \left(2 q_n^{s-j-1} (x_1 + \mathrm{i} x_2)\right)$$

and, since

$$(y_1 + \mathrm{i} y_2) \circ (x_1 + \mathrm{i} x_2) = 0,$$

we can iterate the process obtaining

$$u_1^{k-2j} \circ q_n^{s-j} = 2^{k-2j} q_n^{s-k+j} (x_1 + \mathrm{i} x_2)^{k-2j} \in q_n^{s-k+j} \mathcal{H}_n^{k-2j}.$$

In particular, since $q_n^{s-k+j} \mathcal{H}_n^{k-2j}$ is an irreducible $\mathrm{SO}_n(\mathbb{C})$-module, we must have by Lemma 1.1.22 that

$$q_n^{s-k+j} \mathcal{H}_n^{k-2j} \subseteq \mathrm{Im}\left(\mathrm{Cat}^k_{q_n^s}\right).$$

Therefore, by dimensional reasons, it follows immediately that

$$\mathrm{Cat}^k_{q_n^s}\Big|_{q_n^j \mathcal{H}_n^{k-2j}} : q_n^j \mathcal{H}_n^{k-2j} \to q_n^{s-k+j} \mathcal{H}_n^{k-2j}$$

is an isomorphism of $\mathrm{SO}_n(\mathbb{C})$-modules. □

In a certain sense, the catalecticant maps of q_n^s preserve the decomposition (2.1.12). Indeed, considering the classical coordinates, for every harmonic polynomial $h \in \mathcal{H}_n^k$, we know by formula (2.2.1) that

$$h \circ q_n^s = 2^k \frac{s!}{(s-k)!} q_n^{s-k} h + q_n^{s-k+1} g$$

for some $g \in \mathcal{R}_n^d$. However, since the harmonic decomposition is unique, we have by Proposition 2.2.8 that

$$h \circ q_n^s = 2^k \frac{s!}{(s-k)!} q_n^{s-k} h.$$

In particular, setting

$$q_n^{[s]} = \frac{1}{2^s s!} q_n^s$$

for every $s \in \mathbb{N}$, we obtain

$$h \circ q_n^{[s]} = q_n^{[s-k]} h. \tag{2.2.9}$$

for every $h \in \mathcal{H}_n^k$. So, we can determine a particular basis, for which every catalecticant matrix presents a diagonal form.

Proposition 2.2.10. *Let $\mathcal{B}_{n,d}$ be a basis of $\mathcal{H}_n^d$ for every $n, d \in \mathbb{N}$. Let $A_{n,s,k}$ denote the value given by*

$$A_{n,s,k} = \prod_{j=0}^{k-1} \big(n + 2(s-k+j)\big) = \prod_{j=0}^{k-1} \big(n + 2(s-1-j)\big) = \prod_{j=1}^{k} \big(n + 2(s-j)\big).$$

Let also

$$\mathcal{T}_d = \bigcup_{k=1}^{\lfloor \frac{d}{2} \rfloor} \left\{ \frac{1}{A_{n,s,k}} q_n^k h_{d-2k} \;\middle|\; h_{d-2k} \in \mathcal{B}_{n,k}^{d-2k} \right\}$$

and

$$\mathcal{S}_d = \bigcup_{k=1}^{\lfloor \frac{d}{2} \rfloor} \left\{ q_n^{[k]} h_{d-2k} \;\middle|\; h_{d-2k} \in \left(\mathcal{B}_{n,k}^{d-2k}\right)^* \right\}$$

be basis respectively of $\mathcal{D}_d$ and $\mathcal{R}_d$ for every $d = 1, \dots, 2s$. Then, the entries of the d-th catalecticant matrix $\mathrm{Cat}^d_{q_n^{[s]}}$ with respect to bases $\mathcal{T}_d$ and $\mathcal{S}_{2s-d}$ are

$$\left(\mathrm{Cat}^d_{q_n^{[s]}}\right)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the central catalecticant matrix corresponds to the identity matrix.

Proof. It is sufficient to prove that, for every $1 \le d \le 2s$, the d-th catalecticant map sends elements of $\mathcal{T}_d$ in elements of $\mathcal{S}_{2s-d}$. For every $h_{d-2k} \in \mathcal{H}_n^{d-2k}$, we have by formulas (2.1.7) and (2.2.9) that

$$\begin{aligned} \frac{1}{A_{n,s,k}} q_n^k h_{d-2k} \circ q_n^{[s]} &= \left(\prod_{j=0}^{k-1} \frac{1}{n + 2(s-k+j)} \right) q_n^k h_{d-2k} \circ q_n^{[s]} \\ &= h_{d-2k} \circ q_n^{[s-k]} = q_n^{[s-d+k]} h_{d-2k} \end{aligned}$$

and hence the statement is proved. ◻

2.2.2 Generators of the apolar ideal

The new form of catalecticant matrices of q_n^s allows us to determine more easily how the apolar ideal is made. In particular, by Proposition 2.2.10, we get immediately the following well-known result.

Corollary 2.2.11. *For every $k \in \mathbb{N}$ such that $0 \leq k \leq s$,*

$$\operatorname{Ker}\left(\operatorname{Cat}_{q_n^s}^{s-k}\right) = \left(q_n^s\right)^{\perp}_{s-k} = \{0\}.$$

The fact that the catalecticant matrices of q_n^s are all full-rank has already been proved by B. Reznick, using [Rez92, Theorem 8.15] and referring to [Rez92, Theorem 3.7] and [Rez92, Theorem 3.16]. Another kind of proof is provided by F. Gesmundo and J. M. Landsberg in [GL19, Theorem 2.2]. In addition, again by Proposition 2.2.10, we get also the following corollary, concerning the other components of the catalecticant matrices and apolar ideal.

Corollary 2.2.12. *For every $k \in \mathbb{N}$ such that $1 \leq k \leq s$,*

$$\operatorname{Ker}\left(\operatorname{Cat}_{q_n^s}^{s+k}\right) = \left(q_n^s\right)^{\perp}_{s+k} = \bigoplus_{j=0}^{k-1} q_n^j \mathcal{H}_n^{s+k-2j}. \tag{2.2.13}$$

Now we have all the elements to present our first result, consisting in the determination of the apolar ideal $\left(q_n^s\right)^{\perp}$, which turns out to be generated by harmonic polynomials of degree $s+1$. The proof is based on the determination of a set of generators for each component of the apolar ideal.

Theorem 2.2.14. *The apolar ideal of the form q_n^s is*

$$\left(q_n^s\right)^{\perp} = \left(\mathcal{H}_n^{s+1}\right).$$

Proof. By Corollary 2.2.11 we have seen that $\left(q_n^s\right)^{\perp}_k = 0$ for every $1 \leq k \leq s$. Therefore, to get the statement we simply have to prove that

$$\left(q_n^s\right)^{\perp}_d = \mathcal{H}_n^{s+1} \mathcal{D}_n^{d-s-1}$$

for every $d \geq s+1$. So, let $d = s+k$ for a suitable $k \in \mathbb{N}$. Then by Corollary 2.2.12, if $1 \leq k \leq s+1$, this is the same as proving that

$$\bigoplus_{j=0}^{k-1} q_n^j \mathcal{H}_n^{s+k-2j} = \mathcal{H}_n^{s+1} \mathcal{D}_n^{k-1}. \tag{2.2.15}$$

Corollary 2.2.12 also shows us that

$$\left(q_n^s\right)^{\perp}_{s+1} = \mathcal{H}_n^{s+1},$$

and hence we get immediately the inclusion

$$\bigoplus_{j=0}^{k-1} q_n^j \mathcal{H}_n^{s+k-2j} \supseteq \mathcal{H}_n^{s+1} \cdot \mathcal{D}_n^{k-1}. \tag{2.2.16}$$

The reverse inclusion can be obtained by induction on k separately on odd and even values. If $k = 1$ the equality is clear and we have already seen it. If $k = 2$ we have to show that

$$\mathcal{H}_n^{s+2} \oplus q_n \mathcal{H}_n^s \subseteq \mathcal{H}_n^{s+1} \mathcal{D}_n^1.$$

So, if we consider the polynomials

$$u = y_1 + \mathrm{i} y_2, \quad v = y_1 - \mathrm{i} y_2,$$

then, as we have seen in Lemma 2.2.6, we can consider the polynomial $u^{s+2} \in \mathcal{H}_n^{s+2}$, which can be written as

$$u^{s+1}u \in \mathcal{H}_n^{s+1}\mathcal{D}_n^1.$$

Thus, since $\mathcal{H}_n^{s+2}$ is an irreducible $\mathrm{SO}_n(\mathbb{C})$-module, it follows that

$$\mathcal{H}_n^{s+2} \subseteq \mathcal{H}_n^{s+1}\mathcal{D}_n^1.$$

Now by Remark 2.2.7, we have also that $u^{s+1}v \notin \mathcal{H}_n^{s+2}$ and, in particular, we get by formula (2.2.16) that

$$u^{s+1}v \in \mathcal{H}_n^{s+1}\mathcal{D}_n^1 \subseteq \mathcal{H}_n^{s+2} \oplus q_n\mathcal{H}_n^s.$$

This means that there are unique forms $h_1 \in \mathcal{H}_n^{s+2}$ and $h_2 \in \mathcal{H}_n^s$, with $h_2 \neq 0$ such that

$$u^{s+1}v = h_1 + q_nh_2 \in \mathcal{H}_n^{s+2} \oplus q_n\mathcal{H}_n^s$$

and, in particular,

$$q_nh_2 = u^{s+1}v - h_1 \in \mathcal{H}_n^{s+1}\mathcal{D}_n^1.$$

This implies by irreducibility that

$$q_n\mathcal{H}_n^s \subseteq \mathcal{H}_n^{s+1}\mathcal{D}_n^1,$$

providing the required inclusion. Now, let $3 \leq k \leq s$ and let the equality be true for $k-2$. We have to show that

$$\bigoplus_{j=0}^{k-1} q_n^j\mathcal{H}_n^{s+k-2j} = \mathcal{H}_n^{s+k} \oplus q_n\left(\bigoplus_{j=0}^{k-2} q_n^{j-1}\mathcal{H}_n^{s+k-2-2j}\right) \subseteq \mathcal{H}_n^{s+1}\mathcal{D}_{k-1}.$$

As above, we can consider the polynomial $u^{s+k} \in \mathcal{H}_{s+k}$, which can be written as

$$u^{s+1}u^{k-1} \in \mathcal{H}_n^{s+1}\mathcal{D}_n^{k-1}$$

and conclude, again by irreducibility, that

$$\mathcal{H}_n^{s+k} \subseteq \mathcal{H}_n^{s+1}\mathcal{D}_n^{k-1}.$$

So, since by inductive hypothesis we have

$$\bigoplus_{j=0}^{k-2} q_n^{j-1}\mathcal{H}_n^{s+k-2-2j} \subseteq \mathcal{H}_n^{s+1}\mathcal{D}_n^{k-3},$$

then we have, as $q_n \in \mathcal{D}_n^2$,

$$q_n\left(\bigoplus_{j=0}^{k-2} q_n^{j-1}\mathcal{H}_n^{s+k-2-2j}\right) \subseteq \mathcal{H}_n^{s+1}\mathcal{D}_n^{k-1}.$$

This provides the inclusion

$$\bigoplus_{j=0}^{k-1} q_n^j\mathcal{H}_n^{s+k-2j} \subseteq \mathcal{H}_n^{s+1}\mathcal{D}_n^{k-1},$$

proving equality (2.2.15). It remains to show that for every $d \geq 2(s+1)$

$$\left(q_n^s\right)_d^\perp = \mathcal{H}_n^{s+1}\mathcal{D}_n^{d-s-1}. \tag{2.2.17}$$

In particular, since we have that

$$\left(q_n^s\right)_d^\perp = \mathcal{D}_n^d$$

for every $d \geq 2(s+1)$, we simply have to prove that

$$\mathcal{D}_n^{2s+m+1} = \mathcal{H}_n^{s+1}\mathcal{D}_n^{s+m}$$

for every $m \geq 1$. Now, we have just seen, by formulas (2.2.13) and (2.2.15), that

$$(q_n^s)^{\perp}_{2s+1} = \mathcal{H}_n^{s+1}\mathcal{D}_n^s = \bigoplus_{j=0}^{s} q_n^j \mathcal{H}_n^{2s-2j+1}.$$

This implies, by the decomposition (2.1.12), that

$$\mathcal{H}_n^{s+1}\mathcal{D}_n^s = \mathcal{D}_n^{2s+1},$$

from which we easily get

$$\mathcal{D}_n^{2s+m+1} = \mathcal{D}_n^{2s+1}\mathcal{D}_n^m = \mathcal{H}_n^{s+1}\mathcal{D}_n^s\mathcal{D}_n^m = \mathcal{H}_n^{s+1}\mathcal{D}_n^{s+m}$$

for every $m \geq 1$, which corresponds to equality (2.2.17). □

By Theorem 2.2.14 it is quite easy to obtain a lower bound for the rank of the form q_n^s. Indeed, as a direct consequence of Proposition 1.2.25, we get the following corollary.

Corollary 2.2.18. *For every $n, s \in \mathbb{N}$*

$$\operatorname{rk}(q_n^s) \geq \operatorname{brk}(q_n^s) \geq \binom{s+n-1}{n-1}.$$

CHAPTER 3

Tight decompositions

The lower bound provided by Corollary 2.2.18, i.e. the value

$$\mathrm{T}_{n,s} = \binom{s+n-1}{n-1},$$

leads us to ask in which cases the equality is satisfied, that is, the rank of q_n^s is equal to the rank of the middle catalecticant matrix. B. Reznick analyzes this problem in [Rez92] for real decompositions, calling *tight decompositions* the decomposition having size $T_{n,s}$ and creating a connection with the classical language of spherical designs. Although it is not so easy to extend the whole theory to the complex case, we provide in this chapter some generalizations of results of B. Reznick, which are valid also for the field $\mathbb{C}$.

In section 3.1 we give a summary of results for the real tight decompositions, which seem to preserve some geometric properties if we extend them to the complex case. In particular, we prove that the property of tight decompositions to be formed by points having the same normis valid also over $\mathbb{C}$, not only over $\mathbb{R}$, which is the case proved by B. Reznick. Thanks to this surprising fact, we are able to extend some results exposed by B. Reznick on tight decomposition. One of the most relevant is that many decompositions, resulting to be unique up to real orthogonal transformations, still preserve their uniqueness also extending the set of possible transformation to the complex ones. This represents, in fact, the main subject of the two following sections. First, in section 3.2, we explicitly compute all the possible decompositions in two variables, thanks to Theorem 2.2.14. Then, in section 3.3, we repeat the same procedure for some specific cases in more variables. In this last section, we focus on tight decompositions in the case of lower powers and we also observe that, working on the complex field, there are just a few number of values which can be assumed by n to get tight decompositions. For many cases, the question remains unsolved even for real decompositions.

3.1 Real decompositions and spherical designs

In dealing with minimal real decompositions of the form q_n^s, the number of different values of the norm of the points of the decompositions plays a relevant role. B. Reznick focuses on this in [Rez92, Chapter 8], analyzing the decompositions determined by points having the same norm, also called *first caliber* decompositions, and their one-to-one correspondence with some specific combinatorial objects, known as spherical designs (see [Rez92, Proposition 8.38]). However, although we do not work using spherical designs, we consider these results and generalize some of them to the field of complex numbers, especially for what concerns first caliber decomposition.

Spherical designs can be defined in many different ways. B. Reznick chose in [Rez92] the one provided by P. Delsarte, J.-M. Goethals, and J. J. Seidel in [DGS77, Definition 5.1]. They were the first to define this concept, considering a spherical t-design as a finite set of points A contained in the $(n-1)$-dimensional sphere S^{n-1} such that, for every homogeneous polynomial $f \in \mathcal{R}_n$ with $\deg f \leq t$,

$$\frac{1}{\left|\mathrm{S}^{n-1}\right|} \int_{\mathrm{S}^{n-1}} f(\xi)\,\mathrm{d}\omega(\xi) = \frac{1}{|A|} \sum_{\mathbf{a} \in A} f(\mathbf{a}).$$

More details on spherical designs occur many times in the literature, for which B. Reznick also lists several texts (see [Rez92, p. 113]) which can be consulted to get more information them, as [Ban84], [CS99], [GS79], [GS81a], [GS81b], [Hog90], [Sei84], and [Sei87]. There is another equivalent definition, again provided by P. Delsarte, J.-M. Goethals and J. J. Seidel in [DGS77, Theorem 5.2] and also used by E. Bannai and R. M. Damerell in [BD79] and [BD80]. It tells us that any finite set of points $A \subseteq \mathrm{S}^{n-1}$ is a t-spherical design if and only if

$$\sum_{\mathbf{a} \in A} h(\mathbf{a}) = 0$$

for every $h \in \mathcal{H}_n^k$ and $k = 1, \ldots, t$. Although we can extend this last definition to complex points on a subset A of the complexified sphere

$$\mathrm{S}_{\mathbb{C}}^{n-1} = \left\{ (x_1, \ldots, x_n) \in \mathbb{C}^n \;\middle|\; x_1^2 + \cdots + x_n^2 = 1 \right\},$$

it is not immediate to obtain the same results. This is due to the several hypothesis on the points which are necessary to define another kind of objects, equally important, namely the *spherical codes* (see e.g. [DGS77, Section 4]).

We start by introducing a name to indicate the minimal decomposition of size equal to the rank of central catalecticant matrices. This is the same term used by B. Reznick and it was chosen in relation to the correspondence with tight spherical designs (see [BD79]).

Definition 3.1.1. A decomposition

$$q_n^s = \sum_{k=1}^{m} (a_{k,1}x_1 + \cdots + a_{k,n}x_n)^{2s}$$

is said to be *tight* if $m = T_{n,s}$, where

$$T_{n,s} = \binom{s+n-1}{s}.$$

The work of B. Reznick about the form q_n^s concerns several decompositions, some of which were known from the classical literature and then translated from the language of spherical designs. This can be explained by the fact that the real decompositions of size $m \in \mathbb{N}$ can be associated to a finite set of m real n-tuples $\mathbf{a}_1, \ldots, \mathbf{a}_m$, where

$$\mathbf{a}_k = (a_{k,1}, \ldots, a_{k,n}) \in \mathbb{R}^n,$$

for every $k = 1, \ldots, m$. Then, it is quite natural to identify these as points on several n-dimensional spheres centered in the origin. In particular, we have the following theorem, provided by B. Reznick.

Theorem 3.1.2 ([Rez92, Proposition 9.2])**.** *If q_n^s has a real tight decomposition, then one of the following conditions holds:*

(1) $s = 1$ *or* $n = 2$*;*
(2) $s = 2$ *and* $n = 3$*;*
(3) $s = 2$ *and* $n = m^2 - 2$ *for some odd* $m \in \mathbb{N}$*;*
(4) $s = 3$ *and* $n = 3m^2 - 4$ *for some* $m \in \mathbb{N}$*;*

(5) $s = 5$ *and* $n = 24$.

This result is very powerful in dealing with real numbers, since it shows that there is no tight decomposition for every power $s \geq 6$. Now, given a natural number $r \in \mathbb{N}$, B. Reznick defines a *r-th caliber representation* as the number r of different spheres containing points of such a representation, namely, a real decomposition for which there are r distinct values taken by

$$|\mathbf{a}_k|^{2s} = \left(a_{k,1}^2 + \cdots + a_{k,n}^2\right)^s$$

for $k = 1, \ldots, m$. We can naturally extend this definition to the complex field, including isotropic points.

Definition 3.1.3. For every $r \in \mathbb{N}$, a decomposition

$$q_n^s = \sum_{k=1}^{m} (\mathbf{a}_k \cdot \mathbf{x})^{2s} = \sum_{k=1}^{m} (a_{k,1}x_1 + \cdots + a_{k,n}x_n)^{2s}$$

is said to be an *r-th caliber decomposition* if there are exactly r values $c_1, \ldots, c_r \in \mathbb{C}$ such that

$$\left\{(\mathbf{a}_k \cdot \mathbf{a}_k)^s\right\}_{k=1,\ldots,m} \in \{c_1, \ldots, c_r\}.$$

That is, there are exactly r values taken by

$$(\mathbf{a}_k \cdot \mathbf{a}_k)^s = \left(a_{k,1}^2 + \cdots + a_{k,n}^2\right)^s$$

for $k = 1, \ldots, m$.

The first caliber decompositions own a special role because of their particular symmetry. Most of the results provided by B. Reznick in [Rez92] are based on the construction of a inner product on $\mathcal{R}_n^d$. Given a polynomial $p \in \mathcal{R}_n^d$ and a multi-index $\alpha \in \mathbb{N}^n$ such that $|\alpha| = d$, we denote by $c_\alpha(p)$ the coefficient of the monomial $\mathbf{x}^\alpha$ in the polynomial p, that is,

$$p = \sum_{\alpha \in \mathcal{I}_{n,d}} \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!} c_\alpha(p) \mathbf{x}^\alpha,$$

where

$$\mathcal{I}_{n,d} = \{\, \alpha \in \mathbb{N}^n \mid |\alpha| = d \,\}.$$

In the case of real polynomials, for every $n, d \in \mathbb{N}$, the inner product

$$\langle \, , \rangle \colon S^d\mathbb{R}^n \times S^d\mathbb{R}^n \to \mathbb{R}$$

considered by B. Reznick in [Rez92, pp. 1-2] associates to each pair of polynomials $f, g \in S^d\mathbb{R}^n$ the real value

$$\langle f, g \rangle = \sum_{\alpha \in \mathcal{I}_{n,d}} \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!} c_\alpha(f) c_\alpha(g).$$

This inner product is classically known as *Bombieri inner product* and it is based on the norm in spaces of polynomials known as the *Bombieri norm* and exposed by B. Beauzamy, E. Bombieri, P. Enflo, and H. L. Montgomery in [BBEM90]. For the real case, this concept has been analyzed also by E. Kostlan in [Kos93, Section 4], where he proves (see [Kos93, Theorem 4.1, Theorem 4.2]) that it is invariant under the action of the space $\mathrm{O}_n(\mathbb{R})$ of real orthogonal matrices. We can extend it to the complex case obtaining a complex symmetric bilinear form

$$\langle \, , \rangle \colon \mathcal{R}_n^d \times \mathcal{R}_n^d \to \mathbb{C},$$

defined in the same way.

A nice property about this product, considered by B. Reznick in [Rez92, formula (1.5)], considers the evaluation of a polynomial at a point associated to a linear form and is given by the following proposition.

Proposition 3.1.4. *For every $f \in \mathcal{R}_n^d$ and for every $\mathbf{a} \in \mathbb{C}^n$*

$$\left\langle f, (\mathbf{a} \cdot \mathbf{x})^d \right\rangle = f(\mathbf{a}). \tag{3.1.5}$$

Proof. Since every homogeneous polynomial can be written as a sum of d-powers of linear forms, it is sufficient to prove the statement when f is a power of a linear form $(\mathbf{b} \cdot \mathbf{x})^d$ for some $\mathbf{b} \in \mathbb{C}^n$. Then we have

$$\left\langle (\mathbf{b} \cdot \mathbf{x})^d, (\mathbf{a} \cdot \mathbf{x})^d \right\rangle = \sum_{\alpha \in \mathcal{I}_{n,d}} \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!} \mathbf{b}^\alpha \mathbf{a}^\alpha = (\mathbf{b} \cdot \mathbf{a})^d,$$

which proves the formula. □

We can state, as for the previous one, that this inner product is invariant under the action of complex orthogonal group.

Proposition 3.1.6. *For every $f, g \in \mathcal{R}_n^d$, the equality*

$$\langle A \cdot f, A \cdot g \rangle = \langle f, g \rangle$$

holds for every $A \in \mathrm{O}_n(\mathbb{C})$.

Proof. By linearity, we can just prove the statement for powers of linear forms. So, considering the polynomials

$$f = (\mathbf{a} \cdot \mathbf{x})^d, \quad g = (\mathbf{b} \cdot \mathbf{x})^d,$$

and an orthogonal matrix $A \in \mathrm{O}_n(\mathbb{C})$, i.e. such that ${}^{\mathrm{t}}AA = A{}^{\mathrm{t}}A = I$, we get

$$\begin{aligned}
\left\langle A \cdot (\mathbf{a} \cdot \mathbf{x})^d, A \cdot (\mathbf{b} \cdot \mathbf{x})^d \right\rangle &= \left\langle \left(\mathbf{a} \cdot A\mathbf{x}\right)^d, \left(\mathbf{b} \cdot A\mathbf{x}\right)^d \right\rangle = \left\langle \left({}^{\mathrm{t}}A^{-1}\mathbf{a} \cdot \mathbf{x}\right)^d, \left({}^{\mathrm{t}}A^{-1}\mathbf{b} \cdot \mathbf{x}\right)^d \right\rangle \\
&= \left\langle (A\mathbf{a} \cdot \mathbf{x})^d, (A\mathbf{b} \cdot \mathbf{x})^d \right\rangle = \sum_{\alpha \in \mathcal{I}_{n,d}} \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!} (A\mathbf{a})^\alpha (A\mathbf{b})^\alpha \\
&= (A\mathbf{a} \cdot A\mathbf{b})^d = (\mathbf{a} \cdot \mathbf{b})^d = \left\langle (\mathbf{a} \cdot \mathbf{x})^d, (\mathbf{b} \cdot \mathbf{x})^d \right\rangle
\end{aligned}$$

□

As a particular case of Proposition 3.1.6, we can state (see also [Rez92, formula (8.2) and Corollary 8.18]) that

$$\left\langle q_n^s, q_n^s \right\rangle = \prod_{j=0}^{s-1} \frac{2j+n}{2j+1}. \tag{3.1.7}$$

Furthermore, we observe that, given a decomposition

$$q_n^s = \sum_{k=1}^{m} (\mathbf{a}_k \cdot \mathbf{x})^{2s}, \tag{3.1.8}$$

we get by Proposition 3.1.4 and formula (3.1.7) the equality

$$\prod_{j=0}^{s-1} \frac{2j+n}{2j+1} = \left\langle q_n^s, \sum_{k=1}^{m} (\mathbf{a}_k \cdot \mathbf{x})^{2s} \right\rangle = \sum_{k=1}^{m} \left\langle q_n^s, (\mathbf{a}_k \cdot \mathbf{x})^{2s} \right\rangle = \sum_{k=1}^{m} q_n^s(\mathbf{a}_k) = \sum_{k=1}^{m} (\mathbf{a}_k \cdot \mathbf{a}_k)^s.$$

Thus, we rewrite the following generalization to the complex field for first caliber decompositions.

Proposition 3.1.9. *Let*

$$q_n^s = \sum_{k=1}^{m} (\mathbf{a}_k \cdot \mathbf{x})^{2s}$$

be a first caliber decomposition of size $m \in \mathbb{N}$. Then

$$(\mathbf{a}_k \cdot \mathbf{a}_k)^s = \frac{1}{m} \prod_{j=0}^{s-1} \frac{2j+n}{(2j+1)}$$

for every $k = 1, \dots, m$.

P. D. Seymour and T. Zavlavsky prove in [SZ84] that a first caliber decomposition of q_n^s of size $r \in \mathbb{N}$ always exists for a big enough value of r. Moreover, B. Reznick proves in [Rez92, Corollary 8.17] that every real tight decomposition is first caliber. This result can easily be extended, but in order to prove this, we need the following lemma.

Lemma 3.1.10. *Let $d \geq 2$. Then the following conditions hold:*

(1) for every point $\mathbf{a} \in \mathbb{C}^n$, the d-th power of its associated linear form $l_{\mathbf{a}} = \mathbf{a} \cdot \mathbf{x}$ is harmonic if and only if $\mathbf{a}$ is isotropic in $\mathbb{C}^n$, that is, $\mathbf{a} \cdot \mathbf{a} = 0$;

(2) the space $\mathcal{H}_n^d$ is generated by the d-th powers of linear forms associated to isotropic points; that is,

$$\mathcal{H}_n^d = \left\langle \left\{ l_{\mathbf{a}}^d \in \mathcal{R}_n^d \;\middle|\; \mathbf{a} \in \mathbb{C}^n : \mathbf{a} \cdot \mathbf{a} = 0 \right\} \right\rangle.$$

Proof. Point (1) follows directly by Lemma 1.2.15. Indeed, for every non-zero linear form $l_{\mathbf{a}} \in \mathcal{R}_n^1$ we have

$$\Delta\big(l_{\mathbf{a}}^d\big) = d(d-1)\big(a_1^2 + \cdots + a_n^2\big) l_{\mathbf{a}}^{d-2} = d(d-1)(\mathbf{a} \cdot \mathbf{a}) l_{\mathbf{a}}^{d-2},$$

which is equal to zero if and only if $\mathbf{a} \cdot \mathbf{a} = 0$. To prove point (2), let us consider the space

$$W = \left\langle \left\{ l_{\mathbf{a}}^d \in \mathcal{R}_n^d \;\middle|\; \mathbf{a} \in \mathbb{C}^n : \mathbf{a} \cdot \mathbf{a} = 0 \right\} \right\rangle.$$

We have by point (1) that

$$W \subseteq \mathcal{H}_n^d.$$

Now, for every $A \in \mathrm{SO}_n(\mathbb{C})$, we have

$$A \cdot l_{\mathbf{a}}^d = \big(A \cdot l_{\mathbf{a}}\big)^d = l_{A \cdot \mathbf{a}}^d$$

for every $\mathbf{a} \in \mathbb{C}^n$. Since A is an orthogonal transformation, we have

$$\mathbf{a} \cdot \mathbf{a} = (A \cdot \mathbf{a}) \cdot (A \cdot \mathbf{a}),$$

which implies that

$$A \cdot h \in W$$

for every $h \in W$. Thus, W is a $\mathrm{SO}_n(\mathbb{C})$-module and since $\mathcal{H}_n^d(\mathbb{C})$ is an irreducible $\mathrm{SO}_n(\mathbb{C})$-module, as we have already seen in section 2.1, we must have $W = \mathcal{H}_n^d$. □

Now, in dealing with linear forms of the kind $\mathbf{a} \cdot \mathbf{x}$ for some $\mathbf{a} \in \mathbb{C}^n$, a relevant role is assumed by the associated value $\mathbf{a} \cdot \mathbf{a}$. In particular, from the fact that the middle catalecticant $\mathrm{Cat}_{q_n^s}^s$ is full rank, it immediately follows the following lemma.

Lemma 3.1.11. *For every $n, s \in \mathbb{N}$, let $\mathbf{a} \cdot \mathbf{x}$ be a linear form with $\mathbf{a} \in \mathbb{C}^n$ and let*

$$f = q_n^{[s]} - (\mathbf{a} \cdot \mathbf{x})^{[2s]}.$$

Then

$$T_{n,s} - 1 \leq \operatorname{rk}\big(\mathrm{Cat}_f^s\big) \leq T_{n,s}.$$

Proof. We can select a basis $\{g_1, \dots, g_{T_{n,s}}\}$ of $\mathcal{R}_n^s$ with

$$g_1 = (\mathbf{a} \cdot \mathbf{x})^{[s]}.$$

Then we can also consider a basis $\{h_1, \dots, h_n\}$ of $\mathcal{D}_n^s$ such that

$$h_j \circ q_n^{[s]} = g_j$$

for every $j = 1, \dots, T_{n,s}$. In particular, for any choice of the elements $c_1, \dots, c_{T_{n,s}} \in \mathbb{C}$, let us consider the polynomial

$$h = \sum_{j=1}^{T_{n,s}} c_j h_j \in \mathcal{D}_n^s.$$

Then we have

$$\begin{aligned}
\left(\sum_{j=1}^{T_{n,s}} c_j h_j\right) \circ f &= \left(\sum_{j=1}^{T_{n,s}} c_j h_j\right) \circ q_n^{[s]} - \left(\sum_{j=1}^{T_{n,s}} c_j h_j\right) \circ (\mathbf{a} \cdot \mathbf{x})^{[2s]} \\
&= \sum_{j=1}^{T_{n,s}} c_j g_j - \left(\sum_{j=1}^{T_{n,s}} c_j h_j(\mathbf{a})\right)(\mathbf{a} \cdot \mathbf{x})^{[s]} \\
&= \sum_{j=2}^{T_{n,s}} c_j g_j + \left(c_1 - \sum_{j=1}^{T_{n,s}} c_j h_j(\mathbf{a})\right) g_1.
\end{aligned}$$

Therefore, by linear independence, $h \in \operatorname{Ker}\big(\mathrm{Cat}_f^s\big)$ if and only if

$$c_2 = \cdots = c_{T_{n,s}} = 0$$

either $c_1 = 0$, or $h_1(\mathbf{a}) = 1$. In the first case we clearly have $h = 0$, while in the second one we get

$$\langle h_1 \rangle = \operatorname{Ker}\big(\mathrm{Cat}_f^s\big)$$

and hence

$$\operatorname{rk}\big(\mathrm{Cat}_f^s\big) = T_{n,s} - 1. \qquad \square$$

A quite important fact is related to the middle catalecticant of the form q_n^s and isotropic points. We will use next lemma to prove that any tight decomposition of q_n^s must be first caliber.

Lemma 3.1.12. *For every $n, s \in \mathbb{N}$, let $\mathbf{a} \cdot \mathbf{x}$ be a linear form such that $\mathbf{a} \in \mathbb{C}^n$ is an isotropic point, and let*

$$f = q_n^{[s]} - (\mathbf{a} \cdot \mathbf{x})^{[2s]}.$$

Then the middle catalecticant Cat_f^s of f is full rank.

Proof. Let us consider an element $g \in \operatorname{Ker}(\mathrm{Cat}_f^s)$, which can be written, by decomposition (2.1.12), as

$$g = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} q_n^k h_{s-2k},$$

where h_{s-2k} is a harmonic polynomial for every $k = 1, \dots, \lfloor \frac{s}{2} \rfloor$. Then we have

$$q_n^k h_{s-2k} \circ (\mathbf{a} \cdot \mathbf{x})^{[2s]} = q_n^{k-1} h_{s-2k} \circ \left(q_n \circ (\mathbf{a} \cdot \mathbf{x})^{[2s]}\right) = 0 \tag{3.1.13}$$

for every $k = 1, \dots, \lfloor \frac{s}{2} \rfloor$. Indeed, since $\mathbf{a}$ is isotropic, then it follows by Lemma 3.1.10 that $(\mathbf{a} \cdot \mathbf{x})^{2s}$ is harmonic, that is,

$$q_n \circ (\mathbf{a} \cdot \mathbf{x})^{2s} = 0.$$

Formula (3.1.13) implies that the kernel of the catalecticant map can only contain harmonic polynomials. To see this, let us suppose that $g \circ f = 0$, namely,

$$\left(\sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} q_n^k h_{s-2k}\right) \circ \left(q_n^{[s]} - (\mathbf{a} \cdot \mathbf{x})^{2s}\right) = \left(\sum_{k=1}^{\lfloor \frac{s}{2} \rfloor} A_{n,s,k} q_n^k h_{s-2k}\right) + h_s \circ \left(q_n^{[s]} - (\mathbf{a} \cdot \mathbf{x})^{[2s]}\right) = 0.$$

Then, since the polynomial

$$h_s \circ \left(q_n^{[s]} - (\mathbf{a} \cdot \mathbf{x})^{2s}\right) = h_s - h_s(\mathbf{a})(\mathbf{a} \cdot \mathbf{x})^{[s]}$$

is harmonic, we must have by uniqueness of decomposition (2.1.12) that $h_{s-2k} = 0$ for every $k = 1 \dots, \lfloor \frac{s}{2} \rfloor$. That is,

$$g = h_s = h_s(\mathbf{a})(\mathbf{a} \cdot \mathbf{y})^{[s]},$$

which is harmonic. Now, by Lemma 3.1.10 we know that harmonic polynomials are generated by powers of isotropic linear forms. So, let us consider a basis

$$\mathcal{B} = \left\{(\mathbf{a}_1 \cdot \mathbf{y})^{[s]}, \dots, (\mathbf{a}_m \cdot \mathbf{y})^{[s]}\right\}$$

of the space $\mathcal{H}_n^s \subset \mathcal{D}_n^s$, setting $\mathbf{a}_1 = \mathbf{a}$. Since any polynomial $g \in \operatorname{Ker}(\mathrm{Cat}_f^s)$ must be harmonic, we can write

$$g = \sum_{j=1}^{m} c_j (\mathbf{a}_j \cdot \mathbf{x})^s$$

for some $c_1, \dots, c_m \in \mathbb{C}$ and we have

$$g \circ \left(q_n^{[s]} - (\mathbf{a} \cdot \mathbf{x})^{[2s]}\right) = \left(\sum_{j=1}^{m} c_j (\mathbf{a}_j \cdot \mathbf{y})^s\right) \circ \left(q_n^{[s]} - (\mathbf{a} \cdot \mathbf{x})^{[2s]}\right) = 0.$$

That is, since $\mathbf{a}$ is isotropic, by formula (2.2.9) we get the equalities

$$\begin{aligned} h \circ \left(q_n^{[s]} - (\mathbf{a} \cdot \mathbf{x})^{[2s]}\right) &= \sum_{j=1}^{m} c_j (\mathbf{a}_j \cdot \mathbf{x})^s - \sum_{j=2}^{m} c_j (\mathbf{a} \cdot \mathbf{a}_j)(\mathbf{a} \cdot \mathbf{x})^{[s]} \\ &= \left(c_1 - \sum_{j=2}^{m} c_j (\mathbf{a} \cdot \mathbf{a}_j)\right)(\mathbf{a} \cdot \mathbf{x})^{[s]} + \sum_{j=2}^{m} c_j (\mathbf{a}_j \cdot \mathbf{x})^{[s]} = 0. \end{aligned}$$

This implies, by linear independence, that $c_2 = \cdots = c_m = 0$ and hence also

$$c_1 (\mathbf{a}_1 \cdot \mathbf{x})^s = 0,$$

that is $c_1 = 0$. Therefore, we have $g = 0$, which means that the middle catalecticant Cat_f^s is full rank. □

The importance of Lemma 3.1.12 is that it allows to exclude isotropic points from tight decompositions. In particular, now we can prove that every tight decomposition must be first caliber, even considering complex decompositions.

Theorem 3.1.14. *Every tight decomposition*

$$q_n^s = \sum_{k=1}^{m} (\mathbf{a}_k \cdot \mathbf{x})^{2s},$$

where

$$m = T_{n,s} = \binom{s+n-1}{s},$$

is first caliber. In particular,

$$(\mathbf{a}_k \cdot \mathbf{a}_k)^s = \frac{1}{T_{n,s}} \prod_{j=0}^{s-1} \frac{2j+n}{(2j+1)}.$$

for every $k = 1, \dots, T_{n,s}$.

Proof. Let us suppose that a tight decomposition of $q_n^{[s]}$ contains a summand $(\mathbf{a}_1 \cdot \mathbf{x})^{2s}$, such that $\mathbf{a}_1$ is an isotropic point. This means, in particular, that

$$q_n \circ (\mathbf{a}_1 \cdot \mathbf{x})^{2s} = \Delta\big((\mathbf{a}_1 \cdot \mathbf{x})^{2s}\big) = s(s-1)(\mathbf{a}_1 \cdot \mathbf{a}_1) \cdot (\mathbf{a}_1 \cdot \mathbf{x})^{2s-2} = 0.$$

Then, considering the polynomial

$$f = q_n^{[s]} - (\mathbf{a}_1 \cdot \mathbf{x})^{[2s]},$$

we should have that Cat_f^s is not full rank, but by Lemma 3.1.12 we know that this is not possible. Thus, any tight decomposition cannot contain any power of a linear form which is associated to a isotropic point. Now, it remains to prove that each point of a tight decomposition has the same norm, up to roots of unity. Let

$$q_n^s = \sum_{j=1}^{T_{n,s}} (\mathbf{b}_j \cdot \mathbf{x})^{2s}$$

be a tight decomposition. Now, q_n^s is invariant under the action of $\mathrm{SO}_n(\mathbb{C})$, which acts transitively on the set of non-isotropic points with fixed norm of $\mathbb{C}^n$. We can then suppose that

$$\mathbf{b}_1 = C_0 \mathbf{e}_1 = C_0(1, 0, \dots, 0) \in \mathbb{C}^n$$

for some $C_0 \in \mathbb{C}$. If we consider the polynomial

$$f_1 = q_n^{[s]} - (\mathbf{a}_1 \cdot \mathbf{x})^{[2s]},$$

then we have that $\det\big(\mathrm{Cat}_{f_1}^s\big)$ is a polynomial in the variable C_0 of degree $2s$. Moreover, since the form

$$(\mathbf{b}_1 \cdot \mathbf{x})^{2s} = C_0^{2s} (\mathbf{e}_1 \cdot \mathbf{x})^{2s}$$

would not change by multiplying C_0 by any $2s$-th root of unity, then the roots of $\det\big(\mathrm{Cat}_{f_1}^s\big)$ are given by a unique value up to multiplications by a $2s$-th root of unity. Thus, by the invariance of q_n^s under the action of $\mathrm{SO}_n(\mathbb{C})$, we get that the complex number $(\mathbf{b}_k \cdot \mathbf{b}_k)^s$ is the same for every $k = 1, \dots, T_{n,s}$. In particular, we have by Proposition 3.1.9 that this value is real, namely,

$$(\mathbf{b}_k \cdot \mathbf{b}_k)^s = \frac{1}{T_{n,s}} \prod_{j=0}^{s-1} \frac{2j+n}{(2j+1)}.$$

□

Given any tight decomposition

$$q_n^s = \sum_{k=1}^{T_{n,s}} (\mathbf{b}_k \cdot \mathbf{x})^{2s},$$

we denote by $B_{n,s}$ the value obtained in Theorem 3.1.14, that is

$$B_{n,s} = \frac{1}{T_{n,s}} \prod_{j=0}^{s-1} \frac{2j+n}{2j+1} \tag{3.1.15}$$

for every $k = 1, \dots, T_{n,s}$.

3.2 Tight decomposition in two variables

For the case of two variables, the rank of the powers of the quadric q_n is completely known and exposed by B. Reznick in [Rez92, Theorem 9.5], where he provides all the possible real decompositions, which turn out to be unique up to a real orthogonal transformation. In this section we deal with this fact from the point of view of apolarity and extend it to the complex field, proving that the unique real decomposition is still unique for the complex case.

3.2.1 Real tight decompositions

We know by Theorem 2.2.14 that the apolar ideal of q_2^s is

$$\left(q_2^s\right)^{\perp} = \left(\mathcal{H}_2^{s+1}\right).$$

Hence, we first have to determine a basis of the space $\mathcal{H}_2^{s+1}$. In general, for every $n \in \mathbb{N}$, we have by Proposition 2.1.11 that the dimension of the d-harmonic polynomials in n variables is

$$\dim \mathcal{H}_n^d = \binom{d+n-1}{n-1} - \binom{d+n-3}{n-1},$$

for every $d \in \mathbb{N}$. Therefore, if we restrict to the case of two variables, we obtain that $\dim \mathcal{H}_2^d = 2$. Now, let us consider in $\mathcal{D}_n^1$ the polynomials

$$u = \frac{y_1 + \mathrm{i}y_2}{2}, \quad v = \frac{y_1 - \mathrm{i}y_2}{2},$$

multiples of the polynomials 2.2.4. Then we obtain a new basis of $\mathcal{D}_2^1$ and hence, by a simple change of variables, we have $\mathcal{D}_2 = \mathbb{C}[y_1, y_2] \simeq \mathbb{C}[u, v]$. Thus, it follows by Lemma 2.2.6 that

$$\left(q_2^s\right)^{\perp} = \left(u^{s+1}, v^{s+1}\right).$$

In dealing with complex numbers, we denote by $\Im(z)$ the imaginary part of any number $z \in \mathbb{C}$.

Lemma 3.2.1. *For every $a, b \in \mathbb{C}$, let u_1 and u_2 be the complex values*

$$u_1 = a + \mathrm{i}b, \quad u_2 = a - \mathrm{i}b.$$

Then the following conditions are equivalent:
(1) $a, b \in \mathbb{R}$;
(2) $\overline{u}_1 = u_2$.

Proof. (1) $\Rightarrow$ (2) is trivial. Conversely, if $\overline{u}_1 = u_2$, then we have

$$a + \mathrm{i}b = \overline{a} + \mathrm{i}\overline{b}$$

and hence,

$$2\mathrm{i}\Im(a) = a - \overline{a} = \mathrm{i}(\overline{b} - b) = -2\mathrm{i}^2\Im(b) = 2\Im(b).$$

That is,

$$\Im(a) = \Im(b) = 0,$$

and hence $a, b \in \mathbb{R}$. □

Lemma 3.2.1 can be generalized to projective points. In particular, we can characterize the points in coordinates $\{u, v\}$ such that these correspond to real projective points in coordinates $\{y_1, y_2\}$.

Lemma 3.2.2. *Let $a, b \in \mathbb{C}$ be such that $(a, b) \neq (0, 0)$, let $[a : b] \in \mathbb{P}^1(\mathbb{C})$ be the projective point associated to the pair $(a, b) \in \mathbb{C}^2$ and let $u_1, u_2 \in \mathbb{C}$ be the values*

$$u_1 = a + \mathrm{i}b, \quad u_2 = a - \mathrm{i}b.$$

Then the following conditions are equivalent:
(1) $[a : b] \in \mathbb{P}^1(\mathbb{R})$, that is, b can be written as real multiple of a or $a = 0$;
(2) $[u_1 : u_2] = [u_0 : \overline{u}_0]$ for a suitable $u_0 \in \mathbb{C}$;
(3) $|u_1| = |u_2|$.

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. So, let u_1 and u_2 be such that $|u_1| = |u_2|$. If $a = 0$ or $b = 0$ then the statement (1) is clear. If instead $a, b \neq 0$, then we have

$$\begin{aligned} |u_1|^2 - |u_2|^2 &= (a + \mathrm{i}b)(\overline{a} - \mathrm{i}\overline{b}) - (a - \mathrm{i}b)(\overline{a} + \mathrm{i}\overline{b}) \\ &= |a|^2 + \mathrm{i}\overline{a}b - \mathrm{i}a\overline{b} + |b|^2 - |a|^2 - \mathrm{i}a\overline{b} + \mathrm{i}\overline{a}b \\ &= 2\mathrm{i}(\overline{a}b - a\overline{b}) = 0, \end{aligned}$$

that is

$$a^2 = \left(\frac{|a|}{|b|}\right)^2 b^2.$$

Therefore, we have

$$a = \pm\frac{|a|}{|b|}b$$

and hence a is a real multiple of b. □

As a consequence of Lemma 3.2.2, we get that, given the coordinate function

$$\begin{aligned} v\colon \mathbb{C}^2 &\longrightarrow \mathbb{C} \\ (y_1, y_2) &\longmapsto y_1 - \mathrm{i}y_2 \end{aligned}$$

and the conjugate coordinate function

$$\begin{aligned} \overline{u}\colon \mathbb{C}^2 &\longrightarrow \mathbb{C} \\ (y_1, y_2) &\longmapsto \overline{y}_1 - \mathrm{i}\overline{y}_2, \end{aligned}$$

a projective point $[a : b] \in \mathbb{P}^1(\mathbb{C})$ has a real representative pair if and only if $v(a, b) = \overline{u}(a, b)$. Without loss of generality, we will refer to real roots of a polynomial whenever this condition is satisfied. Now we can provide another proof for the determination of the rank q_2^s, in addition to the one proposed by B. Reznick in [Rez92, Theorem 9.5], exposed in the following theorem. In this case we determine a suitable decomposition by Lemma 1.2.17.

Theorem 3.2.3. *For every $s \in \mathbb{N}$*
$$\mathrm{rk}\left(q_2^s\right) = s+1.$$

Proof. By point (2) of Lemma 1.2.17, to prove the statement we have just to determine a polynomial of degree $s+1$ belonging to the apolar ideal of q_2^s and having $s+1$ distinct roots. So, let us consider the polynomial
$$f = u^{s+1} - v^{s+1}.$$
Then, for every $(u_0, v_0) \in \mathbb{C}^2$, we have that $f(u_0, v_0) = 0$ if and only if $u_0 = v_0 = 0$ or
$$\left(\frac{u_0}{v_0}\right)^{s+1} = 1,$$
that is
$$\frac{u_0}{v_0} = \mathrm{e}^{\mathrm{i}\frac{2(j-1)\pi}{s+1}}$$
for some $j = 1, \dots, s+1$. Thus, we have $s+1$ distinct roots, corresponding to the projective points
$$[u_j : v_j] = \left[\mathrm{e}^{\mathrm{i}\frac{2(j-1)\pi}{s+1}} : 1\right] = \left[\mathrm{e}^{\mathrm{i}\frac{(j-1)\pi}{s+1}} : \mathrm{e}^{-\mathrm{i}\frac{(j-1)\pi}{s+1}}\right]$$
for $j = 1 \dots, s+1$. Now, by Lemma 3.2.2, we can write these roots using the coordinates $\{y_1, y_2\}$, as real points. In particular, considering the value
$$\tau_j = \frac{(j-1)\pi}{s+1}$$
for every $j = 1, \dots, s+1$, we can write the points as
$$\left[y_{1,j} : y_{2,j}\right] = \left[\cos\tau_j : \sin\tau_j\right]$$
for $j = 1, \dots, s+1$. We conclude that f has $s+1$ distinct roots and hence $\mathrm{rk}\left(q_2^s\right) = s+1$. □

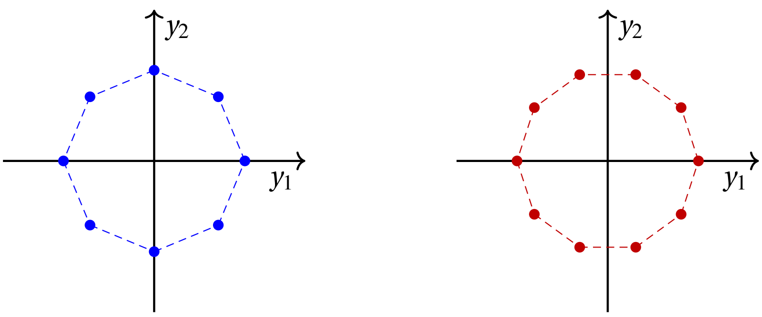

Figure 3.1: Examples of decompositions for the polynomials q_2^3 and q_2^4. The blue octagon on the left represents the 4 projective points obtained by the roots of the polynomial $u^4 - v^4$, while the red decagon on the right represents the 5 projective points obtained by the roots of the polynomial $u^5 - v^5$ (points opposite to the origin represent the same point in $\mathbb{P}^1(\mathbb{C})$).

The roots of the polynomial used in the proof of Theorem 3.2.3 are all real and hence provide a real decomposition of q_2^s, whose elements correspond to the projective classes of the $2(s+1)$-th roots of unity. Equivalently, these points correspond to the vertices of a regular $2(s+1)$-gon (see Figure 3.1), inscribed in a circumference of radius equal to
$$B_{2,s} = (s+1)^{-1} \prod_{j=0}^{s-1} \frac{2j+n}{2j+1} = 2^{2s}(s+1)^{-1}\binom{2s}{s}^{-1}.$$

This last fact is obtained by Theorem 3.1.14.

Analyzing all the polynomials of minimal degree $s+1$ with distinct real roots, which are contained in $(q_2^s)^\perp$, we can then determine all the real decompositions of q_2^s. The procedure simply consists of resolving a real system and so determining, by Lemma 1.2.17, the coefficients of the powers of the linear forms associated to every roots.

Proposition 3.2.4. *Let $f = au^{s+1} + bv^{s+1} \in \left(q_2^s\right)^\perp$ be a non-zero polynomial of degree $s+1$, with $a, b \in \mathbb{C}$. Then f has $s+1$ real distinct roots if and only if $|a| = |b| \neq 0$. In particular, it can be written, up to scalars, as*

$$f = u^{s+1} - \mathrm{e}^{\mathrm{i}\theta} v^{s+1}$$

for some $\theta \in [0, 2\pi)$ and its roots correspond to the projective points

$$[y_{1,j} : y_{2,j}] = [\cos \tau_{\theta,j} : \sin \tau_{\theta,j}],$$

where

$$\tau_{\theta,j} = \frac{2(j-1)\pi + \theta}{2(s+1)}$$

for $j = 1, \ldots, s+1$.

Proof. If the polynomial f has a real root $(y_{1,0}, y_{2,0}) \neq (0,0)$, then

$$u_0 = \overline{v}_0 \neq 0$$

and hence

$$au_0^{s+1} + b\overline{u}_0^{s+1} = 0.$$

This clearly means that $a, b \neq 0$ and $|a| = |b|$. Conversely, if a and b are complex numbers such that $|a| = |b| \neq 0$, then there exists a real number $\theta \in [0, 2\pi)$ such that f can be written, up to a scalar a, as

$$f = u^{s+1} - \mathrm{e}^{\mathrm{i}\theta} v^{s+1}$$

where

$$\mathrm{e}^{\mathrm{i}\theta} = -\frac{b}{a}.$$

Thus, for every pair $(u_0, v_0) \in \mathbb{C}^2$, we have $f(u_0, v_0) = 0$ if and only if $u_0 = v_0 = 0$ or

$$\left(\frac{u_0}{v_0}\right)^{s+1} = \mathrm{e}^{\mathrm{i}\theta}.$$

This means that there are $s+1$ roots $[u_j : v_j] \in \mathbb{PC}^1$ for which we have

$$u_j = \mathrm{e}^{\mathrm{i}\frac{2(j-1)\pi+\theta}{s+1}} v_j$$

for every $j = 1, \ldots, s+1$. That is, the $s+1$ roots of f are

$$[u_j : v_j] = \left[\mathrm{e}^{\mathrm{i}\frac{2(j-1)\pi+\theta}{s+1}} : 1\right] = \left[\mathrm{e}^{\mathrm{i}\frac{2(j-1)\pi+\theta}{2(s+1)}} : \mathrm{e}^{-\mathrm{i}\frac{2(j-1)\pi+\theta}{2(s+1)}}\right],$$

for $j = 1, \ldots, s+1$. So, introducing the notation

$$\tau_{\theta,j} = \frac{2(j-1)\pi + \theta}{2(s+1)}$$

we can write
$$[u_j : v_j] = \left[\mathrm{e}^{\mathrm{i}\tau_{\theta,j}} : \mathrm{e}^{-\mathrm{i}\tau_{\theta,j}}\right],$$
for $j = 1, \dots, s+1$. Hence, by the change of variables, we get the $s+1$ real distinct roots in standard coordinates, namely,
$$[y_{1,j} : y_{2,j}] = \left[\cos\tau_{\theta,j} : \sin\tau_{\theta,j}\right],$$
for $j = 1, \dots, s+1$ □

We observe that in the case of $\theta = 0$ we obtain the same polynomial used for the proof of Theorem 3.2.3. In general, the roots we obtain correspond to the projective classes of the vertices of a regular $2(s+1)$-gon.

Now we have all the elements to obtain all the minimal real decompositions of the form q_2^s, exploiting all the roots of the polynomials considered in Proposition 3.2.4.

Theorem 3.2.5. *The form q_2^s has a unique real decomposition, up to orthogonal real transformations, whose terms correspond to the vertices of a regular $2s$-gon inscribed in a circle of radius equal to*
$$r = 2(s+1)^{-\frac{1}{2s}}\binom{2s}{s}^{-\frac{1}{2s}}.$$
Namely
$$q_2^s = \sum_{j=1}^{s+1}\left(r\cos(\tau_j)x_1 + r\sin(\tau_j)x_2\right)^{2s},$$
where
$$\tau_j = \frac{(j-1)\pi}{s+1}$$
for every $j = 1, \dots, s+1$.

Proof. By Theorem 3.1.14, we know that any decomposition of q_2^s is first caliber and every point has norm equal to
$$r = 2(s+1)^{-\frac{1}{2s}}\binom{2s}{s}^{-\frac{1}{2s}}.$$
Moreover, by Proposition 3.2.4, the polynomials with real distinct roots in the apolar ideal of q_2^s are given by all the linear combinations of the type
$$u^{s+1} - \mathrm{e}^{\mathrm{i}\theta}v^{s+1}, \quad \theta \in [0, 2\pi).$$
Thus, introducing the variables
$$z_1 = x_1 - \mathrm{i}x_2, \quad z_2 = x_1 + \mathrm{i}x_2,$$
and considering the values
$$\tau_{j,\theta} = \frac{(j-1)\pi}{s+1} + \frac{\theta}{2(s+1)}$$
for every $j = 1, \dots, s+1$, we have
$$\begin{aligned} q_2^s = \left(x_1^2 + x_2^2\right)^s = z_1^{2s}z_2^{2s} &= \sum_{j=1}^{s+1}\left(\frac{r}{2}\right)^{2s}\left(\mathrm{e}^{\mathrm{i}\tau_{j,\theta}}z_1 + \mathrm{e}^{-\mathrm{i}\tau_{j,\theta}}z_2\right)^{2s} \\ &= \sum_{j=1}^{s+1}\left(\frac{r}{2}\right)^{2s}\left(2\left(\frac{\mathrm{e}^{\mathrm{i}\tau_{j,\theta}} + \mathrm{e}^{-\mathrm{i}\tau_{j,\theta}}}{2}\right)x_1 + 2\left(\frac{\mathrm{e}^{\mathrm{i}\tau_{j,\theta}} - \mathrm{e}^{-\mathrm{i}\tau_{j,\theta}}}{2\mathrm{i}}\right)x_2\right)^{2s} \end{aligned}$$

$$= \sum_{j=1}^{s+1} r^{2s} \big(\cos(\tau_{j,\theta}) x_1 + \sin(\tau_{j,\theta}) x_2 \big)^{2s}.$$

It remains to prove that all these decompositions are unique up to orthogonal transformation. Since the form q_2^s is invariant under the action of the orthogonal group $\mathrm{O}_2(\mathbb{R})$ (where the field $\mathbb{R}$ of real numbers must be considered for standard coordinates), we can consider the action of the matrix

$$A_\theta = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\theta}{2(s+1)}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\theta}{2(s+1)}} \end{pmatrix}.$$

Then, we observe that

$$\begin{aligned} q_2^s = z_1^{2s} z_2^{2s} &= \sum_{j=1}^{s+1} \left(\frac{r}{2}\right)^{2s} \left(\mathrm{e}^{\mathrm{i}\tau_{j,\theta}} z_1 + \mathrm{e}^{-\mathrm{i}\tau_{j,\theta}} z_2\right)^{2s} \\ &= \sum_{j=1}^{s+1} \left(\frac{r}{2}\right)^{2s} \left(\mathrm{e}^{\mathrm{i}\tau_{j,0}} (A_\theta \cdot z_1) + \mathrm{e}^{-\mathrm{i}\tau_{j,0}} (A_\theta \cdot z_2)\right)^{2s} \\ &= \sum_{j=1}^{s+1} \left(\frac{r}{2}\right)^{2s} \left(\mathrm{e}^{\mathrm{i}\tau_j} z_1 + \mathrm{e}^{-\mathrm{i}\tau_j} z_2\right)^{2s} \\ &= \sum_{j=1}^{s+1} \big(r\cos(\tau_j) x_1 + r\sin(\tau_j) x_2 \big)^{2s}, \end{aligned}$$

where we set $\tau_j = \tau_{j,0}$ for every $j = 1, \ldots, s+1$. □

3.2.2 Complex tight decompositions

It is not so difficult to generalize the results so far obtained to the complex case. In particular, we have to find all the polynomials of minimal degree $s+1$ with distinct roots in the apolar ideal $\left(q_2^s\right)^\perp$. Then we get in the same way all the explicit decompositions of the form q_2^s.

Proposition 3.2.6. *Let $f = au^{s+1} + bv^{s+1} \in \left(q_2^s\right)^\perp$ be a nonzero polynomial such that $a, b \neq 0$. Then f can be written, up to scalars, as*

$$f = u^{s+1} - \mathrm{e}^{\mathrm{i}(\theta + \mathrm{i}k)} v^{s+1},$$

for some $k \in \mathbb{R}$ and some $\theta \in [0, 2\pi)$. Moreover, it has $s+1$ distinct roots corresponding to

$$[y_{1,j} : y_{2,j}] = [\cos(w_{k,\theta,j}) : \sin(w_{k,\theta,j})],$$

where

$$w_{k,\theta,j} = \frac{2(j-1)\pi + \theta + \mathrm{i}k}{2(s+1)},$$

for $j = 1, \ldots, s+1$.

Proof. Since $f \neq 0$, then we can suppose, up to multiplying by a scalar, that

$$f = u^{s+1} - c_0 v^{s+1},$$

where

$$c_0 = -\frac{b}{a} \in \mathbb{C}.$$

Moreover, denoting by $k' \in \mathbb{R}_{>0}$ the norm of c_0, we can write

$$c_0 = k'\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(\theta - \mathrm{i}\log k')} = \mathrm{e}^{\mathrm{i}(\theta+\mathrm{i}k)}$$

for some $\theta \in [0, 2\pi)$, with $k = -\log k' \in \mathbb{R}$. Thus, we get the required form

$$f = u^{s+1} - \mathrm{e}^{\mathrm{i}(\theta+\mathrm{i}k)} v^{s+1}.$$

Now, we proceed exactly as in the proof of Proposition 3.2.4. We observe that $[u_0 : v_0] \in \mathbb{PC}^1$ is a root of f if and only if

$$\left(\frac{u_0}{v_0}\right)^{s+1} = \mathrm{e}^{\mathrm{i}(\theta+\mathrm{i}k)}.$$

Therefore, we can determine $s+1$ distinct roots of f, corresponding to

$$[u_j : v_j] = \left[\mathrm{e}^{\mathrm{i}\frac{2(j-1)\pi+\theta+\mathrm{i}k}{s+1}} : 1\right] = \left[\mathrm{e}^{\mathrm{i}\frac{2(j-1)\pi+\theta+\mathrm{i}k}{2(s+1)}} : \mathrm{e}^{-\mathrm{i}\frac{2(j-1)\pi+\theta+\mathrm{i}k}{2(s+1)}}\right],$$

for $j = 1, \dots, s+1$, which can be written, introducing the term

$$w_{k,\theta,j} = \frac{2(j-1)\pi + \theta + \mathrm{i}k}{2(s+1)}$$

as

$$[u_j : v_j] = \left[\mathrm{e}^{\mathrm{i}w_{k,\theta,j}} : \mathrm{e}^{-\mathrm{i}w_{k,\theta,j}}\right],$$

for every $j = 1, \dots, s+1$. So, rewriting each pair with standard coordinates, we obtain

$$\begin{aligned} y_{1,j} &= u_j + v_j = \mathrm{e}^{\mathrm{i}w_{k,\theta,j}} + \mathrm{e}^{-\mathrm{i}w_{k,\theta,j}} = 2\cos(w_{k,\theta,j}), \\ y_{2,j} &= -\mathrm{i}(u_j - v_j) = -\mathrm{i}\left(\mathrm{e}^{\mathrm{i}w_{k,\theta,j}} + \mathrm{e}^{-\mathrm{i}w_{k,\theta,j}}\right) = 2\sin(w_{k,\theta,j}). \end{aligned}$$

Hence, the roots of the polynomial f are given by

$$[y_{1,j} : y_{2,j}] = [\cos(w_{k,\theta,j}) : \sin(w_{k,\theta,j})],$$

for $j = 1, \dots, s+1$. □

It remains to determine the other decompositions of q_2^s. The proceeding we use is exactly as in the real case.

Theorem 3.2.7. *The form q_2^s has a unique decomposition corresponding, up to complex orthogonal transformations, to the real decomposition whose terms are given by the pairs of opposite vertices of a regular $2s$-gon inscribed in a circle of radius equal to*

$$r = 2(s+1)^{-2s}\binom{2s}{s}^{-2s}.$$

Namely

$$q_2^s = \sum_{j=1}^{s+1} \left(r\cos(\tau_j)x_1 + r\sin(\tau_j)x_2\right)^{2s},$$

where

$$\tau_j = \frac{(j-1)\pi}{s+1}$$

for every $j = 1, \dots, s+1$.

Proof. By Proposition 3.2.6, we can describe every minimal decomposition of q_2^s as a sum of powers of linear forms corresponding to the distinct roots of a polynomial

$$u^{s+1} - \mathrm{e}^{\mathrm{i}(\theta+\mathrm{i}k)} v^{s+1},$$

for some $\theta \in [0, 2\pi)$ and some $k \in \mathbb{R}$. This means that, considering the set of variables $\{z_1, z_2\}$, introduced in the proof of Theorem 3.2.5, and the values

$$w_{k,\theta,j} = \frac{2(j-1)\pi + \theta + \mathrm{i}k}{2(s+1)}$$

for every $j = 1, \dots, s+1$, we can write q_2^s as

$$\begin{aligned}
q_2^s = \left(x_1^2 + x_2^2\right)^s = z_1^{2s} z_2^{2s} &= \sum_{j=1}^{s+1} \left(c_j e^{iw_{k,\theta,j}} z_1 + c_j e^{-iw_{k,\theta,j}} z_2\right)^{2s} \\
&= \sum_{j=1}^{s+1} \left(\frac{r}{2}\right)^{2s} \left(2\left(\frac{\mathrm{e}^{\mathrm{i}w_{k,\theta,j}} + \mathrm{e}^{-\mathrm{i}w_{k,\theta,j}}}{2}\right) x_1 + 2\left(\frac{\mathrm{e}^{\mathrm{i}w_{k,\theta,j}} - \mathrm{e}^{-\mathrm{i}w_{k,\theta,j}}}{2\mathrm{i}}\right) x_2\right)^{2s} \\
&= \sum_{j=1}^{s+1} \left(r\cos(w_{k,\theta,j}) x_1 + r\sin(w_{k,\theta,j}) x_2\right)^{2s},
\end{aligned}$$

for some coefficients $c_1, \dots, c_{s+1} \in \mathbb{R}$. In particular, we observe that

$$q_2^s = z_1^{2s} z_2^{2s} = \sum_{j=1}^{s+1} \left(c_j e^{iw_{k,\theta,j}} z_1 + c_j e^{-iw_{k,\theta,j}} z_2\right)^{2s} = \sum_{j=1}^{s+1} \left(c_j e^{i\tau_j} A_{k,\theta} \cdot z_1 + c_j e^{-i\tau_j} A_{k,\theta} \cdot z_2\right)^{2s}.$$

Therefore, by the invariance of q_2^s under the action of $\mathrm{O}_2(\mathbb{C})$, we conclude by Theorem 3.2.5 that every minimal representation is obtained from the action of a complex orthogonal transformation on the real decomposition

$$q_2^s = \sum_{j=1}^{s+1} \left(r\cos(\tau_j) x_1 + r\sin(\tau_j) x_2\right)^{2s},$$

where

$$r = 2(s+1)^{-2s} \binom{2s}{s}^{-2s}.$$

That is, for every $\theta \in [0, 2\pi)$ and $k \in \mathbb{R}$, we have the equality

$$q_2^s = \sum_{j=1}^{s+1} \left(r\cos(w_{k,\theta,j}) x_1 + r\sin(w_{k,\theta,j}) x_2\right)^{2s}. \qquad \square$$

3.3 General tight decompositions

By the analysis of the central catalecticant matrices we are able to extend, as we have done in the previous section, some of the results obtained for the real case. In particular, for the second and the third power of q_n, we can exclude the existence of tight decompositions for several cases. The strategy consists of trying to determine suitable decompositions by finding the possible points, all with the same norm, contained in the kernel of the central catalecticant map of $q_n^s - B_{n,s} x_1^{2s}$, which must have dimension 1. This is exactly the same strategy adopted by B. Reznick in [Rez92], but thanks to Theorem 3.1.14 we can approach the complex case.

3.3.1 Tight decomposition for exponent $s = 2$

Considering the second power of a quadratic form, our strategy is to subtract a summand from a hypothetical tight decomposition of the form q_n^s, so that the rank of the middle catalecticant of q_n^s decreases by 1. By Theorem 3.1.14, we know that the s-power of the value $\mathbf{a} \cdot \mathbf{a}$ is the same for every point $\mathbf{a} \in \mathbb{C}^n$ of a tight decomposition of q_n^s, namely

$$B_{n,s} = (\mathbf{a} \cdot \mathbf{a})^s = \frac{1}{T_{n,s}} \prod_{j=0}^{s-1} \frac{2j+n}{2j+1} = \frac{s!(n-1)!}{(s+n-1)!} \prod_{j=0}^{s-1} \frac{2j+n}{2j+1}.$$

Therefore, by supposing that a tight decomposition exists, we can consider the form

$$\frac{1}{B_{n,s}} q_n^s,$$

so that every point of the decomposition has norm 1, up to roots of unity. In particular, by the transitivity of the orthogonal group action on the non-isotropic points on the complexified sphere $\mathrm{S}_{\mathbb{C}}^{n-1}$, we can suppose one of them to be the point

$$\mathbf{a}_1 = (1, 0, \ldots, 0).$$

Now, we will use the notation

$$q_n^{[s]} = \frac{1}{2^s s!} q_n^s$$

and we consider the case of exponent $s = 2$. In particular, we have

$$B_{n,2} = \frac{2(n+2)}{3(n+1)}$$

and it follows by Theorem 3.1.14 that, if a tight decomposition of q_n^2 exists, the central catalecticant matrix of the polynomial

$$f_1 = \frac{1}{B_{n,2}} q_n^2 - (\mathbf{a} \cdot \mathbf{x})^4,$$

where $\mathbf{a} \in \mathbb{C}^n$ is such that $\mathbf{a} \cdot \mathbf{a} = 1$, up to roots of unity, must have rank equal to $T_{n,2} - 1$. Moreover, we can determine exactly how the kernel of $\mathrm{Cat}_{f_1}^2$ is made.

Lemma 3.3.1. *Let $n \in \mathbb{N}$ and let*

$$f_1 = \frac{1}{B_{n,2}} q_n^2 - (\mathbf{a} \cdot \mathbf{x})^4,$$

for some $\mathbf{a} \in \mathbb{C}^n$ such that $\mathbf{a} \cdot \mathbf{a} = 1$. Then

$$\mathrm{Ker}\big(\mathrm{Cat}_{f_1}^2\big) = \big\langle (n+2)(\mathbf{a} \cdot \mathbf{y})^2 - q_n \big\rangle.$$

Proof. By Lemma 3.1.11 it is sufficient to prove that the $\mathrm{Ker}\big(\mathrm{Cat}_{f_1}^2\big) \neq 0$. Therefore, using Lemma 1.2.15, formula (2.1.6) and the fact that $\mathbf{a} \cdot \mathbf{a} = 1$, we simply observe that

$$\begin{aligned}
\big((n+2)(\mathbf{a} \cdot \mathbf{y})^2 - q_n\big) \circ f_1 &= \frac{n+2}{B_{n,2}}\big((\mathbf{a} \cdot \mathbf{y})^2 \circ q_n^2\big) - (n+2)\big((\mathbf{a} \cdot \mathbf{y})^2 \circ (\mathbf{a} \cdot \mathbf{x})^4\big) \\
&\quad - \frac{1}{B_{n,2}}\big(q_n \circ q_n^2\big) + \big(q_n \circ (\mathbf{a} \cdot \mathbf{x})^4\big) \\
&= \frac{n+2}{B_{n,2}}\big(4q_n + 8(\mathbf{a} \cdot \mathbf{x})^2\big) - 12(n+2)(\mathbf{a} \cdot \mathbf{x})^2
\end{aligned}$$

$$
\begin{aligned}
&- \frac{4(n+2)}{B_{n,2}} q_n + 12(\mathbf{a}\cdot\mathbf{x})^2 \\
&= \frac{8(n+2)}{B_{n,2}}(\mathbf{a}\cdot\mathbf{x})^2 - 12(n+1)(\mathbf{a}\cdot\mathbf{x})^2 = 0,
\end{aligned}
$$

proving the statement. □

Lemma 3.3.1 guarantees that, assuming the first point to be $\mathbf{a}_1 = (1, 0, \ldots, 0)$, the further points must be roots of the polynomial

$$g_1 = (n+2)y_1^2 - q_n.$$

In particular, this means that, given any other point of the decomposition $\mathbf{a}_2 = (a_{2,1}, \ldots, a_{2,n})$, we must have

$$a_{2,1} = \pm\frac{1}{\sqrt{n+2}}.$$

Moreover, for every $j, k = 1, \ldots, T_{n,2}$ such that $j \neq k$, we have

$$\mathbf{a}_j \cdot \mathbf{a}_k = \pm\frac{1}{\sqrt{n+2}}. \tag{3.3.2}$$

By equation (2.1.4), we can get, using the same strategy adopted by B. Reznick in [Rez92, pp. 130-132], the result of Theorem 3.1.2 for the case of exponent 2, which turns out to be true for complex decompositions as well.

Theorem 3.3.3. *Let $n \geq 3$ and let*

$$\frac{1}{B_{n,2}} q_n^2 = \sum_{j=1}^{T_{n,2}} (\mathbf{a}_j \cdot \mathbf{x})^4$$

be a tight decomposition of q_n^2. Then $n = 3$ or $n = m^2 - 2$ for a suitable odd number $m \in \mathbb{N}$.

Proof. Assuming that a point of the decomposition is

$$\mathbf{a}_1 = (1, 0, \ldots, 0),$$

we can then suppose, by Lemma 3.3.1, that the second point of the decomposition is

$$\mathbf{a}_2 = \left(\frac{1}{\sqrt{n+2}}, \sqrt{\frac{n+1}{n+2}}, 0, \ldots, 0\right).$$

So, given any other point $\mathbf{a}_k$ of the decomposition, which can be written as

$$\mathbf{a}_k = \left(\frac{1}{\sqrt{n+2}}, a_{k,2}, \ldots, a_{k,n}\right),$$

we must have by formula (3.3.2) that

$$\mathbf{a}_2 \cdot \mathbf{a}_k = \frac{1}{n+2} + a_{k,2}\sqrt{\frac{n+1}{n+2}} = \pm\frac{1}{\sqrt{n+2}}$$

and hence

$$a_{k,2} = \frac{-1 \pm \sqrt{n+2}}{\sqrt{(n+1)(n+2)}}$$

for every $k = 3, \dots, T_{n,2}$. Now, we define the natural number $C_1 \in \mathbb{N}$ as the number of elements of the decomposition having the second coordinate equal to

$$\frac{-1-\sqrt{n+2}}{\sqrt{(n+1)(n+2)}}.$$

Consequently, exactly $T_{n,s} - C_1 - 2$ elements must have the second coordinate equal to

$$\frac{-1+\sqrt{n+2}}{\sqrt{(n+1)(n+2)}}.$$

Therefore, developing the form and solving an equation for the coefficients of the monomial x_2^4, we have the equation

$$\frac{(n+1)^2}{(n+2)^2} + (T_{n,s} - C_1 - 2)\frac{\left(-1+\sqrt{n+2}\right)^4}{(n+1)^2(n+2)^2} + C_1\frac{\left(-1-\sqrt{n+2}\right)^4}{(n+1)^2(n+2)^2} = \frac{3(n+1)}{2(n+2)},$$

which we can be written as

$$2(n+1)^4 + 2(T_{n,s} - C_1 - 2)\left(-1+\sqrt{n+2}\right)^4 + 2C_1\left(-1-\sqrt{n+2}\right)^4 = 3(n+1)^3(n+2)$$

and then as

$$-(n+1)^3(n+4) + 2(T_{n,s} - C_1 - 2)\left(-1+\sqrt{n+2}\right)^4 + 2C_1\left(-1-\sqrt{n+2}\right)^4 = 0.$$

Substituting the value

$$T_{n,2} = \binom{n+1}{2},$$

we get

$$-(n+1)^3(n+4) + \left(n(n+1) - 2C_1 - 4\right)\left(-1+\sqrt{n+2}\right)^4 + 2C_1\left(-1-\sqrt{n+2}\right)^4 = 0$$

from which we obtain

$$\begin{aligned}
C_1 &= \frac{-(n+1)^3(n+4) + \left(n^2+n-4\right)\left(-1+\sqrt{n+2}\right)^4}{2\left(-1+\sqrt{n+2}\right)^4 - 2\left(-1-\sqrt{n+2}\right)^4} \\
&= \frac{(n+1)^3(n+4) - \left(n^2+n-4\right)\left((n+2)^2 + 6(n+2) + 1 - 4(n+3)\sqrt{n+2}\right)}{16(n+3)\sqrt{n+2}} \\
&= \frac{(n+1)^3(n+4) - \left(n^2+n-4\right)\left(n^2 + 10n + 17 - 4(n+3)\sqrt{n+2}\right)}{16(n+3)\sqrt{n+2}} \\
&= \frac{-4n^3 - 8n^2 + 36n + 72 + 4\left(n^2+n-4\right)(n+3)\sqrt{n+2}}{16(n+3)\sqrt{n+2}} \\
&= \frac{-n^3 - 2n^2 + 9n + 18 + \left(n^2+n-4\right)(n+3)\sqrt{n+2}}{4(n+3)\sqrt{n+2}} \\
&= \frac{-(n+2)(n-3)(n+3) + \left(n^2+n-4\right)(n+3)\sqrt{n+2}}{4(n+3)\sqrt{n+2}}
\end{aligned}$$

$$= \frac{n^2+n-4-(n-3)\sqrt{n+2}}{4}.$$

Now, since C_1 is a natural value, the equality

$$4C_1 - n^2 - n + 4 = (3-n)\sqrt{n+2}$$

holds if and only if $n = 3$ or $\sqrt{n+2} \in \mathbb{N}$, i.e. $n = m^2 - 2$ for a suitable $m \in \mathbb{N}$. Thus, supposing $n \geq 4$, we determine the value of C_1, obtaining

$$\begin{aligned} C_1 &= \frac{\left(m^2-2\right)^2 + m^2 - 6 - (m^2-5)m}{4} \\ &= \frac{m^4 - 4m^2 - 2 - m(m^2-5)}{4} \\ &= \frac{m^4 - m^3 - 4m^2 + 5m - 2}{4} \\ &= \frac{(m-2)\left(m^3+m^2-2m+1\right)}{4}. \end{aligned}$$

Just substituting modular values, we can check that $m \not\equiv 0 \bmod 4$. We have then proved that there are exactly C_1 points $\mathbf{b}_1, \dots, \mathbf{b}_{C_1}$ such that

$$b_{k,2} = -\frac{m+1}{m\sqrt{m^2-1}}$$

for every $k = 1, \dots, C_1$. Now, again because of the same norm of the points of the decomposition, we can rewrite, after a suitable orthogonal transformation, the first two points as

$$\mathbf{a}_1 = (c_1, c_2, 0, \dots, 0), \quad \mathbf{a}_2 = (c_1, -c_2, 0, \dots, 0).$$

Therefore, since

$$|\mathbf{a}_1| = |\mathbf{a}_2| = \sqrt{c_1^2 + c_2^2} = 1$$

and

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = c_1^2 - c_2^2 = \frac{1}{\sqrt{n+2}} = \frac{1}{m},$$

we easily get

$$c_1^2 = \frac{m+1}{2m}, \quad c_2^2 = \frac{m-1}{2m}$$

and hence

$$\mathbf{a}_1 = \left(\sqrt{\frac{m+1}{2m}}, \sqrt{\frac{m-1}{2m}}, 0, \dots, 0\right), \quad \mathbf{a}_2 = \left(\sqrt{\frac{m+1}{2m}}, -\sqrt{\frac{m-1}{2m}}, 0, \dots, 0\right).$$

By applying Lemma 3.3.1, we observe that the elements in the kernel of the catalecticant of the polynomial

$$\frac{1}{B_{n,2}} q_n^{[2]} - (\mathbf{a}_1 \cdot \mathbf{x})^4 - (\mathbf{a}_2 \cdot \mathbf{x})^4$$

are given by the polynomials

$$(c_1 y_1 \pm c_2 y_2)^2 - \frac{1}{m^2-2} q_n + \frac{2}{m^2\left(m^2-2\right)} q_n,$$

that is
$$\frac{m+1}{2m}y_1^2 \pm \frac{\sqrt{m^2-1}}{m}y_1y_2 + \frac{m-1}{2m}y_2^2 - \frac{1}{m^2}q_n.$$
It is clear that the remaining points must satisfy both the equations. Therefore, since the norm of every point must be equal to one, we obtain for every $k = 3, \dots, T_{n,2}$ that the point
$$\mathbf{a}_k = (a_{k,1}, \dots, a_{k,n})$$
must satisfy the equations of the system
$$\begin{cases} a_{k,1}a_{k,2} = 0, \\ (m+1)a_{k,1}^2 + (m-1)a_{k,2}^2 = \dfrac{2}{m}, \end{cases}$$
that is, we must have
$$a_{k,1} = \sqrt{\frac{2}{m(m+1)}}, \quad a_{k,2} = 0, \tag{3.3.4}$$
or
$$a_{k,1} = 0, \quad a_{k,2} = \sqrt{\frac{2}{m(m-1)}}. \tag{3.3.5}$$
We refer to points satisfying equations (3.3.4) and (3.3.5), respectively, as points of the first and second type. So, considering the equation obtained by equalizing the coefficient of the monomial x_1^4 and denoting by C_1 and C_2 the number of addends respectively of the first and second type, we get
$$\frac{(m+1)^2}{2m^2} + \frac{4C_1}{m^2(m+1)^2} = \frac{3\left(m^2-1\right)}{2m^2}$$
and hence
$$(m+1)^4 + 8C_1 - 3\left(m^2-1\right)(m+1)^2 = 0,$$
that is
$$C_1 = \frac{(m+1)^3(m-2)}{4}.$$
Thus, we also get
$$C_2 = T_{n,2} - C_1 - 2 = \frac{\left(m^2-2\right)\left(m^2-1\right)}{2} - \frac{(m+1)^3(m-2)}{4} - 2 = \frac{(m-1)^3(m+2)}{4}$$
for the elements of the second type. Again up to orthogonal transformations, we can suppose the third point of the decomposition to be
$$\mathbf{a}_3 = \left(\sqrt{\frac{2}{m(m+1)}}, 0, c_3, 0, \dots, 0\right)$$
for some $c_3 \in \mathbb{C}$ and we must have
$$\frac{2}{m(m+1)} + c_3^2 = 1,$$
that is,
$$c_3^2 = 1 - \frac{2}{m(m+1)} = \frac{m^2+m-2}{m(m+1)} = \frac{(m+2)(m-1)}{m(m+1)}.$$

For the successive points of the first type, we must have, instead,

$$\frac{2}{m(m+1)} + a_{4,3}\sqrt{\frac{(m+2)(m-1)}{m(m+1)}} = \pm\frac{1}{m},$$

that is

$$a_{4,3}\sqrt{\frac{(m+2)(m-1)}{m(m+1)}} = \frac{-2 \pm (m+1)}{m(m+1)}$$

and hence

$$a_{4,3} = \frac{-2 \pm (m+1)}{\sqrt{m(m+2)(m^2-1)}}.$$

Now, the summands contributing to the monomial $x_1^2 x_3^2$ are exactly C_1. So, denoting by C_3 the elements with the value

$$\frac{-2-(m+1)}{\sqrt{m(m+2)(m^2-1)}} = -\frac{m+3}{\sqrt{m(m+2)(m^2-1)}}$$

as third coordinate, we get

$$\begin{aligned} &\frac{12(m+2)(m-1)}{m^2(m+1)^2} + 6C_3\frac{2(m+3)^2}{(m-1)m^2(m+1)^2(m+2)} \\ &\quad + 6\left(\frac{(m+1)^3(m-2)}{4} - C_3\right)\frac{2(m-1)^2}{(m-1)m^2(m+1)^2(m+2)} = \frac{3(m^2-1)}{m^2}, \end{aligned}$$

that is,

$$\begin{aligned} &\frac{(m+2)(m-1)}{(m+1)^2} + C_3\frac{(m+3)^2-(m-1)^2}{(m-1)(m+1)^2(m+2)} \\ &\quad + \left(\frac{(m+1)^3(m-2)}{4}\right)\frac{(m-1)^2}{(m-1)(m+1)^2(m+2)} = \frac{m^2-1}{4} \end{aligned}$$

and, just simplifying, we get the equation

$$\frac{(m+2)(m-1)}{(m+1)^2} + C_3\frac{(m+3)^2-(m-1)^2}{(m-1)(m+1)^2(m+2)} + \frac{(m^2-1)(m-2)}{4(m+2)} = \frac{m^2-1}{4}.$$

Proceeding with the resolution, we obtain

$$\frac{(m+2)(m-1)}{(m+1)^2} + C_3\frac{8}{(m-1)(m+1)(m+2)} - \frac{m^2-1}{m+2} = 0$$

and hence

$$\begin{aligned} \frac{8C_3}{(m-1)(m+1)(m+2)} &= \frac{m^2-1}{m+2} - \frac{(m+2)(m-1)}{(m+1)^2} \\ &= \frac{(m-1)(m+1)^3 - (m-1)(m+2)^2}{(m+2)(m+1)^2}, \end{aligned}$$

which corresponds to the equality

$$C_3 = \frac{m(m-1)^2(m+1)^2}{8(m+1)} = \frac{m(m-1)^2(m+1)}{8}.$$

Hence, since C_3 must be an integer value, the modular equation

$$m(m-1)^2(m+1) \equiv 0 \bmod 8$$

must hold. We have already stated that $m \not\equiv 0, 4 \bmod 8$ and hence, if $m \equiv 2, 6 \bmod 8$, we get respectively

$$m(m-1)^2(m+1) \equiv 6 \bmod 8, \quad m(m-1)^2(m+1) \equiv 2 \bmod 8,$$

thus proving that m must be an odd number. □

The problem of establishing for which $n \in \mathbb{N}$ tight decompositions exist is not trivial. We can admire some nice decomposition obtained mainly by classical examples of spherical designs. Apart from the case in two variables, the simplest of these is given by the decomposition

$$q_3^2 = \frac{1}{6} \sum_j^6 \left(x_j \pm \varphi x_{j-1}\right)^4, \tag{3.3.6}$$

where φ is a root of the polynomial $x^2 - x - 1 \in \mathbb{R}[x]$, namely

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

Such decomposition is made by linear forms which geometrically correspond to the vertices of a regular icosahedron, inscribed in a sphere of radius

$$(B_{3,2})^{\frac{1}{4}} = \left(\frac{5}{6}\right)^{\frac{1}{4}},$$

whose coordinates are given by H. S. M. Coxeter in [Cox73]. In particular, we can highlight an essential criterion, classically attributed to J. Haantjes, to obtain the vertices of a regular icosahedron in the three-dimensional space $\mathbb{R}^3$, which represents also an important fact in relation of what we will see later in chapter 5.

Lemma 3.3.7 ([Haa48])**.** *If* 12 *real distinct points* $\mathbf{a}_1, \ldots, \mathbf{a}_{12} \in S^2$ *satisfy the conditions*

$$\mathbf{a}_i \cdot \mathbf{a}_j = \pm \frac{1}{\sqrt{5}}$$

for every $i, j = 1, \ldots, 12$ *with* $i \neq j$*, then they represent the vertices of a regular icosahedron.*

This elegant decomposition of q_3^2, that is represented in Figure 3.2, can be found also in [Rez92, Theorem 9.13], where B. Reznick proves its uniqueness as real decomposition. Considering the language of spherical designs, it was already known (see [DGS77, Example 5.16]), that the vertices of a regular icosahedron represent the unique tight 5-spherical design in $\mathbb{R}^3$. It is not difficult to prove that decomposition (3.3.6) represents also the unique tight decomposition over the field of complex numbers, since the demonstration provided by B. Reznick can be easily extended.

Theorem 3.3.8. *Given* 6 *points* $\mathbf{a}_1, \ldots, \mathbf{a}_6 \in \mathbb{C}^3$*, there is tight decomposition*

$$q_3^2 = \sum_j^6 (a_{j,1}x_1 + a_{j,2}x_2 + a_{j,3}x_3)^4 \tag{3.3.9}$$

if and only if $\mathbf{a}_1, \ldots, \mathbf{a}_6$ *represent the vertices, up to opposite signs, of a regular icosahedron inscribed in a sphere of radius*

$$(B_{3,2})^{\frac{1}{4}} = \left(\frac{5}{6}\right)^{\frac{1}{4}},$$

up to orthogonal complex transformations.

Proof. Using the same argument of Theorem 3.3.3, we have that

$$\mathbf{a}_i \cdot \mathbf{a}_j = \pm\frac{1}{\sqrt{5}}$$

for every $i, j = 1, \dots, 6$ with $i \neq j$ and we can suppose that $|\mathbf{a}_j| = 1$. We can fix, by invariance of the orthogonal group $\mathrm{O}_n(\mathbb{C})$, the first point of the decomposition (3.3.9) as

$$\mathbf{a}_1 = (1, 0, 0),$$

so that

$$a_{j,1} = \frac{1}{\sqrt{5}}$$

for every $j = 2, \dots, 6$. Similarly, we can fix the second point as

$$\mathbf{a}_2 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right),$$

and, consequently, by Lemma 3.3.1 we get that the other points must satisfy the equation

$$5\left(\frac{1}{\sqrt{5}}y_1 + \frac{2}{\sqrt{5}}y_2\right)^2 - q_n = 0.$$

That is,

$$\left(\frac{1}{\sqrt{5}} + 2y_2\right)^2 - 1 = 0$$

and hence

$$5y_2^2 + \sqrt{5}y_2 - 1 = 0,$$

which means that

$$a_{j,2} = \frac{-\sqrt{5} \pm 5}{10}$$

for every $j = 3, \dots, 6$. Finally, we know that decomposition (3.3.6) is first caliber and hence we must have

$$a_{j,1}^2 + a_{j,2}^2 + a_{j,3}^2 = 1.$$

Thus, we get

$$a_{j,3}^2 = \frac{4}{5} - \left(\frac{-\sqrt{5} \pm 5}{10}\right)^2 = \frac{5 \pm \sqrt{5}}{10},$$

that is,

$$a_{j,3} = \pm\sqrt{\frac{5 \pm \sqrt{5}}{10}},$$

for $j = 3, 4, 5, 6$. Since these points are all real, then the condition of Lemma 3.3.7 is satisfied and hence they represent the vertices of a regular icosahedron, up to orthogonal transformations. □

There are also two further tight decompositions representing the successive cases permitted by Theorem 3.3.3, respectively for $n = 7$ and $n = 23$, which are both mentioned as tight spherical designs in [DGS77, p. 371]. The first is given by

$$q_7^2 = \frac{1}{12}\sum_j^{28} (x_j \pm x_{j+1} \pm x_{j+3})^4 \tag{3.3.10}$$

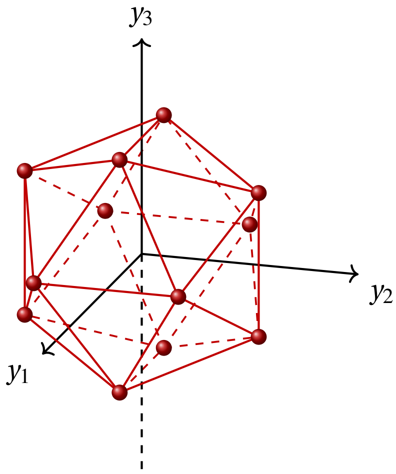

Figure 3.2: Graphical representation of decomposition (3.3.6), whose elements correspond, up to central simmetry, to the vertices of a regular icosahedron.

and represents the maximal set of 28 lines in $\mathbb{R}^7$ having mutual angles equal to the values θ such that

$$\cos^2\theta = \frac{1}{9}.$$

The second one corresponds instead to a set of 276 lines in $\mathbb{R}^{23}$ having mutual angles equal to the values of an angle θ such that

$$\cos^2\theta = \frac{1}{25}.$$

This structure is known as Leech lattice, taking its name by J. Leech, who introduced it in [Lee67]. In particular, decomposition (3.3.10) has been analyzed, as 5-spherical design, also by H. Cuypers in [Cuy05], for which (real) uniqueness is proved.

Another result, always considering real tight spherical designs, is provided by E. Bannai, E. Munemasa, and B. Venkov in [BMV04, Theorem 3.10], where they state that, setting $n = (2m+1)^2 - 2$ for a suitable $m \in \mathbb{N}$, there is no tight spherical design in $\mathbb{R}^n$ in the case of $= 3, 4$ or $m = 2k$ for some $k \in \mathbb{N}$ such that $k \equiv 2 \mod 3$ and both k and $2k+1$ are square-free. This result has been further improved by G. Nebe and B. Venkov in [NV12].

3.3.2 Tight decomposition for exponent $s = 3$

Setting $s = 3$, we can proceed exactly as in the previous case. Supposing the existence of a tight decomposition and considering the form

$$\frac{1}{B_{n,3}} q_n^3,$$

we can take every point of norm 1. So the initial point can be, again,

$$\mathbf{a}_1 = (1, 0, \ldots, 0).$$

We have that

$$B_{n,3} = \frac{6n(n+2)(n+4)}{15n(n+1)(n+2)} = \frac{2(n+4)}{5(n+1)}$$

and, proceeding as in Lemma 3.3.1, we have that the catalecticant of the polynomial

$$f_1 = \frac{1}{B_{n,3}} q_n^3 - (\mathbf{a}\cdot\mathbf{x})^6,$$

for some $\mathbf{a} \in \mathbb{C}^n$ such that $\mathbf{a} \cdot \mathbf{a} = 1$, must have rank equal to $T_{n,3} - 1$.

Lemma 3.3.11. *For every $n \in \mathbb{N}$, the kernel of the polynomial*

$$f_1 = \frac{1}{B_{n,3}} q_n^3 - (\mathbf{a} \cdot \mathbf{x})^6$$

is

$$\operatorname{Ker}\big(\mathrm{Cat}^3_{f_1}\big) = \big\langle (n+4)(\mathbf{a} \cdot \mathbf{y})^3 - 3q_n(\mathbf{a} \cdot \mathbf{y}) \big\rangle.$$

Proof. As in the proof of Lemma 3.3.1, by Lemma 3.1.11 we simply have to prove that the $\operatorname{Ker}\big(\mathrm{Cat}^3_{f_1}\big) \neq 0$. Therefore, using Lemma 1.2.15, formula (2.1.6) and the fact that $\mathbf{a} \cdot \mathbf{a} = 1$, we have

$$\begin{aligned}
\big((n+4)(\mathbf{a} \cdot \mathbf{y})^3 - 3q_n(\mathbf{a} \cdot \mathbf{y})\big) \circ f_1 &= \frac{n+4}{B_{n,3}} \big((\mathbf{a} \cdot \mathbf{y})^3 \circ q_n^3\big) - (n+4)\big((\mathbf{a} \cdot \mathbf{y})^3 \circ (\mathbf{a} \cdot \mathbf{x})^6\big) \\
&\quad - \frac{3}{B_{n,3}} \big(q_n(\mathbf{a} \cdot \mathbf{y}) \circ q_n^3\big) + 3\big(q_n(\mathbf{a} \cdot \mathbf{y}) \circ (\mathbf{a} \cdot \mathbf{x})^6\big) \\
&= \frac{n+4}{B_{n,3}} \big(72 q_n(\mathbf{a} \cdot \mathbf{x}) + 48(\mathbf{a} \cdot \mathbf{x})^3\big) - 120(n+4)(\mathbf{a} \cdot \mathbf{x})^3 \\
&\quad - \frac{72(n+4)}{B_{n,3}} q_n(\mathbf{a} \cdot \mathbf{x}) + 360(\mathbf{a} \cdot \mathbf{x})^3 \\
&= \frac{48(n+4)}{B_{n,3}} (\mathbf{a} \cdot \mathbf{x})^3 - 120(n+1)(\mathbf{a} \cdot \mathbf{x})^3 = 0,
\end{aligned}$$

proving the statement. Proceeding exactly as in the proof of Lemma 3.3.1, we can use Proposition 2.1.11 to write $l^3 = (\mathbf{a} \cdot \mathbf{x})^3$ as

$$l^3 = h + q_n^{[1]}(\mathbf{b} \cdot \mathbf{x})$$

for some $h \in \mathcal{H}_n^3$ and $\mathbf{b} \in \mathbb{C}^n$. In particular, by uniqueness, we have that

$$h = l^3 - q_n^{[1]}(\mathbf{b} \cdot \mathbf{x}) \in \mathcal{H}_n^3,$$

that is, applying the Laplace operator,

$$\Delta\big((\mathbf{a} \cdot \mathbf{x})^3 - q_n^{[1]}(\mathbf{b} \cdot \mathbf{x})\big) = 6(\mathbf{a} \cdot \mathbf{x}) - (n+2)(\mathbf{b} \cdot \mathbf{x}) = 0,$$

namely,

$$(\mathbf{b} \cdot \mathbf{x}) = \frac{6}{n+2} (\mathbf{a} \cdot \mathbf{x}).$$

So, according to Proposition 2.2.10, the element we require is given by the polynomial

$$\begin{aligned}
g_1 &= \left((\mathbf{a} \cdot \mathbf{y})^3 - \frac{6}{n+2} q_n^{[1]}(\mathbf{a} \cdot \mathbf{y})\right) + \frac{6}{(n+2)A_{n,3,1}} q_n(\mathbf{a} \cdot \mathbf{y}) \\
&= (\mathbf{a} \cdot \mathbf{y})^3 - \left(\frac{3}{n+2} - \frac{6}{(n+2)(n+4)}\right) q_n(\mathbf{a} \cdot \mathbf{y}) \\
&= (\mathbf{a} \cdot \mathbf{y})^3 - \left(\frac{3n+12}{(n+2)(n+4)} - \frac{6}{(n+2)(n+4)}\right) q_n(\mathbf{a} \cdot \mathbf{y}) \\
&= (\mathbf{a} \cdot \mathbf{y})^3 - \frac{3}{n+4} q_n(\mathbf{a} \cdot \mathbf{y}),
\end{aligned}$$

which proves the statement. ◻

Now we provide another computation about the values that can be assumed by n to obtain suitable tight decompositions.

Theorem 3.3.12. *Let*

$$\frac{1}{B_{n,3}} q_n^3 = \sum_{j=1}^{T_{n,3}} (\mathbf{a}_j \cdot \mathbf{x})^6$$

be a tight decomposition of q_n^3. Then $n \equiv 2 \bmod 3$.

Proof. Let us consider an hypothetical tight decomposition of q_n^3 and let us suppose that the point

$$\mathbf{a}_1 = (1, 0, \dots, 0)$$

is one point of such decomposition. Then, by Lemma 3.3.11 it follows that, given another point $\mathbf{a}_2$ of this decomposition, we must have

$$a_{2,1} = 0$$

or

$$a_{2,1} = \pm\sqrt{\frac{3}{n+4}}.$$

Supposing that C_1 of these points have $a_{2,1}$ as first coordinate, we can solve the equation

$$\frac{5(n+1)}{2(n+4)} - 1 = C_1 \left(\frac{3}{n+4}\right)^3,$$

that is,

$$\frac{3n-3}{2(n+4)} = C_1 \left(\frac{3}{n+4}\right)^3,$$

obtaining

$$C_1 = \frac{(n-1)(n+4)^2}{18}. \tag{3.3.13}$$

Now, by equality (3.3.13), the elements with the first coordinate equal to 0 are

$$\begin{aligned} C_1' = T_{n,3} - \frac{(n-1)(n+4)^2}{18} - 1 &= \frac{3n(n+1)(n+2) - (n-1)(n+4)^2 - 18}{18} \\ &= \frac{3(n^2+n)(n+2) - (n-1)(n^2+8n+16) - 18}{18} \\ &= \frac{3(n^3+3n^2+2n) - n^3 - 8n^2 - 16n + n^2 + 8n + 16 - 18}{18} \\ &= \frac{2n^3 + 2n^2 - 2n - 2}{18} = \frac{(n-1)(n+1)^2}{9} \end{aligned}$$

and so we must have

$$n \equiv 1, 2, 5, 8 \bmod 9.$$

Since the value C_1' is non-zero for every $n \in \mathbb{N}$, we can suppose that

$$\mathbf{a}_2 = (0, 1, 0, \dots, 0)$$

and, by repeating the same procedure for this last point, we easily see that the second coordinate of each of the other points must have the same possible values assumed by the first coordinate. This means that we must have the same number of points with second coordinate equal to 0. Now, we want to know how many points have the first two coordinates different from zero. In particular, supposing that all the points with non-zero first coordinate have it equal to

$$\sqrt{\frac{3}{n+4}},$$

we must have the coefficient of the monomial $x_1^3 x_2^3$ equal to zero, hence the number of points $\mathbf{a}_k$ such that

$$a_{k,2} = \sqrt{\frac{3}{n+4}}$$

is the same of those having the second coordinate equal to

$$-\sqrt{\frac{3}{n+4}}.$$

Moreover, the coefficient of the monomials $x_1^4 x_2^2$ and $x_1^2 x_2^4$ are

$$\frac{15(n+1)}{2(n+4)} = 15C_2\left(\frac{3}{n+4}\right)^3,$$

that is

$$C_2 = \frac{(n+1)(n+4)^2}{54}.$$

Then, we have

$$n \equiv 2 \bmod 3.$$ □

As in the case of the exponent 2, we can list some cases in which a tight decomposition exist. In particular, for $s = 3$, it exists for $n = 8$ and $n = 23$ and they are presented in [BS81].

CHAPTER 4

General decompositions

As one could have seen through the previous chapters, the structure of the polynomial q_n^s in terms of powers of linear forms, as n and s change, can be quite complicate and it is not so immediate to determine it. Anyway, it is possible to get explicit closed formulas, just depending on n, for some fixed values of the exponent s. One of our main results is related to the forms q_n^2 for $n \in \mathbb{N}$.

In section 4.1 we provide a general survey on this last polynomials, depending on the number n of variables. In particular, after listing some classical and more recent decompositions, we analyze the possible minimal ones. For most of the values of n, indeed, it is possible to determine the exact rank.

In section 4.2 we focus, instead, on the case of three variables. First we introduce a different set of coordinates with respect to the standard ones. After that, we analyze some already known decompositions, whose points represent several polygons set at different heights in three dimensional space and, moreover, we provide new examples presenting the same pattern.

4.1 On the rank of q_n^2

We have already provided an example of a tight decomposition for the exponent 2 in the case of three variables, given by the decomposition (3.3.6), whose points correspond to the vertices of a regular icosahedron. Despite this elegant structure rarely repeats for other values of n, we can see that, at least for every $n \in \mathbb{N}$ such that $n \neq 8$, that rank of q_n^s is at most equal to $T_{n,2} + 1$.

4.1.1 Classical decompositions of q_n^2

There are many examples of decompositions of q_n^2 in the classical literature. We can admire, several simple examples. Fir instance, we can report two of these, due to E. Lucas. The first is given by the equality

$$q_3^2 = \frac{2}{3}\sum\nolimits_j^3 x_j^4 + \frac{1}{12}\sum\nolimits^4 (x_1 \pm x_2 \pm x_3)^4 \tag{4.1.1}$$

and can be found as an exercise in [Hoü77, Question 39, p. 129]. The second one appears in the same article as the preceding exercise, exactly in [Hoü77, Question 38, p. 129], but it had already appeared previously also in [Luc76, p. 101]. It concerns the case of four variables and is given by the equality

$$q_4^2 = \frac{1}{6}\sum\nolimits_{j_1<j_2}^{12} \left(x_{j_1} \pm x_{j_2}\right)^4, \tag{4.1.2}$$

consisting of a decomposition of size 12. Formally, this last one is exactly the same decomposition determined by J. Liouville and exposed by V. A. Lebesgue in [Leb59]. This then corresponds, up to an orthogonal change of coordinates, to

$$q_4^2 = \frac{2}{3}\sum\nolimits_j^4 x_j^4 + \frac{1}{24}\sum\nolimits^8 (x_1 \pm x_2 \pm x_3 \pm x_4)^4. \tag{4.1.3}$$

B. Reznick provides also a formula, which is given in [Rez92, formula (10.35)], which gives us a family of real decompositions for every $n \geq 3$. That is, considering $n \neq 4$, we get

$$q_n^2 = \frac{1}{6}\sum\nolimits_{j_1<j_2}^{\binom{n}{2}} \left(x_{j_1} + x_{j_2}\right)^4 + \frac{1}{6}\sum\nolimits_{j_1<j_2}^{\binom{n}{2}} \left(x_{j_1} - x_{j_2}\right)^4 + \frac{4-n}{3}\sum\nolimits_j^n x_j^4, \tag{4.1.4}$$

which is a decomposition of size n^2. If instead $n = 4$, since the last term of the decomposition becomes equal to zero, we obtain a decomposition of size $4^2 - 4 = 12$, which is exactly the decomposition (4.1.2).

As we can see from all of these decompositions, these examples represent particularly symmetric disposition of points inside the space $\mathbb{R}^n$. The idea emerging from these examples is that the dispositions of points are in a certain sense "balanced" with respect to a center of symmetry. This aspect, as we will see later, is evident in many other decompositions. For instance, besides the perfect symmetry assumed by the points of the icosahedron of decomposition (3.3.6), we can further transform this formula. Indeed, simply by an orthogonal transformation, we can get a disposition of points which are invariant under the action of the permutation group S_3 and symmetrically disposed in the space with respect to the axis identified by the unit vector

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \in \mathbb{R}^3.$$

We will see later that this particular disposition of points can be generalized for all the other values of n.

4.1.2 General decompositions and upper bound for q_n^2

The structure of decomposition (3.3.6) can be generalized to either an arbitrary number or variables $n \geq 3$, distinguishing the cases when n is an odd or an even number. For simplicity, with the symbol x_{n+j}, we will denote in the following theorems the variable x_j, for any $j = 1, \ldots, n$.

Theorem 4.1.5. *For every odd number $n \in \mathbb{N}$ with $n \geq 5$, the form q_n^2 can be described as*

$$6q_n^2 = \sum_{j=1}^{n} \sum_{\substack{1 \leq k \leq \frac{n-1}{2} \\ k \text{ even}}} \left(\varphi_n^{\frac{1}{4}} x_j \pm \varphi_n^{-\frac{1}{4}} x_{j+k}\right)^4 + \sum_{j=1}^{n} \sum_{\substack{1 \leq k \leq \frac{n-1}{2} \\ k \text{ odd}}} \left(\varphi_n^{-\frac{1}{4}} x_j \pm \varphi_n^{\frac{1}{4}} x_{j+k}\right)^4, \tag{4.1.6}$$

which is a decomposition of size $n(n-1)$, where

$$\varphi_n = \frac{3 \pm \mathrm{i}\sqrt{(n-4)(n+2)}}{n-1}.$$

In particular, it is a first caliber decomposition, such that every point **a** *appearing in the summation satisfies the relation*

$$|\mathbf{a}|^4 = \frac{n+2}{3(n-1)}.$$

Proof. By symmetry, we just need to verify the correctness of the coefficients of the monomials x_j^4 and $x_{j_1}^2 x_{j_2}^2$ for $j = 1, \ldots, n$ and $1 \leq j_1 < j_2 \leq n$. In particular, we get that the coefficients of the monomial x_j^4 must be equal to 6 and hence summing such a coefficient for every linear form of the decomposition, we get the equation

$$(n-1)\left(\varphi_n^{-1} + \varphi_n\right) = 6.$$

This means, in particular, that

$$\varphi_n^2 - \frac{6}{n-1}\varphi_n + 1 = 0$$

and hence

$$\varphi_n = \frac{3 \pm \mathrm{i}\sqrt{(n-4)(n+2)}}{n-1}.$$

For the coefficients of $x_{j_1}^2 x_{j_2}^2$, by summing all the powers of the linear forms, we get the value

$$12\varphi_n^{-\frac{1}{2}}\varphi_n^{\frac{1}{2}} = 12,$$

which confirms decomposition (4.1.6). It is clear that such decomposition is first caliber and to get the precise value of the norm raised to $2s$, we simply have to compute

$$\frac{1}{6}\left(\varphi_n^{\frac{1}{2}} + \varphi_n^{-\frac{1}{2}}\right)^2 = \frac{\varphi_n + \varphi_n^{-1} + 2}{6} = \frac{1}{n-1} + \frac{1}{3} = \frac{n+2}{3(n-1)}. \qquad \square$$

We can find the analogous of (4.1.6) for n even, but in this case the decomposition is not first caliber.

Theorem 4.1.7. *For every even number $n \in \mathbb{N}$ with $n \geq 6$, the form q_n^2 can be described as*

$$6q_n^2 = \sum_{j=1}^{n} \sum_{\substack{1\leq k \leq \frac{n-2}{2} \\ k \text{ even}}} \left(\psi_n^{-\frac{1}{4}} x_j \pm \psi_n^{\frac{1}{4}} x_{j+k}\right)^4 + \sum_{j=1}^{n} \sum_{\substack{1\leq k \leq \frac{n-2}{2} \\ k \text{ odd}}} \left(\psi_n^{\frac{1}{4}} x_j \pm \psi_n^{-\frac{1}{4}} x_{j+k}\right)^4 + \sum_{j=1}^{\frac{n}{2}} \left(x_j \pm x_{j+\frac{n}{2}}\right)^4, \qquad (4.1.8)$$

which is a decomposition of size $n(n-1)$, where

$$\psi_n = \frac{3 \pm \mathrm{i}\sqrt{n(n-4)}}{n-2}.$$

Proof. We proceed in the same way as we did in Theorem 4.1.5, by determining first the coefficient of the monomials x_j^4, obtained by solving the equation

$$(n-2)\left(\psi_n + \psi_n^{-1}\right) + 2 = 6,$$

that is,

$$\psi_n^2 - \frac{4}{n-2}\psi_n + 1 = 0,$$

obtaining

$$\psi_n = \frac{3 \pm \mathrm{i}\sqrt{n(n-4)}}{n-2}.$$

For the coefficients of $x_{j_1}^2 x_{j_2}^2$ the equality in both sides of the equations is trivial. $\square$

Despite decomposition (4.1.6) has a quite high size, it proves that, in general, first caliber decompositions can have also complex points as summands. This shows, in particular, that results in [Rez92] about real tight decompositions could be non-valid for the complex ones and hence, some caution is necessary. B. Reznick provides in [Rez92, formula (8.35)] a decomposition of q_n^2 for $3 \leq n \leq 7$, based on a family of integration quadrature formulas of precision 5, exposed by A. H. Stroud in [Str67a], which are essentially real. Besides verifying that these decompositions exist, we can prove that the same formula is valid also for $n \geq 9$, with the only exception, as we will see, of $n = 8$.

In these last cases the decompositions are not real anymore, but the size remains the same, providing summations of size $T_{n,2} + 1$, which clearly are not tight. In particular, these decomposition together with Theorem 3.3.3 provide the exact rank of q_n^2 for many values of n. The set of points forming the decomposition has the special property that it is invariant under the action of the permutation group $\mathfrak{S}_n$ and these points are symmetric with respect to the central axis identified by the vector

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

Theorem 4.1.9. *Let $n \in \mathbb{N}$ be such that $n \geq 3$ and $n \neq 8$. Then the form q_n^2 can be decomposed as*

$$3e^4 q_n^2 = a\Big(\sum\nolimits_j^1 x_j\Big)^4 + \sum\nolimits_k^n \Big(b\Big(\sum\nolimits_j^n x_j\Big) + cx_k\Big)^4 + \sum\nolimits_{k_1,k_2}^{\binom{n}{2}} \Big(d\Big(\sum\nolimits_j^1 x_j\Big) + e\big(x_{k_1} + x_{k_2}\big)\Big)^4, \tag{4.1.10}$$

where, setting $g = (8-n)^{\frac{1}{4}} \in \mathbb{C}$,

$$a = 8(g^4-1)(g^2 \pm 2\sqrt{2}), \quad b = 2g^2 \pm 2\sqrt{2}, \quad c = \mp 2\sqrt{2}g^4 - 8g^2,$$
$$d = 2g, \quad e = \mp 2\sqrt{2}g^3 - 8g.$$

Proof. We begin by solving the equation

$$q_n^2 = a\Big(\sum\nolimits_j^1 x_j\Big)^4 + \sum\nolimits_k^n \Big(b\Big(\sum\nolimits_j^n x_j\Big) + cx_k\Big)^4 + \sum\nolimits_{k_1,k_2}^{\binom{n}{2}} \Big(d\Big(\sum\nolimits_j^1 x_j\Big) + e\big(x_{k_1} + x_{k_2}\big)\Big)^4,$$

that is, determining which values of the coefficients $a, b, c, d, e \in \mathbb{C}$ satisfy the equation. To solve it, we just have to equalize the coefficient of each monomial from both sides of the equation, obtaining in fact a linear system. By explicating the form q_n^2, we obtain

$$q_n^2 = \sum_{j=1}^n x_i^4 + 2\sum_{j_1<j_2} x_{j_1}x_{j_2}$$

and hence, we can expand the sums of powers of the linear forms of the right side of the equation and determine a system in 5 variables. By symmetry, we just have to consider the monomials with different multi-degree. So, for the coefficient of the monomial of the form x_j^4 we have

$$a + (b+c)^4 + (n-1)b^4 + (n-1)(d+e)^4 + \binom{n-1}{2}d^4 = 1,$$

equal to the equation

$$a + nb^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4 + \binom{n}{2}d^4 + (n-1)\big(4d^3e + 6d^2e^2 + 4de^3 + e^4\big) = 1.$$

For the monomial $x_{j_1}^3 x_{j_2}$, we get, instead,

$$a + (b+c)^3b + (b+c)b^3 + (n-2)b^4 + (d+e)^4 + (n-2)(d+e)^3d$$
$$+ (n-2)(d+e)d^3 + \binom{n-2}{2}d^4 = 0,$$

that is,

$$a + nb^4 + 4b^3c + 3b^2c^2 + bc^3 + \binom{n}{2}d^4 + 4(n-1)d^3e + 3nd^2e^2 + (n+2)de^3 + e^4 = 0.$$

From the coefficients of the monomial $x_{j_1}^2 x_{j_2}^2$, we have

$$a + 2b^2(b+c)^2 + (n-2)b^4 + 6(d+e)^4 + 2(n-2)(d+e)^2 d^2 + \binom{n-2}{2} d^4 = \frac{1}{3},$$

from which we get

$$a + nb^4 + 4b^3 c + 2b^2 c^2 + \binom{n}{2} d^4 + 4(n-1)d^3 e + 2(n+1)d^2 e^2 + 4de^3 + e^4 = \frac{1}{3}.$$

Finally, for the coefficients of the monomials of the forms $x_{j_1}^2 x_{j_2} x_{j_3}$ and $x_{j_1} x_{j_2} x_{j_3} x_{j_4}$ we get respectively the equations

$$\begin{aligned} &a + (b+c)^2 b^2 + 2(b+c)b^3 + (n-3)b^4 + 2(d+e)^3 d + (n-2)(d+e)^2 d^2 \\ &+ 2(n-3)(d+e)d^3 + \binom{n-3}{2} d^4 = 0 \end{aligned}$$

and

$$a + 4(b+c)b^3 + (n-4)b^4 + 6(d+e)^2 d^2 + 4(n-4)(d+e)d^3 + \binom{n-4}{2} d^4 = 0,$$

which give the two equations

$$a + nb^4 + 4b^3 c + b^2 c^2 + \binom{n}{2} d^4 + 4(n-1)d^3 e + (n+4)d^2 e^2 + 2de^3 = 0,$$

$$a + nb^4 + 4b^3 c + \binom{n}{2} d^4 + 4(n-1)d^3 e + 6d^2 e^2 = 0.$$

Now, we observe that every equation has a common summand in the first member, namely

$$a + nb^4 + 4b^3 c + \binom{n}{2} d^4 + 4(n-1)d^3 e.$$

Therefore, substituting the last equation, associated to the monomials $x_{j_1} x_{j_2} x_{j_3} x_{j_4}$, to the others, we get

$$a + nb^4 + 4b^3 c + \binom{n}{2} d^4 + 4(n-1)d^3 e = -6d^2 e^2.$$

Then we get the system of 5 equations

$$\begin{cases} 6b^2 c^2 + +6(n-2)d^2 e^2 + 4(n-1)de^3 + (n-1)e^4 + 4bc^3 + c^4 = 1, \\ 3b^2 c^2 + 3(n-2)d^2 e^2 + (n+2)de^3 + e^4 + bc^3 = 0, \\ 2b^2 c^2 + 2(n-2)d^2 e^2 + 4de^3 + e^4 = \dfrac{1}{3}, \\ b^2 c^2 + (n-2)d^2 e^2 + 2de^3 = 0, \\ a + nb^4 + 4b^3 c + \dbinom{n}{2} d^4 + 4(n-1)d^3 e + 6d^2 e^2 = 0. \end{cases}$$

So, we substitute in the same way the fourth equation into the previous ones and we get the value of e^4 and a more simpler system

$$\begin{cases} 4(n-4)de^3 + 4bc^3 + (n-1)\dfrac{1}{3} + c^4 = 1, \\ (n-4)de^3 + bc^3 + \dfrac{1}{3} = 0, \\ e^4 = \dfrac{1}{3}, \\ b^2c^2 + (n-2)d^2e^2 + 2de^3 = 0, \\ a + nb^4 + 4b^3c + \dbinom{n}{2}d^4 + 4(n-1)d^3e + 6d^2e^2 = 0. \end{cases}$$

We can therefore suppose, up to a multiplication by a fourth root of unity, that

$$e = \frac{1}{\sqrt[4]{3}}.$$

Thus, in the same way, summing the second equation to the first and substituting the value of e, we obtain

$$\begin{cases} e = \dfrac{1}{\sqrt[4]{3}}, \\ c^4 = \dfrac{8-n}{3}, \\ \dfrac{n-4}{\sqrt[4]{27}}d + bc^3 + \dfrac{1}{3} = 0, \\ b^2c^2 + \dfrac{n-2}{\sqrt{3}}d^2 + \dfrac{2}{\sqrt[4]{27}}d = 0, \\ a + nb^4 + 4b^3c + \dbinom{n}{2}d^4 + \dfrac{4(n-1)}{\sqrt[4]{3}}d^3 + \dfrac{6}{\sqrt{3}}d^2 = 0. \end{cases}$$

Since by hypothesis we have $n \neq 8$, we have also $c \neq 0$, and hence, introducing the constant value

$$g = (8-n)^{\frac{1}{4}},$$

we get

$$c = \frac{g}{\sqrt[4]{3}}, \quad e = \frac{1}{\sqrt[4]{3}}.$$

It is evident that, if $n \geq 9$, there is no decomposition of q_n^2 with real coefficients maintaining this pattern for the coefficients. Hence, the initial system is now reduced to another one made by 3 equations, which after the suitable substitutions can be written as

$$\begin{cases} g^3b + (4-g^4)d + \dfrac{1}{\sqrt[4]{3}} = 0, \\ g^2b^2 + (6-g^4)d^2 + \dfrac{2}{\sqrt[4]{3}}d = 0, \\ 2a + 2(8-g^4)b^4 + \dfrac{8g}{\sqrt[4]{3}}b^3 + (8-g^4)(7-g^4)d^4 + \dfrac{8}{\sqrt[4]{3}}(7-g^4)d^3 + 4\sqrt{3}d^2 = 0. \end{cases}$$

From the first equation, we get

$$b = \frac{\sqrt[4]{3}(g^4 - 4)d - 1}{\sqrt[4]{3}g^3}$$

and, substituting this value in the second equation, we get

$$\sqrt{3}(g^4 - 4)^2 d^2 - 2\sqrt[4]{3}(g^4 - 4)d + 1 + \sqrt{3}g^4(6 - g^4)d^2 + 2\sqrt[4]{3}g^4 d = 0,$$

that is,

$$2\sqrt{3}(8 - g^4)d^2 + 8\sqrt[4]{3}d + 1 = 0,$$

obtaining

$$d = \frac{-4\sqrt[4]{3} \pm \sqrt{2}\sqrt[4]{3}g^2}{2\sqrt{3}(8 - g^4)}.$$

Therefore, we have

$$b = \frac{2g^2 \pm 2\sqrt{2}}{\sqrt[4]{3}g(-8 \mp 2\sqrt{2}g^2)}, \quad d = \frac{2g}{\sqrt[4]{3}g(-8 \mp 2\sqrt{2}g^2)}.$$

It remains to determine the possible values assumed by a. For simplicity, we can re-scale the values of the coefficients, multiplying both sides of the initial equation by the constant term

$$M = 3(\mp 2\sqrt{2}g^3 - 8g)^4.$$

Thus, redefining all the parameters, we obtain

$$b = 2g^2 \pm 2\sqrt{2}, \quad c = \mp 2\sqrt{2}g^4 - 8g^2, \quad d = 2g, \quad e = \mp 2\sqrt{2}g^3 - 8g,$$

and, in particular, $M = 3e^4$. Now, considering the last equation of the initial system, we see that the value of a is

$$a = \Big((g^4 - 8)b - 4c\Big)b^3 - \left(\frac{(g^4 - 8)(g^4 - 7)}{2}d^2 - 4(g^4 - 7)de + 6e^2\right)d^2.$$

Thus, it remains to substitute the considered values on each term. We first observe that

$$b^3 = 8g^6 \pm 24\sqrt{2}g^4 + 48g^2 \pm 16\sqrt{2}$$

and

$$(g^4 - 8)b - 4c = 2g^6 \pm 10\sqrt{2}g^4 + 16g^2 \mp 16\sqrt{2}.$$

After some computations, we get

$$\Big((g^4 - 8)b - 4c\Big)b^3 = 16(g^{12} \pm 8\sqrt{2}g^{10} + 44g^8 \pm 48\sqrt{2}g^6 + 20g^4 \mp 32\sqrt{2}g^2 - 32).$$

In the same way, observing that

$$d^2 = 4g^2, \quad de = \mp 4\sqrt{2}g^4 - 16g^2, \quad e^2 = 8g^6 \pm 32\sqrt{2}g^4 + 64g^2$$

we can make other computations and obtain

$$\left(\frac{(g^4 - 8)(g^4 - 7)}{2}d^2 - 4(g^4 - 7)de + 6e^2\right)d^2 = 8(g^{12} \pm 8\sqrt{2}g^{10} + 41g^8 \pm 40\sqrt{2}g^6 + 24g^4).$$

Finally, we simply have to substitute this elements to get the values of a, that is

$$\begin{aligned} a &= 8(g^{12} \pm 8\sqrt{2}g^{10} + 47g^8 \pm 56\sqrt{2}g^6 + 16g^4 \mp 64\sqrt{2}g^2 - 64) \\ &= 8(g^4 - 1)(g^8 \pm 8\sqrt{2}g^6 + 48g^4 \pm 64\sqrt{2}g^2 + 64) \\ &= 8(g^4 - 1)(g^2 \pm 2\sqrt{2})^4, \end{aligned}$$

confirming the required decomposition. □

By Theorem 3.3.3, Theorem 4.1.9, decomposition (4.1.8) and the tight decompositions for $n = 7, 23$ mentioned in [DGS77, p. 371], we can summarize the results concerning the rank of q_n^2.

Theorem 4.1.11. *Let $n \geq 3$. Then the following conditions hold:*
(1) if $n = 3, 7, 23$, then $\mathrm{rk}(q_n^2) = T_{n,2}$;
(2) if $n > 23$ and $n = m^2 - 2$ for some odd number $m \in \mathbb{N}$, then $T_{n,2} \leq \mathrm{rk}(q_n^2) \leq T_{n,2} + 1$;
(3) if $n = 8$, then $56 \geq \mathrm{rk}(q_n^2) \geq 37$;
(4) otherwise, $\mathrm{rk}(q_n^2) = T_{n,2} + 1$.

4.2 Decompositions in three variables

The decompositions presented in the previous section are not the only examples of closed formulas which are valid for multiple values of n. In particular, we can find several decompositions also for higher exponents. Again, we can find some easier decompositions of the form q_n^3 in the classical literature. Probably one of the less recent among these is the one for the case of 4 variables and having size 24, presented by A. Kempner in [Kem12, Section 5] and equal to

$$q_4^3 = \frac{8}{15} \sum_j^4 x_j^6 + \frac{1}{15} \sum_{j_1<j_2}^{12} (x_{j_1} \pm x_{j_2})^6 + \frac{1}{120} \sum^8 (x_1 \pm x_2 \pm x_3 \pm x_4)^6. \tag{4.2.1}$$

This last decomposition is a particular case for the set of decompositions which can be obtained by the family of quadrature formulas provided by A. H. Stroud in [Str67b] and that have been exposed by B. Reznick in [Rez92, formula (8.33)]. These are given, for every $n \geq 3$ such that $n \neq 8$, by the formula

$$q_n^3 = \frac{2(8-n)}{15} \sum_j^n x_j^6 + \frac{1}{15} \sum_{j_1<j_2}^{n(n-1)} (x_{j_1} \pm x_{j_2})^6 + \frac{1}{15 \cdot 2^{n-1}} \sum^{2^{n-1}} (x_1 \pm \cdots \pm x_n)^6. \tag{4.2.2}$$

Another formula providing decompositions as functions of n is instead given by J. Buczyński, K. Han, M. Mella, and Z. Teitler in [BHMT18, Section 4.5], where they provide a decomposition of q_n^3 of size

$$4\binom{n}{3} + 2\binom{n}{2} + n,$$

given by the equation

$$60q_n^3 = \sum_{j_1<j_2<j_3}^{4\binom{n}{3}} (x_{j_1} \pm x_{j_2} \pm x_{j_3})^6 + 2(5-n) \sum_{j_1<j_2}^{2\binom{n}{2}} (x_{j_1} \pm x_{j_2})^6 + 2(n^2 - 9n + 38) \sum_j^n x_j^6. \tag{4.2.3}$$

If the case of $n = 3$ B. Reznick provides also two minimal decompositions for the exponents 3 and 4. We can see in [Rez92, Theorem 9.28] that the form q_3^3 can be described by a decomposition of size 11 with real coefficients, given by

$$q_3^3 = \frac{14}{27} x_1^6 + \frac{7}{10} \sum_{j\neq 1}^2 x_j^6 + \frac{1}{540} \sum_{j\neq 1}^4 (2x_1 \pm \sqrt{3}x_j)^6 + \frac{1}{540} \sum^4 (x_1 \pm \sqrt{3}x_2 \pm \sqrt{3}x_3)^6. \tag{4.2.4}$$

With the same strategy of Theorem 3.3.12, B. Reznick proves that this is a minimal decomposition and, in particular, we will prove that it is not unique.

Besides decomposition (4.2.4), B. Reznick analyzes also a minimal decomposition for the successive exponent. Indeed, he proves in [Rez92, formula (8.31)] that the form q_3^4 can be decomposed as

$$140q_3^4 = 3\varphi^{-4} \sum_j^6 (x_j \pm \varphi x_{j-1})^8 + \sum_j^6 (\varphi x_j \pm \varphi^{-1} x_{j-1})^8 + \sum^4 (x_1 \pm x_2 \pm x_3)^8, \tag{4.2.5}$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

Decomposition (4.2.5) consists of a summation of 16 points, corresponding, up to symmetries, to the vertices of a regular icosahedron together with the vertices of a regular dodecahedron (see Figure 4.1)

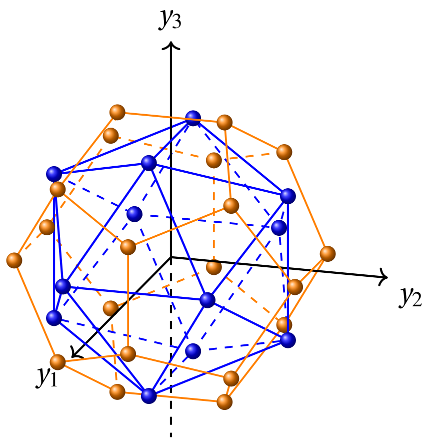

Figure 4.1: Graphical representation of decomposition (4.2.5), whose elements correspond, up to central simmetry, to the vertices of a regular icosahedron (in blue) togheter with the vertices of a regular dodecahedron (in orange).

4.2.1 Set of coordinates and irreducible representations

To analyze the case of three variables, it can be suitable to use another set of coordinates, carrying out a change of basis for the space $\mathcal{D} = \mathbb{C}[y_1, y_2, y_3]$. Indeed, setting

$$u = \frac{y_1 + \mathrm{i}y_2}{2}, \quad v = \frac{y_1 - \mathrm{i}y_2}{2}, \quad z = y_3, \tag{4.2.6}$$

and thus considering the space $\mathcal{D} = \mathbb{C}[u, v, z]$, we can consider the inverse relations

$$y_1 = u + v, \quad y_2 = -\mathrm{i}(u - v), \quad y_3 = z.$$

Furthermore, we can determine the partial derivatives with respect to this new set of coordinates, that is

$$\begin{aligned} \frac{\partial}{\partial u} &= \frac{\partial}{\partial y_1} - \mathrm{i}\frac{\partial}{\partial y_2}, \\ \frac{\partial}{\partial v} &= \frac{\partial}{\partial y_1} + \mathrm{i}\frac{\partial}{\partial y_2}, \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial y_3}. \end{aligned}$$

Consequently, the Laplace operator can be rewritten as

$$\Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} = \frac{\partial^2}{\partial u\, \partial v} + \frac{\partial^2}{\partial z^2}.$$

We have already recalled that the space of harmonic polynomials $\mathcal{H}_n^d$ of degree d is an irreducible $\mathrm{SO}_n(\mathbb{C})$-module for every $d \in \mathbb{N}$ ([GW98, Theorem 5.2.4]). The Lie algebra of the Lie group $\mathrm{SO}_3(\mathbb{C})$ is the space

$$\mathfrak{so}_3\mathbb{C} = \left\{ A \in \mathrm{Mat}_3(\mathbb{C}) \;\middle|\; A = -{}^{\mathrm{t}}A \right\}$$

and, if we consider the three matrices associated to the morphisms $H, E, F \colon \mathbb{C}^3 \to \mathbb{C}^3$ with respect to the canonical basis $\{y_1, y_2, y_3\}$, given by

$$\begin{cases} H(y_1) = 2\mathrm{i}y_2, \\ H(y_2) = -2\mathrm{i}y_1, \\ H(y_3) = 0, \end{cases} \qquad H = \begin{pmatrix} 0 & -2\mathrm{i} & 0 \\ 2\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{cases} E(y_1) = y_3, \\ E(y_2) = \mathrm{i}y_3, \\ E(y_3) = -(y_1 + \mathrm{i}y_2), \end{cases} \qquad E = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -\mathrm{i} \\ 1 & \mathrm{i} & 0 \end{pmatrix},$$

$$\begin{cases} F(y_1) = -y_3, \\ F(y_2) = \mathrm{i}y_3, \\ F(y_3) = y_1 - \mathrm{i}y_2, \end{cases} \qquad F = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\mathrm{i} \\ -1 & \mathrm{i} & 0 \end{pmatrix},$$

then we obtain the equations

$$[H, E] = HE - EH = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2\mathrm{i} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -2\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2\mathrm{i} \\ 2 & 2\mathrm{i} & 0 \end{pmatrix} = 2E,$$

$$[H, F] = HF - FH = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2\mathrm{i} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 2\mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2\mathrm{i} \\ 2 & -2\mathrm{i} & 0 \end{pmatrix} = -2F,$$

$$[E, F] = EF - FE = \begin{pmatrix} 1 & -\mathrm{i} & 0 \\ \mathrm{i} & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & \mathrm{i} & 0 \\ -\mathrm{i} & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2\mathrm{i} & 0 \\ 2\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = H.$$

Since these conditions correspond exactly to equations (1.1.28), we have, in particular, that $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$, justifying the use of the same notations.

Now, by rewriting the matrices H, E and F with respect to the new basis $\{u, v, z\}$, clearly, we do not obtain skew-symmetric matrices, but new ones presenting a quite simpler structure. Indeed, we have

$$\begin{cases} H(u) = 2u, \\ H(v) = -2v, \\ H(z) = 0, \end{cases} \qquad H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{cases} E(u) = 0, \\ E(v) = z, \\ E(z) = -2u, \end{cases} \qquad E = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\begin{cases} F(u) = -z, \\ F(v) = 0, \\ F(z) = 2v, \end{cases} \qquad F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}.$$

Moreover, it follows by the uniqueness of irreducible representations of $\mathfrak{sl}_2\mathbb{C}$ that for every $d \in \mathbb{N}$ the space $\mathcal{H}_3^d$ of harmonic polynomials of degree d in three variable is canonically isomorphic to the $2d$-th symmetric power of the standard representation $\mathbb{C}^2$, that is $\mathcal{H}_3^d \cong S^{2d}\mathbb{C}^2$. We can see it in the following proposition.

Proposition 4.2.7. *For every $d \in \mathbb{N}$, the set*

$$\mathcal{B}_d = \{h_{d,k}\}_{k=-d,\dots,d}$$

composed by the harmonic polynomials

$$h_{d,k} = \binom{2d}{k+d}^{-1} \sum_{j=0}^{\left\lfloor \frac{d-|k|}{2} \right\rfloor} (-1)^{\frac{|k|+k}{2}+j} u^{\left[\frac{|k|+k}{2}+j\right]} v^{\left[\frac{|k|-k}{2}+j\right]} z^{[d-|k|-2j]}, \tag{4.2.8}$$

for $k = -d, \dots, d$, is a basis of the space $\mathcal{H}_3^d$, which corresponds in a canonical way, up to scalars, to the standard basis of $S^{2d}\mathbb{C}^2$, that is the set of monomials $\{x^{2d}, x^{2d-1}y, \dots, xy^{2d-1}, y^{2d}\}$. In particular, for every $k = -d, \dots, d$

$$H(h_{d,k}) = 2k \cdot h_{d,k}$$

and for every $s \in \mathbb{N}$ such that $k - d \leq s \leq d - k$,

$$E^s(h_{d,k}) \in \langle h_{d,k+s} \rangle, \quad F^s(h_{d,k}) \in \langle h_{d,k-s} \rangle.$$

Proof. We need to prove first that for every $k \in \mathbb{N}$ the polynomial $h_{d,k}$ is an eigenvector of the linear map represented by H with eigenvalue $2k$. So, we have

$$H(h_{d,k}) = \binom{2d}{k+d}^{-1} H\left(\sum_{j=0}^{\left\lfloor \frac{d-|k|}{2} \right\rfloor} (-1)^{\frac{|k|+k}{2}+j} u^{\left[\frac{|k|+k}{2}+j\right]} v^{\left[\frac{|k|-k}{2}+j\right]} z^{[d-|k|-2j]} \right)$$

and in the case of $k \geq 0$ we get

$$\begin{aligned}
H(h_{d,k}) &= \binom{2d}{k+d}^{-1} H\left(\sum_{j=0}^{\left\lfloor \frac{d-k}{2} \right\rfloor} (-1)^{k+j} u^{[k+j]} v^{[j]} z^{[d-k-2j]} \right) \\
&= \binom{2n}{k+d}^{-1} \sum_{j=0}^{\left\lfloor \frac{d-k}{2} \right\rfloor} (-1)^{k+j} \frac{H(u^{k+j}) v^j z^{d-k-2j} + u^{k+j} H(v^j) z^{d-k-2j}}{j!(k+j)!(d-k-2j)!} \\
&= \binom{2d}{k+d}^{-1} \sum_{j=0}^{\left\lfloor \frac{d-k}{2} \right\rfloor} (-1)^{k+j} \frac{2(k+j)(u^{k+j} v^j z^{d-k-2j}) - 2j(u^{k+j} v^j z^{d-k-2j})}{j!(k+j)!(d-k-2j)!} \\
&= 2k \binom{2d}{k+d}^{-1} \sum_{j=0}^{\left\lfloor \frac{d-k}{2} \right\rfloor} (-1)^{k+j} u^{[k+j]} v^{[j]} z^{[d-k-2j]} = 2k \cdot h_{d,k}.
\end{aligned}$$

The case of $k < 0$ is obtained in the same way by simply exchanging the roles of u and v. Now, to get the statement, since for the space $S^{2d}\mathbb{C}^2$ we have

$$E(x^{2d}) = 0, \quad E(x^{d+k} y^{d-k}) = (d-k) x^{d+k+1} y^{d-k-1},$$

for every $k = -d, \dots, d-1$, and

$$F(y^{2d}) = 0, \quad F(x^{d+k} y^{d-k}) = (d+k) x^{d+k-1} y^{d-k+1},$$

for every $k = -d+1, \dots, d$, we only have to prove that

$$E(h_{d,d}) = 0, \quad E(h_{d,k}) = (d-k) h_{d,k+1},$$

for every $k = -d, \dots, d-1$, and

$$F(h_{d,-d}) = 0, \quad F(h_{d,k}) = (d+k) h_{d,k-1},$$

for every $k = -d+1, \dots, d$. We can write

$$E(h_{d,k}) = \binom{2d}{k+d}^{-1} E\left(\sum_{j=0}^{\lfloor \frac{d-k}{2} \rfloor} (-1)^{k+j} u^{[k+j]} v^{[j]} z^{[d-k-2j]}\right)$$

and it is quite convenient to analyze separately the different cases in which we have respectively $d-k$ even or odd, due to the presence of an extra term. Again, we only consider the case of $k \geq 0$. In the first case, if we set $d-k = 2t$ for some $t \in \mathbb{N}$, we get

$$\begin{aligned}
E(h_{d,k}) &= \binom{2d}{2d-2t}^{-1} \sum_{j=0}^{t} (-1)^{d-2t+j} \frac{u^{d-2t+j} E(v^j) z^{2t-2j} + u^{d-2t+j} v^j E(z^{2t-2j})}{j!(d-2t+j)!(2t-2j)!} \\
&= \binom{2d}{2d-2t}^{-1} \sum_{j=1}^{t} (-1)^{d-2t+j} \frac{u^{d-2t+j} v^{j-1} z^{2t-2j+1}}{(d-2t+j)!(j-1)!(2t-2j)!)} \\
&\quad - 2\sum_{j=0}^{t-1} (-1)^{d-2t+j} \frac{u^{d-2t+j+1} v^j z^{2t-2j-1}}{(d-2t+j)!j!(2t-2j-1)}.
\end{aligned}$$

by which, setting in the first summation $j' = j-1$, we get

$$\begin{aligned}
E(h_{d,k}) &= \binom{2d}{2d-2t}^{-1} \sum_{j'=0}^{t-1} (-1)^{d-2t+1+j'} \frac{u^{d-2t+1+j'} v^{j'} z^{2t-2j'-1}}{(d-2t+1+j')!j'!(2t-2j'-2)!)} \\
&\quad + 2\sum_{j=0}^{t-1} (-1)^{d-2t+1+j} \frac{u^{d-2t+j+1} v^j z^{2t-2j-1}}{(d-2t+j)!j!(2t-2j-1)!} \\
&= \frac{(2d-2t)!2t!}{(2n)!} (2d-2t+1) \sum_{j=0}^{t-1} (-1)^{d-2t+1+j} u^{[d-2t+j+1]} v^{[j]} z^{[2t-2j-1]} \\
&= 2t \frac{(2d-2t+1)!(2t-1)!}{(2d)!} \sum_{j=0}^{t-1} (-1)^{d-2t+1+j} u^{[d-2t+j+1]} v^{[j]} z^{[2t-2j-1]}.
\end{aligned}$$

Then we finally obtain, considering $k = d-2t$,

$$\begin{aligned}
E(h_{d,k}) &= (d-k) \binom{2d}{d+k+1} \sum_{j=0}^{\lfloor \frac{d-k-1}{2} \rfloor} (-1)^{d-2t+1+j} u^{[k+1+j]} v^{[j]} z^{[d-k-1-2j]} \\
&= (d-k) h_{d,k+1}.
\end{aligned}$$

The case of $k = d$ is trivial, as we have

$$E(u^{[d]}) = u^{[d-1]} E(u) = 0.$$

If we set, instead, $d-k = 2t+1$ for some $t \in \mathbb{N}$ we get something similar, that is

$$E(h_{d,k}) = \binom{2d}{2d-2t-1}^{-1} \sum_{j=0}^{t} (-1)^{d-2t+j-1} \frac{u^{d-2t+j-1} E(v^j) z^{2t-2j+1} + u^{d-2t+j-1} v^j E(z^{2t-2j+1})}{j!(d-2t+j-1)!(2t-2j+1)!}$$

$$= \binom{2d}{2d-2t-1}^{-1} \sum_{j=1}^{t} (-1)^{d-2t+j-1} \frac{u^{d-2t+j-1} v^{j-1} z^{2t-2j+2}}{(d-2t+j-1)!(j-1)!(2t-2j+1)!}$$

$$- 2 \sum_{j=0}^{t-1} (-1)^{d-2t+j-1} \frac{u^{d-2t+j} v^{j} z^{2t-2j}}{(d-2t+j-1)!j!(2t-2j)!}.$$

Then, setting $j' = j-1$ in the first summation, as we have done above for the previous case, we get

$$E(h_{d,k}) = \binom{2d}{2d-2t-1}^{-1} \sum_{j'=0}^{t-1} (-1)^{d-2t+j'} \frac{u^{d-2t+j'} v^{j'} z^{2t-2j'}}{(d-2t+j')!j'!(2t-2j'-1)!)}$$

$$+ 2 \sum_{j=0}^{t-1} (-1)^{d-2t+j} \frac{u^{d-2t+j} v^{j} z^{2t-2j}}{(d-2t+j-1)!j!(2t-2j)!}$$

$$= \frac{(2d-2t-1)!(2t+1)!}{(2d)!} (2d-2t) \sum_{j=0}^{t-1} (-1)^{d-2t+j} u^{[d-2t+j]} v^{[j]} z^{[2t-2j]}$$

$$= (2t+1) \frac{(2d-2t)!(2t)!}{(2d)!} \sum_{j=0}^{t-1} (-1)^{d-2t+j} u^{[d-2t+j]} v^{[j]} z^{[2t-2j]}.$$

Finally, setting $k = d-2t-1$, we have

$$E(h_{d,k}) = (d-k) \binom{2d}{d+k+1} \sum_{j=0}^{\lfloor \frac{d-k-1}{2} \rfloor} (-1)^{k+j+1} u^{[k+1+j]} v^{[j]} z^{[d-k-2j-1]}$$

$$= (d-k) h_{d,k+1}.$$

The statement for the operator F is obtained by exchanging the roles of u with $-v$. $\square$

The spaces generated by the basis of harmonic polynomials defined in formulas (4.2.8), corresponding to the weights of representations of $\mathfrak{so}_3(\mathbb{C})$, can be visualized in Figure 4.2.

$$\underset{h_{2,2}}{\left\langle \frac{1}{2}u^2 \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{2,1}}{\left\langle \frac{1}{4}uz \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{2,0}}{\left\langle \frac{1}{12}(z^2-2uv) \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{2,-1}}{\left\langle \frac{1}{4}vz \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{2,-2}}{\left\langle \frac{1}{2}v^2 \right\rangle}$$

$$\underset{h_{3,3}}{\left\langle \frac{1}{6}u^3 \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{3,2}}{\left\langle \frac{1}{12}u^2z \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{3,1}}{\left\langle \frac{1}{30}u(z^2-uv) \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{3,0}}{\left\langle \frac{1}{120}z(z^2-6uv) \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{3,-1}}{\left\langle \frac{1}{30}v(z^2-uv) \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{3,-2}}{\left\langle \frac{1}{12}v^2z \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{3,-3}}{\left\langle \frac{1}{6}v^3 \right\rangle}$$

$$\vdots$$

$$\underset{h_{n,n}}{\left\langle \frac{1}{d!}u^d \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{d,d-1}}{\left\langle \frac{1}{2d(d-1)!}u^{d-1}z \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \cdots \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{d,-(d-1)}}{\left\langle \frac{1}{2d(d-1)!}v^{d-1}z \right\rangle} \underset{E}{\overset{F}{\rightleftarrows}} \underset{h_{d,-d}}{\left\langle \frac{1}{d!}v^d \right\rangle}.$$

Figure 4.2: Diagram representing the weights of the space of harmonic polynomials defined by formulas (4.2.8).

4.2.2 Decompositions and regular polygons

We have already seen that the minimal decomposition of q_3^2 given by formula (3.3.6) is represented by a configuration of points forming a regular icosahedron in the three-dimensional space. This particular disposition represents a case that, at least for real points, cannot be repeated among the higher powers of q_n. Anyway, we can determine some decompositions which seems to introduce a new pattern of points which suggests a possible repetition of what happens in decomposition (3.3.6) also for the successive cases. This new way of analyzing the decomposition is based on considering the points, just for usefulness, with the new set of coordinates given by relations (4.2.6) and with the new basis given by the elements of the equations (4.2.8). For the next cases, the use of the software system Macaulay2 ([GS]) has been essential in determining decompositions and their associated ideal of points.

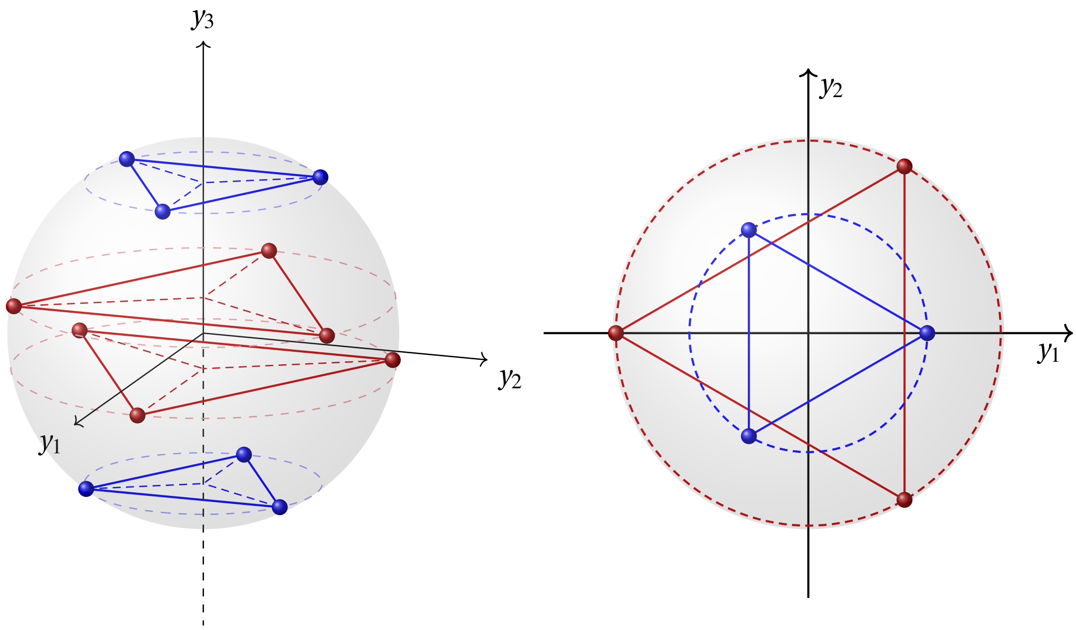

Figure 4.3: Graphical representation of decomposition (4.2.9) in standard coordinates, whose elements correspond, up to central simmetry, to the vertices of two triangles, respectively in red and blue, placed at different heights.

We start by considering decomposition (3.3.6), which, as already said above, results to be made, after a suitable rotation, by points forming two different triangles at different height in the space (see Figure 4.3). Introducing the notation τ_j to denote the j-th root of unity equal to

$$\tau_j = \mathrm{e}^{\frac{2\pi \mathrm{i}}{j}},$$

for every $j \in \mathbb{N}$, we can therefore rewrite decomposition (3.3.6) as

$$54q_3^2 = \sum_j^3 (3-\varphi)\left(\varphi z + \tau_6^{2(j-1)} u + \tau_6^{-2(j-1)} v\right)^4 + \sum_j^3 \varphi\left((3-\varphi) z + \tau_6^{2j-1} u + \tau_6^{-(2j-1)} v\right)^4, \quad (4.2.9)$$

where

$$\varphi = \frac{3+\sqrt{5}}{2}.$$

In particular, the ideal of the points forming such decomposition presents a quite simple set of generators, made by four polynomials symmetric with respect to the variables u and v, that is,

$$\mathcal{I}_{(4.2.9)} = \begin{pmatrix} h_{3,0} - \sqrt{5} h_{3,3}, & h_{3,1} - \sqrt{5} h_{3,-2}, \\ h_{3,0} - \sqrt{5} h_{3,-3}, & h_{3,-1} - \sqrt{5} h_{3,2} \end{pmatrix}, \quad (4.2.10)$$

which effectively presents a symmetric structure with respect to the basis of harmonic polynomials we determine.

If we look now at decomposition (4.2.4), we can observe that the disposition of the points presents some analogies with the disposition of the points of decomposition (4.2.9). Indeed, we observe that the point related to decomposition (4.2.4) are disposed over three square placed at different heights, symmetrically with respect to the central axis y_3. Since one of the squares is placed at height 0, just two points of it must be considered (see Figure 4.4). The decomposition written in new coordinates takes

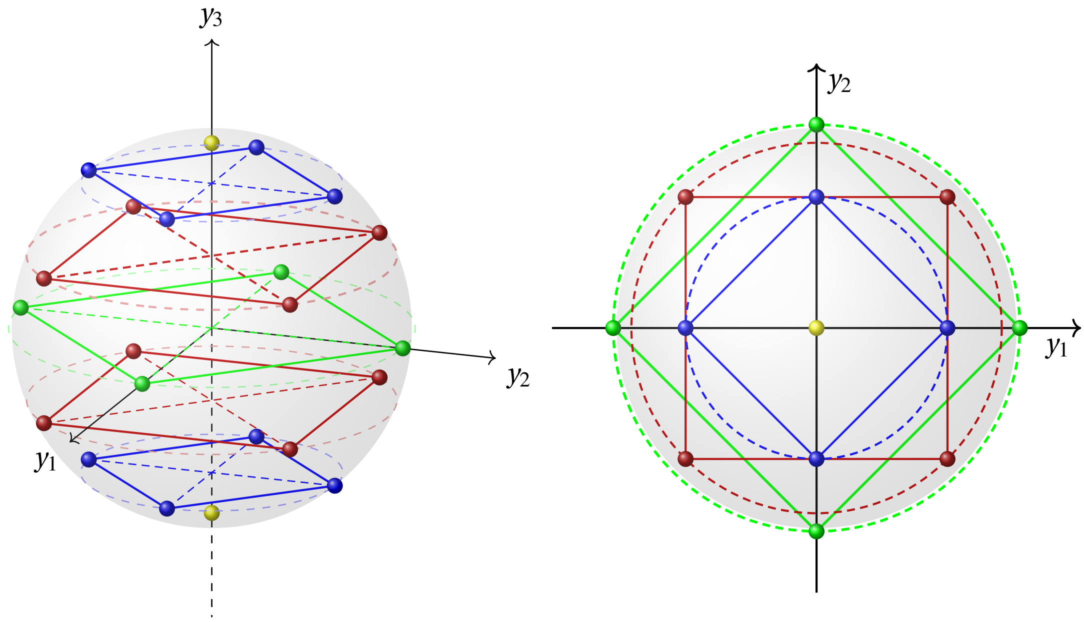

Figure 4.4: Graphical representation of decompositions (4.2.4) and (4.2.11) in standard coordinates, whose elements correspond, up to central simmetry, to the vertices of three squares, respectively in green, red and blue, placed at different heights, with adjuntive and an additional point in the axis y_3, in yellow.

again into considerations the roots of unity, which have size 8 and is given by

$$q_3^3 = \frac{14}{27}z^6 + \frac{7}{640}\sum_{j=1}^{2}(\tau_8^{2(j-1)}u + \tau_8^{-2(j-1)}v)^6 + \frac{1}{1280}\sum_{j=1}^{4}\left(\tau_8^{2(j-1)}u + \tau_8^{-2(j-1)}v + \sqrt{\frac{16}{3}}z\right)^6$$
$$+ \frac{1}{160}\sum_{j=1}^{4}\left(\tau_8^{2j-1}u + \tau_8^{-(2j-1)}v + \sqrt{\frac{2}{3}}z\right)^6. \tag{4.2.11}$$

Again, we observe that ideal associated to such decomposition is structured in a way similar to ideal (4.2.10). Indeed, it presents generators which are obtained by linear combination of elements posed at "distance" 4 in the diagram we have seen in Figure 4.2, that is,

$$\mathcal{I}_{(4.2.11)} = \begin{pmatrix} h_{4,4} - h_{4,-4}, & h_{4,2} - h_{4,-2}, \\ 7h_{4,3} - 3h_{4,-1}, & 7h_{4,-3} - 3h_{3,1} \end{pmatrix}.$$

This provides a symmetric structure with respect to the variables u and v, due to symmetric disposition of the points among the axis y_3.

By Theorem 3.3.12, it follows that decomposition (4.2.11) is minimal. However, we can prove that, in general, minimal decompositions of q_n^s are not unique, not even up to orthogonal tranformations. Indeed,

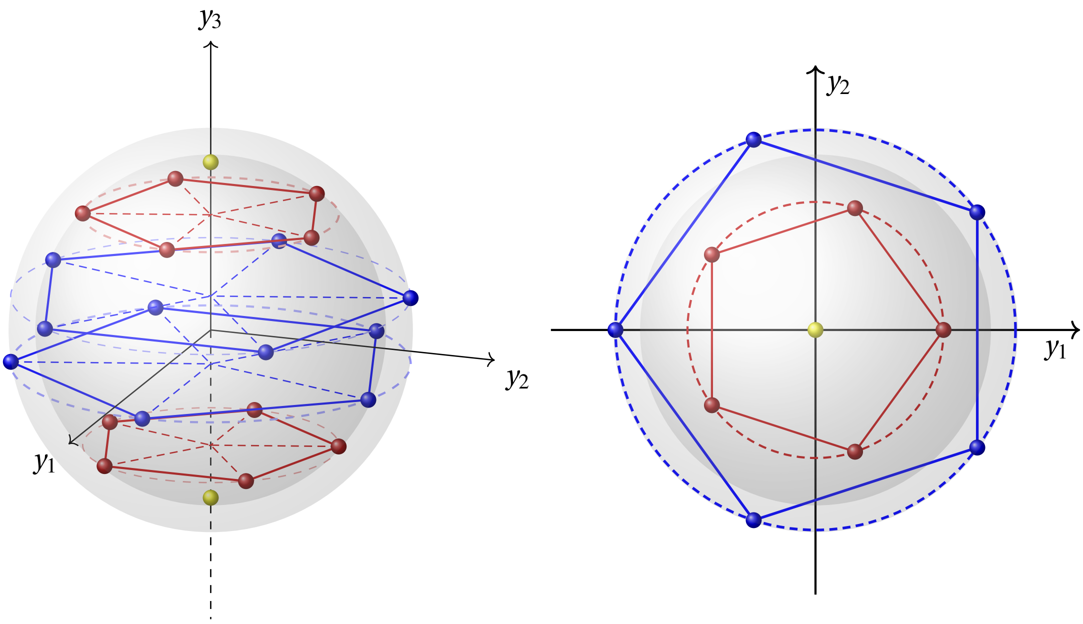

Figure 4.5: Graphical representation of decomposition (4.2.12) in standard coordinates, whose elements correspond, up to central symmetry, to the vertices of two pentagons, respectively in blue, and red, placed at different heights, with an additional point in the axis y_3, in yellow.

by doing some computations, we can obtain another decomposition of q_3^3, which is minimal as well, but which is not in the orbit under the action of $\mathrm{O}_n(\mathbb{C})$ of decomposition (4.2.11). Indeed, it is structured as two set of five points each, forming the vertices of two regular pentagons at different heights, with an additional point (see Figure 4.5). That is

$$q_3^3 = \frac{35}{54}z^6 + \left(\frac{1}{500}\beta^2 + \frac{1}{750}\right)\sum_{j=1}^{5}\left(\tau_{10}^{2(j-1)}u + \tau_{10}^{2(j-1)}v + \alpha z\right)^6$$
$$+ \left(\frac{1}{500}\alpha^2 + \frac{8}{500}\right)\sum_{j=1}^{5}\left(\tau_{10}^{2j-1}u + \tau_{10}^{-(2j-1)}v + \beta z\right)^6, \tag{4.2.12}$$

where $\alpha, \beta \in \mathbb{R}$ are constant values satisfying the equations

$$\alpha\beta = \frac{2}{3}, \quad \alpha^2 + \beta^2 = \frac{11}{3},$$

namely

$$\alpha = \sqrt{\frac{11 \pm \sqrt{105}}{6}}, \quad \beta = \frac{11 \mp \sqrt{105}}{4}\sqrt{\frac{11 \pm \sqrt{105}}{6}}.$$

Again, we get our ideal generated by linear combination of elements having "distance" 5 in diagram of Figure 4.2, namely

$$\mathcal{I}_{(4.2.11)} = \begin{pmatrix} 7h_{4,1} - 8\sqrt{21}h_{4,-4}, & 2h_{4,2} - \sqrt{21}h_{4,-3}, \\ 7h_{4,-1} - 8\sqrt{21}h_{4,4}, & 2h_{4,-2} - \sqrt{21}h_{4,3} \end{pmatrix}.$$

We have thus proved the following proposition.

Proposition 4.2.13. *Minimal decompositions of q_3^3 are not unique, even up to orthogonal transformations.*

Both decompositions (4.2.11) and (4.2.12) are real, but we determine another pattern of non-real points. This decomposition is quite complicated and to determine it, the use of Macaulay2 (see [GS]) has been essential. Despite this, the result is quite elegant and, again, the disposition of points appears balanced in the space. Denoting by α_1, α_2 and α_3 the three roots of the polynomial

$$z^3 - 3z^2 - 3z + 2 \in \mathbb{C}[z],$$

namely

$$\begin{aligned}
\alpha_1 &= 1 + 2\left(\frac{3+\mathrm{i}\sqrt{23}}{2}\right)^{-\frac{1}{3}} + \left(\frac{3+\mathrm{i}\sqrt{23}}{2}\right)^{\frac{1}{3}},\\
\alpha_2 &= 1 - (1-\mathrm{i}\sqrt{3})\left(\frac{3+\mathrm{i}\sqrt{23}}{2}\right)^{-\frac{1}{3}} - \frac{1+\mathrm{i}\sqrt{3}}{2}\left(\frac{3+\mathrm{i}\sqrt{23}}{2}\right)^{\frac{1}{3}},\\
\alpha_3 &= 1 - (1+\mathrm{i}\sqrt{3})\left(\frac{3+\mathrm{i}\sqrt{23}}{2}\right)^{-\frac{1}{3}} - \frac{1-\mathrm{i}\sqrt{3}}{2}\left(\frac{3+\mathrm{i}\sqrt{23}}{2}\right)^{\frac{1}{3}},
\end{aligned}$$

we get the relations

$$\alpha_1 + \alpha_2 + \alpha_3 = 3, \qquad \alpha_1\alpha_2\alpha_3 = -2,$$

by which is possible to determine the decomposition

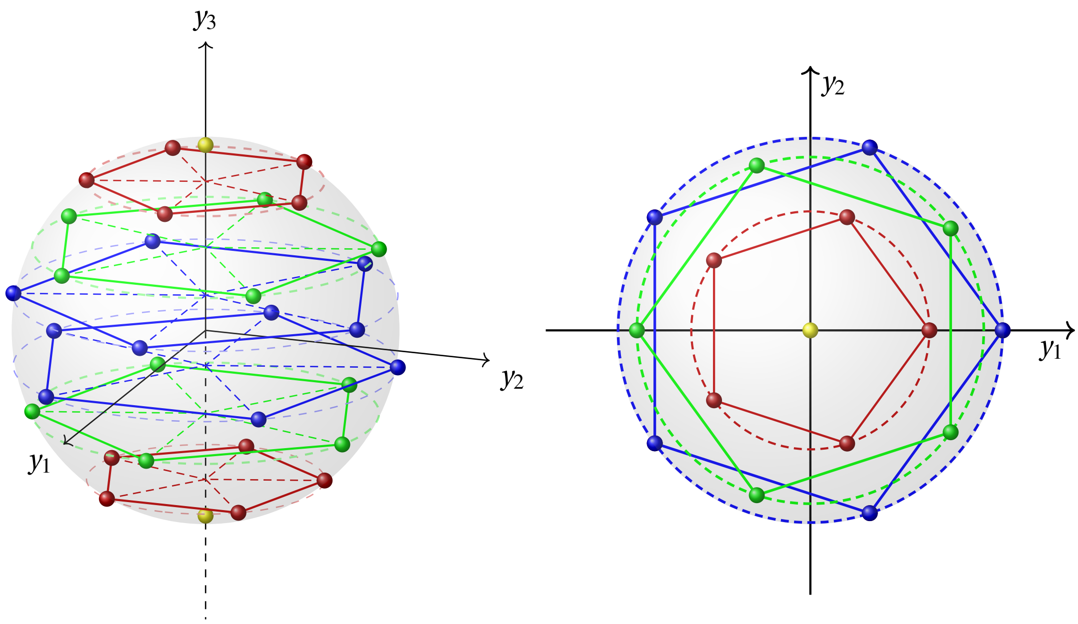

Figure 4.6: Graphical representation of decompositions (4.2.5) and (4.2.15) in standard coordinates, whose elements correspond, up to central simmetry, to the vertices of three pentagons, respectively in blue, green, and red, placed at different heights, with adjuntive and an additional point in the axis y_3, in yellow.

$$\begin{aligned}
q_3^3 = &-\frac{1}{20}u^6 - \frac{1}{20}v^6 + \frac{1}{6210}\left(19\alpha_1^{-1} - 11\alpha_1 + 36\right)\sum_{j=1}^{3}\left(\tau_3^{j-1}u + \tau_3^{-(j-1)}v + \alpha_1 z\right)^6\\
&- \frac{1}{6210}\left(19\alpha_2^{-1} + 11\alpha_2 - 36\right)\sum_{j=1}^{3}\left(\tau_3^{j-1}u + \tau_3^{-(j-1)}v + \alpha_2 z\right)^6
\end{aligned}$$

$$- \frac{1}{6210}\left(19\alpha_3^{-1} - 11\alpha_3 + 36\right) \sum_{j=1}^{3} \left(\tau_3^{j-1} u + \tau_3^{-(j-1)} v + \alpha_3 z\right)^6. \tag{4.2.14}$$

The associated ideal is given, instead, by

$$\mathcal{I}_{(4.2.14)} = \begin{pmatrix} 4h_{4,0} - 7h_{4,3}, & h_{4,1} - 2h_{4,-2}, \\ 4h_{4,0} - 7h_{4,-3}, & h_{4,-1} - 2h_{4,2} \end{pmatrix}.$$

For completeness, we include also a transformation of decomposition (4.2.5), obtained by a rotation for which we set a point on the axis y_3. We obtain points disposed over three different pentagons placed at different heights, symmetrically with respect to the central axis (see Figure 4.6). Thus, we can write

$$\begin{aligned} q_3^4 = \frac{15}{28} z^8 + \frac{1+\varphi^{-1}}{3500} \sum_{j=1}^{5} \left(\tau_{10}^{2(j-1)} u + \tau_{10}^{-2(j-1)} v + \varphi z\right)^8 + \frac{3}{3500} \sum_{j=1}^{5} \left(\tau_{10}^{2j-1} u + \tau_{10}^{-(2j-1)} v + z\right)^8 \\ + \frac{1+\varphi}{3500} \sum_{j=1}^{5} \left(\tau_{10}^{2(j-1)} u + \tau_{10}^{-2(j-1)} v + \varphi^{-1} z\right)^8, \end{aligned} \tag{4.2.15}$$

where

$$\varphi = \frac{3+\sqrt{5}}{2}, \quad \varphi^{-1} = \frac{3-\sqrt{5}}{2}$$

with the ideal of points equal to

$$\mathcal{I}_{(4.2.15)} = \begin{pmatrix} h_{5,1} - 3h_{5,-4} & h_{5,2} - h_{5,-3}, & h_{5,5} - h_{5,-5}, \\ h_{5,-1} - 3h_{5,4}, & h_{5,-2} - h_{5,3} & \end{pmatrix}.$$

CHAPTER 5

On the case of three variables

In this chapter we focus on the decompositions of the powers of ternary quadratic forms. Although the determination of the rank presents some difficulties, we can be more precise in dealing with the border rank. Indeed, using classical apolarity again and the property of smoothability, we can determine the border rank for the powers of quadratic forms in three variables, which turns out to be exactly equal to the rank of the middle catalecticant matrix. In section 5.1 we present the proof of this fact, just determining for every $s \in \mathbb{N}$ a scheme supported on a point, which is apolar to the form q_3^s. The crucial fact is that, thanks to a classical result by J. Fogarty ([Fog68, Theorem 2.4]), every 0-dimensional subscheme in $\mathbb{P}^2$ is smoothable. Then, we can use Lemma 1.2.29 to get the required result.

In section 5.2, we focus instead on the strict inequality between rank and border rank of the ternary form q_3^s. Indeed, we observe that the property of being first caliber decompositions imposes strong conditions between points. A specific analysis on this fact allows to prove that, also for complex numbers,

$$\operatorname{rk} q_3^4 = 16.$$

5.1 Border rank of q_3^s

For every $d \in \mathbb{N}$, let us consider the basis

$$\mathcal{B}_d = \{h_{d,k}\}_{-d \le k \le d},$$

composed by the polynomials we have introduced in Proposition 4.2.7 by formulas (4.2.8). We recall that these are given, up to scalars, by

$$h_{d,k} = \sum_{j=0}^{\lfloor \frac{d-k}{2} \rfloor} (-1)^j u^{[k+j]} z^{[d-k-2j]} v^{[j]}, \quad h_{d,-k} = \sum_{j=0}^{\lfloor \frac{d-k}{2} \rfloor} (-1)^j u^{[k]} z^{[d-k-2j]} v^{[k+j]},$$

for $k = 0, \dots, d$. Our idea is to determine a 0-dimensional scheme which is apolar to q_3^s. In this manner, we will be able to determine, thanks to Lemma 1.2.29, its smoothable rank. Now, we consider the ideal

$$I_d = (h_{d,0}, \dots, h_{d,d})$$

as the candidate ideal to be apolar to q_3^s. In order to see this, we first have to prove that I_d is saturated. We start by next lemma, which provides a basic result related to theory of fat points (see e.g. [Gim89] for further details). Given any ideal I, we denote by $\overline{I}$ its saturation.

Lemma 5.1.1. *For every $d \in \mathbb{N}$, the monomial ideal*

$$J_d = \left(u^d, u^{d-1}z, \dots, z^d\right) \subseteq \mathbb{K}[u, z, v],$$

equal to the ideal generated by all the monomials in the variables u and z of degree d, is saturated.

Proof. Given any polynomial $f \in \overline{J_d}$, we want to prove that $f \in J_d$. Since J_d is a monomial ideal, every monomial appearing in f must belong to J_d and hence we can suppose that f is a monomial. Now, by definition of saturated ideal, there must exists a natural value $m \in \mathbb{N}$ such that

$$v^m f \in J_d,$$

but this means that there is also a natural value $k \leq d$ such that

$$u^{s-k} z^k \mid v^m f.$$

Therefore, we have

$$u^{s-k} z^k \mid f,$$

that is, $f \in J_d$. □

We denote by $\mathrm{LT}(f)$ and $\mathrm{LT}(I)$ respectively the leading term of a polynomial $f \in \mathcal{R}_n^d$ and the leading ideal of an ideal $I \subseteq \mathcal{R}_n$ (see [Eis95, Section 15] for explicit definitions). We want to prove that the ideal J_d defined in Lemma 5.1.1 is actually the leading ideal of I_d. To get this, it is sufficient to prove that the set of generators

$$\mathcal{G}_s = \{h_{s,0}, \dots, h_{s,s}\}$$

forms a Gröbner basis of I_d. The most classical method to prove this is the Buchberger's criterion, which can be found in its standard version [Eis95, Theorem 15.8]. Nevertheless, we consider a particular version of that, described accurately by W. Decker and F.-O. Schreyer in [DS09, Theorem 2.3.9], which allows us to simplify considerably the proof.

Theorem 5.1.2 (Buchberger's criterion)**.** *Let $f_1, \dots, f_r \in \mathbb{K}[x_1, \dots, x_n]$ and let $I = (f_1, \dots, f_r)$. For every $j = 2, \dots, r$, let M_j denote the ideal*

$$M_j = \left(\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_{j-1})\right) : \left(\mathrm{LT}(f_j)\right).$$

Then $\{f_1, \dots, f_r\}$ is a Gröbner basis for I if and only if for every $j = 2, \dots, r$ and each minimal generator $\mathbf{x}^\alpha$ of M_j, the remainder of the division of $\mathbf{x}^\alpha f_j$ by $f_1, \dots, f_r$ is equal to zero.

Then we can use Theorem 5.1.2 to prove that the ideal J_d of Lemma 5.1.1 is effectively the leading ideal of I_d.

Lemma 5.1.3. *For every $d \in \mathbb{N}$, the set*

$$\mathcal{G}_d = \{h_{d,0}, \dots, h_{d,d}\}$$

represents a Gröbner basis of the ideal I_d with respect to the lexicographic monomial order defined by the relation

$$z >_{lex} u >_{lex} v.$$

Proof. We start by observing that for every $k = 0, \dots, s$, we have

$$\left(z^d, \dots, z^{d-k+1}u^{k-1}\right) : \left(z^{d-k}u^k\right) = (z).$$

Thus, by Theorem 5.1.2, it is sufficient to prove that the remainder of $zh_{d,k}$ divided by $h_{d,0}, \dots, h_{d,d}$ is equal to zero for every $k = 0, \dots, s$. Now, if $k = d$, then we have

$$zh_{d,d} = zu^d = uh_{d,d-1}$$

and the property holds. Otherwise, given any $k \leq d-1$, since

$$zh_{d,k} = z \sum_{j=0}^{\lfloor \frac{d-k}{2} \rfloor} (-1)^j u^{[k+j]} z^{[d-k-2j]} v^{[j]} = \sum_{j=0}^{\lfloor \frac{d-k}{2} \rfloor} (-1)^j (d-k-2j+1) u^{[k+j]} z^{[d-k-2j+1]} v^{[j]},$$

we can proceed with the division algorithm, first by considering $\mathrm{LT}(h_{d,k-1}) = z^{[d-k+1]}u^{[k-1]}$. We get then

$$zh_{d,k} = \frac{\mathrm{LT}(zh_{d,k})}{\mathrm{LT}(h_{d,k-1})} h_{d,k-1} + r_1 = \frac{d-k+1}{k} u h_{d,k-1} + r_1,$$

where r_1 corresponds to the first remainder. In particular, we have

$$\begin{aligned} r_1 &= zh_{d,k} - \frac{d-k+1}{k} u h_{d,k-1} \\ &= \sum_{j=0}^{\lfloor \frac{d-k}{2} \rfloor} (-1)^j (d-k-2j+1) u^{[k+j]} z^{[d-k-2j+1]} v^{[j]} \\ &\quad - \frac{d-k+1}{k} u \sum_{j=0}^{\lfloor \frac{d-k+1}{2} \rfloor} (-1)^j u^{[k+j-1]} z^{[d-k-2j+1]} v^{[j]}. \end{aligned}$$

Now, looking at the first summation, if $d-k$ is even, then

$$\left\lfloor \frac{d-k}{2} \right\rfloor = \left\lfloor \frac{d-k+1}{2} \right\rfloor,$$

while if $d-k$ is odd, we get

$$\left\lfloor \frac{d-k}{2} \right\rfloor < \left\lfloor \frac{d-k+1}{2} \right\rfloor.$$

But the factor $d-k-2j+1$ in the summation annihilates for

$$j = \left\lfloor \frac{d-k+1}{2} \right\rfloor.$$

Thus, we can modify the first summation by varying j from 1 to

$$\left\lfloor \frac{d-k+1}{2} \right\rfloor$$

and hence we can rewrite the equality as

$$r_1 = \sum_{j=1}^{\lfloor \frac{d-k+1}{2} \rfloor} (-1)^j \left((d-k-2j+1) - \frac{(d-k+1)(k+j)}{k} \right) u^{[k+j]} z^{[d-k-2j+1]} v^{[j]}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\lfloor \frac{d-k+1}{2} \rfloor} (-1)^j \frac{(d-k-2j+1)k-(d-k+1)(k+j)}{k} u^{[k+j]} z^{[d-k-2j+1]} v^{[j]} \\
&= \sum_{j=1}^{\lfloor \frac{d-k+1}{2} \rfloor} (-1)^{j-1} \frac{(d+k+1)j}{k} u^{[k+j]} z^{[d-k-2j+1]} v^{[j]} \\
&= \frac{d+k+1}{k} v \sum_{j=1}^{\lfloor \frac{d-k+1}{2} \rfloor} (-1)^{j-1} u^{[k+j]} z^{[d-k-2j+1]} v^{[j-1]} \\
&= \frac{d+k+1}{k} v \sum_{j=0}^{\lfloor \frac{d-k-1}{2} \rfloor} (-1)^{j} u^{[k+j+1]} z^{[d-k-2j-1]} v^{[j]} \\
&= \frac{d+k+1}{k} v h_{d,k+1},
\end{aligned}
$$

which guarantees that the last remainder is equal to zero. □

To conclude, we need just another nice property that T. Mańdziuk proves in [Mań22], given in the next lemma.

Lemma 5.1.4 ([Mań22, Lemma 4.12]). *Let I be an ideal in $\mathcal{R}_n$ and let $<$ be a monomial order. Then*

$$\mathrm{LT}_<\left(\overline{I}\right) \subseteq \overline{\mathrm{LT}_<(I)}.$$

In particular, if I is a homogeneous ideal and $\mathrm{LT}_<(I)$ is a saturated ideal, then I is a saturated ideal.

Thus, by Lemma 5.1.1 and Lemma Lemma 5.1.4 together, we immediately obtain the following corollary.

Corollary 5.1.5. *For every $d \in \mathbb{N}$ the ideal I_d is saturated.*

We are now ready to enunciate and prove the result which provides the border rank of q_3^s. An approach focused on classical apolarity is sufficient to reach the aim.

Theorem 5.1.6. *Let $f \in \mathcal{R}_3^s$ be an arbitrary ternary quadratic form. For every $s \in \mathbb{N}$, if f is degenerate, then*

$$\mathrm{smrk}\left(f^s\right) = \mathrm{brk}\left(f^s\right) = \mathrm{rk}\left(f^s\right) = \binom{s + \mathrm{rk}\, f - 1}{\mathrm{rk}\, f - 1}.$$

Otherwise,

$$\mathrm{smrk}\left(f^s\right) = \mathrm{brk}\left(f^s\right) = \binom{s+2}{2} = \frac{(s+1)(s+2)}{2}.$$

Proof. If f is a degenerate form such that $\mathrm{rk}\, f = 1$, then the statement is clear. If instead $\mathrm{rk}\, f = 2$, then the form f^s is equivalent, under the action of $\mathrm{SL}_3(\mathbb{C})$, to the form

$$q_2^s = \left(x_1^2 + x_2^2\right)^s.$$

Thus, we conclude by Theorem 3.2.3 that

$$\mathrm{smrk}\left(f^s\right) = \mathrm{rk}\left(f^s\right) = s + 1.$$

If instead f represents a non-degenerate form, then f^s is equivalent, up to linear transformations, to the form

$$q_3^s = \left(x_1^2 + x_2^2 + x_3^2\right)^s.$$

Now, Lemma 5.1.1 and Lemma 5.1.4 together guarantee that the ideal

$$I_{s+1} = (h_{s+1,s+1}, h_{s+1,s}, \dots, h_{s+1,0}),$$

is saturated for every $s \in \mathbb{N}$. Now let us consider the Hilbert function $\mathrm{HF}_{I_{s+1}}$ of the ideal I_{s+1}, defined as

$$\begin{aligned} \mathrm{HF}_{I_{s+1}} : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ k &\longmapsto \dim\big(\mathcal{D}_k / (I_{s+1})_k\big). \end{aligned}$$

We observe that for every $k \leq s$, since $(I_{s+1})_k = (0)$, we have

$$\mathrm{HF}_{I_{s+1}}(k) = \dim \mathcal{D}_k = \binom{k+2}{2} = \frac{(k+2)(k+1)}{2}.$$

Moreover, since I_{s+1} is generated by $s+1$ elements of degree $s+1$, we also have

$$\mathrm{HF}_{I_{s+1}}(s+1) = \dim\big(\mathcal{D}_{s+1} / (I_{s+1})_{s+1}\big) = \binom{s+3}{2} - (s+1) = \frac{(s+3)(s+2) - 2(s+2)}{2},$$

that is,

$$\mathrm{HF}_{I_{s+1}}(s+1) = \frac{s^2+3s+2}{2} = \binom{s+2}{2} = \mathrm{HF}_{I_{s+1}}(s).$$

This means, since I_{s+1} is a saturated ideal, that

$$\mathrm{HF}_{I_{s+1}}(k) = \mathrm{HF}_{I_{s+1}}(s)$$

for every $k \geq s$. Then, setting

$$r = \binom{s+2}{2},$$

it follows, denoting by Z_s the scheme associated to the ideal I_s, that

$$\deg Z_s = \binom{s+2}{2}.$$

Now, it is well known by a result of J. Fogarty (see [Fog68, Theorem 2.4]) that for every $r \in \mathbb{N}$, the Hilbert scheme $\mathrm{Hilb}_r(\mathbb{P}^2)$ parameterizing r points is smooth. Thus, we obtain that every 0-dimensional subscheme in $\mathbb{P}^2$ is smoothable. Therefore, we can use together Corollary 2.2.18, inequality (1.2.28), and Lemma 1.2.29, to obtain

$$\binom{s+2}{2} \leq \mathrm{brk}\big(q_3^s\big) \leq \mathrm{smrk}\big(q_3^s\big) = \binom{s+2}{2},$$

which proves the statement. □

Example 5.1.7. We can consider the case of exponent $s = 2$, corresponding to the case of the icosahedron determined by B. Reznick. The ideal I_3 is given by

$$I_3 = \Big(u^3, u^2 z, u\big(z^2 - uv\big), z\big(z^2 - 6uv\big)\Big),$$

which represents a scheme supported on the point $(0,0,1)$. Now, observing how this ideal is defined, we can describe it in the affine plane, setting $v = 1$. We can consider a family of radical ideals $\{I_3(t)\}_{t\in\mathbb{C}}$ depending on t, where, for every $t \in \mathbb{C}$,

$$I_3(t) = \big(h_{3,3}(t), h_{3,2}(t), h_{3,1}(t), h_{3,0}(t)\big)$$

is generated by the homogeneous polynomials

$$\begin{aligned} h_{3,3}(t) &= (u - tv)(u + t^2 v)\Big(5u - (t^2 - t)v\Big), \\ h_{3,2}(t) &= z(u - tv)\Big(5u - (t^2 - t)v\Big), \\ h_{3,1}(t) &= (u - tv)(z^2 - uv - t^2 v^2), \\ h_{3,0}(t) &= z(z^2 - 6uv - tv^2). \end{aligned}$$

The corresponding variety is given by the set of six projective points

$$V\big(I_3(t)\big) = \left\{ [-t^2 : 0 : 1], \left[\frac{t^2 - t}{5} : \pm\sqrt{\frac{6t^2 - t}{5}} : 1\right], [t : 0 : 1], \left[t : \pm\sqrt{7t} : 1\right] \right\}.$$

In particular, as $t \to 0$, all of these points tend to the projective point $[0 : 0 : 1]$.

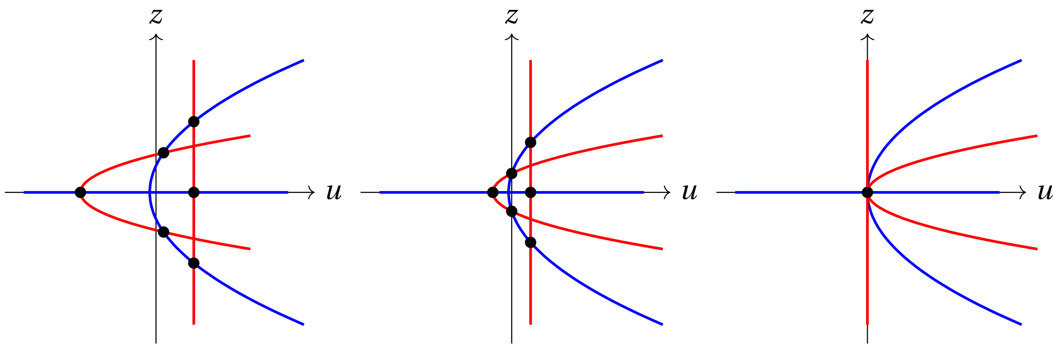

Figure 5.1: Graphical representation of the points associated to the family of radical ideals $\{I_3(t)\}_{t\in\mathbb{C}}$, defined in Example 5.1.7, with $t \to 0$. The red graphic represents the polynomial $h_{3,1}(t)$, and the blue one represents the polynomial $h_{3,0}(t)$.

5.2 Equiangular lines and rank lower bounds

The value of the border rank given in Theorem 5.1.6 represents the first general result about Waring decompositions for every power of ternary quadratic forms. As a noticeable consequence, we can observe that this is one of those cases in which border rank is strictly lower than Waring rank. We can state this by analyzing again the ternary forms, but while for the exponent 3 the conclusion was a bit more easier, as we have seen in Theorem 3.3.12, we cannot say the same for the exponent 4, for which we use another approach. We will consider, indeed, the point of view of angles between points, which can be uniquely defined up to sign. Indeed, there are always two supplementary angles between two lines in the space and the restriction of the first caliber decomposition effectively imposes a strong bound for the disposition of the points inside the sphere.

Definition 5.2.1. Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ two non-isotropic points. Then the two values

$$\theta_1 = \arccos\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right), \qquad \theta_2 = \pi - \arccos\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$$

are called *angles between* $\mathbf{a}$ *and* $\mathbf{b}$.

It is quite easy to verify that the angles between two points are uniquely determined by the formula involving just one of the two angles, defined as the complex value

$$\cos^2\theta_1 = \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{a}|^2|\mathbf{b}|^2} = \frac{1}{|\mathbf{a}|^2|\mathbf{b}|^2}\left(\sum_{j=1}^{n} a_j b_j\right)^2.$$

Clearly, since the value of $\cos^2\theta_1$ is the same also for multiples of $\mathbf{a}$ and $\mathbf{b}$, then it is well defined also for the projective lines generated by $\mathbf{a}$ and $\mathbf{b}$ or, equivalently, for the projective points $[\mathbf{a}], [\mathbf{b}] \in \mathbb{P}^{n-1}$.

The Gram matrix of a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in a vector space V endowed with a bilinear form is the matrix G of order n defined by the entries

$$G_{i,j} = \mathbf{v}_i \cdot \mathbf{v}_j$$

for every $i, j = 1, \dots, n$. The utility of the Gram determinant, in particular, is due to the fact that when computed on the $n(n+1)/2$ scalar products of $n+1$ unitary vectors $\mathbf{a}_1, \dots, \mathbf{a}_{n+1} \in \mathbb{C}^n$, with respect to the Euclidean product, it vanishes identically. Fixing $n = 3$ and considering four points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{C}^3$, the Gram determinant is given by the polynomial in six variables equal to

$$g(c_{12}, \dots, c_{34}) = \det\begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} \\ c_{12} & 1 & c_{23} & c_{24} \\ c_{13} & c_{23} & 1 & c_{34} \\ c_{14} & c_{24} & c_{34} & 1 \end{pmatrix}, \tag{5.2.2}$$

where

$$c_{jk} = \mathbf{a}_j \cdot \mathbf{a}_k \in \mathbb{C}$$

for every $j, k = 1, 2, 3, 4$ with $j \neq k$.

We can check that there is a subgroup of order 24 of the permutation group $\mathfrak{S}_6$ that leaves g invariant. This means that permuting the variables the determinant can assume at most $6!/24 = 30$ possible distinct values and one can detect 30 permutations representing each lateral class to evaluate the determinant. Indeed, the Gram matrix factors in this case as ${}^{t}AA$, where A is the matrix of order 4 having the $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ as columns.

Theorem 5.2.3. *Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{C}^3$ be four distinct nonisotropic lines such that the square cosinus $\cos^2\theta$ of every angle between any two points $\mathbf{a}_j$, $\mathbf{a}_k$ for $j \neq k$ assume the same value. Then*

$$\cos^2\theta \in \left\{\frac{1}{9}, \frac{1}{5}, 1\right\}.$$

Proof. Let $a \in \mathbb{C}$ be the unique complex value assumed by $\cos^2\theta$. Then, supposing

$$|\mathbf{a}_1| = |\mathbf{a}_2| = |\mathbf{a}_3| = |\mathbf{a}_4| = 1,$$

we must have

$$\mathbf{a}_i \cdot \mathbf{a}_j = \pm a$$

for every $i, j = 1, \dots, 4$ such that $i \neq j$. To get the statement, we simply have to compute the Gram determinant given by formula (5.2.2) in the case of all the variables equal to $\pm a$. Now, in case of $\mathbf{a}_i \cdot \mathbf{a}_j = a$ for every $i < j$, we have

$$\det\begin{pmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{pmatrix} = -(a-1)^3(3a+1).$$

By definition of Gram matrix, we simply have to compute matrices in consideration of how many points have mutual angle a or $-a$. So, if just two points have mutual angle equal to $-a$, we can suppose, without loss of generality, that

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = -a$$

and compute the determinant

$$\det\begin{pmatrix} 1 & -a & a & a \\ -a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{pmatrix} = \left(a^2 - 1\right)\left(5a^2 - 1\right).$$

We can follow the same reasoning for the other cases. In the case of two pairs of points with the opposite angle, we have to distinguish the case of these pairs involving the same point and the case of disjoint pairs. That is, we can suppose that

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_1 \cdot \mathbf{a}_3 = -a,$$

getting

$$\det\begin{pmatrix} 1 & -a & -a & a \\ -a & 1 & a & a \\ -a & a & 1 & a \\ a & a & a & 1 \end{pmatrix} = \left(a^2 - 1\right)\left(5a^2 - 1\right),$$

or, respectively, suppose that

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_3 \cdot \mathbf{a}_4 = -a,$$

obtaining

$$\det\begin{pmatrix} 1 & -a & a & a \\ -a & 1 & a & a \\ a & a & 1 & -a \\ a & a & -a & 1 \end{pmatrix} = (a+1)^3(3a-1).$$

Analogously, if we have three pairs of points with mutual angle equal to $-a$, we have to analyze which is the value of the determinant in case that the three pairs involve the same point or not. We have only two possibilities. Indeed supposing that

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_1 \cdot \mathbf{a}_3 = \mathbf{a}_1 \cdot \mathbf{a}_4 = -a,$$

we must also have

$$\mathbf{a}_2 \cdot \mathbf{a}_3 = \mathbf{a}_2 \cdot \mathbf{a}_4 = \mathbf{a}_3 \cdot \mathbf{a}_4 = a,$$

and

$$\det\begin{pmatrix} 1 & -a & -a & -a \\ -a & 1 & a & a \\ -a & a & 1 & a \\ -a & a & a & 1 \end{pmatrix} = -(a-1)^3\left(3a+1\right).$$

If we suppose instead

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_1 \cdot \mathbf{a}_3 = \mathbf{a}_2 \cdot \mathbf{a}_3 = -a,$$

we must also have

$$\mathbf{a}_1 \cdot \mathbf{a}_4 = \mathbf{a}_2 \cdot \mathbf{a}_4 = \mathbf{a}_3 \cdot \mathbf{a}_4 = a,$$

which, up to reversing the roles of $\mathbf{a}_1$ and $\mathbf{a}_4$, is equivalent to the previous case. Thus, it only remains the case in which we can set

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_2 \cdot \mathbf{a}_3 = \mathbf{a}_3 \cdot \mathbf{a}_4 = -a,$$

obtaining

$$\det\begin{pmatrix} 1 & -a & a & a \\ -a & 1 & -a & a \\ a & -a & 1 & -a \\ a & a & -a & 1 \end{pmatrix} = -(a^2-1)(5a^2-1).$$

Thus, we have proved that the Gram determinant can be 0 only if

$$a^2 \in \left\{1, \frac{1}{5}, \frac{1}{9}\right\},$$

which gives the statement. □

Next lemma concerns the structure of the kernel of the catalecticant matrices of the polynomial

$$f_1 = q_n^s - \lambda x_1^{2s}.$$

We have already observed that, for every $n \in \mathbb{N}$, there exists a unique value $\lambda_n \in \mathbb{C}$ such that

$$\dim\Big(\operatorname{Ker}\big(\operatorname{Cat}_{f_1}^s\big)\Big) = 1.$$

In particular, for the cases of $s = 2, 3$, we have stated in proofs of Theorem 3.3.3 and Theorem 3.3.12 that this value must be the norm assumed by each point of any tight decomposition. We can prove that the element generating the kernel is given by a product of quadratic forms, whenever s is odd, multiplied by the variable x_1.

Lemma 5.2.4. *For every $s \in \mathbb{N}$,*

$$\operatorname{Ker}\big(\operatorname{Cat}_{f_1}^s\big) = x_1^{\lfloor \frac{s+1}{2} \rfloor - \lfloor \frac{s}{2} \rfloor} \prod_{j=1}^{\lfloor \frac{s}{2} \rfloor} \left(x_1^2 - a_k \sum_{j=2}^{n} x_j^2\right)$$

for some $a_1, \ldots, a_{\lfloor \frac{s}{2} \rfloor} \in \mathbb{C}$.

Proof. The polynomial f_1 in the kernel of the middle catalecticant must be invariant under the action of $\mathrm{SO}_{n-1}(\mathbb{C})$. We can write the polynomial f_1 as

$$f_1 = \sum_{k=0}^{n} x_1^k g_{n-k}(x_2, \ldots, x_n),$$

where $g_{n-k} \in \mathcal{D}_{n-1}^{n-k}$ for every $k = 0, \ldots, n$. By uniqueness of decomposition (2.1.12), since the polynomials $g_1, \ldots, g_n$ must be invariant, they must be multiples of the powers of q_{n-1}. □

We can now proceed as in Theorem 5.2.3, to highlight which are the values assumed by the angle between a set of four points, in the case of two possible complex values.

Theorem 5.2.5. *Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{C}^3$ be four distinct nonisotropic lines such that the square cosinus $\cos^2\theta$ of every angle between any two points $\mathbf{a}_j$, $\mathbf{a}_k$ for $j \neq k$ assume at most two values a^2 and b^2. Then, either $a^2 = 1$ or $b^2 = 1$ or the pair $(a, b) \in \mathbb{C}^2$ must be a root of at least one of the following polynomials $g_1, \ldots, g_{10} \in \mathbb{C}[x_1, x_2]$:*

$$g_1 = 4x_1^2 \pm x_1x_2 \pm x_1 \pm x_2 - 1, \qquad g_2 = 4x_1^2 + x_2^2 - 1,$$
$$g_3 = x_1^3 - x_1 \pm 4x_1^2x_2 \pm \left(3x_1^2 + 2x_2^2 - 1\right), \quad g_4 = 2x_1 \pm x_2 \pm 1,$$

$$g_5 = 5x_1x_2^2 - x_1 \pm \left(2x_1^2 + 3x_2^2 - 1\right), \qquad g_6 = 2x_1^2 + x_2^2 - 1 \pm 2x_1x_2,$$
$$g_7 = 3x_2^2 - 1 \pm 2x_1, \qquad g_8 = x_1^2 + x_2^2 - 1 \pm x_1x_2 \pm x_1 \pm x_2,$$
$$g_9 = x_1^2 + x_2^2 - 1 \pm 3x_1x_2 \pm x_1 \pm x_2, \qquad g_{10} = x_1^4 + 3x_1^2x_2^2 + x_2^4 - 3x_1^2 - 3x_2^2 + 1 \pm \left(2x_1^3x_2 - 2x_1x_2^3\right).$$

Proof. As in the proof of Theorem 5.2.3, we just have to consider the number of possibly mutual angles between the various points and this allows us to not consider every permutation of the different values in each matrix. Denoting the two possible values of square cosinus between the various angles as $a^2, b^2 \in \mathbb{C}$, we first consider the case of b appearing only one time in the products between the four points, we have several possibilities. Setting

$$|\mathbf{a}_1| = |\mathbf{a}_2| = |\mathbf{a}_3| = |\mathbf{a}_4| = 1,$$

we can suppose, without loss of generality, that

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = b.$$

The first polynomial, equal to the determinant of the Gram matrix in which all the others product equals a, is given by

$$\det\begin{pmatrix} 1 & b & a & a \\ b & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{pmatrix} = (a-1)(b-1)\left(4a^2 - ab - a - b - 1\right).$$

If we suppose, instead, that one of the products is equal to $-a$, then we have to distinguish the case of this product involving $\mathbf{a}_1$ or $\mathbf{a}_2$ from the case of

$$\mathbf{a}_3 \cdot \mathbf{a}_4 = -a.$$

We get respectively

$$\det\begin{pmatrix} 1 & b & -a & a \\ b & 1 & a & a \\ -a & a & 1 & a \\ a & a & a & 1 \end{pmatrix} = \left(a^2 - 1\right)\left(4a^2 + b^2 - 1\right)$$

and

$$\det\begin{pmatrix} 1 & b & a & a \\ b & 1 & a & a \\ a & a & 1 & -a \\ a & a & -a & 1 \end{pmatrix} = (a+1)(b-1)\left(4a^2 + ab + a - b - 1\right).$$

If we have, instead, two of the products equal to $-a$, we have other different cases. If both of the products involve the same point $\mathbf{a}_1$ or $\mathbf{a}_2$, we can suppose this to be $\mathbf{a}_1$ and the polynomial is given by

$$\det\begin{pmatrix} 1 & b & -a & -a \\ b & 1 & a & a \\ -a & a & 1 & a \\ -a & a & a & 1 \end{pmatrix} = (a-1)(b+1)\left(4a^2 + ab - a + b - 1\right).$$

Otherwise, we could have

$$\mathbf{a}_1 \cdot \mathbf{a}_3 = -a, \quad \mathbf{a}_2 \cdot \mathbf{a}_4 = -a,$$

that is,

$$\det\begin{pmatrix} 1 & b & -a & a \\ b & 1 & -a & a \\ -a & -a & 1 & a \\ a & a & a & 1 \end{pmatrix} = (a+1)(b-1)\big(4a^2+ab+a-b-1\big),$$

which is the case of the product equal to $-a$ involving both of the points $\mathbf{a}_1$ and $\mathbf{a}_2$. Finally, if one of the two products is

$$\mathbf{a}_3 \cdot \mathbf{a}_4 = -a,$$

then we get

$$\det\begin{pmatrix} 1 & b & -a & a \\ b & 1 & a & a \\ -a & a & 1 & -a \\ a & a & -a & 1 \end{pmatrix} = \big(a^2-1\big)\big(4a^2+b^2-1\big).$$

We turn now to analyze the various cases with two values of the square cosinus equal to b^2, both involving the same point, and four values equal to a^2. Hence we can suppose that

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_1 \cdot \mathbf{a}_3.$$

Starting by the simplest one, concerning two products equal to b and four products equal to a, we get

$$\det\begin{pmatrix} 1 & b & b & a \\ b & 1 & a & a \\ b & a & 1 & a \\ a & a & a & 1 \end{pmatrix} = (a-1)\big(a^3-4a^2b+3a^2+2b^2-a-1\big).$$

In case of one product equal to $-a$, we have to distinguish the case in which

$$\mathbf{a}_1 \cdot \mathbf{a}_4 = -a,$$

corresponding to the polynomial

$$\det\begin{pmatrix} 1 & b & b & -a \\ b & 1 & a & a \\ b & a & 1 & a \\ -a & a & a & 1 \end{pmatrix} = (a-1)\big(a^3+4a^2b+3a^2+2b^2-a-1\big),$$

and the cases of the product equal to $-a$ not involving $\mathbf{a}_1$. That is, supposing for instance

$$\mathbf{a}_2 \cdot \mathbf{a}_3 = -a,$$

we have

$$\det\begin{pmatrix} 1 & b & b & a \\ b & 1 & -a & a \\ b & -a & 1 & a \\ a & a & a & 1 \end{pmatrix} = (a+1)\big(a^3+4a^2b-3a^2-2b^2-a+1\big).$$

With the same reasoning, we compute the determinant of the Gram matrix for two products equal to $-a$, obtaining respectively, if

$$\mathbf{a}_1 \cdot \mathbf{a}_4 = \mathbf{a}_2 \cdot \mathbf{a}_3 = -a,$$

the polynomial

$$\det\begin{pmatrix} 1 & b & b & -a \\ b & 1 & -a & a \\ b & -a & 1 & a \\ -a & a & a & 1 \end{pmatrix} = (a+1)\big(a^3+4a^2b-3a^2-2b^2-a+1\big)$$

and, if

$$\mathbf{a}_1\cdot\mathbf{a}_4 = \mathbf{a}_2\cdot\mathbf{a}_3 = -a,$$

the polynomial

$$\det\begin{pmatrix} 1 & b & b & a \\ b & 1 & -a & -a \\ b & -a & 1 & a \\ a & -a & a & 1 \end{pmatrix} = (a-1)\big(a^3+4a^2b+3a^2+2b^2-a-1\big).$$

Now, supposing instead the two initial products to be opposite, that is

$$\mathbf{a}_1\cdot\mathbf{a}_2 = -\mathbf{a}_1\cdot\mathbf{a}_3,$$

we can repeat the same computations of the previous cases just replacing one of the coordinates equal to b by $-b$ in the matrices. So we get, in the case of two products equal to a, the polynomial

$$\det\begin{pmatrix} 1 & b & -b & a \\ b & 1 & a & a \\ -b & a & 1 & a \\ a & a & a & 1 \end{pmatrix} = (a-1)\big(a^3+4a^2b+3a^2+2b^2-a-1\big),$$

while in the case of one product equal to $-a$, the polynomials

$$\det\begin{pmatrix} 1 & b & -b & -a \\ b & 1 & a & a \\ -b & a & 1 & a \\ -a & a & a & 1 \end{pmatrix} = (a-1)\big(a^3+4a^2b+3a^2+2b^2-a-1\big)$$

and

$$\det\begin{pmatrix} 1 & b & -b & a \\ b & 1 & -a & a \\ -b & -a & 1 & a \\ a & a & a & 1 \end{pmatrix} = (a+1)\big(a^3+4a^2b-3a^2-2b^2-a+1\big).$$

Finally, in the case of two products equal to $-a$, the polynomials

$$\det\begin{pmatrix} 1 & b & -b & -a \\ b & 1 & -a & a \\ -b & -a & 1 & a \\ -a & a & a & 1 \end{pmatrix} = (a+1)\big(a^3+4a^2b-3a^2-2b^2-a+1\big)$$

and

$$\det\begin{pmatrix} 1 & b & -b & a \\ b & 1 & -a & -a \\ -b & -a & 1 & a \\ a & -a & a & 1 \end{pmatrix} = (a-1)\big(a^3+4a^2b+3a^2+2b^2-a-1\big).$$

Again for the cases of two values of the square cosinus equal to b^2, we now analyze the case in which the corresponding products do not involve the same point. Thus, supposing

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_3 \cdot \mathbf{a}_4 = b$$

if the other four products are all equal to a, we have

$$\det\begin{pmatrix} 1 & b & a & a \\ b & 1 & a & a \\ a & a & 1 & b \\ a & a & b & 1 \end{pmatrix} = -(b-1)^2(2a-b-1)(2a+b+1),$$

while if one of them is equal to $-a$, which we can suppose to be $\mathbf{a}_1 \cdot, \mathbf{a}_3$, we get

$$\det\begin{pmatrix} 1 & b & -a & a \\ b & 1 & a & a \\ -a & a & 1 & b \\ a & a & b & 1 \end{pmatrix} = \left(2a^2-2ab+b^2-1\right)\left(2a^2+2ab+b^2-1\right).$$

For the case of two products equal to $-a$, if

$$\mathbf{a}_1 \cdot \mathbf{a}_3 = \mathbf{a}_1 \cdot \mathbf{a}_4 = -a$$

and the two product involve the same point, we have

$$\det\begin{pmatrix} 1 & b & -a & -a \\ b & 1 & a & a \\ -a & a & 1 & b \\ -a & a & b & 1 \end{pmatrix} = (b-1)(b+1)\left(4a^2+b^2-1\right).$$

Otherwise, supposing

$$\mathbf{a}_1 \cdot \mathbf{a}_4 = \mathbf{a}_2 \cdot \mathbf{a}_3 = -a,$$

we obtain

$$\det\begin{pmatrix} 1 & b & a & -a \\ b & 1 & -a & a \\ a & -a & 1 & b \\ -a & a & b & 1 \end{pmatrix} = -(b+1)^2(2a-b+1)(2a+b-1).$$

Substituting again an element b in the matrices by $-b$, we repeat the same calculations, obtaining in the case of the remaining products equal to a, the polynomial

$$\det\begin{pmatrix} 1 & b & a & a \\ b & 1 & a & a \\ a & a & 1 & -b \\ a & a & -b & 1 \end{pmatrix} = (b-1)(b+1)\left(4a^2+b^2-1\right)$$

and in the case of one product equal to $-a$, the polynomial

$$\det\begin{pmatrix} 1 & b & -a & a \\ b & 1 & a & a \\ -a & a & 1 & -b \\ a & a & -b & 1 \end{pmatrix} = \left(2a^2-2ab+b^2-1\right)\left(2a^2+2ab+b^2-1\right).$$

Finally, in the case of two products equal to $-a$, the polynomials

$$\det\begin{pmatrix} 1 & b & -a & -a \\ b & 1 & a & a \\ -a & a & 1 & -b \\ -a & a & -b & 1 \end{pmatrix} = -(b+1)^2(2a-b+1)(2a+b-1)$$

and

$$\det\begin{pmatrix} 1 & b & a & -a \\ b & 1 & -a & a \\ a & -a & 1 & -b \\ -a & a & -b & 1 \end{pmatrix} = \left(b^2-1\right)^2\left(4a^2+b^2-1\right).$$

It remains to analyze the cases of three values of the square cosinus equal to b^2. We start by supposing

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_1 \cdot \mathbf{a}_3 = \mathbf{a}_1 \cdot \mathbf{a}_4 = b,$$

that is,

$$\det\begin{pmatrix} 1 & b & b & b \\ b & 1 & a & a \\ b & a & 1 & a \\ b & a & a & 1 \end{pmatrix} = -(a-1)^2\left(3b^2-2a-1\right).$$

If instead one of these last three products is opposite, that is, for instance,

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = -b,$$

we get

$$\det\begin{pmatrix} 1 & -b & b & b \\ -b & 1 & a & a \\ b & a & 1 & a \\ b & a & a & 1 \end{pmatrix} = (a-1)\left(5ab^2+2a^2+3b^2-a-1\right).$$

Proceeding in the same way with

$$\mathbf{a}_2 \cdot \mathbf{a}_3 = -a,$$

we get

$$\det\begin{pmatrix} 1 & b & b & b \\ b & 1 & -a & a \\ b & -a & 1 & a \\ b & a & a & 1 \end{pmatrix} = (a+1)\left(5ab^2-2a^2-3b^2-a+1\right)$$

and, even considering both of these cases together, that is

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = -b, \quad \mathbf{a}_2 \cdot \mathbf{a}_3 = -a,$$

we have

$$\det\begin{pmatrix} 1 & -b & b & b \\ -b & 1 & -a & a \\ b & -a & 1 & a \\ b & a & a & 1 \end{pmatrix} = (a+1)\left(5ab^2-2a^2-3b^2-a+1\right).$$

Finally, we have the case of three products equal to $\pm b$ but not involving the same point, that is, we can suppose first

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_2 \cdot \mathbf{a}_3 = \mathbf{a}_3 \cdot \mathbf{a}_4 = b,$$

obtaining

$$\det\begin{pmatrix}1 & b & a & a\\ b & 1 & b & a\\ a & b & 1 & b\\ a & a & b & 1\end{pmatrix} = \left(a^2 - 3ab + b^2 + a + b - 1\right)\left(a^2 + ab + b^2 - a - b - 1\right).$$

If instead one of the three products is equal to $-b$, that is, for instance

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = -b,$$

we get

$$\det\begin{pmatrix}1 & -b & a & a\\ -b & 1 & b & a\\ a & b & 1 & b\\ a & a & b & 1\end{pmatrix} = \left(a^4 - 2a^3b + 3a^2b^2 + 2ab^3 + b^4 - 3a^2 - 3b^2 + 1\right).$$

The last two cases are obtained as previously, setting first

$$\mathbf{a}_1 \cdot \mathbf{a}_3 = -a,$$

and then

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = -a, \quad \mathbf{a}_1 \cdot \mathbf{a}_3 = -a,$$

getting respectively

$$\det\begin{pmatrix}1 & b & -a & a\\ b & 1 & b & a\\ -a & b & 1 & b\\ a & a & b & 1\end{pmatrix} = \left(a^4 + 2a^3b + 3a^2b^2 - 2ab^3 + b^4 - 3a^2 - 3b^2 + 1\right)$$

and

$$\det\begin{pmatrix}1 & -b & -a & a\\ -b & 1 & b & a\\ -a & b & 1 & b\\ a & a & b & 1\end{pmatrix} = \left(a^2 - ab + b^2 - a + b - 1\right)\left(a^2 + 3ab + b^2 + a - b - 1\right).$$

By reversing a and b, or substituting a by $-a$ and b by $-b$, we get the statement. □

By Theorem 3.3.12, we already know that the rank of q_3^3 cannot be equal to its border rank, but, anyway, this result can be obtained also using Theorem 5.2.5, just by doing some computations.

Corollary 5.2.6. $\operatorname{rk}\left(q_3^3\right) > \operatorname{brk}\left(q_3^3\right) = 10.$

Proof. By Theorem 5.2.5 and Lemma 3.3.11, it is sufficient to substitute the values

$$x_1 = 0, \quad x_2 = \pm\sqrt{\frac{3}{7}} \tag{5.2.7}$$

and

$$x_1 = \pm\sqrt{\frac{3}{7}}, \quad x_2 = 0 \tag{5.2.8}$$

in polynomials $g_1, \dots, g_{10} \in \mathbb{C}[x_1, x_2]$, defined in Theorem 5.2.5. We first substitute values given in formula (5.2.7), obtaining

$$g_1\left(0, \pm\sqrt{\frac{3}{7}}\right) = \pm\sqrt{\frac{3}{7}} - 1, \quad g_2\left(0, \pm\sqrt{\frac{3}{7}}\right) = -\frac{4}{7},$$

$$g_3\left(0, \pm\sqrt{\frac{3}{7}}\right) = -\frac{1}{7}, \qquad g_4\left(0, \pm\sqrt{\frac{3}{7}}\right) = \pm\frac{3}{7} \pm 1,$$

$$g_5\left(0, \pm\sqrt{\frac{3}{7}}\right) = \pm\frac{2}{7}, \qquad g_6\left(0, \pm\sqrt{\frac{3}{7}}\right) = \frac{4}{7},$$

$$g_7\left(0, \pm\sqrt{\frac{3}{7}}\right) = \frac{2}{7}, \qquad g_8\left(0, \pm\sqrt{\frac{3}{7}}\right) = -\frac{4}{7} \pm \sqrt{\frac{3}{7}}$$

$$g_9\left(0, \pm\sqrt{\frac{3}{7}}\right) = -\frac{4}{7} \pm \sqrt{\frac{3}{7}}, \quad g_{10}\left(0, \pm\sqrt{\frac{3}{7}}\right) = -\frac{131}{49}.$$

Then, we proceed in the same way with values of formula (5.2.8), obtaining

$$g_1\left(\pm\sqrt{\frac{3}{7}}, 0\right) = \frac{5}{7} \pm \sqrt{\frac{3}{7}}, \qquad g_2\left(\pm\sqrt{\frac{3}{7}}, 0\right) = \frac{5}{7},$$

$$g_3\left(\pm\sqrt{\frac{3}{7}}, 0\right) = -\frac{4}{7}\sqrt{\frac{3}{7}} \pm \frac{2}{7}, \quad g_4\left(\pm\sqrt{\frac{3}{7}}, 0\right) = 2\sqrt{\frac{3}{7}} \pm 1,$$

$$g_5\left(\pm\sqrt{\frac{3}{7}}, 0\right) = \mp\sqrt{\frac{3}{7}} \pm \frac{2}{7}, \qquad g_6\left(\pm\sqrt{\frac{3}{7}}, 0\right) = \frac{2}{7},$$

$$g_7\left(\pm\sqrt{\frac{3}{7}}, 0\right) = \frac{2}{7}, \qquad g_8\left(\pm\sqrt{\frac{3}{7}}, 0\right) = -\frac{4}{7} \pm \sqrt{\frac{3}{7}}$$

$$g_9\left(\pm\sqrt{\frac{3}{7}}, 0\right) = -\frac{4}{7} \pm \sqrt{\frac{3}{7}}, \qquad g_{10}\left(\pm\sqrt{\frac{3}{7}}, 0\right) = -\frac{131}{49}.$$

Since all the values are non-zero, it follows that it cannot exists a tight decomposition of q_3^3. □

In order to proceed in the same way for the form q_3^4, we need to know which are the possible values assumed by the angles between points of a possibly tight decomposition. Thus, we have to analyze the kernel of the catalecticant map of the polynomial

$$f_1 = \frac{1}{B_{n,4}} q_n^4 - (\mathbf{a} \cdot \mathbf{a})^8,$$

where $\mathbf{a} \in \mathbb{C}^n$ is such that $\mathbf{a} \cdot \mathbf{a} = 1$, recalling (see formula (3.1.15) that

$$B_{n,4} = \frac{8(n+4)(n+6)}{35(n+1)(n+3)}.$$

Analogously to Lemma 3.3.1 and Lemma 3.3.11, we get the following lemma for the exponent $s = 4$.

Lemma 5.2.9. *Let $n \in \mathbb{N}$ and let*

$$f_1 = \frac{1}{B_{n,4}} q_n^4 - (\mathbf{a} \cdot \mathbf{x})^8,$$

for some $\mathbf{a} \in \mathbb{C}^n$ such that $\mathbf{a} \cdot \mathbf{a} = 1$. Then

$$\begin{aligned}\operatorname{Ker}\left(\operatorname{Cat}_{f_1}^4\right) &= \left\langle (n+6)(\mathbf{a} \cdot \mathbf{y})^4 - 6q_n(\mathbf{a} \cdot \mathbf{y})^2 + \frac{3}{n+4} q_n^2 \right\rangle \\ &= \left\langle \left((n+6)(\mathbf{a} \cdot \mathbf{y})^2 - \left(3 + \sqrt{\frac{6(n+3)}{n+4}}\right) q_n\right)\left((n+6)(\mathbf{a} \cdot \mathbf{y})^2 - \left(3 - \sqrt{\frac{6(n+3)}{n+4}}\right) q_n\right)\right\rangle.\end{aligned}$$

Proof. By Lemma 3.1.11 it is sufficient to prove that the $\operatorname{Ker}(\mathrm{Cat}^4_{f_1}) \neq 0$. Therefore, using Lemma 1.2.15, formula (2.1.7) and the fact that $\mathbf{a} \cdot \mathbf{a} = 1$, we simply observe that

$$\begin{aligned}
g_1 \circ f_1 &= \left((n+6)(\mathbf{a}\cdot\mathbf{x})^4 - 6q_n(\mathbf{a}\cdot\mathbf{x})^2 + \frac{3}{n+4}q_n^2\right) \circ \left(\frac{1}{B_{n,4}}q_n^4 - (\mathbf{a}\cdot\mathbf{x})^8\right) \\
&= \frac{n+6}{B_{n,4}}\left((\mathbf{a}\cdot\mathbf{y})^4 \circ q_n^4\right) - (n+6)\left((\mathbf{a}\cdot\mathbf{y})^4 \circ (\mathbf{a}\cdot\mathbf{x})^8\right) - \frac{6}{B_{n,4}}\left(q_n(\mathbf{a}\cdot\mathbf{y})^2 \circ q_n^4\right) \\
&\quad + 6\left(q_n(\mathbf{a}\cdot\mathbf{y})^2 \circ (\mathbf{a}\cdot\mathbf{x})^8\right) + \frac{3}{B_{n,4}(n+4)}\left(q_n^2 \circ q_n^4\right) - \frac{3}{n+4}\left(q_n^2 \circ (\mathbf{a}\cdot\mathbf{x})^8\right) \\
&= \frac{48(n+6)}{B_{n,4}}\left(8(\mathbf{a}\cdot\mathbf{x})^4 + 24q_n(\mathbf{a}\cdot\mathbf{x})^2 + 3q_n^2\right) - 1680(n+6)(\mathbf{a}\cdot\mathbf{x})^4 \\
&\quad - \frac{288(n+6)}{B_{n,4}}\left(4q_n(\mathbf{a}\cdot\mathbf{x})^2 + q_n^2\right) + 10080(\mathbf{a}\cdot\mathbf{x})^4 + \frac{144(n+6)}{B_{n,4}}q_n^2 - \frac{5040}{n+4}(\mathbf{a}\cdot\mathbf{x})^4 \\
&= \frac{384(n+6)}{B_{n,4}}(\mathbf{a}\cdot\mathbf{x})^4 - 1680(n+6)(\mathbf{a}\cdot\mathbf{x})^4 + 10080(\mathbf{a}\cdot\mathbf{x})^4 - \frac{5040}{n+4}(\mathbf{a}\cdot\mathbf{x})^4 \\
&= \frac{1680(n+1)(n+3) - 1680n(n+4) - 5040}{n+4}(\mathbf{a}\cdot\mathbf{x})^4 = 0,
\end{aligned}$$

which proves the statement. □

Thus, now we can prove that the rank of q_3^4 is equal to 16. Potentially, this is a technique that can be extended even to higher cases. It would be interesting to determine a general rule in relation to the angles between the various points of the decomposition in all the successive cases. To do this, it would be necessary to determine a general formula to obtain the values of all of the angles between the points of the decomposition.

Corollary 5.2.10. $\operatorname{rk}(q_3^4) = 16$.

Proof. As in the proof of the previous corollary, by substituting $n = 3$ in formula obtained in Lemma 5.2.9, it is sufficient to substitute the values

$$x_1 = \pm\sqrt{\frac{\sqrt{7}+2}{3\sqrt{7}}} = \pm\sqrt{\frac{7+2\sqrt{7}}{21}}, \quad x_2 = \pm\sqrt{\frac{\sqrt{7}-2}{3\sqrt{7}}} = \pm\sqrt{\frac{7-2\sqrt{7}}{21}} \tag{5.2.11}$$

and

$$x_1 = \pm\sqrt{\frac{\sqrt{7}-2}{3\sqrt{7}}} = \pm\sqrt{\frac{7-2\sqrt{7}}{21}}, \quad x_2 = \pm\sqrt{\frac{\sqrt{7}+2}{3\sqrt{7}}} = \pm\sqrt{\frac{7+2\sqrt{7}}{21}} \tag{5.2.12}$$

to polynomial $g_1, \dots, g_{10} \in \mathbb{C}[x_1, x_2]$, defined in Theorem 5.2.5. For computations, it can be useful to observe that, in any case,

$$x_1x_2 = \frac{1}{\sqrt{21}}, \quad x_1^2 + x_2^2 = \frac{2}{3}.$$

Substituting the values (5.2.11), we get

$$g_1\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}}, \pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right) = \frac{7+8\sqrt{7} \pm \sqrt{21} \pm \sqrt{147+42\sqrt{7}} \pm \sqrt{147-42\sqrt{7}}}{21},$$

$$g_2\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}}, \pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right) = \frac{2}{3} + \frac{2\sqrt{7}}{7},$$

$$g_3\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}},\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right)=\frac{2\sqrt{7+2\sqrt{7}}\left(\sqrt{147}-7\sqrt{21}\pm 42\right)\pm 42\left(7+\sqrt{7}\right)}{441},$$

$$g_4\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}},\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right)=\pm 2\sqrt{\frac{7+2\sqrt{7}}{21}}\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\pm 1,$$

$$g_5\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}},\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right)=\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\left(\frac{14-10\sqrt{7}}{21}\right)\pm\frac{14-2\sqrt{7}}{21},$$

$$g_6\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}},\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right)=\frac{2\left(\sqrt{7}\pm\sqrt{21}\right)}{21},$$

$$g_7\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}},\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right)=-\frac{2\sqrt{7}}{7}\pm 2\sqrt{\frac{7+2\sqrt{7}}{21}},$$

$$g_8\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}},\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right)=\frac{-7\pm\sqrt{21}\pm\sqrt{147+42\sqrt{7}}\pm\sqrt{147-42\sqrt{7}}}{21}$$

$$g_9\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}},\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right)=\frac{-7\pm 3\sqrt{21}\pm\sqrt{147+42\sqrt{7}}\pm\sqrt{147-42\sqrt{7}}}{21},$$

$$g_{10}\left(\pm\sqrt{\frac{7+2\sqrt{7}}{21}},\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\right)=\frac{-315+63\sqrt{21}\pm 8\sqrt{147}}{441}.$$

For values (5.2.12), we get instead

$$g_1\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\frac{7-8\sqrt{7}+\sqrt{21}\pm\sqrt{147-42\sqrt{7}}\pm\sqrt{147+42\sqrt{7}}}{21},$$

$$g_2\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\frac{2}{3}-\frac{2\sqrt{7}}{7},$$

$$g_3\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\frac{2\sqrt{7-2\sqrt{7}}\left(-\sqrt{147}-7\sqrt{21}\pm 42\right)\pm 42\left(7-\sqrt{7}\right)}{441},$$

$$g_4\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\pm 2\sqrt{\frac{7-2\sqrt{7}}{21}}\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\pm 1,$$

$$g_5\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\pm\sqrt{\frac{7-2\sqrt{7}}{21}}\left(\frac{14+10\sqrt{7}}{21}\right)\pm\frac{14+2\sqrt{7}}{21},$$

$$g_6\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\frac{2\left(-\sqrt{7}\pm\sqrt{21}\right)}{21},$$

$$g_7\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\frac{2\sqrt{7}}{7}\pm 2\sqrt{\frac{7-2\sqrt{7}}{21}},$$

$$g_8\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\frac{-7\pm\sqrt{21}\pm\sqrt{147-42\sqrt{7}}\pm\sqrt{147+42\sqrt{7}}}{21}$$

$$g_9\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\frac{-7\pm\sqrt{21}\pm\sqrt{147-42\sqrt{7}}\pm\sqrt{147+42\sqrt{7}}}{21},$$

$$g_{10}\left(\pm\sqrt{\frac{7-2\sqrt{7}}{21}},\pm\sqrt{\frac{7+2\sqrt{7}}{21}}\right)=\frac{-315+63\sqrt{21}\pm 8\sqrt{147}}{441}.$$

Since all the values are non-zero, it follows that it cannot exists a tight decomposition of q_3^4. □

Glossary of notations

$B_{n,s}$	Norm of the points of a first caliber decomposition of q_n^s raised to $2s$. 7
L_h	Left multiplication map. 12
$S(V)$	Symmetric algebra of a vector space V. 23
S^dV	d-th symmetric power of a vector space V. 5
T_eG	Tangent space of a Lie group G at the point e. 12
$T_{n,s}$	Size of the middle catalecticant matrix of q_n^s. 6
$V^{\otimes d}$	d-th tensor power of a vector space V. 4
$[X, Y]$	Bracket of the vector fields X and Y. 12
Cat_f	Catalecticant map of a polynomial f. 25
D_ϕ	Differential operator associated to a polynomial ϕ. 24
$\mathrm{End}(V)$	Space of endomorphisms of a vector space V. 17
$\mathrm{GL}(V)$	General linear group of a vector space V. 16
$\mathrm{GL}_n(\mathbb{K})$	General linear group of degree n over a field $\mathbb{K}$. xii
HF_I	Hilbert function of an ideal I. 31
$\mathfrak{I}(z)$	Imaginary part of an element $z \in \mathbb{C}$. 53
$\mathrm{LT}(I)$	Leading ideal of an ideal I. 92
$\mathrm{LT}(f)$	Leading term of a polynomial f. 92
Δ	Laplace operator. 6
$\mathrm{Mat}_n(\mathbb{K})$	Space of the square matrices of order n over a field $\mathbb{K}$. xii
$\mathrm{O}_n(\mathbb{K})$	Orthogonal group of degree n over a field $\mathbb{K}$. xii
$\mathrm{SL}_n(\mathbb{K})$	Special linear group of degree n over a field $\mathbb{K}$. xii
$\mathrm{SO}_n(\mathbb{K})$	Special orthogonal group of degree n over a field $\mathbb{K}$. xii
$\overline{I}$	Saturation of an ideal I. 91
$\mathbf{x}^{[\delta]}$	Divided power monomial. 26
$\mathrm{brk}\, T$	Border rank of a tensor T. 4
$\mathrm{brk}_X(\mathbf{p})$	X-border rank of a point p. 29
$\mathrm{brk_s}\, S$	Symmetric border rank of a symmetric tensor S. 5
$\mathrm{brk}\, f$	Border rank of a polynomial f. 3
$\mathcal{D}_n$	Space of polynomials $\mathbb{K}[y_1, \dots, y_n]$. 24
$\mathcal{H}_n^d$	Space of d-harmonic polynomials in n variables. 34
$\mathcal{R}_n$	Space of polynomials $\mathbb{K}[x_1, \dots, x_n]$. 24
$\circ$	Apolarity action. 25
$\mathrm{crk}_X(\mathbf{p})$	X-rank of a point p. 30
$\mathrm{crk}\, f$	Cactus rank of a polynomial f. 4
$\mathrm{d}f_p$	Differential of a differentiable map f at a point p. 12
$\exp$	Exponential map. 14
$\mathfrak{X}(M)$	Set of vector fields over a manifold M. 12
$\mathfrak{gl}(V)$	Lie algebra of the Lie group $\mathrm{GL}(V)$. 17

$\mathfrak{sl}_2\mathbb{C}$	Lie algebra of the Lie group $\mathrm{SL}_2(\mathbb{C})$. 19
$\mathfrak{so}_3\mathbb{C}$	Lie algebra of the Lie group $\mathrm{O}_3\mathbb{C}$. 81
ν_d	d-Veronese embedding. 4
$\operatorname{rk} T$	Rank of a tensor T. 4
$\operatorname{rk}_X(\mathbf{p})$	X-rank of a point p. 29
$\operatorname{rk}_{\mathbb{R}} f$	Real rank of a polynomial f. 8
$\operatorname{rk}_{\mathrm{s}} S$	Symmetric rank of a symmetric tensor S. 5
$\operatorname{rk} f$	Waring rank of a polynomial f. 1
S^{n-1}	$(n-1)$-dimensional sphere. 46
$\mathrm{S}^{n-1}_{\mathbb{C}}$	Complexified $(n-1)$-dimensional sphere. 46
$\sigma_r(X)$	r-th secant variety of a projective variety X. 30
$\operatorname{smrk}_X(\mathbf{p})$	X-border rank of a point p. 31
$f^\perp$	Apolar ideal of a polynomial f. 25
f_*X	Pushforward of a vector field X by a diffeomorphism f. 12
$l_{\mathbf{a}}^{[d]}$	d-th divided power of the linear form associated to a point **a**. 27
q_n	Quadratic form $x_1^2 + \cdots + x_n^2$. 5

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