From a homogeneous bundle to the quiver representation. Examples of tensor products Informal note

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Let $X = G/P$ be a irreducible Hermitian symmetric variety. In [B-K], [Hi1], [Hi2], [O-R2] it was proved that the category of G-homogeneous bundle is equivalent to the category of a certain quiver $\mathcal{Q}_X$ with relations. We want to discuss how the correspondance

$$\text{representation of } P \to \ \text{representation of } \mathcal{Q}_X$$

depends on the choices made in these papers, and how this influences the relations in $\mathcal{Q}_X$. We will make explicit the case of $\mathbf{P}^2$. In particular we show how some constant appearing in Olver maps can be computed only by the diagram of the weights appearing.

In the second section we will apply these techniques to some examples of tensor products.

1 From a homogeneous bundle to the quiver representation

We recall from [O-R2] that a representation of P which defines a bundle E gives a G-invariant $\theta\colon grE \otimes \Omega^1 \to grE$ which satisfies $\theta \wedge \theta = 0$.

Write $grE = \oplus_\lambda V_\lambda \otimes E_\lambda$. Then $\theta \in \oplus_{\lambda,\mu} Hom(V_\lambda, V_\mu) \otimes Hom(E_\lambda \otimes \Omega^1, E_\mu)^G$.

In order to define morphisms in $Hom(V_\lambda, V_\mu)$ which give the desired quiver representation, it is necessary to choose generators $e_{\lambda\mu}$ in the one dimensional vector spaces $Hom(E_\lambda \otimes \Omega^1, E_\mu)^G$.

In order to choose these generators, Bondal, Kapranov[B-K], Hille[Hi1],[Hi2] and also [O-R2] perform the following construction. We begin to choose maximal vectors $v_\lambda \in E_\lambda$. Then choose also a maximal vector $n_1 \in \Omega^1 = (LieG/LieP)^\vee$, (caution: the dual could be omitted depending on the definition of the bundle associated to a representation) his weight is exactly the negative root $-\alpha$ such that $P = P(\alpha)$. A basis of the vector space $\Omega^1_{[P]}$ is given by n_1 together with $g_{-\beta}(n_1)$ when β moves among the positive roots containing α in their expression in the fundamental system of simple roots. The elements $\{n_1, \ldots, g_{-\beta}(n_1), \ldots\}$ were called $\{n_1, \ldots, n_k\}$ in [O-R2]. We will discuss later how the construction of the quiver representation changes if we change the basis $\{n_1, \ldots, n_k\}$. Assume now that $Hom(E_\lambda \otimes \Omega^1, E_\mu)^G \neq 0$. Then there is a negative root $-\beta = \xi_j$ such that $\mu - \lambda = -\beta = \xi_j$ and we have called $n_j = g_{-\beta}(n_1)$. Now the crucial point is the following

Definition 1.1 *$e_{\lambda\mu} \in Hom(E_\lambda \otimes \Omega^1, E_\mu)^G$ is defined by the condition*

$$e_{\lambda\mu}(v_\lambda \otimes n_j) = v_\mu$$

It is clear that changing the maximal vectors v_λ will give an equivalent representation of the quiver. We perform this construction explicitly in the case $G = SL(3)$ and $\mathbf{P}^2 = G/P$. We will see how the relations in the quiver are forced by the above definition.

There are two simple positive roots α_1, α_2 and $P = P(\alpha_1)$. In terms of the fundamental weights we have

$$\alpha_1 = 2\lambda_1 - \lambda_2$$

$$\alpha_2 = -\lambda_1 + 2\lambda_2$$

We have the maximal vector n_1 with weight $-\alpha_1$ and $n_2 = g_{-\alpha_2}(n_1)$ with weight $-\alpha_1 - \alpha_2$. In matrix notation

$$g_{-\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad g_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$n_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad n_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Indeed $[g_{-\alpha_2}, n_1] = n_2$. Now $E_\lambda = E_{p\lambda_1 + q\lambda_2} = S^qQ(p)$ where Q is the quotient bundle. There are three connected components in the quiver. One is given, as in [O-R1] by

$$\begin{array}{cccc}
 & & & \cdot^{\cdot^{\cdot}} \\
 & & (0,3) & (1,4) \\
 & (0,0) & (1,1) & (2,2) \\
(-3,0) & (-2,1) & (-1,2) & (0,3) \\
\cdot^{\cdot^{\cdot}} & & &
\end{array}$$

where we have labelled the vertex corresponding to $p\lambda_1 + q\lambda_2$ with (p, q). Notice that in this component we have $q - p \equiv 0(mod3)$. The reason of this labelling is that now the actions of n_1, n_2, $g_{-\alpha_2}$ have a nice description in terms of arrows as it follows

$$
\begin{array}{ccccccc}
 & & & & & & \cdots \\
 & & & & (0,3) & \xleftarrow{n_2} & (1,4) \\
 & & & & \downarrow n_1 & \nwarrow g_{\alpha_2} & \downarrow n_1 \\
 & & (0,0) & \xleftarrow{n_2} & (1,1) & \xleftarrow{n_2} & (2,2) \\
 & & \downarrow n_1 & \nwarrow g_{\alpha_2} & \downarrow n_1 & \nwarrow g_{\alpha_2} & \downarrow n_1 \\
(-3,0) & \xleftarrow{n_2} & (-2,1) & \xleftarrow{n_2} & (-1,2) & \xleftarrow{n_2} & (0,3) \\
\cdots & & & & & &
\end{array}
$$

Indeed

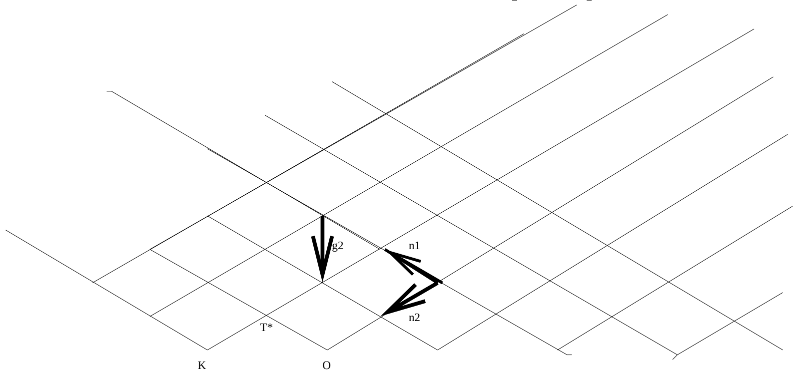
$$n_1(\mathcal{G}_\lambda) = \mathcal{G}_{\lambda-\alpha_1}$$

$$n_2(\mathcal{G}_\lambda) = \mathcal{G}_{\lambda-\alpha_1-\alpha_2}$$

$$g_{-\alpha_2}(\mathcal{G}_\lambda) = \mathcal{G}_{\lambda-\alpha_2}$$

Then in this description the two weights of Ω^1 span the coordinate axis.

Note that bundles with the same slope lie on NW-SE diagonals. It could be convenient to see how the quiver shows in the notations of [O-R2]



In this picture the bundles with the same slope lie on vertical lines (orthogonals to λ_1 which is horizontal). We denote $T^* = \Omega^1$ with weight $(-2,1)$, $K = \Omega^2$ with weight $(-3,0)$. Moreover the Killing form gives exactly the euclidean metric.

Now let be given

$$\theta \in Hom(grE \otimes \Omega^1, grE)^G = \oplus_{\lambda,\mu} Hom(V_\lambda, V_\mu) \otimes Hom(E_\lambda \otimes \Omega^1, E_\mu)^G$$

θ induces vertical maps

$$\begin{array}{c} V_{p,q} \otimes S^q Q(p) \otimes \Omega^1 \\ \downarrow \\ V_{p-2,q+1} \otimes S^{q+1} Q(p-2) \end{array}$$

and respectively horizontal maps

$$V_{p-1,q-1} \otimes S^{q-1} Q(p-1) \leftarrow V_{p,q} \otimes S^q Q(p) \otimes \Omega^1$$

Now fix any maximal vector $v_{p,q} \in S^qQ(p)$ and consider

$e_{p,q}^{p-2,q+1} \in Hom(S^qQ(p) \otimes \Omega^1, S^{q+1}Q(p-2))^G$ such that $e_{p,q}^{p-2,q+1}(v_{p,q} \otimes n_1) = v_{p-2,q+1}$ and

$e_{p,q}^{p-1,q-1} \in Hom(S^qQ(p) \otimes \Omega^1, S^{q-1}Q(p-1))^G$ such that $e_{p,q}^{p-1,q-1}(v_{p,q} \otimes n_2) = v_{p-1,q-1}$

The following result is a particular case of a proposition of Olver(see [O-R2] prop. 6.2). We want to see how it can be proved without any explicit reference to the maximal weights, but only by the Lie algebra structure. It seems conceivable that the same result can proved for the general $\mathbf{P}^n$ without any explicit reference to the maximal weights, but the computations are more difficult even in the case of $\mathbf{P}^3$.

Proposition 1.2

$$(i) e_{p,q}^{p-1,q-1}(v_{p,q} \otimes n_1) = 0$$

$$(ii) e_{p,q}^{p-2,q+1}(v_{p,q} \otimes n_2) = \frac{1}{q+1} g_{-\alpha_2}(v_{p-2,q+1})$$

Lemma 1.3 *Let denote* $e_{p,q}^{p-2,q+1} = e,\ v_{p,q} = v,\ g_{-\alpha_2} = g$

$$(i) e\left(g^k(v) \otimes n_2\right) = g^k e\left(v \otimes n_2\right)$$

$$(ii) e\left(g^k(v) \otimes n_1\right) = ge\left(g^{k-1}(v) \otimes n_1\right) - g^{k-1} e\left(v \otimes n_2\right)$$

Proof of the Lemma (i) follows because e is $LieG$-equivariant and $gn_2 = 0$ (for the adjoint action). For the same reason

$$e\left(g^k(v) \otimes n_1\right) = ge\left(g^{k-1}(v) \otimes n_1\right) - e\left(g^{k-1}(v) \otimes n_2\right)$$

and then (ii) follows by applying (i).

Proof of Prop. 1.2 (i) is obvious by weight reasons. $S^qQ(p)$ is spanned by $v, g(v), \ldots, g^q(v)$ moreover $g^{q+1}(v) = 0$. Now consider that

$$g^q(e(v \otimes n_2) = \text{by Lemma (i)}$$

$$= e(g^q(v) \otimes n_2) = e(g^q(v) \otimes gn_1) = ge(g^q(v) \otimes n_1) = \text{by Lemma (ii)}$$

$$= g^2 e(g^{q-1}(v) \otimes n_1) - g^q e(v \otimes n_2) = \text{again by Lemma (ii)}$$

$$= g^3 e(g^{q-2}(v) \otimes n_1) - 2g^q e(v \otimes n_2) = \text{continuing in this way}$$

$$= g^{q+1} e(v \otimes n_1) - qg^q e(v \otimes n_2)$$

so that

$$g^{q+1} e(v \otimes n_1) = (q+1) g^q e(v \otimes n_1)$$

Since $e(v \otimes n_2)$ is a scalar multiple (true on $\mathbf{P}^2$) of $ge(v \otimes n_1) = gv_{p-2,q+1}$ the proposition follows.

Corollary 1.4

$$\frac{q+2}{q+1}e^{p-3,q}_{p-1,q-1}\wedge e^{p-1,q-1}_{p,q}+e^{p-3,q}_{p-2,q+1}\wedge e^{p-2,q+1}_{p,q}=0$$

in $Hom(S^qQ(p)\otimes\Omega^2, S^qQ(p-3))$.

Proof Apply both sides to $v_{p,q}\otimes(n_1\wedge n_2)$. By the proposition Prop. 1.2 we have

$$e^{p-3,q}_{p-1,q-1}\wedge e^{p-1,q-1}_{p,q}(v_{p,q}\otimes(n_1\wedge n_2))=$$

$$e^{p-3,q}_{p-1,q-1}(e^{p-1,q-1}_{p,q}(v_{p,q}\otimes n_1)\otimes n_2)-e^{p-3,q}_{p-1,q-1}(e^{p-1,q-1}_{p,q}(v_{p,q}\otimes n_2)\otimes n_1)=$$

$$=0-e^{p-3,q}_{p-1,q-1}(v_{p-1,q-1}\otimes n_1)=-v_{p-3,q}$$

$$e^{p-3,q}_{p-2,q+1}\wedge e^{p-2,q+1}_{p,q}(v_{p,q}\otimes(n_1\wedge n_2))=$$

$$e^{p-3,q}_{p-2,q+1}(e^{p-2,q+1}_{p,q}(v_{p,q}\otimes n_1)\otimes n_2)-e^{p-3,q}_{p-2,q+1}(e^{p-2,q+1}_{p,q}(v_{p,q}\otimes n_2)\otimes n_1)=$$

$$=e^{p-3,q}_{p-2,q+1}(v_{p-2,q+1}\otimes n_2)-e^{p-3,q}_{p-2,q+1}(\frac{1}{q+1}g_{-\alpha_2}(v_{p-2,q+1})\otimes n_1)=$$

$$v_{p-3,q}-\frac{1}{q+1}\left(-e^{p-3,q}_{p-2,q+1}(v_{p-2,q+1}\otimes n_2)+g_{-\alpha_2}e^{p-3,q}_{p-2,q+1}(v_{p-2,q+1}\otimes n_1)\right)=$$

$$=v_{p-3,q}+\frac{1}{q+1}(v_{p-3,q}-0)=\frac{q+2}{q+1}v_{p-3,q}$$

Denoting by

$$\begin{array}{l}V_{p,q}\\ \downarrow f^{p-2,q+1}_{p,q}\\ V_{p-2,q+1}\end{array}$$

and

$$V_{p-1,q-1}\overset{f^{p-1,q-1}_{p,q}}{\longleftarrow}V_{p,q}$$

the induced maps in the representation of the quiver we get

Corollary 1.5

$$f^{p-3,q}_{p-1,q-1}f^{p-1,q-1}_{p,q}-\frac{q+2}{q+1}f^{p-3,q}_{p-2,q+1}f^{p-2,q+1}_{p,q}=0$$

Proof The equation $\theta\wedge\theta=0$ induces

$$0=f^{p-3,q}_{p-1,q-1}f^{p-1,q-1}_{p,q}e^{p-3,q}_{p-1,q-1}\wedge e^{p-1,q-1}_{p,q}+f^{p-3,q}_{p-2,q+1}f^{p-2,q+1}_{p,q}e^{p-3,q}_{p-2,q+1}\wedge e^{p-2,q+1}_{p,q}=$$

$$=f^{p-3,q}_{p-1,q-1}f^{p-1,q-1}_{p,q}e^{p-3,q}_{p-1,q-1}\wedge e^{p-1,q-1}_{p,q}-\frac{q+2}{q+1}f^{p-3,q}_{p-2,q+1}f^{p-2,q+1}_{p,q}e^{p-3,q}_{p-1,q-1}\wedge e^{p-1,q-1}_{p,q}$$

So the equation of the corollary is the correct relation we have to define in the quiver in order to have a equivalence of categories according to choices we have done.

It is clear that changing $v_{p,q}$ give a isomorphic representation of the quiver and does not affect the relations. n_1 could be multiplied by a scalar, consequently $n_2 = g_{-\alpha_2}(n_1)$ is multiplied by the same scalar. In this case all the $f^{p-3,q}_{p-1,q-1}$ are multiplied by the same scalar, it is easy to see that we get a isomorphic representation, multiplying the successive maximal vectors for a convenient power of the same scalar (depending on the slope).

A different thing happens if we choose $n_2 = kg_{-\alpha_2}(n_1)$ with a scalar k not necessarily being equal to 1. In this case all the vertical maps are multiplied by the same constant, and the horizontal ones do not change. With this choice we get a representation which is not isomorphic to the one obtained in the case $k = 1$. We get this case by a different choice of $g_{-\alpha_2}$, which is also defined up to scalar multiple. Observe that in this case the relations do not change, so they are really canonical, while the equivalence of categories between $P - mod$ and $\mathcal{Q}_X$ is not canonical. These choices do not affect the relations for any irreducible Hermitian symmetric variety, because scalar multiples of n_i affect always opposite sides of the several parallelograms.

Since we have got a equivalence of categories, there is no problem into changing the relations obtaining an equivalent category.

For example consider in the case of $\mathbf{P}^2$ the quiver $\mathcal{Q}_1$ with the above relations and the quiver $\mathcal{Q}_2$ with commutativity relations (that is forget the factor $\frac{q+2}{q+1}$). It is clear that from any representation of $\mathcal{Q}_1$ we can get a representation of $\mathcal{Q}_2$, just multiply any horizontal map starting from $V_{p,q}$ by $(q+1)$. This was essentially the trick used in the example 5.14 of [O-R2]. This is a functor which gives a equivalence between $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

2 Examples of tensor products

Let $X = G/P$ be a irreducible Hermitian symmetric variety. Let R be the reductive part of P. It is convenient to look at homogeneous bundles as Higgs bundles as in [O-R2]. A homogeneous bundle is described by a pair (F, θ) where F is a R-module and $\theta: F \otimes \Omega^1 \to F$ is G-equivariant and it satisfies $\theta \wedge \theta = 0$. The homogeneous bundle E that we get satisfies $grE = F$. Since $gr(A \otimes B) = (grA) \otimes (grB)$, the tensor product between (F, θ) and (G, ψ) is

$$(F \otimes G, \theta \otimes 1_G + 1_F \otimes \psi)$$

Explicitly let $F = \oplus V_\lambda \otimes E_\lambda$ and $G = \oplus W_\lambda \otimes E_\lambda$ Then

$$F \otimes G = \oplus_{\lambda,\mu} V_\lambda \otimes W_\mu \otimes E_\lambda \otimes E_\mu$$

We have the decomposition $E_\lambda \otimes E_\mu = \oplus_\nu a^\nu_{\lambda\mu} E_\nu$ for certain nonnegative integers $a^\nu_{\lambda\mu}$, they are the Littlewood-Richardson coefficients in the case of $\mathbf{P}^n$. Hence

$$\theta \otimes 1_G + 1_F \otimes \psi: \oplus a^\nu_{\lambda\mu} V_\lambda \otimes W_\mu \otimes E_\nu \otimes \Omega^1 \to \oplus a^{\nu'}_{\lambda'\mu'} V_{\lambda'} \otimes W_{\mu'} \otimes E_{\nu'}$$

In order to understand this cumbersome expression it is convenient to separate the contribution of

$$\theta \otimes 1_G \in Hom(a^{\nu}_{\lambda\mu} a^{\nu'}_{\lambda'\mu} V_\lambda, V_{\lambda'}) \otimes Hom(E_\lambda \otimes \Omega^1, E_{\lambda'})^G$$

from the contribution of

$$1_F \otimes \psi \in Hom(a^{\nu}_{\lambda\mu} a^{\nu'}_{\lambda\mu'} W_\mu, W_{\mu'}) \otimes Hom(E_\mu \otimes \Omega^1, E_{\mu'})^G$$

This means that $E_{\nu'}$ appears in $E_\nu \otimes \Omega^1$ only when the corresponding $E_{\lambda'}$ appears in $E_\lambda \otimes \Omega^1$ or the corresponding $E_{\mu'}$ appears in $E_\mu \otimes \Omega^1$.

In order to perform explicit computations, one has to fix maximal vectors of all the E_ν which appear in the decomposition of $E_\lambda \otimes E_\mu$ in terms of the maximal vectors of E_λ and E_μ.

In case of multiplicities $a^{\nu}_{\lambda\mu} \geq 2$, such maximal vectors should fixed for all the "copies" of E_ν appearing.

We explain how to compute the tensor product with two examples on $\mathbf{P}^2$, where there are no multiplicities, but the same E_ν can appear by the contribution of different $E_\lambda \otimes E_\mu$.

First example: $adV \otimes adQ$

The first example is $adV \otimes adQ$, a bundle of rank 24 on $\mathbf{P}^2$, see example 8.3[O-R2]. We first have to fix maximal vectors for adV, see example 5.14[O-R2].

We have

$$gr(adV) = Q(1) \oplus \mathcal{O} \oplus S^2Q(-1) \oplus Q(-2)$$

We use the notations of the first section, let $g = g_{-\alpha_2} \in LieR$ and n_1 and $g(n_1) = n_2$ be a basis of eigenvectors for Ω^1, n_1 being maximal. Let w_1 be a maximal vector for $Q(1)$, $\{w_1, w_2 = g(w_1)\}$ is a basis for $Q(1)$.

In the same way we denote (the first vector is always the maximal one)

$$S^2Q(-1) =< z_1, z_2 = g(z_1), z_3 = g^2(z_1) >$$

$$\mathcal{O} =< y_1 >$$

$$Q(-2) =< x_1, x_2 = g(z_1) >$$

A graphical representation of these vectors, putting each vector in the corresponding eigenspace is the following

$$\begin{tikzcd}
z_3 \arrow[d, "n_1"] & w_2 \arrow[l, "n_2"'] \arrow[d, "n_1"] & \\
x_2 & y_1 \oplus z_2 \arrow[l, "n_2"'] \arrow[d, "n_1"] \arrow[ul, "g"] & w_1 \arrow[l, "n_2"'] \arrow[d, "n_1"] \arrow[ul, "g"] \\
& x_1 \arrow[ul, "g"] & z_1 \arrow[l, "n_2"'] \arrow[ul, "g"]
\end{tikzcd}$$

The representation of the quiver for adV can be described by

$$\begin{tikzcd}
\mathcal{O} \arrow[d, "b"] & Q(1) \arrow[l, "a"'] \arrow[d, "c"] \\
Q(-2) & S^2Q(-1) \arrow[l, "d"']
\end{tikzcd}$$

where we have the conditions

$$n_2 x_1 = d x_1 \qquad n_1 y_1 = b y_1$$

$$n_1 w_1 = c z_1 \qquad n_2 w_1 = a y_1 + \frac{c}{2} z_2$$

(the first three by definition and the last one by Prop. 1.2 (ii) with $p = q = 1$) and by Cor. 1.5 the relation

$$ba - \frac{3}{2} dc = 0$$

The description of adV is completed.

We fix also the second factor of the tensor product

$$adQ = < v_1, v_2 = g(v_1), v_3 = g^2(v_1) >$$

Of course adQ is isomorphic to $S^2Q(-1)$ but it is better to mantain separate names. Note that in this case θ is given by the four pieces (a, b, c, d) while $\psi = 0$.

We choose now maximal vectors for all the direct summands of $adV \otimes adQ$.

According to the decomposition

$$Q(-2) \otimes adQ = S^3Q(-3) \oplus Q(-2)$$

we have

$S^3Q(-3)$	$Q(-2)$
$x_1 \otimes v_1$	
$x_2 \otimes v_1 + x_1 \otimes v_2$	$2x_2 \otimes v_1 - x_1 \otimes v_2$
$2x_2 \otimes v_2 + x_1 \otimes v_3$	$x_1 \otimes v_2 - x_1 \otimes v_3$
$3x_2 \otimes v_3$	

where g takes each element in its vertical successor (like a cascade, so the maximal vectors are on the top on each column). Here note that $2x \otimes v_1 - x_1 \otimes v_2$ generates in the corresponding two dimensional eigenspace$< x_1 \otimes v_2, x_2 \otimes v_1 >$ the one dimensional $\ker g^2$ so the only choice that we have done is a scalar multiple.

We can repeat these considerations in the other cases.

$$\mathcal{O} \otimes adQ = S^2Q(-1)^I = < y_1 \otimes v_1 >$$

$$S^2Q(-1) \otimes adQ = S^4Q(-2) \oplus S^2Q(-1)^{II} \oplus \mathcal{O}$$

and we have

$S^4Q(-2)$	$S^2Q(-1)^{II}$	$\mathcal{O}$
$z_1 \otimes v_1$		
$z_1 \otimes v_2 + z_2 \otimes v_1$	$-z_1 \otimes v_2 + z_2 \otimes v_1$	
$z_1 \otimes v_3 + 2z_2 \otimes v_2 + z_3 \otimes v_1$	$-z_1 \otimes v_3 + z_3 \otimes v_1$	$z_1 \otimes v_3 - z_2 \otimes v_2 + z_3 \otimes v_1$
$\vdots$	$\vdots$	$\vdots$

$$Q(1) \otimes adQ = S^3Q \oplus Q(1)$$

and we have

S^3Q	$Q(1)$
$w_1 \otimes v_1$	
$w_2 \otimes v_1 + w_1 \otimes v_2$	$2w_2 \otimes v_1 - w_1 \otimes v_2$
$\vdots$	$\vdots$

Now the whole $adV \otimes adQ$ is

$$\begin{tikzcd} \mathcal{O} \arrow[d] & Q(1) \arrow[l] \arrow[d] & \\ Q(-2) & S^2Q(-1)^I \oplus S^2Q(-1)^{II} \arrow[l] \arrow[d] & S^3Q \arrow[l] \arrow[d] \\ & S^3Q(-3) & S^4Q(-2) \arrow[l] \end{tikzcd}$$

and the computation of the maps is a matter of routine. For

$$\begin{tikzcd} Q(1) \arrow[d] \\ S^2Q(-1) \end{tikzcd}$$

we have

$$n_1(2w_2 \otimes v_1 - w_1 \otimes v_2) = c(z_2 \otimes v_1 - z_1 \otimes v_2) - 2ay_1 \otimes v_1$$

so that

$$\begin{tikzcd} Q(1) \arrow[d, "{(-2a,c)}"] \\ S^2Q(-1) \end{tikzcd}$$

For

$$S^2Q(-1) \longleftarrow S^3Q$$

we have

$$n_2(w_1 \otimes v_1) = ay_1 \otimes v_1 + \frac{c}{2} z_2 \otimes v_1$$

and since we have the decomposition

$$z_2 \otimes v_1 = \frac{1}{2}(z_1 \otimes v_2 + z_2 \otimes v_1) + \frac{1}{2}(-z_1 \otimes v_2 + z_2 \otimes v_1)$$

we get

$$S^2Q(-1) \xleftarrow{\begin{pmatrix} a \\ \frac{c}{4} \end{pmatrix}} S^3Q$$

For

$$Q(-2) \longleftarrow S^2Q(-1)^I$$

we have

$$n_2(y_1 \otimes v_1) = b(x_2 \otimes v_1)$$

and since we have the decomposition

$$x_2 \otimes v_1 = \frac{1}{3}(x_1 \otimes v_2 + x_2 \otimes v_1) + \frac{1}{3}(-x_1 \otimes v_2 + 2x_2 \otimes v_1)$$

we get

$$Q(-2) \xleftarrow{\frac{b}{3}} S^2Q(-1)^I$$

For

$$Q(-2) \longleftarrow S^2Q(-1)^{II}$$

we have

$$n_2(-z_1 \otimes v_2 + z_2 \otimes v_1) = -dx_1 \otimes v_2 + dx_2 \otimes v_1$$

and since we have the decomposition

$$x_1 \otimes v_2 = \frac{2}{3}(x_1 \otimes v_2 + x_2 \otimes v_1) - \frac{1}{3}(-x_1 \otimes v_2 + 2x_2 \otimes v_1)$$

we get

$$Q(-2) \xleftarrow{\frac{2d}{3}} S^2Q(-1)^{II}$$

For

$$\begin{array}{c} S^2Q(-1)^I \\ \big\downarrow \\ S^3Q(-3) \end{array}$$

we have

$$n_1(y_1 \otimes v_1) = b(x_1 \otimes v_1)$$

and we get

$$\begin{array}{c} S^2Q(-1)^I \\ \big\downarrow {\scriptstyle b} \\ S^3Q(-3) \end{array}$$

For

$$\begin{array}{c} S^2Q(-1)^{II} \\ \big\downarrow \\ S^3Q(-3) \end{array}$$

we have

$$n_1(-z_1 \otimes v_2 + z_2 \otimes v_1) = -d(x_1 \otimes v_1)$$

and we get

$$\begin{array}{c} S^2Q(-1)^{II} \\ \big\downarrow {\scriptstyle -d} \\ S^3Q(-3) \end{array}$$

Putting together

$$
\begin{tikzcd}
\mathcal{O} \arrow[d] & Q(1) \arrow[d, "{(-2a,c)}"] & \\
Q(-2) & S^2Q(-1)^I \oplus S^2Q(-1)^{II} \arrow[l, "{\begin{pmatrix}\frac{b}{3}\\ \frac{2d}{3}\end{pmatrix}}"'] \arrow[d, "{(b,-d)}"] & S^3Q \arrow[l, "{\begin{pmatrix}a\\ \frac{c}{4}\end{pmatrix}}"'] \arrow[d] \\
 & S^3Q(-3) & S^4Q(-2) \arrow[l]
\end{tikzcd}
$$

Now call

$$
\begin{tikzcd}
\mathcal{O} \arrow[d] & Q(1) \arrow[l] \arrow[d, "f_1"] & \\
Q(-2) & S^2Q(-1) \arrow[l, "f_2"'] \arrow[d, "f_4"] & S^3Q \arrow[l, "f_3"'] \arrow[d] \\
 & S^3Q(-3) & S^4Q(-2) \arrow[l]
\end{tikzcd}
$$

we have

$$
\begin{aligned}
f_2f_1 &= -\frac{2ab}{3} + \frac{2cd}{3} = -\frac{cd}{3} \\
f_2f_3 &= \frac{ab}{3} + \frac{cd}{6} = \frac{2cd}{3} \\
f_4f_1 &= -2ab - cd = -4cd \\
f_4f_3 &= ab - \frac{cd}{4} = \frac{5cd}{4}
\end{aligned}
$$

and the projective invariant

$$\frac{(f_2f_3)(f_4f_1)}{(f_2f_1)(f_4f_3)} = \frac{32}{5}$$

Second example: adC

Our second example is adC where C is the rank 5 bundle defined by

$$Q(-2) \xleftarrow{n_2} S^2Q(-1)^C$$

We denote as above

$$< z_1, z_2 = g(z_1), z_3 = g^2(z_1) >= S^2Q(-1)^I$$

$$< x_1, x_2 = g(x_1) >= Q(-2)$$

and set $n_2z_1 = x_1$. The diagram of these weights is

$$
\begin{tikzcd}
z_3 \arrow[d, "n_1"] & & \\
x_2 & z_2 \arrow[l, "n_2"'] \arrow[ul, "g_2"'] \arrow[d, "n_1"] & \\
 & x_1 \arrow[ul, "g_2"'] & z_1 \arrow[l, "n_2"'] \arrow[ul, "g_2"']
\end{tikzcd}
$$

It is well known that C is a exceptional bundle (it has no deformations) and it is the middle kernel of the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-1) \otimes S^2 V \longrightarrow \mathcal{O} \otimes adV \longrightarrow \mathcal{O}(1) \otimes \Gamma^{2,1,1} \longrightarrow 0$$

Anyway we do not need these properties.

From the Prop. 1.2 we check that

$$n_1 z_2 = -x_1 \qquad n_2 z_2 = x_2$$

The dual diagram for $C^\vee$ given by

$$\begin{tikzcd} Q(1) \arrow[d, "n_1"] \\ S^2Q(-1)^{C*} \end{tikzcd}$$

is (with obvious notations)

$$\begin{tikzcd} v_3 & w_2 \arrow[l] \arrow[d] & \\ & v_2 \arrow[ul] & w_1 \arrow[l] \arrow[ul] \arrow[d] \\ & & v_1 \arrow[ul] \end{tikzcd}$$

Set $n_1 w_1 = v_1$ As before, it is easy to check from the Prop. 1.2 that

$$n_1 w_2 = n_2 w_1 = \frac{v_2}{2} \qquad n_2 v_2 = \frac{w_3}{2}$$

Note that in this case θ is given by the single map

$$S^2Q(-1)^C \otimes \Omega^1 \to Q(-2)$$

while ψ is given by the single map

$$Q(1) \otimes \Omega^1 \to S^2Q(-1)^{C*}$$

We choose maximal vectors for all the direct summands of $C \otimes C^* = adC \oplus \mathcal{O}$. According to the decomposition

$$Q(-1) \otimes S^2Q(-1)^C = S^3Q \oplus Q(1)$$

we have

S^3Q	$Q(1)$
$w_1 \otimes z_1$	
$w_2 \otimes z_1 + w_1 \otimes z_2$	$2w_2 \otimes z_1 - w_1 \otimes z_2$
$\vdots$	$\vdots$

According to the decomposition

$$Q(1) \otimes Q(-2) = S^2Q(-1)^I \oplus \mathcal{O}^I$$

we have

$S^2Q(-1)^I$	$\mathcal{O}^I$
$w_1 \otimes x_1$	
$w_1 \otimes x_2 + w_2 \otimes x_1$	$2w_1 \otimes x_2 - w_2 \otimes x_1$
$2w_2 \otimes x_2$	

According to the decomposition

$$S^2Q(-1)^{C*} \otimes S^2Q(-1)^C = S^4Q(-2) \oplus S^2Q(-1)^{II} \oplus \mathcal{O}^{II}$$

we have

$S^4Q(-2)$	$S^2Q(-1)^{II}$	$\mathcal{O}^{II}$
$v_1 \otimes z_1$		
$v_1 \otimes z_2 + v_2 \otimes z_1$	$-v_1 \otimes z_2 + v_2 \otimes z_1$	
$\vdots$	$\vdots$	$\vdots$

According to the decomposition

$$S^2Q(-1)^{C*} \otimes Q(-2) = S^3Q(-3) \oplus Q(-2)$$

we have

$S^3Q(-3)$	$Q(-2)$
$v_1 \otimes x_1$	
$v_1 \otimes x_2 + v_2 \otimes x_1$	$2v_1 \otimes x_2 - v_2 \otimes x_1$
$\vdots$	$\vdots$

Now exactly as in the first example we have for adC after some computations

$$\begin{array}{ccccc}
\mathcal{O} & & Q(1) & & \\
\downarrow & \left(\begin{array}{c} \longleftarrow \\ -\frac{1}{6} \\ -\frac{2}{3} \end{array}\right) & \Big\downarrow{\scriptstyle (1,1)} & \left(\begin{array}{c} 1 \\ \frac{1}{4} \end{array}\right) & \\
Q(-2) & \longleftarrow & S^2Q(-1)^I \oplus S^2Q(-1)^{II} & \longleftarrow & S^3Q \\
 & & \Big\downarrow{\scriptstyle (1,1)} & & \Big\downarrow \\
 & & S^3Q(-3) & \longleftarrow & S^4Q(-2)
\end{array}$$

and with the same notations of the first example we have

$$f_2 f_1 = -\frac{5}{6}$$

$$f_2 f_3 = -\frac{1}{3}$$
$$f_4 f_1 = 2$$
$$f_4 f_3 = \frac{5}{4}$$

and the projective invariant

$$\frac{(f_2 f_3)(f_4 f_1)}{(f_2 f_1)(f_4 f_3)} = \frac{16}{25}$$

Since the two projective invariants are differents we obtain that $adQ \otimes adV$ and adC are

not isomorphic as G-homogeneous bundles. It can be shown they are not isomorphic even as holomorphic bundles.

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