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The Theorem of Mather on Generic Projections for Singular Varieties

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Abstract. The theorem of Mather on generic projections of smooth algebraic varieties is also proved for the singular ones.

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1. Introduction

In [1], a self-contained proof appeared of the following transversality theorem of Mather on generic projections (see [2]) in the setting of algebraic geometry:

THEOREM 1.1. *Let X be a smooth subvariety of codimension c of the complex projective space $\mathbf{P}^n$. Let T be any linear subspace of $\mathbf{P}^n$ of dimension t such that $T \cap X = \emptyset$ (so $t \leqslant c-1$). For any $i_1 \leqslant t+1$ let $X_{i_1} = \{x \in X \mid \dim [(TX)_x \cap T] = i_1 - 1\}$ (the dimension of $\emptyset$ is -1). When X_{i_1} is smooth, for any $i_2 \leqslant i_1$ define $X_{i_1,i_2} = \{x \in X_{i_1} \mid \dim[(TX_{i_1})_x \cap T] = i_2 - 1\}$ and so on; for $i_k \leqslant ... i_2 \leqslant i_1$ define (when possible) $X_{i_1},\dots,_{i_k}$. For T in a Zariski open set of the Grassmannian $Gr(\mathbf{P}^t, \mathbf{P}^n)$, each $X_{i_1},\dots,_{i_k}$ is smooth (and so the above definitions are possible) until (increasing k) it becomes empty and its codimension υ_I in X can be calculated (where $I = (i_1, i_2 \dots . i_k)$).*

We refer to [1] for the calculation of υ_I and for comments and remarks about the theorem.

This theorem was stated for smooth subvarieties of $\mathbf{P}^n$ but the same proof can also be used for the smooth open set X of a singular algebraic variety Y except for the crucial th. 3.15, (p. 409 of [1]), in which the compactness of X is needed.

In this short note we want to replace the proof in [1] with a little longer proof which also works in the case under examination. We obtain the following theorem:

THEOREM 1.2. *Theorem 1.1 still holds if X is replaced with the smooth open subvariety of a possibly singular projective variety Y.*

2. Background

Let Y be a singular algebraic subvariety of the n-dimensional projective space $\mathbf{P}^n$ over the complex numbers. Let X be the smooth open set of Y. First of all we outline the proof of Mather's theorem given in [1] and we introduce some notation.

Fix an integer t with $0 \leqslant t \leqslant c-1$. Let L be a $(n-t-1)$-dimensional linear subspace of $\mathbf{P}^n$.

Let $F = \{\mathbf{P}^t \in Gr(\mathbf{P}^t, \mathbf{P}^n) | \mathbf{P}^t \cap X = \emptyset \text{ and } \mathbf{P}^t \cap L = \emptyset\}$. For any $f \in F$ let $p_f\colon X \to L$ be the linear projection centered in f and let $j^k p_f$ be its k-jet ($j^k p_f\colon X \to J^k(X, L)$ sends every $x \in X$ into the k-jet of p_f in x, see [1] for the definition of $J^k(X, L)$). Let $I = (i_1, i_2 \cdots, i_k)$ be any sequence of integers with $(i_1 \geqslant i_2 \cdots \geqslant i_k \geqslant 0)$.

Let $g\colon X \times F \to J^k(X, L)$ be given by: $g(x, f) = (j^k p_f)_x$.

The proof of Mather's theorem is divided into two steps:

(1) define in $J^k(X, L)$ some submanifolds Σ^I with the property that $j^k p_f^{-1}(\Sigma^I) = X_I$ (when X_I are defined), this definition is not trivial and it is due to Boardman: Σ^I are the so-called Thom–Boardman singularities, they are smooth, locally closed and of codimension υ_I;
(2) show that there exists a Zariski open set $U \in F$ such that for any $f \in U$, $j^k p_f : X \to J^k(X, L)$ is transversal to Σ^I.

The proof of step (1) runs exactly as in [1].

To prove step (2) first we remark (see [1], prop. 3.13) that for any smooth subvariety $W \subset J^k(X, L)$ there exists a Zariski open set $U \in F$ such that for any $f \in U$, $j^k p_f\colon X \to J^k(X, L)$ is transversal to W if g is transversal to W. Second, we give the following definition: let $\varphi\colon X \to J^k(X, L)$ be a holomorphic map and let $W \subset J^k(X, L)$ be a smooth subvariety, then define:

$$\delta(\varphi, W, x) = 0 \quad \text{if } \varphi(x) \notin W$$

$\delta(\varphi, W, x) = \dim[J^k(X, L)] - \dim[TW_{\varphi(x)} + \mathrm{d}\varphi(TX)_x]$, if $\varphi(x) \in W$, where TW and TX are the tangent spaces and d stands for the usual differential.

Note that $\delta(\varphi, W, x) \geqslant 0$ and that φ is transversal to W at x if and only if $\delta(\varphi, W, x) = 0$.

As in [1], th. 3.10 and 3.11, it can be shown that for $W = \Sigma^I \subset J^k(X, L)$ the following condition (*) is satisfied:

(*) $\delta(g, W, (x, f)) \leqslant \delta(j^k p_f, W, x)$ for any $(x, f) \in X \times F$ and equality holds if and only if $\delta(j^k p_f, W, x) = 0$.

Therefore to prove step (2) all that we need is the following:

THEOREM 2.1. *With the previous notation, assume that condition* $(*)$ *is satisfied for some smooth subvariety* $W \subset J^k(X, L)$*; then there exists a Zariski open set* $U \in F$ *such that for any* $f \in U$, $j^k p_f : X \to J^k(X, L)$ *is transversal to* W.

The proof of this theorem (th. 3.15 in [1]) must be rewritten in our case. In Section 3 we will give this proof and so we will also prove Theorem 1.2.

3. Proof of Theorem 2.1

Let us define $\delta_g = \mathrm{Sup}_{(x,f)\in X\times F}\{\delta(g, W, (x,f))\}$; moreover, let us define $A = \{(x,f) \in X \times F | \delta(g, W, (x,f)) = \delta_g\} \subset X \times F$, A is a Zariski closed set in $X \times F$. Note that Theorem 2.1 is true if $\delta_g = 0$ (see th. 3.13 in [1]), so we can assume $\delta_g \neq 0$ and $A \neq \emptyset$.

Let $\pi_2 : X \times F \to F$ be the natural projection. $X \times F$ is equipped with the induced Zariski topology from $Y \times F$. Let $\bar{A}$ be the Zariski closure of A in $Y \times F$; let $\pi_3 : Y \times F \to F$ be the natural projection, $\pi_3(\bar{A})$ is a Zariski closed set of F. If $\pi_3(\bar{A})$ is a proper subset of F, we can consider $F' = F \backslash \pi_3(\bar{A})$ and $g' = g_{|X\times F'}$. The assumptions of the theorem are true for F' and g' and $\delta_{g'} < \delta_g$. If the corresponding $\pi_3(\bar{A})$ were a proper subset of F', we would get F'' and g'' and so on. After a finite number of steps, we would get F' and g', for which the assumptions would be still true, with $\delta_{g'} = 0$, so the theorem would be proved.

Hence, we have only to prove that $\pi_3(\bar{A})$ is a proper subset of F.

By contradiction, let us assume that $\pi_3(\bar{A}) = F$, then $F = \overline{\pi_2(A)}$.

We can choose $(x_0, f_0) \in A$ and $z_0 = (j^k g)_{(x_0, f_0)} \in W$. As $\delta(g, W, (x_0, f_0))$ is strictly positive, by assumption we get that $\delta(j^k p_{f_0}, W, x_0)$ is strictly positive too, hence $j^k p_{f_0}$ is not transversal to W at x_0.

W is smooth at x_0 so it is a local complete intersection, then it is possible (see [1], proof of th. 3.15) to get a smooth subvariety $W' \subset J^k(X, L)$ and a smooth dense open Zariski set $Z \subset X \times F$ such that: $W \subseteq W'$, $\dim(W') - \dim(W) = \delta_g$, g is transversal to W' at (x,f) for any $(x,f) \in Z$.

The holomorphic map $g_{|Z} : Z \to J^k(X, L)$ is transversal to W' so that $g_{|Z}^{-1}(W') = g^{-1}(W') \cap Z$ is smooth in $X \times F$.

Let us consider

$$\pi = \pi_{2_{|g^{-1}(W')\cap Z}} = \pi_{3_{|g^{-1}(W')\cap Z}} : g^{-1}(W') \cap Z \to F.$$

It is easy to see that

(1) $\overline{\pi_2(A \cap Z)} = F$.

Hence $F = \overline{\pi_2(Z)}$. Moreover $F = \overline{\pi_2(g^{-1}(W'))}$, otherwise there would exist a Zariski open set $B \subset F$ such that $B \cap \overline{\pi_2(g^{-1}(W'))} = \emptyset$, hence for any $f \in B$ and for any $x \in X$, $(x,f) \notin g^{-1}(W')$, i.e. $g(x,f) \notin W'$, i.e. $g(x,f) \notin W$, i.e. for any $f \in B$ and for any $x \in X$, $\delta(g, W, (x,f)) = 0$ and the theorem would be immediately proved (see th. 3.13 of [1]).

It follows:

$$\overline{\pi_2(g^{-1}(W') \cap Z)} \subseteq \overline{\pi_2(g^{-1}(W')) \cap \pi_2(Z)} \subseteq \overline{\pi_2(g^{-1}(W'))} \cap \overline{\pi_2(Z)} = F,$$

therefore:

(2) $\overline{\pi(g^{-1}(W') \cap Z)} = F.$

Now we consider the holomorphic map $\pi: g^{-1}(W') \cap Z \to F$ between smooth manifolds, as (2) holds there exists a Zariski open set $D \subset F$ such that for any $f \in D$ $\pi^{-1}(f)$ is smooth and of the expected codimension.

By (1) $[\pi_2(A \cap Z)] \cap D \neq \emptyset$, then we can choose $f_1 \in [\pi_2(A \cap Z)] \cap D$ such that $\pi^{-1}(f_1)$ is smooth, of the expected codimension and biholomorphic to a Zariski open set of $(j^k p_{f_1})^{-1}(W') \subset X$. We can also choose $x_1 \in X$ such that $(j^k p_{f_1})^{-1}(W')$ is smooth, of the expected codimension and smooth at x_1. This fact implies that $j^k p_{f_1}$ is transversal to W' at x_1, (see [1], th. 1.2), i.e. $\delta(j^k p_{f_1}, W', x_1) = 0$.

On the other hand $f_1 \in \pi_2(A \cap Z)$, hence it is possible to choose $x_1 \in X$ such that $(x_1, f_1) \in A$, i.e. $\delta(g, W, (x_1, f_1)) = \delta_g$.

Let $z_1 = (j^k p_{f_1})_{x_1}$ then:

$$\delta(j^k p_{f_1}, W', x_1) = \dim[J^k(X, L)] - \dim[(TW')_{z_1} + dj^k p_{f_1}(TX)_{x_1}]$$

$$\delta(j^k p_{f_1}, W, x_1) = \dim[J^k(X, L)] - \dim[(TW)_{z_1} + dj^k p_{f_1}(TX)_{x_1}]$$

and

$$0 = \delta(j^k p_{f_1}, W', x_1) \geqslant \delta(j^k p_{f_1}, W, x_1) - \delta_g.$$

But assumption $(*)$ and the fact that $(x_1, f_1) \in A$ imply:

$$0 \geqslant \delta(j^k p_{f_1}, W, x_1) - \delta_g > \delta(g, W, (x_1, f_1)) - \delta_g = \delta_g - \delta_g = 0,$$

a contradiction!

4. Cones

In this brief section we want to remark that when Y is a cone, it is possible to use Mather's theorem (1.1). For instance, let us assume that Y is a cone in $\mathbf{P}^n$ of vertex V on a smooth subvariety B of $\mathbf{P}^n$ whose span is $\mathbf{P}^s$ with $\dim(Y) = \gamma = b + v + 1$, $\dim(B) = b$, $\dim(V) = v$, $n = s + v + 1$.

Let T be a generic t-dimensional subspace of $\mathbf{P}^n$ with: $T \cap Y = \emptyset$, $t \leqslant \gamma - 1$, $\gamma \geqslant (t+1)(n-\gamma)$. Let $Y_{t+1} = \{y \in Y \mid y \text{ is a smooth point}, (TY)_y \supset T\}$.

If Y were smooth Mather's theorem (1.1) would say that, for generic T, Y_{t+1} is a smooth subvariety of Y and $\dim(Y_{t+1}) = \gamma - (t+1)(n-\gamma)$, in our case we have:

PROPOSITION. *The closure of* Y_{t+1} *is a cone of dimension* $\gamma - (t+1)(n-\gamma)$ *with vertex* V *over a smooth variety.*

As Y is a cone we remark that $t \leqslant s-1$ ($t \leqslant n-\gamma-1 = s-b-1$), hence there exists a linear subspace $H \simeq \mathbf{P}^s$ in $\mathbf{P}^n$ such that $H \supset T$ and $H \cap V = \emptyset$. We can assume that $B = H \cap Y$ and we can apply Theorem 1.1 to $\mathbf{P}^s$, T and B as B is smooth, $T \cap B = \emptyset$ and T is generic in $\mathbf{P}^s$ with respect to B. If $t \leqslant b-1$ and $b \geqslant (t+1)(s-b)$ (for instance when $t = 0$ and $2b \geqslant s$) then $B_{t+1} = \{y \in B | (TB)_y \supset T\}$ is a smooth subvariety of B and $\dim(B_{t+1}) = b - (t+1)(s-b)$. On the other hand, $(TB)_y \supset T$ if and only if $(TY)_y \supset T$ as $(TY)_y = \langle V, (TB)_y \rangle$ i.e. $(TB)_y = (TY)_y \cap H$, hence $Y_{t+1} \cap H = B_{t+1}$ and the closure in Y of Y_{t+1} is another cone of vertex V over B_{t+1}. This cone has dimension $b - (t+1)(s-b) + v + 1 = \gamma - (t+1)(n-\gamma)$ which is exactly the expected dimension when Y is smooth.

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