Symplectic bundles on the plane, secant varieties and Lüroth quartics revisited

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Contents

Abstract

Let $X = \mathbf{P}^2 \times \mathbf{P}^{n-1}$ embedded with $\mathcal{O}(1,2)$. We prove that its k-secant variety $\sigma_k(X)$ is a hypersurface for $k = \frac{3n}{2} - 1$, n even, $n \geq 4$, while it is expected that it fills the ambient space. The equation of $\sigma_k(X)$ is the symmetric analog of the Strassen equation. When $n = 4$ the determinantal map takes $\sigma_5(X)$ to the hypersurface of Lüroth quartics, which is the image of the Barth map studied by LePotier and Tikhomirov. This hint allows to obtain some results on the jumping lines and the Brill-Noether loci of symplectic bundles on $\mathbf{P}^2$ by using the higher secant varieties of X.

1. Introduction

A Lüroth quartic is a quartic plane curve which has an inscribed complete pentagon. By a naive dimensional count (see Rem. 6.10), one expects that the general quartic curve has such a inscribed pentagon. The classical theorem of Lüroth, published in 1869, says that a Lüroth quartic has not only one, but infinitely many inscribed pentagons. In equivalent way, Lüroth quartics form a hypersurface in the space of quartics, so destroying the naive numerical expectation. One century later, Barth showed [Bar] the remarkable result that the curve of jumping lines of a rank 2 stable bundle with $(c_1, c_2) = (0, 4)$ on $\mathbf{P}^2$ is a Lüroth quartic. Lüroth's theorem became again popular, and Barth gave in [Bar] a new proof of it by using vector bundles. The equation of the Lüroth hypersurface is a interesting $SL(3)$-invariant, and its degree, which is 54, turned out to be the first nontrivial Donaldson invariant of the plane.

Denote by $M(2, n)$ the moduli space of stable 2-bundles on $\mathbf{P}^2$ with $(c_1, c_2) = (0, n)$. The morphism which associates to any $E \in M(2, n)$ its curve of jumping lines is called the Barth map.

Le Potier and Tikhomirov finally showed [PT] that, conversely, a general Lüroth quartic is the curve of jumping lines of a unique bundle $E \in M(2, 4)$. They generalized this fact by showing that the Barth map from $M(2, n)$ to the variety of plane curves of degree n is generically injective.

The second subject of this paper is the higher secant varieties. For a projective variety $W \subset \mathbf{P}^m$ of dimension n, the k-secant variety $\sigma_k(W)$ is the closure

of the locus spanned by k independent points in W. The expected dimension of $\sigma_k(W)$ is $\min\{m, kn + (k-1)\}$. When $\dim \sigma_k(W) < \min\{m, kn + (k-1)\}$ we say that W is k-defective.

A theorem due to Strassen [S] says that $Y = \mathbf{P}^2 \times \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ with its Segre embedding is k-defective for n odd, $n \geq 3$, $k = \frac{3n-1}{2}$. Indeed, despite of the numerical naive expectation that $\sigma_k(X)$ fills the ambient space, Strassen proved that $\sigma_k(X)$ is a hypersurface and gave an equation for it, see Rem. 3.3. This equation is called the Strassen equation and it was later generalized in the papers [LM] and [LW], where it was put in the setting of invariant theory. In [CGG1] and in [AOP] it was studied the defectivity of Segre varieties, and in [AOP] it was proposed a conjecture about which Segre varieties are indeed defective. The equations of higher secants of Segre varieties are studied, among others, in [LM] and [LW]. The case $\mathbf{P}^2 \times \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ for n odd is probably the most interesting known class of defective Segre varieties, and its description escaped the geometric techniques working in the other cases.

The starting point of this paper was the observation that Strassen equations are (almost) identical to the equations called $\alpha 3$ by Barth in [Bar]. This is a interesting link between secant varieties and vector bundles, and we tried to explore further this connection.

A linear algebra approach to $Y = \mathbf{P}^2 \times \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ in the spirit of [Bar] gives quickly a proof that Strassen equation define the hypersurface $\sigma_k(Y)$ for n odd, $n \geq 3$, $k = \frac{3n-1}{2}$. This is explained in section 3 (see Thm. 3.2). The main argument is the following. Let $\mathbf{P}^2 \times \mathbf{P}^{n-1} \times \mathbf{P}^{n-1} = \mathbf{P}(U) \times \mathbf{P}(V) \times \mathbf{P}(W)$. For any $\phi \in U \otimes V \otimes W$, we consider the contraction operator

$$A_\phi \colon U \otimes V^\vee \longrightarrow \wedge^2 U \otimes W$$

If $\phi \in Y$ then $rkA_\phi = 2$, it follows that if $\phi \in \sigma_k(Y)$ then $rkA_\phi \leq 2k$. So, for $k = \frac{3n-1}{2}$, the tensors $\phi \in \sigma_k(Y)$ have a degenerate A_ϕ and cannot be general. The Strassen equation is simply

$$\det A_\phi = 0$$

A natural generalization to the symmetric case is given in section 4.

The symmetric case corresponds to the Segre-Veronese variety $X = \mathbf{P}^2 \times \mathbf{P}^{n-1}$ embedded by $\mathcal{O}(1,2)$. We prove that it is k-defective for $n \geq 4$, n even, $k = \frac{3n}{2} - 1$, see Thm. 4.1.

Note that the defective cases appear with n odd in the Segre case and with n even in the Segre-Veronese case, adding a touch of intrigue.

Our next result is a proof of Lüroth's theorem (Thm. 6.5) by using the defectivity of the above Segre-Veronese X. In other words, the failure of the numerical expectation in Lüroth theorem and in Strassen theorem are two faces of the same phenomenon.

The vector bundles $E \in M(2, n)$ were studied in [Bar] as cohomology bundles of Barth monads

$$I^\vee \otimes \mathcal{O} \longrightarrow V^\vee \otimes \Omega^1(2) \overset{f}{\longrightarrow} V \otimes \mathcal{O}(1)$$

where V is a vector space of dimension n and I is a vector space of dimension $n - 2$. The Lüroth quartics are defined by the symmetric determinantal morphism Δ of $f \in \mathbf{P}(U \otimes S^2 V)$ when $n = 4$ (see [D] and [DK]). An easy but crucial remark is that the linear map $H^0(f)$ is the symmetric analog of A_f, and it is denoted by S_f. This allows to deepen the link between vector bundles on $\mathbf{P}^2$ and secant varieties to X.

A plane curve of degree n which is circumscribed to a complete $(n+1)$-gon is called a Darboux curve. It is easy to show, by using the theory of secants, that there are only finitely many inscribed complete $(n+1)$-gon in a Darboux curve for $n \geq 5$, see the Cor. 6.9. We point out how a question of Ellingsrud, Le Potier, Stromme and Tikhomirov about the uniqueness of these inscribed $(n+1)$-gons is related to the non weak $(n+1)$-defectivity of the Segre-Veronese variety $\mathbf{P}^2 \times \mathbf{P}^{n-1}$ embedded with $\mathcal{O}(1,2)$ (see Thm. 6.12), by a result of Chiantini and Ciliberto [CC2] .

The Darboux curves correspond, through the Barth map, to $E \in M(2, n)$ such that $E(1)$ has at least one section. We will show (Thm. 8.14) that they are defined through the determinantal morphism Δ from $\sigma_{n+1}(X)$, so generalizing the Lüroth case.

The natural objects that come from this description are the symplectic bundles on $\mathbf{P}^2$. In the section 7 we construct the moduli space $M_{sp}(r, n)$ of

symplectic bundles on $\mathbf{P}^2$ of rank r with $c_2 = n$ and prove that it is irreducible. This result follows the lines of [Hu] and it is certainly well known to experts, but we do not know a reference for it.

In the section 8 we study the Barth map in the higher rank case, and we obtain some numerical results on it and on Brill-Noether loci in $M_{sp}(r, n)$. When $r = 2$ we get an alternative approach to the results of [Bar]. We describe the curves of jumping lines of a general $E \in M_{sp}(n - 2, n)$, see the Thm. 8.11.

This note has the purpose to serve as an introduction to the subject. So we tried to give elementary proofs of several of the results that we use, even in the case where they follow from more advanced results available in the literature. An exception to this phylosophy is the Beilinson Theorem, which we believe is a basic tool, so that it is useful to practice it from the beginning.

I thank Maria Virginia Catalisano for helpful discussions concerning the defectivity of Segre-Veronese varieties and Pietro Pirola and Fabrizio Catanese who provided the examples in Prop. 8.6. Catanese clarified also the proof of Dixon Theorem 5.1. I thank the referee who improved the presentation of this paper.

2. Notations and generalities on higher secant varieties

Let V be a complex vector space of dimension n. We denote by $\mathbf{P}(V)$ the projective space of lines in V, so that $H^0(\mathbf{P}(V), \mathcal{O}(1)) = V^\vee$.

For every $a \in \mathbf{R}$, denote by $\lceil a \rceil$ the smallest integer greater or equal than a.

We recall that for a projective variety $W \subset \mathbf{P}^m$, the k-secant variety $\sigma_k(W)$ is the Zariski closure of the set $\bigcup_{x_1,\ldots,x_k \in W} < x_1, \ldots, x_k >$, where $< x_1, \ldots, x_k >$ denotes the linear span of $x_1, \ldots, x_k$.

A space spanned by $x_1, \ldots, x_k \in W$ is called a k-secant space. The expected dimension of $\sigma_k(W)$ is $\min\{m, kn + (k-1)\}$ and it always holds the inequality $\dim \sigma_k(W) \le \min\{m, kn + (k-1)\}$. When $\dim \sigma_k(W) < \min\{m, kn + (k-1)\}$ we say that W is k-defective. W is called defective if it is k-defective for some k.

We can define the abstract secant variety $\sigma^k(W)$ as the Zariski closure of the incidence variety

$$\sigma_0^k(W) = \{(p, x_1, \ldots, x_k) \in \mathbf{P}^m \times Sym^k W | p \in < x_1, \ldots, x_k >, \dim < x_1, \ldots, x_k >= k\}$$

It is easy to check (see [Ru] or [Z]) that $\sigma_k(W)$ is the image of the projection on the first factor of $\sigma^k(W)$. Since $\sigma_0^k(W)$ is fibered over a open subset of $Sym^k W$ with fibers isomorphic to $\mathbf{P}^{k-1}$, it follows that $\sigma^k(W)$ is irreducible of dimension $kn + (k-1)$.

Proposition 2.1. If $\dim \sigma_k(W) = kn + (k-1) - d$, then the general point in $\sigma_k(W)$ belongs to ∞^d k-secant spaces of dimension k.

PROOF – By assumption the general fibers of the projection $\sigma^k(W) \longrightarrow \sigma_k(W)$ have dimension d. $\qquad\square$

For improvements of the above proposition see [Ru] or [Z].

Definition 2.2. [CC1] W is called k-weakly defective if the general hyperplane which is tangent in k points is tangent along a variety of positive dimension.

The terminology is justified by a result due to Terracini, who proved that k-defective varieties are also k-weakly defective (see[CC1]). In [CC1] are provided counterexamples for the converse and there is a classification of k-defective varieties in small dimension.

Definition 2.3. [CC2] The k-secant order $d_k(W)$ is the number of irreducible components of the general fiber of the morphism $\sigma^k(W) \longrightarrow \sigma_k(W)$.

When $\dim \sigma_k(W) = kn + (k-1)$, then $d_k(W)$ is the degree of the morphism $\sigma^k(W) \longrightarrow \sigma_k(W)$.

d_k measures how many k-secant spaces pass through a general point in $\sigma_k(W)$.

The secant order d_k is called μ_{k-1} in [CC2], due on a different terminology about higher secant varieties. We choosed to change the letter in order to avoid any confusion.

Theorem 2.4 (Chiantini-Ciliberto, [CC2], Corollary 2.7). Let $W \subset \mathbf{P}^m$ be a variety such that $\dim \sigma_k(W) = kn + (k-1) < m$. Then $d_k(W) = 1$ unless it is k-weakly defective.

3. Strassen equations and the tensors $3 \times n \times n$

The aim of this section is to give a simpler proof of the defectivity of $\mathbf{P}^2 \times \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ [S].

Let U, V, V' be vector spaces of dimension respectively $3, n, n$. The variety of the decomposable tensors in the projective space $\mathbf{P}(U \otimes V \otimes V')$ is the Segre variety $Y = \mathbf{P}(U) \times \mathbf{P}(V) \times \mathbf{P}(V')$.

The expected minimal p such that $\sigma_p(Y) = \mathbf{P}^m$ is

$$p = \lceil \frac{3n^2}{2n+1} \rceil$$

It is elementary to check that

(3.1) $$2p = \begin{cases} 3n - 1 & \text{if } n \text{ is odd} \\ 3n & \text{if } n \text{ is even }, \end{cases}$$

For any $\phi \in U \otimes V \otimes V'$, we consider the contraction operator

$$A_\phi : U \otimes V^\vee \longrightarrow \wedge^2 U \otimes V'$$

If u_1, u_2, u_3 is a basis of U and $\phi = u_1 R - u_2 Q + u_3 R$ (so that P, $-Q$, R are the three $n \times n$ slices of ϕ), the matrix representing ϕ with respect to the bases u_1, u_2, u_3 for U and $u_2 \wedge u_3, u_3 \wedge u_1, u_1 \wedge u_2$ for $\wedge^2 U$ is

$$\begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix}$$

Lemma 3.1. *Let Q be invertible. Then*

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & -PQ^{-1} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -Q^{-1}P & I \end{bmatrix} = \begin{bmatrix} 0 & 0 & Q \\ 0 & Z & R \\ -Q & -R & 0 \end{bmatrix}$$

where $Z = PQ^{-1}R - RQ^{-1}P = Q[Q^{-1}P, Q^{-1}R]$. *Moreover* $rk(A_\phi) = 2n + rk(Z) = 2n + rk[Q^{-1}P, Q^{-1}R]$

PROOF – Straightforward. □

From the Lemma 3.1 it follows in particular

$$
\det \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix} = (\det Q)^2 \det \left(PQ^{-1}R - RQ^{-1}P \right)
$$

and hence the formula

$$
\det \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix} (\det Q)^{n-2} = \det \left(P \cdot adj(Q) \cdot R - R \cdot adj(Q) \cdot P \right)
$$

which holds for any P, Q, R.

Theorem 3.2. *(i) If $\phi \in \sigma_k(\mathbf{P}^2 \times \mathbf{P}^{n-1} \times \mathbf{P}^{n-1})$ then $rkA_\phi \leq 2k$.*

(ii) If ϕ is generic and $n \geq 3$, then $rkA_\phi = 3n$, hence $\mathbf{P}^2 \times \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$ is k-defective if n is odd and $k = \frac{3n-1}{2}$.

PROOF – If ϕ is decomposable, say $\phi = u_1 \otimes v_1 \otimes w_1$ then $rkA_\phi = 2$, indeed $Im\ A_\phi$ is $(u_1 \wedge U) \otimes w_1$.

It follows that if $\phi \in \sigma_k(Y)$ then $rkA_\phi \leq 2k$, proving (i).

Let λ_i, μ_j be generic constants. With obvious notations for basis in the vector spaces U, V, V', let $\phi = u_1 \otimes \left(\sum \lambda_i v_i \otimes w_i \right) + u_2 \otimes \left(\sum v_i \otimes w_i \right) + u_3 \otimes \left(\sum \mu_i v_i \otimes w_{i+1} \right)$, where we denote $w_{n+1} = w_1$. Then with the matrix notations of Lemma 3.1 we have $P = diag(\lambda_i)$, $Q = Id$ and

$$
R = \begin{bmatrix} & \mu_1 & & \\ & & \ddots & \\ & & & \mu_{n-1} \\ \mu_n & & & \end{bmatrix}
$$

Hence

$$
Z = [P, R] = \begin{bmatrix} & \mu_1(\lambda_1 - \lambda_2) & & \\ & & \ddots & \\ & & & \mu_{n-1}(\lambda_{n-1} - \lambda_n) \\ \mu_n(\lambda_n - \lambda_1) & & & \end{bmatrix}
$$

It follows that Z is invertible and moreover by Lemma 3.1 $rkA_\phi = 3n$, proving (ii). The k-defectivity for n odd and $k = \frac{3n-1}{2}$ follows, since by (3.1) $sigma_k(Y)$ is expected to fill the ambient space but, on the other side, we have seen that a generic ϕ is not in σ_k (since $(A_\phi) = 3n$). □

Remark 3.3 - $\sigma_k(Y)$ is the hypersurface with equation $\det A_\phi = 0$. This is proved in [S] lemma 4.4 by computing the tangent spaces at k suitably chosen points in Y. By Lemma 3.1 this equation is equivalent to the Strassen one which has the nice commuting shape

$$\det\left(P \cdot adj(Q) \cdot R - R \cdot adj(Q) \cdot P\right) = 0$$

which has to be divided by $(\det Q)^{n-2}$ in order to get a $SL(U) \times SL(V) \times SL(V')$-invariant form with homogeneous weights.

Remark 3.4 - Also $\mathbf{P}^2 \times \mathbf{P}^3 \times \mathbf{P}^3 \subset \mathbf{P}^{47}$ is defective. In [AOP] it is proved that the dimension of $\sigma_5(\mathbf{P}^2 \times \mathbf{P}^3 \times \mathbf{P}^3)$ is 43 instead of 44 by showing that through five generic points in $\mathbf{P}^2 \times \mathbf{P}^3 \times \mathbf{P}^3$ there is a rational normal curve C_8. Note that the argument in the above proof does not work in this case.

Remark 3.5 - Strassen proves in [S] prop. 4.7 that if n is even then $\sigma_{\frac{3n}{2}}(\mathbf{P}^2 \times \mathbf{P}^{n-1} \times \mathbf{P}^{n-1})$ is the ambient space by exhibiting several explicit tensors.

4. The symmetric tensors $3 \times n \times n$

The aim of this section is to extend the results of section 3 to the symmetric cases. The defectivity of Segre-Veronese varieties has been studied in [CGG2], [CarCh] and [CarCa]. Tony Geramita and Enrico Carlini pointed out to me that the defectivity of $X = \mathbf{P}^2 \times \mathbf{P}^3$ embedded with $\mathcal{O}(1,2)$ was classically known, and it essentially appears in a paper of Emil Toeplitz [T](the father of Otto), see [CarCh] theor. 4.3 (and also [CarCa] 6.2), where the reader can find a modern geometrical proof. It turned out that E. Toeplitz already wrote in 1877 ([T], pag. 441) the equation of $\sigma_5(X)$, which is the symmetric analog to the Strassen equation. The approach of the previous section allows to generalize this result to $X = \mathbf{P}^2 \times \mathbf{P}^{n-1}$.

Let U, V be vector spaces of dimension respectively 3, n.

Consider the Segre-Veronese embedding $X = \mathbf{P}(U) \times \mathbf{P}(V) \longrightarrow \mathbf{P}(U \otimes S^2V) = \mathbf{P}^m$ with the linear system $\mathcal{O}(1,2)$, so that $m = 3\binom{n+1}{2} - 1$.

The expected minimal p such that $\sigma_p(X) = \mathbf{P}^m$ is

$$p = \lceil \frac{m+1}{n+2} \rceil$$

It is elementary to check that

(4.1)
$$2p = \begin{cases} 3n - 1 & \text{if } n \text{ is odd} \\ 3n - 2 & \text{if } n \text{ is even }, n \geq 4 \\ 6 & \text{if } n = 2 \end{cases}$$

We can study this embedding in the following way. Let $\phi \in U \otimes S^2V$. It defines the contraction

$$S_\phi : U \otimes V^\vee \longrightarrow \wedge^2 U \otimes V$$

Identifying $\wedge^2 U$ with $U^\vee$ via an orientation of $\wedge^3 U$ ($SL(U)$-invariant), we have

(4.2)
$$S_\phi^t = -S_\phi$$

Indeed, if P, Q, R are the three $n \times n$ symmetric slices of ϕ, the matrix of S_ϕ in the obvious coordinate system is again

$$\begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix}$$

which is now skew-symmetric.

Theorem 4.1. *(i) If $\phi \in \sigma_k(X)$ then $rkS_\phi \leq 2k$.*

(ii) If ϕ is generic and $n \geq 2$, n is even, then $rkS_\phi = 3n$. Hence we get that X is k-defective for $n \geq 4$, n even, $k = \frac{3n}{2} - 1$.

PROOF – It is analogous to the proof of Thm. 3.2. If ϕ is decomposable then $rkS_\phi = 2$. The variety of the decomposable tensors in the projective space $\mathbf{P}(U \otimes S^2 V)$ is the Segre-Veronese variety $X = \mathbf{P}(U) \times \mathbf{P}(V)$.

It follows that if $\phi \in \sigma_k(X)$ then $rkS_\phi \leq 2k$, proving (i).

Let $n = 2h$, λ_i, μ_j be generic constants for $1 \leq i \leq n$, $1 \leq j \leq h$. With obvious notations for basis in the vector spaces U, V let $\phi = u_1 \otimes (\sum v_i^2) + u_2 \otimes (\sum \lambda_i v_i^2) + u_3 \otimes [\sum \mu_i (v_i + v_{i+h})^2]$. Then $rkS_\phi = 3n$, proving (ii) thanks to (4.1). $\qquad\square$

Remark 4.2 - It is surprising that in the Thm. 3.2 and Thm. 4.1 the odd and even cases exchange. If n is even, $n \geq 4$, the secant variety $\sigma_{\frac{3n-2}{2}}(X)$ is a hypersurface of degree $\frac{3n}{2}$, with equation $Pf(S_\phi) = 0$. This can be proved like in [S] lemma 4.4.

If n is odd then $\sigma_{\frac{3n-1}{2}}(X)$ fills the ambient space, indeed S_ϕ is always singular. Following [S] lemma 4.4, the list of tensors such that their tangent spaces span the ambient space is the following. Let $v_1, \ldots, v_n$ be a basis of V, let u_i, for $1 \leq i \leq n$, and $\tilde{u}_\nu$, for $1 \leq \nu \leq n-2$, ν odd be $\frac{3n-1}{2}$ vectors in U such that their coefficients are algebraically independent over $\mathbf{Q}$.

Then the list of tensors is

$u_i \otimes v_i^2$, for $1 \leq i \leq n$

$\tilde{u}_\nu \otimes (v_\nu + v_{\nu+1} + v_{\nu+2})^2$, for $1 \leq \nu \leq n-2$, ν odd.

5. Generalities on plane curves as linear symmetric determinants

For any $f \in \mathbf{P}(U \otimes S^2 V)$ consider the map $\tilde{f} \colon V^\vee \to V \otimes U$ as a map from $V^\vee$ to V with coefficients in U. Fixed bases of V, $V^\vee$, it can be represented as a $n \times n$ matrix with coefficient linear forms on $\mathbf{P}(U^\vee)$.

Its determinant $\Delta(f)$ gives a morphism

$$\mathbf{P}(U \otimes S^2 V) \setminus Z(\Delta) \xrightarrow{\Delta} \mathbf{P}(S^n U)$$

where $Z(\Delta)$ is the locus $\{f | \Delta(f) \equiv 0\}$. The elements of $\mathbf{P}(S^n U)$ can be regarded as degree n curves in $\mathbf{P}(U^\vee)$.

The morphism Δ was classically studied as discriminant locus of a linear system of quadrics. All the coefficients of Δ are $SL(U)$-invariant.

Assume now $\dim U = 3$, hence $\Delta(f)$, for $f \notin Z(\Delta)$, is the equation of a plane curve C of degree n. The classical Theorem of Dixon, that we review in this section, says that the general plane curve C is in the image of Δ, and each symmetric determinantal representation of C corresponds to a theta characteristic θ on C such that $h^0(\theta) = 0$. In [Ca] and in [Bea1] the Theorem of Dixon is proved as a consequence of a more general result about ACM bundles.

A general $f \in \mathbf{P}(U \otimes S^2 V)$ gives a subspace $\mathbf{P}(U^\vee) \subset \mathbf{P}(S^2 V)$, regarding f as an injective map. The space of quadrics $\mathbf{P}(S^2 V)$ contains the discriminant hypersurface

$$\mathcal{D} := \{g \in S^2 V \mid \det(g) = 0\}$$

and C is identified with the intersection $\mathbf{P}(U^\vee) \cap \mathcal{D}$. Note that $Sing(\mathcal{D})$ consists of the symmetric matrices of rank $\leq n - 2$ and it has codimension three. The general immersion of $\mathbf{P}(U^\vee)$ does not meet $Sing(\mathcal{D})$, hence the curve C is in general smooth by Bertini Theorem. We follow [D], as in section 1 of Chap. 4. We denote by $\phi \colon C \to \mathbf{P}(S^2 V)$ the previous immersion, hence every $x \in C$ defines (up to scalar multiplication) a symmetric morphism $\phi(x) \colon V^\vee \to V$. We consider also the map

$$
\begin{array}{rcl}
\psi \colon & C & \to & (V^\vee) \\
& x & \mapsto & \ker \phi(x)
\end{array}
$$

and the composition

$$\Phi \colon C \xrightarrow{(\psi,\psi)} \mathbf{P}(V^\vee) \times \mathbf{P}(V^\vee) \xrightarrow{s_2} \mathbf{P}(V^\vee \otimes V^\vee)$$

where s_2 is the Segre map. $\Phi(x)$, up to constants, is represented by the matrix product of the column vector defined by $\ker(\phi(x))$ with the row vector defined by $\ker(\phi^t(x))$ and it is easily proven (cf.[D]) that, up to multiplicative constants, $\Phi(x)$ is exactly $Ad(\phi(x))$, the adjoint matrix of $\phi(x)$. Since $\phi(x)$ is symmetric, $\Phi(x) \in \mathbf{P}(S^2 V^\vee)$ and its entries are polynomials of degree $n - 1$ in x.

If we define $L := \psi^*(\mathcal{O}_{\mathbf{P}(V^\vee)}(1))$, then $\Phi^*(\mathcal{O}_{\mathbf{P}(S^2 V^\vee)}(1)) = L^2$ and, by the above argumentation $\deg L^2 = n(n-1)$, $\quad L^2 = \mathcal{O}_C(n-1)$, and it follows that

$$\deg L = \binom{n}{2}$$

Set $\theta = L(-1)$, we get

$$\theta^2 = \mathcal{O}_C(n-3) = K_C$$

that is θ is a theta-characteristic.

Moreover L is generated by the n sections given by the embedding ψ, that is we have a surjection

$$\mathcal{O}_{\mathbf{P}(U^\vee)} \otimes V \longrightarrow L \longrightarrow 0$$

We could compute now the kernel of this morphism, but we do not pursue this because we will follow in a while another road, by using the Beilinson Theorem.

Conversely, given a theta-characteristic θ on a smooth plane curve $C \subset \mathbf{P}(U^\vee)$ of degree n such that $h^0(\theta) = 0$, we can construct θ as symmetric linear determinant.

Indeed let $L = \theta(1)$, then we define $V = H^0(L)$ which has dimension n and we get the embedding $\psi \colon C \to \mathbf{P}(V)$. By composing with the Veronese embedding we get C in $\mathbf{P}(S^2 V)$ with associated line bundle $L^2 = K_C(2) = \mathcal{O}(n-1)$. Since $\mathcal{O}(n-1)$ is the restriction of $\mathcal{O}_{\mathbf{P}(U^\vee)}(n-1)$, it follows that this last embedding is the restriction of an embedding $\mathbf{P}(U^\vee) \subset \mathbf{P}(S^2 V)$ given by a linear system of plane curves of degree $n-1$, which associates to x the symmetric matrix $\phi(x)$. An explicit computation (see [D], proof of theor. 4.1.3) shows that, if F is the equation of C in $\mathbf{P}(U^\vee)$, then $f := F^{n-2} Ad(\phi) \in \mathbf{P}(U \otimes S^2 V)$ and $F = \Delta(f)$.

The general result is the following

Theorem 5.1 (Dixon)**.** *Let C be a smooth plane curve defined by a polynomial F of degree n and θ be a theta-characteristic on C such that $h^0(\theta) = 0$. There is a symmetric map M such that*

$$0 \longrightarrow \mathcal{O}(-2)^n \overset{M}{\longrightarrow} \mathcal{O}(-1)^n \longrightarrow \theta \longrightarrow 0$$

and $\det M = F$. Two maps M, M' define the same θ if and only if they lie in the same $SL(V)$-orbit. In particular Δ is dominant.

Conversely, the cokernel of a injective symmetric map $\mathcal{O}(-2)^n \xrightarrow{M} \mathcal{O}(-1)^n$ is a theta-characteristic θ on the curve C defined by $\det M = 0$, such that $h^0(\theta) = 0$.

The quickest proof of the Thm. 5.1 is probably obtained as an application of the Beilinson Theorem .

We recall the Beilinson theorem in the form given in [AO].

Theorem 5.2 (Beilinson). *Let F be a coherent sheaf on $\mathbf{P}^n$ and let Q be the quotient bundle. There is a complex*

$$\ldots \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \ldots$$

on $\mathbf{P}^n$ such that

(i) $C^h = \oplus_{j+h=i} \wedge^j Q^\vee \otimes H^i(F(-j))$

(ii) the horizontal maps extracted from d_i

$$\wedge^j Q^\vee \otimes H^i(F(-j)) \longrightarrow \wedge^{j-1} Q^\vee \otimes H^i(F(-j+1))$$

are the natural multiplication maps

(iii) the cohomology is

$$\frac{\operatorname{Ker} d_h}{\operatorname{Im} d_{h-1}} = \begin{cases} 0 & \text{if } h \neq 0 \\ F & \text{if } h = 0 \end{cases}$$

PROOF OF THM. 5.1 - Consider θ as a coherent sheaf on $\mathbf{P}^2$ extended to zero.

The Beilinson table for $\theta(1)$

$$\begin{array}{ccc} H^2(\theta(-1)) & H^2(\theta) & H^2(\theta(1)) \\ H^1(\theta(-1)) & H^1(\theta) & H^1(\theta(1)) \\ H^0(\theta(-1)) & H^0(\theta) & H^0(\theta(1)) \end{array}$$

is

$$\begin{array}{ccc} 0 & 0 & 0 \\ V^\vee & 0 & 0 \\ 0 & 0 & V \end{array}$$

hence we get from Thm. 5.2 the resolution

$$0 \longrightarrow V^\vee \otimes \mathcal{O}(-2) \xrightarrow{M} V \otimes \mathcal{O}(-1) \longrightarrow \theta \longrightarrow 0$$

Applying the functor $\underline{Hom}(-, \mathcal{O})$ we get

$$0 \longrightarrow V^\vee \otimes \mathcal{O}(1) \xrightarrow{M^t} V \otimes \mathcal{O}(2) \longrightarrow \underline{Ext}^1(\theta, \mathcal{O})$$

and by Grothendieck duality ([FGA] pag. 149-08)

$$\underline{Ext}^1(\theta, \mathcal{O}) = \theta^\vee(3)$$

so that twisting by $\mathcal{O}(-3)$ we get the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V^\vee \otimes \mathcal{O}(-2) & \xrightarrow{M} & V \otimes \mathcal{O}(-1) & \xrightarrow{p} & \theta & \longrightarrow & 0 \\
& & \downarrow{A} & & \downarrow{B} & & \downarrow{1} & & \\
0 & \longrightarrow & V^\vee \otimes \mathcal{O}(-2) & \xrightarrow{M^t} & V \otimes \mathcal{O}(-1) & \xrightarrow{q} & \theta & \longrightarrow & 0
\end{array}
$$

where B has been found because $Ext^1(V(-1), V^\vee(-2)) = 0$. Applying the functor $\underline{Hom}(-, \mathcal{O})$ (and Grothendieck duality) again we get

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V^\vee \otimes \mathcal{O}(-2) & \xrightarrow{M} & V \otimes \mathcal{O}(-1) & \xrightarrow{p} & \theta & \longrightarrow & 0 \\
& & \downarrow{B^t} & & \downarrow{A^t} & & \downarrow{1} & & \\
0 & \longrightarrow & V^\vee \otimes \mathcal{O}(-2) & \xrightarrow{M^t} & V \otimes \mathcal{O}(-1) & \xrightarrow{q} & \theta & \longrightarrow & 0
\end{array}
$$

Hence $(A^t - B)q = 0$ and it follows $A^t = B$. Denoting $M' = A^t M$ we get $M' = M'^t$, hence the map is symmetric.

A morphism between θ and θ' lifts to a morphism between the two resolutions. The converse follows again by the Grothendieck duality. $\square$

Corollary 5.3. Let $f \in \mathbf{P}(U \otimes S^2 V)$ such that $\Delta(f)$ is a smooth plane curve in $\mathbf{P}(U^\vee)$. Then the fiber of Δ containing f has dimension equal to $\dim SL(V) = n^2 - 1$.

PROOF – By the Thm. 5.1 the fiber is a union of orbits, hence their dimension is $\leq n^2 - 1$. Since the map Δ is dominant every fiber has dimension $\geq n^2 - 1$. $\square$

Remark 5.4 - Moreover *every* smooth plane curve is in the image of Δ ([Bea1], Remark 4.4) but we will do not need this fact.

Wall studied in [W] the map Δ in the setting of invariant theory. He remarked, as a consequence of Dixon theorem, that the field of invariants of $SL(V)$ acting on $\mathbf{P}(U \otimes S^2V)$ is a finite extension of the field generated by the coefficients of Δ. In other terms, the semistable points for the action of $SL(V)$ on $\mathbf{P}(U \otimes S^2V)$ are exactly given by the locus where Δ is not defined. Wall also proves that if $\Delta(f)$ is a reduced curve, then the stabilizer of f is finite. In particular if $\Delta(f)$ is smooth then the stabilizer of f is finite. In other terms

Proposition 5.5 (Wall)**.** *There is a factorization through the GIT quotient*

$$
\begin{array}{ccc}
\mathbf{P}(U \otimes S^2V)^{ss} & & \\
\downarrow{\scriptstyle\pi} & \searrow{\scriptstyle\Delta} & \\
\mathbf{P}(U \otimes S^2V)//SL(V) & \xrightarrow{g} & \mathbf{P}(S^nU)
\end{array}
$$

where g is generically finite.

6. Lüroth quartics revisited

Definition 6.1. *A complete n-gon is the union of n lines in $\mathbf{P}^2 = \mathbf{P}(U^\vee)$ meeting in $\binom{n}{2}$ distinct vertices.*

Definition 6.2. *A Lüroth quartic is a smooth quartic which has a inscribed complete pentagon, that is it contains its ten vertices. More generally a plane curve of degree n which has a inscribed a complete $(n+1)$-gon is called a Darboux curve.*

Let us identify, again with a slight abuse of notations, the Segre Veronese variety $X \simeq \mathbf{P}(U) \times \mathbf{P}(V)$ embedded by $\mathcal{O}(1,2)$ with its cone in $U \otimes S^2V$. As we saw in section 4, the elements in $U \otimes S^2V$ are stratified by the (border) rank, denoting by rank one the elements of X and by rank k the elements of $\sigma_k(X) \setminus \sigma_{k-1}(X)$.

Note that if $f \in X$ then $\mathrm{rank}S_f = 2$ so that if $f \in \sigma_k(X)$ then $\mathrm{rank}S_f \leq 2k$. So we are exactly in the setting of Thm. 4.1.

Proposition 6.3. *(i) If $f \in \sigma_n(X) \setminus \sigma_{n-1}(X)$ is general then the plane curve given by $\Delta(f) = 0$ consists of n lines.*

(ii) If $f \in \sigma_{n+1}(X) \setminus \sigma_n(X)$ is general then the plane curve of degree n given by $\Delta(f) = 0$ is a Darboux curve (so for $n = 4$ is a Lüroth quartic). The divisor given by the $\binom{n+1}{2}$ vertices of the complete inscribed $(n+1)$-gon has the form $\theta + 2H$ where θ is the theta characteristic defined in Thm. 5.1 and H is the hyperplane divisor.

PROOF – (i) Assume $f = \sum_{i=1}^{n} u_i \otimes v_i^2$. The vectors $u_i \in U$ defines lines L_i in the plane $\mathbf{P}(U^\vee)$ by the equation $u_i(-) = 0$. Then the matrix corresponding to f evaluated on the line L_i has rank $\leq n-1$ and it is degenerate. This means that $\cup_i L_i \subseteq \{\Delta(f) = 0\}$ and the other inclusion holds by degree reasons.

(ii) Assume $f = \sum_{i=1}^{n+1} u_i \otimes v_i^2$. Then the matrix corresponding to f evaluated in the point of intersection $L_p \cap L_q = 0$ is a symmetric matrix of rank $\leq n-1$ (because two summands vanish), hence it is degenerate and the point belongs to the curve $\{\Delta(f) = 0\}$. Let D be the divisor corresponding to the vertices of the $(n+1)$-gon, so that $\deg D = \binom{n+1}{2}$. Set $\theta = D - 2H$. Since every vertex contains two edges we get that $2D = (n+1)H$ and it follows that $2\theta = 2D - 4H = (n-3)H = K$, hence θ is a theta-characteristic. An explicit computation (see the next example (6.1) shows that θ is the theta characteristic defined in Thm. 5.1. □

We did not prove that for a general $f \in \sigma_{n+1}(X) \setminus \sigma_n(X)$ the curve given by $\Delta(f) = 0$ is smooth. This is well known. Indeed the following determinantal representation gives a smooth curve of degree n with an inscribed complete $(n+1)$-gon: taken general linear forms $l_1, \ldots, l_n, l_{n+1} = l$ consider

$$(6.1) \quad \begin{vmatrix} l_1 + l & l & l & \ldots & l \\ l & l_2 + l & l & \ldots & l \\ \vdots & \vdots & & & \vdots \\ l & l & \ldots & \ldots & l_n + l \end{vmatrix} = \sum_{i=1}^{n+1} l_1 l_2 \cdots \hat{l_i} \cdots l_{n+1} = 0$$

The following theorem was proved in [Bar, Lemma2] in a more general setting. For the convenience of the reader we repeat here Barth's proof in our case.

Lemma 6.4. *Given a general complete* $(n + 1)$*-gon, the space of curves of degree* n *passing through its* $\binom{n+1}{2}$ *vertices has projective dimension* n*, that is the* $\binom{n+1}{2}$ *vertices impose independent conditions to curves of degree* n.

PROOF – Let Z be the scheme given by the $\binom{n+1}{2}$ vertices and let L by a general line.

We have the exact sequence

$$0 \longrightarrow H^0(\mathcal{I}_Z(n-1)) \longrightarrow H^0(\mathcal{I}_Z(n)) \longrightarrow H^0(\mathcal{I}_Z(n)_{|L})$$

The first space is zero dimensional because a curve of degree $n - 1$ through Z vanishes on all the edges of the $(n + 1)$-gon. Since $\mathcal{I}_Z(n)_{|L} \simeq \mathcal{O}_L(n)$ it follows $h^0(\mathcal{I}_Z(n)) \leq n + 1$. Moreover $h^0(\mathcal{I}_Z(n)) \geq \binom{n+2}{2} - \binom{n+1}{2} = n + 1$, as we wanted. $\square$

Theorem 6.5. *(Lüroth, 1869) If a plane curve of degree 4 has a inscribed complete pentagon, then it has* ∞^1 *inscribed complete pentagons. Equivalently, the (closure of the) locus of Lüroth quartics is a hypersurface in* $\mathbf{P}(S^4 U)$.

PROOF – By Prop. 6.3 every $f \in \sigma_5(X) \setminus \sigma_4(X)$ defines a Lüroth quartic with equation $\Delta(f)$. By Thm. 4.1 the higher secant variety $\sigma_5(X)$ is defective and has dimension one less then expected. By Prop. 2.1 f belongs to ∞^1 5-secant spaces. The quartics defined in this way are the image through the determinantal morphism of the $SL(V)$-invariant variety $\sigma_5(X)$, so by Thm. 5.1 (since there are smooth Lüroth quartics) they give a irreducible hypersurface in $\mathbf{P}(S^4 U)$.

This is the core of the argument but it does not yet conclude the proof. In order to prove that the general (and hence any) Lüroth quartic has ∞^1 inscribed pentagons, we need the following easy dimensional count.

Consider the product $\mathbf{P}(S^4 U) \times (\mathbf{P}(U))^5$ and the incidence variety I given by the closure of

$$I^0 = \{(f, l_1, \ldots, l_5) | \ l_i \text{ define a complete pentagon inscribed in the smooth } f\}$$

By Lemma 6.4 the general fiber of the second projection is a linear space of projective dimension 4, hence I is irreducible and $\dim I = 14$. Clearly the projection of I on $\mathbf{P}(S^4 U)$ gives the closure of the locus of Lüroth quartics.

Our first argument showed that I contains a dense open subset of pairs given by a Lüroth quartic and its inscribed pentagons, hence the projection of I is a hypersurface. This concludes the proof. $\qquad\square$

Remark 6.6 - The above proof is close to Frahm one ([Fr]), see also the Remark 4.5 in [CarCh], although in these sources the approach is a bit different. We hope it is still useful to review this subject in the modern language and to make direct the link with the higher secant variety of X. The original proof of Lüroth was based on the polarity with Clebsch quartics, and it has been reviewed in [DK].

Remark 6.7 - It can happen that $f \notin \sigma_5(X)$ but still $\Delta(f)$ is a Lüroth quartic. Indeed $f \notin \sigma_5(X)$ picks only one of the 36 connected components of a general fiber of Δ.

The hypersurface of Lüroth quartics was classically studied. F. Morley (the same of trisector theorem) proved in 1918 ([Mo]) that it has degree 54, by the classical Aronhold construction of plane quartics from seven given points.

Theorem 6.8. *Let $X = \mathbf{P}^2 \times \mathbf{P}^{n-1}$ embedded with the linear system $\mathcal{O}(1,2)$. For $n \geq 5$ the secant variety $\sigma_{n+1}(X)$ has the expected dimension $(n+1)^2 + n$.*

PROOF – For $n = 5$ it is enough to pick six generic points , compute their tangent spaces and apply the Terracini lemma. For $n \geq 6$ we can use the standard inductive technique, which goes back to Terracini.

Let consider the divisor $X' = \mathbf{P}^2 \times \mathbf{P}^{n-2} \subset X$. Given $n + 1$ P_i points on X let specialize the first n of them on X'.

Let $Z = \{P_1, \dots, P_n\}$, $\qquad Z^2 = \{P_1^2, \dots, P_n^2\}$.

We get the exact sequence

$$0 \longrightarrow I_{Z \cup P_{n+1}^2, X}(1,1) \longrightarrow I_{Z^2 \cup P_{n+1}^2, X}(1,2) \longrightarrow I_{Z^2, X'}(1,2)$$

The first space has always the expected dimension. The last space has the expected dimension by the inductive assumption. Hence also the middle space has the expected dimension. $\qquad\square$

The following theorem says that the statement of the Lüroth Thm. 6.5 is false for $n \geq 5$.

Corollary 6.9. *Let $n \geq 5$. The generic Darboux curve of degree n has only finitely many inscribed complete $(n+1)$-gons.*

PROOF – Let $M_n^{sm} \subset \mathbf{P}(S^n U) = M_n$ be the variety of smooth plane curves of degree n, let G_n be the open (smooth) part of $Sym^{n+1}\mathbf{P}^2$ parametrizing complete $(n+1)$-gons. Let R_n be the closure of the incidence variety in $M_n \times G_n$, namely

$$(6.2) \qquad R_n = \overline{R_n^0} \qquad R_n^0 = \{(m, g) \in M_n^{sm} \times G_n | \; g \text{ is inscribed in } m\}$$

Let p, q be the restriction to R_n of the two projections respectively on M_n and G_n. By Prop. 6.4 the generic fiber of q is isomorphic to $\mathbf{P}^n$, hence R_n is irreducible and $\dim R_n = 3n + 2$. From Thm. 6.8, the examples (6.1) and the Cor. 5.3 the image $p(R_n)$ has dimension $\geq (n+1)^2 + n - (n^2 - 1) = 3n + 2$. It follows that p is generically finite, as we wanted. $\square$

Remark 6.10 - The first part of the above proof still says that R_4 is irreducible of dimension 14. But in this case the Thm. 6.8 fails and R_4 is contracted by p to the Lüroth hypersurface. We remark also that a open subset of R_n comes from determinantal curves in $\sigma_{n+1}(X)$ by the construction in the examples (6.1).

It follows that

Proposition 6.11. *Let $n \geq 4$. $\Delta(\sigma_{n+1}(X))$ is the Darboux locus, that is the closure of the variety consisting of Darboux plane curves of degree n. Its dimension is 13 for $n = 4$ and $3n + 2$ for $n \geq 5$.*

Denote by $\pi_1 \colon X \to \mathbf{P}^2$ the projection on the first factor. The abstract secant variety $\sigma^{n+1}(X)$ has a dominant morphism

$$\Delta \times \pi_1^{n+1} \colon \sigma^{n+1}(X) \to R_n$$

which identifies with the $SL(V)$-quotient $\sigma^{n+1}(X)//SL(V)$. To prove this claim, let consider, like in Prop. 5.5, the factorization

$$\sigma^{n+1}(X)^{ss}$$
$$\downarrow \pi \qquad \searrow {\scriptstyle \Delta \times \pi_1^{n+1}}$$
$$\sigma^{n+1}(X)//SL(V) \xrightarrow{\ g\ } R_n$$

where g is finite. Since the inscribed $(n+1)$-gon determines by Prop. 6.3 the theta divisor, from the Thm. 5.1 it follows that g is a isomorphism, so proving the claim.

It is a natural question if the morphism $p\colon R_n \to M_n$ (see (6.2)) is injective, posed by Ellinsgrud and Stromme in [ES] and settled again in [PT]. This question is equivalent to ask how many complete $(n+1)$-gons are inscribed in the general Darboux curve of degree n.

Now observe that the secant order $d_{n+1}(X)$ (see def. 2.3) can be computed as degree of the induced morphism on quotients

$$R_n = \sigma^{n+1}(X)//SL(V) \longrightarrow \sigma_{n+1}(X)//SL(V)$$

We have the factorization

$$R_n$$
$$\downarrow \pi \qquad \searrow {\scriptstyle p}$$
$$\sigma_{n+1}(X)//SL(V) \xrightarrow{\ h\ } M$$

and it follows that

$$\deg p = d_{n+1}(X) \cdot \deg h$$

$\deg h$ measures how many different theta divisors defined by inscribed $(n+1)$-gons (like in Prop. 6.3) lie over the generic Darboux curve.

The degree of h is not known, although it is expected that it is one.

The secant order $d_{n+1}(X)$ measures how many $(n+1)$-gons give linearly equivalent θ divisors like in Prop. 6.3.

The following reformulation of the Thm. 2.4 in this setting looks interesting.

Theorem 6.12. *Let* $n \geq 5$. *$d_{n+1}(X) = 1$ unless $X = \mathbf{P}^2 \times \mathbf{P}^{n-1}$ embedded with the linear system $\mathcal{O}(1,2)$ is $(n+1)$-weakly defective.*

PROOF – By Thm. 2.4 and Thm. 6.8. $\qquad\square$

Problem 6.13. *Is the Segre-Veronese variety $X = \mathbf{P}^2 \times \mathbf{P}^{n-1}$ embedded with the linear system $\mathcal{O}(1,2)$ $(n+1)$-weakly defective? The Thm. 6.8 says only that X is not $(n+1)$-defective.*

M. Toma [Tom] has shown a uniqueness result for curves such that their inscribed $(n+1)$-gon is also circumscribed to a smooth conic (and $n \geq 5$), they are called Poncelet curves.

7. The moduli space of symplectic vector bundles on the plane

Let $M(r,n)$ be the moduli space of stable bundles of rank $r \geq 2$ on $\mathbf{P}^2$ with $(c_1, c_2) = (0, n)$. It is known that $M(r,n)$ is not empty if and only if $r \leq n$. $M(r,n)$ is a smooth irreducible variety of dimension $2rn - r^2 + 1$ ([Hu]).

Moreover for the general $E \in M(2,4)$ we have $H^0(E(1) = 2$, while for $n \geq 5$ the Brill-Noether locus $H_n = \{E \in M(2,n)|h^0(E(1)) \geq 1\} \subseteq M(2,n)$ is a irreducible subvariety of dimension $3n+2$, which is proper if $n \geq 6$ ([Bar]). The bundles $E \in H_n$ are called Hulsbergen bundles.

In this and in next section we will reprove the main results of [Bar] and [Hu] in the case of symplectic bundles, by connecting them with the higher secant varieties.

Definition 7.1. *A vector bundle is called symplectic if there is a isomorphism*

$$\alpha \colon E \to E^\vee$$

such that $\alpha^t = -\alpha$.

In particular it follows that if E is symplectic then $c_1(E) = 0$, r is even and $\wedge^2 E$ contains $\mathcal{O}$ as a direct summand. If moreover E is stable we have that $h^0(\wedge^2 E) = \mathbf{C}$, so that the isomorphism α is unique up to scalar multiplication.

Let now E be a stable vector bundle of rank r on $\mathbf{P}^2$ with $c_1(E) = 0$, $c_2(E) = n$. A simple computation shows that $\chi(E(-1)) = \chi(E(-2)) = -n$ is independent by r.

Theorem 7.2. *Let E be a symplectic bundle of rank r on $\mathbf{P}^2 = \mathbf{P}(U)$ with $c_2(E) = n$ such that $H^0(E) = 0$. Denote $V = H^1(\mathbf{P}^2, E(-1))$ which is a vector space of dimension n. Then E is the cohomology bundle of the following Barth monad*

$$I \otimes \mathcal{O} \xrightarrow{g} V^\vee \otimes \Omega^1(2) \xrightarrow{f} V \otimes \mathcal{O}(1)$$

that is $E = \operatorname{Ker} f / \operatorname{Im} g$ where $f \in U \otimes S^2 V$ is the natural (symmetric) multiplication map and $I = H^1(E(-3)) = H^1(E)^\vee$ has dimension $n - r$.

Conversely the cohomology bundle $E(f)$ of such a monad where $f \in U \otimes S^2 V$ is a symplectic bundle of rank r with $c_2(E) = n$ such that $H^0(E) = 0$.

PROOF – The Beilinson table for $E(-1)$ is

$$
\begin{array}{ccc}
0 & 0 & 0 \\
I & V^\vee & V \\
0 & 0 & 0
\end{array}
$$

hence by twisting by $\mathcal{O}(1)$ we get from Thm. 5.2 the monad in the statement. Note that $f \in Hom(V^\vee \otimes \Omega^1(2), V \otimes \mathcal{O}(1)) = U \otimes V \otimes V$.

Since Serre duality is induced by cup product which is skew commutative in odd dimension, it is well known that $f = f^t$ (for details see [Bar] Prop. 1), hence $f \in U \otimes S^2 V$. The converse is trivial. $\qquad\qquad\square$

Proposition 7.3. *Two simple bundles $E(f)$, $E(f')$ as in Thm. 7.2 are isomorphic if and only if f, f' are $SL(V)$-equivalent.*

PROOF – It is easy to check that a morphism between bundles lifts to a morphism between the corresponding Barth monads. The details are left to the reader. $\qquad\qquad\square$

From the Barth monad of Thm. 7.2 we get the cohomology map

$$H^0(f)\colon V^\vee \otimes H^0(\Omega^1(2)) \to V \otimes H^0(\mathcal{O}(1))$$

It is important to remark that $H^0(f)$ is identified with $S_f\colon U \otimes V^\vee \to U^\vee \otimes V$ in formula (4.2) of section 4, indeed $H^0(\Omega^1(2)) = \wedge^2 U^\vee = U$. Note that rank $H^0(f) = 2n + r$, indeed the kernel of $H^0(f)$ contains I which has dimension

$n - r$, and it cannot be bigger, otherwise $H^0(E) \neq 0$ contradicting the stability of E.

We denote $M_{sp}(r, c_2) \subseteq M(r, c_2)$ the moduli space of stable symplectic vector bundles (note that r is even). We recall that the adjoint representation for the symplectic group $Sp(\mathbf{C}^r)$ is isomorphic to the symmetric power $S^2\mathbf{C}^r$.

Since any $E \in M_{sp}(r, c_2)$ is simple, we get $h^0(S^2E) = 0$, moreover $h^2(S^2E) = h^0(S^2E(-3)) = 0$.

We have $c_2(S^2E) = n(r + 2)$ and we get by Hirzebruch-Riemann-Roch theorem

$$h^1(S^2E) = -\chi(S^2E) = (r + 2)n - \binom{r + 1}{2}$$

It follows that $M_{sp}(r, n)$ (when nonempty) is smooth of dimension $(r+2)n - \binom{r+1}{2}$.

Following [Hu], denote $M_{sp}^0(r, n) = \{E \in M_{sp}(r, n) | E_{|l} = \mathcal{O}^r \text{for some line } l\}$. We remark that, by semicontinuity, if $E_{|l}$ is trivial on a line l, then it is trivial on the general line l. This can be seen by observing that $E_{|l}$ is trivial if and only if $h^0(E_{|l}(-1)) = 0$, as we do after the following proposition.

Proposition 7.4.

$$\overline{M_{sp}^0(r, n)} = M_{sp}(r, n)$$

PROOF – (Hirschowitz) Let $E \in M_{sp}(2h, n)$. For a line l we have a small deformation of $E_{|l}$ such that $E_{|l}$ is trivial. We have the exact sequence

$$0 \longrightarrow S^2E(-1) \longrightarrow S^2E \longrightarrow S^2E_{|l} \longrightarrow 0$$

which yields

$$H^1(S^2E) \longrightarrow H^1(S^2E_{|l}) \longrightarrow 0$$

Hence the deformation on l lifts to a deformation on $\mathbf{P}^2$. $\square$

Let $E = E(f)$ like in the statement of Thm. 7.2. The exact sequence

$$0 \longrightarrow E(-2) \longrightarrow E(-1) \longrightarrow E(-1)_{|l} \longrightarrow 0$$

yields

$$0\longrightarrow H^0(E(-1)_{|l})\longrightarrow H^1(E(-2))\longrightarrow H^1(E(-1))$$

hence $E_{|l}$ is trivial if and only if the determinant of the morphism

$$H^1(E(-2))\longrightarrow H^1(E(-1))$$

is nonzero, that is if and only if $\Delta(f)$ evaluated at l is nonzero.

Hence $E(f)$ is trivial on the general line l if and only if $\Delta(f)$ is not identically zero, that is if and only if f corresponds to a semistable point in $\mathbf{P}(U\otimes S^2V)$ for the $SL(U)\times SL(V)$-action.

Definition 7.5.

$$K_{r,n}=\{f\in P(U\otimes S^2V)|\text{rank }H^0(f)=2n+r\}$$

The general element in $\sigma_{n+(r/2)}(X)$ belongs to K_r (so it is nonempty).

In order to prove the irreducibility of $M_{sp}(r,n)$ we need the following auxiliary result from [BPV] Cor. 3.6 (see also [Bas] Cor. 2.6 for a elementary proof in the spirit of [Hu]).

Proposition 7.6 (Brennan-Pinto-Vasconcelos). *Let V be a vector space of dimension n and let r be even. The subvariety*

$$J_{r,n}:=\{(P,Q)\in S^2V\times S^2V|\text{rank }[P,Q]=r\}$$

is irreducible of codimension $\binom{n-r}{2}$, and moreover its reduced equations are the pfaffians of order $r+2$ of $[P,Q]$.

Theorem 7.7. *The moduli space $M_{sp}(r,n)$ is irreducible of dimension*

$$(r+2)n-\binom{r+1}{2}$$

PROOF – We have the open subvariety $\tilde{K}_{r,n}\subseteq K_{r,n}$ defined by $f\in K_{r,n}$ such that the corresponding morphism

$$V^\vee\otimes\Omega^1(2)\overset{f}{\longrightarrow}V\otimes\mathcal{O}(1)$$

is surjective. Any such f defines $E(f)$ as cohomology bundle of the corresponding Barth monad. It is easy to see that there is a universal bundle $\mathcal{E}$ over $\mathbf{P}(U) \times \tilde{K}_{r,n}$ such that $\mathcal{E}_{\mathbf{P}^2 \times \{f\}} = E(f)$, by constructing a universal monad like in [Hu] prop. 1.6.1. Since stability is a open property, we have a open subvariety $\tilde{K}^s_{r,n} \subseteq \tilde{K}_{r,n}$ consisting of f such that $E(f)$ is stable. By the universal property of moduli space we have a surjective morphism $\tilde{K}^s_{r,n} \xrightarrow{\pi} M_{sp}(r,n)$. By Prop. 7.4 it is enough to prove that $M^0_{sp}(r,n)$ it is irreducible, hence it is enough to prove that $\pi^{-1}(M^0_{sp}(r,n)) = \tilde{K}^s_{r,n} \setminus Z(\Delta)$ is irreducible. We will prove that $\tilde{K}_{r,n} \setminus Z(\Delta)$ is irreducible. For any $x \in \mathbf{P}(U)$ define

$$\tilde{K}_{r,n,x} = \{f \in K_{r,n} | \Delta(f)(x) \neq 0\}$$

These are open subsets in $\tilde{K}_{r,n}$ such that, for any $x, y \in \mathbf{P}(U)$, $\tilde{K}_{r,n,x} \cap \tilde{K}_{r,n,y}$ is a non empty open subsariety of $\tilde{K}_{r,n}$ and moreover

$$\bigcup_{x \in \mathbf{P}(U)} \tilde{K}_{r,n,x} = \tilde{K}_{r,n} \setminus Z(\Delta)$$

Finally we prove (thanks to the $SL(U)$-action) that for $z = (0,1,0)$ $\tilde{K}_{r,n,z}$ is irreducible. Let $C\tilde{K}_{r,n,z}$ be the affine cone over $\tilde{K}_{r,n,z}$. In the matrix representation of Lemma 3.1 this means that the slice called Q of S_f is invertible, hence we have a fibration $C\tilde{K}_{r,n,z} \to S(n)$ sending f to Q, where $S(n)$ is the variety of symmetric invertible matrices of order n. This fibration is $GL(V)$-equivariant and all its fibers are isomorphic to $J_{r,n}$ defined in Prop. 7.6 (cf. Lemma 3.1). Hence the result follows by Prop. 7.6. Note that

$$\dim S(n) + \dim J_{r,n} - \dim GL(V) = 3\binom{n+1}{2} - \binom{n-r}{2} - n^2 = (r+2)n - \binom{r+1}{2}$$

$\square$

Problem 7.8. *Are the moduli spaces of orthogonal bundles on $\mathbf{P}^2$ irreducible?*

8. The Barth map and Brill-Noether loci

We keep all the notations from the previous section.

Let remark the following result of LePotier

Proposition 8.1 (LePotier)**.** *Assume $f \in \mathbf{P}(U \otimes S^2 V)$ is a semistable point (see Prop. 5.5) for the $SL(V)$-action. If $E(f)$ is a stable bundle, then f is a stable point.*

PROOF – The proof is essentially the same as in [P2] Lemma 2, and appears in different forms also in [Bar] and [Hu], so we omit it. $\square$

Now we consider the GIT quotient $\tilde{K}^s_{r,n}//SL(V)$. Since all points not lying in $Z(\Delta)$ are $SL(V)$-semistable, by Prop. 7.3 and the above construction we have a surjective morphism $\tilde{K}^s_{r,n}//SL(V) \xrightarrow{\tau} M^0_{sp}(r,n)$ which is birational. Consider the closure

$$\overline{K_{r,n}} = \{f \in P(U \otimes S^2 V) | \text{rank } H^0(f) \leq 2n + r\} = \bigcup_{s \leq r} K_{s,n}$$

We get the projective scheme

$$M^{mon}_{sp}(r,n) := \overline{K_{r,n}}//SL(V)$$

(by definition of GIT-quotient only the semistable points in $\overline{K_{r,n}}$ are considered) where the suffix *mon* reads for monads, which has the following properties

(i) it is birational to the Maruyama moduli space $\overline{M_{sp}(r,n)}$

(ii) the morphism Δ factors through

$$
\begin{array}{ccc}
\overline{K_{r,n}} \setminus Z(\Delta) & & \\
\downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \Delta} & \\
M^{mon}_{sp}(r,n) & \xrightarrow{b_{r,n}} & \mathbf{P}(S^n U)
\end{array}
$$

(iii) The following diagram commutes

$$
\begin{array}{ccc}
\tilde{K}^s_{r,n}//SL(V) & \xrightarrow{\tau} & M^0_{sp}(r,n) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle J} \\
M^{mon}_{sp}(r,n) & \xrightarrow{b_{r,n}} & \mathbf{P}(S^n U)
\end{array}
$$

where i is an open embedding and $J(E)$ is the degree n curve of jumping lines of E supported by

$$\{l \in P(U^\vee) | E_{|l} \neq \mathcal{O}^r\}$$

where l represents a line of $\mathbf{P}(U)$.

Definition 8.2. *The morphism* $b_{r,n}\colon M_{sp}^{mon}(r,n)\longrightarrow \mathbf{P}(S^nU)$ *which factors in (ii) above is called the Barth map.*

Its degree and the degree of its image are equal to the ones for Barth maps as defined in [PT] in the case $r = 2$. Note that from property (ii) above the Barth map can be computed by the symmetric determinantal morphism Δ. See [P2] for the connection with Donaldson invariants.

Proposition 8.3. *The Barth map is generically finite, the image of the Barth map is irreducible and has codimension* $1 + \frac{(n+1-r)(n-2-r)}{2} = \binom{n-r}{2}$ *in the projective space of plane curves of degree* n.

PROOF – By Thm. 5.1 the map Δ is dominant and Prop. 7.6 states that J_n is irreducible of codimension $\binom{n-r}{2}$. The theorem follows then by (ii) above (and the discussion at the end of the proof of Thm. 7.7). $\qquad\square$

Corollary 8.4. *The Barth maps* $b_{n,n}$ *(n even) and* $b_{n-1,n}$ *(n odd) are dominant.*

The case $r = n - 2$, n even looks particularly interesting because the image of $b_{n-2,n}$ is a hypersurface in M_n. For $n = 4$ this hypersurface is the Lüroth hypersurface of section 6. By Thm. 4.1 we get that , with n even and $k = \frac{3n}{2} - 1$

$$(8.1) \qquad\qquad \overline{K_{n-2,n}} = \sigma_k(X)$$

We remark that $\sigma_{n+1}(X) = \overline{K_{2,n}}$ for $n = 4, 5$, while for $n \geq 6$ we have $\dim \overline{K_{2,n}} = \dim \sigma_{n+1}(X) + (n - 5)$ by Thm. 6.8 and the proof of Thm. 7.7. Note also that $\sigma_{n-1}(X) \subseteq Z(\Delta)$ but this inclusion is strict for $n \geq 3$.

Consider the diagram

$$
\begin{array}{ccc}
M_{sp}^{mon}(r,n) & & \\
\Big\downarrow{\scriptstyle i} & \searrow{\scriptstyle b_{r,n}} & \\
\mathbf{P}(U \otimes S^2V)//SL(V) & \xrightarrow{\;g\;} & \mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}(n)))
\end{array}
$$

By Prop. 5.5 and by Thm. 5.1 it follows that g is generically finite, in particular the fiber over a smooth $C \in M_n^{sm}$ is given by the theta characteristic $\{L \in Pic(C) | L^2 = K_C, h^0(L) = 0\}$.

Proposition 8.5 (Beauville, Catanese). *Let be given a general smooth plane curve C and genus $g = \binom{n-1}{2}$. Then the set $\{L \in Pic(C)|L^2 = K_C, h^0(L) = 0\}$ has cardinality*

$$\begin{cases} 2^{g-1}(2^g+1) & \text{if } n \text{ is even or if } n \equiv 3,5 \bmod 8 \\ 2^{g-1}(2^g+1)-1 & \text{if } n \equiv 1,7 \bmod 8 \end{cases}$$

PROOF – We just sketch the proof, for details see for example [Bea2] prop. 3. Let n be even. The moduli space $\mathcal{T}_n$ of pairs (C, θ) where C is a smooth plane curve of degree n and θ is a theta characteristic has exactly two irreducible components $\mathcal{T}_n^0$ and $\mathcal{T}_n^1$, corresponding to the parity of θ, which are both ètale covering of the space $U_n \subset \mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}(n)))$ of smooth plane curves.

By the Thm. 5.1, the generic plane curve C has a θ such that $h^0(\theta) = 0$. It follows that the subvariety in $\mathcal{T}_n^0$ of pairs (C, θ) such that $h^0(\theta) \geq 2$ has codimension at least one in $\mathcal{T}_n^0$, and the same is true for its projection on U_n. The number of sheets of $\mathcal{T}_n^0$ is classical, for a moden reference see [At]. For n odd $\mathcal{T}_n$ has a third irreducible component corresponding to $\mathcal{O}(\frac{n-3}{2})$ which has always $h^0 \geq 2$ and it is even if $n \equiv 1,7 \bmod 8$. $\qquad\square$

It is well known that, in the moduli space M_g of curves of genus g, the locus of curves which have a even theta characteristic such that $h^0(\theta) \geq 2$ has pure codimension one, see [Teix] theor. 2.16. This is called the *theta locus*.

For $g = 3$ this divisor coincides with the hyperelliptic locus and so it does not meet the locus of plane curves of degree 4.

For plane curves of degree ≥ 5 the situation changes, thanks to the following interesting examples, that can be found in the prop. 2.28 of [Ca]. The case $n = 6$ was first communicated to me by Pirola. See also the Theorem B in [Bea1].

Proposition 8.6 (Catanese). *Let $n \geq 6$. The general symmetric morphism on $\mathbf{P}^2$*

$$\mathcal{O}(-3)^2 \oplus \mathcal{O}(-2)^{n-6} \xrightarrow{M} \mathcal{O}(-1)^{n-6} \oplus \mathcal{O}^2$$

degenerate on a plane curve C of degree n and $\mathrm{Coker}\, M = \theta$ is a theta-characteristic on C which satisfies $h^0(\theta) = 2$. For $n = 5$ the same conclusion

holds by looking at

$$\mathcal{O}(-3)^2 \oplus \mathcal{O}(-1) \xrightarrow{M} \mathcal{O}(-2) \oplus \mathcal{O}^2$$

Plane curves constructed in this way give a divisor in the space of plane curves of degree n.

PROOF – To compute the dimension of the family, subtract from the space of symmetric morphisms, which has dimension $3 \cdot 10 + 2(n-6) \cdot 6 + 3 \cdot \binom{n-5}{2}$, the dimension of the group

$$GL(2) \times GL(n-6) \times (\mathbf{C}^2 \otimes \mathbf{C}^{n-6} \otimes U)$$

and get $\binom{n+2}{2} - 2$. The computation for $n = 5$ is analogous. $\square$

These examples show that the theta locus actually meets the variety of smooth plane curves of degree n for any $n \geq 5$.

Remark 8.7 - It is well known that every smooth plane quartic has exactly 36 theta characteristic such that $h^0(\theta) = 0$ and 28 theta characteristic such that $h^0(\theta) = 1$. Indeed, by Clifford's theorem, on a smooth plane quartic every theta-characteristic satisfies $h^0(\theta) \leq 1$.

The degree of $b_{2,n}$ is one by [PT]. The degree of the image of $b_{2,n}$ are known, see e.g. [EG].

The *proper transform* $\Delta^*(W)$ of a subvariety $W \subseteq M_n$ through the morphism

$$\Delta \colon \mathbf{P}(U \otimes S^2V) \setminus Z(\Delta) \longrightarrow M_n$$

is defined as the Zariski closure of $\Delta^{-1}(W) \setminus Z(\Delta)$. The degree of the proper transform is difficult to compute, with the exception of hypersurfaces. Assume W is a hypersurface. Since $Z(\Delta)$ has bigger codimension, we get that $\Delta^*(W)$ is a hypersurface too and

$$\deg \Delta^*(W) = n \deg W$$

Theorem 8.8. *Let E be a symplectic vector bundle on $\mathbf{P}^2$ and let $r = n - 2$, hence the image of the Barth map is a hypersurface. Let $g = \binom{n-1}{2}$. Two mutually exclusive cases are possible.*

(i) The image of the Barth map $b_{n,n-2}$ is contained in the even theta locus

(ii) (degree of $b_{n,n-2}$) · (degree of image of $b_{n,n-2}$) = $3 \cdot 2^{g-2}(2^g + 1)$

PROOF – Assume that (i) does not hold. Then the generic curve in the image of the Barth map has $2^{g-1}(2^g+1)$ theta-characteristic θ such that $h^0(\theta) = 0$.

Denote $a = $ (degree of $b_{n,n-2}$), $b = $ (degree of image of $b_{n,n-2}$). The proper transform of the image of Barth map is a hypersurface in $\mathbf{P}(U \otimes S^2 V)$ of degree nb which contains $\sigma_{\frac{3n-2}{2}}(X)$ which has degree $\frac{3n}{2}$ by the remark at the end of section 4.

Consider the intersection with the proper transform of a general line. It is given by b fibers (over smooth curves), each of them is the union of a $SL(V)$-orbits, which have the same degree. Hence we have the equation

$$\frac{nb}{3n/2} = \frac{2^{g-1}(2^g + 1)}{a}$$

which gives the thesis. □

We do not know which of the two possibilities hold, except for $n = 4$ where we have the following corollary, that was proved first in [PT] for any c_2. Our approach is different.

Corollary 8.9. *The Barth map $b_{2,4}$ is generically injective.*

PROOF – We know that by the Remark 8.7 the case (i) of Thm. 8.8 cannot occur. By [Mo] we know that the degree of the Lüroth hypersurface, which is the image of $b_{2,4}$, is 54. The result follows. □

Remark 8.10 - Although we do not know the explicit expression of the Lüroth invariant L of degree 54, we can say that its pullback $\Delta^* L$ has degree 216 and it contains $Pf(S_f)$ of degree 6 as irreducible factor.

Theorem 8.11. *Let $E \in M_{sp}(n - 2, n)$ be general and let $C = J(E)$ be its curve of jumping lines, so that C lies in the image of $b_{n-2,n}$. Then there are $r_1, \ldots, r_k$ lines where $k = \frac{3n}{2} - 1$ and linear forms h_i such that the equation of C can be written as*

$$\Delta(\sum_{i=1}^{k} r_i h_i^2) = 0$$

Moreover the varieties of lines $r_1, \ldots, r_t$ which describe C in the above equation has dimension $\frac{n}{2} - 1$.

PROOF – By (8.1) we find the k lines and the k linear forms. We have that $\dim(\mathbf{P}(U) \times \mathbf{P}(V)) = n + 1$ and $\dim \sigma_k(X) = 3\binom{n+1}{2} - 2$ by Remark 4.2. Thus by Prop. 2.1 we conclude. $\qquad\square$

When $n = 4$ the Thm. 8.11 reduces to the result of [Bar] that the jumping lines of a general $E \in M_{sp}(2, 4)$ are Lüroth quartics.

The linear forms h_i define a $k \times n$ matrix H. Let I be any subset of n rows and let h_I be the corresponding minor of H.

Then the equation of C can be written as

$$\sum_I h_I^2 \prod_{j \in I} r_j = 0$$

Problem 8.12. *What is the geometric interpretation of the curves of degree n lying in the image of the Barth map $b_{n-2,n}$? For $n = 4$ they are the Lüroth quartics. How is θ related to the data of the equation ? For $r = 3$ they are sextics with a determinantal representation arising from 9 lines.*

The last case we are interested regards the bundles defined from a general $f \in \sigma_{n+(r/2)}(X)$.

Let $E = E(f) \in M_{sp}(r, n)$ on $\mathbf{P}(U)$. We remark that from the Barth monad we have

$$h^0(E(1)) = \dim \ker \left[V^\vee \otimes H^0(\Omega^1(3)) \xrightarrow{H^0(f(1))} V \otimes H^0(\mathcal{O}(2)) \right] - 3(n - r)$$

It is well known that $\Omega^1(3)$ is the tangent bundle, then $H^0(\Omega^1(3)) = \mathrm{ad}\, U^\vee$, moreover $H^0(\mathcal{O}(2)) = S^2 U^\vee$.

Lemma 8.13. *On $\mathbf{P}(U)$ we have the exact $SL(U)$-homogeneous sequence*

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow S^2 U \xrightarrow{q} \mathrm{ad}\, U(1) \longrightarrow U(2) \longrightarrow 0$$

PROOF – It is a particular case of the Four Term Lemma (Lemma 26 in [OR]), it can also be found by an explicit computation. $\qquad\square$

Theorem 8.14. *Let $f \in \sigma_{n+(r/2)}(X)$ general and $E = E(f) \in M_{sp}(r,n)$ on $\mathbf{P}(U)$. Then $h^0(E(1)) \geq r/2$.*

PROOF – Consider first $f = u \otimes v^2 \in X$. The $n \times n$ matrix corresponding to the map $V^\vee \otimes \Omega^1(3) \longrightarrow V \otimes \mathcal{O}(2)$ has only one non zero coefficient, which is u, say at the entry $(1,1)$. At level of H^0, the contraction by u corresponds at the evaluation of q^t of Lemma 8.13 at u. From Lemma 8.13 it follows that codim ker $H^0(f(1)) = 5$.

If $f = \sum_{i=1}^{n+(r/2)} f_i$ with $f_i \in X$ we get that

$$\cap_i \ker H^0(f_i(1)) \subseteq \ker H^0(f(1))$$

hence

$$\text{codim } \ker H^0(f(1)) \leq \sum_i \text{codim } \ker H^0(f_i(1)) = 5\,[n + (r/2)]$$

It follows that

$$h^0(E(1)) \geq 8n - 5\,[n + (r/2)] - 3(n - r) = r/2$$

as we wanted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 8.15 - A simple variation of the Thm. 8.14 is the following. If $f \in \sigma_{n+s}(X)$ general and $E = E(f) \in M_{sp}(r,n)$ then $h^0(E(1)) \geq 3r - 5s$.

Remark 8.16 - The Thm. 8.14 is meaningful only when $n > \frac{5r}{2}$, otherwise all $E \in M_{sp}(r,n)$ satisfy the inequality $h^0(E(1)) \geq r/2$, because $\chi(E(1)) = 3r - n$.

Remark 8.17 - When $r = 2$ the bundles constructed in the Thm. 8.14 are exactly the Hulsbergen bundles in [Bar]. Their curve of jumping lines is a Darboux curve by Prop. 6.11. When $n = 4$ the intersection computed in the above proof is not transversal (the reader can recognize here the flavour of the Terracini lemma), indeed in such a case the general E satisfies $h^0(E(1)) = 2$. It is easy to show that their section vanishes exactly on the vertices of the inscribed $(n+1)$-gon.

Let $M_{sp}(r,n)^k := \{E \in M_{sp}(r,n) | h^0(E(1) \geq k\}$ be the Brill-Noether locus and consider $E \in M_{sp}(r,n)^k$. The Brill-Noether theory says that the tangent space to $M_{sp}(r,n)^k$ at E is the kernel of the natural morphism

$$H^1(S^2 E) \longrightarrow H^0(E(1))^\vee \otimes H^1(E(1))$$

If $h^0(E(1)) = r/2$, such tangent space has dimension bigger or equal than

$$(r+2)n - \binom{r+1}{2} - (r/2)\left[n - (5r/2)\right] = n\left[2 + (r/2)\right] + \frac{3r^2 - 2r}{4}$$

which is equal to

$$\dim \Delta(\sigma_{n+(r/2)}(X)) = n\left[2 + (r/2)\right] + r$$

only when $r = 2$, otherwise it is bigger. This means that the general bundle in $M_{sp}(r,n)^{(r/2)}$ comes from $\sigma_{n+(r/2)}(X)$ only if $r = 2$, which is the case of Hulsbergen bundles.

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