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REGULARITY OF THE MODULI SPACE OF INSTANTON BUNDLES $MI_{\mathbf{P}^3}(5)$

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Abstract. We prove that the moduli space of mathematical instanton bundles on $\mathbf{P}^3$ with $c_2 = 5$ is smooth.

Introduction

Instanton bundles were defined by Atiyah, Drinfeld, Hitchin and Manin [ADHM] in order to construct all the self-dual solutions of the Yang–Mills equation over S^4. A mathematical instanton bundle E on $\mathbf{P}^3 := \mathbf{P}^3(\mathbf{C})$ can be defined as the cohomology bundle of a monad

$$\mathcal{O}(-1)^k \longrightarrow \mathcal{O}^{2k+2} \longrightarrow \mathcal{O}(1)^k$$

on $\mathbf{P}^3$. This is equivalent to the condition that E is a stable bundle of rank 2 on $\mathbf{P}^3$ such that $c_1(E) = 0$, $c_2(E) = k$, and $H^1(E(-2)) = 0$. If E is a mathematical instanton bundle, then it is easy to check by using the Hirzebruch–Riemann–Roch Theorem that $h^1(S^2E) - h^2(S^2E) = 8k - 3$. By deformation theory, $h^1(S^2E) = \dim(T_E MI(k)) \geqslant \dim_E MI(k) \geq 8k - 3$ and in case of equality, $MI(k)$ is smooth at E. So $8k - 3$ is the expected dimension of the moduli space of mathematical instanton bundles $MI_{\mathbf{P}^3}(k) = MI(k)$. It is not known if the moduli space $MI(k)$ is a regular variety of pure dimension $8k - 3$. It is evident in the case $k = 1$. In the cases $2 \leqslant k \leqslant 4$ it was proved in [H], [ES] and [LeP]. In [Ch] and later in [NT] it was proved that $MI(k)$ is regular at bundles E with $h^0(E(1)) \neq 0$. In [R2] (see also [S]) it was proved that $MI(k)$ is regular at bundles with a jumping line of maximal order. In this article we give a general proof of the regularity of $MI(k)$ for the cases $2 \leqslant k \leqslant 5$.

Theorem 1. *For $2 \leqslant k \leqslant 5$ the moduli space $MI(k)$ of mathematical instantons is a regular variety of pure dimension $8k - 3$.*

Our result should be compared with [AO2] (see also [R1]), where it was proved that the closure of $MI(5)$ in the Maruyama scheme of vector bundles of rank 2 with $c_1 = 0$, $c_2 = 5$ contains singular points. Our proof requires tools both from multilinear algebra and algebraic geometry.

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An invariant theoretical description of $MI(k)$

Our first goal is to describe the moduli space $MI(k)$ in terms of invariant theory. The group $\mathbf{SL}_{2k+2}$ acts canonically on the space $\mathbf{C}^{2k+2}$. Let ω be a nondegenerated 2-form on $\mathbf{C}^{2k+2}$ and $\mathbf{Sp}_{2k+2}$ the stabilizer of ω in the group $\mathbf{SL}_{2k+2}$. The 2-form ω defines canonically the $\mathbf{Sp}_{2k+2}$-isomorphism $\mathbf{C}^{(2k+2)*} \simeq \mathbf{C}^{2k+2}$. We have the canonical actions of the group $\mathbf{SL}_4 \times \mathbf{SL}_k \times \mathbf{Sp}_{2k+2}$ on the spaces $\mathbf{C}^4, \mathbf{C}^{4*}, \mathbf{C}^{2k+2}, \mathbf{C}^k, \mathbf{C}^{k*}, \mathbf{C}^4 \otimes \mathbf{C}^k, \ldots$.

We have the canonical quadratic $\mathbf{SL}_4 \times \mathbf{SL}_k \times \mathbf{Sp}_{2k+2}$-morphism

$$\gamma : \ \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2} \longrightarrow S^2\mathbf{C}^{4*} \otimes \wedge^2 \mathbf{C}^{k*}.$$

$\gamma(A)$ is the symmetrization in the two indices corresponding to $\mathbf{C}^{4*}$ and the full contraction in the indices corresponding to $\mathbf{C}^{2k+2}$ of the tensor product $A \otimes A \otimes \omega$. Also consider the canonical bilinear $\mathbf{SL}_4 \times \mathbf{SL}_k \times \mathbf{Sp}_{2k+2}$-morphisms

$$\beta : \ \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2} \ \times \ \mathbf{C}^{2k+2} \longrightarrow \mathbf{C}^{4*} \otimes \mathbf{C}^{k*}$$

and

$$\varepsilon : \ \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2} \ \times \ \mathbf{C}^{4} \otimes \mathbf{C}^{k} \longrightarrow \mathbf{C}^{2k+2}.$$

Consider the following conditions for an element $A \in \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2}$:

(E_1) $\varepsilon(A, f \otimes b) \neq 0$ for all $0 \neq f \in \mathbf{C}^4$, $0 \neq b \in \mathbf{C}^k$,

(E_2) $\gamma(A) = 0$,

(E_3) $\beta(A, h) \neq 0$ for all $0 \neq h \in \mathbf{C}^{2k+2}$.

An element $A \in \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2}$ defines the sheaf morphism $\mathcal{O}^{2k+2} \xrightarrow{f_A} \mathcal{O}(1)^k$. f_A is the composition $\mathbf{C}^{2k+2} \otimes \mathcal{O} \to H^0(\mathcal{O}(1)) \otimes \mathbf{C}^{k*} \otimes \mathcal{O} \to \mathbf{C}^{k*} \otimes \mathcal{O}(1)$, where $H^0(\mathcal{O}(1)) = \mathbf{C}^{4*}$, the left map is given by A, and the right map is the evaluation of $H^0(\mathcal{O}(1))$ at points of $\mathbf{P}^3$. The morphism f_A and the symplectic structure over $\mathcal{O}^{2k+2}$ define the sequence

$$\mathcal{O}(-1)^k \xrightarrow{f_A^\top} \mathcal{O}^{2k+2} \xrightarrow{f_A} \mathcal{O}(1)^k. \tag{1}$$

The condition (E_1) means that f_A is surjective or that $\operatorname{Ker} f_A$ is locally free. The condition (E_2) means that the above sequence is a complex. Therefore, (E_1) and (E_2) together mean that (1) is a monad according to [BH]. The condition (E_3) means moreover that the cohomology bundle E of the monad is a stable vector bundle. It is well known (see e.g., [AO1], Th. 2.8) that the conditions (E_1) and (E_2) imply (E_3). Set

$$I_i = \{A \in \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2} \mid \text{the condition } (E_i) \text{ holds for } A\},$$
$$I := I_1 \cap I_2 \cap I_3 = I_1 \cap I_2,$$

and consider the canonical mapping $\pi : I \longrightarrow I/G$, where $G = \mathbf{SL}_k \times \mathbf{Sp}_{2k+2} \times \mathbf{C}^*$ and I/G is the *set* of G-orbits in I.

Remark 1. In [CO] it was proved that there exists a structure of an affine variety on I/G such that the mapping π is the invariant-theoretical factorization. Moreover, the factor I/G is the geometrical factor.

Lemma 2. *For any $A \in I$ we have*

$$\dim(T_{\pi(A)}MI(k)) = \dim(T_A I) - 3k^2 - 5k - 3.$$

Therefore, $\dim(T_{\pi(A)}MI(k)) \geqslant 8k - 3$ *and*

$$\dim(T_{\pi(A)}MI(k)) = 8k - 3 \text{ if and only if } \dim(T_A I) = 3k^2 + 13k.$$

This is a well known result (see [O], Pr. 1.4 for example). For the convenience of the reader here is the sketch of the proof. Let K be the kernel of f_A in (1). From (1) we get the two sequences:

$$0 \longrightarrow \wedge^2(\mathcal{O}(-1)^k) \longrightarrow K(-1)^k \longrightarrow S^2K \longrightarrow S^2E \longrightarrow 0$$

and

$$0 \longrightarrow S^2K \longrightarrow S^2(\mathcal{O}^{2k+2}) \longrightarrow \mathcal{O}(1)^{k(2k+2)} \xrightarrow{g_A} \wedge^2(\mathcal{O}(1)^k) \longrightarrow 0.$$

From the first sequence it follows that $h^1(S^2E) = h^1(S^2K) - k^2$.

From the second sequence it follows that $h^1(S^2K) = \dim\ker(H^0(g_A)) - (2k+3)(k+1)$. Now observe that $H^0(g_A)$ is $d\gamma|_A$, hence $\ker(H^0(g_A))$ can be identified with $T_A I$ and this concludes the proof. $\square$

Theorem 3. *Suppose that E is an instanton bundle on $\mathbf{P}^3$ and H is a plane. Then $h^0(E|_H) \leqslant 1$.*

Proof. (Trautmann) From the sequence

$$0 \longrightarrow E(-2) \longrightarrow E(-1) \longrightarrow E|_H(-1) \longrightarrow 0$$

we have $H^0(E|_H(-1)) = 0$. If s is any section of $E|_H$, then its cokernel is the ideal sheaf I_Z of a 0-dimensional subscheme Z in H because if Z contains a divisorial component, then $H^0(E|_H(-1)) \neq 0$. Obviously, $H^0(I_Z) = 0$ hence s must span $H^0(E|_H)$. $\square$

Definition 1. $W(E) = \{H \in \mathbf{P}^{3*} \mid h^0(E|_H) \neq 0\}$ is called the variety (scheme) of unstable planes of E. Its scheme structure is defined as the degeneracy locus of the mapping

$$H^1(E(-1)) \otimes \mathcal{O} \longrightarrow H^1(E) \otimes \mathcal{O}(1)$$

over $\mathbf{P}^{3*}$ (Theorem 3 shows that this map drops rank at most by one).

For an element $A \in \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2}$ define the subvariety

$$X_A = \{(\overline{f^*}, \overline{b^*}) \in \mathbf{P}^{3*} \times \mathbf{P}^{k-1*} \mid f^* \otimes b^* \in \mathrm{Im}(\beta(A, \cdot))\}.$$

Lemma 4. *Let q_1 be the projection of $\mathbf{P}^{3*} \times \mathbf{P}^{k-1*}$ on $\mathbf{P}^{3*}$. We have $W(E) = q_1(X_A)$ and the fiber of the projection $X_A \longrightarrow q_1(X_A)$ over H is isomorphic to $\mathbf{P}(H^0(E|_H))$.*

Proof. We have $H \in W(E)$ iff $h^0(K|_H) \neq 0$, where $K := \operatorname{Ker} f_A$. We have $H^0(K|_H) = \operatorname{Ker}(\mathbf{C}^{2k+2} \longrightarrow (\mathbf{C}^{4*}/\overline{f^*}) \otimes \mathbf{C}^{k*})$, where the line $\overline{f^*} = \mathbf{C}f^*$ corresponds to H. Then the existence of a nonzero $\alpha \in H^0(K|_H)$ is equivalent to $\beta(A, \alpha) = f^* \otimes b^*$, where $(\overline{f^*}, \overline{b^*}) \in \mathbf{P}^{3*} \times \mathbf{P}^{k-1*}$. □

Corollary 5. *The morphism* $X_A \longrightarrow q_1(X_A)$ *is bijective, in particular* $\dim X_A = \dim q_1(X_A)$. □

Recall that special 't Hooft bundles are the instanton bundles such that $h^0(E(1)) = 2$. They can be defined through the Serre correspondence by $k+1$ skew lines lying on a smooth quadric surface [H]. We need the following special case of a theorem of J. Coanda [Co].

Theorem 6. *If E is an instanton bundle such that* $\dim W(E) \geqslant 2$*, then E is a special 't Hooft bundle and $W(E)$ is a quadric surface.* □

It is known [H] that special 't Hooft bundles are smooth points with expected local dimension in the moduli space.

Corollary 7. *If* $A^0 \in I$ *and* $\dim X_{A^0} \geqslant 2$*, then*

$$\dim(T_{\pi(A^0)} MI(k)) = 8k - 3. \quad \square$$

Lemma 8. *Suppose* $A^0 \in I$ *and* $\dim(T_{A^0} I) > 3k^2 + 13k$*; then there exists* $0 \neq S^0 \in S^2\mathbf{C}^4 \otimes \wedge^2 \mathbf{C}^k$ *such that* $\xi(A^0, S^0) = 0$*, where*

$$\xi : \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2} \times S^2\mathbf{C}^4 \otimes \wedge^2 \mathbf{C}^k \longrightarrow \mathbf{C}^4 \otimes \mathbf{C}^k \otimes \mathbf{C}^{2k+2}$$

is the canonical bilinear $\mathbf{SL}_4 \times \mathbf{SL}_k \times \mathbf{Sp}_{2k+2}$*-morphism.*

Proof. From $\dim(T_{A^0} I) > 3k^2 + 13k$ it follows that the differential $d\gamma|_{A^0}$ is nonsurjective. The differential $d\gamma|_{A^0}$ is nonsurjective iff $(d\gamma|_{A^0})^*$ is noninjective, i.e., $(d\gamma|_{A^0})^*(S^0) = 0$ for some $0 \neq S^0 \in S^2\mathbf{C}^4 \otimes \wedge^2 \mathbf{C}^k$. It can be easily checked that

$$(d\gamma|_A)^*(S) \equiv 2\xi(A, S).$$

Hence, $\dim(T_{A^0} I) > 3k^2 + 13k$ implies that $\xi(A^0, S^0) = 0$ for some element $0 \neq S^0 \in S^2\mathbf{C}^4 \otimes \wedge^2 \mathbf{C}^k$. □

For the convenience of the reader we give a cohomological interpretation of Lemma 8. Let E^0 be the instanton bundle defined by $A^0 \in I$ as the cohomology bundle of monad (1). By Lemma 2 and deformation theory the assumption that $\dim(T_{A^0} I) > 3k^2 + 13k$ is equivalent to $h^1(S^2 E^0) = \dim(T_{\pi(A^0)} MI(k)) > 8k - 3$. Therefore, the assumption of Lemma 8 is equivalent to $H^2(S^2 E^0) \neq 0$. The second symmetric power of the left-hand side of (1) gives $H^2(S^2 E^0) \simeq H^2(S^2(\operatorname{Ker} f_{A^0}))$. The second symmetric power of the right hand side of (1) gives

$$H^2(S^2(\operatorname{Ker}\ f_{A^0})) \simeq \operatorname{Coker}\left[H^0(\mathcal{O}(1)) \otimes \mathbf{C}^{k*} \otimes \mathbf{C}^{2k+2*} \xrightarrow{\Phi} H^0(\mathcal{O}(2)) \otimes \wedge^2(\mathbf{C}^{k*})\right].$$

Lemma 8 follows because the dual of Φ can be identified with $\xi(A^0, \cdot)$.

Algebraic lemmas

In this section we prove some algebraic lemmas which we use in the proof of our main result.

Lemma 9. *Suppose R is a nonzero block-matrix:*

$$R = \begin{bmatrix} R^1 \\ R^2 \end{bmatrix},$$

where R^i is a skew-symmetric matrix of size $k \times k$; then there exists a column v_0 of height k such that

$$Rv_0 \ = \ \begin{bmatrix} \lambda_1 u_0 \\ \lambda_2 u_0 \end{bmatrix} \neq 0$$

for some column u_0 of height k, $\lambda_1, \lambda_2 \in \mathbf{C}$.

Proof. Suppose that $\det(R^1) \neq 0$. In this case set $v_0 \in \mathrm{Ker}(R^2 - \mu_0 R^1)$, where μ_0 is a root of the equation $\det(R^2 - \mu R^1) = 0$.

Suppose that $\det(R^1) = 0$. One can assume that

$$R^1 \ = \ \begin{bmatrix} R^1_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad R^2 = \begin{bmatrix} R^2_{11} & R^2_{12} \\ R^2_{21} & R^2_{22} \end{bmatrix},$$

where R^1_{11} is a skew-symmetric matrix of size $k' \times k'$, $k' < k$, $\det(R^1_{11}) \neq 0$ and R^2_{11} is a skew-symmetric matrix of size $k' \times k'$. If $R^2_{12} \neq 0$ or $R^2_{22} \neq 0$, then we set $v_0 = \begin{bmatrix} 0 \\ v'_0 \end{bmatrix}$ for some v'_0 such that $R^2_{12} v'_0 \neq 0$ or $R^2_{22} v'_0 \neq 0$. If $R^2_{12} = 0$ and $R^2_{22} = 0$, then $R^2_{21} = 0$ and we set $v_0 = \begin{bmatrix} v'_0 \\ 0 \end{bmatrix}$, where

$$\begin{bmatrix} R^1_{11} \\ R^2_{11} \end{bmatrix} v'_0 = \begin{bmatrix} \lambda_1 u'_0 \\ \lambda_2 u'_0 \end{bmatrix} \neq 0. \qquad \square$$

Consider the linear spaces $\mathbf{C}^4$ and $\mathbf{C}^k$. Let $f_1, \ldots, f_4$ be the standard basis of $\mathbf{C}^4$ and let $f_1^*, \ldots, f_4^*$ be the dual basis of the dual space $\mathbf{C}^{4*}$. Let $b_1, \ldots, b_k$ be the standard basis of $\mathbf{C}^k$ and let $b_1^*, \ldots, b_k^*$ be the dual basis of the dual space $\mathbf{C}^{k*}$. The group $\mathbf{SL}_4$ acts canonically on the space $\mathbf{C}^4$ and the group $\mathbf{SL}_k$ acts canonically on the space $\mathbf{C}^k$. So the actions of the group $\mathbf{SL}_4 \times \mathbf{SL}_k$ are defined on the spaces $\mathbf{C}^4, \mathbf{C}^{4*}, \mathbf{C}^k, \mathbf{C}^{k*}, \mathbf{C}^4 \otimes \mathbf{C}^k, \ldots$.

Consider the linear space $S^2\mathbf{C}^4 \otimes \wedge^2\mathbf{C}^k$. For an element $S \in S^2\mathbf{C}^4 \otimes \wedge^2\mathbf{C}^k$ define

$$\mathrm{rk}(S) = \dim(\mathrm{Im}(\rho(S, \cdot))),$$

where

$$\rho : S^2\mathbf{C}^4 \otimes \wedge^2\mathbf{C}^k \ \times \ \mathbf{C}^{4*} \otimes \mathbf{C}^{k*} \longrightarrow \mathbf{C}^4 \otimes \mathbf{C}^k$$

is the canonical bilinear $\mathbf{SL}_4 \times \mathbf{SL}_k$-morphism. Note that $\mathrm{rk}(S)$ is an even number.

Lemma 10. *Suppose $2 \leqslant k \leqslant 5$ and consider $S \in S^2\mathbf{C}^4 \otimes \wedge^2\mathbf{C}^k$ such that $2 \leqslant \mathrm{rk}(S) \leqslant 2k - 2$. Then one of the following conditions holds:*

(1) $\rho(S, B^{*0}) = f^0 \otimes b^0 \neq 0$ *for some* $B^{*0} \in \mathbf{C}^{4*} \otimes \mathbf{C}^{k*}$, $f^0 \in \mathbf{C}^4$, $b^0 \in \mathbf{C}^k$.
(2) $\mathrm{rk}(S) = 6$ *and there exists* $0 \neq f^{*0} \in \mathbf{C}^{4*}$ *such that* $\rho(S, f^{*0} \otimes b^*) = 0$ *for all* $b^* \in \mathbf{C}^{k*}$.
(3) $\mathrm{rk}(S) = 8$ *and* $\dim(Z_S) \geqslant 2$, *where*

$$Z_S = \{(\overline{f^*}, \overline{b^*}) \in \mathbf{P}^{3*} \times \mathbf{P}^{k-1*} \mid \rho(S, f^* \otimes b^*) = 0\},$$
$$\mathbf{P}^{3*} = P\mathbf{C}^{4*}, \mathbf{P}^{k-1*} = P\mathbf{C}^{k*}.$$

Proof. Consider the coordinate expression of S in the bases $\{f_i\}$ and $\{b_i\}$:

$$S = \sigma_{lp}^{ij} f_l f_p \otimes b_i \wedge b_j.$$

We get a block matrix σ defined by

$$\sigma = (\sigma^{ij})_{1 \leqslant i,j \leqslant k} = \begin{bmatrix} 0 & \sigma^{12} & \dots & \sigma^{1k} \\ \sigma^{21} & 0 & \dots & \sigma^{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^{k1} & \sigma^{k2} & \dots & 0 \end{bmatrix},$$

where $\sigma^{ij} = (\sigma_{lp}^{ij})_{1 \leqslant l,p \leqslant 4}$ is a symmetric matrix of size 4×4, $\sigma^{ij} = -\sigma^{ji}$.

Rewrite the coordinate expression of S as

$$S = \widehat{\sigma}_{lp}^{ij} f_i f_j \otimes b_l \wedge b_p.$$

Then we get a second block matrix $\widehat{\sigma}$ defined by

$$\widehat{\sigma} = (\widehat{\sigma}^{ij})_{1 \leqslant i,j \leqslant 4} = \begin{bmatrix} \widehat{\sigma}^{11} & \widehat{\sigma}^{12} & \widehat{\sigma}^{13} & \widehat{\sigma}^{14} \\ \widehat{\sigma}^{21} & \widehat{\sigma}^{22} & \widehat{\sigma}^{23} & \widehat{\sigma}^{24} \\ \widehat{\sigma}^{31} & \widehat{\sigma}^{32} & \widehat{\sigma}^{33} & \widehat{\sigma}^{34} \\ \widehat{\sigma}^{41} & \widehat{\sigma}^{42} & \widehat{\sigma}^{43} & \widehat{\sigma}^{44} \end{bmatrix},$$

where $\widehat{\sigma}^{ij} = (\widehat{\sigma}_{lp}^{ij})_{1 \leqslant l,p \leqslant k}$ is a skew-symmetric matrix of size $k \times k$, $\widehat{\sigma}^{ij} = \widehat{\sigma}^{ji}$. Let r be the maximal rank of full contractions of $S \otimes b^* \otimes b'^*$ over all $b^*, b'^* \in \mathbf{C}^{k*}$. Transform the basis $\{b_i\}$ and obtain

$$r = \mathrm{rk}(\sigma^{12}). \tag{2}$$

We have

$$2k - 2 \geqslant \mathrm{rk}(S) = \mathrm{rk}(\sigma) = \mathrm{rk}(\widehat{\sigma}) \geqslant 2\,\mathrm{rk}(\sigma^{12}) = 2r.$$

Therefore one of the following cases holds:

(a) $r = 1$ or 2,
(b) $r = 3$, $\mathrm{rk}(\sigma) = 6$, and $k \geqslant 4$,
(c) $r = 4$, $\mathrm{rk}(\sigma) = 8$, and $k = 5$,
(d) $r = 3$, $\mathrm{rk}(\sigma) = 8$, and $k = 5$.

Transform the basis $\{f_i\}$ and obtain

$$\sigma^{12}_{lp} = \begin{cases} 1 & \text{if } 1 \leqslant l = p \leqslant r, \\ 0 & \text{if } l \neq p \text{ or } l = p > r. \end{cases} \tag{3}$$

From (2) it follows that $\sigma^{ij}_{lp} = 0$ for $l, p > r$, whence

$$\widehat{\sigma}^{ij} = 0 \quad \text{for } i, j > r. \tag{4}$$

(a) Consider the case (a).

In this case we prove that the condition (1) holds, i.e., we prove that there exists a column f^0 of height 4 and columns $b^0, B^{*01}, \dots, B^{*04}$ of height k such that

$$\widehat{\sigma} \begin{bmatrix} B^{*01} \\ \vdots \\ B^{*04} \end{bmatrix} = \begin{bmatrix} f^0_1 b^0 \\ \vdots \\ f^0_4 b^0 \end{bmatrix} \neq 0.$$

Suppose that $\widehat{\sigma}^{ij} \neq 0$ for some $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant 4$. In this case we set $B^{*0k} = 0$ for all $k \neq j$ and choose B^{*0j} by using (4) and Lemma 9.

Suppose that $\widehat{\sigma}^{ij} = 0$ for all $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant 4$. We have $\widehat{\sigma}^{lp} \neq 0$ for some $1 \leqslant l \leqslant 2$ and $1 \leqslant p \leqslant 2$. In this case we set $B^{*0k} = 0$ for all $k \neq p$ and choose B^{*0p} by using (4) and Lemma 9.

(b) Consider the case (b).

In this case we prove that the condition (2) holds, i.e. we prove that there exists a column f^{*0} of height 4 such that

$$\sigma \begin{bmatrix} b^*_1 f^{*0} \\ \vdots \\ b^*_k f^{*0} \end{bmatrix} = 0 \tag{5}$$

for any column b^* of height k.

From the condition $\mathrm{rk}(\sigma) = 6$ and 2 it follows that

$$\sigma^{ij} = \begin{bmatrix} \sigma^{ij}_{11} & \sigma^{ij}_{12} & \sigma^{ij}_{13} & 0 \\ \sigma^{ij}_{21} & \sigma^{ij}_{22} & \sigma^{ij}_{23} & 0 \\ \sigma^{ij}_{31} & \sigma^{ij}_{32} & \sigma^{ij}_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From this, for

$$f^{*0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

it easily follows (5).

(c) Consider the case (c).

In this case we prove that the condition (3) holds. We have

$$Z_S = \left\{ (\overline{f^*}, \overline{b^*}) = (\overline{\begin{bmatrix} f_1^* \\ \vdots \\ f_4^* \end{bmatrix}}, \overline{\begin{bmatrix} b_1^* \\ \vdots \\ b_5^* \end{bmatrix}}) \mid \sigma \begin{bmatrix} b_1^* f^* \\ \vdots \\ b_5^* f^* \end{bmatrix} = 0 \right\}.$$

Consider the matrix

$$\widetilde{\sigma} = \begin{bmatrix} 0 & \mathrm{E}_4 & \sigma^{13} & \sigma^{14} & \sigma^{15} \\ -\mathrm{E}_4 & 0 & \sigma^{23} & \sigma^{24} & \sigma^{25} \end{bmatrix},$$

where E_4 is the identity matrix of size 4×4. The 8 rows of the matrix $\widetilde{\sigma}$ are the first 8 rows of the matrix σ. Since $\mathrm{rk}(\sigma) = 8 = \mathrm{rk}(\widetilde{\sigma})$ and for a matrix P of size $20 \times p$ we have:

$$\sigma P = 0 \quad \text{iff} \quad \widetilde{\sigma} P = 0. \tag{6}$$

For $3 \leqslant i \leqslant 5$ consider the following matrix P_i of size 20×4:

$$P_i = \begin{bmatrix} -\sigma^{2i} \\ \sigma^{1i} \\ P_{i3} \\ P_{i4} \\ P_{i5} \end{bmatrix},$$

where $P_{ii} = -\mathrm{E}_4$ and $P_{ij} = 0$ for $j \neq i$. We see that $\widetilde{\sigma} \cdot P_i = 0$.

From (6) it follows that $\sigma \cdot P_i = 0$ or

$$\sigma^{ji} = \sigma^{1j}\sigma^{2i} - \sigma^{2j}\sigma^{1i}, \quad 3 \leqslant j \leqslant 5.$$

From this we obtain

$$\begin{aligned} 0 = \sigma^{ji} + \sigma^{ij} &= \sigma^{1j}\sigma^{2i} - \sigma^{2j}\sigma^{1i} + \sigma^{1i}\sigma^{2j} - \sigma^{2i}\sigma^{1j} \\ &= [\sigma^{1j}, \sigma^{2i}] + [\sigma^{1i}, \sigma^{2j}], \quad 3 \leqslant i, j \leqslant 5. \end{aligned}$$

One can rewrite these equations into the following compact form:

$$[t_1\sigma^{13} + t_2\sigma^{14} + t_3\sigma^{15}, t_1\sigma^{23} + t_2\sigma^{24} + t_3\sigma^{25}] = 0 \tag{7}$$

for all $t_1, t_2, t_3 \in \mathbf{C}$.

Claim 11. *For every $(b_3^*, b_4^*, b_5^*) \neq (0,0,0)$ there exists (b_1^*, b_2^*) and a nonzero column f^* of height 4 such that*

$$\sigma \begin{bmatrix} b_1^* f^* \\ \vdots \\ b_5^* f^* \end{bmatrix} = 0.$$

Proof of Claim 11. From (7) it follows that the symmetric matrices

$$b_3^*\sigma^{13} + b_4^*\sigma^{14} + b_5^*\sigma^{15}, \quad b_3^*\sigma^{23} + b_4^*\sigma^{24} + b_5^*\sigma^{25}$$

commute. Therefore they have a common eigenvector f^* with the eigenvalues $-b_2^*$, b_1^*, respectively. We have

$$\widetilde{\sigma}\begin{bmatrix} b_1^* f^* \\ \vdots \\ b_5^* f^* \end{bmatrix} = 0$$

and from this and (6) Claim 11 follows. □

From Claim 11 it follows that $\dim(Z_S) \geqslant 2$.

(d) Consider the case (d).

In this case we prove that the condition (3) holds, i.e., we prove that $\dim(Z_S) \geqslant 2$.

Claim 12. *Suppose $N \subset P\mathbf{C}^{5*}$ is a line in general position; then there exists $0 \neq f^{*0} \in \mathbf{C}^{4*}$, $0 \neq b^{*0} \in N$ such that $\rho(S, f^{*0} \otimes b^{*0}) = 0$.*

Proof of Claim 12. One can assume that $N = \overline{\langle b_1^*, b_2^* \rangle}$, where b_i^* are basic vectors of $\mathbf{C}^{5*}$. We have to prove that there exists a column f^{*0} of height 4 and $\lambda_1, \lambda_2 \in \mathbf{C}$, $(\lambda_1, \lambda_2) \neq (0, 0)$ such that

$$\sigma \begin{bmatrix} \lambda_1 f^{*0} \\ \lambda_2 f^{*0} \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0. \tag{8}$$

Consider the 4th and 8th rows of the matrix σ:

$$\mathbf{row}_4(\sigma) = (0, \dots, 0, \sigma_{41}^{13}, \sigma_{42}^{13}, \dots, \sigma_{43}^{15}, \sigma_{44}^{15}),$$
$$\mathbf{row}_8(\sigma) = (0, \dots, 0, \sigma_{41}^{23}, \sigma_{42}^{23}, \dots, \sigma_{43}^{25}, \sigma_{44}^{25}).$$

We want to show that $\mathbf{row}_4(\sigma)$ and $\mathbf{row}_8(\sigma)$ are linearly dependent. Suppose that $\mathbf{row}_4(\sigma)$ and $\mathbf{row}_8(\sigma)$ are linearly independent. Then the first 8 rows of the matrix σ are linearly independent. Since $\mathrm{rk}(\sigma) = 8$, we see that every row of σ is a linear combination of the first 8 rows. From $\mathbf{row}_4(\sigma) \neq 0$ it follows that $\sigma_{4j}^{1i} \neq 0$ for some $3 \leqslant i \leqslant 5$, $1 \leqslant j \leqslant 4$. Since $\sigma_{j4}^{i1} = -\sigma_{4j}^{1i} \neq 0$, we see that the $(4(i-1)+j)$th row

$$\mathbf{row}_{4(i-1)+j}(\sigma) = (\sigma_{j1}^{i1}, \sigma_{j2}^{i1}, \sigma_{j3}^{i1}, \sigma_{j4}^{i1}, \sigma_{j1}^{i2}, \sigma_{j2}^{i2}, \sigma_{j3}^{i2}, \sigma_{j4}^{i2}, \dots)$$

of the matrix σ is *not* a linear combination of the first 8 rows. This contradiction proves that $\mathbf{row}_4(\sigma)$ and $\mathbf{row}_8(\sigma)$ are linearly dependent.

Finally, to obtain (8) we take

$$f^{*0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and λ_1, λ_2 such that $(\lambda_1, \lambda_2) \neq (0, 0)$ and $\lambda_1 \mathbf{row}_4(\sigma) + \lambda_2 \mathbf{row}_8(\sigma) = 0$. □

From Claim 12 it follows that $\dim(Z_S) \geqslant 3 > 2$. □

The proof of Theorem 1

We suppose that there exists $A^0 \in I$ such that $\dim(T_{\pi(A^0)}MI(k)) > 8k-3$ and obtain a contradiction.

From Corollary 7 it follows that

$$\dim(X_{A^0}) \leqslant 1 \tag{9}$$

and by Lemma 2 we have $\dim(T_{A^0}I) > 3k^2+13k$. Hence, by Lemma 8 there exists $0 \neq S^0 \in S^2\mathbf{C}^4 \otimes \wedge^2\mathbf{C}^k$ such that

$$\xi(A^0, S^0) = 0. \tag{10}$$

Claim 13. (1) *Consider the following composition of linear mappings*

$$\rho(S^0,\cdot)\circ\beta(A^0,\cdot): \ \mathbf{C}^{2k+2} \longrightarrow \mathbf{C}^4\otimes\mathbf{C}^k, \quad h \mapsto \rho(S^0,\beta(A^0,h)).$$

Then we have $\rho(S^0,\cdot)\circ\beta(A^0,\cdot)=0$. (2) *Consider the following composition of linear mappings*

$$\varepsilon(A^0,\cdot)\circ\rho(S^0,\cdot): \ \mathbf{C}^{4*}\otimes\mathbf{C}^{k*} \longrightarrow \mathbf{C}^{2k+2}, \quad B^* \mapsto \varepsilon(A^0,\rho(S^0,B^*)).$$

Then we have $\varepsilon(A^0,\cdot)\circ\rho(S^0,\cdot))=0$.

Proof of Claim 13. Consider the following nontrivial trilinear $\mathbf{SL}_4\times\mathbf{SL}_k\times\mathbf{Sp}_{2k+2}$-morphism:

$$\tau: \ \mathbf{C}^{4*}\otimes\mathbf{C}^{k*}\otimes\mathbf{C}^{2k+2}\times S^2\mathbf{C}^4\otimes\wedge^2\mathbf{C}^k\times\mathbf{C}^{2k+2}\longrightarrow\mathbf{C}^4\otimes\mathbf{C}^k,$$
$$(A,S,h)\mapsto\kappa(\xi(A,S),h),$$

where

$$\kappa: \mathbf{C}^{4*}\otimes\mathbf{C}^{k*}\otimes\mathbf{C}^{2k+2}\times\mathbf{C}^{2k+2}\longrightarrow\mathbf{C}^4\otimes\mathbf{C}^k$$

is the canonical bilinear $\mathbf{SL}_4\times\mathbf{SL}_k\times\mathbf{Sp}_{2k+2}$-morphism. Note that

$$\tau(A^0,S^0,h)=\kappa(\xi(A^0,S^0),h)\equiv 0. \tag{11}$$

The $\mathbf{SL}_4\times\mathbf{SL}_k\times\mathbf{Sp}_{2k+2}$-module

$$(\mathbf{C}^{4*}\otimes\mathbf{C}^{k*}\otimes\mathbf{C}^{2k+2})\otimes(S^2\mathbf{C}^4\otimes\wedge^2\mathbf{C}^k)\otimes\mathbf{C}^{2k+2}$$

contains the irreducible $\mathbf{SL}_4\times\mathbf{SL}_k\times\mathbf{Sp}_{2k+2}$-module $\mathbf{C}^4\otimes\mathbf{C}^k$ with multiplicity 1. Therefore, there exists a unique, up to a scalar factor, nontrivial trilinear $\mathbf{SL}_4\times\mathbf{SL}_k\times\mathbf{Sp}_{2k+2}$-morphism

$$\mathbf{C}^{4*}\otimes\mathbf{C}^{k*}\otimes\mathbf{C}^{2k+2}\times(S^2\mathbf{C}^4\otimes\wedge^2\mathbf{C}^k)\times\mathbf{C}^{2k+2}\longrightarrow\mathbf{C}^4\otimes\mathbf{C}^k.$$

Therefore,

$$(\rho(S,\cdot)\circ\beta(A,\cdot))(h)\equiv c_1\tau(A,S,h) \tag{12}$$

for some $c_1 \in \mathbf{C}$ and

$$(\varepsilon(A,\cdot)\circ\rho(S,\cdot))^*(h) \equiv c_2\tau(A,S,h) \tag{13}$$

for some $c_2 \in \mathbf{C}$.

From (12) and (11) we have

$$(\rho(S^0,\cdot)\circ\beta(A^0,\cdot))(h) = c_1\tau(A^0,S^0,h) \equiv 0.$$

This gives us statement (1). From (13) and (11) we have

$$(\varepsilon(A^0,\cdot)\circ\rho(S^0,\cdot))^*(h) = c_2\tau(A^0,S^0,h) \equiv 0.$$

From this statement (2) follows. □

From Claim 13 (1) we have

$$\operatorname{Im}(\beta(A^0,\cdot)) \subset \operatorname{Ker}(\rho(S^0,\cdot)). \tag{14}$$

On the other hand, by (E3) we have $\operatorname{rk}(\beta(A^0,\cdot)) = 2k+2$ and with (14) this gives us

$$\operatorname{rk}(\rho(S^0,\cdot)) \leqslant 2k-2. \tag{15}$$

From (15) it follows that one of the conditions (1)–(3) of Lemma 10 holds for $S = S^0$.

I. Consider the case when the condition (1) of Lemma 10 holds for $S = S^0$.

By the condition (1) of Lemma 10 there exists $B^{*0} \in \mathbf{C}^{4*}\otimes\mathbf{C}^{k*}$ such that $\rho(S^0,B^{*0}) = f^0\otimes b^0 \neq 0$. Thus, we have $\varepsilon(A^0, f^0\otimes b^0) = \varepsilon(A^0,\rho(S^0,B^{*0})) = 0$ by Claim 13 (2) and, therefore, $A^0 \notin I_1$. But this contradicts to $A^0 \in I$.

II. Consider the case when the condition (2) of Lemma 10 holds for $S = S^0$.

From (15) it follows that $k = 4$ or $k = 5$. By the condition (2) of Lemma 10 we have $\{f^{*0}\}\times\mathbf{C}^{k*} \subset \operatorname{Ker}(\rho(S^0,\cdot))$. On the other hand, we have (14) and

$$\dim(\operatorname{Ker}(\rho(S^0,\cdot))) - \dim(\operatorname{Im}(\beta(A^0,\cdot))) = \begin{cases} 0 & \text{if } k=4,\\ 2 & \text{if } k=5. \end{cases}$$

Therefore, $\operatorname{Im}(\beta(A^0,\cdot) \supset \{f^{*0}\}\times M$ for some linear subspace $M \subset \mathbf{C}^{k*}$ of dimension $\geqslant 3$. But this contradicts (9).

III. Consider the case when the condition (3) of Lemma 10 holds for $S = S^0$.

From (15) it follows that $k = 5$. Thus,

$$\dim(\operatorname{Im}(\beta(A^0,\cdot))) = 12 = \dim(\operatorname{Ker}(\rho(S^0,\cdot)))$$

and from this together with (14) it follows that $\operatorname{Im}(\beta(A^0,\cdot)) = \operatorname{Ker}(\rho(S^0,\cdot))$. Therefore, $X_{A^0} = Z_{S^0}$. From this and the condition (3) of Lemma 10 we obtain $\dim(X_{A^0}) = \dim(Z_{S^0}) \geqslant 2$. But this again contradicts (9).

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