QUIVERS AND THE COHOMOLOGY OF HOMOGENEOUS VECTOR BUNDLES

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Abstract

We describe the cohomology groups of a homogeneous vector bundle E on any Hermitian symmetric variety $X = G/P$ of ADE-type as the cohomology of a complex explicitly described. The main tool is the equivalence (introduced by Bondal, Kapranov, and Hille) between the category of homogeneous bundles and the category of representations of a certain quiver $\mathcal{Q}_X$ with relations. We prove that the relations are the commutative ones on projective spaces, but they involve additional scalars on general Grassmannians. In addition, we introduce moduli spaces of homogeneous bundles.

Contents

1. Introduction

The Borel-Weil-Bott theorem computes the cohomology groups of an irreducible homogeneous bundle on a rational homogeneous variety X. In this article we compute the cohomology groups of any homogeneous bundle (including the reducible ones) on a symmetric Hermitian variety of ADE-type. This class of varieties includes Grassmannians, quadrics of even dimension, spinor varieties, two exceptional cases, and products among all of them.

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In order to compute the cohomology groups (see Th. 6.11), we have to describe the homogeneous bundles as representations of a certain quiver $\mathcal{Q}_X$. The moduli spaces of such representations give moduli spaces of homogeneous bundles, which are introduced in §7 and seem to have an intrinsic interest.

We describe now in detail the background of this article.

Let $X = G/P$ be a rational homogeneous variety. It is known that the category of G-homogeneous bundles on X is equivalent to the category P-mod of representations of P, and also to the category $\mathscr{P}$-mod, where $\mathscr{P} = \text{Lie } P$ (see, e.g., [BK]). Since P is not reductive, its representations are difficult to describe. In fact, if E is a homogeneous bundle, it has a filtration $0 \subset E_1 \subset \cdots \subset E_k = E$ where E_i/E_{i-1} is irreducible, but the filtration does not split in general.

Let $P = R \cdot N$ be the Levi decomposition, where R is reductive and N is nilpotent. At the level of Lie algebras, this amounts to $\mathscr{P} = \mathscr{R} \oplus \mathscr{N}$ as vector spaces. Considering E as R-module (and hence as $\mathscr{R}$-module), we get the graded bundle $\text{gr } E = \bigoplus_i E_i/E_{i-1}$. The nilpotent radical $\mathscr{N}$ is an $\mathscr{R}$-module itself, with the adjoint action, corresponding to the bundle Ω^1_X. The action of $\mathscr{P}$ over E induces a G-equivariant map

$$\theta\colon \Omega^1_X \otimes \text{gr } E \longrightarrow \text{gr } E. \tag{$*$}$$

Our first result is that when X is a Hermitian symmetric variety, a morphism of $\mathscr{R}$-modules $\theta\colon \Omega^1_X \otimes F \longrightarrow F$ is induced by a $\mathscr{P}$-action if and only if $\theta \wedge \theta = 0$ (see Th. 3.1).

In analogy with [S], we call a completely reducible bundle F endowed with such θ satisfying $\theta \wedge \theta = 0$ a (*homogeneous*) *Higgs bundle*. So the category of G-homogeneous bundles turns out to be equivalent to the category of Higgs bundles. In the pair (F, θ), F encodes the discrete part and θ encodes the continuous part.

By using the Bott theorem, we can prove that $\text{Hom}(\text{gr } E \otimes \Omega^1_X, \text{gr } E)^G$ is isomorphic to $\text{Ext}^1(\text{gr } E, \text{gr } E)^G$ (see Th. 4.3). In this setting, a reformulation of Theorem 3.1 implies that the set of $\mathscr{P}$-modules E such that $\text{gr } E = F$ is in natural bijection with the set of $e \in \text{Ext}^1(F, F)^G$ such that $m(e) = 0$, where m is the quadratic Yoneda morphism $\text{Ext}^1(F, F)^G \longrightarrow \text{Ext}^2(F, F)^G$.

Bondal and Kapranov had the remarkable idea that quivers are the appropriate tool to manage P-modules; indeed, we state our results in the framework of quivers.

A quiver $\mathcal{Q}_X$ is associated to any rational homogeneous variety X. The points of $\mathcal{Q}_X$ are the dominant weights of R, and the arrows correspond to the weights of $\mathscr{N}$ in the action $(*)$. Bondal and Kapranov [BK] and Hille [H1] proved that the category of G-homogeneous bundles on X is equivalent to the category of representations of $\mathcal{Q}_X$ with certain relations to be determined (see also [Ki]). Hille in [H1] proved that the relations in $\mathcal{Q}_X$ are quadratic if X is Hermitian symmetric and found that the relations of the quiver constructed in [BK] were not properly stated in the case of the Grassmannian of lines in $\mathbf{P}^3$ (see Exam. 5.11). Then Hille showed that in $\mathcal{Q}_{\mathbf{P}^2}$, the relations correspond to the commutativity of all square diagrams. If X is Hermitian symmetric, we see that the relations are consequences of the condition $\theta \wedge \theta = 0$. This allows us to extend Hille's result to $\mathcal{Q}_{\mathbf{P}^n}$ (see Cor. 8.5). We have found that the relations for general Grassmannians involve some additional scalars (see Prop. 8.4).

The second part of this article is devoted to the computation of the cohomology. The Borel-Weil-Bott theorem computes the cohomology groups of an irreducible bundle E on X. In particular, it says that $H^*(E)$ is an irreducible G-module. It follows that for any G-homogeneous bundle E, there is a spectral sequence constructed by the filtration $\operatorname{gr} E$ abutting to the cohomology groups of E. The main problem is that the maps occurring in the spectral sequence, although they are equivariant, are difficult to control. In fact, most of the main open problems about rational homogeneous varieties, like the computation of syzygies of their projective embeddings, reduce to the computation of cohomology groups of certain homogeneous bundles (see the book [W]).

Assume now that X is Hermitian symmetric of ADE-type. Thanks to the Borel-Weil-Bott theorem, and to the results of Kostant in [Ko], we can divide the points of $\mathcal{Q}_X$ into several chambers, separated by the hyperplanes containing the singular weights, which we call *Bott chambers*. We consider the segments connecting any point of $\mathcal{Q}_X$ with its mirror images in the adjacent Bott chambers, and we define certain linear maps $c_i\colon H^i(\operatorname{gr} E) \to H^{i+1}(\operatorname{gr} E)$, by composing the maps associated to the representation of $\mathcal{Q}_X$ corresponding to E, along these segments. We get a sequence

$$\cdots \longrightarrow H^i(\operatorname{gr} E) \xrightarrow{c_i} H^{i+1}(\operatorname{gr} E) \xrightarrow{c_{i+1}} \cdots .$$

In Theorem 6.11 we prove that this sequence is a complex, and its cohomology (as a G-module) is the usual cohomology $H^i(X, E)$.

The proof of this result is obtained by comparing the maps c_i with the boundary maps. In the case of projective spaces, the computation of c_i can be done quite easily. The advantage of this approach with respect to the spectral sequence is that the maps c_i are defined at once, while in the spectral sequence we need an iterative construction. Note that the spectral sequence can degenerate after a large number of steps. At present it is not clear if our approach will be useful for the computation of syzygies of G/P. It is worth noting that the derived category of homogeneous bundles was described by Kapranov in the last section of [K]. The quivers allow us to refine that approach.

It turns out from our proof that the cohomology modules $H^i(E)$ are equipped with a natural filtration

$$0 \subset H^i[1](E) \subset H^i[2](E) \subset \cdots \subset H^i[N](E) = H^i(E).$$

The third part of this article deals with moduli spaces. There is a notion of semistability of representations of quivers introduced in [Ki] (see also [M]) which is suitable to construct moduli spaces according to Mumford's geometric invariant theory (GIT). This notion of semistability turns out to be equivalent to the Mumford-Takemoto semistability of the bundle, and we get moduli spaces of G-homogeneous semistable bundles with fixed $\operatorname{gr} E$. More precisely, the choice of an $\mathscr{R}$-module F is equivalent to the choice of a dimension vector α as in [Ki]. All semistable P-modules E such that $\operatorname{gr} E = F$ are parametrized by a projective moduli space $M_X(\alpha)$. The properties of such moduli spaces probably deserve further study.

In the last part of this article we compute explicitly the relations on any Grassmannians by using Olver maps. As one of the referees pointed out, in order to generalize our result to other Hermitian symmetric varieties it would be necessary to find the substitutes for the Olver maps.

Finally, we want to mention that some applications of this approach to the case of homogeneous bundles on $\mathbf{P}^2$ appear in [OR].

We now sketch the content of the sections. In §3 we describe the equivalence of categories between G-homogeneous bundles and Higgs bundles. In §4 we recall the Borel-Weil-Bott theorem, in the form found by Kostant [Ko], which is suitable for our purposes. In §5 we construct in detail the quiver $\mathscr{Q}_X$ with its relations, and we prove the equivalence between the category of homogeneous bundles and the category of representations of $\mathscr{Q}_X$. In §6 we prove our main result about the cohomology groups. In §7 we introduce the moduli spaces $M_X(\alpha)$, and we compare some different notions of stability. In §8 we make explicit for Grassmannians the relations stated in §5 by using Olver maps.

2. Notation and preliminaries

Throughout this article, let G be a semisimple complex Lie group. We fix a Cartan subalgebra $\mathscr{H}$ in $\operatorname{Lie} G$. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a *fundamental system of simple roots* for $\operatorname{Lie} G$. A positive root is a linear combination with nonnegative integral coefficients of the simple roots. The Killing product allows us to identify $\mathscr{H}$ with $\mathscr{H}^\vee$ and thus to define the Killing product also on $\mathscr{H}^\vee$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the *fundamental weights* corresponding to $\{\alpha_1, \dots, \alpha_n\}$, that is, the elements of $\mathscr{H}^\vee$ such that $2(\lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$, where $(\ ,\ )$ is the Killing product. Let Z be the lattice generated by the fundamental weights. The elements in Z which are a linear combination with nonnegative coefficients of the fundamental weights are called the *dominant weights for* G, and they are the maximal weights of the irreducible representations of $\operatorname{Lie} G$. In the ADE-case, all roots have length $\sqrt{2}$.

For any W-representation of G, we denote by W^G its invariant part, that is, the subspace of W where G acts trivially. If V is an irreducible representation, we denote $W^V := \operatorname{Hom}(V, W)^G \otimes V$.

If $\lambda \in Z$, we denote by V_λ the irreducible representation of G with highest-weight λ. In the case $G = \mathrm{SL}(n+1)$, to any λ is associated a Young diagram. Precisely if we have $\lambda = \sum_{i=1}^{n} n_i \lambda_i$, then we set $a_i = \sum_{j \geq i} n_j$, and we get the Young diagram with a_i boxes in the ith row. We use the notation where the first row is the top row. The n-tuple $a = (a_1, \ldots, a_n)$ is a partition of $\sum a_i$, and it is customary to denote V_λ by $\mathscr{S}^a V$. In particular, $\mathscr{S}^2 V = \mathrm{Sym}^2 V$ and $\mathscr{S}^{1,1} V = \bigwedge^2 V$.

Let $X = G/P$ be a rational homogeneous variety, where P is a parabolic subgroup (see [Ko], [FH]). We fix a splitting $\mathrm{Lie}\, P = \mathrm{Lie}\, R \oplus \mathrm{Lie}\, N = \mathscr{R} \oplus \mathscr{N}$, where $\mathscr{R}$ is reductive and $\mathscr{N}$ is the nilpotent radical. A representation of P is completely reducible if and only if it is trivial on N (see [I] or [Ot]). In this case the representations are determined by their restriction on R.

Homogeneous vector bundles. The group G is a principal bundle over $X = G/P$ with fiber P. Denote by z the point of X which is fixed by P, corresponding to the lateral class $P \in G/P$. Any G-homogeneous vector bundle E with fiber $E(z)$ over z is induced by this principal bundle via a representation $\rho \colon P \to \mathrm{GL}(E(z))$. We denote $E = E_{[\rho]}$. Equivalently, $E_{[\rho]}$ can be defined as the quotient $G \times_\rho E(z)$ of $G \times E(z)$ via the equivalence relation $\sim$, where $(g, v) \sim (g', v')$ if and only if there exists $p \in P$ such that $g = g'p$ and $v = \rho(p^{-1})v'$.

We denote by E_λ the homogeneous bundle corresponding to the irreducible representation of P with maximal weight λ. Here λ belongs to the fundamental Weyl chamber of the reductive part of P (see the beginning of §4).

Hermitian symmetric varieties. We recall that the tangent bundle of X is defined by the adjoint representation over $\mathrm{Lie}\, G/\mathrm{Lie}\, P$. According to Kostant, we say that X is a Hermitian symmetric variety if the above adjoint representation is trivial on N. This is equivalent to asking if $[\mathscr{N}, \mathscr{N}] = 0$. The Hermitian symmetric varieties were classified by Cartan, and their list is well known. They are the product of irreducible ones. The irreducible ones are Grassmannians, quadrics, spinor varieties, maximal Lagrangian Grassmannians, and two varieties of exceptional type of dimensions 16 and 27 (see Th. 5.12 for the precise list). For a modern treatment, see [Ko] or [LM]. According to the corresponding Dynkin diagram, an irreducible Hermitian symmetric variety is called of type ADE if $G = \mathrm{SL}(m)$, $\mathrm{Spin}(2m)$, E_6, or E_7. Only odd quadrics and maximal Lagrangian Grassmannians are left, which are called of type BC. A Hermitian symmetric variety is called of ADE-type if it is the product of irreducible Hermitian symmetric varieties of ADE-type. The reason for which we have to restrict to the ADE-type in the computation of cohomology is explained in Propositions 6.4 and 6.5. In all the irreducible cases we have $\mathrm{Pic}(X) = \mathbf{Z}$. Thus on irreducible Hermitian symmetric varieties the first Chern class $c_1(E)$ of a bundle E can be identified with an integer, and the slope is by definition $\mu(E) = c_1(E)/\mathrm{rk}(E) \in \mathbf{Q}$. On any Hermitian symmetric variety $X = X_1 \times \cdots \times X_r$, where X_i are irreducible, there are several possible choices of slopes. With obvious notation, if $c_1(E) = (c_1^1, \ldots, c_1^r) \in \mathbf{Z}^r$ and $a = (a_1, \ldots, a_r) \in \mathbf{Q}^r$, then we define $\mu_a(E) = \left(\sum c_1^i a_i\right)/\mathrm{rk}(E) \in \mathbf{Q}$.

It is easy to check (see, e.g., [R, §5.2]) that $\mu_a(E_{\sum n_i \lambda_i}) = \sum n_i \mu_a(E_{\lambda_i})$.

The Hasse quiver. Quivers are recalled in §5. For this paragraph it is enough to know that a quiver is just an oriented graph. If X is a rational homogeneous variety, the cohomology $H^*(X, \mathbf{Z})$ can be organized in a quiver in the following way. Consider the action of a Borel subgroup $B \subset P$ on X. Then it is well known that X is divided in a finite union of orbits; their closures are called the *Schubert cells* and form an additive basis $H^*(X, \mathbf{Z})$. The vertices of the Hasse quiver $\mathscr{H}_X$ are the Schubert cells; we draw an arrow between $X_\omega \in H^{2p}(X, \mathbf{Z})$ and $X_{\omega'} \in H^{2p+2}(X, \mathbf{Z})$ if $X_\omega \supset X_{\omega'}$. If X is a Hermitian symmetric variety, the additive basis of $H^{2p}(X, \mathbf{Z})$ corresponds to the direct summands of Ω^p. If X is Hermitian symmetric, the degrees of the Schubert cycles in the homogeneous minimal embedding are computed as the number of paths in the Hasse quiver which start from the corresponding vertex. We learned this fact from L. Manivel (see [IM]).

The filtration of a homogeneous bundle and the functor gr. Let E be a homogeneous bundle on an irreducible Hermitian symmetric variety.

We define $\mathrm{gr}\, E = \bigoplus_i E_i/E_{i-1}$ for any filtration $0 \subset E_1 \subset \cdots \subset E_k = E$ such that E_i/E_{i-1} is completely reducible. The graded bundle $\mathrm{gr}\, E$ does not depend on the filtration; in fact, it is given by the restriction of the representation giving E to the reductive part R of P.

For example, the Euler sequence on $\mathbf{P} = \mathbf{P}(V)$ tells us that $\mathrm{gr}(\mathscr{O}(1)_\mathbf{P} \otimes V) = \mathscr{O}_\mathbf{P} \oplus T\mathbf{P}$.

The functor $E \mapsto \mathrm{gr}\, E$ from P-mod to R-mod (which in the literature is often denoted as Ind_R^P) is exact. It is easy to check the formulas

$$(\mathrm{gr}\, E)^* = \mathrm{gr}(E^*), \qquad \mathrm{gr}(E \oplus F) = \mathrm{gr}\, E \oplus \mathrm{gr}\, F, \qquad \mathrm{gr}(E \otimes F) = \mathrm{gr}\, E \otimes \mathrm{gr}\, F.$$

The spectral sequence abutting to the cohomology. The Borel-Weil-Bott theorem describes the cohomology of the irreducible homogeneous bundles E. It says that $H^*(E)$ is an irreducible G-module. For any homogeneous bundle and for any filtration, there is a spectral sequence abutting to the cohomology of the bundle. Precisely, if $\mathrm{gr}\, E = \bigoplus_{i=1}^k A_i$ as before, we have $E_{p,q}^1 = H^{p+q}(A_{k-p})$ abutting to $E_{p,q}^\infty$, where $H^i(E) = \bigoplus_{p+q=i} E_{p,q}^\infty$. Theorem 6.11 gives a more efficient way to compute $H^i(E)$.

Yoneda product. We recall the Yoneda product on Ext according to [E, Exer. A3.27]. For any homogeneous bundles E, F, and K, there is an equivariant Yoneda product

$$\mathrm{Ext}^i(E,F)\otimes\mathrm{Ext}^j(F,K)\to\mathrm{Ext}^{i+j}(E,K),$$

and this product is associative. In particular, in the case where $E=F=K$ and $i=j=1$, we get a (nonsymmetric) bilinear map whose symmetric part induces a quadratic morphism

$$\mathrm{Ext}^1(E,E)\to\mathrm{Ext}^2(E,E).$$

In particular, since it preserves the invariant part, it gives

$$m\colon \mathrm{Ext}^1(E,E)^G\to\mathrm{Ext}^2(E,E)^G.$$

Tensor product of two irreducible representations. Let λ and ν be two weights in the fundamental Weyl chamber of a Lie algebra K. The tensor product of the corresponding representations $V_\lambda\otimes V_\nu$ can be expressed as a sum $\bigoplus c_{\lambda\nu\kappa}V_\kappa$, where $c_{\lambda\nu\kappa}$ are integers (counting the multiplicities). When $K=\mathrm{Lie}\,\mathrm{SL}(n)$, the integers $c_{\lambda\nu\kappa}$ can be computed by the so-called Littlewood-Richardson rule (see [FH]). A more conceptual algorithm was later conjectured by Weyman and proved by Littelmann in [L]; this algorithm holds for arbitrary simple Lie groups. Let $\nu_1=\nu,\nu_2,\dots,\nu_k$ be all the weights of V_ν. Littelmann proves that

$$V_\lambda\otimes V_\nu=\bigoplus\nolimits_{i\in I}V_{\lambda+\nu_i},\tag{1}$$

where I is a subset of $\{1,\dots,k\}$ such that the weights ν_i for $i\in I$ correspond exactly to the standard Young tableaux of the form corresponding to ν which are λ-dominant (see [L] for the precise definitions). A particularly interesting case is when $\lambda+\nu_i$ are all dominant for $i=1,\dots,k$; this is true when $\lambda\gg 0$. In this case we have the whole decomposition

$$V_\lambda\otimes V_\nu=\bigoplus\nolimits_{i=1}^k V_{\lambda+\nu_i}$$

(see also [FH, Exer. 25.33]). Formula (1) applied to vector bundles gives

$$E_\lambda\otimes E_\nu=\bigoplus\nolimits_{i\in I}E_{\lambda+\nu_i},$$

where all the direct summands in the right-hand side have the same slope (see [R] or [Ot]).

3. P-mod and the category of Higgs bundles

Let X be a Hermitian symmetric variety. We recall that $\mathcal{N}$ is an $\mathcal{R}$-module with the adjoint action. Our starting point is the following.

THEOREM 3.1

(i) *Given a $\mathcal{P}$-module E on X, the action of $\mathcal{N}$ over E induces a morphism of $\mathcal{R}$-modules*

$$\theta : \mathcal{N} \otimes \operatorname{gr} E \longrightarrow \operatorname{gr} E$$

such that $\theta \wedge \theta = 0$ in $\operatorname{Hom}\left(\bigwedge^2 \mathcal{N} \otimes \operatorname{gr} E, \operatorname{gr} E\right)$.

(ii) *Conversely, given an $\mathcal{R}$-module F on X and a morphism of $\mathcal{R}$-modules*

$$\theta : \mathcal{N} \otimes F \longrightarrow F$$

such that $\theta \wedge \theta = 0$, we have that θ extends uniquely to an action of $\mathcal{P}$ over F, giving a bundle E such that $\operatorname{gr} E = F$.

Proof

(i) For every $r \in \mathcal{R}$, $n \in \mathcal{N}$, $f \in F$, since E is a $\mathcal{P}$-module we have

$$r \cdot (n \cdot f) = n \cdot (r \cdot f) + [r, n] \cdot f;$$

that is,

$$r \cdot \big(\theta(n \otimes f)\big) = \theta\big(n \otimes (r \cdot f)\big) + \theta([r, n] \otimes f) = \theta\big(r \cdot (n \otimes f)\big),$$

so that θ is $\mathcal{R}$-equivariant. Moreover, for any $n_1, n_2 \in \mathcal{N}$,

$$\theta \wedge \theta\big((n_1 \wedge n_2) \otimes f\big) = n_1 \cdot (n_2 \cdot f) - n_2 \cdot (n_1 \cdot f) = [n_1, n_2] \cdot f = 0$$

because $[\mathcal{N}, \mathcal{N}] = 0$, and this is equivalent to $\theta \wedge \theta = 0$.

(ii) We have, for any $r + n \in \mathcal{R} \oplus \mathcal{N} = \mathcal{P}$,

$$(r + n) \cdot f := r \cdot f + \theta(n \otimes f),$$

and we have to prove that for any $p_1, p_2 \in \mathcal{P} = \mathcal{R} \oplus \mathcal{N}$, we have

$$[p_1, p_2] \cdot f = p_1 \cdot (p_2 \cdot f) - p_2 \cdot (p_1 \cdot f). \tag{2}$$

We distinguish three cases.

- If $p_1, p_2 \in \mathcal{R}$, then (2) is true because F is an $\mathcal{R}$-module.
- If $p_1, p_2 \in \mathcal{N}$, then $[p_1, p_2] = 0$ and (2) is true because $\theta \wedge \theta = 0$.

- If $p_1 \in \mathscr{R}$, $p_2 \in \mathscr{N}$, we have $[p_1, p_2] \in \mathscr{N}$ and

  $$[p_1, p_2] \cdot f + p_2 \cdot (p_1 \cdot f) = \theta\big(p_1 \cdot (p_2 \otimes f)\big) = p_1\theta(p_2 \otimes f) = p_1 \cdot (p_2 \cdot f)$$

  because θ is $\mathscr{R}$-equivariant. □

Theorem 3.1 allows us to construct a $\mathscr{P}$-module in two steps. The first step consists of giving the $\mathscr{R}$-module F, which encodes the discrete part; the second step consists of giving θ, which encodes the continuous part. This is made precise in §7 about moduli spaces. At present, it is convenient to reformulate Theorem 3.1 in terms of vector bundles.

We have seen in the introduction that on a Hermitian symmetric variety, the $\mathscr{P}$-module $\mathscr{N}$ corresponds to Ω^1_X. Since $[\mathscr{N}, \mathscr{N}] = 0$, Ω^1_X is completely reducible. Let E be a G-homogeneous bundle E. The action of $\mathscr{N}$ over the $\mathscr{R}$-module $\operatorname{gr} E$ induces by Theorem 3.1 an $\mathscr{R}$-equivariant morphism of completely reducible representations $\mathscr{N} \otimes \operatorname{gr} E \to \operatorname{gr} E$; hence we get a G-equivariant morphism $\theta \in \operatorname{Hom}(\operatorname{gr} E, \operatorname{gr} E \otimes T_X)^G$ such that $\theta \wedge \theta = 0$. To any E we can associate the pair $(\operatorname{gr} E, \theta)$. Such pairs are analogous to what is called in [S] a Higgs bundle. The pairs $(\operatorname{gr} E, \theta)$ are the natural extension of Higgs bundles for rational homogeneous varieties, where T_X is globally generated; so we maintain the terminology of Higgs bundles also in this case.

More precisely, we have the following.

Definition 3.2
Let X be a Hermitian symmetric variety. A Higgs bundle on X is a pair (F, θ), where F is an R-module and $\theta\colon F \longrightarrow F \otimes T_X$ is G-equivariant and satisfies $\theta \wedge \theta = 0$.

Higgs bundles form an abelian category, where a morphism between two Higgs bundles (F_1, θ_1) and (F_2, θ_2) is a G-equivariant morphism $f\colon F_1 \longrightarrow F_2$ such that $(f \otimes \mathrm{id})\theta_1 = \theta_2 f$. Hence Theorem 3.1 can be reformulated in the following way.

THEOREM 3.3
Let $X = G/P$ be a Hermitian symmetric variety. There is an equivalence of categories between
(i) *G-homogeneous bundles over X and*
(ii) *Higgs bundles (F, θ) over X.*

Remark. On any rational homogeneous variety, the category of G-homogeneous bundle is equivalent to the category of pairs (F, θ), where F is an R-module and $\theta\colon F \longrightarrow F \otimes T_X$ is G-equivariant and satisfies certain relations.

4. The Borel-Weil-Bott theorem

It is well known that the hyperplanes orthogonal to the roots of G divide $\mathscr{H}^\vee$ into regions called Weyl chambers. The fundamental Weyl chamber D of G is

$$D = \left\{ \sum x_i \lambda_i \,\middle|\, x_i \geq 0 \right\},$$

and it contains exactly the dominant weights. The Weyl group W acts in a simple transitive way as a group of isometries on the Weyl chambers. Following [Ko], we denote $g = \sum \lambda_i$. Any homogeneous variety with $\mathrm{Pic} = \mathbf{Z}$ is the quotient $X = G/P(\alpha_j)$ for some j, where the Lie algebra of $P(\alpha_j)$ is spanned by the Cartan subalgebra, by the eigenspaces of the negative roots, and by the eigenspaces of the positive roots $\alpha = \sum n_i \alpha_i$ such that $n_i \geq 0$ for any i and $n_j = 0$.

The reductive part of $P(\alpha_j)$ has its own fundamental Weyl chamber $D_1 \supset D$ defined by

$$D_1 = \left\{ \sum x_i \lambda_i \,\middle|\, x_i \geq 0 \text{ for } i \neq j \right\}.$$

D_1 contains exactly the maximal weights of the irreducible representations of $P(\alpha_j)$. Let

$$W^1 = \{ w \in W \mid wD \subset D_1 \}$$

(see [Ko, Rem. 5.13]). The cardinality of W^1 divides the order of W.

Let H_ϕ be the hyperplane orthogonal to the root ϕ, and let r_ϕ be the reflection with respect to H_ϕ. It is well known that the reflections r_{α_i} generate the Weyl group.

Let $Y_\phi = H_\phi - g$.

Let $\xi_1, \ldots, \xi_m$ be the weights of the representation giving the bundle Ω^1_X, where $m = \dim X$. Let s_j for $j = 1, \ldots, m$ be the reflection through Y_{ξ_j}. Note that for any weight λ,

$$s_j(\lambda) = r_{\xi_j}(\lambda + g) - g; \tag{3}$$

thus s_j and r_{ξ_j} are conjugate elements in $\mathrm{Iso}(\mathscr{H}^\vee)$. It follows that if $w = r_{\xi_1} \cdot \ldots \cdot r_{\xi_p}$, then $w(\lambda + g) - g = s_1 \cdot \ldots \cdot s_p(\lambda)$.

An element $\nu \in Z$ is called *regular* if $(\nu, \phi) \neq 0$ for any root ϕ; otherwise, it is called *singular*. Observe that ν is singular if and only if $\nu \in H_\phi$ for some root ϕ.

Denote (see [Ko, Rem. 6.4])

$$D_1^0 = \{ \xi \in D_1 \mid g + \xi \text{ is regular} \}.$$

D_1^0 consists of the subset of D_1 obtained by removing exactly the Y_{ξ_j}. Hence a convenient composition of s_j brings D into the several "chambers" in which D_1^0 is

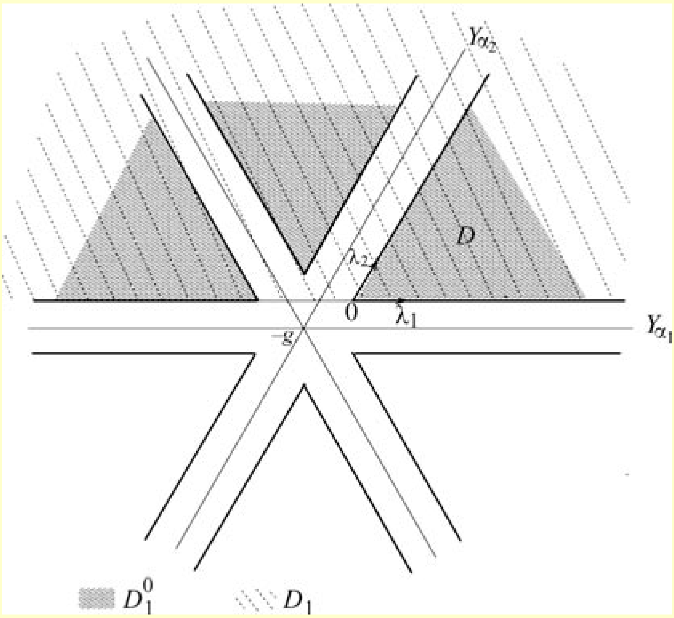

Figure 1

divided, which we call *Bott chambers*. (Do not confuse them with the usual Weyl chambers.) The Bott chambers are obtained by performing a slight "separation" on the Weyl chambers (see Fig. 1 in the case of $\mathbf{P}^2 = \mathrm{SL}(3)/P(\alpha_1)$, where the three Bott chambers are shadowed).

Now for any $w \in W$ the *length* $l(w)$ is defined as the minimum number of reflections r_α (with α a root) needed to obtain w. Any Bott chamber has its own length. Two Bott chambers are said to be adjacent if they have a common hyperplane in their boundary. The lengths of two Bott chambers are consecutive integers.

We state the Bott theorem (cf. [Ko, Th. 5.14]).

THEOREM 4.1 (Bott)
If $\lambda \in D_1$, then $\exists!\ w \in W$ s.t. $w^{-1} \in W^1$ and $w(\lambda + g) \in D$.

(i) *If $w(\lambda + g)$ belongs to the interior of D, then setting $\nu = w(\lambda + g) - g$ we have $H^{l(w)}(E_\lambda) = V_\nu$ and $H^j(E_\lambda) = 0$ for $j \neq l(w)$. In particular, if $\lambda \in D$ (thus w is the identity), then $H^0(E_\lambda) = V_\lambda$ and $H^i(E_\lambda) = 0$ for $i > 0$.*

(ii) *If $w(\lambda + g)$ belongs to the boundary of D, then $H^j(E_\lambda) = 0,\ \forall j$.*

We recall the following result of Kostant (see [Ko, Cor. 8.2]):

$$\#\left\{w \in W^1 \,\middle|\, l(w) = i\right\} = \dim H^{2i}(X, \mathbf{C});$$

in particular,

$$\#W^1 = \chi(X, \mathbf{C}). \tag{4}$$

We now explain the relation of the previous result with the Bott theorem. By Hodge-Deligne theory, $H^{2i}(X, \mathbf{C})$ is isomorphic to $H^i(X, \Omega^i_X) = H^i(X, \Omega^i_X)^G$. Moreover, for any irreducible Hermitian symmetric varieties the bundle Ω^1 is irreducible and Ω^i splits as a sum of direct summands; the number of these summands is equal to $\dim H^{2i}(X, \mathbf{C})$. Moreover, on $X = X_1 \times \cdots \times X_r$ with projections p_i, we have $\Omega^1_X = \bigoplus p_i^* \Omega^1_{X_i}$. The vertices λ of the Bott chambers correspond exactly to the direct summands of Ω^i for some i. Indeed, for any such a vertex λ there exists w as in the Bott theorem (thus $w^{-1} \in W^1$) such that $w(\lambda + g) - g = 0$ (i.e., $\lambda = w^{-1}(g) - g$) and $l(w) = i$.

We note the following consequences of the results of Bott and Kostant.

COROLLARY 4.2
Let E be a completely reducible bundle on X Hermitian symmetric variety. Then $H^j(E)^G$ is isomorphic to $\mathrm{Hom}(\Omega^j, E)^G$. *This means that when E is irreducible, $H^j(E)^G \neq 0$ if and only if E is a direct summand of Ω^j.*

Proof
We may suppose that $E = E_\lambda$. We have $\mathrm{Hom}(\Omega^j, E_\lambda)^G \neq 0$ if and only if E_λ is a direct summand of Ω^j, and in this case it is isomorphic to $\mathbf{C}$. By the Bott theorem, we have $H^j(E_\lambda)^G \neq 0$ if and only if $H^j(E_\lambda) = \mathbf{C}$, and this is true only if $w(\lambda + g) - g = 0$ (w as in the Bott theorem) and $l(w) = j$. These cases are exactly when E_λ is a direct summand of Ω^j. □

THEOREM 4.3
Let $X = G/P$ be a Hermitian symmetric variety.

(i) *There is a natural isomorphism* $\mathrm{Hom}(E_\lambda \otimes \Omega^i_X, E_\nu)^G \to \mathrm{Ext}^i(E_\lambda, E_\nu)^G$, $\forall \lambda, \nu \in D_1$. *Both spaces are isomorphic to* $\mathbf{C}$ *or to zero for $i = 1$.*

(ii) *If X is irreducible and* $\mathrm{Ext}^i(E_\lambda, E_\nu)^G \neq 0$, *then $\mu(E_\nu) = \mu(E_\lambda) + i\,\mu(\Omega^1)$.*

(iii) *If $X = X_1 \times \cdots \times X_r$, product of irreducible ones, and* $\mathrm{Ext}^i(E_\lambda, E_\nu)^G \neq 0$ *define $a_i = 1/(\mu(\Omega^1_{X_i}))$, then with this choice for any i we have $\mu_a(\Omega^1_X) = \mu_a(p_i^* \Omega^1_{X_i}) = 1$ (see §2), and we get $\mu_a(E_\nu) = \mu_a(E_\lambda) + i$.*

Proof

(i) By Corollary 4.2, only the last statement needs an explanation. In fact, all the irreducible components of $E_\lambda \otimes \Omega^1_X$ have multiplicity one. Indeed, look at (1) and observe that eigenspaces of the roots of G have dimension 1.

(ii) All direct summands of $E_\lambda \otimes \Omega^i_X$ have the same μ equal to $\mu(E_\lambda \otimes \Omega^i_X) = \mu(E_\lambda) + i\mu(\Omega^1_X)$.

(iii) This follows immediately as in (ii). □

Remark. For $i \geq 2$, there are some irreducible components of $E_\lambda \otimes \Omega^i_X$ which appear with multiplicity greater than 1. For example, in the Grassmannian $\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^3) = \mathrm{SL}(4)/P(\alpha_2)$, let T be the tangent bundle. We have that $\mathrm{Ext}^2(T, T(-2))^G$ contains $H^2(\Omega^2) = \mathbf{C}^2$, and correspondingly, $T \otimes \Omega^2$ contains two copies of $T(-2)$. Indeed, Ω^2 splits into two irreducible summands, and there is a copy of $T(-2)$ for each of these summands. In the case of quadrics Q_n with $n \geq 5$, the list of weights of the irreducible Ω^2 contains a weight of multiplicity $[(n-1)/2]$; in this case, for $\lambda \gg 0$, the tensor product $E_\lambda \otimes \Omega^2$ contains a direct summand with multiplicity $[(n-1)/2]$. In the case $X = \mathbf{P}^n$, all irreducible summands of $E_\lambda \otimes \Omega^2$ appear with multiplicity one by the formula (see [FH, (6.9)]); indeed, in this case all the weights of Ω^2 are distinct.

COROLLARY 4.4

If E is an irreducible bundle on a Hermitian symmetric variety, then $\mathrm{Ext}^i(E, E)^G = 0$ *for $i > 0$.*

Proof

Apply Theorem 4.3 for $\lambda = \nu$. □

COROLLARY 4.5

For every $i < \dim X$ and $\lambda \in D_1$, there are λ' and s_j such that $\lambda' = s_j(\lambda)$ and $H^i(E_\lambda) = H^{i+1}(E_{\lambda'})$ or $H^i(E_\lambda) = H^{i-1}(E_{\lambda'})$. In particular, λ and λ' differ by a multiple of ξ_j. There is exactly one of such λ' in every Bott chamber having a common boundary with the chamber containing E_λ.

Proof

Consider the vertex λ_0 of the Bott chamber containing λ. Then consider all the s_j such that $s_j(\lambda_0)$ is the maximal weight of a summand of Ω^{i+1}. Such s_j's work. □

Remark. From Corollary 4.5, λ' and s_j are unique in the case of $\mathbf{P}^n$, but they are not unique for general Grassmannians.

In Figures 2 and 3 we list all the vertices of the Bott chambers in the cases $\mathbf{P}^4$ and $\mathrm{Gr}(1, 4)$. The 4-tuple (x_1, x_2, x_3, x_4) denotes the weight $\sum x_i\lambda_i$. An arrow labeled with the root β means the reflection

$$\cdot \mapsto r_\beta(\cdot + g) - g,$$

so the arrow labeled with $-\xi_j$ means the reflection s_j. For example, $(-2, 1, 0, 0) = r_{\alpha_1}((0, 0, 0, 0) + g) - g = s_1(0, 0, 0, 0)$. To check Figures 2 and 3, Lemma 4.7 can be useful.

$$\begin{tikzcd}[row sep=large]
\bullet\ (0,0,0,0) \arrow[d, "\alpha_1"] \\
\bullet\ (-2,1,0,0) \arrow[d, "\alpha_1+\alpha_2"] \\
\bullet\ (-3,0,1,0) \arrow[d, "\alpha_1+\alpha_2+\alpha_3"] \\
\bullet\ (-4,0,0,1) \arrow[d, "\alpha_1+\alpha_2+\alpha_3+\alpha_4"] \\
\bullet\ (-5,0,0,0)
\end{tikzcd}$$

Figure 2. $\mathbf{P}^4$

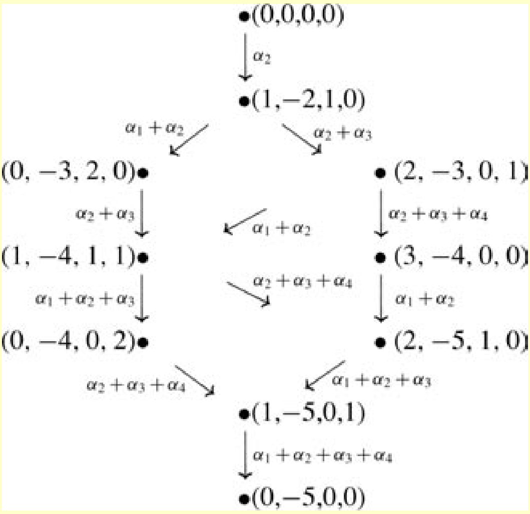

Figure 3. Gr(1, 4)

Of course, Figures 2 and 3 are exactly the Hasse quivers $\mathscr{H}_{\mathbf{P}^4}$ and $\mathscr{H}_{\mathrm{Gr}(1,4)}$.

On $\mathbf{P}^n$ we have a simplification of the Bott theorem. In this case, Ω^p are irreducible for all p.

PROPOSITION 4.6 (Bott on $\mathbf{P}^n$)

Let $X = \mathbf{P}^n = \mathrm{SL}(n+1)/P(\alpha_1)$.

(i) *If* λ *is any weight and* $\exists i \in \mathbf{N}$ *s.t.* $\nu := r_{\alpha_i} \cdots r_{\alpha_1}(\lambda + g) - g \in D$, *then* $H^i(E_\lambda) = V_\nu$ *and* $H^j(E_\lambda) = 0$ *for* $j \neq i$. *In particular, if* $\lambda \in D$, *then* $H^0(E_\lambda) = V_\lambda$ *and* $H^i(E_\lambda) = 0$ *for* $i > 0$.

(ii) *In the remaining cases,* $H^j(E_\lambda) = 0$, $\forall j$.

Proof

It is sufficient, by Theorem 4.1, to prove that $W^1 = \{r_{\alpha_1} \cdot \ldots \cdot r_{\alpha_i} \mid i \in \{1 \cdots n\}\} \cup \{1\}$. It is well known that $r_{\alpha_i}(\lambda_j)$ is equal to λ_j if $j \neq i$, and to $\lambda_{j-1} - \lambda_j + \lambda_{j+1}$ if $j = i$ (with the convention that $\lambda_0 = \lambda_{n+1} = 0$). It holds that

$$r_{\alpha_1} \cdot \ldots \cdot r_{\alpha_i}\Big(\sum_{j=1}^{n} p_j\lambda_j\Big) = \Big(-\sum_{j=1}^{i} p_i\Big)\lambda_1 + \sum_{j=1}^{i} p_j\lambda_{j+1} + \sum_{j=i+1}^{n} p_j\lambda_j.$$

(To check it, prove that $r_{\alpha_1} \cdots r_{\alpha_i}(\lambda_j)$ is equal to λ_j if $j > i$ and that it is equal to $r_{\alpha_1} \cdots r_{\alpha_j}(\lambda_j) = -\lambda_1 + \lambda_{j+1}$ if $j \leq i$.)

Hence the elements $r_{\alpha_1} \cdot \ldots \cdot r_{\alpha_i}$ belong to W^1 for $i = 1$ to n, so these elements, together with the identity, fill W^1 by (4). The last remark is that $(r_{\alpha_1} \cdots r_{\alpha_i})^{-1} = r_{\alpha_i} \cdots r_{\alpha_1}$. □

Point (iv) of the following lemma gives an alternative way to express point (i) of the Bott theorem.

LEMMA 4.7

On $\mathbf{P}^n$ *we have, for* $i = 1 \cdots n$,

(i) $\xi_i = -\alpha_1 + \cdots - \alpha_i$,

(ii) $\alpha_1 + \cdots + \alpha_{i+1} = (r_{\alpha_1} \cdots r_{\alpha_i})(\alpha_{i+1})$,

(iii) $r_{\xi_{i+1}} = (r_{\alpha_i} \cdot \ldots \cdot r_{\alpha_1})^{-1} r_{\alpha_{i+1}} (r_{\alpha_i} \cdot \ldots \cdot r_{\alpha_1})$,

(iv) $r_{\xi_1} \cdot \ldots \cdot r_{\xi_i} = r_{\alpha_i} \cdot \ldots \cdot r_{\alpha_1}$.

Proof

This is straightforward. (For (iii), observe that by (ii), $r_{\alpha_1+\cdots+\alpha_{i+1}} = r_{r_{\alpha_1}\cdots r_{\alpha_i}(\alpha_{i+1})}$.) □

COROLLARY 4.8

On $\mathbf{P}^n$ *if* $\lambda = s_{i+1}(\lambda')$, *then* $H^i(E_\lambda) = H^{i+1}(E_{\lambda'})$. *The converse holds if* $H^i(E_\lambda) \neq 0$.

In particular, λ *and* λ' *differ by a multiple of* $\alpha_1 + \cdots + \alpha_{i+1}$. *Precisely, if* $\lambda = \sum_{j=1}^{n} p_j\lambda_j$, *then* $\lambda' - \lambda = -\sum_{j=1}^{i+1}(p_j + 1)(\alpha_1 + \cdots + \alpha_{i+1})$.

Proof

By Proposition 4.6, only the converse needs to be proved. If $H^{i+1}(E_{\lambda'}) = H^i(E_\lambda) \neq 0$, then by the Bott theorem $h(\lambda + g) = r_{\alpha_{i+1}} h(\lambda' + g)$, where $h = r_{\alpha_i} \cdots r_{\alpha_1}$, and this implies that $h(\lambda + g) - h(\lambda' + g)$ is parallel to α_{i+1}, that is, that $\lambda - \lambda'$ is parallel to $h^{-1}\alpha_{i+1} = \alpha_1 + \cdots + \alpha_{i+1}$ (by Lem. 4.7(ii)). Moreover, the last formula holds because $(\lambda + g, \xi_{i+1}) = -\sum_{j=1}^{i+1}(p_j + 1)$. □

5. The quiver and its relations

For a quick introduction to the theory of quivers and their representations, we refer to [Ki]. More details about quivers with relations can be found in [GR] or in [H2].

Definition 5.1

A *quiver* is an oriented graph $\mathcal{Q}$ with the set $\mathcal{Q}_0$ of points and the set $\mathcal{Q}_1$ of arrows. There are two maps $h, t\colon \mathcal{Q}_1 \to \mathcal{Q}_0$ which indicate, respectively, the head (sink) and the tail (source) of each arrow.

A *path* in $\mathcal{Q}$ is a formal composition of arrows $\beta_m \cdots \beta_1$, where the tail of an arrow is the head of the previous one. Paths can be summed and composed in a natural way, defining the *path algebra* $\mathbf{C}\mathcal{Q}$. It is graded by pairs in $\mathcal{Q}_0$.

A *relation* in $\mathcal{Q}$ is a linear form $\lambda_1 c_1 + \cdots + \lambda_m c_m$, where c_i are paths in $\mathcal{Q}$ with a common tail and a common head and $\lambda_i \in \mathbf{C}$.

A *representation of a quiver* $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is the couple of a set of vector spaces $\{X_i\}_{i \in \mathcal{Q}_0}$ and of a set of linear maps $\{\varphi_\beta\}_{\beta \in \mathcal{Q}_1}$, where $\varphi_\beta : X_i \to X_j$ if β is an arrow from i to j.

Let $\mathcal{R}$ be a homogeneous ideal in the path algebra. A *representation of a quiver* $\mathcal{Q}$ *with relations* $\mathcal{R}$ is a representation of the quiver such that

$$\sum_j \lambda_j \varphi_1^j \cdots \varphi_{m_j}^j = 0$$

for every $\sum_j \lambda_j \beta_1^j \cdots \beta_{m_j}^j \in \mathcal{R}$.

Let $(X_i, \varphi_\beta)_{i \in \mathcal{Q}_0,\, \beta \in \mathcal{Q}_1}$ and $(Y_i, \psi_\beta)_{i \in \mathcal{Q}_0,\, \beta \in \mathcal{Q}_1}$ be two representations of the quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$. A *morphism* f from $(X_i, \varphi_\beta)_{i \in \mathcal{Q}_0,\, \beta \in \mathcal{Q}_1}$ to $(Y_i, \psi_\beta)_{i \in \mathcal{Q}_0,\, \beta \in \mathcal{Q}_1}$ is a set of linear maps $f_i : X_i \to Y_i$, $i \in \mathcal{Q}_0$, such that for every $\beta \in \mathcal{Q}_1$, β arrow from i to j, the following diagram is commutative:

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\varphi_\beta"'] & Y_i \arrow[d, "\psi_\beta"] \\ X_j \arrow[r, "f_j"] & Y_j \end{tikzcd}$$

It is well known (and easy to prove) that the category of representations of $\mathcal{Q}$ with relations $\mathcal{R}$ is equivalent to the category of $(\mathbf{C}\mathcal{Q}/\mathcal{R})$-modules.

A quiver $\mathscr{Q}$ is called *leveled* if there exists a function $s\colon \mathscr{Q}_0 \to \mathbf{Q}$ such that for any arrow $i \longrightarrow j$, we have $s(i) = s(j) + 1$.

Let $X = G/P$ be a Hermitian symmetric variety. In order to describe all G-homogeneous bundles on X, we define a quiver $\mathscr{Q}_X$.

Definition 5.2
Let $\mathscr{Q}_X$ be the following quiver. The points of $\mathscr{Q}_X$ are the irreducible representations of R, which we identify with irreducible G-homogeneous bundles over $X = G/P$ or with the corresponding elements in $\mathscr{H}^\vee$. Let E_λ and E_μ be irreducible representations with maximal weights $\lambda, \mu \in D_1$. There is an arrow in $\mathscr{Q}_X$ from E_λ to E_μ if and only if $\mathrm{Ext}^1(E_\lambda, E_\mu)^G \neq 0$. The ideal of relations in $\mathscr{Q}_X$ are defined in Definition 5.7.

Observe that if $\mathrm{Ext}^1(E_\lambda, E_\mu)^G \neq 0$, then this group is isomorphic to $\mathbf{C}$ by Theorem 4.3.

COROLLARY 5.3
If there is an arrow from E_λ to E_μ, then $\mu(E_\mu) = \mu(E_\lambda) + \mu(\Omega^1)$. In particular, the quiver is leveled (see Def. 5.1) by μ_a of Theorem 4.3(iii) (see [H1], [H2]).

Proof
The corollary is proved by Theorem 4.3. □

COROLLARY 5.4
The arrows (modulo translation) between elements of the quiver can be identified with the weights of Ω^1 (which are negative roots).

Proof
From (1), it follows that $E_\lambda \otimes \Omega^1 \subset \bigoplus E_{\lambda+\xi_i}$; then we conclude by Theorem 4.3. □

We postpone the description of the relations in the quiver after we have defined the representation associated to a bundle.

Definition 5.5
We associate to a G-homogeneous bundle E the following *representation* of $\mathscr{Q}_X$. Let $\mathrm{gr}\, E = \bigoplus_\lambda E_\lambda \otimes V_\lambda$, where $V_\lambda = \mathbf{C}^k$ and k is the number of times E_λ occurs.

To the point λ, we associate the vector space V_λ.

For any $\lambda \in \mathscr{Q}_0$, let us fix a maximal vector $v_\lambda \in E_\lambda$. For any ξ_i root of $\mathscr{N}$, let us fix an eigenvector $n_i \in \mathscr{N}$. We have

$$\mathrm{Ext}^1(\mathrm{gr}\, E, \mathrm{gr}\, E) = \bigoplus\nolimits_{\lambda,\mu} \mathrm{Hom}(V_\lambda, V_\mu) \otimes \mathrm{Ext}^1(E_\lambda, E_\mu). \tag{5}$$

We know that $\mathrm{Ext}^1(E_\lambda, E_\mu)^G = \mathrm{Hom}(E_\lambda \otimes \Omega^1, E_\mu)^G$ is equal to $\mathbf{C}$ or zero, and when it is equal to $\mathbf{C}$, then $\mu - \lambda = \xi_j$ for some j. We fix the generator $m_{\mu\lambda}$ of $\mathrm{Hom}(E_\lambda \otimes \Omega^1, E_\mu)^G$ which takes $v_\lambda \otimes n_j$ to v_μ; indeed, $E_\lambda \otimes \Omega^1$ contains a unique summand of multiplicity one isomorphic to E_μ. This normalization appears already in [BK, p. 48]. Hence in order to define an element of $\mathrm{Hom}(V_\lambda, V_\mu)$, $\forall\lambda, \mu$, it is enough to give an element $[E] \in \mathrm{Ext}^1(\mathrm{gr}\, E, \mathrm{gr}\, E)^G$, and this is the element corresponding to θ of Theorem 3.1(i), according to the isomorphism of Theorem 4.3.

The correspondence $E \mapsto [E]$ is functorial; indeed, a G-equivariant map $E \to F$ induces first a morphism $\mathrm{gr}\, E \mapsto \mathrm{gr}\, F$ and then a morphism of representations of $\mathcal{Q}_X$ given by $[E] \mapsto [F]$.

A direct consequence of Theorem 3.1 is the following.

THEOREM 5.6
Let G/P be a Hermitian symmetric variety.

(i) *For any G-homogeneous bundle E, we have $m([E]) = 0$, where m is the invariant Yoneda morphism recalled in §2,*

$$m\colon \mathrm{Ext}^1(\mathrm{gr}\, E, \mathrm{gr}\, E)^G \to \mathrm{Ext}^2(\mathrm{gr}\, E, \mathrm{gr}\, E)^G.$$

(ii) *Conversely, for any R-module F and any $e \in \mathrm{Ext}^1(F, F)^G$ such that $m(e) = 0$, there exists a G-homogeneous bundle E such that $\mathrm{gr}\, E = F$ and $e = [E]$.*

Remark. It is well known, although we do not need it, that for any bundle F the usual Yoneda morphism $\mathrm{Ext}^1(F, F) \to \mathrm{Ext}^2(F, F)$ is the quadratic part of the Kuranishi morphism. In particular, the invariant Yoneda morphism $\mathrm{Ext}^1(F, F)^G \to \mathrm{Ext}^2(F, F)^G$ is the invariant piece of the quadratic part of the Kuranishi morphism.

Remark. We recall that the functor $E \mapsto \mathrm{gr}\, E$ from P-mod to R-mod is exact. Our description of the quiver and Theorem 5.6 can be thought of roughly as an additional structure on R-mod which allows us to invert the functor gr.

The theorem shows how to define relations in $\mathcal{Q}_X$ in order to get an equivalence of categories. The relations have to reflect the vanishing $m(e) = 0$. We have to note that since in Definition 5.5 we have fixed a normalization, the relations in $\mathcal{Q}_X$ can be changed up to scalar multiplications of the maps involved (see Cor. 8.5).

Definition 5.7
Write $e \in \mathrm{Ext}^1(\mathrm{gr}\, E, \mathrm{gr}\, E)^G$ as

$$e = \sum g_{\mu\lambda} m_{\mu\lambda},$$

where $m_{\mu\lambda} \in \mathrm{Ext}^1(E_\lambda, E_\mu)^G$ were fixed in Definition 5.5 and $g_{\mu\lambda} \in \mathrm{Hom}(V_\lambda, V_\mu)$ come from the isomorphism (5). The equation $m(e) = 0$ becomes

$$\sum_{\nu,\lambda} \Big(\sum_{\mu} (g_{\nu\mu} g_{\mu\lambda})(m_{\nu\mu} \wedge m_{\mu\lambda}) \Big) = 0,$$

where $m_{\nu\mu} \wedge m_{\mu\lambda} \in \mathrm{Ext}^2(E_\lambda, E_\mu)^G$ is the Yoneda product of $m_{\nu\mu}$, $m_{\mu\lambda}$ and $g_{\nu\mu} g_{\mu\lambda} \in \mathrm{Hom}(V_\lambda, V_\nu)$ are the composition maps. For any fixed λ and ν, the equation

$$\sum_{\mu} (g_{\nu\mu} g_{\mu\lambda})(m_{\nu\mu} \wedge m_{\mu\lambda}) = 0 \tag{6}$$

gives a system of at most $\dim \mathrm{Ext}^2(E_\lambda, E_\mu)^G$ quadratic equations in the unknowns $g_{\nu\mu}$ and $g_{\mu\lambda}$.

We define the *relations* in $\mathcal{Q}_X$ as the ideal generated by all these quadratic equations for any pair λ and ν.

THEOREM 5.8

(i) *For any homogeneous bundle E on X Hermitian symmetric variety, $[E]$ satisfies these relations; hence it gives a representation of the quiver $\mathcal{Q}_X$ with relations.*

(ii) *Conversely, given a representation e of the quiver $\mathcal{Q}_X$ with relations, there exists a homogeneous bundle E such that $e = [E]$.*

Proof

By definition, the relations are equivalent to $\theta \wedge \theta = 0$. Hence the statement is equivalent to Theorem 3.1 and to Theorem 5.6 (see also Exam. 5.13). □

The isomorphism class of $[E]$ lives in $\mathrm{Ext}^1(\mathrm{gr}\, E, \mathrm{gr}\, E)^G / \mathrm{Aut}^G(\mathrm{gr}\, E)$. We note that in each case, the isomorphism class of the bundle determines the isomorphism class of the representation of $\mathcal{Q}_X$ (by the functoriality). Hence Theorem 5.6 can be reformulated in the following way.

THEOREM 5.9 (Reformulation of Th. 5.8)

Let $X = G/P$ be a Hermitian symmetric variety. There is an equivalence of categories among

(i) *G-homogeneous bundles over X;*

(ii) *finite-dimensional representations of the quiver (with relations) $\mathcal{Q}_X$ (associating zero to all but a finite number of points of $\mathcal{Q}_X$);*

(iii) *Higgs bundles (F, θ) over X.*

Subquivers and quotient quivers. Since there is no danger of confusion, we denote by $\mathbf{C}\mathcal{Q}_X$ the path algebra of the quiver with relations $\mathcal{Q}_X$, meaning that the algebra has been divided by the ideal of relations. There are two basic constructions for quiver representations that we need.

Definition 5.10
Let $\operatorname{gr} E = \bigoplus V_\lambda \otimes E_\lambda$, so that $V = \bigoplus V_\lambda$ is a $\mathbf{C}\mathcal{Q}_X$-module. For any subspace $V' \subset V$, the submodule generated by V' defines a homogeneous subbundle of E. In the case of $V' = V_{\lambda'}$ for some λ', we call this subbundle the bundle defined by all arrows starting from λ'.

Also, $(V' : \mathbf{C}\mathcal{Q}_X) := \{v \in V \mid fv \in V',\ \forall f \in \mathbf{C}\mathcal{Q}_X\}$ is a submodule, and the quotient $V/(V' \colon \mathbf{C}\mathcal{Q}_X)$ defines a homogeneous quotient of E. Let $\pi_{\lambda'} \colon V \to V_{\lambda'}$ be the projection; in the case of $V' = \operatorname{Ker} \pi_{\lambda'}$, we have $V/(V' \colon \mathbf{C}\mathcal{Q}_X) = V/\{v \in V \mid \pi_{\lambda'} f v = 0,\ \forall f \in \mathbf{C}\mathcal{Q}_X\}$, and we call this quotient bundle the bundle defined by all arrows arriving in λ'.

Example 5.11 (cf. [H1])
Let $\mathbf{P}^3 = \mathbf{P}(V)$. The bundle $E = \bigwedge^2 V$ on $X = \operatorname{Gr}(\mathbf{P}^1, \mathbf{P}^3)$ has graded bundle $\operatorname{gr} E = \mathcal{O}(-1) \oplus \Omega^1(1) \oplus \mathcal{O}(1)$. The corresponding representation of the quiver associates to

$$\begin{tikzcd} & \mathcal{O}(1) \arrow[d] \\ \mathcal{O}(-1) & \Omega^1(1) \arrow[l] \end{tikzcd}$$

the diagram of linear maps

$$\begin{tikzcd} & \mathbf{C} \arrow[d, "\theta_1"] \\ \mathbf{C} & \mathbf{C} \arrow[l, "\theta_2"'] \end{tikzcd}$$

Equivalently, θ splits into the two summands

$$\theta_1 \colon \mathcal{O}(1) \otimes \Omega^1 \longrightarrow \Omega^1(1)$$

and

$$\theta_2 \colon \Omega^1(1) \otimes \Omega^1 \longrightarrow \mathcal{O}(-1)$$

and satisfies $\theta \wedge \theta = 0$ because

$$\operatorname{Ext}^2\big(\mathcal{O}(1), \mathcal{O}(-1)\big)^G = \operatorname{Hom}\big(\mathcal{O}(1) \otimes \Omega^2, \mathcal{O}(-1)\big)^G = 0.$$

In fact, in $\mathcal{Q}_X$ the commutativity of the diagram

$$\begin{array}{ccc} 0 & \longleftarrow & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ \mathcal{O}(-1) & \longleftarrow & \Omega^1(1) \end{array}$$

is not a relation.

THEOREM 5.12
Let X be an irreducible Hermitian symmetric variety. The number of connected components of $\mathcal{Q}_X$ is given by Table 1.

Table 1

Grassmannians	*Odd Quadrics*	*Even Quadrics*	*Spinor Varieties*
$\mathrm{SL}(n+1)/P(\alpha_{k+1})$ $\mathrm{Gr}(\mathbf{P}^k, \mathbf{P}^n)$	$\mathrm{Spin}(2n+1)/P(\alpha_1)$ $Q_{2n-1}, \quad n \geq 2$	$\mathrm{Spin}(2n+2)/P(\alpha_1)$ $Q_{2n}, \quad n \geq 2$	$\mathrm{Spin}(2n+2)/P(\alpha_{n+1})$ $(1/2)\mathrm{Gr}(\mathbf{P}^n, Q_{2n}), \quad n \geq 3$
$n+1$	2	4	4

Lagrangian Grassmannians	*Cayley Plane*	X_{27}
$\mathrm{Sp}(2n)/P(\alpha_n)$ $\mathrm{Grn}(\mathbf{P}^{n-1}, \mathbf{P}^{2n-1}), \quad n \geq 2$	$E_6/P(\alpha_1)$ $\mathbf{OP}^2$	$E_7/P(\alpha_1)$
2	3	2

Proof
The number of connected components is equal to the index of the lattice $\langle \xi_1, \ldots, \xi_m \rangle_{\mathbf{Z}}$ in $\langle \lambda_1, \ldots, \lambda_n \rangle_{\mathbf{Z}}$. It is easy to check in any case that

$$\langle \xi_1, \ldots, \xi_m \rangle_{\mathbf{Z}} = \langle \alpha_1, \ldots, \alpha_n \rangle_{\mathbf{Z}}$$

by the shape of the roots. (The list in the exceptional cases is in [Sn].) Hence the number of connected components is given in any case by the determinant of the corresponding Cartan matrix, and these are well known (see, e.g., [FH, Exer. 21.18]). □

Every homogeneous bundle E on X splits as $E = \bigoplus E^{(i)}$, where the sum is over the connected components of $\mathcal{Q}_X$ and $\mathrm{gr}(E^{(i)})$ contains only irreducible bundles corresponding to points of the connected component labeled by i. We analyze separately each of the irreducible Hermitian symmetric varieties. The decomposition of $E_\lambda \otimes \Omega^1$ in the cases where G is of type A, D, or E appears already in [BK, Prop. 2].

• When $G = \mathrm{SL}(n+1)$, then $X = G/P(\alpha_{k+1})$ is the Grassmannian $\mathrm{Gr}(\mathbf{P}^k, \mathbf{P}^n)$. In this case, all the roots Ω^1_X are $\beta_{ij} = -\sum_{t=i}^{j} \alpha_t$ for $1 \leq i \leq k+1 \leq j \leq n$. If U

and Q are the universal and the quotient bundle, it is well known that $\Omega^1 = U \otimes Q^*$, $\Omega^2 = \left[\mathrm{Sym}^2 U \otimes \bigwedge^2 Q^*\right] \oplus \left[\bigwedge^2 U \otimes \mathrm{Sym}^2 Q^*\right]$.

Here $\mu(\Omega^1) = -(n+1)/((k+1)(n-k))$. Every irreducible bundle on X can be described by $E = \mathscr{S}^\alpha U \otimes \mathscr{S}^\beta Q^*(t)$ for some partitions α, β and for some $t \in \mathbf{Z}$. The $n+1$ connected components are distinguished by the class of $(|\alpha|, |\beta|) \in \mathbf{Z} \times \mathbf{Z}$ modulo the lattice $\langle(-1, 1), (k+1, n-k)\rangle_{\mathbf{Z}}$. If $\gcd(n+1, (k+1)(n-k)) = 1$, the components are distinguished more easily by $(k+1)(n-k)\mu(E) = 0, 1, \ldots, n \pmod{n+1}$.

• When $k = 0$, we get $X = \mathbf{P}^n$. Due to the importance of this case in the applications, we stress our attention on it. We saw in Lemma 4.7 the corresponding roots $\xi_1, \ldots, \xi_n$. We have the simple formulas (of course, some summands can be zero)

$$E_\lambda \otimes \Omega^1 = \bigoplus_{i=1}^{n} E_{\lambda+\xi_i},$$

$$E_\lambda \otimes \Omega^2 = \bigoplus_{1 \le i < j \le n} E_{\lambda+\xi_i+\xi_j}.$$

In Corollary 8.5 we see that the relations in the quiver $\mathscr{Q}_{\mathbf{P}^n}$ can be summed up by saying that for any weight $\lambda \in D_1$ and any $1 \le i < j \le n$, all diagrams

$$\begin{array}{ccc} E_{\lambda+\xi_i} & \longleftarrow & E_\lambda \\ \downarrow & & \downarrow \\ E_{\lambda+\xi_i+\xi_j} & \longleftarrow & E_{\lambda+\xi_j} \end{array}$$

have to be commutative. This fits with [BK]. The quiver $\mathscr{Q}_{\mathbf{P}^n}$ is isomorphic to the half-space of $\mathbf{Z}^n$ defined by the inequalities $x_1 \ge x_2 \ge \cdots \ge x_n$ for $(x_1, \ldots, x_n) \in \mathbf{Z}^n$ with arrows following the standard basis (with the directions reversed). Here $\mu(\Omega^1) = -(n+1)/n$. The $n+1$ connected components are distinguished by $n\mu(E) = 0, 1, \ldots, n \pmod{n+1}$ for an irreducible E.

• In the case of odd-dimensional quadrics $\mathrm{Spin}(2n+1)/P(\alpha_1) = Q_{2n-1} \subset \mathbf{P}^{2n}$, we have that Ω^1 has maximal weight $-\alpha_1 = 2\lambda_2 - 2\lambda_1$ for $n = 2$ and has maximal weight $-\alpha_1 = \lambda_2 - 2\lambda_1$ for $n \ge 3$, while Ω^2 has maximal weight $2\lambda_2 - 3\lambda_1$ for $n = 2$, $2\lambda_3 - 3\lambda_1$ for $n = 3$, and $\lambda_3 - 3\lambda_1$ for $n \ge 4$. Denote again by $\xi_1, \ldots, \xi_m$ ($m = 2n-1$) the roots of Ω^1. We have

$$E_\lambda \otimes \Omega^1 = \bigoplus_{i=1}^{m} E_{\lambda+\xi_i},$$

while $E_\lambda \otimes \Omega^2$ is contained in $\bigoplus_{1 \le i < j \le m} E_{\lambda+\xi_i+\xi_j}$ and can be determined according to λ by the explicit algorithm in [L]. When $\lambda \gg 0$, then we have the equality. Here $\mu(\Omega^1) = -1$ and $\mu(S) = -1/2$ for the spinor bundle. The two connected components are distinguished by $2\mu(E) = 0, 1 \pmod 2$ for an irreducible E.

• In the case of even-dimensional quadrics $\mathrm{Spin}(2n+2)/P(\alpha_1) = Q_{2n} \subset \mathbf{P}^{2n+1}$ (λ_n and λ_{n+1} correspond to the two spinor bundles), we have that Ω^1 has maximal weight $\lambda_2+\lambda_3-2\lambda_1$ for $n=2$ and $\lambda_2-2\lambda_1$ for $n\geq 3$, while Ω^2 splits with two maximal weights $2\lambda_2-3\lambda_1$ and $2\lambda_3-3\lambda_1$ for $n=2$ (this is the Grassmannian of lines in $\mathbf{P}^3$ already considered); it has maximal weight $\lambda_3+\lambda_4-3\lambda_1$ for $n=3$ and $\lambda_3-3\lambda_1$ for $n\geq 4$. Here $\mu(\Omega^1)=-1$ and $\mu(S)=-1/2$ for the two spinor bundles. The knowledge of μ is not enough to distinguish the several components. If $E=E_{\sum p_i\lambda_i}$, the four components are distinguished by $[(p_n, p_{n+1})]\in\mathbf{Z}_2\times\mathbf{Z}_2$.

• In the case of spinor variety $\mathrm{Spin}(2n+2)/P(\alpha_{n+1})$, we have the universal bundle U of rank $n+1$, and it is well known that $\Omega^1=\bigwedge^2 U$ and $\Omega^2=\bigwedge^2\left(\bigwedge^2 U\right)=\mathscr{S}^{2,1,1}U$. Let $m=\binom{n+1}{2}$, and let $\xi_1,\dots,\xi_m$ be the roots of Ω^1. Then it is easy to check that

$$E_\lambda\otimes\Omega^1=\bigoplus_{i=1}^m E_{\lambda+\xi_i},$$

while $E_\lambda\otimes\Omega^2$ is contained in $\bigoplus_{1\leq i<j\leq m}E_{\lambda+\xi_i+\xi_j}$ which can be determined according to λ by the classical Littlewood-Richardson rule (because the semisimple part of $P(\alpha_{n+1})$ is $\mathrm{SL}(n+1)$). When $\lambda\gg 0$, then we have the equality. Here $\mu(\Omega^1)=-4/(n+1)$. If $\gcd(4,n+1)=1$, then the four connected components are distinguished by $(n+1)\mu(E)=0,1,2,3 \pmod 4$. Otherwise, the knowledge of μ is not enough to distinguish the several components. Every irreducible bundle on X can be described by $E=\mathscr{S}^\alpha U\otimes\mathscr{O}(t)$ for some partition α and some integer t. The four connected components are distinguished by the class of $(|\alpha|,t)\in\mathbf{Z}_2\times\mathbf{Z}_2$.

• In the case of maximal Lagrangian Grassmannians $\mathrm{Sp}(2n)/P(\alpha_n)$, we have the universal bundle U of rank n, and it is well known that $\Omega^1=\mathrm{Sym}^2U$ and $\Omega^2=\bigwedge^2(\mathrm{Sym}^2U)=\mathscr{S}^{3,1}U$. Let $m=\binom{n+1}{2}$, and let $\xi_1,\dots,\xi_m$ be the roots of Ω^1. In this case, $E_\lambda\otimes\Omega^1$ is contained in $\bigoplus_{i=1}^m E_{\lambda+\xi_i}$ and the inclusion can be strict. Indeed, this computation can also be done by using the classical Littlewood-Richardson rule. Note that we can write the ξ_i as $\gamma_j+\gamma_k$, where γ_j are the weights of U. A fortiori, $E_\lambda\otimes\Omega^2$ is contained in $\bigoplus_{1\leq i<j\leq n}E_{\lambda+\xi_i+\xi_j}$, and it can be determined according to λ by the classical Littlewood-Richardson rule. Here $\mu(\Omega^1)=-2/n$. The two connected components are distinguished by $n\mu(E)=0,1 \pmod 2$ for an irreducible E.

• In the case of the Cayley plane $E_6/P(\alpha_1)=\mathbf{OP}^2$ (see [LM], [IM]), the semisimple part of $P(\alpha_1)$ is $\mathrm{Spin}(10)$. E_{λ_2} is a twist of one of the two spinor bundles, and $\Omega^1=E_{\lambda_2}(-2)$.

Hence $\Omega^2=E_{\lambda_3}(-3)$ is irreducible. Here $\mu(\Omega^1)=-3/4$. The three connected components are distinguished by $4\mu(E)=0,1,2 \pmod 3$ for an irreducible E. The Cayley plane has an intrinsic interest because it is a Severi variety.

• Also, the 27-dimensional case $E_7/P(\alpha_1)$ has $\Omega^1=E_{\lambda_2}(-2)$ and $\Omega^2=E_{\lambda_3}(-3)$, both irreducible. Here $\mu(\Omega^1)=-2/3$. The two connected components are distinguished by $3\mu(E)=0,1 \pmod 2$ for an irreducible E.

The case of the projective plane $\mathbf{P}^2$ allows an explicit description of some interest. Let $(x, y) \in \mathcal{N} \simeq \mathbf{C}^2$. Consider the linear maps given by matrices with coefficients in $\bigwedge^* \mathcal{N}$,

$$C_k = \frac{1}{k+1} \cdot \begin{bmatrix} x & y & & \\ & \ddots & \ddots & \\ & & x & y \end{bmatrix} \quad \text{of size } k \times (k+1),$$

$$B_k = \frac{1}{k} \cdot \begin{bmatrix} -ky & & & \\ x & -(k-1)y & & \\ & \ddots & \ddots & \\ & & (k-1)x & -y \\ & & & kx \end{bmatrix} \quad \text{of size } (k+1) \times k.$$

Now it is easy to check that

$$C_k \wedge C_{k+1} = 0, \qquad B_{k+1} \wedge B_k = 0, \qquad C_{k+1} \wedge B_{k+1} + B_k \wedge C_k = 0. \tag{7}$$

The interpretation in terms of representations is the following. The parabolic subgroup $P(\alpha_1) \subset \mathrm{SL}(3)$ has the form

$$P(\alpha_1) = \left\{ \begin{bmatrix} e & x & y \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix} \middle| \, e \det A = 1 \right\}.$$

The irreducible representation of $P(\alpha_1)$ corresponding to $\mathrm{Sym}^p Q(t)$ is defined by $\mathrm{Sym}^p A e^{-t}$. Consider the derivative $\mathcal{P} = \mathrm{Lie}\, P(\alpha_1) \to \mathrm{gl}(\mathrm{Sym}^p \mathbf{C}^2)$, and call it (with a slight abuse of notation) $\mathrm{Sym}^p A - teI$. The extension $w \in \mathrm{Ext}^1(\mathrm{Sym}^k Q, \mathrm{Sym}^{k-1} Q(-1))^G = \mathbf{C}$ defines a bundle with representation

$$\begin{bmatrix} \mathrm{Sym}^{k-1} A + eI & wC_k \\ 0 & \mathrm{Sym}^k A \end{bmatrix}, \tag{8}$$

where w is a scalar multiple and $w = 0$ if and only if the extension splits.

Analogously, the extension $w \in \mathrm{Ext}^1(\mathrm{Sym}^k Q(2), \mathrm{Sym}^{k+1} Q)^G = \mathbf{C}$ defines a bundle with representation

$$\begin{bmatrix} \mathrm{Sym}^{k+1} A & w B_k \\ 0 & \mathrm{Sym}^k A - 2eI \end{bmatrix}, \tag{9}$$

where w is a scalar multiple, which is zero if and only if the extension splits. By Theorem 3.1, several extensions as in (8) and (9) fit together to give a representation ρ of $\mathscr{P}$ if and only if $\rho_{|N} \wedge \rho_{|N} = 0$ (see Exam. 5.13). We note that (7) are equivalent due to the fact that the only relations in $\mathscr{Q}_{\mathbf{P}^2}$ are the commutative ones (see Cor. 8.5) in all the square diagrams and the relation $a_2 b_1 = 0$ in the diagrams

$$\begin{tikzcd} & \mathscr{O}(t) \arrow[d, "b_1"] \\ \mathscr{O}(t-3) & Q(t-2) \arrow[l, "a_2"'] \end{tikzcd}$$

for any $t \in \mathbf{Z}$. These last relations can be seen as the commutativity in the diagrams

$$\begin{tikzcd} 0 \arrow[d] & \mathscr{O}(t) \arrow[l] \arrow[d, "b_1"] \\ \mathscr{O}(t-3) & Q(t-2) \arrow[l, "a_2"'] \end{tikzcd}$$

Example 5.13
We describe explicitly the homogeneous bundle on $\mathbf{P}^2 = \mathbf{P}(V)$ corresponding to the representation that associates to

$$\begin{tikzcd} \mathscr{O} \arrow[d] & Q(1) \arrow[l] \arrow[d] \\ Q(-2) & \mathrm{Sym}^2 Q(-1) \arrow[l] \end{tikzcd}$$

the diagram of linear maps

$$\begin{tikzcd} \mathbf{C}^a \arrow[d, "\beta_1"] & \mathbf{C}^b \arrow[l, "\gamma_1"'] \arrow[d, "\beta_2"] \\ \mathbf{C}^c & \mathbf{C}^d \arrow[l, "\gamma_2"'] \end{tikzcd}$$

where a, b, c, d are positive integers. We get

$$\rho \begin{bmatrix} e & x & y \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} A^c + 2eI & \gamma_2 \otimes C_2 & \beta_1 \otimes B_1 & 0 \\ 0 & (\mathrm{Sym}^2 A)^d + eI & 0 & \beta_2 \otimes B_2 \\ 0 & 0 & 0 & \gamma_1 \otimes C_1 \\ 0 & 0 & 0 & A^b - eI \end{bmatrix},$$

and this is a P-module if and only if (by Th. 3.1)

$$\begin{bmatrix} 0 & \gamma_2\otimes C_2 & \beta_1\otimes B_1 & 0 \\ 0 & 0 & 0 & \beta_2\otimes B_2 \\ 0 & 0 & 0 & \gamma_1\otimes C_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & \gamma_2\otimes C_2 & \beta_1\otimes B_1 & 0 \\ 0 & 0 & 0 & \beta_2\otimes B_2 \\ 0 & 0 & 0 & \gamma_1\otimes C_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0,$$

which is equivalent by (7) to

$$\gamma_2\cdot\beta_2-\beta_1\cdot\gamma_1=0,$$

confirming the commutative relations. In the special case of $a=b=c=d=1$ and all the maps given by the identity, this bundle is $\operatorname{ad} V$.

The isomorphism classes of representations are equivalent to the orbits in $m^{-1}(0)$ with respect to the $\operatorname{Aut}_G(\operatorname{gr} E)$-action.

6. Computation of cohomology

In all of this section, X is a Hermitian symmetric variety of ADE-type.

We now want to describe how to compute the cohomology of a homogeneous bundle E on X from the representation of the quiver.

We need the following easy lemma.

LEMMA 6.1 ([CE, Chap. XV, Lem. 1.1])
Let the following diagram be commutative:

$$\begin{tikzcd} & & C \arrow[d, "\phi"] \arrow[drr, "\psi"] & & \\ A' \arrow[rr, "\phi'"'] \arrow[urr] & & A \arrow[rr, "\eta"'] & & A'' \end{tikzcd}$$

and let the row be exact. Then

$$\operatorname{Im}\phi/\operatorname{Im}\phi' \simeq \operatorname{Im}\psi.$$

Let

$$0=E_0\subset E_1\subset E_2\subset\cdots\subset E_r=E$$

be a filtration of a vector bundle (not necessarily homogeneous).

Now, let

$$Z_j^p := \operatorname{Ker}\bigl(H^j(E_{p+1}/E_p) \xrightarrow{\partial} H^{j+1}(E_p)\bigr),$$
$$B_j^p := \operatorname{Im}\bigl(H^{j-1}(E/E_{p+1}) \xrightarrow{\partial} H^j(E_{p+1}/E_p)\bigr),$$

where the maps are the boundary maps of the two obvious exact sequences.

The next theorem follows from the discussion at the beginning of [CE, Chap. XV]. For the convenience of the reader, we sketch the proof.

THEOREM 6.2
$B_j^p \subset Z_j^p$, and

$$H^j(E) \simeq \bigoplus\nolimits_{p=0}^{r-1} Z_j^p / B_j^p .$$

Proof
We have the commutative diagram

$$\begin{tikzcd}
 & H^j(E_{p+1}) \arrow[d, "\phi"] \arrow[dr, "\psi"] & \\
H^{j-1}(E/E_{p+1}) \arrow[ur] \arrow[r, "\phi'"] & H^j(E_{p+1}/E_p) \arrow[r, "\eta"] & H^j(E/E_p)
\end{tikzcd}$$

hence $B_j^p \subset Z_j^p$, and from Lemma 6.1 we get

$$\operatorname{Im}\bigl(H^j(E_{p+1}) \xrightarrow{\psi} H^j(E/E_p)\bigr) \simeq \operatorname{Im}\phi / \operatorname{Im}\phi' = \operatorname{Im}\phi / \operatorname{Ker}\eta = Z_j^p / B_j^p . \tag{10}$$

Consider also the diagram

$$\begin{tikzcd}
 & H^j(E_{p+1}) \arrow[d, "\phi_p"] \arrow[dr, "\psi"] & \\
H^j(E_p) \arrow[ur] \arrow[r, "\phi_{p-1}"] & H^j(E) \arrow[r, "\eta_p"] & H^j(E/E_p)
\end{tikzcd}$$

We get again from Lemma 6.1,

$$\operatorname{Im}\bigl(H^j(E_{p+1}) \xrightarrow{\psi} H^j(E/E_p)\bigr) \simeq \operatorname{Im}(\phi_p)/\operatorname{Im}(\phi_{p-1}), \tag{11}$$

and since we have the graduation

$$H^j(E) \simeq \bigoplus\nolimits_p \operatorname{Im}(\phi_p)/\operatorname{Im}(\phi_{p-1}) \overset{(10)(11)}{=} \bigoplus\nolimits_p Z_j^p / B_j^p ,$$

we get the result. □

We return now to the case of homogeneous bundles.

We need a short digression about homogeneous bundles whose quiver representation has support on an A_n-type quiver; that is, $\operatorname{gr} E = \bigoplus V_\lambda \otimes E_\lambda$ and V_λ is zero outside a path connecting the vertices $\{\lambda + p\xi_j \mid 0 \le p \le k\}$.

The following theorem is well known since the former work on quivers by P. Gabriel (see [GR]).

THEOREM 6.3
Every representation of the A_m-quiver is the direct sum of irreducible representations with dimension vector

$$(0, 0, \dots, 0, 1, 1, \dots, 1, 0, \dots, 0),$$

where the nontrivial linear maps are isomorphisms.

The reader can deduce the previous theorem as a consequence of Theorem 5.9 for $X = \mathbf{P}^1$ and the Segre-Grothendieck theorem, which says that every bundle on $\mathbf{P}^1$ splits as the sum of line bundles.

PROPOSITION 6.4
Let E_λ and E_μ be in two adjacent Bott chambers with $H^i(E_\lambda) \simeq H^{i+1}(E_\mu) \simeq W$; then $\mu - \lambda = k\xi_j$ for some integer k and some root ξ_j of Ω^1. We have

$$\dim \operatorname{Hom}(E_\lambda \otimes \operatorname{Sym}^k \Omega^1, E_\mu)^G = 1.$$

Proof
By (1), it is enough to show that there are no other weights among $\left\{a_1\xi_{i_1} + \cdots + a_h\xi_{i_h} \,\middle|\, \sum a_i = k\right\}$ which are equal to $\mu - \lambda$. With the ADE-assumption, ξ_j is a vertex of the convex polytope containing the weights of Ω^1 because all the roots have the same length. Hence $k\xi_j$ is a vertex of the convex polytope containing the weights of $\operatorname{Sym}^k \Omega^1$. □

PROPOSITION 6.5
Let ξ_j be a weight of Ω^1. We have

$$\operatorname{Ext}^2(E_\lambda, E_{\lambda+2\xi_j})^G = \operatorname{Hom}\big((E_\lambda \otimes \Omega^2, E_{\lambda+2\xi_j})^G\big) = 0.$$

Proof
Since ξ_j is a vertex of the convex polytope containing the weights of Ω^1, there are no distinct weights ξ_p, ξ_q of Ω^1 such that $\xi_j = (1/2)(\xi_p + \xi_q)$. Then apply Theorem 4.3. □

Remark. Without the ADE-assumption, Propositions 6.4 and 6.5 are false. For example, if $X = Q_3$, the weights of Ω^1 are α_1, $\alpha_1 + \alpha_2$, $\alpha_1 + 2\alpha_2$. The weight $\alpha_1 + \alpha_2$ is shorter; indeed, $2(\alpha_1 + \alpha_2)$ coincides with the sum of the two vertices $(\alpha_1) + (\alpha_1 + 2\alpha_2)$. In particular, $\dim \mathrm{Hom}(\mathrm{Sym}^2\Omega^1, E_{2(\alpha_1+\alpha_2)})^G = 2$, and $\dim \mathrm{Ext}^2(\mathscr{O}, E_{2(\alpha_1+\alpha_2)})^G = 1$. Hence there is no indecomposable homogeneous bundle E with support A_2 such that $\mathrm{gr}\, E = \bigoplus_{i=0}^{2} E_{i(\alpha_1+\alpha_2)}$.

With the assumption of Propositions 6.4 and 6.5, note that the distinguished elements in $\mathrm{Hom}(E_{\lambda+p\xi_j} \otimes \Omega^1, E_{\lambda+(p+1)\xi_j})^G$ which were chosen in Definition 5.5 give a distinguished element in $\mathrm{Hom}(E_\lambda \otimes \mathrm{Sym}^k\Omega^1, E_\mu)^G$, which is one dimensional by Proposition 6.4. These elements allow us to define extensions of the form

$$0 \longrightarrow E_{\lambda+(p+1)\xi_j} \longrightarrow Z_p \longrightarrow E_{\lambda+p\xi_j} \longrightarrow 0$$

which fit together (by Th. 5.6(ii) since the corresponding Ext^2 vanish by Prop. 6.5), giving a bundle P' with $\mathrm{gr}\, P' = \bigoplus_{p=0}^{k-1} E_{\lambda+p\xi_j}$ and two exact sequences (this argument is similar to the one in [De])

$$0 \longrightarrow Z' \longrightarrow P' \longrightarrow E_\lambda \longrightarrow 0, \tag{12}$$

$$0 \longrightarrow E_\mu \longrightarrow Z' \longrightarrow Z'/E_\mu \longrightarrow 0. \tag{13}$$

THEOREM 6.6
We have

$$H^j(P') = 0, \quad \forall j.$$

We need a short preparation in order to prove Theorem 6.6. Let λ' (resp., μ') be the vertex of the Bott chamber containing λ (resp., μ). Let A be the unique indecomposable bundle in the extension

$$0 \longrightarrow E_{\mu'} \longrightarrow A \longrightarrow E_{\lambda'} \longrightarrow 0.$$

PROPOSITION 6.7
We have

$$H^i(A) = 0, \quad \forall i.$$

Proof
The boundary map $H^i(E_{\lambda'}) \longrightarrow H^{i+1}(E_{\mu'})$ can be seen as the cup product of the class of the Schubert cell corresponding to $E_{\lambda'}$ as subbundle of Ω^i (by Hodge theory) with the hyperplane class in $H^1(\Omega^1)$, and it is nonzero by [Hi, Chap. V, Cor. 3.2]. $\square$

PROPOSITION 6.8
We have that $\mathrm{gr}(E_{\mu'} \otimes W)$ *contains only* E_μ *as a direct summand with* $H^* \simeq W$.

Proof
Let E_α be the irreducible bundle such that $H^0(E_\alpha) = W$. The weights of W as G-module lie in a convex polytope P_W whose vertices are the reflections of α through the hyperplanes H_ϕ (for any root ϕ of G) which separate the Weyl chambers of G (see [FH, p. 204]). The weights of $E_{\mu'} \otimes W$ lie inside $P_W + \mu'$.

Let $\tilde{P}_W$ be the convex polytope whose vertices are the reflections of $g + \alpha$ through the hyperplanes H_ϕ. Note that P_W is strictly contained in $\tilde{P}_W$ and that there is a natural bijective correspondence f between the vertices of P_W and the vertices of $\tilde{P}_W$ such that if $\beta\gamma$ is an edge of P_W of length $d\sqrt{2}$, then $f(\beta)f(\gamma)$ is a parallel edge of $\tilde{P}_W$ of length $(d+1)\sqrt{2}$. Precisely, the corresponding vertices $\tilde{\beta}$ and β, respectively, of $\tilde{P}_W$ and P_W differ by $w_\beta(g)$ for a composition of reflections w_β defined by $\beta = w_\beta(\alpha)$. The point of P_W of least distance from $\tilde{\beta}$ is β.

We have that $\mu' = w(g) - g$ for some w. Let $\overline{\mu} = w(\alpha)$; then $w = w_{\overline{\mu}}$. Then $\mu = w_{\overline{\mu}}(\alpha + g) - g = \overline{\mu} + \mu'$ is a vertex of $P_W + \mu'$; hence it is a maximal weight of $E_{\mu'} \otimes W$.

By the Bott theorem, all the weights ν such that $H^i(E_\nu) = W$ for some i are obtained from α after reflecting through the hyperplanes that separate the Bott chambers of G. All these weights are some of the vertices of $\tilde{P}_W - g$.

It is enough to show that the vertices of $\tilde{P}_W - g$ meet $P_W + \mu'$ only in the point μ.

The distance of $\tilde{\beta} - g$ from $P_W - g + (\mu' + g) = P_W - g + w_{\overline{\mu}}(g)$ vanishes only when $\tilde{\beta} - w_{\overline{\mu}}(g) \in P_W$, and this happens if and only if $w_\beta(g) = w_{\overline{\mu}}(g)$ (since the point of least distance between $\tilde{\beta}$ and P_W is $\tilde{\beta} - w_\beta(g)$); thus $\beta = \overline{\mu}$. Then $\tilde{\beta} - g = \beta + w_\beta(g) - g = \beta + w_{\overline{\mu}}(g) - g = \overline{\mu} + \mu' = \mu$. $\square$

Proof of Theorem 6.6
Let K be the submodule in $A \otimes W$ generated by the direct summands isomorphic to E_λ. (It can be shown that there is only one, but we do not need this fact.) We have the exact sequence

$$0 \longrightarrow K \longrightarrow A \otimes W \longrightarrow Q \longrightarrow 0.$$

By Proposition 6.8, we have that $H^j(K)^W$ and $H^j(Q)^W$ are nonzero at most for $j = i$ or $j = i + 1$.

We claim that $\mathrm{gr}\, K$ contains all the direct summands isomorphic to E_μ; otherwise, $E_\mu \subset \mathrm{gr}\, Q$, and we would have $H^{i+1}(Q)^W \neq 0$. Hence by Proposition 6.7, $H^{i+2}(K)^W \neq 0$, which is a contradiction. Hence we get $H^j(Q)^W = 0$, $\forall j$, and it follows that

$$H^j(K)^W = 0, \quad \forall j.$$

At last, let S' be the quotient of K obtained by restricting the quiver representation to the path joining the vertices corresponding to E_λ and E_μ.

We have

$$0 \longrightarrow K' \longrightarrow K \longrightarrow S' \longrightarrow 0.$$

Now $H^j(\mathrm{gr}\, K')^W = 0,\ \forall j$; hence $H^j(K')^W = 0,\ \forall j$, and it follows that $H^j(S') = 0,\ \forall j$. Decompose S' into its irreducible components (see Th. 6.3); we get that S' is isomorphic to the direct sum of several copies of P' by the definition of K. □

From the sequence (12) and Theorem 6.6 we have the isomorphism

$$H^j(E_\lambda) \xrightarrow{\partial} H^{j+1}(Z')^W,$$

and from (13) we have an isomorphism

$$H^{j+1}(E_\mu) \xrightarrow{\simeq} H^{j+1}(Z');$$

hence we get a distinguished isomorphism

$$j_{\mu\lambda} \colon H^j(E_\lambda) \longrightarrow H^{j+1}(E_\mu). \tag{14}$$

LEMMA 6.9

Let E_λ and E_μ be in two adjacent Bott chambers with $H^{j-1}(E_\lambda) \simeq H^j(E_\mu) \simeq W$. Denote by P the homogeneous bundle corresponding to the A_n-type, starting from E_λ and arriving in E_μ, with the same representation quiver maps as for E. (It exists by Th. 5.6(ii) again, by the same argument as before.) Then the boundary map

$$W \otimes V_\lambda = H^{j-1}(P/V_\mu E_\mu)^W \xrightarrow{\partial} H^j(V_\mu E_\mu) = W \otimes V_\mu$$

is the tensor product of the distinguished isomorphism in (14) and the composition of the maps of the quiver representation.

Proof

We first prove the theorem for P irreducible. We may assume that $\dim V_{\lambda+p\xi_j} = 1$ for $0 \leq p \leq k$ and $\lambda + k\xi_j = \mu$; moreover, P defines nonzero elements in the one-dimensional spaces

$$\begin{aligned}&\mathrm{Hom}(V_{\lambda+p\xi_j}\otimes E_{\lambda+p\xi_j}\otimes\Omega^1,\, V_{\lambda+(p+1)\xi_j}\otimes E_{\lambda+(p+1)\xi_j})^G\\ &\quad=\mathrm{Hom}(V_{\lambda+p\xi_j},\, V_{\lambda+(p+1)\xi_j})\otimes\mathrm{Hom}(E_{\lambda+p\xi_j}\otimes\Omega^1,\, E_{\lambda+(p+1)\xi_j})^G.\end{aligned}$$

There is a natural isomorphism between

$$\bigotimes_{i=0}^{k-1}\mathrm{Hom}(V_{\lambda+p\xi_j}\otimes E_{\lambda+p\xi_j}\otimes\Omega^1,\, V_{\lambda+(p+1)\xi_j}\otimes E_{\lambda+(p+1)\xi_j})^G$$

and

$$\mathrm{Hom}(V_\lambda\otimes E_\lambda\otimes\mathrm{Sym}^k\Omega^1,\, V_\mu\otimes E_\mu)^G=\mathrm{Hom}(V_\lambda,\, V_\mu)\otimes\mathrm{Hom}(E_\lambda\otimes\mathrm{Sym}^k\Omega^1,\, E_\mu)^G,$$

where in $\mathrm{Hom}(V_\lambda, V_\mu)$ we perform the composition of the quiver representation maps.

It is clear that the element obtained in $\mathrm{Hom}(V_\lambda, V_\mu)\otimes\mathrm{Hom}(E_\lambda\otimes\mathrm{Sym}^k\Omega^1, E_\mu)^G$ is sufficient to reconstruct P.

Now we consider the two exact sequences

$$0\longrightarrow Z\longrightarrow P\longrightarrow V_\lambda\otimes E_\lambda\longrightarrow 0,$$

$$0\longrightarrow V_\mu\otimes E_\mu\longrightarrow Z\longrightarrow P'\longrightarrow 0.$$

From the first sequence we have

$$H^j(E_\lambda\otimes V_\lambda)\xrightarrow{\partial} H^{j+1}(Z)^W,$$

and from the second one an isomorphism (by Th. 6.6)

$$H^{j+1}(E_\mu\otimes V_\mu)\xrightarrow{\simeq} H^{j+1}(Z);$$

hence we get a map

$$c_{\mu\lambda}\colon H^j(E_\lambda\otimes V_\lambda)\longrightarrow H^{j+1}(E_\mu\otimes V_\mu) \tag{15}$$

which by construction is the tensor product of the distinguished isomorphism $j_{\mu\lambda}$ constructed in (14) and the composition of the maps of the quiver representation, as we wanted.

In general, we have $P=\bigoplus P_i$, where P_i are irreducible by Theorem 6.3. Moreover, we have $V_\lambda=\bigoplus V_\lambda^i$, $V_\mu=\bigoplus V_\mu^i$, where every V_λ^i and V_μ^i has dimension one or zero, and for each i the morphism $W\otimes V_\lambda^i=H^{j-1}(P_i/V_\mu^iE_\mu)^W\xrightarrow{\partial} H^j(V_\mu^iE_\mu)=W\otimes V_\mu^i$ coincides again with the tensor product of the distinguished isomorphism $j_{\mu\lambda}$ constructed in (14) and the composition of the maps of the quiver representation. □

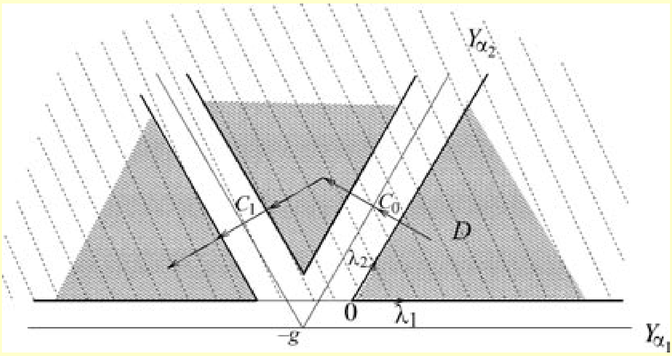

Figure 4

We now construct maps $H^j(\mathrm{gr}\, E) \xrightarrow{c_j} H^{j+1}(\mathrm{gr}\, E)$ by patching together the maps $c_{\mu\lambda}$ already constructed in (15); that is, we have the following.

Definition 6.10
We have

$$c_j := \sum c_{\mu\lambda},$$

where the sum is over all pairs λ, μ in two adjacent Bott chambers such that $H^j(E_\lambda) \simeq H^{j+1}(E_\mu)$.

Although separately the isomorphism $j_{\mu\lambda}$ in (14) and the composition of the quiver representation maps depend on the choices made in Definition 5.5, it is easy to check that their tensor product does not depend on these choices. (The numbers for which one has to multiply cancel together.) Moreover, the construction in Definition 6.10 is functorial; that is, given a morphism $E \longrightarrow F$, we get a map $H^*(\mathrm{gr}\, E) \longrightarrow H^*(\mathrm{gr}\, F)$. We now see that $H^*(\mathrm{gr}\, E)$ is a complex and that it gives a way to compute the cohomology.

In the case of $\mathbf{P}^n$, this construction can be made more explicit. We have maps given by $g_{\lambda,i} : W_\lambda \to W_{\lambda+\alpha_1+\cdots+\alpha_{i+1}} = W_{\lambda'}$. Let $\lambda = \sum_{i=1}^n p_i\lambda_i$. Let $p_i(\lambda) = -\sum_{j=1}^{i+1}(p_j + 1)$. Composing the maps $W_{\lambda+j(\alpha_1+\cdots+\alpha_{i+1})} \to W_{\lambda+(j+1)(\alpha_1+\cdots+\alpha_{i+1})}$ for i fixed and $j = 0, \ldots, p_i - 1$, we get $W_\lambda \to W_{\lambda+p_i(\lambda)(\alpha_1+\cdots+\alpha_{i+1})}$ and $W_\lambda \xrightarrow{g'_{\lambda,i}} W_{\lambda'}$, where $H^i(E_\lambda) = H^{i+1}(E_{\lambda'})$ and $g'_{\lambda,i} = \prod_{j=1}^{p_i(\lambda)} g_{\lambda+(j-1)(\alpha_1+\cdots+\alpha_{i+1}),i}$. The corresponding maps c_0, c_1 in the case of $\mathbf{P}^2$ are shown in Figure 4.

Remark. We warn the reader that the use of the distinguished isomorphism (14) is not a formal and superfluous addition, but it determines the correct signs that are necessary in concrete computations. For example, assume that we have λ, μ, ν in three consecutive adjacent Bott chambers such that $H^j(E_\lambda) \simeq H^{j+1}(E_\mu) \simeq H^{j+2}(E_\nu)$ and assume that we have λ, μ', ν in the same situation (at most two μ's exist between λ and ν); it may be shown as an application of the well-known relation in [CE, Chap. III, Prop. 4.1] that we have the anticommutative relation

$$j_{\nu\mu} j_{\mu\lambda} = - j_{\nu\mu'} j_{\mu'\lambda}.$$

The next theorem implies in this case that

$$c_{\nu\mu} c_{\mu\lambda} = - c_{\nu\mu'} c_{\mu'\lambda};$$

hence it follows by the construction of $c_{\mu\lambda}$ that the corresponding composition of the quiver representation maps is commutative for the square

$$\begin{array}{ccc} \lambda & \longrightarrow & \mu \\ \big\downarrow & & \big\downarrow \\ \mu' & \longrightarrow & \nu \end{array}$$

taken from the Hasse quiver. In the last section about Olver maps, we give more information in the case of Grassmannians.

THEOREM 6.11
$(H^*(\mathrm{gr}\, E), c_*)$ is a complex, and its cohomology is given by

$$\frac{\operatorname{Ker} c_i}{\operatorname{Im} c_{i-1}} = H^i(E).$$

Proof
Let W be any irreducible G-module, and let $n = \dim X$. It is enough to compute that

$$H^j(E)^W = \frac{\operatorname{Ker}(H^i(\mathrm{gr}\, E)^W \xrightarrow{c_i} H^{i+1}(\mathrm{gr}\, E)^W)}{\operatorname{Im}(H^{i-1}(\mathrm{gr}\, E)^W \xrightarrow{c_{i-1}} H^i(\mathrm{gr}\, E)^W)}.$$

We consider the filtration of E defined in the following way.

E_1 is defined by taking all arrows starting from any $F \in \mathrm{gr}\, E$ such that $H^n(F)^W \neq 0$ (see Def. 5.10).

E_2 is defined by taking all arrows starting from any $F \in \mathrm{gr}\, E$ such that

$$H^n(F)^W \oplus H^{n-1}(F)^W \neq 0.$$

In general, E_i is defined by taking all arrows starting from any $F \in \mathrm{gr}\, E$ such that

$$\bigoplus_{j=0}^{i-1} H^{n-j}(F)^W \neq 0.$$

We get

$$H^j(\mathrm{gr}\, E_{i+1}/E_i)^W = \begin{cases} H^{n-i}(\mathrm{gr}\, E)^W & \text{if } j = n-i, \\ 0 & \text{if } j \neq n-i; \end{cases}$$

hence by the spectral sequence,

$$H^j(E_{i+1}/E_i)^W = \begin{cases} H^{n-i}(\mathrm{gr}\, E)^W & \text{if } j = n-i, \\ 0 & \text{if } j \neq n-i. \end{cases}$$

We have the commutative diagram

$$\begin{tikzcd}
H^{i-1}(E_{n-i+2}/E_{n-i+1})^W \arrow[d, equal] & & \\
H^{i-1}(\mathrm{gr}\, E/E_{n-i+1})^W \arrow[d, "f"] \arrow[dr, "\partial"] & & \\
H^{i-1}(E/E_{n-i+1})^W \arrow[r, "\partial"] & H^i(E_{n-i+1}/E_{n-i})^W \arrow[r, "\partial"] \arrow[dr, "\partial"] & H^{i+1}(E_{n-i})^W \arrow[d, "g"] \\
& & H^{i+1}(\mathrm{gr}\, E_{n-i})^W \arrow[d, equal] \\
& & H^{i+1}(E_{n-i}/E_{n-i-1})^W
\end{tikzcd}$$

where f is the projection given by the spectral sequence $(H^i(\mathrm{gr}\, E/E_{n-i+1})^W = 0)$ and g is injective (because $H^i(\mathrm{gr}\, E_{n-i})^W = 0$). Moreover, we note that the central term is

$$H^i(E_{n-i+1}/E_{n-i})^W = H^i(\mathrm{gr}\, E)^W.$$

It follows from this diagram and Theorem 6.2 that

$$H^i(E)^W = Z_i^{n-i}/B_i^{n-i} = \frac{\mathrm{Ker}(H^i(E_{n-i+1}/E_{n-i})^W \xrightarrow{\partial} H^{i+1}(E_{n-i}/E_{n-i-1})^W)}{\mathrm{Im}(H^{i-1}(E_{n-i+2}/E_{n-i+1})^W \xrightarrow{\partial} H^i(E_{n-i+1}/E_{n-i})^W)}.$$

Now it is enough to show that the boundary map

$$H^{i-1}(E_{n-i+2}/E_{n-i+1})^W \xrightarrow{\partial} H^i(E_{n-i+1}/E_{n-i})^W$$

induced by the exact sequence

$$0 \longrightarrow E_{n-i+1}/E_{n-i} \longrightarrow E_{n-i+2}/E_{n-i} \longrightarrow E_{n-i+2}/E_{n-i+1} \longrightarrow 0$$

is the composition of the quiver representation maps tensored with $j_{\mu\lambda}$ in (14).

Lemma 6.9 tells us that this is true in the particular case of the quiver representations with support A_m, and we reduce to that case. Pick $V_\lambda \otimes E_\lambda \subset \operatorname{gr} E_{n-i+2}/E_{n-i+1}$ and $V_\mu \otimes \operatorname{gr} E_\mu \subset E_{n-i+1}/E_{n-i}$ such that $W \simeq H^{i-1}(E_\lambda) \simeq H^i(E_\mu)$.

We have to show that the composition

$$H^{i-1}(V_\lambda \otimes E_\lambda) \xrightarrow{l} H^{i-1}(E_{n-i+2}/E_{n-i+1})^W \xrightarrow{\partial} H^i(E_{n-i+1}/E_{n-i})^W \longrightarrow H^i(V_\mu \otimes E_\mu)$$

is obtained by composing the maps appearing in the quiver representation from V_λ to V_μ.

Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \cap (E_{n-i+1}/E_{n-i}) \arrow[r] \arrow[d] & E_{n-i+1}/E_{n-i} \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & E_{n-i+2}/E_{n-i} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & & E_{n-i+2}/E_{n-i+1} \arrow[r] \arrow[d] & Q'' \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where Q is the quotient of E_{n-i+2}/E_{n-i} obtained by taking all arrows arriving in E_μ (see Def. 5.10) and the other bundles are defined from the diagram itself.

This diagram induces the diagram

$$\begin{tikzcd}
 & H^i(V_\lambda \otimes E_\lambda) \arrow[d] \arrow[dr] & \\
 & H^i(E_{n-i+2}/E_{n-i+1})^W \arrow[r] \arrow[d, "\partial"] & H^i(Q'')^W \arrow[d, "\partial"] \\
H^{i+1}\big(K \cap (E_{n-i+1}/E_{n-i})\big)^W \arrow[r, "f"] & H^{i+1}(E_{n-i+1}/E_{n-i})^W \arrow[r] \arrow[d, "h"] & H^{i+1}(Q')^W \\
 & H^{i+1}(V_\mu \otimes E_\mu) &
\end{tikzcd}$$

The composition hf is zero because E_μ is not a vertex of K; then the map h lifts to

$$\begin{tikzcd}
H^i(V_\lambda \otimes E_\lambda) \arrow[d] \arrow[dr, "r"] & & & \\
H^i(E_{n-i+2}/E_{n-i+1})^W \arrow[r] \arrow[d, "\partial"] & H^i(Q'')^W \arrow[d, "\partial"] & = & H^i(V_\lambda \otimes E_\lambda) \\
H^{i+1}(E_{n-i+1}/E_{n-i})^W \arrow[r] \arrow[d, "h"] & H^{i+1}(Q')^W \arrow[dl, "g"] & & \\
H^{i+1}(V_\mu \otimes E_\mu) & & &
\end{tikzcd}$$

The last step consists of constructing the subbundle P of Q taking all arrows starting from λ (see Def. 5.10); hence P is as in the assumptions of Lemma 6.9. We get the commutative diagram

$$\begin{tikzcd}
H^i(P/P \cap Q')^W \simeq V_\lambda \otimes W \arrow[r, "r"] \arrow[d, "\partial"] & H^i(Q'')^W \arrow[d, "\partial"] \\
H^{i+1}(V_\mu \otimes E_\mu)^W \arrow[r, "k"] & H^{i+1}(Q')^W \arrow[d, "g"] \\
& H^{i+1}(V_\mu \otimes E_\mu)
\end{tikzcd}$$

where k and r are induced by the inclusions. By the construction of Q, we have $H^{i+1}(\mathrm{gr}\, Q')^W = H^{i+1}(V_\mu \otimes E_\mu)$; hence it follows that $H^{i+1}(Q')^W = H^{i+1}(V_\mu \otimes E_\mu)$, where the equality is given by g and the composition gk is the identity.

By Lemma 6.9, the map ∂ in the first column of the last diagram is the composition of the quiver representation maps tensored with $j_{\mu\lambda}$ in (14); then by chasing in the above two diagrams, the claim is proved and the proof is complete. □

Remark. In principle, the fact that $(H^*(\mathrm{gr}\, E), c)$ is a complex should be a consequence of the relation $\theta \wedge \theta = 0$. Conversely, Theorem 6.11 shows that the relation $\theta \wedge \theta = 0$, which is quite difficult to be handled directly, has simpler consequences. The reader may find some more information on this topic in the last section about Olver maps.

Remark. The computation of cohomology allows an interpretation involving the Hasse quiver $\mathscr{H}_X$ (see §2). $\mathscr{H}_X$ is obviously leveled according to Definition 5.1. Let $\lambda \longrightarrow \mu \longrightarrow \nu$ be any composition of arrows in $\mathscr{H}_X$. We define quadratic relations in $\mathscr{H}_X$ by asking that the sum of all the compositions of two arrows between λ and ν is zero for all λ and ν. Now given a homogeneous bundle E and an irreducible G-module W, we define a representation of $\mathscr{H}_X$ in the following way. Let $\mathrm{gr}\, E = \bigoplus V_\lambda \otimes E_\lambda$.

Given the vertex μ in $\mathscr{H}_X$, there is a unique λ in the Bott chamber with vertex μ such that $H^*(E_\lambda) \simeq W$. Then we associate to this vertex the G-module $W \otimes V_\lambda$. The maps c_i of the complex $H^*(\mathrm{gr}\, E)$ give the maps of this representation. The direct sum of all these representations for any irreducible G-module gives a representation of $\mathscr{H}_X$, which satisfies the relations we have defined just because $H^*(\mathrm{gr}\, E)$ is a complex.

So we have constructed a functor from representations of $\mathscr{Q}_X$ (in finite-dimensional vector spaces) to representations of $\mathscr{H}_X$ (in finite-dimensional G-modules). This functor is not injective on the objects because the singular weights give zero contribution. Also, it is easy to see that this functor is not surjective, so that the representations that are in the image of the functor make an interesting subcategory.

We have that for any homogeneous bundle E (on X Hermitian symmetric variety), the Yoneda product with $[E] \in \mathrm{Ext}^1(\mathrm{gr}\, E, \mathrm{gr}\, E)^G$ defines a complex

$$\cdots \longrightarrow H^i(\mathrm{gr}\, E) \xrightarrow{c_i[1]} H^{i+1}(\mathrm{gr}\, E) \longrightarrow \cdots .$$

It is a complex because $m([E]) = 0$. We get a functor from P-mod to the (abelian) category $\mathrm{Kom}(G\text{-mod})$ of complexes of G-modules

$$E \mapsto H^*(\mathrm{gr}\, E).$$

It is straightforward to check, by using the properties of the Yoneda product, that it is an exact functor. So it is natural to ask about the cohomology of the above complex. It turns out that, in the ADE-case, it gives only the first step of a filtration of the cohomology $H^*(E)$. In fact, for any integer n, we can consider the map $H^i(\mathrm{gr}\, E) \xrightarrow{c_i[n]} H^{i+1}(\mathrm{gr}\, E)$ which considers the summands of c_i which are compositions of at most n arrows. The $(n = 1)$-case is given by the Yoneda product, while when n is big enough, we get the whole c_i. Correspondingly, we have a filtration

$$0 \subset H^i[1](E) \subset H^i[2](E) \subset \cdots \subset H^i(E).$$

Remark. The hypercohomology module of the complex

$$\mathrm{gr}\, E \xrightarrow{\theta\wedge} \mathrm{gr}\, E \otimes T_X \xrightarrow{\theta\wedge} \mathrm{gr}\, E \otimes \bigwedge^2 T_X \xrightarrow{\theta\wedge} \cdots$$

is another interesting invariant of E (cf. [S, p. 24]). The computation in the case of $E = K_X$ shows that this should be related to the filtration above if we twist by $\mathscr{O}(t)$ and sum over $t \in \mathbf{Z}$.

7. Moduli and stability

For simplicity, we restrict in this section to the case when X is an irreducible Hermitian symmetric variety. We now consider the moduli problem of homogeneous bundles E

on X with the same gr E. Any $\mathscr{R}$-module $F=\bigoplus V_\lambda\otimes E_\lambda$ corresponds to the dimension vector $\alpha=(\alpha_\lambda)\in\mathbf{Z}^{(\mathscr{Q}_X)_0}$, where $\alpha_\lambda=\dim V_\lambda$. The group

$$\mathrm{GL}(\alpha):=\prod_{\lambda\in(\mathscr{Q}_X)_0}\mathrm{GL}(V_\lambda)$$

acts over

$$\mathscr{K}(\mathscr{Q}_X,\alpha):=\bigoplus\nolimits_{a\in(\mathscr{Q}_X)_1}\mathrm{Hom}(V_{ta},V_{ha})$$

and over the closed subvariety

$$V_X(\alpha)\subset\mathscr{K}(\mathscr{Q}_X,\alpha)$$

defined by the relations in $\mathscr{Q}_X$. The affine quotient $\mathrm{Spec}(\mathbf{C}[V_X(\alpha)]^{\mathrm{GL}(\alpha)})$ is a single point, represented by F itself. King [Ki] considers the characters of $\mathrm{GL}(\alpha)$ which are given by

$$\chi_\sigma(g)=\prod_{\lambda\in(\mathscr{Q}_X)_0}\det(g_\lambda)^{\sigma_\lambda}$$

for $\sigma\in\mathbf{Z}^{(\mathscr{Q}_X)_0}$ such that $\sum_\lambda\sigma_\lambda\alpha_\lambda=0$. The element σ can also be interpreted as a homomorphism $K_0(R\text{-mod})\to\mathbf{Z}$ which applied to E_λ gives σ_λ. A function $f\in\mathbf{C}[V_X(\alpha)]$ is called a relative invariant of weight σ if $f(g\cdot x)=\chi_\sigma(g)f(x)$, and the space of such relatively invariant functions is denoted by $\mathbf{C}[V_X(\alpha)]^{\mathrm{GL}(\alpha),\sigma}$.

There is a natural character, which is convenient to denote by $\mu(\alpha)$, defined by

$$\mu(\alpha)_\lambda=c_1(F)\,\mathrm{rk}(E_\lambda)-\mathrm{rk}(F)c_1(E_\lambda).$$

Observe that $\mu(\alpha)(F)=\sum_\lambda\alpha_\lambda\mu(\alpha)_\lambda=0$. For any subrepresentation E' of $E\in M_X(\alpha)$, let $\mathrm{gr}\,E'=\bigoplus V'_\lambda\otimes E_\lambda$ with $\dim V'_\lambda=\alpha'_\lambda$; then let

$$\mu(\alpha)(E')=\sum_\lambda\alpha'_\lambda\mu(\alpha)_\lambda=\mathrm{rk}\,E'\,\mathrm{rk}\,F\,\big(\mu(F)-\mu(E')\big)\,. \tag{16}$$

Then we define

$$M_X(\alpha):=\mathrm{Proj}\Big(\bigoplus\nolimits_{n\geq 0}\mathbf{C}[V_X(\alpha)]^{\mathrm{GL}(\alpha),n\mu(\alpha)}\Big),$$

which is projective over $\mathrm{Spec}(\mathbf{C}[V_X(\alpha)]^{\mathrm{GL}(\alpha)})$; hence it is a projective variety. The moduli space $M_X(\alpha)$ is the GIT quotient of the open set $V_X(\alpha)^{\mathrm{ss}}$ of $\chi_{\mu(\alpha)}$-semistable points (see [Ki]). Different characters give moduli spaces that are birationally equivalent to $M_X(\alpha)$.

We collect the known results about this topic in the following theorems. We saw that E is determined by $\theta_E \in \mathrm{Hom}(\mathrm{gr}\, E, \mathrm{gr}\, E \otimes T_X)$ such that $\theta_E \wedge \theta_E = 0$ (see Th. 3.1).

THEOREM 7.1

Let E be a homogeneous bundle on X irreducible Hermitian symmetric variety, and let α be the dimension vector corresponding to gr E. *The following facts are equivalent:*

(i) *for every G-invariant subbundle K, we have $\mu(K) \leq \mu(E)$* (equivariant semistability)*;*

(ii) *for every subbundle K such that $\theta_E(\mathrm{gr}\, K) \subset \mathrm{gr}\, K \otimes T_X$, we have $\mu(K) \leq \mu(E)$* (Higgs semistability)*;*

(iii) *the representation $[E]$ of $\mathcal{Q}_X$ is $\mu(\alpha)$-semistable, according to [Ki, Def. 1.1]* (quiver semistability)*;*

(iv) *E is a $\chi_{\mu(\alpha)}$-semistable point in $V_X(\alpha)$ for the action of* GL(α) *(see [Ki, Def. 2.1])* (GIT semistability)*;*

(v) *for every subsheaf K, we have $\mu(K) \leq \mu(E)$* (Mumford-Takemoto semistability*; see [OSS]).*

Proof

The equivalence (i) $\Longleftrightarrow$ (ii) follows from the fact that $F \subset E$ is G-invariant if and only if $\theta_E(\mathrm{gr}\, F) \subset \mathrm{gr}\, F \otimes T_X$. The equivalence (ii) $\Longleftrightarrow$ (iii) is straightforward from Theorem 5.9 and (16). The equivalence (iii) $\Longleftrightarrow$ (iv) is proved in [Ki, Prop. 3.1, Th. 4.1]. The equivalence (i) $\Longleftrightarrow$ (v) is proved in [M] and independently in [Ro] (in the last one, only in the case of $\mathbf{P}^n$, but his proof extends in a straightforward way to any G/P; see [Ot]). □

Remark. Migliorini shows in [M] in the analytic setting that conditions from (i) – (v) are equivalent to the existence of an approximate Hermite-Einstein metric, which can be chosen invariant for a maximal compact subgroup of G. He also relates the stability to the image of the moment map.

THEOREM 7.2

Let E be a homogeneous bundle on X irreducible Hermitian symmetric variety, and let α be the dimension vector corresponding to gr E. *The following facts are equivalent:*

(i) *for every G-invariant proper subbundle K, we have $\mu(K) < \mu(E)$* (equivariant stability)*;*

(ii) *for every proper subbundle K such that $\theta_E(\mathrm{gr}\, K) \subset \mathrm{gr}\, K \otimes T_X$, we have $\mu(K) < \mu(E)$* (Higgs stability)*;*

(iii) *the representation* $[E]$ *of* $\mathcal{Q}_X$ *is* $\mu(\alpha)$*-stable, according to [Ki, Def. 1.1]* (quiver stability)*;*

(iv) E *is a* $\chi_{\mu(\alpha)}$*-stable point in* $V_X(\alpha)$ *for the action of* $\mathrm{GL}(\alpha)$ *(see [Ki, Def. 1.2])* (GIT stability)*;*

(v) $E \simeq W \otimes E'$*, where* W *is an irreducible* G*-module, and for every proper subsheaf* $K \subset E'$*, we have* $\mu(K) \leq \mu(E')$ (Mumford-Takemoto stability *of* E'*; see [OSS]*).

Proof
The equivalences (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) are proved as above. The equivalence (i) $\Longleftrightarrow$ (v) is proved in [F]. □

Remark. The equivalence (i) $\Longleftrightarrow$ (v) holds in Theorems 7.1 and 7.2 over any rational homogeneous variety X (for any slope μ_a).

Remark. Theorem 7.1 and 7.2 extend in a straightforward way to any $\sigma\colon K_0(R\text{-}\mathrm{mod}) \to \mathbf{Z}$ such that $\sigma(\mathrm{gr}\, E) = 0$ at the place of $\mu(\alpha)$.

Remark. Theorem 7.2 shows that Mumford-Takemoto stability is a stronger condition than stability in $\mathcal{Q}_X$. The Euler sequence on $\mathbf{P}^n$ just explains this fact. Indeed, $\mathcal{O} \otimes V$ corresponds to a stable representation of $\mathcal{Q}_{\mathbf{P}^n}$, but it is not a Mumford-Takemoto stable bundle. The points in $M_X(\alpha)$ parametrize S-equivalent classes of semistable homogeneous bundles E with the same $\mathrm{gr}\, E$ corresponding to α. The closed orbits in $V_X(\alpha)^{ss}$ correspond to direct sums $\bigoplus_j W_j \otimes F_j$, where W_j are irreducible G-modules and F_j are Mumford-Takemoto stable homogeneous bundles.

When E is a Mumford-Takemoto homogeneous stable bundle, we get $W = \mathbf{C}$ in condition (v), and an open set containing the corresponding point in $M_X(\alpha)$ embeds in the corresponding Maruyama scheme of stable bundles (see the construction of families in [Ki, §5]). The tangent space at this point is $H^1(\mathrm{End}\, E)^G$.

Observe that the irreducible bundles do not deform as homogeneous bundles, and their corresponding moduli space in the sense above is a single point (see Cor. 4.4).

Example 7.3
We describe an example of a homogeneous bundle on $\mathbf{P}^2$ with a continuous family of homogeneous deformations. This example appears already in [H2, Exam. 1.8.7, Prop. 4.2.4].

Such an example is $E = \mathrm{Sym}^2 Q(-1) \otimes \mathcal{S}^{2,1} V$, whose rank is 24. It is easy to compute that $H^1(\mathrm{End}\, E)^G = \mathbf{C}$. The corresponding representation of the quiver associates to

$$\begin{tikzcd} \mathcal{O} \arrow[d] & Q(1) \arrow[l] \arrow[d] & \\ Q(-2) & \mathrm{Sym}^2 Q(-1) \arrow[l] \arrow[d] & \mathrm{Sym}^3 Q \arrow[l] \arrow[d] \\ & \mathrm{Sym}^3 Q(-3) & \mathrm{Sym}^4 Q(-2) \arrow[l] \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \mathbf{C} \arrow[d] & \mathbf{C} \arrow[l] \arrow[d, "f_1"] & \\ \mathbf{C} & \mathbf{C}^2 \arrow[l, "f_4"'] \arrow[d, "f_3"] & \mathbf{C} \arrow[l, "f_2"'] \arrow[d] \\ & \mathbf{C} & \mathbf{C} \arrow[l] \end{tikzcd}$$

The four arrows starting from or ending to the middle $\mathbf{C}^2$ determine four one-dimensional spaces (two kernels and two images) which correspond to four marked points in $\mathbf{P}^1$. The cross-ratio of these four points describes the deformation. The generic deformation is Mumford-Takemoto stable. If we fix the dimension vector $\alpha = (1, 1, 1, 2, 1, 1, 1)$ according to the diagram above, then

$$M_{\mathbf{P}^2}(\alpha) = \mathbf{P}^1.$$

Indeed, the character $\mu(\alpha)$ is $72(0, -1, 1, 0, -2, 2, 0)$. We can divide by 72, and the coordinate ring

$$\bigoplus_{n\geq 0} \mathbf{C}[V_X(\alpha)]^{\mathrm{GL}(\alpha), n\mu(\alpha)}$$

is generated by

$$S = (f_4 f_1)(f_3 f_2)^2 \qquad \text{and} \qquad T = (f_4 f_2)(f_3 f_2)(f_3 f_1).$$

(Both correspond to $n = 1$.) If we do not divide by 72, then the two generators are S^{72} and T^{72}.

There are three distinguished points. We get the first one (corresponding to $S = 0$) when $\operatorname{Im} f_1 = \operatorname{Ker} f_4$. In this case, there are three different orbits where the S-equivalence class contains $\mathcal{O}$ as a direct summand. We have the second one (corresponding to $S = T$) when $\operatorname{Im} f_1 = \operatorname{Im} f_2$ or when $\operatorname{Ker} f_3 = \operatorname{Ker} f_4$. In this case, there are three different orbits where the S-equivalence class contains $\mathrm{Sym}^2 Q(-1)$ as a direct summand. We have the third one (corresponding to $T = 0$) when $\operatorname{Im} f_1 = \operatorname{Ker} f_3$

or when $\operatorname{Im} f_2 = \operatorname{Ker} f_4$. Also, in this case, there are three different orbits where the S-equivalence class contains $\operatorname{ad} V$ as a direct summand. Observe that $\operatorname{Im} f_2 = \operatorname{Ker} f_3$ gives a nonstable situation where the middle row $\mathbf{C} \xleftarrow{f_4} \operatorname{Im} f_2 \xleftarrow{f_2} \mathbf{C}$ destabilizes.

There are other two particular points in $M_{\mathbf{P}^1}(\alpha)$ which correspond, respectively, to $\operatorname{Sym}^2 Q(-1) \otimes \mathscr{S}^{2,1} V$ and to $\operatorname{ad} C$ where C is the rank 5 exceptional bundle defined by the sequence

$$0 \longrightarrow Q(-1) \longrightarrow C \longrightarrow \operatorname{Sym}^2 Q \longrightarrow 0.$$

Remark. It seems an interesting open question to understand when $M_X(\alpha)$ is nonempty or irreducible.

8. Olver maps and explicit relations for Grassmannians

The aim of this section is to make explicit in the case of Grassmannians the relations coming from $\theta \wedge \theta = 0$.

We restrict to the case $G = \mathrm{SL}(V)$. Let a be the Young diagram associated to a weight λ; that is, let $\mathscr{S}^a V$ be the representation with maximal weight λ. Let a' be obtained by adding one box to a. Let λ' be the weight corresponding to a'. In the unpublished preprint [O], Olver gave a nice description of the Pieri maps $\mathscr{S}^a V \otimes V \longrightarrow \mathscr{S}^{a'} V$. These maps are defined up to a nonzero scalar multiple. This description was used in [D]; then a proof of the correctness of Olver's description appeared in [MO] in the more general setting of skew Young diagrams.

It is well known that $\mathscr{S}^a V$ can be obtained as a quotient of $\operatorname{Sym}^a V := \operatorname{Sym}^{a_1} V \otimes \cdots \otimes \operatorname{Sym}^{a_n} V$ (see [DEP] or [FH]); namely, there is the quotient map (see [D, §2.6])

$$\rho_a : \operatorname{Sym}^a V \longrightarrow \mathscr{S}^a V.$$

Olver's idea is to consider Pieri maps at the level of $\operatorname{Sym}^a V$ and then factor through the quotient.

We follow here [D], where a different notation is used; in particular, $\operatorname{Sym}^{\tilde{a}} V$ in [D] is our $\operatorname{Sym}^a V$. We refer to [D] for the definition of the linear map $\chi_a^{a'} : \operatorname{Sym}^{a'} V \longrightarrow \operatorname{Sym}^a V \otimes V$. This is called an *Olver map*.

THEOREM 8.1 (Olver; [D, Th. 2.14])
Consider the diagram

$$\begin{tikzcd}
\operatorname{Sym}^{a'} V \arrow[r, "\chi_a^{a'}"] \arrow[d, "\rho_{a'}"] & \operatorname{Sym}^a V \otimes V \arrow[d, "\rho_a \otimes 1"] \\
\mathscr{S}^{a'} V & \mathscr{S}^a V \otimes V
\end{tikzcd}$$

Then $\chi_a^{a'}(\ker \rho_{a'}) \subset \ker(\rho_a \otimes 1)$, *and* $\chi_a^{a'}$ *induces the nonzero* $\mathrm{SL}(V)$*-equivariant*

$$\psi_a^{a'} : \mathscr{S}^{a'} V \longrightarrow \mathscr{S}^a V \otimes V,$$

making the above diagram commutative.

A *tableau* on the Young diagram a is a numbering of the boxes with the integers between 1 and $n+1$. A tableau is called *standard* if the entries of the rows are weakly increasing from the left to the right and the entries of the columns are strictly increasing from the top to the bottom. The *content* of a tableau T is the function $C_T : \{1, \dots, n\} \to \mathbf{N}$ such that $C_T(p)$ is the number of times p occurs in T. After a basis $e_1, \dots, e_{n+1}$ of V has been fixed, to any tableau T is associated in the natural way a tensor T^S in $\mathrm{Sym}^a V$ by symmetrizing the basis vectors labeled by each row. The eigenvectors for the action of the diagonal subgroup of $\mathrm{SL}(V)$ over $\mathscr{S}^a V$ correspond to $\rho_a(T^S)$ for T chosen among the standard tableaux. They form a basis of $\mathscr{S}^a V$.

Let K^a be the tableau obtained by filling the ith row with entries equal to i (it is called *canonical* in [DEP]); K^a is the only standard tableau among those with the same content. The projection $\rho_a(K^{aS})$ is a maximal eigenvector for $\mathscr{S}^a V$, and we denote it by κ^a. Let a' be obtained from a by adding a box to the ith row, and let a'' be obtained from a' by adding a box to the jth row. Consider the map $\chi_a^{a''} : \mathrm{Sym}^{a''} V \longrightarrow \mathrm{Sym}^a V \otimes V \otimes V$ defined as the composition

$$\mathrm{Sym}^{a''} V \xrightarrow{\chi_{a'}^{a''}} \mathrm{Sym}^{a'} V \otimes V \xrightarrow{\chi_a^{a'} \otimes 1} \mathrm{Sym}^a V \otimes V \otimes V;$$

$\chi_a^{a''}$ induces the nonzero $\mathrm{SL}(V)$-equivariant morphism

$$\psi_a^{a''} : \mathscr{S}^{a''} V \longrightarrow \mathscr{S}^a V \otimes V \otimes V.$$

Let $K_{i,j}^{a'}$ be the tableau on a' obtained by adding a box filled with j at the ith row of K^a. We denote the element $\rho_{a'}(K_{i,j}^{aS})$ by $\kappa_{i,j}^{a'}$.

PROPOSITION 8.2

(i) *If* $i > j$, *then*

$$\psi_{a'}^{a''}(\kappa^{a''}) = (a_j + 1)\kappa^{a'} \otimes e_j + \sum_{h \neq j,i} \tau_h \otimes e_h$$

for some τ_h.

(ii) *If* $i = j$, *then*

$$\psi_{a'}^{a''}(\kappa^{a''}) = (a_j + 2)\kappa^{a'} \otimes e_j + \sum_{h \neq j} \tau_h \otimes e_h$$

for some τ_h.

(iii) *If $i < j$, then*

$$\psi_{a'}^{a''}(\kappa^{a''}) = \Big(-\frac{(a_j+1)(a_i+1)}{a_i-a_j+j-i}\kappa_{i,j}^{a'}+\tau\Big)\otimes e_i+(a_j+1)\kappa^{a'}\otimes e_j+\sum_{h\neq i,j}\tau_h\otimes e_h$$

for some τ, τ_h, where $\psi_a^{a'}(\tau)$ has the coefficient of $e_j \otimes \kappa^a$ equal to zero.

Proof

In (i) and (ii) the summand $\kappa^{a'} \otimes e_j$ is obtained with $J = (0, j)$ (see [D, §2.12]). In (iii) the summand $\kappa_{i,j}^{a'} \otimes e_i$ is obtained with $J = (0, i, j)$, while the summand $\kappa^{a'} \otimes e_j$ is obtained with $J = (0, j)$. □

COROLLARY 8.3

(i) *If $i > j$, then*

$$\psi_a^{a''}(\kappa^{a''}) = (a_i+1)(a_j+1)\kappa^a \otimes e_i \otimes e_j$$
$$+\cdots+ \textit{(linear combination of other basis vectors}$$
$$\textit{different from } \kappa^a \otimes e_j \otimes e_i\textit{)}.$$

(ii) *If $i = j$, then*

$$\psi_a^{a''}(\kappa^{a''}) = (a_j+1)(a_j+2)\kappa^a \otimes e_j \otimes e_j$$
$$+\cdots+ \textit{(linear combination of other basis vectors)}.$$

(iii) *If $i < j$, then*

$$\psi_a^{a''}(\kappa^{a''}) = (a_i+1)(a_j+1)\kappa^a \otimes \Big(e_i\otimes e_j - \frac{1}{a_i-a_j+j-i}e_j\otimes e_i\Big)$$
$$+\cdots+ \textit{(linear combination of other basis vectors)}.$$

Remark. The case (i) of Corollary 8.3 does not appear if $i = j+1$ and $a_i = a_j$. In such a case, a'' is obtained from a by adding two boxes to the same column, and the only possibility is to add first the highest box and then the lowest one.

Now consider a bundle $E_\lambda = S^\alpha U \otimes S^\beta Q^*(t)$ (as in §5) in the Grassmannian $\mathrm{Gr}(\mathbf{P}^k, \mathbf{P}^n)$, where $\lambda = \sum_{i=1}^n c_i\lambda_i$. Let $p, q \in \mathbf{N}$.

Let

$$n_{p,q} := -\sum_{i=-(p-1)}^{q-1} \alpha_{k+1+i},$$

$$\lambda_{p,q} := \lambda + n_{p,q},$$

$$\lambda_q := \lambda_{1,q} = \lambda - \sum_{i=0}^{q-1} \alpha_{k+1+i}.$$

We denote the corresponding morphism as

$$m_{\lambda,p,q} \colon E_\lambda \otimes \Omega^1 \to E_{\lambda_{p,q}},$$

normalized according to Definition 5.5.

Then $E_{\lambda_{p,q}} = S^{\alpha'} U \otimes S^{\beta'} Q^*(t)$, where α' is obtained from α by adding a box to row p and β' is obtained from β by adding a box to row q.

In the following proposition we make the relations (see Def. 5.7) explicit for $\mathcal{Q}_{\mathrm{Gr}(\mathbf{P}^k,\mathbf{P}^n)}$. We consider $E_{\lambda''} = S^{\alpha''} U \otimes S^{\beta''} Q^*(t)$, where α'' is obtained from α by adding two box to the rows p_1, p_2 and β'' is obtained from β by adding two boxes to the rows q_1, q_2. If $p_1 = p_2$ and $q_1 = q_2$, then $\mathrm{Ext}^2(E_\lambda, E_{\lambda''})^G = 0$. By the symmetry, we may assume that $p_1 \le p_2, q_1 < q_2$. Let

$$\tilde{p} := \sum_{i=p_1}^{p_2-1} c_{k+1-i} + p_2 - p_1 = \alpha_{p_1} - \alpha_{p_2} + p_2 - p_1,$$

$$\tilde{q} := \sum_{i=q_1}^{q_2-1} c_{k+1+i} + q_2 - q_1 = \beta_{q_1} - \beta_{q_2} + q_2 - q_1.$$

Note that $\tilde{p} = 1$ if and only if $p_2 = p_1 + 1$ and $c_{k+1-p_1} = 0$. In the same way, $\tilde{q} = 1$ if and only if $q_2 = q_1 + 1$ and $c_{k+1+q_1} = 0$.

PROPOSITION 8.4 (Explicit relations for $\mathcal{Q}_{\mathrm{Gr}(\mathbf{P}^k,\mathbf{P}^n)}$)

(i) *If $p_1 < p_2$, we have the following subcases:*

(i1) $\tilde{p} \neq 1, \tilde{q} \neq 1$*; in this case, we have the two equations*

$$g_{\lambda_{p_1,q_1},p_2q_2} g_{\lambda,p_1,q_1} \left(\frac{1}{\tilde{q}} - \frac{1}{\tilde{p}}\right) - g_{\lambda_{p_1,q_2},p_2q_1} g_{\lambda,p_1,q_2} + g_{\lambda_{p_2,q_1},p_1q_2} g_{\lambda,p_2,q_1} = 0,$$

$$g_{\lambda_{p_1,q_1},p_2q_2} g_{\lambda,p_1,q_1} \left(\frac{1}{\tilde{p}\tilde{q}} - 1\right) - g_{\lambda_{p_1,q_2},p_2q_1} g_{\lambda,p_1,q_2} \left(\frac{1}{\tilde{p}}\right)$$

$$-g_{\lambda_{p_2,q_1},p_1q_2} g_{\lambda,p_2,q_1} \left(\frac{1}{\tilde{q}}\right) + g_{\lambda_{p_2,q_2},p_1q_1} g_{\lambda,p_2,q_2} = 0;$$

(i2) $\tilde{p}=1$ *and* $\tilde{q}\neq 1$*; in this case,* $\lambda_{p_2,q_1}, \lambda_{p_2,q_2}$ *do not exist and we have the single equation*

$$g_{\lambda_{p_1,q_1},p_2q_2}g_{\lambda,p_1,q_1}\left(\frac{1}{\tilde{q}}-1\right)-g_{\lambda_{p_1,q_2},p_2q_1}g_{\lambda,p_1,q_2}=0;$$

(i3) $\tilde{p}\neq 1$ *and* $\tilde{q}=1$*; in this case,* $\lambda_{p_1,q_2}, \lambda_{p_2,q_2}$ *do not exist and we have the single equation*

$$g_{\lambda_{p_1,q_1},p_2q_2}g_{\lambda,p_1,q_1}\left(1-\frac{1}{\tilde{p}}\right)+g_{\lambda_{p_2,q_1},p_1q_2}g_{\lambda,p_2,q_1}=0;$$

(i4) $\tilde{p}=\tilde{q}=1$*; in this case, only* λ_{p_1,q_1} *survives and there are no equations at all. The Hille counterexample (see Exam. 5.11) fits this case.*

(ii) *If* $p_1=p_2$*, we have the following subcases:*

(ii1) $\tilde{q}\neq 1$*; in this case, we have the equation*

$$g_{\lambda_{p_1,q_1},p_1q_2}g_{\lambda,p_1,q_1}\left(\frac{1+\tilde{q}}{\tilde{q}}\right)-g_{\lambda_{p_1,q_2},p_1q_1}g_{\lambda,p_1,q_2}=0;$$

(ii2) $\tilde{q}=1$*; in this case, we have the equation*

$$g_{\lambda_{p_1,q_1},p_1q_2}g_{\lambda,p_1,q_1}=0.$$

Proof
Let $p_1<p_2$. Consider that

$$\begin{aligned}m_{\lambda_{p_1,q_1},p_2q_2}\wedge m_{\lambda,p_1,q_1}(n_{p_1q_2}\wedge n_{p_2q_1}\otimes v_\lambda) &= m_{\lambda_{p_1,q_1},p_2q_2}\left(n_{p_1q_2}\otimes m_{\lambda,p_1,q_1}(n_{p_2q_1}\otimes v_\lambda)\right)\\ &\quad -m_{\lambda_{p_1,q_1},p_2q_2}\left(n_{p_2q_1}\otimes m_{\lambda,p_1,q_1}(n_{p_1q_2}\otimes v_\lambda)\right)\\ &= \left(-\frac{1}{\tilde{p}}+\frac{1}{\tilde{q}}\right)v_{\lambda''}.\end{aligned}$$

(The last equality is by Cor. 8.3.) In the same way, if $\tilde{q}\neq 1$,

$$m_{\lambda_{p_1,q_2},p_2q_1}\wedge m_{\lambda,p_1,q_2}(n_{p_1q_2}\wedge n_{p_2q_1}\otimes v_\lambda)=-v_{\lambda''}.$$

Moreover, if $\tilde{p}\neq 1$,

$$m_{\lambda_{p_2,q_1},p_1q_2}\wedge m_{\lambda,p_2,q_1}(n_{p_1q_2}\wedge n_{p_2q_1}\otimes v_\lambda)=v_{\lambda''}.$$

Besides,

$$m_{\lambda_{p_2,q_2},p_1q_1}\wedge m_{\lambda,p_2,q_2}(n_{p_1q_2}\wedge n_{p_2q_1}\otimes v_\lambda)=0.$$

Now by computing the left-hand side of relation (6) on $n_{p_1q_2} \wedge n_{p_2q_1} \otimes v_\lambda$, we get the first equation of (i1).

In the same way, by computing the left-hand side of relation (6) on $n_{p_1q_1} \wedge n_{p_2q_2} \otimes v_\lambda$, we get the second equation of (i1). The other subcases of (i) are particular cases of (i1). Case (ii) is analogous. □

Remark. The number of equations obtained in Proposition 8.4 measures exactly the dimension of $\mathrm{Ext}^2(E_\lambda, E_{\lambda''})^G$, which can be 2, 1, or 0. An interesting consequence of Proposition 8.4 is that (with the assumptions in (i)) there is no indecomposable homogeneous bundle on $\mathrm{Gr}(\mathbf{P}^k, \mathbf{P}^n)$ such that its quiver representation has support equal to the parallelogram with vertices E_λ, $E_{\lambda_{p_1,q_2}}$, $E_{\lambda_{p_2,q_1}}$, $E_{\lambda''}$. The first consequence is that on $\mathrm{Gr}(\mathbf{P}^1, \mathbf{P}^3)$, every homogeneous bundle E such that $\mathrm{gr}\, E = \Omega^1 \oplus \Omega^2 \oplus \Omega^3$ decomposes. On the other hand, there exists an indecomposable homogeneous bundle such that its quiver representation has support equal to the parallelogram with vertices E_λ, $E_{\lambda_{p_1,q_1}}$, $E_{\lambda_{p_2,q_2}}$, $E_{\lambda''}$ if and only if $\tilde{p} = \tilde{q}$. The first nontrivial example is, on the Grassmannian of lines in $\mathbf{P}^3 = \mathbf{P}(V)$, the cohomology bundle E of the monad

$$\mathcal{O}(-2) \longrightarrow \mathcal{S}^{2,2} V \longrightarrow \mathcal{O}(2),$$

whose graded bundle is $\mathrm{gr}\, E = \mathcal{O} \oplus \Omega^1 \oplus \Omega^1(2) \oplus (\mathrm{Sym}^2 U \otimes \mathrm{Sym}^2 Q)$.

COROLLARY 8.5 (Explicit relations for $\mathcal{Q}_{\mathbf{P}^n}$)
In the case of $\mathbf{P}^n$, the category of homogeneous bundles is equivalent to the category of representations of $\mathcal{Q}_{\mathbf{P}^n}$ with the commutative relations.

Proof
Put $p_1 = p_2 = 1$ in Proposition 8.4; we get

$$g_{\lambda_{q_1},q_2} g_{\lambda,q_1} \left(\frac{1+\tilde{q}}{\tilde{q}} \right) - g_{\lambda_{q_2},q_1} g_{\lambda,q_2} = 0,$$

unless $\tilde{q} = 1$.

Denoting

$$h_{\lambda,i} := (c_i + 1)(c_{i-1} + c_i + 2) \cdots (c_2 + \cdots + c_i + i - 1) f_{\lambda,i},$$

we get a functor from the quiver $\mathcal{Q}_{\mathbf{P}^n}$ with the relations that we have defined to the same quiver with the commutative relations

$$h_{\lambda_{q_1},q_2} h_{\lambda,q_1} - h_{\lambda_{q_2},q_1} h_{\lambda,q_2} = 0.$$

This functor gives the desired equivalence. □

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