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On partial polynomial interpolation

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ABSTRACT

The Alexander–Hirschowitz theorem says that a general collection of k double points in $\mathbf{P}^n$ imposes independent conditions on homogeneous polynomials of degree d with a well known list of exceptions. We generalize this theorem to arbitrary zero-dimensional schemes contained in a general union of double points. We work in the polynomial interpolation setting. In this framework our main result says that the affine space of polynomials of degree $\leq d$ in n variables, with assigned values of any number of general linear combinations of first partial derivatives, has the expected dimension if $d \neq 2$ with only five exceptional cases. If $d = 2$ the exceptional cases are fully described.

1. Introduction

Let $R_{d,n} = K[x_1, \ldots, x_n]_d$ be the vector space of polynomials of degree $\leq d$ in n variables over an infinite field K. Note that $\dim R_{d,n} = \binom{n+d}{d}$. Let $p_1, \ldots, p_k \in K^n$ be k general points and assume that over each of these points a general affine proper subspace $A_i \subset K^n \times K$ of dimension a_i is given. Assume that $a_1 \geqslant \cdots \geqslant a_k$. Let $\Gamma_f \subseteq K^n \times K$ be the graph of $f \in R_{d,n}$ and $T_{p_i}\Gamma_f$ be its tangent space at the point $(p_i, f(p_i))$. Note that $\dim T_{p_i}\Gamma_f = n$ for any i. Consider the conditions

$$A_i \subseteq T_{p_i}\Gamma_f, \quad \text{for } i = 1, \ldots, k \tag{1}$$

When $a_i = 0$, the assumption (1) means that the value of f at p_i is assigned. When $a_i = n$, (1) means that the value of f at p_i and the values of all first partial derivatives of f at p_i are assigned. In the

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intermediate cases, (1) means that the value of f at p_i and the values of some linear combinations of first partial derivatives of f at p_i are assigned.

Consider now the affine space

$$V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k) = \{f \in R_{d,n} | A_i \subseteq T_{p_i}\Gamma_f,\ i = 1, \ldots, k\} \tag{2}$$

The polynomials in this space solve a partial polynomial interpolation problem. The conditions in (1) correspond to $(a_i + 1)$ affine linear conditions on $R_{d,n}$. Our main result describes the codimension of the above affine space. Since the description is different for $d = 2$ and $d \neq 2$, we divide the result in two parts.

Theorem 1.1. *Let $d \neq 2$ and char $(K) = 0$. For a general choice of points p_i and subspaces A_i, the affine space $V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$ has codimension in $R_{d,n}$ equal to*

$$\min\left\{\sum_{i=1}^{k}(a_i + 1), \dim R_{d,n}\right\}$$

with the following list of exceptions

$(a)\ n = 2,\ d = 4,\ k = 5, a_i = 2\quad \text{for } i = 1, \ldots, 5$
$(b)\ n = 3,\ d = 4,\ k = 9, a_i = 3\quad \text{for } i = 1, \ldots, 9$
$(b')\ n = 3,\ d = 4,\ k = 9, a_i = 3\quad \text{for } i = 1, \ldots, 8 \text{ and } a_9 = 2$
$(c)\ n = 4,\ d = 3,\ k = 7, a_i = 4\quad \text{for } i = 1, \ldots, 7$
$(d)\ n = 4,\ d = 4,\ k = 14, a_i = 4\quad \text{for } i = 1, \ldots, 14$

In particular when $\sum_{i=1}^{k}(a_i + 1) = \binom{n+d}{d}$ there is a unique polynomial f in $V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$, with the above exceptions (a), (b′), (c), (d). In the exceptional cases the space $V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$ is empty.

The "general choice" assumption means that the points can be taken in a Zariski open set (i.e. outside the zero locus of a polynomial) and for each of these points the space A_i can be taken again in a Zariski open set. On the real numbers this assumption means that the choices can be done outside a set of measure zero. Our result is not constructive but it ensures that in the case $\sum_{i=1}^{k}(a_i + 1) = \binom{n+d}{d}$ the linear system computing the interpolating polynomial with general data has a unique solution. Hence any algorithm solving linear systems can be successfully applied. Actually our proof shows that Theorem 1.1 holds on any infinite field, with the possible exception of finitely many values of char K (see the appendix). For finite fields the genericity assumption is meaningless.

The case in which $a_i = n$ for all i was proved by Alexander and Hirschowitz in [1,2], see [4] for a survey. The most notable exception is the case of seven points with seven tangent spaces for cubic polynomials in four variables, as in c). This example was known to classical algebraic geometers and it was rediscovered in the setting of numerical analysis in [11]. The case of curvilinear schemes was proved as a consequence of a more general result by [5] on $\mathbf{P}^2$ and by [8] in general.

The case $d = 1$ follows from elementary linear algebra. The case $n = 1$ is easy and well known: in this case the statement of Theorem 1.1 is true with the only requirement that the points p_i are distinct and the spaces A_i are not vertical, that is their projections $\pi(A_i)$ on K^n satisfy $\dim A_i = \dim \pi(A_i)$.

Assume now $d = 2$. We set $a_i = -1$ for $i > k$. For any $1 \leq i \leq n$ we denote

$$\delta_{a_1,\ldots,a_k}(i) = \max\left\{0, \sum_{j=1}^{i} a_j - \sum_{j=1}^{i}(n + 1 - j)\right\}$$

Theorem 1.2. *Let K be an infinite field. For a general choice of points p_i and subspaces A_i, the affine space $V_{2,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$ has codimension in $R_{2,n}$ equal to*

$$\min\left\{\sum_{i=1}^{k}(a_i + 1), \dim R_{2,n}\right\}$$

if and only if one of the following conditions takes place:

(1) *either* $\delta_{a_1,\ldots,a_k}(i) = 0$ *for all* $1 \leqslant i \leqslant n$;
(2) *or* $\sum_i (a_i + 1) \geqslant \binom{n+2}{2} + \max\{\delta_{a_1,\ldots,a_k}(i) : 1 \leqslant i \leqslant n\}$.

In particular when $\sum_{i=1}^{k}(a_i+1) = \binom{n+2}{2}$ *there is a unique polynomial* f *in* $V_{2,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$ *if and only if, for any* $1 \leq i \leq n$, *we have*

$$\sum_{j=1}^{i} a_j \leq \sum_{j=1}^{i}(n+1-j).$$

The first nontrivial example which explains Theorem 1.2 is the following. Consider $k = 2$ and $(a_1, a_2) = (n, n)$. Then the affine space $V_{2,n}(p_1, p_2, A_1, A_2)$ is given by quadratic polynomials with assigned tangent spaces A_1, A_2 at two points p_1, p_2. This space is not empty if and only if the intersection space $A_1 \cap A_2$ is not empty and its projection on K^n contains the midpoint of p_1p_2, which is a codimension one condition. In order to prove this fact restrict to the line through p_1 and p_2 and use a well known property of the tangent lines to the parabola. In this case $\delta_{n,n}(i) = \begin{cases} 0 & i \neq 1 \\ 1 & i = 1 \end{cases}$ and the two conditions of Theorem 1.2 are not satisfied. In Section 3 we will explain these two conditions with more details.

Let $\pi(A_i)$ be the projection of A_i on K^n. For $i = 1, \ldots, k$ we consider the ideal

$$I_i = \left\{ f \in K[x_1, \ldots, x_n] | f(p_i) + \sum_{j=1}^{n}(x_j - (p_i)_j)\frac{\partial f}{\partial x_j}(p_i) = 0 \text{ for any } x \in \pi(A_i) \right\}$$

Notice that we have $m_{p_i}^2 \subseteq I_i \subseteq m_{p_i}$ and the ring $K[x_1, \ldots, x_n]/I_i$ corresponds to a zero-dimensional scheme ξ_i of length a_i+1, supported at p_i and contained in the double point p_i^2. When $V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$ is not empty, its associated vector space (that is its translate containing the origin) consists of the hypersurfaces of degree d through $\xi_1, \ldots, \xi_k$. Moreover, when this vector space has the expected dimension, it follows that $V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$ has the expected dimension too.

The space K^n can be embedded in the projective space $\mathbf{P}^n$. Since the choice of points is general, we can always avoid the "hyperplane at infinity". In order to prove the above two theorems, we will reformulate them in the projective language of hypersurfaces of degree d through zero-dimensional schemes. More precisely we refer to Theorem 3.2 for $d = 2$, Theorem 4.1 for $d = 3$ and Theorem 5.6 for $d \geqslant 4$. This reformulation is convenient mostly to rely on the wide existing literature on the subject. In this setting Alexander and Hirschowitz proved that a general collection of double points imposes independent conditions on the hypersurfaces of degree d (with the known exceptions) and our result generalizes to a general zero-dimensional scheme contained in a union of double points. It is possible to degenerate such a scheme to a union of double points only in few cases, in such cases of course our result is trivial from [1].

Our proof of Theorem 5.6, and hence of Theorem 1.1, is by induction on n and d. Since it is enough to find a particular zero-dimensional scheme which imposes independent condition on hypersurfaces of degree d, we specialize some of the points on a hyperplane, following a technique which goes back to Terracini. We need a generalization of the Horace method, like in [1], that we develop in the proof of Theorem 5.6. The case of cubics, which is the starting point of the induction, is proved by generalizing the approach of [4], where we restricted to a codimension three linear subspace. This case is the crucial step which allows to prove the Theorem 1.1. Section 4 is devoted to this case, which requires a lot of effort and technical details, in the setting of discrete mathematics. Compared with the quick proof we gave in [4], here we are forced to divide the proof in several cases and subcases. While the induction argument works quite smoothly for $n, d \gg 0$, it is painful to cover many of the initial cases. In the case $d = 3$ we need the help of a computer, by a Montecarlo technique explained in the appendix.

A further remark is necessary. In [1,4] the result about the independence of double points was shown to be equivalent, through Terracini lemma, to a statement about the dimension of higher secant varieties of the Veronese varieties, which in turn is related to the Waring problem for polynomials. Here the assumption that K is algebraically closed of zero characteristic is necessary to translate safely the results, see also Theorem 6.1 and Remark 6.3 in [10]. For example, on the real numbers, the closure in the euclidean topology of the locus of secants to the twisted cubic is a semi-algebraic set, corresponding to the cubic polynomials which have not three distinct real roots, and it is defined by the condition that the discriminant is nonpositive. Indeed a real cubic polynomial can be expressed as the sum of two cubes of linear polynomials (Waring problem) if and only if it has two distinct complex conjugate roots or a root of multiplicity three.

2. Preliminaries

Let X be a scheme contained in a collection of double points of $\mathbf{P}^n$. We say that the type of X is $(m_1, \ldots, m_{n+1})$ if X contains exactly m_i subschemes of a double point of length i, for $i = 1, \ldots, n+1$. For example the type of k double points is $(0, \ldots, 0, k)$. The degree of X is $\deg X = \sum im_i$. A scheme of type $(m_1, \ldots, m_{n+1})$ corresponds to a collection of m_i linear subspaces $L_i \subseteq \mathbf{P}^n$ with $\dim L_i = i-1$ and with a marked point on each L_i.

Algebraic families of such schemes can be defined over any field K with the Zariski topology.

Any irreducible component ζ of length k contained in a double point supported at the point p corresponds to a linear space L of projective dimension $k-1$ passing through p. The hypersurfaces containing ζ are exactly the hypersurfaces F such that $T_pF \supseteq L$.

This description allows to consider a *degeneration* (or collision) of two components as the span of the corrisponding linear spaces. More precisely, consider two irreducible schemes ζ_0, ζ_1, supported respectively at p_0, p_1, of length respectively k_0, k_1 and consider the space $V(\zeta_0, \zeta_1)$ of the hypersurfaces containing ζ_0 and ζ_1. Let L_i be the space corresponding to ζ_i. By the above remark this space consists of the hypersurfaces F such that $T_{p_0}F \supseteq L_0$ and $T_{p_1}F \supseteq L_1$.

Let $L = \langle L_0, L_1\rangle$ be the projective span of L_0 and L_1, that is the smallest projective space containing L_0 and L_1. If L_0 and L_1 are general, and if moreover $k_0 + k_1 - 1 \le n$, then $\dim L = k_0 + k_1 - 1$ and L corresponds to an irreducible scheme ζ of length $k_0 + k_1$ supported at p_0 (or at p_1). It is not difficult to construct a degeneration of $\zeta_0 \cup \zeta_1$ which has ζ as limit.

This implies, by semicontinuity, that $\dim V(\zeta_0 \cup \zeta_1) \le \dim V(\zeta)$.

In particular if we prove that $V(\zeta)$ has the expected dimension, the same is true for $V(\zeta_0 \cup \zeta_1)$. We will use often this remark through the paper.

We recall now some notation and results from [4].

Given a zero-dimensional subscheme $X \subseteq \mathbf{P}^n$, the corresponding ideal sheaf $\mathcal{I}_X$ and a linear system $\mathcal{D}$ on $\mathbf{P}^n$, the Hilbert function is defined as follows:

$$h_{\mathbf{P}^n}(X, \mathcal{D}) := \dim \mathrm{H}^0(\mathcal{D}) - \dim \mathrm{H}^0(\mathcal{I}_X \otimes \mathcal{D}).$$

If $h_{\mathbf{P}^n}(X, \mathcal{D}) = \deg X$, we say that X is $\mathcal{D}$-independent, and in the case $\mathcal{D} = \mathcal{O}_{\mathbf{P}^n}(d)$, we say d-independent.

A zero-dimensional scheme is called curvilinear if it is contained in the smooth part of a curve. Notice that a curvilinear scheme contained in a double point has length 1 or 2.

Lemma 2.1 (Curvilinear Lemma [6,4]). *Let X be a zero-dimensional scheme of finite length contained in a union of double points of $\mathbf{P}^n$ and $\mathcal{D}$ a linear system on $\mathbf{P}^n$. Then X is $\mathcal{D}$-independent if and only if every curvilinear subscheme of X is $\mathcal{D}$-independent.*

For any scheme $X \subset L$ in a projective space L, we denote $\mathcal{I}_X(d) = \mathcal{I}_X \otimes \mathcal{O}_L(d)$ and $I_{X,L}(d) = \mathrm{H}^0(\mathcal{I}_X(d))$. The expected dimension of the vector space $I_{X,\mathbf{P}^n}(d)$ is $\mathrm{expdim}(I_{X,\mathbf{P}^n}(d)) = \max\left(\binom{n+d}{n} - \deg X, 0\right)$.

For any scheme $X \subset \mathbf{P}^n$ and any hyperplane $H \subseteq \mathbf{P}^n$, the residual of X with respect to H is denoted by $X : H$ and it is defined by the ideal sheaf $\mathcal{I}_{X:H} = \mathcal{I}_X : \mathcal{I}_H$. We have, for any d, the well known *Castelnuovo sequence*

$$0 \to I_{X:H,\mathbf{P}^n}(d-1) \to I_{X,\mathbf{P}^n}(d) \to I_{X\cap H,H}(d).$$

Remark 2.2. If $Y \subseteq X \subseteq \mathbf{P}^n$ are zero-dimensional schemes, then

- if X is d-independent, then so is Y,
- if $h_{\mathbf{P}^n}(Y,d) = \binom{d+n}{n}$, then $h_{\mathbf{P}^n}(X,d) = \binom{d+n}{n}$.

It follows that if any zero-dimensional scheme $X \subseteq \mathbf{P}^n$ with $\deg X = \binom{d+n}{n}$ is d-independent, then any scheme contained in X imposes independent conditions on hypersurfaces of degree d in $\mathbf{P}^n$.

Remark 2.3. Fix $n \geqslant 2$ and $d \geqslant 3$. Assume that if a scheme X with degree $\binom{d+n}{n}$ does not impose independent conditions on hypersurfaces of degree d in $\mathbf{P}^n$, then it is of type $(m_1,\ldots,m_{n+1})$ for some given m_i. It follows that any subscheme of X is d-independent. Indeed any proper subscheme Y of X is also a subscheme of a scheme X' with degree $\binom{d+n}{n}$ and of type $(m'_1,\ldots,m'_{n+1}) \neq (m_1,\ldots,m_{n+1})$, for some m'_i and since X' is d-independent, so is Y. Moreover any scheme Z containing X impose independent conditions on hypersurfaces of degree d if it contains a scheme X'' with degree $\binom{d+n}{n}$ and of type $(m''_1,\ldots,m''_{n+1}) \neq (m_1,\ldots,m_{n+1})$ for some m''_i. Indeed since X'' imposes independent conditions on hypersurfaces of degree d, also Z does.

3. Quadratic polynomials

Assume that X is a general scheme of type $(m_1,\ldots,m_{n+1})$. Let us fix an order on the irreducible components $\xi_1,\ldots,\xi_m$ of X (where $m = \sum m_i$) such that

$$\text{length}(\xi_1) \geqslant \cdots \geqslant \text{length}(\xi_m)$$

and for any $1 \leq i \leq m$ let us denote by l_i the length of ξ_i and by p_i the point where ξ_i is supported. Set $l_i = 0$ for $i > m$. For any $1 \leq i \leq n$ let us denote

$$\delta_X(i) = \max\left\{0, \sum_{j=1}^{i} l_j - \sum_{j=1}^{i}(n+2-j)\right\}.$$

Note that $\delta_X(1) = 0$ for any scheme X. Clearly $\delta_X(2) = 0$ unless X is the union of two double points and in this case $\delta_X(2) = 1$. If $\delta_X(2) = 0$, then $\delta_X(3) = 0$ unless $l_1 = n+1, l_2 = l_3 = n$, where $\delta_X(3) = 1$. If $\delta_X(2) = \delta_X(3) = 0$, then $\delta_X(4) = 0$ unless either $l_1 = n+1, l_2 = n, l_3 = l_4 = n-1$, where $\delta_X(4) = 1$, or $l_1 = l_2 = l_3 = n$ and $l_4 \geqslant n-1$, where $1 \leqslant \delta_X(4) \leqslant 2$.

Lemma 3.1. *If $\delta_X(i) > 0$ for some $1 \leq i \leq n$, then the quadrics containing $\{\xi_1,\ldots,\xi_i\}$ are exactly the quadrics singular along the linear space spanned by $p_1,\ldots,p_i$.*

Proof. Let us denote $\mathbf{P}^n = \mathbf{P}(V)$, fix a basis $\{e_0,\ldots,e_n\}$ of V and assume that $p_j = [e_{n+2-j}]$ for all $j = 1,\ldots,i$. Let A be the symmetric matrix defining a quadric $\mathcal{Q}$ in $\mathbf{P}(V)$ passing through the scheme $\{\xi_1,\ldots,\xi_i\}$. Therefore $\mathcal{Q}$ is defined in V by the equation $\{v \in V : v^T A v = 0\}$ and the condition that the quadric contains ξ_j means that $e^T_{n+2-j}Aw = 0$ for any $w \in W$, where W is a general subspace of V of dimension l_j. Then, it is easy to see that the condition $\sum_{j=1}^{i} l_j \geqslant \sum_{j=1}^{i}(n+2-j)$ implies that the elements of the last i columns and rows of the matrix A are all equal to 0. This implies that the quadric $\mathcal{Q}$ is singular along the linear space spanned by $\{p_1,\ldots,p_i\}$. $\square$

From the previous lemma it follows that if $\delta_X(i)$ is positive for some $1 \leq i \leq n$, then the scheme $\{\xi_1, \ldots, \xi_i\}$ does not impose independent conditions on quadrics. Indeed the scheme $\{\xi_1, \ldots, \xi_i\}$ has degree $\sum_{j=1}^{i} l_j$, but imposes only $\sum_{j=1}^{i}(n+2-j) = \binom{n+2}{2} - \binom{n-i+2}{2}$ conditions on quadrics.

The following result describes all the schemes which impose independent conditions on quadrics, giving necessary and sufficient conditions.

Theorem 3.2. *A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points of type $(m_1, \ldots, m_{n+1})$ imposes independent conditions on quadrics if and only if one of the following conditions takes place:*

(1) *either $\delta_X(i) = 0$ for all $1 \leqslant i \leqslant n$;*
(2) *or $\deg X \geqslant \binom{n+2}{2} + \max\{\delta_X(i) : 1 \leqslant i \leqslant n\}$.*

Proof. First we prove that if X does impose independent conditions on quadrics, then either condition 1 or 2 hold. Assume that both conditions are false and let us prove that $I_X(2)$ has not the expected dimension $\max\left\{0, \binom{n+2}{2} - \deg(X)\right\}$. In particular assume that there is an index $i \in \{1, \ldots, n\}$ such that $\delta_X(i) > 0$ and $\deg(X) < \binom{n+2}{2} + \delta_X(i)$. Consider the family $\mathcal{C}$ of quadratic cones with vertex containing the linear space $\mathbf{P}^{i-1}$ spanned by $p_1, \ldots, p_i$. Of course we have

$$\dim I_X(2) \geqslant \dim(\mathcal{C}) - \left(\deg(X) - \sum_{j=1}^{i} l_j\right) = \binom{n-i+2}{2} - \deg(X) + \sum_{j=1}^{i} l_j =: c$$

Now, using $\binom{n+2}{2} - \binom{n-i+2}{2} = \sum_{j=1}^{i}(n+2-j)$, we compute

$$\dim I_X(2) - \mathrm{expdim} I_X(2) \geqslant \min\left\{c, \sum_{j=1}^{i} l_j - \binom{n+2}{2} + \binom{n-i+2}{2}\right\} = \min\{c, \delta_X(i)\}$$

By assumption $\delta_X(i) > 0$ and

$$c > \binom{n-i+2}{2} - \binom{n+2}{2} - \delta_X(i) + \sum_{j=1}^{i} l_j = \sum_{j=1}^{i} l_j - \sum_{j=1}^{i}(n+2-j) - \delta_X(i) = 0$$

Hence the dimension of $I_X(2)$ is higher than the expected dimension and we have proved that X does not impose independent conditions on quadrics.

Now we want to prove that if either condition 1 or condition 2 hold, then X imposes independent conditions on quadrics. We work by induction on $n \geqslant 2$. If $n = 2$ it is easy to check directly our claim.

Consider a scheme X in $\mathbf{P}^n$ which satisfies condition 1 and fix a hyperplane $H \subset \mathbf{P}^n$. We specialize all the components of X on H in such a way that the residual of each of the components $\xi_1, \ldots, \xi_n$ is 1 (if the component is not empty) and the residual of the remaining components is zero. Indeed the vanishing $\delta_X(2) = 0$ implies that $l_j \leqslant n$ for all $j \geqslant 2$, and so such a specialization is possible. Then we get the Castelnuovo sequence

$$0 \to I_{X:H,\mathbf{P}^n}(1) \to I_{X,\mathbf{P}^n}(2) \to I_{X\cap H,H}(2)$$

where $X : H$ is the residual given by at most n simple points and $X \cap H$ is the trace in H. Hence we conclude by induction once we have proved that the trace $X \cap H$ satisfies condition 1 or 2.

Note that in order to compute $\delta_{X\cap H}(i)$ we need to choose an order on the components $\xi_i \cap H$ of $X \cap H$ such that the sequence of their lengths is not increasing. If

$$\text{length}(\xi_n) - 1 = \text{length}(\xi_n \cap H) \geqslant \text{length}(\xi_{n+1} \cap H) = \text{length}(\xi_{n+1}) \tag{3}$$

then we can choose the same order on the components of $X \cap H$ chosen for the components of X. In this case it is easy to prove that $X \cap H$ satisfies condition 1. Indeed for any $i \geq 1$, let us denote by $l'_i = \text{length}(\xi_i \cap H)$. Recall that m is the number of components of X. By construction we have that $l'_i = l_i - 1$ for any $1 \leqslant i \leqslant \min\{n, m\}$. Then for all $1 \leqslant i \leqslant \min\{n-1, m\}$ we have

$$\sum_{j=1}^{i} l'_j - \sum_{j=1}^{i}(n+1-j) = \sum_{j=1}^{i} l_j - i - \sum_{j=1}^{i}(n+2-j) + i = \sum_{j=1}^{i} l_j - \sum_{j=1}^{i}(n+2-j)$$

from which we have

$$\delta_{X\cap H}(i) = \max\left\{0, \sum_{j=1}^{i} l'_j - \sum_{j=1}^{i}(n+1-j)\right\} = \max\left\{0, \sum_{j=1}^{i} l_j - \sum_{j=1}^{i}(n+2-j)\right\} = \delta_X(i) = 0$$

Now assume that (3) does not hold. This implies in particular that $l_n = l_{n+1}$, and so when we compute $\delta_{X\cap H}(i)$ we have to change the order on the components. In order to better understand the situation, let us consider the following example: X in $\mathbf{P}^5$ given by 9 components of length 4. Note that $\delta_X(i) = 0$ for any $1 \leq i \leq 5$. After the specialization described above we get a scheme $X \cap H$ in $H \cong \mathbf{P}^4$ given by 5 components of length 3 and 4 components of length 4. We easily compute that $\delta_{X\cap H}(4) = 2 > 0$.

Now we will prove that if $X \cap H$ does not satisfy condition 1, then it satisfies 2. Assume that for X in $\mathbf{P}^n$ we have $\delta_X(i) = 0$ for all $1 \leqslant i \leqslant n$, while for $X \cap H$ in H we have $\delta_{X\cap H}(i) > 0$ for some $1 \leqslant i \leqslant n-1$.

Let us denote $l := l_n = l_{n+1}$ and let $1 \leq k < n$ be the index such that $l_k > l_{k+1} = \cdots = l_n = l_{n+1} = l$. Let h be the index such that $\delta_{X\cap H}(h) = \max\{\delta_{X\cap H}(i)\}$ and note that $h > k$.

As above we denote by l'_i the lenghts of the components of $X \cap H$ ordered in a not increasing way. Hence we have,

$$l'_1 = l_1 - 1, \ldots, l'_k = l_k - 1,\ l'_{k+1} = l, \ldots, l'_h = l, \ldots$$

and this implies that $l_k > l_{k+1} = \cdots = l_n = \cdots = l_{n+h-k} = l$.

Now since $\delta_{X\cap H}(h) > 0$ we obtain

$$\sum_{i=1}^{h} l'_i = \sum_{i=1}^{k} l_i - k + (h-k)l > \sum_{i=1}^{h}(n+1-i) = \sum_{i=1}^{h}(n+2-i) - h$$

and since $\delta_X(k) = 0$ we have $\sum_{i=1}^{k} l_i \leqslant \sum_{i=1}^{k}(n+2-i)$, and combining these two inequalities we have

$$(h-k)l > \sum_{i=k+1}^{h}(n+2-i) - (h-k) > (h-k)(n+1-h)$$

from which it follows:

$$l \geqslant (n+2-h). \tag{4}$$

Now in order to prove that $X \cap H$ satisfies 2 we need to show that

$$\deg(X \cap H) \geqslant \binom{n+1}{2} + \delta_{X\cap H}(h).$$

Notice that

$$\deg(X \cap H) \geqslant \deg X - n \geqslant \sum_{i=1}^{n+h-k} l_i - n,$$

hence if we prove the following inequality we are done:

$$\sum_{i=1}^{n+h-k} l_i - n \geqslant \binom{n+1}{2} + \delta_{X\cap H}(h)$$

i.e.

$$\sum_{i=1}^{k} l_i + (n+h-2k)l - n \geqslant \binom{n+1}{2} + \sum_{i=1}^{k} l_i - k + (h-k)l - \sum_{i=1}^{h}(n+1-i)$$

which reduces to

$$(n-k)l \geqslant \binom{n+1}{2} + n - k - h(n+1) + \binom{h+1}{2}$$

By using inequality (4) it is enough to prove, for any $n \geqslant 2$, any $1 \leqslant k < h \leqslant n-1$, the inequality

$$(n-k)(n+2-h) \geqslant \binom{n+1}{2} + (n-k) - h(n+1) + \binom{h+1}{2} \tag{5}$$

and we prove this inequality by induction on $h \leqslant n-1$. First fix n, k and choose $h = n-1$. In this case (5) becomes

$$3(n-k) \geqslant \binom{n+1}{2} + (n-k) - (n^2-1) + \binom{n}{2} = n-k+1$$

which is true. Now if we assume that (5) is verified for $h' \leqslant n-1$, it is easy to check it for $h = h'-1$, thus completing the proof of (5).

It remains to prove that if X satisfies condition 2, then the system of quadrics $|\mathcal{I}_X(2)|$ containing X is empty. If $\delta_X(i) = 0$ for all $1 \leqslant i \leqslant n$ then we are in the previous case. We may assume that there exists i such that $\delta_X(i) > 0$.

Assume that the sequence $\{\delta_X(i)\}$ is nondecreasing Then $\delta_X(n) > 0$ and by Lemma 3.1 we know that the quadrics containing the first n components $\{\xi_1, \ldots, \xi_n\}$ are singular along the hyperplane $H = \langle p_1, \ldots, p_n\rangle$, so the only existing quadric is the double hyperplane H^2. By assumption $\deg X > \left[\binom{n+2}{2} - 1\right] + \delta_X(n) = \sum_{j=1}^{n} l_j$, hence there is at least an extra condition given by another component ξ_{n+1} of X and so $|\mathcal{I}_X(2)| = \emptyset$ as we wanted.

Therefore we may assume that there exists $1 \leqslant i < n$ such that $\delta_X(i+1) < \delta_X(i)$ and we pick the first such i. In particular it follows

$$l_{i+1} < n+1-i \tag{6}$$

As above, by Lemma 3.1 all the quadrics containing $X_0 = \{\xi_1, \ldots, \xi_i\}$ are singular along the linear space $L_0 =< p_1, \ldots, p_i >$. Let $X_1 = X \setminus X_0$. By definition $\deg X_0 = \sum_{j=1}^{i} l_j = \binom{n+2}{2} - \binom{n+2-i}{2} + \delta_X(i)$.

Let π be a general projection from L_0 on a linear space $L_1 \simeq \mathbf{P}^{n-i}$. By (6) we have $\deg X_1 = \deg \pi(X_1)$. Hence there is a bijective correspondence between $|\mathcal{I}_X(2)|$ and $|\mathcal{I}_{\pi(X_1)}(2)| \subseteq |\mathcal{O}_{L_1}(2)|$. By generality we may assume that X_1 is supported outside L_0.

Note that

$$\deg \pi(X_1) - \binom{n-i+2}{2} = \deg X - \binom{n+2}{2} - \delta_X(i) \geq \max_h\{\delta_X(h)\} - \delta_X(i) \geq 0$$

hence if $\delta_{\pi(X_1)}(h) = 0$ for $h = 1, \dots, n-i$ we conclude again by the first case. If there exists $1 \leqslant j \leqslant n-i$ such that $\delta_{\pi(X_1)}(j) > 0$, notice that in this case we have

$$\delta_{\pi(X_1)}(j) = \delta_X(j+i) - \delta_X(i),$$

hence

$$\max_p\{\delta_{\pi(X_1)}(p)\} = \max_h\{\delta_X(h)\} - \delta_X(i).$$

So we proved that

$$\deg \pi(X_1) - \binom{n+2-i}{2} \geq \max_p\{\delta_{\pi(X_1)}(p)\}$$

This means that $\pi(X_1)$ satisfies the assumption 2 on L_1 and then by (complete) induction on n we get that $|\mathcal{I}_{\pi(X_1)}(2)| = \emptyset$ as we wanted. $\square$

A straightforward consequence of the previous theorem is the following corollary.

Corollary 3.3. *A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points with $\deg X = \binom{n+2}{2}$ imposes independent conditions on quadrics if and only if $\delta_X(i) = 0$ for all $1 \leqslant i \leqslant n$.*

Theorem 3.2 provides a classification of all the types of general subschemes X of a collection of double points of $\mathbf{P}^n$ which do not impose independent conditions on quadrics. For example in $\mathbf{P}^2$, the only case is X given by two double points. In Tables 1 and 2 below we list the subschemes which do not impose independent conditions on quadrics in $\mathbf{P}^3$ and $\mathbf{P}^4$.

4. Cubic polynomials

In this section we generalize the approach of [4, Section 3] to our setting and we prove the following result.

Theorem 4.1. *A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on cubics with the only exception of $n = 4$ and X given by 7 double points.*

First we give the proof of the previous theorem in cases $n = 2, 3, 4$.

Lemma 4.2. *Let be $n = 2, 3$ or 4. Then a general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on cubics with the only exception of $n = 4$ and X given by 7 double points.*

Proof. By Remark 2.2 it is enough to prove the statement for X with degree $\binom{n+3}{3}$. Note that if X is a union of double points the statement is true by the Alexander–Hirschowitz theorem.

Let $n = 2$ and X a subscheme of a collection of double points with $\deg X = 10$. Fix a line H in $\mathbf{P}^2$ and consider the Castelnuovo exact sequence

$$0 \to I_{X:H,\mathbf{P}^2}(2) \to I_{X,\mathbf{P}^2}(3) \to I_{X\cap H}(3)$$

It is easy to prove that it is always possible to specialize some components of X on H so that $\deg(X \cap H) = 4$ and that the residual $X : H$ does not contain two double points. The last condition ensures that $\delta_{X:H}(i) = 0$ for $i = 1, 2$. Hence we conclude by Corollary 3.3.

In the case $n = 3$, the scheme X has degree 20. Since there are no cubic surfaces with five singular points (in general position) we can assume that X contains at most three double points. Indeed if X contains 4 double points we can degenerate it to a collection of 5 double points, in general position.

Table 1
List of exceptions in $\mathbf{P}^3$.

X	$\deg X$	$\max\{\delta_X(i)\}$	$(m_1, \ldots, m_4)$	$\dim I_X(2)$
4, 4, 4	12	3	(0, 0, 0, 3)	1
4, 4, 3	11	2	(0, 0, 1, 2)	1
4, 4, 2	10	1	(0, 1, 0, 2)	1
4, 4, 1, 1	10	1	(2, 0, 0, 2)	1
4, 4, 1	9	1	(1, 0, 0, 2)	2
4, 4	8	1	(0, 0, 0, 2)	3
4, 3, 3	10	1	(0, 0, 2, 1)	1

We fix a plane H in $\mathbf{P}^3$ and we want to specialize some components of X on H so that $\deg(X \cap H) = 10$ and that the residual $X : H$ imposes independent conditions on quadrics. By looking at Table 1, since $\deg(X : H) = 10$, it is enough to require that $X : H$ is not of the form $(0, 1, 0, 2)$, $(2, 0, 0, 2)$ or $(0, 0, 2, 1)$. It is easy to check that this is always possible: indeed specialize on H the components of X starting from the ones with higher length and keeping the residual as minimal as possible until the degree of the trace is 9 or 10. If the degree of the trace is 9 and there is in X a component with length 1 or 2 we can obviously complete the specialization. The only special case is given by X of type $(0, 0, 4, 2)$ and in this case we specialize on H the two double points and two components of length 3 so that each of them has residual 1.

If $n = 4$ the case of 7 double points is exceptional. Assume that X has degree 35 and contains at most 6 double points. We fix a hyperplane H of $\mathbf{P}^4$ and we want to specialize some components of X on H so that $\deg(X \cap H) = 20$ and that the residual $X : H$ imposes independent conditions on quadrics. By looking at Table 2, it is enough to require that $X : H$ does not contain two double points, does not contain one double point and two components of length 4 and it is not of the form $(0, 0, 1, 3, 0)$. It is possible to satisfy this conditions by specializing the components of X in the following way: we specialize the components of X on H starting from the ones with higher length and keeping the residual as minimal as possible until the degree of the trace is maximal and does not exceed 20. Then we add some components allowing them to have residual 1 in order to reach the degree 20. It is possible to check that this construction works, except for the case $(0, 0, 5, 0, 4)$ where we have to specialize on H all the double points and 2 of the components with length 3 so that both have residual 1. It is easy also to check that following the construction above the residual has always the desired form, except for X of the form $(0, 0, 1, 8, 0)$, where the above rule gives a residual of type $(0, 0, 1, 3, 0)$. In this case we make a specialization ad hoc: for example we can put on H six components of length 4 and the unique component of length 3 in such a way that all them have residual 1 and we obtain a residual of type $(7, 0, 0, 2, 0)$ which is admissible.

Now we have to check the schemes either contained in 7 double points or containing 7 double points. But this follows immediately by Remark 2.3. $\square$

We want to restrict a zero dimensional scheme X of $\mathbf{P}^n$ to a given subvariety L. We could define the residual $X : L$ as a subscheme of the blow-up of $\mathbf{P}^n$ along L as in [3], but we prefer to consider $\deg(X : L)$ just as an integer associated to X and L. More precisely given a subvariety $L \subset \mathbf{P}^n$, we denote $\deg(X : L) = \deg X - \deg(X \cap L)$. In particular we will use this notion in the following cases:

$$\deg(X : L), \quad \deg(X : (L \cup M)), \quad \deg(X : (L \cup M \cup N))$$

where $L, M, N \subset \mathbf{P}^n$ are three general subspaces of codimension three. We also recall that

$$\deg(X \cap (L \cup M)) = \deg(X \cap L) + \deg(X \cap M) - \deg(X \cap (L \cap M))$$

and

$$\begin{aligned}\deg(X \cap (L \cup M \cup N)) = {} & \deg(X \cap L) + \deg(X \cap M) + \deg(X \cap N) - \deg(X \cap L \cap M) \\ & - \deg(X \cap L \cap N) - \deg(X \cap M \cap N) + \deg(X \cap L \cap M \cap N).\end{aligned}$$

The proof of Theorem 4.1 relies on a preliminary description, which is inspired to the approach of [4]. More precisely the proof is structured as follows:

Table 2
List of exceptions in $\mathbf{P}^4$.

X	$\deg X$	$\max\{\delta_X(i)\}$	$(m_1, \ldots, m_5)$	$\dim I_X(2)$
5, 5, 5, 5	20	6	(0, 0, 0, 0, 4)	1
5, 5, 5, 4	19	5	(0, 0, 0, 1, 3)	1
5, 5, 5, 3	18	4	(0, 0, 1, 0, 3)	1
5, 5, 5, 2	17	3	(0, 1, 0, 0, 3)	1
5, 5, 5, 1, 1	17	3	(2, 0, 0, 0, 3)	1
5, 5, 5, 1	16	3	(1, 0, 0, 0, 3)	2
5, 5, 5	15	3	(0, 0, 0, 0, 3)	3
5, 5, 4, 4	18	4	(0, 0, 0, 2, 2)	1
5, 5, 4, 3	17	3	(0, 0, 1, 1, 2)	1
5, 5, 4, 2	16	2	(0, 1, 0, 1, 2)	1
5, 5, 4, 1, 1	16	2	(2, 0, 0, 1, 2)	1
5, 5, 4, 1	15	2	(1, 0, 0, 1, 2)	2
5, 5, 4	14	2	(0, 0, 0, 1, 2)	3
5, 5, 3, 3	16	2	(0, 0, 2, 0, 2)	1
5, 5, 3, 2	15	1	(0, 1, 1, 0, 2)	1
5, 5, 3, 1, 1	15	1	(2, 0, 1, 0, 2)	1
5, 5, 3, 1	14	1	(1, 0, 1, 0, 2)	2
5, 5, 3	13	1	(0, 0, 1, 0, 2)	3
5, 5, 2, 2, 1	15	1	(1, 2, 0, 0, 2)	1
5, 5, 2, 2	14	1	(0, 2, 0, 0, 2)	2
5, 5, 2, 1, 1, 1	15	1	(3, 1, 0, 0, 2)	1
5, 5, 2, 1, 1	14	1	(2, 1, 0, 0, 2)	2
5, 5, 2, 1	13	1	(1, 1, 0, 0, 2)	3
5, 5, 2	12	1	(0, 1, 0, 0, 2)	4
5, 5, 1, 1, 1, 1, 1	15	1	(5, 0, 0, 0, 2)	1
5, 5, 1, 1, 1, 1	14	1	(4, 0, 0, 0, 2)	2
5, 5, 1, 1, 1	13	1	(3, 0, 0, 0, 2)	3
5, 5, 1, 1	12	1	(2, 0, 0, 0, 2)	4
5, 5, 1	11	1	(1, 0, 0, 0, 2)	5
5, 5	10	1	(0, 0, 0, 0, 2)	6
5, 4, 4, 2	15	1	(0, 1, 0, 2, 1)	1
5, 4, 4, 1, 1	15	1	(2, 0, 0, 2, 1)	1
5, 4, 4, 1	14	1	(1, 0, 0, 2, 1)	2
5, 4, 4	13	1	(0, 0, 0, 2, 1)	3
4, 4, 4, 4	16	2	(0, 0, 0, 4, 0)	1
4, 4, 4, 3	15	1	(0, 0, 1, 3, 0)	1

- in Proposition 4.3 below we generalize [4, Proposition 5.2],
- in Proposition 4.7 and Proposition 4.8 we generalize [4, Proposition 5.3],
- the analogue of [4, Proposition 5.4] is contained in Lemma 4.9, Lemma 4.10, Lemma 4.11 and Proposition 4.12.

Proposition 4.3. *Let $n \geq 8$ and let $L, M, N \subset \mathbf{P}^n$ be general subspaces of codimension 3. Let $X = X_L \cup X_M \cup X_N$ be a general scheme contained in a union of double points, where X_L (resp. X_M, X_N) is supported on L (resp. M, N), such that the triple $(\deg(X_L : L), \deg(X_M : M), \deg(X_N : N))$ is one of the following*

(i) (6, 9, 12)
(ii) (3, 12, 12)
(iii) (0, 12, 15)
(iv) (6, 6, 15)
(v) (0, 9, 18)

then there are no cubic hypersurfaces in $\mathbf{P}^n$ which contain $L \cup M \cup N$ and which contain X.

Proof. For $n = 8$ it is an explicit computation, which can be easily performed with the help of a computer (see the appendix).

For $n \geq 9$ the statement follows by induction on n. Indeed if $n \geq 8$ it is easy to check that there are no quadrics containing $L \cup M \cup N$. Then given a general hyperplane $H \subset \mathbf{P}^n$ the Castelnuovo sequence induces the isomorphism

$$0 \longrightarrow I_{L\cup M\cup N,\mathbf{P}^n}(3) \longrightarrow I_{(L\cup M\cup N)\cap H,H}(3) \longrightarrow 0$$

hence specializing the support of X on the hyperplane H, since the space $I_{L\cup M\cup N,\mathbf{P}^n}(2)$ is empty, we get

$$0 \longrightarrow I_{X\cup L\cup M\cup N,\mathbf{P}^n}(3) \longrightarrow I_{(X\cup L\cup M\cup N)\cap H,H}(3)$$

then our statement immediately follows by induction. □

Remark 4.4. It seems likely that the previous proposition holds with much more general assumption. Anyway the general assumption $\deg(X_L : L) + \deg(X_M : M) + \deg(X_N : N) = 27$ is too weak, indeed the triple $(0, 6, 21)$ cannot be added to the list of the Proposition 4.3. Indeed there are two independent cubic hypersurfaces in $\mathbf{P}^8$, containing L, M, N, two general double points on M and seven general double points on N, as it can be easily checked with the help of a computer (see the appendix). Quite surprisingly, the triple $(0, 0, 27)$ could be added to the list of the Proposition 4.3, and we think that this phenomenon has to be better understood. In Proposition 4.3 we have chosen exactly the assumptions that we will need in the following propositions, in order to minimize the number of the initial checks.

For the specialization technique we need the following two easy remarks.

Remark 4.5. Let L, N be two codimension three subspaces of $\mathbf{P}^n$, for $n \geq 5$. Let ξ be a general scheme contained in a double point p^2 supported on L such that $\deg(\xi : L) = a$, $0 \leq a \leq 3$. Then there is a specialization η of ξ such that the support of η is on $L \cap N$, $\deg(\eta : L) = a$ and $\deg((\eta \cap N) : (L \cap N)) = a$.

Remark 4.6. Let L be a codimension three subspace of $\mathbf{P}^n$. Let X be a scheme contained in a double point p^2.

(i) If $\deg X = n + 1$ then there is a specialization Y of X which is supported at $q \in L$ such that $\deg(Y : L) = 3$.
(ii) If $\deg X = n$ then there are two possible specializations Y of X which are supported at $q \in L$ such that $\deg(Y : L) = 3$ or 2.
(iii) If $\deg X = n - 1$ then there are three possible specializations Y of X which are supported at $q \in L$ such that $\deg(Y : L) = 3, 2$ or 1.
(iv) If $\deg X \leq n - 2$ then there are four possible specializations Y of X which are supported at $q \in L$ such that $\deg(Y : L) = 3, 2, 1$ or 0.

Proposition 4.7. *Let $n \geq 5$ and let $L, M \subset \mathbf{P}^n$ be subspaces of codimension three. Let $X = X_L \cup X_M \cup X_O$ be a scheme contained in a union of double points such that X_L (resp. X_M) is supported on L (resp. M) and it is general among the schemes supported on L (resp. M) and X_O is general. Assume that the following further conditions hold:*

$$\deg(X_L : L) + \deg(X_M : M) + \deg X_O = 9(n-1),$$

$$n - 2 \leq \deg(X_L : L) \leq \deg(X_M : M) \leq 4n - 6,$$

$$3n + 3 \leq \deg X_O \leq 3n + 6.$$

Then there are no cubic hypersurfaces in $\mathbf{P}^n$ which contain $L \cup M$ and which contain X.

Proof. For $n = 5, 6, 7$ it is an explicit computation (see the appendix).

For $n \geq 8$, the statement follows by induction from $n-3$ to n. Indeed given a third general codimension three subspace N, we get the exact sequence

$$0 \longrightarrow I_{L\cup M\cup N,\mathbf{P}^n}(3) \longrightarrow I_{L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{(L\cup M)\cap N,N}(3) \longrightarrow 0$$

where the dimensions of the three spaces in the sequence are respectively $27, 9(n-1)$ and $9(n-4)$.

We will specialize now some components of X_L on $L \cap N$ and some components of X_M on $M \cap N$. We denote by X_L' the union of the components of X_L supported on $L \setminus N$ and by X_L'' the union of the components of X_L supported on $L \cap N$. Since $n \geq 5$ we may assume also that $\deg(X_L'' : (L \cup N)) = 0$. Analogously let X_M' and X_M'' denote the corresponding subschemes of X_M. Now we describe more explicitly the specialization.

From the assumption

$$3n + 3 \leq \deg X_O \leq 3n + 6$$

it follows that in particular X has at least three irreducible components and so we may specialize all the components of X_O on N in such a way that $\deg(X_O : N) = 9$.

Notice that the degree of the trace $X_O \cap N = \deg X_O - 9$ satisfies the same inductive hypothesis

$$3(n-3) + 3 \leq \deg(X_O \cap N) \leq 3(n-3) + 6$$

and we have

$$6n - 15 \leq \deg(X_L : L) + \deg(X_M : M) \leq 6n - 12$$

If $\deg(X_M : M) \leq 3n$, by using that

$$\deg(X_L : L) \leq \frac{1}{2}\left(\deg(X_L : L) + \deg(X_M : M)\right) \leq \deg(X_M : M)$$

we get

$$3n - 7 \leq \deg(X_M : M) \leq 3n$$

$$3n - 15 \leq \deg(X_L : L) \leq 3n - 6$$

then we can specialize X_M and X_L in such a way that $\deg(X_M' : M) = 12$ and $\deg(X_L' : L) = 6$, indeed the conditions

$$n - 5 \leq \deg(X_M : M) - 12 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) - 6 \leq 4n - 18$$

are true for $n \geq 8$ and guarantee that the inductive assumptions are true on the trace.

Now if $\deg(X_M : M) \geq 3n + 1$, we have

$$3n + 1 \leq \deg(X_M : M) \leq 4n - 6$$

$$2n - 9 \leq \deg(X_L : L) \leq 3n - 13$$

and we can specialize in such a way that $\deg(X_L' : L) = 0$ and $\deg(X_M' : M) = 18$. Indeed we have, for $n \geq 6$

$$n - 5 \leq \deg(X_M : M) - 18 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) \leq 4n - 18$$

In any of the previous cases, the residual satisfies the assumptions of Proposition 4.3, while the trace $(X \cup L \cup M) \cap N$ satisfies the inductive assumptions on $N = \mathbf{P}^{n-3}$. In conclusion by using the sequence

$$0 \longrightarrow I_{X\cup L\cup M\cup N,\mathbf{P}^n}(3) \longrightarrow I_{X\cup L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{(X\cup L\cup M)\cap N,N}(3)$$

we complete the proof. $\square$

The following proposition is analogous to the previous one, with a different assumption on $\deg X_O$. In this case we need an extra assumption on X_L and X_M, namely that in one of them there are enough irreducible components with residual different from 2. The reason for this choice is that it makes possible to find a suitable specialization with residual 3, 9 or 15, by the Remark 4.5 (if all the components have residual 2, this should not be possible).

From now on we denote by X_L^i (resp. X_M^i) for $i = 1, 2, 3$ the union of the irreducible components ξ of X_L (resp. X_M) such that $\deg(\xi : L) = i$ (resp. $\deg(\xi : M) = i$).

Proposition 4.8. *Let $n \geq 5$ and let $L, M \subset \mathbf{P}^n$ be subspaces of codimension three. Let $X = X_L \cup X_M \cup X_O$ be a scheme contained in a union of double points such that X_L (resp. X_M) is supported on L (resp. M) and it is general among the schemes supported on L (resp. M) and X_O is general. Assume that either the number of the irreducible components of $X_L^1 \cup X_L^3$, or that the number of the irreducible components of $X_M^1 \cup X_M^3$ is at least $\frac{n-2}{3}$. Assume that the following further conditions hold:*

$$\deg(X_L : L) + \deg(X_M : M) + \deg X_O = 9(n-1),$$

$$n - 2 \leq \deg(X_L : L) \leq \deg(X_M : M) \leq 4n - 6,$$

$$3n + 7 \leq \deg X_O \leq 5n + 2.$$

Then there are no cubic hypersurfaces in $\mathbf{P}^n$ which contain $L \cup M$ and which contain X.

Proof. For $n = 5, 6, 7$ it is an explicit computation (see the appendix), and the thesis is true even without the assumption on $X_L^1 \cup X_L^3$.

For $n \geq 8$ the statement follows by induction from $n-3$ to n, by using possibly also Proposition 4.7. As in the previous proof, given a third general codimension three subspace N, we get the exact sequence

$$0 \longrightarrow I_{L\cup M\cup N,\mathbf{P}^n}(3) \longrightarrow I_{L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{(L\cup M)\cap N,N}(3) \longrightarrow 0$$

We will specialize now some components of X_L on $L \cap N$ and some components of X_M on $M \cap N$. We use the same notations as in the previous proof, and we describe more precisely the specialization in the following two cases.

1. Assume first that

   $$3n + 7 \leq \deg X_O \leq 4n + 7$$

   In particular X has at least four irreducible components and we may specialize all the components of X_O on N in such a way that

   $$\deg((X_O \cap N) : N) = 12$$

   and so we have

   $$5n - 16 \leq \deg(X_L : L) + \deg(X_M : M) \leq 6n - 16$$

   In particular it follows

   $$\frac{5n}{2} - 8 \leq \deg(X_M : M) \leq 4n - 6$$

   $$n - 2 \leq \deg(X_L : L) \leq 3n - 8$$

   We divide into two subcases.

   In the first one we assume that the number of the irreducible components of $X_L^1 \cup X_L^3$ is at least $\frac{n-2}{3}$. In this case we can specialize X_M and X_L in such a way that $\deg(X_M' : M) = 12$ and $\deg(X_L' : L) = 3$. Moreover there exists a specialization such that X_L'' has at least $\frac{n-5}{3} = \frac{n-2}{3} - 1$ components with residual 1 or 3. Indeed in X_L' we keep at most one of these components, and if we are forced to keep three components of length one, it means that there are no components of length 2 in X_L, which implies our claim.

Notice that the conditions

$$n-5 \leq \deg(X_M : M) - 12 \leq 4n-18$$

$$n-5 \leq \deg(X_L : L) - 3 \leq 4n-18$$

are true for $n \geq 10$. They are also true for $n \geq 8$ as soon as $\deg(X_M : M) \geq n+7$, so we need only to check the cases $8 \leq n \leq 9$ and $\deg(X_M : M) \leq n+6$, which implies $\deg(X_L : L) \geq 4n-22$.

In this case we specialize X_M and X_L in such a way that $\deg(X'_M : M) = 6$, $\deg(X'_L : L) = 9$ and X''_L has at least $\frac{n-5}{3} = \frac{n-2}{3} - 1$ components with residual 1 or 3. The conditions

$$n-5 \leq \deg(X_M : M) - 6 \leq 4n-18$$

$$n-5 \leq \deg(X_L : L) - 9 \leq 4n-18$$

are true if $n = 9$ or if $n = 8$ and $\deg(X_L : L) \geq n+4$.

So the remaining cases to be considered are when $n = 8$, $\deg(X_M : M) \leq n+6 = 14$, and $\deg(X_L : L) \leq n+3 = 11$, that is when the triple

$$(\deg(X_L : L), \deg(X_M : M), \deg X_O)$$

is one of the following: $(10, 14, 39)$, $(11, 13, 39)$, $(11, 14, 38)$, which have been checked with random choices (see the appendix) with a computer.

In the second subcase, we know that the number of the irreducible components of $X^1_M \cup X^3_M$ is at least $\frac{n-2}{3}$. Then we can specialize X_M and X_L in such a way that $\deg(X'_M : M) = 9$ and $\deg(X'_L : L) = 6$. As above it is easy to check that there exists a specialization such that X''_M has at least $\frac{n-5}{3} = \frac{n-2}{3} - 1$ components with residual 1 or 3.

Notice that the conditions

$$n-5 \leq \deg(X_M : M) - 9 \leq 4n-18$$

$$n-5 \leq \deg(X_L : L) - 6 \leq 4n-18$$

are true for $n \geq 8$ as soon as one of the following conditions is satisfied

(a) $\deg(X_M : M) \leq 4n-17$, which implies $\deg(X_L : L) \geq n+1$.

(b) $n = 8$, $\deg(X_L : L) \geq n+1 = 9$, which implies $\deg(X_M : M) \leq 5n-17 = 23$

Assume then that (a) and (b) are not satisfied.

We have $4n-16 \leq \deg(X_M : M) \leq 4n-6$ and we specialize X_M and X_L in such a way that $\deg(X'_M : M) = 15$ and $\deg(X'_L : L) = 0$. The conditions

$$n-5 \leq \deg(X_M : M) - 15 \leq 4n-18$$

$$n-5 \leq \deg(X_L : L) \leq 4n-18$$

are true for $n \geq 9$ or if $n = 8$ and $\deg(X_M : M) \geq n+10$.

So the remaining cases to be considered are when $n = 8$, $4n-16 = 16 \leq \deg(X_M : M) \leq n+9 = 17$ and (by case (b)) $\deg(X_L : L) \leq 8$. The only remaining case are

$$(\deg(X_L : L), \deg(X_M : M), \deg X_O) = (7, 17, 39), (8, 16, 39), (8, 17, 38)$$

which we have checked with a computer.

2. Assume now that

$$4n+8 \leq \deg X_O \leq 5n+2$$

which implies

$$4n-11 \leq \deg(X_L : L) + \deg(X_M : M) \leq 5n-17$$

In particular X has at least five irreducible components and we may specialize all the components of X_O on N in such a way that $\deg((X_O \cap N) : N) = 15$.

In this case we have

$$2n - 5 \leq \deg(X_M : M) \leq 4n - 6$$

$$n - 2 \leq \deg(X_L : L) \leq \frac{5n - 17}{2}$$

and we can specialize X_M and X_L in such a way that $\deg(X'_M : M) = 12$ and $\deg(X'_L : L) = 0$. Notice that the conditions

$$n - 5 \leq \deg(X_M : M) - 12 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) \leq 4n - 18$$

are true for $n \geq 12$ and also for $n \geq 8$ as soon as $\deg(X_M : M) \geq n + 7$.

Assume now that $8 \leq n \leq 11$ and $\deg(X_M : M) \leq n+6$, which implies $\deg(X_L : L) \geq 3n-17$.

In this case we specialize X_M and X_L in such a way that $\deg(X'_M : M) = 6$ and $\deg(X'_L : L) = 6$. The conditions

$$n - 5 \leq \deg(X_M : M) - 6 \leq 4n - 18$$

$$n - 5 \leq \deg(X_L : L) - 6 \leq 4n - 18$$

are true for $n \geq 9$ and also for $n = 8$ if $\deg(X_L : L) \geq n + 1$.

The only remaining cases to be considered are then

$n = 8, 7 \leq \deg(X_L : L) \leq 8$, and $\deg(X_M : M) \leq n + 6 = 14$ that is when the triple

$$(\deg(X_L : L), \deg(X_M : M), \deg X_O)$$

is one of the following: $(7, 14, 42)$, $(8, 13, 42)$, $(8, 14, 41)$ which we have checked with a computer.

In conclusion in any previous case we conclude by using the sequence

$$0 \longrightarrow I_{X \cup L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{X \cup L \cup M, \mathbf{P}^n}(3) \longrightarrow I_{(X \cup L \cup M) \cap N, N}(3)$$

since the trace $(X \cup L \cup M) \cap N$ satisfies the inductive assumptions on $N = \mathbf{P}^{n-3}$ and the residual satisfies the hypotheses of Proposition 4.3. $\square$

Let $X_O \subset \mathbf{P}^n$ be a scheme, contained in a union of double points, of degree $(n + 1)^2 + \alpha$ with $0 \leq \alpha \leq n - 1$ and M be a subspace of codimension three. Assume that $n \geq 8$ and that X_O contains at most one component of degree ≤ 3. Let h_i be the number of components of X_O of degree i for $i = 4, \dots, n + 1$ and let h $(0 \leq h \leq 3)$ be the degree of the component of X_O of degree ≤ 3. Note that $\sum_{i=4}^{n+1} ih_i + h = (n + 1)^2 + \alpha$. Let us choose an order on the irreducible components of X_O in such a way the length of any component is non increasing.

We consider one of the following two specializations $X_O = X'_O \cup X_M$ where X_M is supported on M and it contains the possible component of degree ≤ 3, and X'_O is supported outside M:

(a) we choose as X'_O the union of the irreducible components of X_O, starting from the ones with maximal length, in such a way that $\deg X'_O = 3(n + 1) + \beta \geq 3(n + 1) + \alpha$ and it is minimal. By construction $0 \leq \beta - \alpha \leq n$. Let a_i be the number of components of $X_M = X_O \setminus X'_O$ of degree i for $i = 4, \dots, n + 1$. Then

$$\sum_{i=4}^{n+1} ia_i + h = \deg(X_M) = (n + 1)(n - 2) + \alpha - \beta$$

($\widehat{a}$) we choose as X'_O the union of the irreducible components of X_O, starting from the ones with maximal length, in such a way that $\deg X'_O = 3(n+1)+\widehat{\beta} \geq 3(n+1)$ and it is minimal. By construction

$0 \leq \widehat{\beta} \leq n-1$. Let $\widehat{a}_i$ be the number of components of $X_M = X_O \setminus X'_O$ of degree i for $i = 4, \ldots, n+1$. Then

$$\sum_{i=4}^{n+1} i\widehat{a}_i + h = \deg(X_M) = (n+1)(n-2) + \alpha - \widehat{\beta}$$

In both the specializations let us denote: $\gamma = \deg(X_M \cap M) - (n-2)^2$ and note that we have some freedom to specialize X_M on M, according to Remark 4.6. If we have a specialization with $\deg(X_M \cap M) = p$ and another specialization with $\deg(X_M \cap M) = q$ then for any value between p and q there is a suitable specialization such that $\deg(X_M \cap M)$ attains that value. We will use often this technique by evaluating the maximum (resp. the minimum) possible value of $\deg(X_M \cap M)$ under a specialization.

Lemma 4.9. *If in the specialization* (a) *we have*

$$a_n + 2a_{n-1} + 3\sum_{i=4}^{n-2} a_i \leq 1$$

then we have $a_{n+1} \neq 0$ *and there exists a specialization of type* (a) *such that* $\gamma = \alpha \leq n-4$.

Proof. From the assumptions it follows that $a_j = 0$ for any $j = 4, \ldots, n-1$ and $a_n = i$ with $0 \leq i \leq 1$. Then X_M consists of points of maximal length $n+1$ with at most one component of of length h and at most one component of length n. Hence X'_O consists only of double points and this implies that β is a multiple of $n+1$. Hence we have $a_{n+1} = \frac{(n+1)(n-2)+\alpha-\beta-h-in}{n+1}$, which is an integer, so that $\frac{\alpha-h-i(n+1)+i}{n+1}$ is an integer, so that $\alpha = h - i \leq n-4$.

It follows that $a_{n+1} = n-2-i$, hence the maximum degree of $X_M \cap M$ is $(n-2)^2 + h$, the minimum degree is $(n-2-i)(n-2) + i(n-3) + (h-1) = (n-2)^2 + (h-i-1)$, and we can choose $\gamma = h - i = \alpha$. $\square$

Lemma 4.10. *If in the specialization* (a) *we have*

$$3a_{n+1} + 2a_n + a_{n-1} \geq 3n - 7 + \alpha - \beta$$

then there exists a specialization of type $(\widehat{a})$ *such that either* $\gamma = \alpha \leq n-4$ *or* $\gamma = \alpha - 3 \leq n-4$.

Proof. Assume first $a_{n+1} = 0$. Since $\alpha - \beta \geq -n$, from the assumption it follows

$$2a_n + a_{n-1} \geq 2n - 7$$

Notice also that

$$a_n + a_{n-1} \leq \frac{(n+1)(n-2) + \alpha - \beta}{n} \leq n - 2 + \frac{n-2}{n}$$

hence

$$a_n + a_{n-1} \leq n-2.$$

These two conditions imply that we have only the following possibilities:

$$(a_n, a_{n-1}) \in \{(n-2, 0)(n-3, 0), (n-4, 1), (n-3, 1), (n-4, 2), (n-5, 3)\}$$

In all these cases, by performing the specialization of type $(\widehat{a})$, we have $n-3 \leq \widehat{\beta} \leq n-1$ or $\widehat{\beta} = 0$. Moreover it is easy to check that $\widehat{a}_n = a_n$ if $\alpha \leq \widehat{\beta}$, $\widehat{a}_n = a_n + 1$ if $\alpha > \widehat{\beta}$, and $\widehat{a}_j = a_j$ for any $j \leq n-1$. In any case the difference δ between the maximum degree of the trace $X_M \cap M$ and the minimum degree satisfies

$$\delta \geq \widehat{a}_n + 2\widehat{a}_{n-1} + 3\sum_{i=4}^{n-2} \widehat{a}_i + \max\{h-1, 0\}.$$

We have $\deg(X_M) = \sum_{i=4}^{n} i\widehat{a}_i + h = (n+1)(n-2) + \alpha - \widehat{\beta}$ and so

$$\sum_{i=4}^{n-2} i\widehat{a}_i + h \geq (n+1)(n-2) - \widehat{\beta} - n\widehat{a}_n - (n-1)\widehat{a}_{n-1}.$$

In the first two cases, where $(a_n, a_{n-1}) = (a, 0)$ and $n-3 \leq a \leq n-2$, we assume first $n-3 \leq \widehat{\beta}$, then the maximal degree of the trace $X_M \cap M$ is

$$(n-2)^2 + \alpha + 1 \leq (n+1)(n-2) + \alpha - \widehat{\beta} - 2\widehat{a}_n \leq (n-2)^2 + \alpha + 3$$

since $\widehat{a}_n \geq n-3$, moreover $\delta \geq n-2 \geq 6$ and so we have that either $\gamma = \alpha$, or $\gamma = \alpha - 3$ work. It remains the case $\widehat{\beta} = 0$ where we get that in X'_0 we have three points of length $n+1$, then either $\beta = 0$ and $\alpha = 0$, or $\beta = n$ and $\alpha > 0$. By substituting in the hypothesis of our lemma the values $(a_{n+1}, a_n, a_{n-1}) = (0, a, 0)$ we get $\beta = n$ and $0 < \alpha \leq 3$. In this case the maximal degree $\mathcal{M}$ of the trace $X_M \cap M$ satisfies

$$(n+1)(n-2) + \alpha + (n-4) \leq \mathcal{M} \leq (n-2)^2 + \alpha + (n-2)$$

and, since $\delta \geq n-2$, the choice $\gamma = \alpha$ works.

Now consider the case $(a_n, a_{n-1}) = (a, 1)$, where $n-4 \leq a \leq n-3$. Assume first $n-3 \leq \widehat{\beta}$, then the maximal degree of the trace $X_M \cap M$ is

$$(n-2)^2 + \alpha \leq (n+1)(n-2) + \alpha - \widehat{\beta} - 2\widehat{a}_n - 1 \leq (n-2)^2 + \alpha + 4$$

since $n-4 \leq \widehat{a}_n \leq n-2$, moreover $\delta \geq n-1 \geq 7$ so that either $\gamma = \alpha$, or $\gamma = \alpha - 3$ work. It remains the case $\widehat{\beta} = 0$, where we have either $\beta = 0$ and $\alpha = 0$, or $\beta = n$ and $\alpha > 0$. By substituting in the hypothesis of our lemma the values $(a_{n+1}, a_n, a_{n-1}) = (0, a, 1)$, for $n-4 \leq a \leq n-3$, we get $\beta = n$ and $0 < \alpha \leq 2$. Then we have $\widehat{a}_n = n-2$ and so the maximal degree of the trace $X_M \cap M$ is

$$(n+1)(n-2) + \alpha - 2(n-2) - 1 = (n-2)^2 + \alpha + (n-3)$$

and since the difference $\delta \geq n-1$, the choice $\gamma = \alpha$ works.

In the case $(a_n, a_{n-1}) = (n-4, 2)$, if $n-3 \leq \widehat{\beta}$, then the maximal degree of the trace $X_M \cap M$ is

$$(n-2)^2 + \alpha + 1 \leq (n+1)(n-2) + \alpha - \widehat{\beta} - 2\widehat{a}_n - 2 \leq (n-2)^2 + \alpha + 3$$

and since $\delta \geq n \geq 6$ it follows that either $\gamma = \alpha$, or $\gamma = \alpha - 3$ work. It remains the case $\widehat{\beta} = 0$ where $\beta = 0$ or $\beta = n$. By substituting in the hypothesis of our lemma the values $(a_{n+1}, a_n, a_{n-1}) = (0, n-4, 2)$ we get $\beta = n$ and $\alpha = 1$. In this case the maximal degree of the trace $X_M \cap M$ is

$$(n+1)(n-2) + 1 - 2(n-3) - 2 = (n-2)^2 - 1 + n$$

and since $\delta \geq n+1$ we can choose $\gamma = \alpha = 1$.

In the last case $(a_n, a_{n-1}) = (n-5, 3)$, if $n-3 \leq \widehat{\beta}$, then the maximal degree of the trace $X_M \cap M$ is

$$(n-2)^2 + \alpha \leq (n+1)(n-2) + \alpha - \widehat{\beta} - 2\widehat{a}_n - 3 \leq (n-2)^2 + \alpha + 4$$

and since $\delta \geq n \geq 7$ it follows that either $\gamma = \alpha$, or $\gamma = \alpha - 3$ work. It remains the case $\widehat{\beta} = 0$ where $\beta = 0$ or $\beta = n$. By substituting in the hypothesis of our lemma the values $(a_{n+1}, a_n, a_{n-1}) = (0, n-5, 3)$ we get $\beta = n$ and $\alpha = 0$, which is a contradiction. Then this case is impossible.

Now assume that $a_{n+1} \neq 0$. In this case we have also $\beta = 0$, hence it follows $\widehat{\beta} = 0$ and $\widehat{a}_j = a_j$ for any $4 \leq j \leq n+1$. By assumption we have

$$3a_{n+1} + 2a_n + a_{n-1} \geq 3n - 7$$

and, as in the first case, we also have

$$a_{n+1} + a_n + a_{n-1} \leq n - 2$$

These two inequalities imply that (a_{n+1}, a_n, a_{n-1}) lies in the tetrahedron with vertices $(n-2, 0, 0)$, $(n-3, 1, 0)$, $(n-\frac{7}{3}, 0, 0)$, $(n-\frac{5}{2}, 0, \frac{1}{2})$. The only integer points in this tetrahedron are $(n-2, 0, 0)$ and $(n-3, 1, 0)$.

In the case $(n-2, 0, 0)$ the maximal degree of the trace $X_M \cap M$ is

$$(n+1)(n-2) + \alpha - 3(n-2) = (n-2)^2 + \alpha$$

and clearly the minimal degree is $(n-2)^2$, thus one of the choices $\gamma = \alpha$ or $\gamma = \alpha - 3$ works. In the case $(n-3, 1, 0)$ the maximal degree of the trace $X_M \cap M$ is

$$(n+1)(n-2) + \alpha - 3(n-3) - 2 = (n-2)^2 + \alpha + 1$$

and the minimal degree is obviously $(n-2)^2$, so that one of the choices $\gamma = \alpha$ or $\gamma = \alpha - 3$ works. $\square$

Lemma 4.11. *If all the assumptions of Lemma 4.9 and Lemma 4.10 are not satisfied, then there exists $\gamma' \geq 0$ satisfying $\gamma' + 2 \leq n - 4$, and every $\gamma \in [\gamma', \gamma' + 2]$ can be attained by a convenient specialization of type (a).*

Proof. The maximal degree of the trace $X_M \cap M$ is

$$\mathcal{M} := (n+1)(n-2) + \alpha - \beta - 3a_{n+1} - 2a_n - a_{n-1}$$

Since the assumption of Lemma 4.10 are not satisfied, we have $\mathcal{M} \geq (n-2)^2 + 2$.

The minimal possible degree of the trace $X_M \cap M$ is

$$\begin{aligned} m &:= \sum_{i=4}^{n+1} (i-3)a_i + \min\{1, h\} = (n+1)(n-2) + \alpha - \beta - 3\sum_{i=4}^{n+1} a_i + \min\{1-h, 0\} \\ &\leq (n+1)(n-2) - 3\sum_{i=4}^{n+1} a_i \leq (n+1)(n-2) - 3(n-2) = (n-2)^2 \end{aligned}$$

where we use the fact that $\sum_{i=4}^{n+1} a_i \geq n-2$. This is true because either $a_{n+1} = n-2$ or $a_{n+1} \leq n-3$ and we have

$$\sum_{i=4}^{n} a_i \geq \frac{(n+1)(n-2-a_{n+1}) + \alpha - \beta}{n} > n - 2 - a_{n+1} - 1.$$

Hence if $\mathcal{M} \leq n-4$ we choose $\gamma' = \mathcal{M} - (n-2)^2 - 2$. Otherwise if $\mathcal{M} \geq n-3$ we choose $\gamma' = n-6$.

Both cases work because of the assumption

$$\mathcal{M} - m = a_n + 2a_{n-1} + 3\sum_{i=4}^{n-2} a_i - \min\{1-h, 0\} \geq 2. \quad \square$$

We can now prove the last preliminary proposition. Recall that we denote by X_L^i for $i = 1, 2, 3$ the union of the irreducible components ξ of X_L such that $\deg(\xi : L) = i$.

Proposition 4.12. *Let $n \geq 5$ and let $L \subset \mathbf{P}^n$ be a subspace of codimension three. Let $X = X_L \cup X_O$ be a scheme contained in a union of double points such that X_L is supported on L and is general among the schemes supported on L and X_O is general. Assume that $\deg(X_L : L) + \deg X_O = \binom{n+3}{3} - \binom{n}{3} = \frac{3}{2}n^2 + \frac{3}{2}n + 1$, and that $\deg X_O = (n+1)^2 + \alpha$, for $0 \leq \alpha \leq n-1$. We also assume that the number of the irreducible components of $X_L^1 \cup X_L^3$ is $\geq \frac{n}{3}$. Then there are no cubic hypersurfaces in $\mathbf{P}^n$ which contain L and which contain X.*

Proof. For $n = 5, 6, 7$ it is a direct computation (see the appendix).

For $n \geq 8$ the statement follows by induction, and by the sequence

$$0 \longrightarrow I_{L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{L,\mathbf{P}^n}(3) \longrightarrow I_{L\cap M,M}(3) \longrightarrow 0$$

where M is a general codimension three subspace. We get

$$0 \longrightarrow I_{X\cup L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{X\cup L,\mathbf{P}^n}(3) \longrightarrow I_{(X\cup L)\cap M,M}(3).$$

First by Lemmas 4.9, 4.10, 4.11 we can specialize $X_O = X'_O \cup X_M$ in such a way that $\deg X'_O = 3(n+1)+\beta$ (we will call in the following $\widehat{\beta} = \beta$), X_M is supported on M and $\deg(X_M \cap M) = (n-2)^2+\gamma$, where $0 \leq \beta \leq 2n-1$, $0 \leq \gamma \leq n-4$, $\gamma = \alpha \pmod 3$ and $\alpha - \beta - n \leq \gamma \leq \alpha$. Notice also that we have $\alpha - \beta - \gamma \geq -2n+4$. It follows that

$$n-2 \leq \deg(X_M : M) = 3(n-2)+\alpha-\beta-\gamma \leq 4n-6$$

Moreover let us specialize $X_L = X'_L \cup X''_L$ where X'_L is supported on $L \setminus M$ and X''_L is supported on $L \cap M$. We may also assume that the number of irreducible components of $(X''_L)^1 \cup (X''_L)^3$ is $\geq \frac{n-3}{3}$. We may assume that

$$2n-5 \leq \deg(X'_L : L) = 3(n-2)+\gamma-\alpha \leq 3(n-2)$$

indeed note that $3(n-2)+\gamma-\alpha = 0 \pmod 3$ and there exist at least $\frac{n}{3}$ irreducible component in $(X'_L)^1 \cup (X'_L)^3$. Note that by using the minimal number of irreducible component in $(X'_L)^1 \cup (X'_L)^3$, at least $\frac{n}{3}-1$ components remain in X''_L, preserving our inductive assumption. It follows that

$$\deg(X'_L : L) + \deg(X_M : M) + \deg X'_O = 9(n-1)$$

moreover we have clearly

$$4n-11 \leq \deg(X'_L : L) + \deg(X_M : M) \leq 6n-12$$

and we may apply Proposition 4.7 and Proposition 4.8, since the scheme $X'_L \cup X_M \cup X'_O$ satisfies the corresponding assumptions. Then we conclude by induction, indeed the scheme $(X_M \cup X''_L) \cap M$ satisfies our assumptions with respect to the spaces M and $M \cap L \subset M$. Precisely we have (by subtraction)

$$\deg((X''_L \cap M) : (L\cap M)) + \deg(X_M \cap M) = \frac{3}{2}(n-3)^2 + \frac{3}{2}(n-3)+1,$$

and $\deg(X_M \cap M) = (n-2)^2+\gamma$, where $0 \leq \gamma \leq n-4$ $\square$

We are finally in position to give the proof of the main theorem.

Proof of Theorem 4.1. We fix a codimension three linear subspace $L \subset \mathbf{P}^n$ and we prove the statement by induction by using the exact sequence

$$0 \longrightarrow I_{L,\mathbf{P}^n}(3) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^n}(3)) \longrightarrow H^0(\mathcal{O}_L(3)).$$

We prove the claim by induction on n from $n-3$ to n. By Lemma 4.2 we know that the theorem holds for $n = 2, 3, 4$. Let X be a general scheme contained in a collection of double points and with $\deg X = \binom{n+3}{3}$

Since $n \geqslant 5$ we can assume that X contains at most one component of length ≤ 3. Fix a codimension three linear subspace $L \subset \mathbf{P}^n$ and consider the exact sequence

$$0 \longrightarrow I_{X\cup L,\mathbf{P}^n}(3) \longrightarrow I_{X,\mathbf{P}^n}(3) \longrightarrow I_{X\cap L,L}(3) \tag{7}$$

We want to specialize on L some components of X so that $\deg(X \cap L) = \binom{n}{3}$ and apply Proposition 4.12.

We keep outside L the irreducible components of X starting from the ones with maximal length in such a way that $\deg X_O = (n+1)^2 + \alpha \geq (n+1)^2$ and it is minimal. We get by construction that $\alpha \leq n-1$. Let a_i be the number of components of $X_L = X \setminus X_O$ of degree i for $i = 4, \ldots, n+1$ and let h be the degree of the component of X of length ≤ 3. Then $\sum_{i=4}^{n+1} ia_i + h = \binom{n+3}{3} - (n+1)^2 - \alpha$.

After the specialization, the minimum degree of the trace $X_L \cap L$ is

$$\sum_{i=4}^{n+1}(i-3)a_i + 1 = \binom{n+3}{3} - (n+1)^2 - \alpha - h - 3\sum_{i=4}^{n+1} a_i + 1$$

if $h \geq 1$ or

$$\sum_{i=4}^{n+1}(i-3)a_i = \binom{n+3}{3} - (n+1)^2 - \alpha - 3\sum_{i=4}^{n+1} a_i$$

if $h = 0$. On the other hand the maximum degree of the trace $X_L \cap L$ is

$$\binom{n+3}{3} - (n+1)^2 - \alpha - 3a_{n+1} - 2a_n - a_{n-1}$$

We want to prove that $\binom{n}{3}$ belongs to the range between the minimum and the maximum of $\deg(X_L \cap L)$. This is implied by the inequalities

$$\alpha + 3a_{n+1} + 2a_n + a_{n-1} \leq \frac{n(n-1)}{2} \tag{8}$$

and

$$\frac{n(n-1)}{2} \leq \alpha + h + 3\sum_{i=4}^{n+1} a_i - 1, \quad \text{or} \quad \frac{n(n-1)}{2} \leq \alpha + 3\sum_{i=4}^{n+1} a_i \tag{9}$$

In order to prove the inequality (8), consider first the case $a_{n+1} \neq 0$. Then $\alpha = 0$ and we have

$$a_{n+1} + \frac{2}{3}a_n + \frac{1}{3}a_{n-1} \leq \frac{1}{n+1}\sum_{i=4}^{n+1} ia_i = \frac{1}{n+1}\left[\binom{n+3}{3} - (n+1)^2 - h\right]$$
$$= \frac{n(n-1)}{6} - \frac{h}{n+1} \leq \frac{n(n-1)}{6}$$

as we wanted. If $a_{n+1} = 0$ we get

$$2a_n + a_{n-1} + \alpha \leq \frac{2}{n}\sum_{i=4}^{n+1} ia_i + \alpha = \frac{2}{n}\left[\binom{n+3}{3} - (n+1)^2 - h - \alpha\right]$$
$$+\alpha \leq \frac{2}{n}\left[\binom{n+3}{3} - (n+1)^2\right] + (n-1)\left(1 - \frac{2}{n}\right)$$

which is $\leq \frac{n(n-1)}{2}$ if $n \geq 6$, as we wanted. In order to prove the inequality (9), notice that

$$\sum_{i=4}^{n+1} a_i \geq \frac{1}{n+1}\sum_{i=4}^{n+1} ia_i = \frac{n(n-1)}{6} - \frac{\alpha+h}{n+1}$$

then if $h = 0$ we conclude since $\alpha\left(1 - \frac{3}{n+1}\right) \geq 0$, while if $h \geq 1$ we conclude by the inequality $(\alpha + h)(1 - \frac{3}{n+1}) \geq 1$, which is true if $\alpha + h \geq 2$, in particular if $\alpha \geq 1$.

Consider the last case $\alpha = 0$ and $h \geq 1$. If $n \neq 2 \pmod 3$, so that $\frac{n(n-1)}{6}$ is an integer, then $X \setminus X_O$ contains at least $\frac{n(n-1)}{6} + 1$ irreducible components and this confirms the inequality. If $n = 2 \pmod 3$, even $\lfloor \frac{n(n-1)}{6} \rfloor$ double points and one component of length 3 are not enough to cover all $X \setminus X_O$. Then $X \setminus X_O$ contains at least $\lfloor \frac{n(n-1)}{6} \rfloor + 2$ irreducible components and again the inequality is confirmed.

Then a suitable specialization of X_L exists such that $\deg(X_L \cap L) = \binom{n}{3}$. We denote again by X_L^i for $i = 1, 2, 3$ the union of irreducible components ξ of X_L such that $\deg(\xi : L) = i$.

In order to apply Proposition 4.12 we need only to show that the irreducible components of $X_L^1 \cup X_L^3$ are at least $\frac{n}{3}$. If this condition is not satisfied, we show now that it is possible to choose another suitable specialization such that again $\deg(X_L \cap L) = \binom{n}{3}$ but the number of irreducible components of $X_L^1 \cup X_L^3$ is $\geq \frac{n}{3}$. We assume that the number of irreducible components of $X_L^1 \cup X_L^3$ is $\leq \frac{n}{3}$. Indeed we may perform the following operations, that leave the degree of the trace and of the residual both constant.

- Pull out a component from X_L^2 to X_L^3 and push down another component from X_L^2 to X_L^1.
- Pull out a component from X_L^2 to X_L^3 and push down a component of X_L^1 .
- Pull out two components from X_L^2 to X_L^3 and push down a component from X_L^3 to X_L^1.

After such operations have been performed, we get that X_L is still a specialization of a subscheme of X, allowing our semicontinuity argument.

If none of the above operations can be performed, then X_L^1 contains only a_{n-1} components of length $n-1$, X_L^2 contains only a'_n components of length n, X_L^3 contains only a''_n components of length n and a_{n+1} components of length $n+1$.

Then we get

$$\deg(X_L : L) = a_{n-1} + 2a'_n + 3a''_n + 3a_{n+1} = \frac{n(n-1)}{2} - \alpha$$

hence

$$a'_n = \frac{n(n-1)}{4} - \frac{\alpha}{2} - \frac{a_{n-1}}{2} - \frac{3a''_n}{2} - \frac{3a_{n+1}}{2} \geq \frac{n(n-1)}{4} - \frac{\alpha}{2} - \frac{3}{2}\left(a_{n-1} + a''_n + a_{n+1}\right)$$

On the other hand, we have also

$$\begin{aligned}
\deg(X_L \cap L) = \binom{n}{3} &\geq (n-2)\left(a_{n-1} + a'_n + a''_n + a_{n+1}\right) \\
&\geq (n-2)\left[\frac{n(n-1)}{4} - \frac{\alpha}{2} - \frac{1}{2}(a_{n-1} + a''_n + a_{n+1})\right] \\
&> (n-2)\left[\frac{n(n-1)}{4} - \frac{n-1}{2} - \frac{n-1}{6}\right] \geq \binom{n}{3}
\end{aligned}$$

where the last inequality is true for $n \geq 8$. This contradiction concludes the proof. $\square$

5. Induction

In order to prove Theorem 1.1 we will work by induction on the dimension and the degree. In the following lemmas we describe case by case the initial and special instances, while in Theorem 5.6 below we present the general inductive procedure, which involves the differential Horace method.

Lemma 5.1. *A general zero-dimensional scheme $X \subset \mathbf{P}^2$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^2}(d)$ for any $d \geqslant 4$, with the only exception of $d = 4$ and X given by the collection of 5 double points.*

Proof. Assume that X is a general subscheme of a union of double points with $\deg(X) = \binom{d+2}{2}$. If X is a collection of double points the statement follows from the Alexander–Hirschowitz theorem on $\mathbf{P}^2$ (for an easy proof see for example [4, Theorem 2.4]).

If X is not a collection of double points, fix a hyperplane $\mathbf{P}^1 \subset \mathbf{P}^2$. Note that since $\deg(X) = \binom{d+2}{2}$ and $d \geq 4$, then X has at least $d+1$ components. Since X contains at least a component of length 1 or 2, it is clearly always possible to find a specialization of X such that the trace has degree exactly $d+1$. Then we conclude by induction from the Castelnuovo sequence:

$$0 \to I_{X:\mathbf{P}^1,\mathbf{P}^2}(d-1) \to I_{X,\mathbf{P}^2}(d) \to I_{X\cap\mathbf{P}^1,\mathbf{P}^1}(d).$$

Notice that any subscheme of 5 double points and any scheme containing 5 double points impose independent conditions on quartics, by Remark 2.3. $\square$

We give now an easy technical lemma that we need in the following.

Lemma 5.2. *Assume that X is a general zero-dimensional scheme contained in a union of double points of $\mathbf{P}^n$, which contains at least $n-1$ components of length less than or equal to n. Then if $\deg(X) = \binom{n+d}{n}$ it is possible to specialize some components of X on a fixed hyperplane $\mathbf{P}^{n-1}$ in such a way that $\deg(X \cap \mathbf{P}^{n-1}) = \binom{n-1+d}{n-1}$.*

Proof. By assumption there exist at least $n-1$ components $\{\eta_1, \ldots, \eta_{n-1}\}$ with $\mathrm{length}(\eta_i) \leqslant n$. Specialize $\eta_1, \ldots, \eta_{n-1}$ on the hyperplane $\mathbf{P}^{n-1}$ in such a way that the residual of each component is zero. Then specialize other components so that

$$\delta = \binom{n-1+d}{n-1} - \deg(X \cap \mathbf{P}^{n-1}) \geqslant 0$$

is minimal. If $\delta = 0$ the claim is proved, so assume $\delta \geqslant 1$. Obviously we have $\delta < k-1 \leqslant n$, where k is the minimal length of the components of X which lie outside $\mathbf{P}^{n-1}$. Let ζ be a component with length k. Now we make the first components $\eta_1, \ldots, \eta_{k-1-\delta}$ having residual 1 with respect to $\mathbf{P}^{n-1}$ and we specialize ζ on $\mathbf{P}^{n-1}$ with residual 1. Notice that this is possible since $0 < k-1-\delta \leqslant n-1$. $\square$

Lemma 5.3. *Fix $3 \leqslant n \leqslant 4$. A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(4)$, with the following exceptions:*

- *$n = 3$ and either X is the union of 9 double points, or X is the union of 8 double points and a component of length 3;*
- *$n = 4$ and X is the union of 14 double points.*

Proof. If X is a collection of double points, the statement holds by the Alexander–Hirschowitz theorem. We may assume that X is a scheme with degree $\binom{n+4}{4}$ which is not a union of double points. Let us denote by D the number of double points in X and by C the number of the components with length less than or equal to n.

If $n = 3$ and $C = 1$, then $D = 8$ and X is one exceptional case of the statement. If $n = 3$ and $C = 2$, then $D = 8$ and the two components η_1 and η_2 with length less than or equal to 3 have necessarily length 1 and 2. In this case we specialize X on $\mathbf{P}^2$ in such a way that the trace is given exactly by the union of η_1, η_2 and by the intersection of 4 of the 8 double points with $\mathbf{P}^2$. Hence we conclude by the Castelnuovo sequence

$$0 \to I_{X:\mathbf{P}^2,\mathbf{P}^3}(3) \to I_{X,\mathbf{P}^3}(4) \to I_{X\cap\mathbf{P}^2,\mathbf{P}^2}(4) \qquad (10)$$

and by induction. If $C \geqslant 3$, then we denote by η the component of X with minimal length. We specialize η on $\mathbf{P}^2$ in such a way that its residual is 1 if $\mathrm{length}(\eta) \geqslant 2$, and 0 if η is a simple point. Then we apply the construction of Lemma 5.2 on $X \setminus \eta$ (which has at least two components with length less than or equal to 3) and we obtain a trace different from 5 double points. Hence we conclude again by the Castelnuovo sequence (10) and by induction.

If $n = 4$ and $C = 2$, then X is given either by the union of 13 double points, a component of length 3 and one of length 2, or by the union of 13 double points, a component of length 4 and a simple point. In the first case we specialize X obtaining a trace given by 8 double points, a component of length 2 and a simple point. Then we conclude by induction as before. In the second case we cannot use the Castelnuovo sequence since we would obtain an exceptional case. In order to conclude we prove that a general union of 13 double points and a component of length 4 imposes independent conditions on quartics. Indeed we know by the Alexander–Hirschowitz theorem that there exists a unique quartic hypersurface through 14 double points supported at $p_1, \ldots, p_{14}$. This implies that for any $i = 1, \ldots, 14$ there is a unique line r_i through p_i such that $r_1, \ldots, r_{14}$ are contained in a hyperplane. Then we consider the scheme Y given by the union of 13 double points supported at $\{p_1, \ldots, p_{13}\}$ and the component of length 4 corresponding to a linear space of dimension 3 which does not contain r_{14}. It is clear that the scheme Y imposes independent conditions on quartics, then also the scheme given by the union of Y and a general simple point does the same.

Assume now that $n = 4$ and $C = 3$. If $D = 13$, then we can degenerate X to one of the previous cases where the components with length less than or equal to 4 are two. If $D = 12$, then the remaining three components have length either 3, 3, 4, or 2, 4, 4. In these cases we can obtain as a trace 7 double points and three components of length either 2, 2, 3, or 1, 3, 3, and we conclude by the Castelnuovo sequence.

If $n = 4$ and $C \geqslant 4$, we denote by η the component of X with minimal length. If length$(\eta) = 1$ we can degenerate X to a scheme X' where the components with length less than or equal to 4 are one less and we apply the argument to X'. If $2 \leqslant \text{length}(\eta) \leqslant 3$, then we specialize η on $\mathbf{P}^3$ in such a way that the residual of η is 1. Then we apply the construction of Lemma 5.2 on $X \setminus \eta$ (which has at least three components with length less than or equal to 3) and we obtain a trace different from 8 double points and a component of length 3. Moreover with this construction we always avoid a residual given by 7 double points. Hence we conclude by the Castelnuovo sequence. If length$(\eta) = 4$, we have only the following possibilities: 5 components of length 4 and 10 double points, 10 components of length 4 and 6 double points, 15 components of length 4 and 2 double points. In the first two cases we can obtain trace on $\mathbf{P}^3$ given by 5 components of length 3 and 5 double points, while in the third case we can obtain a trace equal to 9 components of length 3 and 2 double points. Then we conclude by the Castelnuovo sequence. $\square$

Lemma 5.4. *Fix* $5 \leqslant n \leqslant 9$. *A general zero-dimensional scheme* $X \subset \mathbf{P}^n$ *contained in a union of double points imposes independent conditions on* $\mathcal{O}_{\mathbf{P}^n}(4)$.

Proof. If X is a collection of double points, the statement holds by the Alexander–Hirschowitz theorem. We may assume that X is a scheme with degree $\binom{n+4}{4}$ which is not a union of double points. Let us denote by D the number of double points in X and by C the number of the components with length less than or equal to n.

If $n \in \{5, 6, 8\}$ and $C = 2$, then we conclude by degenerating X to a union of double points, avoiding special cases.

If $n = 5$ and $C = 3$, then we get either $D = 20$, or $D = 19$. In the first case we conclude degenerating X to the union of 21 double points. In the second case the remaining three components have length 2, 5, 5, or 3, 4, 5, or 4, 4, 4. Then we can obtain a trace equal to 12 double points and three components of length respectively 2, 4, 4 in the first case, or 3, 3, 4 in the second and third cases. Then we conclude by induction.

If $n = 5$ and $C = 4$, then we have $D \in \{20, 19, 18\}$. In the first case we can degenerate X to a union of 21 double points. If X can be degenerate to a scheme which contains only three components with length less than or equal to 5, we conclude by using the previous results. Then we have to consider only the two cases where X is given by 18 double points and four components of length either 3, 5, 5, 5, or 4, 4, 5, 5. In these cases we can obtain a trace equal to 12 double points and three components of length respectively 2, 4, 4 in the first case, and 3, 3, 4 in the second case. Hence we conclude by induction.

If $n = 5$ and $C \geqslant 5$, we denote by η the component with minimal length. Then we specialize η on $\mathbf{P}^4$ in such a way that the residual of η is 1 if η if $\text{length}(\eta) \geqslant 2$, and 0 if η is a simple point. Then we apply the construction of Lemma 5.2 on $X \setminus \eta$ (which has at least four components with length less than or equal to 5) and we obtain a trace different from 14 double points. Hence we conclude by the Castelnuovo sequence and by induction.

If $n = 6$ and $D \geqslant 21$, we specialize 21 double points on $\mathbf{P}^5$ and we conclude by the Castelnuovo sequence. If $D < 21$, then we have $C \geqslant 5$ and we can apply Lemma 5.2, concluding by the Castelnuovo sequence.

If $n = 7$ and $D \geqslant 30$, we specialize 30 double points on $\mathbf{P}^6$ and we conclude by the Castelnuovo sequence. If $D < 30$, then we have $C \geqslant 6$ and we can apply Lemma 5.2.

If $n = 8$ and $C = 3$, then either $D = 58$ and X can be degenerated to the union of 59 double points, or $D = 57$. In this case the remaining three components can have length 5, 5, 8, or 5, 6, 7, or 6, 6, 6. In all these case we can obtain a trace on $\mathbf{P}^7$ given by 40 double points and two components of total degree 10.

If $n = 8$ and $C = 4$ and X can be degenerated to a scheme with less than 4 components with length less than or equal to 8, then we conclude. Then we have only to consider the case where $D = 56$ and the remaining four components of X have length 3, 8, 8, 8, or 4, 7, 8, 8, or 5, 6, 8, 8, or 5, 7, 7, 8, or 6, 6, 7, 8, or 6, 7, 7, 7. In all these cases we obtain a trace on $\mathbf{P}^7$ given by 40 double points and two components of total degree 10, with the exception of the last case, where we can obtain a trace given by 39 double points and three components of total degree 18.

If $n = 8$ and $C = 5$ and X can be degenerated to a scheme with less than 5 components with length less than or equal to 8, then we conclude. Hence we have only to consider the cases $D = 56$ or $D = 55$. Listing all the possible lengths of the remaining five components we easily notice that we can always obtain a trace on $\mathbf{P}^7$ given either by 40 double points and two components of total degree 10, or by 39 double points and three components of total degree 18.

If $n = 8$ and $C = 6$ and X can be degenerated to a scheme with less than 6 components with length less than or equal to 8, then we conclude. Hence we have only to consider the cases $D = 55$ or $D = 54$. Listing all the possible lengths of the remaining six components, we easily notice, as before, that we can always obtain a trace on $\mathbf{P}^7$ given either by 40 double points and two components of total degree 10, or by 39 double points and three components of total degree 18.

If $n = 8$ and $C \geqslant 7$, we apply Lemma 5.2 and we conclude by the Castelnuovo sequence.

If $n = 9$ and $D \geqslant 59$, we specialize 59 double points on $\mathbf{P}^8$ and we conclude by the Castelnuovo sequence. If $D < 59$, then we get $C \geqslant 8$ and we conclude by applying Lemma 5.2 and by the Castelnuovo sequence. $\square$

Lemma 5.5. *Fix $3 \leqslant n \leqslant 4$ and $5 \leqslant d \leqslant 6$. A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$.*

Proof. If X is a collection of double points, the statement holds by the Alexander–Hirschowitz theorem. Assume that X is a scheme with degree $\binom{n+d}{n}$ which is not a union of double points.

If $(n, d) \neq (4, 5)$ and X has only 2 components with length less than or equal to n, we conclude by degenerating X to a union of double points.

If $(n, d) = (3, 5)$ and X contains at least 7 double points, we specialize them on the trace and we conclude by the Castelnuovo sequence, since the residual contains 7 simple points. If X has less than 7 double points, then X has obviously at least 3 components with length less than or equal to 3. In this case we specialize a component with minimal length making it having residual 1, then we apply the construction of Lemma 5.2 on the remaining components and we conclude by the Castelnuovo sequence, since the residual contains at least a simple point.

If $(n, d) = (4, 5)$ and X contains at least 14 double points, we specialize them on the trace and we conclude by the Castelnuovo sequence, since the residual contains 14 simple points. If X has less than 14 double points, then X has obviously at least 4 components with length less than or equal to 4. In this case we specialize a component with minimal length making it having residual 1, then we apply the construction of Lemma 5.2 on the remaining components and we conclude by the Castelnuovo sequence, since the residual contains at least a simple point.

If either $(n, d) = (3, 6)$, or $(n, d) = (4, 6)$ and X has at least 3 components with length less than or equal to 3, we conclude by Lemma 5.2 and by induction. □

We are now in position to give the general inductive argument which completes the proof of Theorem 1.1.

Given a scheme $X \subseteq \mathbf{P}^n$ of type $(m_1, \ldots, m_{n+1})$ and a fixed hyperplane $\mathbf{P}^{n-1} \subseteq \mathbf{P}^n$, we denote for any $1 \leqslant i \leqslant n+1$:

- by $m_i^{(1)}$ the number of component of length i completely contained in $\mathbf{P}^{n-1}$,
- by $m_i^{(2)}$ the number of component of length i supported on $\mathbf{P}^{n-1}$ and with residual 1 with respect to $\mathbf{P}^{n-1}$, and
- by $m_i^{(3)}$ the number of component of length i whose support does not lie in $\mathbf{P}^{n-1}$.

Obviously we have $m_i^{(1)} + m_i^{(2)} + m_i^{(3)} = m_i$, and $m_{n+1}^{(1)} = 0, m_1^{(2)} = 0$. We denote $t_i = m_i^{(1)} + m_{i+1}^{(2)}$, for $i = 1, \ldots, n+1$, $r_1 = m_1^{(3)} + \sum m_i^{(2)}$, and $r_i = m_i^{(3)}$ for $i = 2, \ldots, n+1$. Note that, for any i, t_i is the number of components of length i in the scheme $X \cap \mathbf{P}^{n-1}$, while r_i is the number of components of length i in the scheme $X : \mathbf{P}^{n-1}$.

Theorem 5.6. *Fix the integers $n \geqslant 2$ and $d \geqslant 4$. A general zero-dimensional scheme $X \subset \mathbf{P}^n$ contained in a union of double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$ with the following exceptions*

- *$n = 2$, $d = 4$ and X is the union of 5 double points;*
- *$n = 3$ and either X is the union of 9 double points, or X is the union of 8 double points and a component of length 3;*
- *$n = 4$ and X is the union of 14 double points.*

Proof. We prove the statement by induction on n and d. In Lemma 5.1 we have proved the statement for $n = 2, d \geqslant 4$, in Lemma 5.3 and Lemma 5.4 for $d = 4, 3 \leqslant n \leqslant 9$ and in Lemma 5.5 for $d = 5, n = 3, 4$ and $d = 6, n = 3, 4$. Then we need to prove the remaining cases. Assume $n \geqslant 3$ and in particular when $d = 4$ assume $n \geqslant 10$, and when $5 \leqslant d \leqslant 6$ assume $n \geqslant 5$.

The proof by induction is structured as follows:

- for $d = 4$ and $n \geqslant 10$, we assume that any scheme in $\mathbf{P}^n$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^{n-1}}(4)$. Recall that any scheme in $\mathbf{P}^n$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(3)$ (by Theorem 4.1) and any scheme of degree greater than or equal to $(n+1)^2$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(2)$ (by Theorem 3.2). Then we prove the statement for $d = 4, n \geqslant 10$;
- for $d \geqslant 5$ we assume that any scheme in $\mathbf{P}^a$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^a}(b)$ for $(a, b) \in \{(n-1, d), (n, d-1), (n, d-2)\}$ and we prove it for $(a, b) = (n, d)$.

It is enough to prove the statement for a scheme X with degree $\deg X = \binom{d+n}{n}$.

Let $X \subseteq \mathbf{P}^n$ be a scheme of type $(m_1, \ldots, m_{n+1})$ contained in a union of double points and suppose $\deg X = \sum i m_i = \binom{d+n}{n}$. Fix a hyperplane $\mathbf{P}^{n-1}$ in $\mathbf{P}^n$. In order to apply induction, we want to degenerate X so that some of the components fall in the hyperplane $\mathbf{P}^{n-1}$. By abuse of notation we call again X the scheme after the degeneration.

Now if there exists a degeneration such that

$$\deg(X \cap \mathbf{P}^{n-1}) = \sum i t_i = \binom{d+n-1}{n-1}$$

where $m_i^{(1)}, m_i^{(2)}, m_i^{(3)}$ and t_i, r_i are defined as above, then we can conclude by the Castelnuovo sequence

$$0 \to I_{X:\mathbf{P}^{n-1}}(d-1) \to I_X(d) \to I_{X\cap\mathbf{P}^{n-1}}(d)$$

and by induction. Then we may assume that such a degeneration does not exist. Let us choose a degeneration of X such that $\binom{d+n-1}{n-1} - \sum it_i > 0$ is minimal and define

$$\varepsilon := \binom{d+n-1}{n-1} - \sum it_i. \tag{11}$$

Obviously $0 < \varepsilon < n$ and $\varepsilon < \min\left\{i : m_i^{(3)} \neq 0\right\} - 1$. By the minimality assumption we have $m_1^{(3)} = m_2^{(3)} = 0$ and we have also $m_i^{(2)} = 0$ for all $i \neq n+1$.

Now let us define

$$\varepsilon_{n+1} = \min\left\{\varepsilon, m_{n+1}^{(3)}\right\}, \quad \varepsilon_n = \min\left\{\varepsilon - \varepsilon_{n+1}, m_n^{(3)}\right\}$$

and, for any $i = n-1, \ldots, 1$,

$$\varepsilon_i = \min\left\{\varepsilon - \sum_{k=i+1}^{n+1} \varepsilon_k, m_i^{(3)}\right\}.$$

Obviously we have $\varepsilon_1 = \varepsilon_2 = 0$ and $\sum_{i=3}^{n+1} \varepsilon_i = \varepsilon$.

Step 1: Let $\Gamma \subseteq \mathbf{P}^{n-1}$ be a general scheme of type $(0, \varepsilon_3, \ldots, \varepsilon_{n+1}, 0)$ supported on a collection $\{\gamma_1, \ldots, \gamma_\varepsilon\} \subseteq \mathbf{P}^{n-1}$ of points and $\Sigma \subseteq \mathbf{P}^n$ a general scheme of type $(0, 0, m_3^{(3)} - \varepsilon_3, \ldots, m_{n+1}^{(3)} - \varepsilon_{n+1})$ supported at points which are not contained in $\mathbf{P}^{n-1}$.

By induction we know that

$$h_{\mathbf{P}^n}(\Gamma \cup \Sigma, d-1) = \min\left(\deg(\Gamma \cup \Sigma), \binom{n+d-1}{n}\right)$$

where $\deg(\Gamma \cup \Sigma) = \sum (i-1)\varepsilon_i + \sum i(m_i^{(3)} - \varepsilon_i) = \sum im_i^{(3)} - \varepsilon$.

Recall that $\binom{n+d-1}{n} = \binom{n+d}{n} - \binom{n+d-1}{n-1}$. From the definition of ε it follows that $\binom{n+d-1}{n} = \binom{n+d}{n} - \sum it_i - \varepsilon = m_{n+1}^{(2)} + \sum im_i^{(3)} - \varepsilon$ and since of course $m_{n+1}^{(2)} \geqslant 0$, we obtain $h_{\mathbf{P}^n}(\Gamma \cup \Sigma, d-1) = \sum im_i^{(3)} - \varepsilon$

Step 2: Now we want to add a collection Φ of $m_{n+1}^{(2)}$ simple points in $\mathbf{P}^{n-1}$ to the scheme $\Gamma \cup \Sigma$ and we want to obtain a $(d-1)$-independent scheme. From the previous step it is clear that $\dim I_{\Gamma\cup\Sigma}(d-1) = m_{n+1}^{(2)}$. Hence we have only to prove that there exist no hypersurfaces of degree $d-2$ through Σ. Let us show that for $d \geqslant 5$ we have

$$\deg(\Sigma) = \sum i(m_i^{(3)} - \varepsilon_i) \geqslant \binom{n+d-2}{n} \tag{12}$$

and for $d = 4$ and $n \geqslant 10$ we have

$$\deg(\Sigma) = \sum i(m_i^{(3)} - \varepsilon_i) \geqslant (n+1)^2 \geqslant \binom{n+2}{n} \tag{13}$$

Indeed by definition of ε, we have

$$\sum i(m_i^{(3)} - \varepsilon_i) = \binom{n+d-1}{n} + \varepsilon - \sum i\varepsilon_i - m_{n+1}^{(2)}$$

and since

$$\sum i\varepsilon_i - \varepsilon = \sum (i-1)\varepsilon_i \leqslant n\varepsilon \leqslant (n-1)n \quad \text{and} \quad m_{n+1}^{(2)} \leqslant \frac{1}{n}\binom{n+d-1}{n-1}$$

we obtain

$$\sum i(m_i^{(3)} - \varepsilon_i) \geqslant \binom{n+d-1}{n} - (n-1)n - \frac{1}{n}\binom{n+d-1}{n-1} =: S(n,d).$$

It is easy to check that for any $d \geqslant 5$ and $n \geqslant 3$ we have $S(n,d) > \binom{n+d-2}{n}$, which proves inequality (12). On the other hand one can also check that $S(n,4) > (n+1)^2$ for any $n \geqslant 10$, proving thus inequality (13).

Then by induction we know that Σ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d-2)$, and so we get $\dim I_\Sigma(d-2) = 0$. Thus we obtain

$$h_{\mathbf{P}^n}(\Gamma \cup \Sigma \cup \Phi, d-1) = \sum i m_i^{(3)} - \varepsilon + m_{n+1}^{(2)} = \binom{n+d-1}{n}.$$

Step 3: Let us choose a family of general points $\{\delta_{t_1}^1, \ldots, \delta_{t_\varepsilon}^\varepsilon\} \subseteq \mathbf{P}^n$, with parameters $(t_1, \ldots, t_\varepsilon) \in K^\varepsilon$, such that for any $i = 1, \ldots, \varepsilon$ we have $\delta_0^i = \gamma_i \in \mathbf{P}^{n-1}$ and $\delta_{t_i}^i \notin \mathbf{P}^{n-1}$ for any $t_i \neq 0$.

Now let us consider a family of schemes $\Delta_{(t_1,\ldots,t_\varepsilon)}$ of type $(\varepsilon_2, \ldots, \varepsilon_{n+1}, 0)$ supported at the points $\{\delta_{t_1}^1, \ldots, \delta_{t_\varepsilon}^\varepsilon\}$. Note that $\Delta_{(0,\ldots,0)}$ is the scheme Γ defined in Step 1. Moreover let $\Psi \subseteq \mathbf{P}^{n-1}$ be a scheme of type $(m_1^{(1)}, \ldots, m_n^{(1)}, 0)$ supported at general points of $\mathbf{P}^{n-1}$, and recall that in Step 2 we have introduced the scheme $\Phi \subset \mathbf{P}^{n-1}$. Let Φ^2 be the union of double points, supported on the scheme Φ.

By induction the scheme $(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup \Gamma) \subseteq \mathbf{P}^{n-1}$ has Hilbert function

$$h_{\mathbf{P}^{n-1}}(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup \Gamma, d) = \sum i m_i^{(1)} + n m_{n+1}^{(2)} + \varepsilon = \sum i t_i + \varepsilon = \binom{d+n-1}{n-1}$$

i.e. it is d-independent.

We will work now with the following schemes:

- $\Delta_{(t_1,\ldots,t_\varepsilon)}$ the family of schemes introduced in Step 3, of type $(\varepsilon_2, \ldots, \varepsilon_{n+1}, 0)$ supported at the points $\{\delta_{t_1}^1, \ldots, \delta_{t_\varepsilon}^\varepsilon\}$ and such that $\Delta_{(0,\ldots,0)} = \Gamma$;
- $\Psi \subseteq \mathbf{P}^{n-1}$ the scheme introduced in Step 3, of type $(m_1^{(1)}, \ldots, m_n^{(1)}, 0)$ supported at general points of $\mathbf{P}^{n-1}$;
- Φ^2 of type $(0, \ldots, 0, m_{n+1}^{(2)})$, that is the union of double points supported on the scheme $\Phi \subset \mathbf{P}^{n-1}$ introduced in Step 2;
- $\Sigma \subseteq \mathbf{P}^n$, the scheme defined in Step 1, of type $(0, 0, m_3^{(3)} - \varepsilon_3, \ldots, m_{n+1}^{(3)} - \varepsilon_{n+1})$.

In order to prove that X imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$, it is enough to prove the following claim.

Claim. There exist $(t_1, \ldots, t_\varepsilon)$ such that the scheme $\Delta_{(t_1,\ldots,t_\varepsilon)}$ is $\mathcal{D}$-independent, where $\mathcal{D}$ is the linear system determined by the vector space $I_{\Psi \cup \Phi^2 \cup \Sigma}(d)$.

Assume by contradiction that the claim is false. Then by Lemma 2.1 for any $(t_1, \ldots, t_\varepsilon)$ there exist pairs $(\delta_{t_i}^i, \eta_{t_i}^i)$ for all $i = 1, \ldots, \varepsilon$, with $\eta_{t_i}^i$ a curvilinear scheme supported at $\delta_{t_i}^i$ and contained in $\Delta_{(t_1,\ldots,t_\varepsilon)}$ such that

$$h_{\mathbf{P}^n}(\Psi \cup \Phi^2 \cup \Sigma \cup \eta_{t_1}^1 \cup \ldots, \eta_{t_\varepsilon}^\varepsilon, d) < \binom{d+n}{n} - \sum (i-2)\varepsilon_i. \tag{14}$$

Let η_0^i be the limit of $\eta_{t_i}^i$, for $i = 1, \ldots, \varepsilon$.

Suppose that $\eta_0^i \not\subset \mathbf{P}^{n-1}$ for $i \in F \subseteq \{1, \ldots, \varepsilon\}$ and $\eta_0^i \subset \mathbf{P}^{n-1}$ for $i \in G = \{1, \ldots, \varepsilon\} \setminus F$.

Given $t \in K$, let us denote $Z_t^F = \cup_{i \in F}(\eta_t^i)$ and $Z_t^G = \cup_{i \in G}(\eta_t^i)$. Denote by $\widetilde{\eta_0^i}$ for $i \in F$ the residual of η_0^i with respect to $\mathbf{P}^{n-1}$ and by f and g the cardinalities respectively of F and G.

By the semicontinuity of the Hilbert function and by (14) we get

$$h_{\mathbf{P}^n}(\Psi \cup \Phi^2 \cup \Sigma \cup Z_0^F \cup Z_t^G, d) \leqslant h_{\mathbf{P}^n}(\Psi \cup \Phi^2 \cup \Sigma \cup Z_t^F \cup Z_t^G, d) < \binom{d+n}{n} - \sum (i-2)\varepsilon_i.$$

On the other hand, by the semicontinuity of the Hilbert function there exists an open neighborhood O of 0 such that for any $t \in O$

$$h_{\mathbf{P}^n}(\Phi \cup \Sigma \cup (\cup_{i \in F}\widetilde{\eta_0^i}) \cup Z_t^G, d-1) \geqslant h_{\mathbf{P}^n}(\Phi \cup \Sigma \cup (\cup_{i \in F}\widetilde{\eta_0^i}) \cup Z_0^G, d-1)$$

Since the scheme $\Phi \cup \Sigma \cup (\cup_{i \in F}\widetilde{\eta_0^i}) \cup Z_0^G$ is contained in $\Phi \cup \Sigma \cup \Gamma$, which is $(d-1)$-independent by Step 2, we have

$$h_{\mathbf{P}^n}(\Phi \cup \Sigma \cup (\cup_{i \in F}\widetilde{\eta_0^i}) \cup Z_0^G, d-1) = m_{n+1}^{(2)} + \sum i(m_i^{(3)} - \varepsilon_i) + f + 2g.$$

Since $\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup (\cup_{i \in F}\gamma_i)$ is a subscheme of $\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup \Gamma$, which is d-independent by Step 3, it follows that

$$h_{\mathbf{P}^{n-1}}(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup (\cup_{i \in F}\gamma_i), d) = \sum i m_i^{(1)} + n m_{n+1}^{(2)} + f$$

Hence for any $t \in O$, by applying the Castelnuovo exact sequence to the scheme $\Psi \cup \Phi^2 \cup \Sigma \cup Z_0^F \cup Z_t^G$, we get

$$\begin{aligned}
&h_{\mathbf{P}^n}(\Psi \cup \Phi^2 \cup \Sigma \cup Z_0^F \cup Z_t^G, d) \\
&\geqslant h_{\mathbf{P}^n}(\Phi \cup \Sigma \cup (\cup_{i \in F}\widetilde{\eta_0^i}) \cup Z_t^G, d-1) + h_{\mathbf{P}^{n-1}}(\Psi \cup \Phi^2|_{\mathbf{P}^{n-1}} \cup (\cup_{i \in F}\gamma_i), d) \\
&\geqslant (m_{n+1}^{(2)} + \sum i(m_i^{(3)} - \varepsilon_i) + f + 2g) + \left(\sum i m_i^{(1)} + n m_{n+1}^{(2)} + f\right) \\
&= \sum i m_i - \sum i\varepsilon_i + 2\varepsilon = \binom{d+n}{n} - \sum (i-2)\varepsilon_i
\end{aligned}$$

contradicting (14). This completes the proof of the claim. □

6. Appendix

Here we explain how to compute the dimension of the space

$$V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$$

defined in (2) in the introduction.

These computations are performed in characteristic 31991 using the program Macaulay2 [9], and consist essentially in checking that several square matrices, randomly chosen, have maximal rank. We underline that if an integer matrix has maximal rank in positive characteristic, then it has also maximal rank in characteristic zero. Very likely Theorem 1.1 should be true on any infinite field, but a finite number of values for the characteristic (not including 31991) require further and tedious checks, that we have not performed.

Assume that $\dim A_i = a_i$ are given and that $\sum_{i=1}^k (a_i + 1) = \binom{n+d}{n} = \dim R_{d,n}$. Consider the monomial basis for $R_{d,n}$ as a matrix T of size $\binom{n+d}{n} \times 1$. Consider the jacobian matrix J computed at p_i, which has size $\binom{n+d}{n} \times (n+1)$. Choose a random $(n+1) \times a_i$ integer matrix A. We concatenate T

computed at p_i with $J \cdot A$. It results a matrix of size $\binom{n+d}{n} \times (a_i + 1)$. When $a_i = n$ (this is the case of Alexander–Hirschowitz theorem) there is no need to use a random matrix, and by Euler identity we can simply take the jacobian matrix J computed at p_i. By repeating this construction for every point, and placing side by side all these matrices, we get a square matrix of order $\binom{n+d}{n}$. This is the matrix of coefficients of the system (1), which corresponds to our interpolation problem. Then there is a unique polynomial f satisfying (1) if and only if the above matrix has maximal rank. We emphasize that this Montecarlo technique provides a proof, and not only a probabilistic proof. Indeed consider the subset $\mathcal{S}$ of points $(p_1, \ldots, p_k, A_1, \ldots, A_k)$ (lying in a Grassmann bundle, which locally is isomorphic to the product of affine spaces and Grassmannians, hence irreducible) such that the corresponding matrix has maximal rank. The subset $\mathcal{S}$ is open and if it is not empty, because it contains a random point, then it is dense.

In Proposition 4.3, Proposition 4.7, Proposition 4.8, Proposition 4.12 we need a modification of the above strategy, since the points are supported on some given codimension three subspaces.

As a sample we consider the case considered in Proposition 4.8 where $n = 8$, `l` $= \deg(X_L : L) = 10$, `m` $= \deg(X_M : M) = 14$, and `F` $= \deg(X_O) = 39$ and we list below the Macaulay2 script. Given monomial subspaces L and M, we first compute the cubic polynomials containing L and M, finding a basis of 63 monomials. Then we compute all the possible partitions of 10 and 14 in integers from 1 to 3 (which are the possible values of $\deg(\xi : L)$, resp. $\deg(\xi : M)$, where ξ is an irreducible component of X_L, resp. X_M), and of 39 in integers from 1 to 9 (which are the possible lengths of a subscheme of a double point in $\mathbf{P}^8$), by excluding the cases which can be easily obtained by degeneration. We collect the results in the matrices `tripleL`, `tripleM` and `XO`, each row corresponds to a partition. Then for any combination of rows of the three matrices the program computes a matrix `mat` of order 63 and its rank. If the rank is different from 63 the program prints the case. Running the script we see that the output is empty, as we want.

```
KK=ZZ/31991;
E=KK[e_0..e_8];
--coordinates in P8

f=ideal(e_0..e_8);
g=ideal(e_0..e_2);
h=ideal(e_3..e_5);
T1=f*g*h;
T=gens gb(T1)
--basis for the space of cubics containing
--L (e_0=e_1=e_2=0) and M (e_3=e_4=e_5=0)
--T is a (63x1) matrix

J=jacobian(T);
-- J is a (63x9) matrix

--first case: for the other cases of Proposition 4.8 it is enough
--to change to following line
l=10;m=14;F=39;

---start program
tripleL=matrix{{0,0,0}};
for t from 0 to ceiling(l/3) do
for d from 0 to ceiling(l/2) do
for u from 0 to 1 do
     (if (3*t+2*d+u==l) then tripleL=(tripleL||matrix({{t,d,u}})));

tripleM=matrix{{0,0,0}},
for t from 0 to ceiling(m/3) do
for d from 0 to ceiling(m/2) do
for u from 0 to 1 do
     (if (3*t+2*d+u==m) then tripleM=(tripleM||matrix({{t,d,u}})));

XO=matrix{{0,0,0,0,0,0,0,0,0}};
for n from 0 to ceiling(F/9) do
     (if (9*n+1==F) then XO=(XO||matrix({{n,0,0,0,0,0,0,0,1}})));
(for n from 0 to ceiling(F/9) do
(for o from 0 to ceiling(F/8) do
     (if (9*n+8*o+2==F) then XO=(XO||matrix({{n,o,0,0,0,0,0,1,0}})))));
(for n from 0 to ceiling(F/9) do
(for o from 0 to ceiling(F/8) do
(for s from 0 to ceiling(F/7) do
```

```
     (if (9*n+8*o+7*s+3==F) then XO=(XO||matrix({{n,o,s,0,0,0,1,0,0}}))))));
(for n from 0 to ceiling(F/9) do
(for o from 0 to ceiling(F/8) do
(for s from 0 to ceiling(F/7) do
(for e from 0 to ceiling(F/6) do
(for c from 0 to ceiling(F/5) do
     (if (9*n+8*o+7*s+6*e+5*c==F)
          then XO=(XO||matrix({{n,o,s,e,c,0,0,0,0}}))))))));

k=1;
for a from 1 to (numgens(target(tripleL))-1) do
for b from 1 to (numgens(target(tripleM))-1) do
for c from 1 to (numgens(target(XO))-1) do
     (k=k+1,
mat=random(E^1,E^63)*0,
for i from 1 to tripleL_(a,0) do
(q1=(matrix(E,{{0,0,0}})|random(E^1,E^6)), mat=mat||random(E^3,E^9)*sub(J,q1)),
for i from 1 to tripleL_(a,1) do
(q1=(matrix(E,{{0,0,0}})|random(E^1,E^6)), mat=mat||random(E^2,E^9)*sub(J,q1)),
for i from 1 to tripleL_(a,2) do
(q1=(matrix(E,{{0,0,0}})|random(E^1,E^6)), mat=mat||random(E^1,E^9)*sub(J,q1)),
for i from 1 to tripleM_(b,0) do
(r1=(random(E^1,E^3)|matrix(E,{{0,0,0}})|random(E^1,E^3)),mat=mat||random(E^3,E^9)*sub(J,r1)),
for i from 1 to tripleM_(b,1) do
(r1=(random(E^1,E^3)|matrix(E,{{0,0,0}})|random(E^1,E^3)),mat=mat||random(E^2,E^9)*sub(J,r1)),
for i from 1 to tripleM_(b,2) do
(r1=(random(E^1,E^3)|matrix(E,{{0,0,0}})|random(E^1,E^3)),mat=mat||random(E^1,E^9)*sub(J,r1)),
for i from 1 to XO_(c,0) do
(p1=random(E^1,E^9), mat=mat||sub(J,p1)),
for i from 1 to XO_(c,1) do
(p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(8-1),E^9)*sub(J,p1)),
for i from 1 to XO_(c,2) do
(p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(7-1),E^9)*sub(J,p1)),
for i from 1 to XO_(c,3) do
(p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(6-1),E^9)*sub(J,p1)),
for i from 1 to XO_(c,4) do
(p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(5-1),E^9)*sub(J,p1)),
for i from 1 to XO_(c,5) do
(p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(4-1),E^9)*sub(J,p1)),
for i from 1 to XO_(c,6) do
(p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(3-1),E^9)*sub(J,p1)),
for i from 1 to XO_(c,7) do
(p1=random(E^1,E^9), mat=mat||sub(T,p1)||random(E^(2-1),E^9)*sub(J,p1)),
for i from 1 to XO_(c,8) do mat=mat||sub(T,random(E^1,E^9)),
if (rank(mat)!=63)
then (print(tripleL_(a,0),tripleL_(a,1),tripleL_(a,2),tripleM_(b,0),tripleM_(b,1),tripleM_(b,2),
XO_(c,0),XO_(c,1),XO_(c,2),XO_(c,3),XO_(c,4),XO_(c,5),XO_(c,6),XO_(c,7),XO_(c,8))),
          if (mod(k,29)==0) then print(k));
```

All the others scripts are available at the page <http://web.math.unifi.it/users/brambill/homepage/macaulay.html>.

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