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On Moduli of Stable 2-Bundles with Small Chern Classes on Q_3 (*).

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with an appendix by NICOLAE MANOLACHE

Abstract. - *Let $M(c_1, c_2)$ be the moduli space of stable rank-2 vector bundles with Chern classes c_1, c_2 over the smooth quadric $Q_3 \subset \mathbb{P}^4$. The main result of the paper consists in a description of $M(0,2)$ by studying the interplay between the quadrics determined by the jumping lines and the null-correlation over the spinor variety $\mathbb{P}^3 = \mathrm{Gr}(\mathbb{P}^1, Q_3)$. We describe also $M(-1,2)$, $M(-1,3)$ and $M(0,4)$. The irreducibility of $M(0,4)$ relies on the classification of curves $Y \subset Q_3$ of degree 6 with $w_Y = \mathcal{O}_Y(-1)$, achieved by Manolache in the appendix.*

In the current paper, the moduli spaces of stable holomorphic vector bundles, with small Chern numbers on a smooth quadric $\mathbb{Q}_3 \subset \mathbb{P}^4$ are studied. A research on such bundles was started in [SSW]. Most of the results has a geometric interpretation and as algebraic manifolds the moduli spaces are as follows

— $M(0,2) = \mathbb{P}^9 \setminus V_4$, where V_4 is a normal degree-4 hypersurface—moreover, there is a locally trivial fibration of $M(0,2)$ over $\mathbb{P}^4 \setminus \mathbb{Q}_3$ with fibre $\mathbb{P}^5 \setminus \mathbb{Q}_4$;

— $M(-1,2)$ is a locally trivial fibration over $\mathbb{Q}_4 \setminus \mathbb{Q}_3$ with fibres $\mathbb{P}^2 \setminus \mathbb{Q}_1$;

— $M(-1,3)$ is irreducible, unirational and reduced of dimension 12;

— $M(0,4)$ is irreducible, unirational and reduced of dimension 21.

For the sake of some completeness, let us point out that $M(0, 2k-1)$ is empty (Schwarzenberger relations) while $M(-1,1) = \{S\}$ where S is the spinor bundles, see e.g. [AS1].

As for bundles over $\mathbb{P}^3$ the «instanton» condition, i.e., vanishing of $H^1(E(-2))$ comes out in a natural way when we strive for constructing the moduli of stable bundles with the given rank and Chern classes.

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— We study the action of the automorphism group $\mathrm{Aut}(\mathbb{Q}_3)$ on $M(0,2)$ and $M(-1,2)$. It turns out (Theorem (2.22)) that there is a 1-parameter family of non isomorphic bundles in $M(0,2)$ up to automorphisms of $\mathbb{Q}_3$, while $\mathrm{Aut}(\mathbb{Q}_3)$ acts transitively on $M(-1,2)$, see (4.12).

— We study also the configuration of jumping lines of the bundles of $M(0,2)$ and $M(-1,2)$. In particular, every bundle of $M(0,2)$ is uniquely determined by its jumping lines, see (2.13), while in the fibration π: $M(-1, 2) \to \mathbb{Q}_4 \setminus \mathbb{Q}_3$ we have that the jumping lines of bundles E and E' are the same if and only if $\pi(E) = \pi(E')$, see (4.10).

The paper is organized as follows. In the Sections 0 and 1 we collect all necessary facts of homological character (dimensions of cohomology groups, monads).

Section 2 is the core of the paper. Here we study the moduli space $M(0,2)$ in four (not entirely different) ways, namely:

1) by a «geometric» method we show that $M(0,2)$ is a fibre bundle over $\mathbb{P}^4 \setminus \mathbb{Q}_3$ with fibres $\mathbb{P}^5 \setminus \mathbb{Q}_4$ and then we give three descriptions of the fibration (Propositions (2.1), (2.11) and (2.17)). The fibration is not trivial (6.17);

2) by examining the appropriate monad (1.2) we show that $M(0,2)$ is $\mathbb{P}^9 \setminus V_4$ where $\mathbb{P}^9$ is viewed at as the space of all 5×5 skew-symmetric matrices and V_4 is a quartic hypersurface with the equation given by the quadratic form of $\mathbb{Q}_3$ evaluated on the pfaffians as in (2.14);

3) by looking at $\mathbb{P}^9$ as the space of all quadrics in $\mathbb{P}^3$, we explain which quadrics belong to V_4; this depends on the type of the quadric as well as on its position with respect to the configuration of lines which are isotropic with respect to the null-correlation determined on $\mathbb{P}^3$ by fixing an isomorphism between the manifold of lines on $\mathbb{Q}_3$ and $\mathbb{P}^3$ (Proposition (2.15));

4) replacing the quadrics by symmetric matrices 4×4 allows us to utilize another approach to study the bundles from $M(0,2)$. Namely, we define an invariant $\sigma(A)$ of the «skew-characteristic polynomial» of the symmetric matrix A, such that the pair

$$\{\mathrm{rk}(A), \det(A)/(\sigma^2(A) - 4\det(A))\}$$

determines the orbit of the bundle from $M(0,2)$ under the action of $\mathrm{Aut}(\mathbb{Q}_3)$. Now the equation of V_4 is a discriminant as in (2.21).

In Sections 3, 4 and 5 we study respectively the moduli spaces $M(0,4)$, $M(-1,2)$ and $M(-1,3)$.

The methods we start with to have the first description of $M(0,2)$ and other moduli consist in studying curves where sections of (an appropriate twist) of the bundles vanish and then in reconstructing the bundles from such curves. This is a very natural method and was applied e.g. in [H] and [HaS] to study the moduli of rank-2 bundles on $\mathbb{P}^3$.

After the preprint of the current paper was distributed, IGNACIO SOLS pointed out that he had worked on extending the results from [HaS] to the bundles on quadrics already in 1980, but the results were never published. The authors are indebted to IGNACIO SOLS for this remark and appreciate his work.

The detailed study of the space $M(0,2)$ requires some special approach and new techniques. For $M(-1,3)$ we use also Kapranov's spectral sequence, which is the quadric analogue of the Beilinson sequence on projective spaces. It should pointed out, however, that the results from the Klein quadric Q_4 do not carry over onto Q_3 automatically here.

The proof of the irreducibility of $M(0,4)$ relies heavily on the classification of degree 6 curves Y with $w_Y = \mathcal{O}(-1)$. This is achieved by MANOLACHE in the long appendix, with a detailed study of all possible multiple structures arising in this problem. In particular MANOLACHE shows that every family of bundles coming from such curves has dimension ≤ 20.

Another tool we use is the classification of the bundles with no intermediate cohomology on Q_n, se e.g. [AS2].

Finally, Section 6 is devoted to a study of topology of $M(0,2)$ and $M(-1,2)$.

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0. – Preliminaries.

(0.1) *Basic facts from the geometry of* $\mathbb{Q}_n$.

The cohomology ring $H^*(\mathbb{Q}_3, \mathbb{Z})$ is generated by the class of a hyperplane section $H \in H^2(\mathbb{Q}_3, \mathbb{Z})$, a line $L \in H^4(\mathbb{Q}_3, \mathbb{Z})$ and a point $P \in H^6(\mathbb{Q}_3, \mathbb{Z})$ with the following relations

$$H^2 = 2L, \qquad H \cdot L = P, \qquad H^3 = 2P.$$

(0.2) *Formulas for the Chern classes and the Euler-Poincaré characteristic of bundles on* $\mathbb{Q}_3$.

We identify Chern classes with integers.

The following formulas can be checked in a standard way

For a coherent sheaf F of rank r on $\mathbb{Q}_3$:

$$c_1(F(k)) = c_1 + kr;$$

$$c_2(F(k)) = c_2 + 2k(r-1)c_1 + 2k^2\binom{r}{2};$$

$$c_3(F(k)) = c_3 + k(r-2)c_2 + 2k^2\binom{r-1}{2} + 2k^3\binom{r}{3};$$

$$\chi(F) = (2c_1^3 - 3c_1c_2 + 3c_3)/6 + 3(c_1^2 - c_2)/2 + 13c_1/6 + rk(F),$$

hence if F is a bundle with $c_1(F) = 0$, then c_2 is even.

We may easily calculate that for rank-2 bundles on $\mathbb{Q}_3$ with $c_1 = 0$ there is

$$\chi(E(t)) = 2t^3/3 + 2t^2 + (-c_2 + 13/3)t + (2 - 3c_2/2) =$$

$$= 4\binom{t+3}{3} - 2\binom{t+2}{2} - c_2\binom{t+1}{1} - c_2/2.$$

In particular

$$\chi(E) = 2 - 3c_2/2, \qquad \chi(E(1)) = 10 - 5c_2/2,$$

$$\chi(E(-1)) = -c_2/2, \qquad \text{and also } \chi(\mathrm{End}(E)) = 4 - 6c_2,$$

while if $c_1 = -1$, there is

$$\chi(E(t)) = 2t^3/3 + 2t^2 + (-c_2 + 7/3)t + (1 - c_2) =$$

$$= 4\binom{t+3}{3} - 4\binom{t+2}{2} + (1 - c_2)\binom{t+1}{1},$$

hence

$$\chi(E) = 1 - c_2, \qquad \chi(E(1)) = 6 - 2c_2,$$

$$\chi(E(-1)) = 0 \quad \text{for every } c_2; \qquad \text{also } \chi(\mathrm{End}(E)) = 7 - 6c_2.$$

(0.3) *Lines on* $\mathbb{Q}_3$.

The n-dimensional quadrics $\mathbb{Q}_n$ in $\mathbb{P}^{n+1}$ have many lines. The variety which parametrizes all lines on a given X is called *the Fano variety* of X. It is well known that the family of lines on $\mathbb{Q}_3$ is isomorphic to $\mathbb{P}^3$. One of the best ways to see this is the following: consider a linear embedding $\mathbb{Q}_3 \subset \mathbb{Q}_4 = \mathrm{Gr}(1,3)$. Every line l is contained is exactly one α-plane and in exactly one β-plane of $\mathrm{Gr}(1,3)$ and each of the two families parametrizing linear $\mathbb{P}^2$'s in $\mathbb{Q}_4$ is isomorphic to $\mathbb{P}^3$.

Moreover, $\{\mathbb{P}^1 \mid \mathbb{P}^1 \subset \mathbb{Q}_3\} = \mathbb{P}^3$ is endowed with a null-correlation $N: \mathbb{P}^3 \to \mathbb{P}^{3*}$ given by $\overline{\mathbb{P}^1} \to \{\mathbb{P}^1 \mid \mathbb{P}^1 \cap \overline{\mathbb{P}^1} \neq \emptyset\}$. The natural embedding $\{\mathbb{P}^1 \mid \mathbb{P}^1 \subset \mathbb{Q}_3\} \to \to \mathrm{Gr}(1,4) \subset \mathbb{P}^9$ is the 2-Veronese embedding and the bundle $\mathcal{N}$ corresponding to the null-correlation N is the pullback of the universal 2-bundle on the Grassmannian, see e.g. [Ta], Section 6.

A ruling of a smooth quadric in $\mathbb{P}^3$ describes a conic in $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3) \cong \mathbb{Q}_4$ and the isotropic lines of the ruling are given by the intersection points of the conic with a hyperplane in $\mathbb{P}^5$. There are three possibilities in a ruling:

i) two isotropic lines,

ii) one isotropic line (tangency),

iii) all lines are isotropic.

Two rulings of the same smooth quadric correspond to two conics such that the planes that they span are polar. The conic corresponding to the ruling of a quadric cone lies in a plane contained in $\mathbb{Q}_4$. An isotropic line in $\mathbb{P}^3 \cong \mathrm{Gr}(\mathbb{P}^1, \mathbb{Q}_3)$ corresponds to the ruling of a cone $C \subset \mathbb{Q}_3$. A pair of antipolar lines in $\mathbb{P}^3$ correspond to the two rulings of a smooth quadric $\mathbb{Q}_2 \subset \mathbb{Q}_3$. Such facts about lines on $\mathbb{Q}_3$ have been known for a hundred years.

(0.4) *The spectrum of a stable 2-bundle on* $\mathbb{Q}_3$.

In [ES], the notion of a spectrum of a 2-bundle on $\mathbb{Q}_3$ is introduced. The spectrum of a stable, rank-2 vector bundle on $\mathbb{Q}_3$ with $c_1 = 0$ is a sequence of integers $a_1, \ldots, a_s$, with $s = c_2$ such that for the bundle $\mathcal{H} = \mathcal{O}(a) \oplus \ldots \oplus \mathcal{O}(a_s)$ there holds

$$h^1(E(j)) = h^0(\mathcal{H}(j+1)) \quad \text{for } j \leqslant -1\,.$$

The spectrum has the following properties:

#1) if $k > 0$ is in the spectrum, so are $0, 1, \ldots k-1$;

#2) if $k < 0$ is in the spectrum, so are $k+1, \ldots, -1$;

#3) the sequence $a_1, \ldots, a_s$ is symmetric with respect to $-1/2$;

#4) $\sum a_i = -c_2/2$.

Similarly, if $c_1(E) = -1$, then the spectrum is symmetric with respect to -1 and $\sum a_i = -c_2 + 1$. It consists of $c_2 - 1$ elements. EIN and SOLS prove in [ES] that if E is stable 2-bundle on $\mathbb{Q}_3$, we have

$$h^1(E(t)) = 0 \quad \text{for } t \leqslant -1 - c_2/4 \quad \text{if } c_1 = 0$$

and

$$h^1(E(t)) = 0 \quad \text{for } t \leqslant -c_2/4 \quad \text{if } c_1 = -1\,.$$

Later on we will need a lemma on the (non)existence of some spectrum, giving a slightly sharper estimate than that in [ES].

(0.5) LEMMA. - The spectrum of a stable, rank-2 bundle on $\mathbb{Q}_3$ with $c_1 = 0$, $c_2 > 2$ does not contain $-1 + c_2/2$ (and hence starts from $d \geqslant 1 - c_2/2$).

PROOF. – Assume the contrary. By the property #3) above, the appearance of $-1+c_2/2$ in the spectrum is equivalent to that of $-c_2/2$. From properties #1), #2), #4) we see easily that the smallest number $-c_2/2$ does not occur twice, i.e., the spectrum begins with $\{-c_2/2, 1-c_2/2\}$. Since $c_2 \geqslant 4$, the dimensions of $H^1(E(-c_2/2))$ and $H^1(E(1-c_2/2))$ can be calculated from the spectrum and the result is $h^1(E(-c_2/2)) = 1$, $h^1(E(1-c_2/2)) = 3$. By *the bilinear lemma* [Ha1], Lemma 5.1, applied to

$$H^0(\mathcal{O}(1)) \times H^1(E(-c_2/2)) \to H^1(E(1-c_2/2))$$

we see that $x\colon H^1(E(-c_2/2)) \to H^1(E(1-c_2/2))$ has a non-trivial kernel for some $x \in H^0(\mathcal{O}(1))$. Let $\mathbb{Q}_2$ be the corresponding hyperplane section. From the sequence

$$0 \to E(-c_2/2) \to E(1-c_2/2) \to E(1-c_2/2)|\mathbb{Q}_2 \to 0$$

we then obtain $h^0(E(1-c_2/2)|\mathbb{Q}_2) \geqslant 1$ and hence for every (not necessarily smooth) conic $C \subset \mathbb{Q}_2$ we have also the exact sequence

$$0 \to E(1-c_2/2)|\mathbb{Q}_2 \to E(2-c_2/2)|\mathbb{Q}_2 \to E(2-c_2/2)|C \to 0\,.$$

The corresponding cohomology sequence is

$$0 \to H^0(E(1-c_2/2)|\mathbb{Q}_2) \to H^0(E(2-c_2/2)|\mathbb{Q}_2) \to H^0(E(2-c_2/2)|C)$$

with $h^0(E(2-c_2/2)|\mathbb{Q}_2) \geqslant 1$. Assume $h^0(E(2-c_2/2)|\mathbb{Q}_2) = 1$. Then the unique non-zero section $s \in H^0(E(2-c_2/2)|\mathbb{Q}_2)$ comes from $H^0(E(1-c_2/2)|\mathbb{Q}_2)$, i.e., vanishes on a conic. We may, however, pick a conic different from that one. Hence we have shown that $\dim H^0(E(2-c_2/2)|\mathbb{Q}_2) \geqslant 2$. The exact sequence

$$0 \to E(1-c_2/2) \to E(2-c_2/2)| \to E(2-c_2/2)|\mathbb{Q}_2 \to 0$$

then yields a contradiction.

(0.6) COROLLARY. – For a bundle as in (0.5), $h^1(E(j)) = 0$ for $j \leqslant -c_2/2$.

It follows directly from the properties of the spectrum.

(0.7) *Kapranov spectral sequences.*

These sequences are the analogoues of Beillinson's ones for bundles over projective spaces. For the convenience of the reader we recall here the formulas from [AO1] and [Ka]. First, sheaves ψ_i on a smooth quadric $\mathbb{Q}_n$ are introduced: $\psi_0 = \mathcal{O}$, $\psi_1 = \Omega^1(1)|\mathbb{Q}_n$ and for $i \geqslant 2$ ψ_i is the only non-splitting element in the extension

$$0 \to \Omega^i(i)|\mathbb{Q}_n \to \psi_i \to \psi_{i-2} \to 0\,,$$

where $\Omega^i(i)$ = is the sheaf of twisted holomorphic i-forms on $\mathbb{P}^n$. The main properties of ψ_i are

$$rk\,\psi_i = \sum_{j=0}^{i}\binom{n}{j}, \qquad \dim H^0(\psi_i(1)) = \sum_{j=0}^{i+1}\binom{n+1}{j}, \qquad c_1(\psi_i) = -\sum_{j=0}^{i-1}\binom{n-1}{j},$$

$\psi_n = \psi_{n+1} = S^{\oplus 2^{(n+1)/2}}$. In particular, on $\mathbb{Q}_3$ we have

$$\psi_1 = \Omega^1(1)\,|\,\mathbb{Q}_3,$$

ψ_2 given by

$$0 \to \Omega^2(2)\,|\,\mathbb{Q}_3 \to \psi_2 \to \mathcal{O} \to 0,$$

and $\psi_3 = S^4$.

The sheaves ψ_i are, together with the spinor bundles, like building blocks, out of which all vector bundles on quadrics can be constructed. In terms of the spectral sequences this can be stated as the following:

THEOREM ([Ka], [AO1]). – Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{Q}_n$. The resolution of the diagonal in $\mathbb{Q}_n \times \mathbb{Q}_n$ gives the spectral sequences with E_1^{pq} and $'E_1^{pq}$:

$$E_1^{pq} = H^q(\mathbb{Q}_n, \mathcal{F}(p)) \otimes \psi_{-p} \qquad \text{if } p > -n$$

and

$$E_1^{nq} = \begin{cases} H^q(Q_n, \mathcal{F} \otimes S^*(-n)) \otimes S & \text{if } n \text{ is odd}, \\ (H^q(Q, \mathcal{F} \otimes S'^*(-n)) \otimes S') \oplus (H^q(Q, \mathcal{F} \otimes S''^*(-n)) \otimes S'') & \text{otherwise} \end{cases}$$

$$'E_1^{pq} = H^q(\mathbb{Q}_n, \mathcal{F} \otimes \psi_{-p}) \otimes \mathcal{O}(-p) \qquad \text{if } p > -n$$

and

$$'E_1^{nq} = \begin{cases} H^q(Q_n, \mathcal{F} \otimes S) \otimes S^*(-n) & \text{if } n \text{ is odd}, \\ (H^q(Q_n, \mathcal{F} \otimes S') \otimes S'^*(-n)) \oplus (H^q(Q_n, \mathcal{F} \otimes S'') \otimes S''^*(-n)) & \text{otherwise}. \end{cases}$$

For the two sequences there is $E_\infty^{pq} = 0$ for $p + q \neq 0$ and $\oplus E_\infty^{-p,p}$ is the associated graded sheaf of a filtration of $\mathcal{F}$.

It is worthwhile to «unzip» the above theorem. We will do it over $\mathbb{Q}_3$ for the first spectral sequence. It then says that any coherent sheaf $\mathcal{F}$ on $\mathbb{Q}_3$ can be realized as the cohomology of a complex involving the (normalized) spinor bundle S. Precisely, we have a complex

$$0 \to L^{-3} \xrightarrow{d_{-3}} L^{-2} \xrightarrow{d_{-2}} L^{-1} \xrightarrow{d_{-1}} L^{-0} \xrightarrow{d_0} L^1 \xrightarrow{d_1} L^2 \xrightarrow{d_2} L^3 \to 0$$

with $L^k = \bigoplus_{j+k=i} X_j^i$, where $X_j^i = \psi_j^{\oplus h^i(\mathcal{F}(-j))}$ for $j = 0, 1, 2$ and $X_3^i = S^{\oplus h^i(\mathcal{F} \otimes S(-2))}$ such that

$$\frac{\ker d^i}{\operatorname{Im} d_{i-1}} = \begin{cases} \mathcal{F} & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

The bundles L^k are constructed by summing up the «NW-SE» diagonals in the matrix

$$\begin{matrix} S & \psi_2 & \psi_1 & \mathcal{O} \\ S & \psi_2 & \psi_1 & \mathcal{O} \\ S & \psi_2 & \psi_1 & \mathcal{O} \\ S & \psi_2 & \psi_1 & \mathcal{O} \end{matrix}$$

where each bundle is taken the number of times corresponding to the same entry in the following «Kapranov diagram»

$$\begin{matrix} h^3(\mathcal{F} \otimes S(-2)) & h^3(\mathcal{F}(-2)) & h^3(\mathcal{F}(-1)) & h^3(\mathcal{F}) \\ h^2(\mathcal{F} \otimes S(-2)) & h^2(\mathcal{F}(-2)) & h^2(\mathcal{F}(-1)) & h^2(\mathcal{F}) \\ h^1(\mathcal{F} \otimes S(-2)) & h^1(\mathcal{F}(-2)) & h^1(\mathcal{F}(-1)) & h^1(\mathcal{F}) \\ h^0(\mathcal{F} \otimes S(-2)) & h^0(\mathcal{F}(-2)) & h^0(\mathcal{F}(-1)) & h^0(\mathcal{F}). \end{matrix}$$

(0.8) *Bundles with no intermediate cohomology.*

(0.8) THEOREM. – Let E be a bundle on $\mathbb{Q}_n$, $n \geq 3$. Let E be a bundle with $H^i(E(j)) = 0$ for $i = 1, \ldots, n-1$ and all $j \in \mathbb{Z}$. Then E is a direct sum of line bundles and twisted spinor bundles.

This was proved first in [Kn], see also [So]. In [AO1] it is shown that such a characterization can be obtained from Kapranov's sequence. The paper [AS2] contains a more elementary proof.

(0.9) *The Castelnuovo-Mumford criterion.*

The well-known criterion for a sheaf to be globally generated ([Mu], Th. 2, p. 41) can be translated word for word for the sheaves on quadrics [HeS].

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{Q}_3$ such that $H^i(\mathcal{F}(-i)) = 0$ for $i > 0$. Then $\mathcal{F}$ is globally generated and $H^i(\mathcal{F}(-i+j)) = 0$ for $i > 0$, $j \geq 0$.

(0.10) Finally, to construct a monad for stable 2-bundles on $\mathbb{Q}_3$ we use Horrocks' killing technique [Hor].

1. - Bundles with $H^1(E(-2)) = 0$ on Q_3.

Such bundles are similar to instanton bundles on $\mathbb{P}^3$. Let us notice that a bundle on $\mathbb{Q}_3$ with $H^1(E(-2)) = 0$ is either stable or trivial. The proof can be obtained easily from the proof of the analogous property on $\mathbb{P}^3$, see [OSS], Sect. 3.4. Furthermore, we have

(1.1) PROPOSITION. - For a rank-2 bundle E with $c_1(E) = 0$ on $\mathbb{Q}_3$ the following three conditions are equivalent

i) E is stable with a minimal spectrum, i.e.

$$\{-1, -1, \ldots, -1, 0, 0, \ldots, 0\},$$

ii) E is stable with $H^1(E(-2)) = 0$,

iii) E is the cohomology of a monad

$$\mathcal{O}(-1)^{c_2/2} \to \mathcal{O}^{c_2+2} \to \mathcal{O}(1)^{c_2/2}. \tag{1.2}$$

We prove Proposition (1.1) in several steps. The equivalence i)$\Leftrightarrow$ii) is easy: since the spectrum of such a bundle is symmetric with respect to $-1/2$, any appearance of a number $c \leqslant 2$ is equivalent to that of a number $d \geqslant 1$. Any spectrum different from the minimal one gives then

$$H^1(E(-2)) = H^0(\mathcal{O}(d-1) \oplus \ldots) \neq 0.$$

To prove ii)$\Rightarrow$iii), i.e., to construct the monad we need to know some cohomology. First of all, we have immediately

$$\begin{array}{cccc|l}
0 & 0 & 0 & 0 & \uparrow i \\
 & & 0 & 0 & \qquad h^i(E(j)) \\
0 & 0 & & & \\
0 & 0 & 0 & 0 & \\
\hline
 & & j=-1 & j=0 & \rightarrow j
\end{array}$$

The remaining values of $h^i(E(j))$ for $-3 \leqslant j \leqslant 0$ can be calculated from the Euler-Poincaré characteristic. We get

$$\begin{array}{cccc|l}
0 & 0 & 0 & 0 & \uparrow i \\
a & b & 0 & 0 & \qquad h^i(E(j)) \\
0 & 0 & a & b & \\
0 & 0 & 0 & 0 & \\
\hline
 & & j=-1 & j=0 & \rightarrow j
\end{array}$$

where $a = 3c_2/2 - 2$, $b = c_2/2$. We then apply Horrocks' killing-H^1 technique. Namely, for a suitably chosen extension

$$0 \to E \to B \to \mathcal{O}^a \oplus \mathcal{O}(1)^b \to 0 \tag{1.3}$$

the induced maps

$$H^0(\mathcal{O}^a \oplus \mathcal{O}(1)^b) \to H^1(E),$$

$$H^0(\mathcal{O}(-1)^a \oplus \mathcal{O}^b) \to H^1(E(-1)),$$

are onto and therefore $B(j)$ has no first cohomology for $j = -1, 0$.

CLAIM. – The bundle B has rank $2c_2$, its first Chern class is $c_1(B) = b = c_2/2$ and its cohomology is as follows

$$\begin{array}{cccc|c}
a & 0 & 0 & 0 & \uparrow i \\
a & b & 0 & 0 & \quad h^i(B(j)) \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 5b & \\
\hline
 & & j=-1 & j=0 & \to j
\end{array}$$

PROOF. – $H^1(B)$ being killed, $h^0(B)$ is equal to $h^0(E) + h^0(\mathcal{O}^a \oplus \mathcal{O}(1)^b) - h^1(E) = 5b$. Then $h^2(B(-2)) = h^2(E(-2)) = b$, $h^2(B(-3)) = h^2(E(-3)) = a$, $h^3(B(-3)) = h^3(\mathcal{O}(-3)^a \oplus \mathcal{O}(-2)^b) = a$ and $H^1(B(-1)) = 0$, since it has been just killed.

All remaining zeros follow from (1.3) immediately.

For B^* we have, in an obvious way

$$\begin{array}{cccc|c}
5b & 0 & 0 & 0 & \uparrow i \\
0 & 0 & 0 & 0 & \quad h^i(B^*(j)) \\
0 & 0 & b & a & \\
0 & 0 & 0 & a & \\
\hline
 & & j=-1 & j=0 & \to j
\end{array}$$

We now kill $H^1(B^*)$ and $H^1(B^*(-1))$ is a similar way as above. For a suitably chosen extension

$$0 \to B^* \to F \to \mathcal{O}^a \oplus \mathcal{O}(1)^b \to 0 \tag{1.4}$$

we have $H^1(F) = H^1(F(-1)) = 0$. All other groups $H^i(F(j))$, $i = 1, 2, 3$, $j = 0, -1, -2, -3$ vanish, too. Then, by the «Castelnuovo-Mumford' criterion» [HeS], see also (0.9) of the present paper, applied to F and F^* we conclude that F is a bundle of rank $4c_2 - 2$ with no intermediate cohomology and with

$$h^0(F) = 4c_2 - 2,$$

$$h^0(F(-j)) = 0 \quad \text{for } j < 0,$$

$$h^3(F(-1)) = h^3(F(-2)) = 0.$$

From the characterization of bundles with no intermediate cohomology on quadrics given in (0.8), we then obtain the crucial

LEMMA. – $F = \mathcal{O}^{4c_2-2}$.

CLAIM. – $B^* = \mathcal{O}^a \oplus K$ with K being the kernel of a map τ in

$$0 \to K \to \mathcal{O}^{5b-a} \to \mathcal{O}(1)^b \to 0\,. \tag{1.5}$$

PROOF. – Since F is trivial, (1.4) reduces to

$$0 \to B^* \to \mathcal{O}^{5c_2/2} \to \mathcal{O}(1)^b \to 0\,. \tag{1.6}$$

We have also, by (1.3), an embedding $0 \to \mathcal{O}^a \oplus \mathcal{O}(-1)^b \to B^*$. Hence a copy of $\mathcal{O}^a$ must factor out of B^*. What remains, is the kernel of an epimorphism τ, as stated. Then the sequence (1.3) becomes

$$0 \to E \to K^* \to \mathcal{O}(1)^b \to 0 \tag{1.7}$$

and the monad is as we have claimed.

Finally, we prove that condition iii) of Proposition (1.1) implies ii). Let K be the kernel of the latter morphism in the monad (1.2), i.e., the sequence

$$0 \to K \to \mathcal{O}^{c_2+2} \to \mathcal{O}(1)^{c_2/2} \to 0 \tag{1.9}$$

is exact.

(1.10) CLAIM. – *a*) K is stable and uniquely determined by E;

b) when $H^0(E(1)) = 0$, then conversely, E is also uniquely determined by K.

PROOF OF CLAIM (based on the idea from [AO2], 2.8). – Let us recall the criterion of Hoppe [Ho], (2.6): if for any integer $i = 1, \ldots, \operatorname{rank}(E) - 1$ holds

$$H^0((\Lambda^i(E)) = 0\,,$$

then the bundle E is stable.

In our situation we have $c_1(K^*) = -c_2/2$, $\operatorname{rank}(K) = (c_2/2) + 2$, hence $\mu(K^*) = c_2/(c_2+4)$ and $K^*_{\text{norm}} = K^*(-1)$. From the sequence dual to (1.9) we calculate that $H^0(K^*_{\text{norm}}) = H^0(K^*(-1)) = 0$. For $p \geq 1$ there is

$$\mu((\Lambda^{p+2}K^*)(-p)) = (p+2)\mu(K^*) - p = \frac{2c_2 - 4p}{c_2+4}\,,$$

hence if $p + 2 \leq \operatorname{rank}(K) - 1 = (c_2/2) + 1$, then $\mu((\Lambda^{p+2}K^*)(-p)) > 0$ and to check the stability of K^* it is sufficient to know that

$$H^0((\Lambda^{p+2}K^*)(-p-1)) = 0 \qquad \text{for } p = 0, \ldots, \frac{c_2}{2}+1 .$$

The case $p=0$ is easy. For $p \geq 1$, taking the second exterior power of (1.7) we get, after a twist

$$(\Lambda^{p+2}E)(-p-1) = 0 \to (\Lambda^{p+2}K^*)(-p-1) \to (\Lambda^{p+1}K^*)(-p)^{c_2/2} \to \ldots$$

hence the stability of K follows by induction on p. In particular,

$$H^0((\Lambda^{1+c_2/2}K^*)(-c_2/2)) = H^0(K) = 0 .$$

We now prove the second part of the first statement of (1.10). Assume that there exist two exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(-1)^b \arrow[r] & K \arrow[r, "p"] \arrow[dr, dashed, "p"] & E \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & \mathcal{O}(-1)^b \arrow[r] & K' \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

From (1.5) we see that $H^1(K^*(-1)) = 0$. The morphism p can be then lifted to a non-zero morphism $q\colon K \to K'$. Because K and K' are stable bundles of the same ranks and Chern classes, q must be an isomorphism.

To prove the second assertion of (1.10), let us notice first that if $H^0(E(1)) = 0$, then from (1.7), twisted and dualized, it follows that $H^0(K(1)) = \mathbb{C}^b$, hence there is only one immersion of $\mathcal{O}^b$ into $K(1)$.

We now may easily conclude the proof of Proposition (1.2). From the dual to (1.7) we get $H^0(E) = 0$ (since E is autodual) and from (1.7) twisted by -2 we calculated that $H^1(E(-2)) = 0$.

(1.11) PROPOSITION. - If E is as above, i.e., stable with $c_1(E) = 0$ and $H^1(E(-2)) = 0$ on $\mathbb{Q}_3$, then $E(c_2/2)$ is globally generated.

PROOF. - With the notation as above, $E(c_2/2)$ is an image of $K(c_2/2) = K \otimes \det(K)^{-1} = \Lambda^{b+1}(K^*)$ and K^* is globally generated as an image of $\mathcal{O}^{b-a}$.

(1.12) COROLLARY. - Let E be a stable, rank-2 bundle with $H^1(E(1)) = H^1(E(-2)) = 0$. Then $H^2(\mathrm{End}(E)) = 0$.

PROOF. - From $H^1(E(1)) = 0$, we calculate, tensoring (1.9) by E^*, that $H^2(E^* \otimes K) = 0$. Then, tensoring (1.7) again by $E = E^*$, we conclude that $H^2(E \otimes E^*) = 0$.

(1.13) PROPOSITION. - The stable bundles K arising as kernels as in (1.9) make up a smooth and Zariski open subset U of dimension $(5/4)c_2^2 + c_2 - 3$ of the moduli space of rank-$(c_2/2+2)$ bundles with Chern classes $-c_2/2$, $(c_2^2/2)$, $-(c_2^3/4)$ on Q_3. The set U is irreducible and unirational.

PROOF. – We proved in (1.10) that K is stable. From the sequence (1.9), its twists and its tensor product with K we calculate, in a standard way that $h^2(K\otimes K^*) = h^3(K\otimes K^*) = 0$. Thus the corresponding points of the moduli space are smooth. The stability of K implies $h^0(K\otimes K^*) = 1$ hence, by similar tricks we may easily calculate that

$$h^1(K\otimes K^*) = 1 - \chi(K\otimes K^*) = 1 - (c_2+2)\cdot\chi(K) + (c_2/2)\cdot\chi(K(-1)) =$$
$$= 1 - (c_2+2)(c_2+2) + (c_2/2)[5(c_2+2) - (c_2/2)] = \frac{5}{4}c_2^2 + c_2 - 3\,.$$

In order to see that the property to be a kernel in an exact sequence (1.9) is open in the moduli space, we first calculate the cohomology of such a K

$$\begin{array}{cccc|l}
c & 0 & 0 & 0 & \uparrow i \\
0 & 0 & 0 & 0 & \quad h^i(K(j)) \\
0 & 0 & b & a & \\
0 & 0 & 0 & 0 & \\
\hline
 & & j=-1 & j=0 & \rightarrow j
\end{array}$$

with $a = 3c_2/2 - 2$, $b = c_2/2$, $c = 5c_2/2$. The cohomology is natural, hence it remains the same on an open subset of the moduli space. On the other hand, having K with the cohomology as above, we may find a vector bundle

$$0 \to K \to F \to \mathcal{O}^a \oplus \mathcal{O}(1)^b \to 0\,, \tag{1.14}$$

defined by generator of $\mathrm{Ext}^1(\mathcal{O}^a \oplus \mathcal{O}(1)^b, K)$. For the bundle F we then have

$$\begin{array}{cccc|l}
c & 0 & 0 & 0 & \uparrow i \\
0 & 0 & 0 & 0 & \quad h^i(F(j)) \\
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & c & \\
\hline
 & & j=-1 & j=0 & \rightarrow j
\end{array}$$

and $h^0(F) = c$. By the characterization of bundles with no intermediate cohomology, see (0.8), F must be a trivial bundle, hence (1.14) reduces to (1.9), as we wanted. Now, an open subset of $\mathrm{Hom}(\mathcal{O}^{c_2+2}, \mathcal{O}(1)^{c_2/2})$, namely the one corresponding to stable bundles, surjects on U, hence U is irreducible and unirational. This concludes the proof of (1.13).

To study the moduli $M(0, c_2)$, we distinguish two types of bundles, these with $H^0(E(1)) \neq 0$ and those with $H^0(E(1)) = 0$. We are interested in the case $c_2 \leqslant 4$, but some things can be stated in a general set-up, as well.

(1.15) PROPOSITION. – If E is a stable rank-2 bundle on $\mathbb{Q}_3$ with $c_1(E)=0$ and $H^1(E(-2))=0$, but $H^0(E(1))\neq 0$, then E arises as an extension

$$0\to \mathcal{O}\to E(1)\to I_Z(2)\to 0\,, \tag{1.16}$$

where Z is a locally complete intersection curve with

$$H^0(\mathcal{O}_Z)=1+c_2/2\,.$$

If Z is smooth, then it is the sum of $1+c_2/2$ disjoint conics.

PROOF. – It is similar to that in [SSW] for $c_2=4$. Namely, let Z be the zero of a generic section of $E(1)$. Z is neither a surface nor empty nor zero-dimensional, hence a curve. By the adjunction formula we obtain $\omega_Z=\mathcal{O}(-1)|Z$, hence no connected component of Z is a single line. From the exact sequences (1.16) and

$$0\to I_Z(2)\to \mathcal{O}_{\mathbb{Q}_3}(2)\to \mathcal{O}_Z(2)\to 0 \tag{1.17}$$

we calculate (knowing the cohomology of E) that $H^0(I_Z)=0$ and $\dim H^1(I_Z)=c_2/2$. Moreover, $\deg Z=c_2(E(1))=c_2+2$. This concludes the proof.

(1.18) COROLLARY. – If in the above proposition Z is smooth, then the moduli space $M(0,\,c_2)$ is smooth at the corresponding points.

PROOF. – To prove the smoothness of the moduli at these points, let us notice that (1.16) tensored with $\mathcal{O}_Z$ gives

$$E(1)|Z=\mathfrak{N}_{Z/\mathbb{Q}_3}=\mathcal{O}_Z(1)\oplus\mathcal{O}_Z(1)\,,$$

where $\mathfrak{N}_{Z/\mathbb{Q}_3}$ is the normal bundle. We know already by the properties of the spectrum that $H^1(E(-2))=0$. Then $H^2(E(1))=H^1(E(-4))=0$. The vanishing of $H^2(\operatorname{End}E)$ now follows by tensoring (1.16) and (1,17) with $E(-1)$.

(1.19) PROPOSITION. – The family of bundles in $M(0,\,c_2)$ coming from disjoint conics is of dimension $(7/2)c_2+6$ if $c_2\geqslant 4$ and of dimension 9 if $c_2=2$.

PROOF. – The dimension of the family is equal to

$$\dim \operatorname{Ext}^1(I_Z(2),\,\mathcal{O})+(1+c_2/2)\cdot\dim G(2,\,4)-\dim H^0(E(1))\,,$$

the first dimension being $1+c_2/2=\dim H^0(\mathcal{O}_Z)$, [OSS], ch. 1 § 5 and the last one being generically 1 if $c_2\geqslant 4$ and 5 if $c_2=2$.

By (0.2), the component of $M(0,\,c_2)$ containing bundles arising from disjoint conics is of dimension $6c_2-3$. Hence, as already noticed in [SSW], for $c_2\geqslant 4$ the generic bundle does not come from disjoint conics. We may notice also (compare with Prop. 1.13) that $(5/4)c_2^2+c_2-3>6c_2-3$ for $c_2\geqslant 6$, but these two expressions are equal for $c_2=4$. This means that in general there exist some K's which do not come from any E and that this does not happen for $c_2=4$.

2. - The moduli space $M(0,2)$.

It is natural to expect some analogy between $M(0,2)$ on $\mathbb{Q}_3$ and $M_{\mathbb{P}^3}(0, 1)$. To some extent it is so. Let us recall that a stable, rank-2 bundle on $\mathbb{P}^3$ with $c_1 = 0$, $c_2 = 1$ is a null-correlation bundle and that the moduli space $M_{\mathbb{P}^3}(0, 1)$ is isomorphic to $\mathbb{P}^5 \setminus \mathbb{Q}_4 = \mathbb{P}(\Lambda^2 V) \setminus G(1, 3)$, where V is the linear 4-space with $\mathbb{P}(V) = \mathbb{P}^3$. We showed, (0.6), (1.11), that for this case bundles $E(1)$ are spanned, that $\dim H^0(E(1)) = 5$ and hence, by (1.16), any such bundle arises from two disjoint conics. By (1.19), the moduli is then 9-dimensional. It was shown in [SSW] that each bundle from $M_{\mathbb{Q}_3}(0, 2)$ is a pull-back of a null-correlation bundle on $\mathbb{P}^3$. To obtain a detailed description of the moduli $M(0,2)$, we present three different approaches. The first one is more geometric while in the second one we work on skew-symmetric matrices and in the third one on symmetric matrices.

(2.1) THEOREM. - The moduli space $M_{\mathbb{Q}_3}(0, 2)$ is a locally trivial algebraic fibration over $\mathbb{P}^4 \setminus \mathbb{Q}_3$ with fibre $\mathbb{P}^5 \setminus \mathbb{Q}_4$.

PROOF. - Let us consider the bundles K defined in the monad (1.2) as the kernels of the maps $\mathcal{O}^4 \to \mathcal{O}(1)$. We saw in (1.10) that any such K is uniquely determined by E. We then obtain a holomorphic map υ: $M(0, 2) \to M'$ where M' is a component of $M(-1, 2, 2)$ and will show that this gives a fibration as stated in (2.1); the stability of K was proved in (1.10). This is the main difference between the present case and the case of $M_{\mathbb{P}^3}(0, 1)$, where the kernel K is $\Omega^1_{\mathbb{P}^3}(1)$ whose moduli is a one-point space.

From the monad (1.1) we calculate

$$h^1(K \otimes K^*) = 4\,, \qquad h^2(K \otimes K^*) = 0\,, \qquad K(1) = \Lambda^2 K^*\,, \qquad h^0(K(1)) = 6$$

and hence the moduli of K (i.e. families of K's arising as kernels in

$$0 \to K \to \mathcal{O}^4 \to \mathcal{O}^4(1) \to 0)$$

is the space of all quadruples of hyperplanes in $\mathbb{P}^4 := \mathbb{P}(H^0(\mathcal{O}(1)))$ meeting outside the quadric $\mathbb{Q}_3$, in other words, $\mathbb{P}^4 \setminus \mathbb{Q}_3$. Geometrically, the map υ associates to each bundle E the point of intersection of planes which contain two skew conics—the zero locus of a section of $E(1)$. In particular, this point is determined uniquely by E (compare (3.7) below). Projection from this point down onto a hyperplane transversal to $\mathbb{Q}_3$ shows that the fibre of υ is the moduli space $M_{\mathbb{P}^3}(0, 1)$, i.e. $\mathbb{P}^5 \setminus \mathbb{Q}_4$ and also that the fibration is locally trivial.

(2.2) PROPOSITION. - The moduli space $M_{\mathbb{Q}_3}(0, 2)$ is fine.

PROOF. - A universal family can be constructed by pulling back the natural universal family $\{\mathfrak{U}\}$ on $\mathbb{P}^4 \setminus \mathbb{Q}_3 = \{$moduli space for cokernels of $\mathcal{O} \to \mathcal{O}(1)^4\}$ which is given by

$$0 \to \alpha^* \mathcal{O} \xrightarrow{f} \alpha^* \mathcal{O}(1)^4 \to \mathfrak{U} \to 0$$

with $\alpha: (\mathbb{P}^4 \setminus \mathbb{Q}_3) \times \mathbb{Q}_3 : \to \mathbb{Q}_3$ and over the point (s, x) which corresponds to $s: \mathcal{O}_{Q_3} \to \to \mathcal{O}_{Q_3}(1)^4$ we have $f(s, x) = s(x)$.

We may, however, prove (2.2) directly from a criterion of Maruyama [Mar]: the moduli space of stable bundles with the Hilbert polynomial $\chi(E(t)) = \sum_{i=0}^{3} a_i \binom{t+i}{i}$ is fine if $GCD(a_i) = 1$.

COROLLARY. – $M(0, c_2)$ is fine if $c_2 \equiv 2 \pmod 4$ and $M(-1, c_2)$ is fine if $GCD(c_2 - - 1, 4) = 1$.

PROOF. – It suffices to apply this criterion to the formulas given in (0.2).

We show the second method to study the moduli space $M(0,2)$ on $\mathbb{Q}_3$. Of course

(2.3) The space of all non-zero 5×5 skew-symmetric matrices modulo proportionality is the projective space $\mathbb{P}^9$. The projective coordinates $c_{12}, c_{13}, \dots, c_{45}$ in this $\mathbb{P}^9$ originate in the upper-right entries of such matrices. The subset of rank-4 matrices is isomorphic to $\mathbb{P}^9 \setminus \mathrm{Grass}(1, 4)$.

Let now

$$C_0 = [C_{11}, C_{22}, C_{33}, C_{44}, C_{44}, C_{55}],$$

where C_{ii} are the pfaffians of the matrix $[c_{ij}]$, i.e.

$$c_{11} = \sum (-1)^{\mathrm{sgn}(\sigma)} \cdot c_{\sigma(2)\sigma(3)} c_{\sigma(4)\sigma(5)},$$

σ running over permutations of the set $\{1, 2, 3, 4, 5\}$, etc. Let the quadric $\mathbb{Q}_3$ be given in $\mathbb{P}^4$ by $x^T Q x = 0$, where Q is a symmetric matrix of rank 5. Then we have

(2.4) THEOREM. – The moduli space $M(0, 2)$ is equal to $\mathbb{P}^9 \setminus V_4$, where V_4 is a quartic hypersurface given by $C_0^T \cdot Q \cdot C_0 = 0$.

We prove (2.4) in several simple steps.

(2.5) LEMMA. – The 4×4 determinants of a 5×5 skew-symmetric matrix belong to the ideal generated by the pfaffians C_{ii} in the polynomial ring $\mathbb{C}[c_{ij}]$.

PROOF. – A straightforward check.

(2.6) LEMMA. – A skew-symmetric 5×5 matrix C has rank 4 if and only if $C_0 = = [C_{11}, C_{22}, C_{33}, C_{44}, C_{55}]$ is not zero. If this is the case, then C_0^T is the only solution of $C \cdot x = 0$.

PROOF. – Follows from (2.5). This shows also that the space of all skew-symmetric 5×5 matrices of the maximal rank 4 is $\mathbb{P}^9 \setminus \mathrm{Grass}(1, 4)$.

(2.7) LEMMA. – Let $A, B, \widetilde{A}, \widetilde{B}$ be 5×4 matrices of rank 4. Then $AB^T = \widetilde{A}\widetilde{B}^T$ if and only if there exists an invertible D such that $\widetilde{A} = AD$, $\widetilde{B} = B(D^{-1})^T$.

PROOF. – Let $\widetilde{A} = AD$, $\widetilde{B} = BE$. Then $\widetilde{A}\widetilde{B}^T = ADE^TB^T = AB^T$. Since A and B are of maximal ranks, we may cancel both sides of the latter equality by A and B^T and the lemma follows.

(2.8) LEMMA. – Let the maps $g\colon \mathcal{O}(-1) \to \mathcal{O}^4$ and $f\colon \mathcal{O}^4 \to \mathcal{O}(1)$ in the monad (1.2) be given by $f = \sum a_i x_i$ and $g = \sum b_i^T x_i$, with a_i and b_i matrices of the size 1×4. If $A = [a_0, a_1, a_2, a_3, a_4]$, $B = [b_0, b_1, b_2, b_3, b_4]^T$ are 5×4 matrices, then the conditions for A and B to make up a monad are

a) AB^t is a skew-symmetric matrix of the maximal rank 4.

b) If $AB^t y = 0$ (i.e. if y is given by the pfaffians of AB^t) then $yQy^t \neq 0$.

PROOF. – The conditions to have a monad are

i) A, B have maximum rank.

ii) The points a_0, b_0 such that $A^t a_0 = 0$, $B^t b_0 = 0$ satisfy $a_0^t Q a_0 \neq 0$ $b_0^t Q b_0 \neq 0$.

iii) $x^t AB^t x = 0$ if x satisfies $x^t Q x = 0$.

By iii) we have $AB^t + BA^t = \lambda Q$ for some $\lambda \in \mathbb{C}$. It follows

$$\lambda a_0^t Q a_0 = a_0^t AB^t a_0^t + a_0^t BA^t a_0 = 0 + 0 = 0$$

and then $\lambda = 0$ and AB^t is a skew-symmetric matrix of rank 4. In particular $a_0 = b_0$. This proves the theorem.

(2.9) LEMMA. – Let f, g be as above and f', g' be another pair of maps like in Lemma (2.8), with the corresponding matrices A', B'. Then (f, g) give the same bundle as (f', g') if and only if $A' = AD$, $B' = B(D^{-1})^T$ with a non-singular D.

PROOF. – Directly from (2.7).

Hence, the map

$$M(0, 2) \to \mathbb{P}^9 \setminus V_4 \tag{2.10}$$

which takes a bundle E onto its Kronecker module AB^T (in the sense e.g. of [OSS]), is an embedding and in fact, an isomorphism. Indeed, let $C \in \mathbb{P}^9 \setminus V_4$ be a 5×5 skew-symmetric matrix of rank 4, whose rows are for example $r_1, r_2, r_3, r_4, \lambda_1 r_1 + \lambda_2 r_2 +$

$+\lambda_3 r_3 + \lambda_4 r_4$. Then $C = AB^T$ with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}, \qquad B^T = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix},$$

and A, B satisfy the monad condition.

(2.11) REMARK. – The fibration υ: $\mathbb{P}^9 \setminus V_4 \to \mathbb{P}^4 \setminus \mathbb{Q}_3$ with fibres $\mathbb{P}^5 \setminus \mathbb{Q}_4$ can be seen directly. Let Q be the identity matrix, i.e. $\mathbb{Q}_3 = (\sum x_i^2 = 0)$ in $\mathbb{P}^4$. Then

$$\mathbb{P}^9 \setminus V_4 \ni M = [e_{ij}] \xrightarrow{\upsilon} x = [C_{11}, C_{22}, C_{33}, C_{44}, C_{55}] \in \mathbb{P}^4 \setminus \mathbb{Q}_3,$$

where C_{ii} are the pfaffians. The equation of V_4 is now $\sum C_{ii}^2 = 0$.

We shall now discuss the jumping line variety of bundles from $M(0,2)$.

(2.12) PROPOSITION. – Let $\langle x, y \rangle = l \subset \mathbb{Q}_3$ be a line and $E \in M(0,2)$ be a bundle defined by the monad corresponding to the matrices A, B. Then the Kronocker module AB^T of E is nondegenerate on l, i.e., $x^T AB^T y \neq 0$, if and only if the restriction $E|l$ is trivial.

PROOF. – Since $E|l$ is the cohomology of the monad

$$\mathcal{O}(-1)|l \to \mathcal{O}^4|l \to \mathcal{O}(1)|l,$$

we can translate the proof of Lemma (4.2.3) in [OSS] almost word for word.

(2.13) COROLLARY. – The variety of jumping lines of an $E \in M(0,2)$ is a quadric $\mathbb{Q}_2$ in $\mathbb{P}^3 := \mathrm{Fano}(\mathbb{Q}_3)$. The bundle is determined uniquely by its variety of jumping lines.

PROOF. – The condition for a line $l = \langle x, y \rangle$ to be a jumping one is $x^T AB^T y = 0$, which is a linear equation in the Plücker coordinates $p_{ij} = \begin{vmatrix} x_i x_j \\ y_i y_j \end{vmatrix}$ in $\mathbb{P}^9 \supset \mathrm{Grass}(1,4) \supset \mathbb{P}^3 =$ the variety of lines in $\mathbb{Q}_3$, the embedding $\mathbb{P}^3 \subset \mathbb{P}^9$ being 2-Veronese.

(2.14) All quadratic surfaces in $\mathbb{P}^3 = \mathbb{P}(W)$ form a $\mathbb{P}^9$ and it is worth to remark that the proof of (2.13) gives an explicit correspondence between this $\mathbb{P}^9$ and that of (2.3).

From the above discussion a clear geometric picture emerges. Any skew symmetric matrix $A \in \mathbb{P}^9 \setminus V_4$ determines a «partial» null-correlation on $\mathbb{P}^4 \supset \mathbb{Q}_3$, namely by $x \to \ker(x^T A)$. Such a null-correlation associates to a point a hyperplane passing through a point $C_0 = [C_{11}, C_{22}, C_{33}, C_{44}, C_{55}]$, where C_{ii} are the pfaffians. The jumping lines through a point $x \in \mathbb{Q}_3$ are now the lines on the 2-quadric which is the intersection of the null-hyperplane of x with our quadric $\mathbb{Q}_3$. It may happen that the intersection is a cone, not a smooth $\mathbb{Q}_2$. These «lucky» points have a whole $\mathbb{P}^1$ of jumping lines passing through them, not only two. The following examples show that the configuration of jumping lines may be different for various bundles.

EXAMPLE 1. – Let the quadric $\mathbb{Q}_3$ be $2x_0x_1 + 2x_2x_3 + x_4^2 = 0$ and the bundle E be the cohomology of the monad

$$\mathcal{O}(-1) \xrightarrow{x_0,\, x_1,\, x_2,\, x_3} \mathcal{O}^4 \xrightarrow{x_1,\, -x_0,\, x_3,\, -x_2} \mathcal{O}(-1).$$

Then the line $\langle x, y\rangle$ is a jumping one iff

$$x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0 .$$

Consider the lines $l_1 = \{x_1 = x_3 = x_4\}$, $l_2 = \{x_0 = x_2 = x_4\}$. Then

$$l \text{ is a jumping line iff } l \text{ meets } l_1 \text{ or } l_2 .$$

Hence the family of jumping lines consists of two planes in $\mathbb{P}^3$, namely the union of the apolar planes to the points $[l_1]$, $[l_2]$ in $\mathbb{P}^3 = \mathrm{Fano}(\mathbb{Q}_3)$. Indeed, through any point $x \notin l_1 \cup l_2$ there pass two jumping lines: the one joining $x = [x_0, x_1, x_2, x_3, x_4]$ with $p = [-x_3, 0, x_1, 0, 0]$ on l_1 and the other joining x with $[0, -x_2, 0, x_0, 0] \in l_2$. Let us also notice that $l_1 \cup l_2$ is the degeneracy locus on $\mathbb{Q}_3$ of the matrix

$$\begin{bmatrix} x_1 & -x_0 & x_3 & -x_2 & 0 \\ x_1 & x_0 & x_3 & -x_2 & x_4 \end{bmatrix}.$$

EXAMPLE 2. – Let $\mathbb{Q}_3$ be $\sum_{i=0}^{4} x_i^2 = 0$ and the monad be given by

$$\mathcal{O}(-1) \xrightarrow{x_0,\, x_1,\, x_2,\, x_3} \mathcal{O}^4 \xrightarrow{f_0, f_1, f_2, f_3} \mathcal{O}(-1)$$

with $[f_0, f_1, f_2, f_3] = [-x_1 - ix_3/2,\ x_0 - 3ix_2/2,\ 3ix_1/2 - x_3,\ ix_0/2 + x_2]$. Then the degeneracy locus on $\mathbb{Q}_3$ of the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 - ix_3/2 & x_0 - 3ix_2/2 & 3ix_1/2 - x_3 & ix_0/2 + x_2 \end{bmatrix}$$

consists of two points $p_1 = [i, i, 1, -1, 0]$, $p_2 = [i, -i, 1, 1, 0]$ and $\langle x, y\rangle$ is a jumping line if and only if

$$x_0(-y_1 - iy_3/2) + x_1(y_0 - 3iy_2/2) + x_2(3iy_1/2 - y_3) + x_3(iy_0/2 + y_2) = 0 .$$

Hence for every $x \in \mathbb{Q}_3$, $x \neq p_1, p_2$ there pass two jumping lines and lines through one of the p_1, p_2 are jumping. The quadric which parametrizes the jumping lines in the $\mathbb{P}^3$ of lines is a cone with vertex in $[\langle p_1, p_2\rangle]$.

We have $M_{\mathbb{Q}_3}(0, 2) \cong \mathbb{P}^9 \setminus V_4$. The following theorem describes V_4 geometrically, looking at $\mathbb{P}^9$ as the variety of quadrics in $\mathbb{P}^3$ (see (2.14)), equipped with the nullcorrelation N.

(2.15) THEOREM. – *a*) No double plane belongs to $\mathbb{P}^9 \setminus V_4$, that is the 2-Veronese embedding of $\mathbb{P}^3$ lies in V_4.

b) A pair of intersecting planes belongs to $\mathbb{P}^9 \setminus V_4$ if and only if the common line of the two planes is not isotropic.

c) A quadric cone belongs to $\mathbb{P}^9 \setminus V_4$ if and only if in its ruling there are two distinct isotropic lines.

d) A smooth quadric belongs to $\mathbb{P}^9 \setminus V_4$ if and only if it has four isotropic lines, (two in each ruling).

PROOF. – Consider a smooth quadric in $\mathbb{P}^3 \cong \mathrm{Gr}(\mathbb{P}^1, \mathbb{Q}_3)$ with exactly four isotropic lines, two in each ruling. These lines meet in four points in a configuration ($^4 2$). We have then four quadric cones in $\mathbb{Q}_3$ whose vertices are joined by a dual configuration ($^4 2$). Let $P_i \in \mathbb{P}^3$ $(i = 1, \ldots, 4)$ be the four vertices of the cones and z_i be the vectors of their homogeneous coordinates. Let C be the skew- symmetric matrix corresponding to the smooth quadric we consider, see (2.14). Let Q be the symmetric matrix defining $\mathbb{Q}_3$.

We see that Cz_i is proportional to Qz_i, hence $rk(C) = 4$. Let $c_0 \in \mathbb{P}^3$ be the point with coordinates given by the five principal pfaffians of C. We get $c_0^t C z_i = 0 z_i = 0$, hence $c_0^t Q z_i = 0$ $\forall i$, that is c_0 is the pole of the hyperplane $Z = \langle P_1, P_2, P_3, P_4 \rangle$. $Z \cap \mathbb{Q}_3$ is a smooth quadric because a cone cannot contain four lines in a configuration ($^4 2$). Hence $c_0 \notin \mathbb{Q}_3$, that is $C \notin V_4$ by the definition of V_4. It follows that no quadric with four isotropic lines belong to V_4. Now it is sufficient to check that the degree of the closure of the variety of quadrics without four isotropic lines is four. It is easy and can be also seen from the tables I and II.

Hence the case *d*) of the theorem is proved. The cases *a*), *b*) and *c*) follow from the case *d*) by a degeneration argument.

(2.16) REMARK. – If $p(l)$ is a point(line) in $\mathbb{P}^4$, we denote by π_p (π_l) the polar hyperplane (plane) with respect to $\mathbb{Q}_3$. Referring to the above theorem the corresponding configurations of jumping lines in $\mathbb{Q}_3$ are:

b) There are two disjoint lines $l_1, l_2 \subset \mathbb{Q}_3$ such that a line $l \subset \mathbb{Q}_3$ is jumping, if and only if $l \cap l_1 \neq \emptyset$ or $l \cap l_2 \neq \emptyset$.

In this case $\{x \in \mathbb{Q}_3 \mid \text{ all lines through } x \text{ are jumping}\} = l_1 \cup l_2$.

c) There are a line $l \subset \mathbb{Q}_3$ and a smooth conic $C \subset \pi_l$ meeting l in two disjoint points. A line l' is jumping if and only if there is a point $p \in C$ such that l' belongs to the same ruling of l in the quadric $\pi_p \cap \mathbb{Q}_3$.

In this case $C \cap l = \{P_1, P_2\}$ and

$$\{x \in \mathbb{Q}_3 \mid \text{all lines through } x \text{ are jumping}\} = \{\mathrm{P}_1, \mathrm{P}_2\}.$$

d) There are a line l not contained in $\mathbb{Q}_3$ meeting $\mathbb{Q}_3$ in P_1, P_2, a plane $\pi \supset l$ meeting $\mathbb{Q}_3$ in a smooth conic D and a smooth conic C tangent to D in P_1, P_2. For

TABLE I. – *Orbits of* $Sp(4) = \{G \mid GJG^t = J\}$ *on* $\mathbb{P}^9$, *see* (2.3) *and* (2.20).

Orbit	Projective Jordan form of AJ^{-1}	Dimension	Degree of closure	Description of closure
A_λ $\lambda \neq 0, 1$	$\begin{bmatrix} \sqrt{\lambda} & & & \\ & -\sqrt{\lambda} & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$	8	4 for $\lambda \neq 1$ 2 for $\lambda = -1$	for $\lambda = -1$ it is $\sigma(A) = 0$, otherwise a hypersurface $\dfrac{\det(A)}{\sigma^2(A) - 4\det(A)} = \dfrac{\lambda}{(\lambda-1)^2}$
B	$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	8	4	$\det(A) = 0$
C	$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$	6	5	$Gr(\mathbb{P}^1, \mathbb{P}^4)$; $\operatorname{rk}(A) \leq 2$; equations: $AJ^{-1}A = \mu J$
D	$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$	6	10	$\operatorname{Sec}(\mathbb{P}^3_{\mathcal{O}(2)})$; $\operatorname{rk}(A) \leq 2$
E	$\begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix}$	8	4	V_4
F	$\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	7	8	$\{\sigma = 0\} \cap \{\det(A) = 0\}$
G	$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$	5	10	Image of $\mathbb{P}(T\mathbb{Q}_3)$ through $\mathcal{O}_{\mathbb{P}}(1)$ {tangent lines to $\mathbb{Q}_3$} equations: $AJ^{-1}A = 0$.
H	$\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$	3	8	$\mathbb{P}^3_{\mathcal{O}(2)}$; $\operatorname{rk}(A) = 1$; spinor variety of lines in $\mathbb{Q}_3$

TABLE II. – *Orbits of* Spin (5) *on* $\mathbb{P}^9 = \{$quadrics in $\mathbb{P}^9\}$, *see* (2.14), (2.20).

Orbit	Geometrical description in terms of quadric surfaces	Closure	Singular locus of closure
A_λ	smooth quadrics with four isotropic lines	$A_\lambda \cup F \cup G \cup H$	$\overline{G}$ for $\lambda \neq -1$ $\emptyset$ for $\lambda = -1$
B	cones with two isotropic lines	$B \cup D \cup F \cup G \cup H$	D
C	smooth quadrics with all lines of a ruling isotropic two isotropic in the other ruling	$C \cup G \cup H$	$\emptyset$
D	two planes with non-isotropic intersection	$D \cup G \cup H$	H
E	smooth quadrics with one line of a ruling isotropic two isotropic in the other ruling	$C \cup E \cup F \cup G \cup H$	$\overline{C}$
F	cones with one isotropic line	$F \cup G \cup H$	$\overline{G}$
G	two planes with isotropic intersection	$G \cup H$	H
H	doubles planes	H	$\emptyset$

every $p \in C$ there are two rulings l_p^1, l_p^2 in the smooth quadric $\pi_p \cap \mathbb{Q}_3$ which form two distinguished families l^1, l^2 parametrized by C. A line l' is jumping if and only if there is a point $p \in C$ such that l' belongs to the ruling l_p^1. The choice l_p^2 corresponds to another smooth quadric whose lines are antipolar with respect to the first ones. Hence we have an involution i on the open part of $M(0,2)$ given by $rk-4$ quadrics. In terms of symmetric matrices the involution i is given by $A \to JA^{-1}J^t$, where J is the skew-symmetric matrix defining the nullcorrelation in $\mathbb{P}^3$. The variety of fixed points of i is C in the notation of the tables I and II, (its closure is $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^4)$). One can check that every E is isomorphic to $i(E)$ up to automorphisms of $\mathbb{Q}_3$ (see (2.21)).

Let P_0 be the pole of the line l with respect to C. Let $\bar{l}$ be the polar line of l with respect to the smooth quadric $\pi_{P_0} \cap \mathbb{Q}_3$ (or equivalently the polar line of π with respect to $\mathbb{Q}_3$).

We have $\bar{l} \cap \mathbb{Q}_3 = \{P_3, P_4\}$ and in this case $\{x \in \mathbb{Q}_3 \mid$ all lines through x are jumping$\} = \{P_1, P_2, P_3, P_4\}$. The four points P_i are joined by four lines in $\pi_{P_0} \cap \mathbb{Q}_3$ in a configuration ([4]2). We remark that the other ruling of the smooth quadric in $\mathbb{P}^3$ defines a conic $\overline{C}$ in the plane $\pi_l \supset \bar{l}$.

Making use of the isomorphism stated in (2.10), we denote the fibrations (2.1) and (2.11) by the same symbol φ. We have then a third «very geometric» description of the fibration.

(2.17) PROPOSITION. – If $p \in \mathbb{P}^9 \setminus V_4$ corresponds to a rk 2 quadric (case b) then $\varphi(p)$ is the pole of the hyperplane $\langle l_1, l_2 \rangle$.

If $p \in \mathbb{P}^9 \setminus V_4$ corresponds to a rk 3 quadric (case c) then $\varphi(p)$ is the pole of l with respect to C.

If $p \in \mathbb{P}^9 \setminus V_4$ corresponds to a rk 4 quadric (case d) then $\varphi(p)$ is the pole of l with respect to C (or equivalently the pole of $\bar{l}$ with respect to $\overline{C}$), or the pole of $\langle l, \bar{l} \rangle$ with respect to $\mathbb{Q}_3$.

PROOF. – In order to see in the case (d) that P_0 is polar to $l = \langle P_1, P_2 \rangle$ with respect to C consider that, as P_0 is polar to $\{P_1, P_2, P_3, P_4\}$, all the lines P_0P_1, P_0P_2, P_0P_3, P_0P_4 are tangent to $\mathbb{Q}_3$. Then the polar hyperplane of any point $Q \in P_0P_1$ must contain P_1, P_3, P_4. If $Q \in C$ then the quadric $\pi_Q \cap \mathbb{Q}_3$ must contain the lines P_1P_3 and P_1P_4, hence at least one of them should be jumping. But in the smooth quadric in $\mathbb{P}^3$ of jumping lines two lines of the same family never intersect, hence the situation above can happen only for one Q, that is for $Q = P_1$. This means that P_0P_1 is tangent to C. In the same way we can show that P_0P_2 is tangent to C.

Now a degeneration argument again proves the case (c). The case (b) is clear from the proof of (2.15). The fact that the cases (b) and (c) really occur is clear from the Examples 1 and 2 above.

(2.18) COROLLARY. – The subvariety of $\mathbb{P}^9 \setminus V_4$ given by rk-2 quadrics (it is $\overline{D}$ in the notation of the tables) cuts every fibre of the fibration φ: $\mathbb{P}^9 \setminus V_4 \to \mathbb{P}^4 \setminus \mathbb{Q}_3$ in a variety isomorphic to $S^2\mathbb{P}^1 \setminus \{\text{diagonal}\}$ whose closure in $\mathbb{P}^5 \setminus \mathbb{Q}_4$ is the Veronese surface. The subvariety of $rk \leq 3$-quadrics (it is $\overline{B}$ is the notations of the tables) cuts every fiber of φ in a subvariety whose closure is the secant variety of the Veronese surface.

We may also explain what the unstable planes for E are. Here the description of $M_{\mathbb{Q}_3}(0, 2)$ as a fibre space is more convenient.

(2.19) PROPOSITION. – Let $E \mapsto \varphi(E) = p \in \mathbb{P}^4 \setminus \mathbb{Q}_3$ be the fibration as in (2.1) and H be a hyperplane in $\mathbb{P}^4$ such that $\mathbb{Q}_3 \cap H$ is smooth. Then $E \mid \mathbb{Q}_3 \cap H$ is stable (with respect to $\mathcal{O}(1)$) iff $p \notin H$.

PROOF. – Let $p \in H$. The projection π: $\mathbb{Q}_3 \to \mathbb{P}^3$ from p induces

$$\pi_H: \mathbb{Q}_3 \cap H \to \mathbb{P}^3 \cap H = \mathbb{P}^3.$$

There exists a null-correlation bundle E' on $\mathbb{P}^3$ with $\pi^*(E') = E$, $\pi_H^*(E' \mid \mathbb{P}^2) =$

$= E|\mathbb{Q}_3 \cap H$, [SSW]. Since $h^0(E'|\mathbb{P}^2) = 1$ (a property of the null-correlation bundles), we have $h^0(E|\mathbb{Q}_3 \cap H) = 1$. Let $p \notin H$ and $s \in H^0(E(1))$ be a section with the zero set Y given by two disjoint conics C_1 and C_2. The planes the conics are contained in meet at p and $Z = H \cap Y$ is a 4-point scheme in $\mathbb{Q}_3 \cap H$. Tensoring (1.16) with $\mathcal{O}_{\mathbb{Q}_3 \cap H}$ we get

$$0 \to \mathcal{O}_{\mathbb{Q}_3 \cap H} \to E(1)|\mathbb{Q}_3 \cap H \to I_Z(2) \to 0 .$$

The lines $H \cap C_1$ and $H \cap C_2$ are skew, hence Z is not contained in a plane, that is $h^0(I_Z(1)) = 0$, otherwise the plane of the two conics meet in H. It follows that $h^0(E|\mathbb{Q}_3 \cap H) = 0$, too.

(2.20) Let J be the nondegenerate 4×4 skew-symmetric matrix which defines the nullcorrelation N on $\mathbb{P}^3$. We have the action of $\mathrm{Sp}(4) = \{G | GJG^t = J\}$ over $\mathbb{P}(S^2 V) \cong \mathbb{P}^9$ given by $A \to GAG^t$ (A is a symmetric matrix).

Now, looking at the 2:1 covering

$$\mathrm{Sp}(4) \cong \mathrm{Spin}(5) \to SO(5) \cong \mathrm{Aut}(\mathbb{Q}_3)$$

we see that the action in (2.20) induces the natural action of $\mathrm{Aut}(\mathbb{Q}_3)$ on the open subset $\mathbb{P}^9 \setminus V_4 \cong M(0, 2)$. We are interested in the orbits of this action. The formula $(GAG^t)J^{-1} = G(AJ^{-1})G^{-1}$ shows that we reduce to look at the possible projective Jordan forms of AJ^{-1}. In the tables I, II we list the results obtained by WILLIAMSON ([Wil], pag. 162) with a geometrical interpretation. For the computations of singular loci the program [BS] was useful.

It is easy to check that the characteristic polynomial of AJ^{-1} is an even polynomial, that is

$$\det(AJ^{-1} - tI) = t^4 + \sigma(A)t^2 + \det(A). \tag{2.21}$$

This defines the quadratic form $\sigma(A)$ whose zero locus is a smooth quadric hypersurface in $\mathbb{P}^9$. In particular $\sigma(A)$ and $\det(A)$ are affine $\mathrm{Sp}(4)$-invariant and the expression $\sigma^2(A)/\det(A)$ is a projective $\mathrm{Sp}(4)$-invariant. A look at the tables I and II proves the following

(2.22) THEOREM. – (i) The equation of V_4 in $\mathbb{P}^9$ is $\sigma^2(A) - 4 \cdot \det(A) = 0$,

(ii) $\mathrm{Sing}\ V_4 \cong \mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^4)$,

(iii) Let E_A be the bundle corresponding to the symmetric matrix $A \in \mathbb{P}^9 \setminus V_4$. There exists an automorphism $g \in \mathrm{Aut}(\mathbb{Q}_3)$ such that $g^* E_A \cong E_{A'}$ if and only if

$$\begin{cases} \det(A)/(\sigma^2(A) - 4\det(A)) = \det(A')/(\sigma^2(A') - 4\det(A')) \\ \mathrm{rk}(A) = \mathrm{rk}(A'). \end{cases}$$

(2.23) REMARK. – The adjoint representation of $\mathrm{Spin}(5)$ (which is $\Lambda^2 W$ where $\mathbb{P}^4 = \mathbb{P}(W)$) acts on $\mathbb{P}^9 = \mathbb{P}(\Lambda^2 W)$. If $B \in \mathbb{P}(\Lambda^2 W)$ is represented by a 5×5 skew-symmetric matrix and Q is the matrix defining $\mathbb{Q}_3$, then the polynomial

$$\det(BQ^{-1} - tI) = -(t^5 + s(B)t^3 + c(B)t)$$

is a Spin(5)-invariant and the analogue of $\det(AJ^{-1} - tI)$ considered above. We could have worked directly on this polynomial and obtain the same results. In the correspondence described in (2.14) we have

$$\frac{16\det(A)}{\sigma^2(A) - 4\det(A)} = \frac{s^2(B) - 4c(B)}{c(B)}.$$

In particular, the equation of V_4 is $c(B) = 0$. Also, $\sigma(A) = 0$ and $s(B) = 0$ define the same quadric. Moreover, with the help of [BS] one can check that (with the notation of the table I), the equation of H in matrix form is $BQ^{-1}B = 0$ and the equation of $\overline{G}$ is (up to multiple structures) $BQ^{-1}BQ^{-1}B = 0$.

3. – The moduli space $M(0,4)$.

Let us now discuss the case $c_2 = 4$. Some such bundles arise from three disjoint conics and make up a family of dimension 20, [SSW].

(3.1) PROPOSITION. – Let E be a bundle from $M(0,4)$ whose generic section vanishes on three disjoint conics. Then $\dim H^0(E(1)) = 2$ or 1 depending whether E is or is not a pullback of a bundle E' on $\mathbb{P}^3$ under a double covering $\mathbb{Q}_3 \to \mathbb{P}^3$.

In order to prove proposition (3.1) we need the following lemma.

(3.2) LEMMA. – Let P_1, P_2, P_3 be three 2-planes in $\mathbb{P}^4$, let $P = P_1 \cap P_2 \cap P_3$. Then

$$H^0(I_{P,\mathbb{P}^4}(2)) = \begin{cases} 0 & \text{if } P_1 \cap P_2 \cap P_3 = \emptyset, \\ 1 & \text{if } P_1 \cap P_2 \cap P_3 \text{ is one point}. \end{cases}$$

PROOF. – Let us assume that the planes which contain the conics are disjoint. There is no smooth quadric $\mathbb{Q}_3 \subset \mathbb{P}^4$ containing them. Let us then suppose that they are contained in a cone through a point q. Take a generic $\mathbb{P}^3$ not containing q. A smooth quadric $\mathbb{P}^3 \cap C$ then contains three disjoint lines. This is not possible. Of course no cone with vertex a line can contain a plane not containing the line, otherwise it contains all $\mathbb{P}^4$. If the vertex line meets the three planes at three points, we obtain the cone containing three different $\mathbb{P}^3$'s, hence a variety of degree at least 3, a contradiction again.

If $P_1 \cap P_2 \cap P_3 = \{p\}$ then we may pick three lines on our planes that lie in one $\mathbb{P}^3$ and find a smooth 2-quadric K containing them. It is easy to see that the cone over K

with vertex p is the only quadric in $\mathbb{P}^4$ which contains $P_1 \cap P_2 \cap P_3$. This proves the lemma.

(3.3) COROLLARY. – Three conics in a general position in $\mathbb{P}^4$ do not lie on a 3-quadric.

PROOF OF PROPOSITION (3.1). – Let us assume that $\dim H^0(E(1)) = 1$. Then $\dim H^1(E(1)) = 1$, too. Let Z be the union of the three conics. From (1.16) we get $\dim H^0(I_{Z, Q_3}(2)) = 0$, $H^1(I_{Z, Q_3}(2)) = 1$. Since $I_{Q_3, \mathbb{P}^4}(2) = \mathcal{O}_{\mathbb{P}^4}$, from

$$0 \to I_{Q_3, \mathbb{P}^4}(2) \to I_{Z, \mathbb{P}^4}(2) \to I_{Z, Q_3}(2) \to 0$$

and from the preceding lemma we get

$$\dim H^0(I_{Z, \mathbb{P}^4}(2)) = \dim H^1(I_{Z, \mathbb{P}^4}(2)) = 1\,.$$

Hence there is one 3-quadric in Q_4 which contains Z. The planes must then be disjoint, since otherwise apart from a smooth quadric passing through the conics, there is one described in the proof of the Lemma (3.2).

The bundles whose conics' planes meet at one point p have then $\dim H^0(E(1)) = 2$ and the above discussion shows also the converse. It is easy to see that such E's are pullbacks of bundles on $\mathbb{P}^3$ by a double projection of Q_3 from p and that they make up a family of dimension 17.

A local deformation of bundles discussed in (3.1) need not be such, [SSW], and if it does not arise from three conics, then $H^0(E(1)) = 0$.

This argument generalizes using Manolache's result in the appendix, and we have:

(3.4) THEOREM. – $M(0,4)$ is irreducible, unirational and reduced.

PROOF. – Let E be a bundle from $M(0,4)$. Suppose first we have $h^0(E(1)) \neq 0$. Then every nonzero section of $E(1)$ vanishes on a degree 6 curve Y with $\omega_Y = \mathcal{O}_Y(-1)$ (see (1.15)). In the appendix Manolache shows that every family of bundles coming from such curves has dimension ≤ 20. By (0.2) $\chi(\operatorname{End}(E)) = -20$, hence $h^1(\operatorname{End} E) - h^2(\operatorname{End} E) = 21$ and every component of $M(0,4)$ must be of dimension ≥ 21. It follows that the generic bundle E in every component has $h^0(E(1)) = 0$. By the formula to the Euler-Poincaré characteristic (see (0.2)), we infer that $h^1(E(1)) = 0$. Since for $c_2 = 4$ we have $(5/4)c_2^2 + c_2 - 3 = 6c_2 - 3 = 21$ the claim follows from (1.10), (1.12) and (1.13).

(3.5) PROPOSITION. – Let E be a generic bundle in $M(0,4)$

(*a*) $E(2)$ is globally generated;

(*b*) a generic section of $E(2)$ vanishes on a curve of genus 7 and degree 12 on Q_3.

PROOF. - (a) follows by (1.11). Then we tensor (1.16) and (1.17) with $\mathcal{O}(-2)$ and check, plugging in the cohomology of E, that $h^0(\mathcal{O}_Z) = \mathbb{C}$. Then, by the adjunction formula, $\omega_Z = \mathcal{O}(1)|_Z$, hence $g(Z) = 5 + (c_2/2)$ by the «$2g-2$»-formula. This proves (b).

4. - The moduli space $M(-1,2)$.

The main result of this section is the following

(4.1) THEOREM. - The moduli space $M := M_{\mathbb{Q}_3}(-1, 2)$ is a locally trivial algebraic fibration over $B := \mathbb{P}^4 \setminus \mathbb{Q}_3$ with fibre being two disjoint copies of $\mathbb{P}^2 \setminus \mathbb{Q}_1$. In particular, it is a Stein manifold of dimension 6, rational, irreducible and smooth.

REMARK. - Contrary to the case $c_1 = 0$, no bundle in $M_{\mathbb{Q}_3}(-1, 2)$ is a pullback of a bundle from $\mathbb{P}^3$ by a double covering. Indeed there is no rank-2 stable bundle on $\mathbb{P}^3$ with $c_1 = -1$, $c_2 = 1$, [OSS].

The main idea of our proof of (4.1) consists in showing that all our bundles come either from disjoint lines in $\mathbb{Q}_3$ or from a double line living on some smooth $\mathbb{Q}_2 \subset \mathbb{Q}_3$. Then we study such bundles: We start with some preparatory lemmas. Let E be from $M_{\mathbb{Q}_3}(-1, 2)$.

(4.2) PROPOSITION. - The cohomology of $E(j)$ is as follows

$$
\begin{array}{cccc|cl}
0 & 0 & 0 & & 0 & \\
1 & 0 & 0 & & 0 & h^i(E(j)) \\
0 & 0 & 1 & & 0 & \\
0 & 0 & 0 & & 2 & \\
\hline
j=-2 & j=-1 & j=0 & & j=1 & j
\end{array}
$$

PROOF. - The spectrum of E consists of -1 only. Then $h^1(E(-j)) = 0$ for $j \leq -1$ and $h^1(E) = 1$. From the Riemann-Roch formula for the Euler-Poincaré characteristic, see (0.2), we calculate that $\chi(E(1)) = 2$. In a standard way (using Serre's duality and stability) we also find that the groups $H^0(E(j))$ and $H^3(E(j))$, $j \leq 0$, are as we claim.

Since $\chi(E(1)) = 2$, the bundle $E(1)$ has at least two sections. Let Y be the zero set of such a generic section. Then $\deg Y = c_2(E(1)) = 2$, and by adjunction formula $\omega_Y = \mathcal{O}(-2)|Y$. Moreover, this section gives rise to an extension

$$0 \to \mathcal{O} \to E(1) \to I_Y(1) \to 0\,, \tag{4.3}$$

hence $h^0(I_Y(1)) \geq 1$, this means that at least one linear form vanishes on Y. Then $Y \subset \mathbb{Q}_3 \cap H$ where H is a hyperplane in $\mathbb{P}^4$. If $h^0(I_Y(1))$ were ≥ 2, there would exist two independent forms that vanish on Y, hence $Y \subset \mathbb{Q}_3 \cap P$ with a two-dimensional plane $P \subset \mathbb{P}^4$. This, by adjunction, would imply $\omega_Y = \mathcal{O}(-1)|Y$, a contradiction. Hence $h^0(E(1)) = 2$, which proves (4.2).

(4.4) PROPOSITION. – The zero set Y of a section of $E(1)$ is a divisor of type $(2, 0)$ on a smooth hyperplane section $\mathbb{Q}_2 \subset \mathbb{Q}_3$ (and hence is either a disjoint sum of two lines or a double line). The zero sets Y_s, Y_t of two sections of $E(1)$ lie on the same smooth 2-quadric $\mathbb{Q}_2$ and cut a system g_2^1 without base points.

PROOF. – From the preceding discussion it follows easily that there exists a hyperplane H such that $Y \subset \mathbb{Q}_3 \cap H$. If $\mathbb{Q}_3 \cap H$ were a cone, Y would be a curve of degree 2 on the cone, which is a complete intersection with a plane. Hence $\omega_Y = \mathcal{O}(-1)$, what is a contradiction. Thus $\mathbb{Q}_3 \cap H$ is smooth, and in a similar way we exclude Y of type $(1,1)$.

Let s, t be the two sections of $E(1)$ with s vanishing on Y_S. As in lemma 9.3 of [Ha] it follows that Y_s and Y_t lie on the same 2-quadric. The zero sets Y_s and Y_t determine then a system g_2^1. We show that it has no base points. To this end, it is sufficient to prove that for a given line L, there exists $s \in H^0(E(1))$ not vanishing identically on L. Hence we have to show that $h^0(I_L \otimes (E(1))) \leq 1$. Tensoring the Koszul complex

$$0 \to \mathcal{O}(-1) \to S \to I_L \to 0$$

by $E(1)$ we obtain

$$0 \to E \to E \otimes S(1) \to I_L \otimes E(1) \to 0\,.$$

If L' is generic, we have $h^0(I_{L'} \otimes E(1)) = 0$, hence by the above exact sequence with L' in place of L we get $h^0(E \otimes S(1)) = 0$. Thus the cohomology sequence of the above sequence gives

$$0 \to H^0(I_L \otimes E(1)) \to H^1(E) = \mathbb{C}$$

as we wanted.

We may now prove our description of the moduli $M(-1, 2)$ as stated in (4.1). The quadric surfaces on $\mathbb{Q}_3$ form a $\mathbb{P}^4$. Let

$$u\colon M \to \mathbb{P}^4 \setminus \mathbb{Q}_3 \tag{4.5}$$

be the map sending a bundle E to its corresponding quadric $\mathbb{Q}_2$—the envelope of zeros of sections of $E(1)$. The fibres of this map are the base-point-free systems of type $(2, 0)$ on $\mathbb{Q}_2$, up to proportionality, i.e., two disjoint copies of $\mathbb{P}^2 \setminus \mathbb{Q}_1$, which correspond to the two rulings of the quadric $\mathbb{Q}_2$ defined by E. Any ruling defines a conic in $G(1, 3)$ and any system g_2^1 on a conic is defined by all the lines through a fixed point p in the plane of the conic, having base points if and only if p lies on the conic. This proves (4.1).

(4.6) REMARK. – Later on we show (see 6.7) that the moduli space $M(-1, 2)$ has also another fibration, namely that it is a $(\mathbb{P}^2 \setminus \mathbb{Q}_1)$-bundle over $\mathbb{Q}_4 \setminus \mathbb{Q}_3$.

(4.7) REMARK. – In [Ot], it is shown that E extends to a vector bundle on $\mathbb{Q}_4$ and even to one on $\mathbb{Q}_5$, but not further. Moreover, the inclusion $\mathbb{Q}_4 \to \mathbb{Q}_5$ induces an isomorphism between the moduli of stable, rank-2 bundles with Chern classes $(-1, 2)$ on $\mathbb{Q}_5$ and on $\mathbb{Q}_4$ with $(-1, (1, 1))$, which are both isomorphic to $P^7 \setminus \mathbb{Q}_6$. We saw that this was not the case for the inclusion $\mathbb{Q}_3 \to \mathbb{Q}_4$.

(4.8) REMARK. – As in the case $c_1 = 0$, we may construct a monad for stable 2-bundles with $c_1 = -1$, $c_2 = 2$:

Every stable 2-bundle with $c_1 = -1$, $c_2 = 2$ on $\mathbb{Q}_3$ is the cohomology of a monad

$$\mathcal{O} \to (S \oplus S)(1) \to \mathcal{O}(1)$$

where S is the spinor bundle.

PROOF. – The cohomology of E being as in Proposition (3.2), we consider an extension

$$0 \to E(1) \to B \to \mathcal{O}(1) \to 0 ,$$

killing $H^1(E)$, then we take an extension

$$0 \to B^* \to C \to \mathcal{O} \to 0$$

which kills $H^1(B^*)$. The bundle C^* has no intermediate cohomology and

$$c_1(C^*) = 2 , \qquad h^0(C^*) = 8 , \qquad \operatorname{rank}(C^*) = 4 .$$

Hence C^* must be $S \oplus S^*$ by (0.8) and the monad follows. The first map of the monad correspond to a choice of two skew lines on $\mathbb{Q}_3$, whereas the second epimorphism is in fact a choice of a $\mathbb{Q}_2$. We may have then got (4.1) from the monad.

We now pass to the study of the jumping behaviour of bundles E from $M_{\mathbb{Q}_3}(-1, 2)$ on lines, conics and planes. Let us call the ruling determined by E on a two-dimensional quadric $\mathbb{Q}_2$ (see 4.4), the first ruling. Then

(4.9) PROPOSITION. – A line l is a jumping line for E if and only if it belongs to the second ruling of the 2-quadric determined by E. The bundle has an exceptional splitting type on a conic C iff $C \subset \mathbb{Q}_2$.

PROOF. – It is easy to see that there exist lines with the splitting type $(0, -1)$. Hence it must be a generic type. Therefore a line l is jumping iff every section of $E(1)|l$ vanishes at $\geqslant 2$ points, i.e., meets zeros of all sections of $E(1)$ twice. Similar arguments apply to smooth conics as well.

(4.10) PROPOSITION. – For every smooth $\mathbb{Q}_2 \subset \mathbb{Q}_3$, the bundle $E \in M_{\mathbb{Q}_3}(-1, 2)$ is stable on $\mathbb{Q}_2$.

PROOF. – Straightforward from the divisorial sequences.

(4.11) COROLLARY. – The manifold of jumping lines of a bundle $E \in M_{\mathbb{Q}_3}(-1, 2)$ is a non isotropic line in $\mathbb{P}^3 = \mathrm{Fano}(\mathbb{Q}_3)$ and does not determine E uniquely. The morphism $p' : M(-1.2) \to \mathbb{Q}_4 \setminus \mathbb{Q}_3$ (see (5.7)) can be interpreted as

$$E \mapsto \{\text{variety of jumping lines of } E\}.$$

PROOF. – Easy from (4.9) and the information from (0.3).

(4.12) THEOREM. – $\mathrm{Aut}(\mathbb{Q}_3)$ acts transitively on $M(-1, 2)$.

PROOF. – First observe that we can perform an automorphism which moves any point of $\mathbb{P}^4 \setminus \mathbb{Q}_3$ to any other one. Hence the proof is reduced to a consideration of the orbits of the action of $\mathrm{Spin}(4)$ on pairs of lines of a quadric $\mathbb{Q}_2 \subset \mathbb{P}(V)$. The two half-spin representations of $\mathrm{Spin}(4) = SL(2) \times SL(2)$ act on the two rulings of $\mathbb{Q}_2$. The isotropy subgroup that fixes a point contains $GL(2)$ and this acts transitively over $\mathbb{C}^*$.

5. – The moduli space $M(-1, 3)$.

As in the previous sections, the first thing to be explained is what happens when $E(1)$ has sections.

(5.1) LEMMA. – The bundles $E \in M_{\mathbb{Q}_3}(-1, 3)$ with $H^0(E(1)) \neq 0$ make up families of dimensions ≤ 11.

PROOF. – Any section $s \in H^0(E(1))$ vanishes on a locally complete intersection curve of degree $c_2(E(1)) = 3$, with $\omega_Z = \mathcal{O}(-2)$. We get three possibilities only:

a) Z is given by three disjoint smooth lines or

b) Z is the disjoint union of a smooth line with a double one or

c) Z is a triple line.

It seems likely that the cases *b)* and *c)* are in the closure of the case *a)*, but we avoid a detailed discussion by a simpler dimensional count. In the first case the family of curves depends on $3 + 3 + 3 = 9$ parameters. Applying [BF], we can compute the dimensions of the families occurring in the other two cases. In case *b)*, the double structures Z' on a fixed line L with $\omega_{Z'} = \mathcal{O}(-2)$ are given by the Ferrand construction and they are in a bijective correspondence with the surjective maps

$$\mathcal{N}^*_{L, Q_3} = \mathcal{O} \oplus \mathcal{O}(-1) \to \mathcal{O}$$

modulo $\mathbb{C}^*$, hence depending on two parameters. The family of curves Z from *b)* depends then on $3 + 3 + 2 = 8$ parameters. In case *c)* we have triple structures Z on a fixed line and they arise in two consecutive steps

$$0 \to \mathcal{O}_L \to \mathcal{O}_{Z'} \to \mathcal{O}_L \to 0$$

$$0 \to \mathcal{O}_L \to \mathcal{O}_Z \to \mathcal{O}_{Z'} \to 0$$

and by Corollary (2.5) in [BF] they depend on

$$h^0(\mathcal{N}^*_{L, Q_3}) + h^0(\det(\mathcal{N}^*_{L, Q_3})) - 1 = 3 + 2 - 1 = 4$$

parameters. Hence the family of curves Z of case *c)* depends on $3 + 4 = 7$ parameters. In the cases *a)*, *b)*, *c)* we have the exact sequence

$$0 \to \mathcal{O} \to E(1) \to J_Y(1) \to 0$$

and we can compute $h^0(\mathcal{E}xt^1(J_Y(1), \mathcal{O})) = h^0(\omega_Z(2)) = h^0(\mathcal{O}_Z) = 3$.

Hence the bundles coming from three disjoint lines consist of a family of dimension $9 + 3 - 1 = 11$ ($h^0(E(1)) = 1$) if the three lines do not lie in a hyperplane). In the other two cases the dimension is smaller.

(5.2) THEOREM. – The moduli space $M_{Q_3}(-1, 3)$ is irreducible, unirational, reduced of dimension 12. The generic bundle has $H^0(E(1)) = 0$ and $E(2)$ is globally generated.

PROOF. – Let E be such a bundle. From the formulas given in Section 0, we calculate $\chi(E(-1)) = \chi(E(-2)) = 0$, $\chi(E) = -2$ and

$$h^1(\operatorname{End} E) - h^2(\operatorname{End} E) = 12\,.$$

This implies that every irreducible component of $M_{Q_3}(-1, 3)$ has dimension at least 12. By Lemma (5.1) it follows that in every component of $M_{Q_3}(-1, 3)$ the generic bundle has $H^0(E(1)) = 0$. The data for the Kapranov diagram of such a bundle $E(1)$ are then

$$\begin{array}{cccc} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 2 & 0 \\ * & 0 & 0 & 0 \end{array}$$

$E(1)$ has rank 2 and is the cohomology of the corresponding Kapranov sequence, hence the diagram becomes (this is the only possibility to make the output to be a bundle of rank 2)

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

and we see easily that $E(1)$ has a resolution

$$0 \to S^3 \to \psi_1^2 \to E(1) \to 0 . \tag{5.3}$$

It follows that we have a morphism from an open subset of the vector space $\mathrm{Hom}(S^3, \psi_1^2)$ to the open subset of $M_{Q_3}(-1, 3)$ consisting of bundles with $H^0(E(1)) = 0$. This shows that $M_{Q_3}(-1, 3)$ is irreducible and unirational. To prove that it is reduced of dimension 12 we compute $h^2(\mathrm{End}\, E)$ for bundles corresponding to three lines in a generic position. From the exact sequences like (1.16) and (1.17) above, where 2 in latter one is to be replaced by 1, taking into consideration that the normal bundle to a line on $\mathbb{Q}_3$ is $\mathcal{O} \oplus \mathcal{O}(1)$ and using the spectrum, we calculate very easily (the argument is the same of the proof of Corollary (1.18)) that $h^2(\mathrm{End}\, E) = 0$, hence $\dim H^1(\mathrm{End}\, E) = 12$ and the moduli is smooth at such E. $E(2)$ is globally generated by (0.9).

6. - The topology of $M(0,2)$ and of $M(-1,2)$.

In this section we study the topology of the moduli spaces. The reason for writing this in a separate section is that the methods we use are purely topological. We follow the ideas from [Ne]. Let us first collect tools. First, we shall need the following version of

The Lefschetz duality theorem. Let X be an irreducible projective variety of complex dimension n and Y a subvariety such that $X \setminus Y$ is smooth. Then, for every integer i

$$H^i(X, Y) = H_{2n-i}(X \setminus Y) .$$

Since every projective variety can be triangulated as a finite polyhedron, this result follows from [Sp], 6.2.19 and 6.1.11. Then

(6.1) The long cohomology sequence of a pair $\mathbb{Q}_{n-1} \subset \mathbb{P}^n$ gives the long exact sequence

$$\dots \to H_{2n-i}(\mathbb{P}^n \setminus \mathbb{Q}_{n-1}) \to H^i(\mathbb{P}^n) \xrightarrow{r} H^i(\mathbb{Q}_{n-1}) \to H_{2n-i-1}(\mathbb{P}^n \setminus \mathbb{Q}_{n-1}) \to \dots$$

where r is the restriction map. r is an isomorphism for i odd and for i even, $i \leqslant n-2$ and it is the multiplication by two for i even, $n \leqslant i \leqslant 2n$. For $i = n-1$ even r takes the generator $1 \in H^{n-1}(\mathbb{P}^n)$ to the element $(1, 1) \in H^{n-1}(\mathbb{Q}_{n-1}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. From these facts ti follows that $H_i(\mathbb{P}^n \setminus \mathbb{Q}_{n-1})$ is isomorphic to $\mathbb{Z}$ for $i = 0$ or $i = n$ odd, to $\mathbb{Z}_2$ for $i \leq n-1$, i odd and to 0 otherwise.

(6.2) Absolute Hurewicz theorem. If X is a simply connected topological space and there is $m \geq 2$ such that $H_q(X) = 0$ for $1 \leq q < m$, then $\pi_q(X) = 0$ for $q < m$ and $\pi_m(X) = H_m(X)$.

(6.3) Lifting criterion. Consider a commutative triangle of continuous maps

$$\begin{tikzcd} & (E, e_0) \arrow[d, "d"] \\ (Y, y_0) \arrow[ur, dashed, "f'"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Then there exists a lifting f' of f if and only if

$$f_*(\pi_1(Y, y_0)) \subset d_* \pi_1(E, e_0).$$

(6.4) The long homotopy sequence for a fibering $X \to Y$ with fibre F is an exact sequence

$$\ldots \to \pi_{i+1}(Y) \to \pi_i(F) \to \pi_i(X) \to \pi_i(Y) \to \pi_{i-1}(F) \to \ldots$$

(6.5) $\pi_1(\mathbb{P}^n \setminus \mathbb{Q}_{n-1}) = \mathbb{Z}_2$.

PROOF. – Let us show first that $\pi_1(\mathbb{Q}_2 \setminus \mathbb{Q}_1) = 0$. Fix a ruling on $\mathbb{Q}_2$ and a smooth $\mathbb{Q}_1 \subset \mathbb{Q}_2 \subset \mathbb{P}^3$. Let p: $\mathbb{Q}_2 \setminus \mathbb{Q}_1 \to \mathbb{Q}_1$ be a projection which associates to each point of $\mathbb{Q}_1$ the point on the chosen ruling through it and lying on $\mathbb{Q}_1$. Hence p is a fibering with fibre $\mathbb{C}$ and the long homotopy sequence gives

$$\pi_1(\mathbb{C}) \to \pi_1(\mathbb{Q}_2 \setminus \mathbb{Q}_1) \to \pi_1(\mathbb{P}^1).$$

Since $\pi_1(\mathbb{C}) = \pi_1(\mathbb{P}^1) = 0$, we have $\pi_1(\mathbb{Q}_2 \setminus \mathbb{Q}_1) = 0$, too. The standard double covering $\mathbb{Q}_2 \setminus \mathbb{Q}_1 \to \mathbb{P}^2 \setminus \mathbb{Q}_1$ is a 2:1 map, hence $\pi_1(\mathbb{P}^2 \setminus \mathbb{Q}_1) = \mathbb{Z}_2$. The formula (6.5) now follows from the following theorem of Zariski, [Ch], (1.1):

Let H be an algebraic hypersurface of the complex projective space $\mathbb{P}^n$. If $n \geq 3$, then for any hyperplane L from a Zariski open and non-empty subset of the space of all hyperplanes in $\mathbb{P}^n$, the canonical injection of $L \setminus H$ into $\mathbb{P}^n \setminus H$ induces an isomorphism

$$\pi_1(L \setminus H, e) \to \pi_1(\mathbb{P}^n \setminus H, e)$$

for $e \in L \setminus H$.

(6.6) The double covering d: $\mathbb{Q}_n \setminus \mathbb{Q}_{n-1} \to \mathbb{P}^n \setminus \mathbb{Q}_{n-1}$ is the covering space map.

PROOF. – Follows from (6.5).

Now we want to show that the moduli $M = M(-1, 2)$ has another fibration then that given in (4.1).

(6.7) The fibering of $M: = M(-1, 2)$ given in Section 3 lifts to $M \to \mathbb{Q}_4 \setminus \mathbb{Q}_3$ and the fibres are $\mathbb{P}^2 \setminus \mathbb{Q}_1$:

$$\begin{tikzcd} & Q_4 \setminus Q_3 \arrow[d, "d"] \\ M \arrow[ur, dashed, "p'"] \arrow[r, "p"] & \mathbb{P}^4 \setminus Q_3 \end{tikzcd}$$

PROOF. – We want to apply (6.3). The appropriate piece of the long homotopy sequence

$$\ldots \to \pi_1(M) \to \pi_1(\mathbb{P}^4 \setminus \mathbb{Q}_3) \to \pi_0((\mathbb{P}^2 \setminus \mathbb{Q}_1)) \coprod ((\mathbb{P}^2 \setminus \mathbb{Q}_1)) \to \pi_0(M)$$

specifies to

$$\ldots \to \pi_1(M) \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0\,,$$

hence the image of $\pi_1(M)$ in $\pi_1(\mathbb{P}^4 \setminus \mathbb{Q}_3)$ is zero. The lifting condition is therefore fulfilled. Recall that p associates to each bundle E of M the quadric containing zeros of sections of $E(1)$ and the fibre corresponds to the two rulings. The covering $\mathbb{Q}_4 \setminus \mathbb{Q}_3 \to \mathbb{P}^4 \setminus \mathbb{Q}_3$ (where $\mathbb{Q}_4$ is the Klein quadric) splits the rulings and hence the fibres of p' are $\mathbb{P}^2 \setminus \mathbb{Q}_1$.

(6.8) The homology of $\mathbb{Q}_n \setminus \mathbb{Q}_{n-1}$ is

$$H_i(\mathbb{Q}_n \setminus \mathbb{Q}_{n-1}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = n \\ 0 & \text{otherwise}\,. \end{cases}$$

PROOF. – Looking at the analogue of (5.1), we get that the restriction map $H^i(\mathbb{Q}_n) \to H^i(\mathbb{Q}_{n-1})$ is an isomorphism if $i \neq n$, $i \neq 2n$. For $i = 2n$ it is zero and has kernel $\mathbb{Z}$. It has kernel $\mathbb{Z}$ for $i = n$ even and it has cokernel $\mathbb{Z}$ for $i = n$ odd.

(6.9) $\pi_2(\mathbb{Q}_4 \setminus \mathbb{Q}_3) = \pi_3(\mathbb{Q}_4 \setminus \mathbb{Q}_3) = 0$, $\pi_4(\mathbb{Q}_4 \setminus \mathbb{Q}_3) = \mathbb{Z}$.

PROOF. – Immediate from (6.8) and the Absolute Hurewicz Theorem.

(6.10) $\pi_1(M(-1, 2)) = \mathbb{Z}_2$.

PROOF. – The appropriate piece of the long homotopy sequence of the fibering $M \to \to \mathbb{Q}_4 \setminus \mathbb{Q}_3$ is

$$\ldots \to \pi_2(\mathbb{Q}_4 \setminus \mathbb{Q}_3) \to \pi_1(\mathbb{P}^2 \setminus \mathbb{Q}_1) \to \pi_1(M) \to \pi_1(\mathbb{Q}_4 \setminus \mathbb{Q}_3) = 0 .$$

Since $\pi_2(\mathbb{Q}_4 \setminus \mathbb{Q}_3) = 0$ and $\pi_1(\mathbb{P}^2 \setminus \mathbb{Q}_1) = \mathbb{Z}_2$, (6.10) follows.

We may also calculate higher homotopy groups and some homology of the moduli space $M = M(-1, 2)$. Namely

(6.11) Let M be any fibration over $\mathbb{Q}_4 \setminus \mathbb{Q}_3$ with fibre $\mathbb{P}^2 \setminus \mathbb{Q}_1$. Then

$$H_i(M, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 4 , \\ \mathbb{Z}_2 , & \text{for } i = 1, 5 \\ 0 & \mathrm{i} = 2, 3, 6 , \end{cases}$$

and $\pi_2(M) = \pi_3(M) = 0$, $\pi_4(M) = \mathbb{Z}_2$.

PROOF. – We have a Serre spectral sequence with

$$E_2^{p,q} = H_p(\mathbb{Q}_4 \setminus \mathbb{Q}_3, H_q(\mathbb{P}^2 \setminus \mathbb{Q}_1))$$

abutting to $H_*(M)$. In our case $E_2^{p,q}$ is (see also (6.1)):

$$\begin{array}{c|cccccc} q\uparrow & \mathbb{Z}_2 & 0 & 0 & 0 & \mathbb{Z}_2 & 0 \\ & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0 \\ \hline & & & & & & \to p \end{array}$$

hence $E_2^{p,q} = E_\infty^{p,q}$ so the homology is as we claimed. To calculate the homotopy, we use Hurewicz Absolute Theorem again.

(6.12) COROLLARY. – If $\mathbb{Q}$ denotes the field of rational number, then

$$H_i(M(-1, 2), \mathbb{Q}) = \mathbb{Q} \quad \text{for } i = 0, 4 \text{ and zero otherrwise} .$$

(6.13) COROLLARY. – The topological Euler-Poincaré characteristic is $\chi(M(-1, 2), \mathbb{Q}) = 2$.

For the bundles from $M{:} = M(0, 2)$ we have

(6.14) PROPOSITION. – The fundamental group $\pi_1(M)$ is $\mathbb{Z}_4$.

PROOF. – By (2.21), the quartic hypersurface V_4 discussed in Section 2 is normal and with the singular locus of dimension 6. The best way to check this is to use [BS]. The generic one-dimensional section of V_4 by hyperplanes is then a smooth quartic curve $C_4 \subset \mathbb{P}^2$. Hence $\pi_1(M) = \pi_1(\mathbb{P}^9 \setminus V_4) = \pi_1(\mathbb{P}^2 \setminus C_4) = \mathbb{Z}_4$ by the theorem of Zariski and [Sh], Ch. IX, Sect. 4, Ex. 1.

(6.16) COROLLARY. – $\pi_2(M(0,2)) = \pi_3(M(0,2) = 0$, $\pi_4(M(0,2)) = \mathbb{Z}$. $H_i(M(0,2), \mathbb{Q}) = \mathbb{Q}$ for $i = 0, 5$ and 0 otherwise. The Euler-Poincaré characteristic $\chi(M(0,2), \mathbb{Q})$ is 0.

PROOF. – Similar to that for $M(-1,2)$.

(6.17) The fibration $\varphi\colon \mathbb{P}^9 \setminus V_4 \to \mathbb{P}^4 \setminus \mathbb{Q}_3$, seen in (2.1), (2.11), (2.17), is not trivial.

PROOF. – Otherwise $\pi_1(M) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

APPENDIX.

The curve Y of degree 6 with $\omega_Y = \mathcal{O}_Y(-1)$ on a smooth quadric $\mathbb{Q}_3$.

NICOLAE MANOLACHE

In this appendix we give a complete list of the curves as in the title.

THEOREM. – Curves Y of degree 6 with $\omega_Y = \mathcal{O}_Y(-1)$ on a smooth quadric $\mathbb{Q}_3$ are of the following types

1) three disjoint conics (maybe degenerate),

2) a disjoint union of a conic and a double conic (maybe degenerate),

3) a double structure of a connected curve consisting of a union of three lines,

4) a triple structure on a conic (maybe degenerate),

5) a double structure on a union of a line and a conic meeting at one point,

6) a double structure on a union Y_0 of a simple line and a double line, the Hilbert polynomial of Y_0 being $3n + 1$,

7) certain «quasiprimitive» multiplicity-6 structures on a line,

8) a double structure on the first infinitesimal neighbourhood of a line,

9) a double structure on a twisted cubic.

Hence there are nine families (not disjoint) $\mathcal{C}_1, \dots, \mathcal{C}_9$ of curves, which cover the Hilbert scheme of curves in $\mathbb{Q}_3$ of degree 6 with $\omega_Y = \mathcal{O}_Y(-1)$. Corresponding to them are nine families $\mathcal{F}_1, \dots, \mathcal{F}_9$ of vector bundles on $\mathbb{Q}_3$, of rank 2, stable and with Chern classes $c_1 = 0$, $c_2 = 4$, given by extensions $0 \to \mathcal{O} \to E(1) \to I_Y(2) \to 0$. We calculate (or at least evaluate) the dimensions of the families $\mathcal{C}_1, \dots, \mathcal{C}_9$ and $\mathcal{F}_1, \dots, \mathcal{F}_9$. As they are all less than 21, it follows that the moduli space $M_{Q_3}(0, 4)$ is irreducible (see the Theorem 3.4 of the paper for the details).

The classification done here resembles very much that given in [BM] for curves in $\mathbb{P}^3$ with $\omega_Y = \mathcal{O}_Y(-1)$ and of degree 6, done in the connection with the study of the moduli space $M(-1, 4)$ on $\mathbb{P}^3$.

The current work was done during the author's stay at the Sonderforschungsbereich 170 «Geometrie and Analysis» in Göttingen, Germany, in 1992.

1. - Proof of the theorem.

We organize the classification upon the number of connected components and then upon the number of irreducible components of a curve with the properties as in the title. We begin with some easy remarks.

REMARK 1. - Because of the equality

$$2p_a(Y_1) - 2 = \deg \omega_{Y_1} = \deg \mathcal{O}_{Y_1}(-1) = -\deg Y_1,$$

any connected component Y_1 of Y should have even degree.

REMARK 2. - Let $Y_1 \subset Y$ be a connected l.c.i. reduced curve, such that $Y_{\text{red}} \setminus Y_1$ and Y_1 have no common irreducible component. Then $Y = Y_1 \cup Y_2$ (primary decomposition) and Y_1, Y_2 are Cohen-Macalauy curves with no common component. It follows that Y_1 and Y_2 are locally algebraically linked (see [M2] or [M3]) and we have the exact sequences (dual to each other)

$$0 \to \omega_{Y_1}(1) \to \mathcal{O}_Y \to \mathcal{O}_{Y_2} \to 0, \tag{1}$$

$$0 \to \omega_{Y_2}(1) \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \to 0. \tag{2}$$

By restricting the first one to Y_1 one obtains

$$0 \to \omega_{Y_1}(1) \to \mathcal{O}_{Y_1} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0. \tag{3}$$

REMARK 3. - If Y_1 in Remark 2 is a line, then it follows from the above that $Y_1 \cap Y_2$ is, as a scheme, a reduced point. Then, using the fact that $Y_1 \cup Y_2$ is a complete intersection in $Y_1 \cap Y_2$ and that Y_1 is nonsingular in $Y_1 \cap Y_2$, we see that Y_2 is non- singular in $Y_1 \cap Y_2$, too. If we restrict the exact sequence (2) to Y_2, we obtain $\omega_{Y_2'} = \mathcal{O}_{Y_2'}(-2)$, Y_2' being the connected component of Y_2 which meets Y_1. Due to the first remark, the degree of Y_2' should be 1, 3 or 5. When the degree is 1, then Y_1 and another line meeting it, make up a connected component of Y. However, degrees 3 and 5 are not possible. Indeed, if $\deg Y_2' = 3$, then $Y_1 \cup Y_2'$ would be a curve of degree 4 with $\omega = \mathcal{O}(-1)$, hence with the Hilbert polynomial $4n + 2$ and then Y_2' would have the Hilbert polynomial $3n+3$. These data contradict the exact sequence

$$0 \to \mathcal{O}_{Y_1 \cup Y_2'} \to \mathcal{O}_{Y_1 \times Y_2'} \to \mathcal{O}_{Y_1 \cap Y_2'} \to 0 .$$

In a similar way one excludes the degree 5.

Thus we showed that Y contains a line as an irreducible component iff it contains a degenerate conics as a connected component and the line is a component of the conic.

REMARK 4. – If Y_1 from Remark 2 is a conic (possibly degenerate), then the exact sequence (3) shows that Y_1 is a connected component of Y.

From the above remarks it follows that Y may have at most three connected components. We discuss the three cases separately.

I) *Three connected components.* From the above remarks it follows that the components are conics, maybe degenerate.

II) *Two connected components.* They should have degrees 2 or 4 and that of degree 2 should be a conic. The other component has degree 4 with $\omega = \mathcal{O}(-1)$. The curves in $\mathbb{P}^3$ of degree 4 and with $\omega = \mathcal{O}(-1)$ were classified in [M1] and it was shown there that they are either a union of two conics (maybe degenerate) or a double structure on a conic (smooth or not). This remains true also in our case. We shall sketch the proof.

Let Y be a connected curve of degree 4, $\omega = \mathcal{O}(-1)$. Then Y cannot have more than two irreducible components. Indeed, let us assume this is the case. Then they are two double lines, since the single lines and the conics have been excluded by the Remarks 3 and 4. Then the residue of $X = Y_{\mathrm{red}}$ in Y, in the sense of the locally algebraic linkage, is X (see [M2] or [M3] and also [BM], Lemma 8). The condition $\omega_Y = \mathcal{O}_Y(-1)$ shows that the doubling is made with the invertible sheaf $\mathcal{O}_X$, i.e., we have an exact sequence of the form

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \to \mathcal{O}_X \to 0 .$$

When Y_{red} is irreducible, Y is either a double conic or a structure of multiplicity 4 on a line. We show now that these multiplicity-4 structures are also double structures on a conic degenerated to a double line. Indeed, suppose that the structure on Y is quasiprimitive in the sense of [BF1] and [BF2]. Then $\omega_Y | X = \omega_X \otimes L^{-3}(-D)$, where L is an invertible sheaf on $Y_{\mathrm{red}} = X$ and D is a divisor on X (cf. [BF2] or [M2], [M3]). The condition $\omega_Y = \mathcal{O}_Y(-1)$ allows only $L = \mathcal{O}_X(-1)$ and $\deg D = 2$. The double structure on X with $L = \mathcal{O}_X(-1)$ as the associated line bundle is a degenerate conic and Y is a double structure on it (cf. [BM], the remark on p. 333).

When Y is not quasiprimitive, by [BF2], § 4, the ideal I_Y of Y in $\mathbb{Q}_3$ is given by the exact sequence

$$0 \to I_Y/I_X^3 \to I_X/I_X^3 \xrightarrow{p} \mathcal{O}_X(-1) \to 0\,, \tag{4}$$

where I_X is the ideal of X in $\mathbb{Q}_3$ and the discriminant of p does not vanish anywhere. Let us note that the middle term in the sequence equals $S^2(I_X/I_X^2)$.

We want to show now that also these structures can be obtained by doubling a conic degenerate to a double line. If X is a line on $\mathbb{Q}_3$, then it is easy to see that we can choose homogeneous coordinates in $\mathbb{P}^4$ such that X is described by the ideal (x, y, z) in the quadric $\mathbb{Q}_3$ given by the equation $q = 0$, where $q = ux - vy + \phi(x, y, z)$, ϕ being a quadratic form in x, y, z. Then the conormal bundle of X in $\mathbb{Q}_3$ is

$$\nu_{X, Q_3} = \frac{(x, y, z)}{(x, y, z)^2 + (q)} =$$

$$= \frac{(x, y) + (x, y, z)^2 + (ux - vy)}{(x, y, z)^2 + (ux - xy)} \oplus \frac{(z) + (x, y, z)^2 + (ux - vy)}{(x, y, z)^2 + (ux - xy)} = \mathcal{O}_X \oplus \mathcal{O}_X(-1)$$

and one has

$$S^2(\nu_{X, Q_3}) = \frac{(x, y, z)^2 + (q)}{(x, y, z)^3 + (q)} = \frac{(x, y)^2 + (x, y, z)^3 + (q)}{(x, y, z)^3 + (q)} \oplus$$

$$\oplus \frac{(xz, yz)^2 + (x, y, z)^3 + (q)}{(x, y, z)^3 + (q)} \oplus \frac{(z)^2 + (x, y, z)^3 + (q)}{(x, y, z)^3 + (q)} = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)\,.$$

Then, I_Y defined by an exact sequence like (4), is of the form $I_Y = (x^2, xy, y^2, z^2, q)$ in suitable new coordinates. If we take $Y_1 \subset \mathbb{Q}_3$ to be the subscheme of $\mathbb{P}^4$ given by the ideal (x, y, z^2), then we see directly that $I_{Y_1}^2 \subset I_Y \subset I_{Y_1}$, as ideals in $\mathcal{O}_{\mathbb{Q}_3}$. This proves our claim.

III) *One connected component.* By the remarks made at the beginning, the curve Y with one connected component cannot have more than three irreducible components, and if this is the case, these components are necessarily three double lines. Then Y is a doubling of a curve X consisting of three lines. They cannot lie in the same plane in a $\mathbb{P}^4$ containing $\mathbb{Q}_3$, hence X is a curve with the Hilbert polynomial $3n + 1$. The ideal of Y is given as the kernel of a surjective map $I_X \to \omega_X(1)$. This case can be also interpreted as a degeneration of the one that will appear later on, namely a double structure on the sum of a line and a conic meeting at one simple point.

When Y has two irreducible components Y_1 and Y_2 there are two possibilities: they have degrees $(3, 3)$ or $(2, 4)$ and they are necessarily either nilpotent structures on lines or nilpotent structures on a line and a conic (this can happen only in the second case).

Let us consider the case $(3, 3)$. Let $X_1 = (Y_1)_{\mathrm{red}}$, $X_2 = (Y_2)_{\mathrm{red}}$. Then, in appropriate coordinates x, y, z of the point $X_1 \cap X_2$ in $\mathbb{Q}_3$ we have $X_1 = (z, x)$, $X_2 = (z, y)$ and Y is locally one of the following

$$I_Y = (z, x^3y^3), \qquad I_Y = (z^3, xy), \qquad I_Y = (z^3, xy + z^2)$$

or

$$I_Y = ((z + y^2)(z + xy - x^2), (z + y)z + xy^2)$$

according to [BM], Lemma 9.

We show that the last form of I_Y cannot occur. Indeed, Y_1, Y_2 are then triple l.c.i. structures, because in the only point where they may be not such, namely in the commont point of X_1 and X_2, we have

$$I_{Y_1} = (z + xy - x^2, x^3), \qquad I_{Y_2} = (z + y^2, y^3).$$

Then Y_1, Y_2 are primitive triple structures given by certain invertible sheaves $\mathcal{O}_{X_1}(r_1)$, $\mathcal{O}_{X_2}(r_2)$, hence of the Hilbert polynomials $3n + 3r_1 + 3$ and $3n + 3r_2 + 3$, respectively. As Y_1 and Y_2 are locally algebraically linked by Y, we have the exact sequences

$$0 \to \omega_{Y_2}(1) \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \to 0$$

and then the Hilbert polynomial of Y is

$$\chi_Y(n) = 6n + 3 + 3(r_1 - r_2),$$

which shows $r_1 = r_2 =: r$. On the other hand, the natural exact sequence:

$$0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \times \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0$$

gives $2r = 1$, which is impossible.

If we denote by J the ideal of Y in $\mathbb{Q}_3$ and by I the ideal of $X = Y_{\text{red}} = X_1 \cup X_2$ in $\mathbb{Q}_3$, then one sees by some natural local calculations that $J: I = I_2$ defines a l.c.i. double structure on X and that $J: I^2 = I$. Moreover, as the «direct filtration» from above coincides with the «inverse one» (i.e., $J:(J: I) = I$, $J:(J: I^2) = I_2$), we have an algebra structure (cf. [M3]) on $\mathcal{A}(Y) = \mathcal{O}_X \oplus I/I_2 \oplus I_2/J$, where I/I_2, I_2/J are invertible sheaves on X. In particular, one sees that the map

$$I/I_2 \otimes I/I_2 \to I^2/(I \cdot I_2) \to I_2/J$$

is an isomorphism.

Thus we have proved that Y is a triple structure on X, given by exact sequences of the form

$$0 \to I_2/I_X^2 \to I_X/I_X^2 \to L \to 0\,,$$

$$0 \to J/(I_X \cdot I_2) \to I_2/(I_X \cdot I_2) \to L^2 \to 0\,,$$

where $L = I/I_2$, $L^2 = I_2/J$.

In particular, $\omega_Y | X = \omega_X \otimes L^{-2} = L^{-2}(-1)$; the condition $\omega_Y = \mathcal{O}_Y(-1)$ gives

$L = \mathcal{O}_X$. Because $H^1(L) = H^1(L^2) = 0$, we have $\operatorname{Pic} Y \simeq \operatorname{Pic} X$ and so the above construction really gives triple structures on X with $\omega = \mathcal{O}(-1)$.

Suppose now that $Y = Y_1 \cup Y_2$ where Y_1 is a double line and Y_2 a degree-4 curve. Then Y_2 is either a double conic or a multiplicity-4 structure on a line. When $(Y_2)_{\mathrm{red}} = X_2$ is a conic, then Y is a doubling by $\omega_X(1)$ of a curve $X = X_1 \cup X_2$, where X_1 is a line, X_2 is a conic and $X_1 \cap X_2$ is a simple point.

The case where Y_2 is a multiplicity-4 line is discussed in the following

LEMMA 1. – If $Y_1 \cup Y_2$ is a l.c.i. structure on $X = X_1 \cup X_2$, where X_1, X_2 are two meeting lines, $\deg Y_1 = 2$, $\deg Y_2 = 4$ and $\omega_Y = \mathcal{O}(-1)$, then Y_2 is a quasiprimitive structure with the associated line bundle $\mathcal{O}_{X_2}(-1)$ or $\mathcal{O}_{X_2}$ and Y is a double structure on $X_1 \cup Y_2''$, where Y_2'' is the double structure on X_2 in the canonical filtration of Y_2. The Hilbert polynomial of $X_1 \cup Y_2''$ is $3n + 1$.

PROOF. – Assume the contrary, i.e. that Y_2 is not quasiprimitive. Then Y_2 contains the first infinitesimal neighbourhood X_2'' of X_2 in $\mathbb{Q}_3$ and the residue of X_2'' in Y_2 is X_2. Then $X_1 \cup Y_2''$ and $X_1 \cup X_2$ are locally algebraically linked by Y, so that we have an exact sequence

$$0 \to \omega_{X_1 \cup Y_2''}(1) \to \mathcal{O}_Y \to \mathcal{O}_{X_1 \cup X_2} \to 0 .$$

It follows that the Hilbert polynomial of $X_1 \cup X_2''$ is $4n + 2$. From the exact sequence

$$0 \to \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_2}(-1) \to \mathcal{O}_{Y_2''} \to \mathcal{O}_{X_2} \to 0 .$$

we infer that the Hilbert polynomial of Y_2'' is $3n + 2$ and from the exact sequence

$$0 \to \mathcal{O}_{X_1 \cup Y_2''} \to \mathcal{O}_{X_1} \times \mathcal{O}_{Y_2''} \to \mathcal{O}_{X_1 \cap Y_2''} \to 0$$

it follows that $X_1 \cap Y_2''$ has length 1. On the other hand, a direct computation shows that $l(\mathcal{O}_{X_1 \cap Y_2''}) = 2$; a contradiction. Then Y_2 is a quasiprimitive structure on X_2 and there exist exact sequences:

$$0 \to \mathcal{O}_{X_2}(r) \to \mathcal{O}_{Y_2''} \to \mathcal{O}_{X_2} \to 0 ,$$

$$0 \to \mathcal{O}_{X_2}(2r + d_1) \to \mathcal{O}_{Y_2'''} \to \mathcal{O}_{Y_2''} \to 0 ,$$

$$0 \to \mathcal{O}_{X_2}(3r + d_1 + d_2) \to \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_2'''} \to 0 ,$$

where d_1 and d_2 are the degree of some divisors D_1, D_2 on X_2, D_2 concentrated at $X_1 \cap X_2$. As $X_1 \cup X_2$ and $X_1 \cup Y_2'''$ are locally algebraically linked by Y, one has an exact sequence

$$0 \to \omega_{X_1 \cup Y_2'''}(1) \to \mathcal{O}_Y \to \mathcal{O}_{X_1 \cup X_2} \to 0 .$$

which shows that the Hilbert polynomial of $X_1 \cup Y_2'''$ is $4n + 2$. As $X_{Y_2'''}(n) = 3n + 3r + d_1 + 3$, we easily calculate that $l(X_1 \cap Y_2''') = 3r + d_1 + 2$. As this length is an integer between 1 and 3 and $d_1 \geqslant 0$, the only possibilities are $r = -1$, $r = 0$. The residue of $X_1 \cup Y_2''$ being $X_1 \cup Y_2''$ itself, Y is a doubling of $X_1 \cup Y_2''$. Notice that $\chi(n) = \chi_{X_1 \cup Y_2''}(n) = \chi_{X_1}(n) + \chi_{Y_2''}(n) - l(X_1 \cap Y_2'') = 3n + r + 3 - l$, where l is the length of $X_1 \cap Y_2''$. From here it follows

$$\chi(n) = 3n \qquad \text{for } r = -1\,, \quad l = 2\,;$$

$$\chi(n) = 3n + 1 \qquad \text{for } r = -1\,, \quad l = 1 \text{ or } r = 0, \quad l = 2$$

$$\chi(n) = 3n + 2 \qquad \text{for } r = 0, \quad l = 1\,.$$

We show now that $\chi(n) = 3n + 1$ is the only possibility. Indeed, $r = -1$, $l = 2$ implies that the double line is a degenerate conic, hence it lies in a plane and that the simple line lies in the same plane. But $\mathbb{Q}_3$ does not contain a cubic plane curve. This excludes $\chi(n) = 3n$.

We show now that $l = 1$ implies $r = -1$. Indeed, we have $l = 1$ only when locally around the point $X_1 \cap X_2$ the ideals of X_1 of Y_2'' and of $X_1 \cup Y_2''$ are $I_{X_1} = (y, z)$, $I_{Y_2''} = (x, z^2)$, $I_{X_1 \cup Y_2''} = (xy, xz, z^2)$.

Introducing a double structure Y on $X_1 \cup Y_2'' := Y_0$ with $\omega_Y = \mathcal{O}_Y(-1)$, is equivalent to giving a surjection $I_{Y_0}/I_{Y_0}^2 \to \omega_{Y_0}(1)$. Restricting this surjection to X_2 one obtains a surjection $I_{Y_0}/(I_{X_2} \cdot I_{Y_0}) \to \omega_{Y_0}(1)|X_2$. From the diagram with the exact line and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \arrow[d] & \\
 & \mathbb{C} \arrow[d] & & & \mathbb{C} \arrow[d] & \\
0 \arrow[r] & I_{Y_0} \cap I_{X_2}^2/(I_{X_2} \cdot I_{Y_0}) \arrow[r] \arrow[d] & I_{Y_0}/(I_{X_2} \cdot I_{Y_0}) \arrow[r] & I_{X_2}/I_{X_2}^2 \arrow[r] & I_{X_2}/(I_{Y_0} + I_{X_2}^2) \arrow[r] \arrow[d] & 0 \\
 & I_{X_2}^2/(I_{Y_2''} \cdot I_{X_2}) \arrow[d] & & & I_{X_2}/I_{Y_2''} \simeq \mathcal{O}_X(r) \arrow[d] & \\
 & 0 & & & 0 &
\end{tikzcd}$$

where $\mathbb{C}$ is the skyscraper sheaf at $X_1 \cap X_2$, one obtains

$$I_{Y_0}/(I_{X_2} \cdot I_{Y_0}) \simeq \mathcal{O}_{X_2}(2r) \oplus \mathcal{O}(-r-2) \oplus \mathbb{C}\,.$$

To compute $\omega_{Y_0}|X_2$ let us observe that we have the exact sequence (by a Cohen-Macalay l.a.l., cf. [M2], which works for curves):

$$0 \to \mathcal{O}_{X_1}(-2) \to \mathcal{O}_{Y_0} \to \mathcal{O}_{Y_2''} \to 0$$

which gives, by dualization

$$0 \to \omega_{Y_2''} \to \omega_{Y_0} \to \omega_{X_1}(2) \to 0 .$$

But, as $\omega_{Y_2''} \simeq \mathcal{O}_{Y_2''}(-r-2)$, by restricting the above sequence to X_2 one obtains

$$0 \to \mathcal{O}_{X_2}(-r-2) \to \omega_{Y_0}|X_2 \to \mathbb{C} \to 0 .$$

A local computation shows that this sequence in fact splits, and hence $\omega_{Y_0}|X_2 \simeq \mathcal{O}_{X_2}(-r-2) \oplus \mathbb{C}$. Then a surjection $I_{Y_0}/(I_{X_2} \cdot I_{Y_0}) \to \omega_{Y_0}(1)|X_2$ is possible for $r = -1$ only. Hence the proof is finished.

In what follows we consider curves Y of degree 6 with $\omega_Y = \mathcal{O}_Y(-1)$ and such that $Y_{\mathrm{red}} = X$ is irreducible. X can be a line, a conic or a twisted cubic (plane cubics do not lie in $\mathbb{Q}_3$).

Let us take first X to be a line and Y a quasiprimitive structure on it. Then the associated graded $\mathcal{O}_X$-algebra has the form

$$\mathcal{O}_X \oplus \mathcal{O}_X(r) \oplus \mathcal{O}_X(2r+e+f_1) \oplus \mathcal{O}_X(3r+d+e+f_1+f_2) \oplus$$
$$\oplus \mathcal{O}_X(4r+d+2e+2f_1+f_2) \oplus \mathcal{O}_X(5r+d+2e+2f_1+f_2)$$

where d, e, f_1, f_2 are the degrees of some effective divisors D, E, F_1, F_2 such that D, E, $F_1 + F_2$ are pairwise disjoint (cf. [M3]). Then $\chi(\mathcal{O}_Y(n)) = 6n + 15r + 3d + 6e + 6f_1 + 3f_2 + 6 = 6n + 3$ implies $r = -1$ and $3d + 6e + 6f_1 + 3f_2 = 12$. From here $e + f_1 \leq 2$.

By a general theory, Y is a double structure on the triple structure Y_3 in the canonical filtration. The case $e + f_1 = 0$ cannot occur, because such a triple structure would be a plane curve of degree 3 and such curves do not exist on $\mathbb{Q}_3$. Then Y_3 is a triple line with the Hilbert polynomial $3n+1$ or $3n+2$ and Y is a doubling of it.

If Y is not quasiprimitive, there are two possibilities: the numerical character is $(1,2,2,1)$ or $(1,2,1,1,1)$, cf. [M3]. In the first case the canonical filtration will contain the first infinitesimal neighbourhood $X^{(1)} = Y_2$ of X in $\mathbb{Q}_3$ (this is a triple structure of the Hilbert polynomial $3n+2$) and as $I_{Y_2}^2 \subset I_Y$ (one checks this locally, using the local structure of I_Y, cf. [M3]), Y is a double structure on Y_2, given by a surjection $I_{Y_2}/I_{Y_2}^2 \to \omega_{Y_2}(1)$.

LEMMA. – There is no multiplicity-6 structure Y on a line X in $\mathbb{Q}_3$ with $\omega_Y = \mathcal{O}_Y(-1)$ and numerical characters $(1,2,1,1,1)$.

PROOF. – Adapting the Theorem 4.15 from [M3] to our situation, we see that such a structure would yield two exact sequences

$$0 \to L^3(D) \to F \to L^2 \to 0\,,$$

$$0 \to L^2(D) \to \nu \to L \to 0\,,$$

where F is a rank-2 vector bundle on X, ν is the conormal bundle of X in $\mathbb{Q}_3$, L is a line bundle on X and D is an effective divisor. Also, in this case $\omega_Y|X \simeq \simeq L^{-4}(-D) \otimes \omega_X \simeq \mathcal{O}_X(-4r-d-2)$, if $L = \mathcal{O}_X(r)$ and $d = \deg D \geq 0$. But $\nu = \mathcal{O}_X(-1) \oplus \mathcal{O}_X$, so that $r = -1$, $d = 2$ are the only possibilities in the second exact sequence above. This implies $\omega_Y|X = \mathcal{O}_X$, which contradicts $\omega_Y = \mathcal{O}_Y(-1)$.

REMARK. – One can show that in $\mathbb{P}^3$ the l.c.i. structures Y on a line X of degree 6, of numerical character $(1,2,1,1,1)$ and $\omega_Y = \mathcal{O}_X(-1)$ do not exist, either. The requirement about the character is equivalent to a certain structure of the local rings of Y, cf. [M3].

When $Y_{\mathrm{red}} = X$ is a conic, then Y is a primitive structure on it. The associated line bundle should satisfy $\omega_Y|X = \omega_X \otimes L^{-2}$ hence $L = \mathcal{O}_X$. When $Y_{\mathrm{red}} = X$ is a twisted cubic, then Y is a double structure on it, corresponding to a surjection $I_X/I_X^2 \to \omega_X(1)$.

2. – Computation of the dimension of the families of curves and bundles.

We can now compute, or at least evaluate from above, the dimensions of the families of curves $\mathcal{C}_1, \ldots, \mathcal{C}_9$ and of the corresponding families of vector bundles $\mathcal{F}_1, \ldots, \mathcal{F}_9$, given as non-trivial extensions

$$0 \to \mathcal{O} \to E(1) \to I_Y(2) \to 0$$

with Y in $\mathcal{C}_1, \ldots, \mathcal{C}_9$. For a generic E and Y we have

$$\dim \mathcal{F}_i = \dim \mathcal{C}_i + h^0(\mathcal{O}_Y) - h^0(E(1))$$

Let us also notice that $h^0(E(1)) = 1 + h^0(I_Y(2))$.

#1) As $\dim \mathcal{C}_1 = 18$, it follows that $\dim \mathcal{F}_1 = 20$, because $h^0(\mathcal{O}_Y) = 3$ and $h^0(E(1)) = 1$ generically.

#2) Here the doubling of a conic X is given by surjections $I_X/I_X^2 \to \mathcal{O}_X$ and $I_X/I_X^2 \simeq 2\mathcal{O}_X(-1)$ and hence by 5 parameters. Then $\dim \mathcal{C}_2 = 17$ and hence $\dim \mathcal{F}_2 \leq \leq 19$.

#3) $\mathcal{C}_3$ and $\mathcal{F}_3$ consists of curves, resp. vector bundles, which are degenerations of $\mathcal{C}_5$, resp. $\mathcal{F}_5$.

#4) To give a conic X means to give 6 parameters, to give a doubling Y_2 with $\mathcal{O}_X$

means 5 more parameters, to give a tripling which extends the given doubling means to give a splitting of the natural exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_X^2/(I_X\cdot I_Y) & \longrightarrow & I_{Y_2}/(I_X\cdot I_Y) & \longrightarrow & I_{Y_2}/I_X^2 & \longrightarrow & 0\,. \\ & & \| & & & & \| & & \\ & & \mathcal{O}_X & & & & \mathcal{O}_X(-2) & & \end{array}$$

Such splitting are in a 1-1 correspondence with $\operatorname{Hom}(\mathcal{O}_X(-2), \mathcal{O}_X) = H^0(\mathcal{O}_X(2))$ hence 5 more parameters are required. This shows that $\dim \mathcal{C}_4 = 16$ and so

$$\dim \mathcal{F}_4 \leqslant 18\,.$$

#5) A conic C and a line X such that $C \cap X$ is a simple point determine a $\mathbb{P}^3$. For the generic situation it follows that the curve $Y_0 = X \cup C$ is contained in a smooth quadric $\mathbb{Q}_2$ of dimension 2. Then to give such an Y_0, we need 8 parameters (4 to give a $\mathbb{P}^3$ plus 3 to give a conic in $\mathbb{P}^3 \cap \mathbb{Q}_3 = \mathbb{Q}_2$ plus 1 to give a point on the conic). To give a doubling requires

$$\dim \operatorname{Hom}(I_{Y_0}/I_{Y_0}^2, \omega_{Y_0}(-1)) - 1 = h^1(I_{Y_0}/I_{Y_0}^2(-1)) - 1$$

parameters. From the exact sequence of conormal sheaves

$$0 \to \nu_{\mathbb{Q}_2,\,\mathbb{Q}_3} \otimes \mathcal{O}_{Y_0} \to \nu_{Y_0,\,\mathbb{Q}_3} \to \nu_{Y_0,\,\mathbb{Q}_2} \to 0\,,$$

and the exact sequences

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_{Y_0} \to \mathcal{O}_C \to 0$$

$$0 \to \mathcal{O}_{\mathbb{Q}_2}(-2,\ -4) \to \mathcal{O}_{\mathbb{Q}_2}(-1,\ -2) \to \nu_{Y_0,\,\mathbb{Q}_2} \to 0\,,$$

the latter coming from the fact that Y_0 is a divisor of type $(1,2)$ in $\mathbb{Q}_2$, one computes that $h^1(\nu_{Y_0,\,\mathbb{Q}_3}(-1)) = 11$. Hence $\dim \mathcal{C}_5 = 18$. To compute $\dim \mathcal{F}_5$, we need $h^0(\mathcal{O}_Y)$ and $h^0(I_Y(2))$. The first one we calculate easily from the sequence

$$0 \to \omega_{Y_0}(1) \to \mathcal{O}_Y \to \mathcal{O}_{Y_0} \to 0\,,$$

and the sequence which relates $\mathcal{O}_{Y_0}$ to $\mathcal{O}_X(-1)$ and $\mathcal{O}_c$ and the result is $h^0(\mathcal{O}_Y) = 3$. To evaluate $h^0(I_Y(2))$, we have to study the structure of Y more closely. We may look at Y as a union of a double line X_2 and a double conic C_2. The two double structures give rise to exact sequences

$$(5)\qquad \begin{gathered} 0 \to \mathcal{O}_X(r) \to \mathcal{O}_{X_2} \to \mathcal{O}_X \to 0 \\ 0 \to L \to \mathcal{O}_{C_2} \to \mathcal{O}_c \to 0 \end{gathered}$$

where we assume that $i^*L = \mathcal{O}_{\mathbb{P}^1}(s)$ for the embedding i of $\mathbb{P}^1$ as the conic C in $\mathbb{Q}_3$. Then the Hilbert polynomials are:

$$\chi_{X_2}(n) = 2n + r + 2\,, \qquad \chi_{c_2}(n) = 4n + s + 2\,.$$

The exact sequence coming from local algebraic linkage

$$0 \to \omega_{X_2}(1) \to \mathcal{O}_Y \to \mathcal{O}_{c_2} \to 0$$

implies

$$(6)\qquad s - r = 1\,.$$

We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & \omega_X \otimes \omega_Y^{-1} \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d, equal] & \mathcal{O}_{C_2 \cup X} \arrow[r] \arrow[d] & 0 \\
 & \omega_{X_2} \otimes \omega_Y^{-1} \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] & \mathcal{O}_{C_2} \arrow[r] & 0 \\
 & \omega_X(-r) \otimes \omega_Y^{-1} \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

The first row is the exact sequence of the linkage of X and $C_2 \cup X$ and the first column is the dual of (5) tensored with ω_Y^{-1}. From here we obtain

$$\omega_X(-r) \otimes \omega_Y^{-1} \simeq I_{C_2}/I_{C_2 \cup X}\,.$$

On the other hand, the exact sequence

$$0 \to I_{C_2}/(I_{C_2 \cup X}) \to \mathcal{O}/I_X \to \mathcal{O}/I_{C_2} + I_X \to 0$$

shows that $I_{C_2}/(I_{C_2 \cup X}) = \mathcal{O}_X(-D_1)$ with D_1 the divisor on X associated to $C_2 \cap X$ as a subscheme in X. In this way we proved that

$$\omega_Y|X \simeq \omega_X(-r) \otimes \mathcal{O}_X(D_1)\,.$$

Completely similarly one proves

$$\omega_Y|C \simeq \omega_C \otimes L^{-1} \otimes \mathcal{O}_C(D_2)\,.$$

where D_2 is $C \cap X_2$ as a divisor on C.

These two formulas above were firstly proved in the non-published manuscript [BF1]. Hence we provided here another proof.

From the exact sequence

$$0 \to \mathcal{O}_{X \cup C_2} \to \mathcal{O}_{X_2} \times \mathcal{O}_{C_2} \to \mathcal{O}_{X \cap C_2} \to 0$$

one obtains

$$l(\mathcal{O}_{X \cap C_2}) = r + s + 1 . \tag{7}$$

Around the point $P = C \cap X$ there are only two possibilities for Y:

A) $I_Y = (z, x^2 y^2)$, $I_{C_2} = (z, x^2)$, $I_{X_2} = (z, y^2)$ or

B) $I_Y = (z^2, xy)$, $I_{C_2} = (z^2, x)$, $I_{X_2} = (z^2, y)$.

We then make use of the exact sequences

$$0 \to I_Y \to I_{C_2} \to \omega_{X_2}(1) \to 0 ,$$

$$0 \to I_{C_2} \to I_C \to L \to 0 .$$

In the situation *A)* we have

$$r + s + 1 = 4 . \tag{7'}$$

The equation (6) and (7′) give $s = 2$, $r = 1$ and $D_1 = 2P$, $D_2 = 2P$, so that $\omega_{X_2} = \mathcal{O}_{X_2}(-3)$. In this situation $h^0(I_Y(2)) = h^0(I_{C_2}(2)) \geq 2$. In the case *B)* we have

$$r + s + 1 = 2 \tag{7''}$$

and hence $r = 0$, $s = 1$, $D_1 = P$, $D_2 = P$ and finally $\omega_{X_2} \simeq \mathcal{O}_{X_2}(-2)$. This gives $h^0(I_Y(2)) = h^0(I_{C_2}(2)) \geq 3$. From all of this we obtain

$$\dim \mathcal{F}_5 \leq 18 .$$

#6) According to Lemma 1 and with the notation from there, a curve Y in $\mathcal{C}_6$ is a doubling on $Y_0 = X_1 \cup Y_2''$, where X_1 is a simple line, Y_2'' is a double structure on a line X_2 meeting X_1 and with the invertible sheaf $\mathcal{O}_{X_2}(r)$ where $r = -1$ or $r = 0$. When $r = -1$, we have $l(X_1 \cap Y_2'') = 1$, so that this case has been already discussed at #5)—being its degeneration. Here we consider the case $r = 0$, $l(X_1 \cap Y_2'') = 2$. Let $\mathcal{C}_6'$ be the corresponding family of curves. We choose homogeneous coordinate x, y, z, u, v in $\mathbb{P}^4$ such that $I_{X_1} = (x, y, u)$, $I_{X_2} = (x, y, z)$ and the equation of $\mathbb{Q}_3$ takes the form $\phi(x, y, z) + yv + zu = 0$ where the quadratic form ϕ does not contain a term in z^2. As the doubling Y_2'' of X_2 is done with $L = \mathcal{O}_{X_2}$ and $\nu_{X_2, \mathbb{Q}_3} = \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_2}(-1)$, one sees that the ideal of Y_2 in $\mathbb{P}^4$ is of the form $I_{Y_2''} = (x - ky - mz, (x, y, z)^2, yv + uz)$ where k, m are constants. The condition $l(X_1 \cap Y_2'') = 2$ gives $m = 0$. If we change $x - ky$ with X and substitute $X = x$ afterwards, we obtain $I_{X_1} = (x, y, u)$, $I_{X_2} = (x, y, z)$, $I_{Y_2''} = (x, (y, z)^2, yv + zu)$, $I_{X_1 \cup Y_2''} = (x, y^2, yz, yv + uz)$. We showed that $X_1 \cup Y_2'' = Y_0$ is a l.c.i. curve in $\mathbb{Q}_3$, contained in the smooth quadric surface of the equation $x = 0$ in $\mathbb{Q}_3$. From the exact sequences

$$0 \to \mathcal{O}_{X_1}(-2) \to \mathcal{O}_{Y_0} \to \mathcal{O}_{Y_2''} \to 0,$$

$$0 \to \mathcal{O}_{X_2} \to \mathcal{O}_{Y_2''} \to \mathcal{O}_{X_2} \to 0,$$

$$0 \to \mathcal{O}_{Y_0}(-1) \to \nu_{Y_0, \mathbb{Q}_3} \to \nu_{Y_0, \mathbb{Q}_2} \to 0,$$

one gets $h^1((I_{Y_0}/I_{Y_0}^2)(-1)) = 11$. Summing up:

$$\dim \mathcal{C}_6' = 3 \text{ (a line } X_1) + 1\text{(a point on } X_1) + 1\text{(a line } X_2 \text{ through the point)} +$$
$$+1\text{(a doubling of } X_2 \text{ with } r = 0,\ l(X_1 \cap Y_2'') = 2) +$$
$$+10\text{(a doubling on } Y_0 = X_1 \cup Y_2'') = 16.$$

In a standard way one computes $h^0(\mathcal{O}_Y) = 3$ and then $h^0(I_Y(2)) \geq 1$, because at least the term x^2 is in the ideal. Then

$$\dim \mathcal{F}_6 \leq 17.$$

#7) As we saw, in this case we have in fact two families of curves, the first one corresponding to $r = -1$, $e + f_1 = 1$, the other one to $r = -1$, $e + f_1 = 2$. From the equality $3d + 6e + 6f_1 + 3f_2 = 12$ one sees that $e + f_1 = 1$ gives $d + f_2 = 2$ and $e + f_1 = 2$ yields $d = f_2 = 0$.

According to [BF2], the dimension of all quasiprimitive structures of fixed type (i.e., r and a divisor fixed) is

$$3\text{(to give a line } X \text{ in } \mathbb{Q}_3) + 0 \text{ (doubling with } \mathcal{O}_X(-1)) +$$
$$+h^0(X, (\det N_X^*) \otimes \mathcal{O}_X(-3) \otimes \mathcal{O}_X(E + F_1) \otimes \mathcal{O}_X(D_2')) +$$
$$+h^0(X, (\det N_X^*) \otimes \mathcal{O}_X(-4) \otimes \mathcal{O}_X(D + E + F_1 + F_2) \otimes \mathcal{O}_X(D_3')) +$$
$$+h^0(X, (\det N_X^*) \otimes \mathcal{O}_X(-5) \otimes \mathcal{O}_X(D + 2E + 2F_1 + F_2) \otimes \mathcal{O}_X(D_4')) +$$
$$+h^0(X, (\det N_X^*) \otimes \mathcal{O}_X(-6) \otimes \mathcal{O}_X(D + 2E + 2F_1 + F_2) \otimes \mathcal{O}_X(D_5')),$$

where N_X is the normal bundle of X in $\mathbb{Q}_3$, D, E, F_1 and F_2 are the divisors associated to the quasiprimitive structure and D_i' are divisors on X given by $\mathcal{O}_{D_i'} = J_2/(J_i + I^2)$ with J_i being the ideals in the canonical filtration. Using [M3], Theorem 3.3, one computes D_i':

$$D_2' = E + F_1, \qquad D_3' = D + E + F_1 + F_2,$$

$$D_4' = D + E + 2F_1 + F_2, \qquad D_5' \leq D + E + 2F_1 + F_2.$$

In this way we have: $\dim \mathcal{C}_7(e + f_1 = 1) \leq 17$ and then, as $h^0(\mathcal{O}_Y) = 3$ and $h^0(E(1)) \geq 1$, we obtain $\dim \mathcal{F}_7(e + f_1 = 1) \leq 19$. In fact, we can show that $h^0(I_{Y_3}(1)) = 1$ and then $h^0(I_Y(2)) \geq 1$ so that $h^0(E(1)) \geq 2$, but this is not important for the whole classification and we will omit the proof.

Using the same technique as above, we calculate that $\dim \mathcal{C}_7(e+f_1=2) \leq 13$ and $\dim \mathcal{F}_7(e+f_1=2) \leq 15$. Altogether

$$\dim \mathcal{F}_7 \leq 19 .$$

#8) As the conormal bundle of a line X in $\mathbb{Q}_3$ is $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$, the conormal sheaf of the first infinitesimal neighbourhood Y_0 will be $\mathcal{O}_X \oplus \mathcal{O}_X(-1) \oplus \mathcal{O}_X(-2)$ and then, denoting by I_X the ideal of X in $\mathbb{Q}_3$:

$$\dim \mathcal{C}_8 = 3\,(\text{a line}) + h^0(\omega_{Y_0}(1)) + h^0(\omega_{Y_0}(2)) + h^0(\omega_{Y_0}(3)) - 1 =$$

$$= 2 + h^2(I_X^2(-1)) + h^2(I_X^2(-2)) + h^2(I_X^2(-3)) .$$

From the exact sequence $0 \to I_X^2 \to I_X \to \mathcal{O}_X \oplus \mathcal{O}_X(-1) \to 0$ we now see immediately that $\dim \mathcal{C}_8 \leq 11$ and

$$\dim \mathcal{F}_8 \leq 13 .$$

#9) Finally, we discuss the dimension of the family of double twisted cubics. Let us notice that in the moduli space $M_{\mathbb{P}^3}(-1, 4)$ such curves give rise to a family of bundles which is dense in one of the components of in $M_{\mathbb{P}^3}(-1, 4)$, [BM].

Denoting now a twisted cubic by X, we see that the dimension of the family of double twisted cubics on $\mathbb{Q}_3$ equals

$$9(\text{to give a twisted cubic}) + h^1((I_X/I_X^2)(-1)) - 1 .$$

As is well known, a twisted cubic is a divisor of type $(1,2)$ on a smooth 2-quadric. Then one computes $h^1((I_X/I_X^2)(-1)) = 11$, so that $\dim \mathcal{C}_9 = 19$. From the exact sequence

$$0 \to I_Y/I_X^2 \to I_X/I_X^2 \to \omega_X(1) \to 0$$

and from the fact that $i^*(I_X/I_X^2) \simeq \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-4)$ for an embedding i of $\mathbb{P}^1$ as a twisted cubic, one sees that $i^*(I_Y/I_X^2) = \mathcal{O}_{\mathbb{P}^1}(-8)$. Hence the exact sequence $0 \to I_X^2 \to I_Y \to I_Y/I_X^2 \to 0$ shows $h^0(I_Y(2)) = h^0(I_X^2(2)) = 1$. Putting the things together, we see that

$$\dim \mathcal{F}_9 = 20 .$$

REFERENCES

[AO1] V. ANCONA - G. OTTAVIANI, *Some applications of Beilinson's theorem to projective spaces and quadrics,* Forum Math., **3** (1991), pp. 157-176.

[AO2] V. ANCONA - G. OTTAVIANI, *On the stability of special instanton bundles on* $\mathbb{P}^{2n+1}$, to appear in Trans. Amer. Math. Soc.

[AS1] E. ARRONDO - I. SOLS, *Classification of smooth congruence of low degree,* J. Reine Angew. Math., **393** (1989), pp. 199-219.

[AS2] E. ARRONDO - I. SOLS, *On congruences of lines in the projective space*, Soc. Math. de France, Mémoire n° 50, Supplément au Bulletin de la S.M.F., Tome 120, 1992.

[BF] C. BĂNICĂ - O. FORSTER, *Multiplicity structures on space curves, The Lefschetz Centennial Conference*, part I, contemporary Mathematics, vol. 58, pp. 47-64.

[BPV] W. BARTH - C. PETERS - A. VAN DE VEN, *Compact Complex Surfaces*, Springer-Verlag, Berlin (1984).

[BS] D. BAYER - M. STILLMANN, *MACAULAY, a computer algebra system for algebraic geometry*.

[Ch] D. CHENIOT, *Une démonstration du théorème de Zariski sur les sections hyperplanes d'une hypersurface projective et du théorème de van Kampen sur le groupe fondamental du complémentaire d'une courbe projective plane*, Comp. Math., **27** (1973), pp. 141-158.

[ES] L. EIN - I. SOLS, *Stable vector bundles on quadric hypersurfaces*, Nagoya Math. J., **96** (1984), pp. 11-22.

[Ha] R. HARTSHORNE, *Stable vector bundles of rank* 2 *on* $\mathbb{P}^3$, Math. Ann., **238** (1978), pp. 229-280.

[Ha1] R. HARTSHORNE, *Stable reflexive sheaves*, Math. Ann., **254** (1980), pp. 121-176.

[HaS] R. HARTSHORNE - I. SOLS, *Stable vector bundles on P^3 with $c_1 = -1$, $c_2 = 2$*, J. Reine Angew. Math., **325** (1981), pp. 145-152.

[HeS] R. HERNANDEZ - I. SOLS, *On a family of rank-* 3 *bundles on* $\mathrm{Gr}(1,3)$, J. Reine Angew. Math., **360** (1985), pp. 125-135.

[Ho] H.-J. HOPPE, *Generischer Spaltungstyp und zweite Chernklasse stabilier Vektorraumbündel von Rang* 4 *auf* $\mathbb{P}_4$, Math. Zeitschrift, **187** (1984), pp. 345-360.

[Hor] G. HORROCKS, *Construction of bundles on* $\mathbb{P}^n$, Seminaire Douady-Verdier ENS 77/78, Asterisque, **71-72**, Les equations de Yang-Mills (1980), pp. 197-203.

[Ka] M. M. KAPRANOV, *On the derived categories of coherent sheaves on some homogeneous spaces*, Inv. Math., **92** (1988), pp. 479-508.

[Kn] H. KNÖRRER, *Cohen-Macaulay modules of hypersurface singularities I*, Inv. Math., 88 (1987), pp. 153-164.

[Ma] M. MARUYAMA, *Moduli of stable sheaves II*, J. Math. Kyoto Univ., 18 (1978), pp. 557-614.

[Ne] P. E. NEWSTEAD, *A note on quadratic complexes*, J. London Math. Soc. (2), 5 (1972), pp. 748-752.

[MM] Y. MATSUSHIMA - A. MORIMOTO, *Sur certain fibrés holomorphes sur une variété de Stein*, Bull. Soc. Math. France, 88 (1960), pp. 137-155.

[Mu] D. MUMFORD, *Varieties defined by quadratic equations*, C.I.M.E. (1969).

[OSS] CH. OKONEK - M. SCHNEIDER - H. SPINDLER, *Vector Bundles on Complex Projective Spaces*, Birkhäuser (1981).

[Ot] G. OTTAVIANI, *On Cayley bundles on the five- dimensional quadric*, Boll. Un. Mat. Ital. (7), 4-A (1990), pp. 87-100.

[Sh] I. R. SHAFAREVICH, *Osnovy algebraičeskoj geometrii*, Moskva (1972); Izd. «Mir» (Russian).

[So] I. SOLS, *On spinor bundles*, J. Pure Appl. Algebra, **35** (1985), pp. 85-94.

[SSW] I. SOLS - M. SZUREK - J. A. WIŚNIEWSKI, *On Fano bundles over a smooth 3-dimensional quadric*, Pacific J. Math., **148** (1991), pp. 153-160.

[Sp] E. H. SPANIER, *Algebraic Topology*, McGraw-Hill (1966).

[Ta] H. TANGO, *On projective spaces embedded in Grassmannians*, J. Math. Kyoto Univ., 14 (1974), pp. 415-460.

[Wil] J. WILLIAMSON, *On an algebraic problem, concerning the normal forms of linear dynamical systems*, Amer. J. Math., 58 (1936), pp. 141-163.

REFERENCES FOR THE APPENDIX

[BF1] C. BĂNICĂ - O. FORSTER, *Sur les structures multiples,* manuscript (1981).

[BF2] C. BĂNICĂ - O. FORSTER, *Multiplicity structures on space curves. The Lefschetz Centennial Conference,* part I, Contemporary Mathematics, 58 (1986), pp. 47-64.

[BM] C. BĂNICĂ - N. MANOLACHE, *Rank-2 stable vector bundles on* $\mathbb{P}^3(\mathbb{C})$ *with Chern classes* $c_1 = -1$, $c_2 = 4$, Math. Z., **190** (1985), pp. 315-339.

[M1] N. MANOLACHE, *Rank-2 stable vector bundles on* $\mathbb{P}^3$ *with Chern classes* $c_1 = -1, c_2 = 2$, Rev. Roum. Math. Pures Appl., **26** (1981), pp. 1203-1209.

[M2] N. MANOLACHE, *Cohen-Macaulay nilpotent structures,* Rev. Roum. Math. Pures Appl., **31** (1986), pp. 563-575.

[M3] N. MANOLACHE, *Multiple structures on smooth supports,* to appear in Mathematische Nachrichten.