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THE THEOREM OF MATHER ON GENERIC PROJECTIONS IN THE SETTING OF ALGEBRAIC GEOMETRY

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We present the proof of the theorem of Mather on generic projections, stated in the setting of algebraic geometry. The main tools used are the Thom-Boardman singularities in the jet space. This theorem has been applied in the study of codimension two submanifold of $\mathbf{P}^n$ and it seems that it could have further applications.

0. Introduction and statement of theorems 1 and 2

Around the seventies J. Mather completed an intensive study on the stability of C^∞-mappings(see [7]). Applying the techniques arised from this work he was able to prove in 1973 [9] the following remarkable theorem on generic projections (although not explicitly stated in this form).

Theorem 1 (Mather) and definition of X^i *Let* $X=X^1$ *be a smooth subvariety of the complex projective space* $\mathbf{P}^n$ *of codimension* d. *For each point* $p\in\mathbf{P}^n$ *define* $X^2=\{x\in X^1|p\in T_xX^1\}$ *and when* X^2 *is smooth define* $X^3=\{x\in X^2|p\in T_xX^2\}$. *When* X^3 *is smooth define* X^4 *and so on.*

For a generic choice of p <u>*each* X^i *is smooth*</u>*(so the definitions above are possible) and when is not empty is* <u>*of codimension* d *in* X^{i-1}</u>.

This theorem has been applied by the authors of this note in the work around the Hartshorne conjecture about the projective varieties of small codimension [2].

The statement of the theorem seems quite natural. In fact it is easy to prove the smoothness of X^2 using the theorem of generic smoothness ([6], cor. III.10.7). This does not work for X^i with $i\geq 3$ because if one changes the point p then all the varieties X^i change (not only the last!). By this fact one is forced to work with jet spaces, which take into account the behaviour of all the derivatives together. Nevertheless, the smoothness of X^i with $i\geq 3$ remains a deep result. The main reason is that the projections are "too few" in the space of all C^∞-

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mappings to use standard results about transversality (see for ex. the transversality theorem of Thom[5] and the results of Kleiman, Laksov and Speiser[10]).
Mather's proof requires a careful local study of the Thom-Boardman singularities in the jet space and it is not easy to read for a non-specialist in the field. In fact it is divided in four or five different papers and in the assumptions are often considered C^{∞} real mappings instead of complex holomorphic (although Mather himself points out that the argument works also in the complex case). Furthermore, the results are stated sometimes in full generality (e.g. multi-jets, non compact manifolds), which is not necessary for our purposes.

C.Peskine encouraged us to write down a self-contained account of Mather's proof in the setting of (complex)algebraic geometry and this is exactly what we try to do in this note. Our feeling is that Mather's theorem is promising for further applications in algebraic geometry.

Theorem 1 can be seen as a corollary(take $t=0$, $i_1=\ldots=i_k=1, X^i=X_{1,\ldots,1}$(i-1 times)) of the following more general

Theorem 2 and definition of $X_{i_1,\ldots,i_k}$ *Let* X= *be a smooth subvariety of the complex projective space* $\mathbf{P}^n$ *of codim* d. *Let* T *be any linear subspace of* $\mathbf{P}^n$ *of dimension* t *such that* $T\cap X=\emptyset$ *(so* $t<d$*). For any* $i_1\leq t+1$ *let* $X_{i_1}=\{x\in X|\dim[T_xX_1\cap T]=i_1-1\}$[2]. *When* X_{i_1} *is smooth for any* $i_2\leq i_1$ *define* $X_{i_1,i_2}=\{x\in X_{i_1}|\dim[T_xX_{i_1}\cap T]=i_2-1\}$ *and so on for* $i_1\geq i_2\geq\ldots\geq i_k$ *define (when possible)* $X_{i_1,\ldots,i_k}$. *For* T *in a Zariski open set of the grassmannian* $Gr(\mathbf{P}^t,\mathbf{P}^n)$ *we have that each* $X_{i_1,\ldots,i_k}$ *is smooth (and so the definitions above are possible) until (increasing* k*) it becomes empty and its codimension in* X_1 *is equal to the number* $\nu_{i_1,\ldots,i_k,0,\ldots,0}$ *defined below.*

The varieties X_i as in theorem 1 are closed in the Zariski topology. Every variety $X_{i_1,\ldots,i_k}$ as in theorem 2 is locally closed but in general not closed. The closure $\overline{X_{i_1,\ldots,i_k}}$ is obtained by asking dim$\geq$ instead of = in the definition given in the theorem. It does not seem that $\overline{X_{i_1,\ldots,i_k}}$ is smooth in general, except in the case $i_1=\ldots=i_k=t+1$.

Definition of μ_I and ν_I *Let* $I=(i_1,\ldots,i_k)$ *be any sequence of integers satisfying* $i_1\geq i_2\geq\ldots\geq i_k\geq 0$. *Define first* μ_I=*number of sequences* $(j_1,\ldots,j_k)$ *such that* $j_1\geq j_2\geq\ldots\geq j_k$, $i_r\geq j_r$ *for* $1\leq r\leq k$, $j_1>0$. *Then* $\nu_I:=(d-t-1+i_1)\mu_{i_1,\ldots,i_k}-(i_1-i_2)\mu_{i_2,\ldots,i_k}-\ldots-(i_{k-1}-i_k)\mu_{i_k}$.

Examples $\mu_q=\mu_{q,0,\ldots,0}=q$ so that $\mathrm{codim}_X X_q=q(d-t-1+q)$. When $q=1$ the cycles $\overline{X_q}$ are Poincaré dual to the Chern classes $c_{d-t}(N_{X_1,\mathbf{P}^n}(-1))$. For $q\geq 1$ the cycles X_q represent the loci where a projection of X centered in T drops rank by q and their closures are expressed by Porteous formula.

$\mu_{1,\ldots,1(q\text{ times})}=q$ so that $\mathrm{codim}\,_X X_{1,\ldots,1(q+1\text{ times})}=q(d-t)$ (when $t=0$ this is the formula of theorem 1)

[2] by convention $\dim\emptyset=-1$

Alternative definition of X^i *When X^i are smooth they can also be defined as*

$$X^i=\{x\in X \mid \textit{multiplicity of intersection of the line } \overline{px} \textit{ with } X \textit{ at } x \textit{ is } \geq i\}.$$

In fact if X is the transverse intersection of the hypersurfaces $\{f_1=0\},\ldots,\{f_r=0\}$ (f_j homog. polynomial, $r\geq\text{codim } X$) the condition above on the multiplicity means that $t=0$ is a root of multiplicity$\geq i$ of $F_j(t)=f_j(pt+x)$ for $j=1,\ldots,r$. Set $p=(p_0,\ldots,p_n)$, $x=(x_0,\ldots,x_n)$. We have $\frac{d^m}{dt^m}F_j(t)|_{t=0}=\sum\frac{\partial^m f_j}{\partial x_{i_1}\ldots\partial x_{i_m}}p_{i_1}\ldots p_{i_m}$. In this case X^i is given by the intersection of the hypersurfaces

$$\sum\frac{\partial^m f_j}{\partial x_{i_1}\ldots\partial x_{i_m}}p_{i_1}\ldots p_{i_m}=0 \text{ for } 0\leq m\leq i-1,\ 1\leq j\leq r.$$

Theorem 1 implies that for generic p this intersection is transverse. The tangent space to X^i at x_0 is given by

$$\sum\frac{\partial^{m+1} f_j}{\partial x_{i_1}\ldots\partial x_{i_m}\partial x_s}p_{i_1}\ldots p_{i_m}x_s=0 \text{ for } 0\leq m\leq i-1,\ 1\leq j\leq r,\ 0\leq s\leq n$$

hence the condition that TX^i contains p gives exactly X^{i+1}.

The outline of Mather's proof of theorem 2 is the following:

Let L be a $(n-t-1)$-dimensional linear subspace of $\mathbf{P}^n$. For any $T\in Gr(\mathbf{P}^t,\mathbf{P}^n)$ disjoint from X_1 and from L let $f_T:X_1\to L$ be the linear projection centered in T and jf_T its jet. Let $I=(i_1,\ldots,i_n)$ be any sequence of integers with $i_1\geq i_2\geq\ldots\geq i_k\geq 0$.

1) Now define in the jet space $J(X_1,L)$ some submanifolds Σ^I with the property that $(jf_T)^{-1}\Sigma^I=X_I$ (when X_I are defined). This definition is not trivial and it is due to Boardman [3],[8]. Σ^I are the so called Thom-Boardman singularities, they are smooth, locally closed and of codim ν_I.

2) Show that, for a generic choice of T, jf_T is transversal to Σ^I. This is proved by Mather in [9] using results of [7] and [8].

This note does not want to be a substitute to the original papers, but only a guide to them, although we give all the details with few exceptions where a clear reference is given. Anyway, for the convenience of the reader, in some cases we have preferred to repeat almost word for word some arguments of [8].

Although we have tried to translate many steps of the proof in the algebraic language, we underline that the proof that we present here is not purely algebraic. In fact we need several times some local analytic neighborhoods in order to integrate some vector fields.

We wish to thank J.Mather for some helpful letters.

1. Background

Our policy will be to recall only the background that we strictly need.

1.a Transversality

Let X,Y be complex manifolds. If $f:X \to Y$ is an holomorphic map, denote by $df:TX \to TY$ its derivative (consequently we have on the fibers $df_x:TX_x \to TY_{f(x)}$).

Definition 1.1 *Let* $W \subset Y$ *be a submanifold. An holomorphic map* $f:X \to Y$ *is said to be* *__transversal__ to* W *at* $x \in X$ *if we have*

i) $f(x) \notin W$ *or*

ii) $df_x(TX_x)+TW_{f(x)}=TY_{f(x)}$

f *is said to be transversal to* W *if* i) *or* ii) *hold* $\forall x \in X$.

For the first properties of transversality we refer to [5], where is treated the real case, which is analogous to ours.

The following theorem gives a characterization of transversality in the algebraic case. By an algebraic variety we mean a separated reduced scheme of finite type over **C**.

Theorem 1.2 *Let* X,Y *be nonsingular algebraic varieties and* $W \subset Y$ *be a nonsingular algebraic subvariety. A morphism* $f:X \to Y$ *is transversal to* W *at* $x \in X$ *if and only if the fibred product* $f^{-1}(W)= X \times_Y W$ *(as scheme) does not contain* x *or is smooth of codimension equal to* codimW at x.

Proof [4] §17.13.2.

The "only if" part of the theorem holds also for complex manifolds (it follows from the implicit function theorem, see [5]p.52), but the converse is false when $f^{-1}(W)$ is considered as set (forgetting the structure of scheme). A counterexample is the map $\mathbf{C}^2 \to \mathbf{C}$ at the origin given by $(x,y) \mapsto x^2$.

1.b Jet spaces

Let X,Y be complex manifolds (dimX=n, dimY=m) and $(x,y) \in X \times Y$. Two holomorphic maps f,g from a (common) neighborhood of x to Y such that $f(x)=g(x)=y$ are said *to have* k*-order contact* if their derivatives of order $\leq k$ in local coordinates coincide. Precisely, if $x_1,\dots,x_n$ is a local system of coordinates centered in x, and $y_1,\dots,y_m$ is a local system of coordinates centered in y we must

have $\frac{\partial^{|I|}(y_j \circ f)}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}(x) = \frac{\partial^{|I|}(y_j \circ g)}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}(x)$ for any j and for any multiindex $I=(i_1,\dots,i_n)$ with

$\sum_t i_t \leq k$ (the derivatives containing $\frac{\partial}{\partial \bar{x}_i}$ are zero by the assumption that f,g are holomorphic). In this case we write $f \sim_k g$ at x. This is equivalent to ask that the Taylor expansions of f and g up to degree k coincide. An equivalence class of holomorphic germs under $\sim_k$ at x is called a (holomorphic) k-*jet*. The equivalence class of f at x is written as $j^k f_x$. x and y are called respectively the *source* and the *target* of $j^k f_x$.

$J^k(X,Y)$ as set is the union of all k-jets for all $(x,y) \in X \times Y$. It is possible to give to $J^k(X,Y)$ the structure of a complex manifold in such a way that the natural projection $J^k(X,Y) \to X \times Y$ is a fiber bundle (for the real case we refer to [5]). Precisely if $x \in U \subset X$, $y \in V \subset Y$ are local neighborhoods of some atlas for X and Y and x,y are the source and the target of $j^k f \in J^k(U,V)$ then $J^k(U,V) \subset J^k(X,Y)$ is a local neighborhood in $j^k f_x$ with the chart given by

$T: J^k(U,V) \longrightarrow U \times V \times \mathbf{C}^N$ where $N = m[\binom{n+k}{k} - 1]$

$\sigma \mapsto$ (source of σ, target of σ, derivatives of order $\leq k$ of a representative of σ)

It is easy to check that $(x_1, \ldots, x_n, y_1, \ldots, y_m, \ldots, \frac{\partial^s (y_i \circ f)}{\partial x_{i_1} \ldots \partial x_{i_s}}(x), \ldots)$ for $1 \leq i_1, \ldots, i_s \leq n$, $1 \leq s \leq k$ are the local coord. of $j^k f_x$.

The fibers of $J^k(X,Y)$ over $X \times Y$ are vector spaces $\mathbf{C}^N$ but $J^k(X,Y)$ is not a vector bundle for $k \geq 2$ (for $k=1$ $J^1(X,Y) \simeq Hom(TX,TY) \simeq TX^* \otimes TY$).

Any fiber $\mathbf{C}^N$ is identified with $J^k(n,m)$, space of k-jets from $\mathbf{C}^n$ to $\mathbf{C}^m$ sending the origin to the origin.

Example: $J^2(\mathbf{P}^1, \mathbf{P}^1)$ We have the natural projection $J^2(\mathbf{P}^1, \mathbf{P}^1) \xrightarrow{\pi} \mathbf{P}^1 \times \mathbf{P}^1$. Let $(x_0, x_1; y_0, y_1)$ be homogeneous coord. on $\mathbf{P}^1 \times \mathbf{P}^1$, which is covered by the four affine sets $U_{ij} = \{x_i \neq 0,\ y_j \neq 0\}$. On $\pi^{-1}(U_{00})$ we have coordinates $X_0 = \frac{x_1}{x_0}$, $Y_0 = \frac{y_1}{y_0}$, Y_0', Y_0'' in such a way that if f is a local holomorphic function from some neighborhood $U \subset \mathbf{P}^1$ containing (1,0) to $V \subset \mathbf{P}^1$ containing $F_0(1,0) = \frac{x_1 \circ f(1,0)}{x_0 \circ f(1,0)} = F_0(1)$ (with a slight abuse of notation) the coordinates of $j^2 f((1,0)) \in \pi^{-1}(U_{00})$ are $X_0 = 1$, $Y_0 = F_0(1)$, $Y_0' = F_0'(1)$, $Y_0'' = F_0''(1)$. In an analogous way X_0, Y_1, Y_1', Y_1'' are local coordinates on $\pi^{-1}(U_{01})$ and on $\pi^{-1}(U_{00}) \cap \pi^{-1}(U_{01})$ the change of coordinates is given by

$$X_0 = X_0$$
$$Y_1 = \frac{1}{Y_0}$$
$$Y_1' = -\frac{Y_0'}{(Y_0)^2}$$
$$Y_1'' = \frac{-Y_0'' Y_0 + 2(Y_0')^2 Y_0}{(Y_0)^4}.$$

Thus on the fibers of π (which are $\mathbf{C}^2$) the change $(Y_0', Y_0'') \to (Y_1', Y_1'')$ is quadratic (and not linear!). If we consider $J^k(\mathbf{P}^1, \mathbf{P}^1)$ then powers of degree k are involved in the changes of coordinates.

If X,Y are algebraic also $J^k(X,Y)$ is an algebraic variety.

Note also that for every holomorphic map f: $X \to Y$ we have an holomorphic section $j^k f: X \to J^k(X,Y)$ and $j^k f(x)=j^k f_x$.

Let E be a vector bundle on a complex manifold X. Let $\mathcal{E}$ be the associated sheaf of sections and $\forall x \in X$ let $\mathcal{M}_x \subset \mathcal{O}_X$ be the maximal ideal in the structure sheaf.

The vector bundle with stalk in x equal to $\mathcal{E}_x / \mathcal{M}_x^{k+1} \cdot \mathcal{E}_x$ is denoted by $J^k(E)$ and is called the k-*jet bundle* of E. In particular $J^0(E)=E$.

$J^k(E)$ can be constructed considering the local sections of E and identifying any two of them when their Taylor expansions up to degree k coincide. $J^k(E)$ can be seen as a sort of $J^k(X,E)$ where we take germs of sections f: $X \to E$ (not of every map!). In fact the structure of vector bundle in $J^k(E)$ comes from the addition of sections in E.

Every vector bundle map $\phi: E \to E'$ lifts in a natural way to a vector bundle map $J^k\phi: J^kE \to J^kE'$.

1.c Integral curves on complex manifolds

For further use we prefer to state the following known theorem.

Theorem 1.3 Let $U \subset \mathbf{C}^n$ be an open set and $z_i = x_i + \sqrt{-1}y_i$ be the coordinate functions. Let V be a holomorphic vector field on U, that is $V = \sum a_i \frac{\partial}{\partial z_i}$ with a_i holomorphic functions (or $V = \sum \alpha_i \frac{\partial}{\partial x_i} + \beta_i \frac{\partial}{\partial y_i}$ with $\alpha_i + \sqrt{-1}\beta_i$ holomorphic). Then there exists an open neighborhood $W \subset \mathbf{R} \times U$ such that $\{0\} \times U \subset W$ and a map $F: W \to U$ such that

$$F(0,z)=z$$

$$\frac{\partial F(t,z)}{\partial t} = V(F(t,z)) \quad \left(\text{i.e. } \frac{\partial F_i(t,z)}{\partial t} = a_i(F(t,z))\right)$$

F(t,z) is holomorphic in z.

This theorem can be quickly derived from the C^∞case. In fact if the a_i satisfy the Cauchy-Riemann equations then the conditions $\frac{\partial F_i(t,z)}{\partial t} = a_i(F(t,z))$ and $F(0,z)=z$ imply that also the F_i satisfy the Cauchy-Riemann equations.

Of course this theorem can be applied when we have a local holomorphic vector field on a complex manifold. The curve $t \to F(t,z)$ is called an integral curve of V.

2. Definition of Thom-Boardman singularities

The Thom-Boardman singularities are some submanifolds of the jet space that were first defined by Boardman [3], by means of the "Jacobian extensions". We follow the approach of Arnold and Mather [8]. Their main properties are expressed by theorems 2.28 and 2.31 (see prop. 2 and 3 of section 4 and prop.1 of section 6 in [8]).

We recall some facts about formal power series algebras (we can consider formal series instead of convergent series because we are interested in k-jets and not in the infinite jet space, see also remark 2.17).

Let $A_n=\mathbb{C}[[x_1,\dots,x_n]]$ be the C-algebra of formal power series in n variables. It is a local ring with maximal ideal $\mathcal{M}_n=(x_1,\dots,x_n)$. By a *derivation* D we mean a linear operator on A_n of the form $D=\sum u_i\frac{\partial}{\partial x_i}$ with $u_i\in A_n$. It satisfies the Leibniz rule $D(uv)=uDv+vD(u)$ $\forall u,v\in A_n$, hence $D(u^p)=pu^{p-1}D(u)$. Two ideals $\mathfrak{I}_1,\mathfrak{I}_2\subset A_n$ are said to be *equivalent* if $f(\mathfrak{I}_1)=\mathfrak{I}_2$ where $f:A_n\to A_n$ is an automorphism of C-algebras. Every surjective morphism $f:A_p\to A_n$ satisfies $f(\mathcal{M}_p)\subset\mathcal{M}_n$ (proof: if $f(x_1)\notin\mathcal{M}_n$ then $\exists a\in\mathbb{C}^*$ such that $f(x_1-a)\in\mathcal{M}_n$. This is a contradiction because $f(x_1-a)$ is invertible, in fact $\frac{1}{x_1-a}$ is a power series). f induces a surjective morphism of vector spaces $\bar{f}:\mathcal{M}_p/\mathcal{M}_p^2\to\mathcal{M}_n/\mathcal{M}_n^2$. Conversely a surjective morphism of vector spaces $\bar{f}$ can be lifted to a (not unique!) surjective morphism of algebras $f:A_n\to A_p$, this follows from Nakayama's lemma, see for example the prop. 2.8 in [1].

Let $\mathfrak{I}$ be an ideal in A_n and s a positive integer.

2.1 Definition of Jacobian extension *By $\Delta_s\mathfrak{I}$ we denote the ideal $\mathfrak{I}+\mathfrak{I}'$ where $\mathfrak{I}'$ is the ideal generated by all* s×s *determinants $|\Delta_if_j|$ where $\Delta_1,\dots,\Delta_s$ are derivations and $f_1,\dots,f_s\in\mathfrak{I}$. $\Delta_s\mathfrak{I}$ is called a Jacobian extension of $\mathfrak{I}$.*

The proofs of the following facts are trivial

1) $\mathfrak{I}=\Delta_m\mathfrak{I}$ for $m>n$ (in fact the rows of the matrix $\left(\Delta_if_j\right)$ are dependent because the space of derivations has dimension n over A_n).

2) $\mathfrak{I}\subset\Delta_n\mathfrak{I}\subset\Delta_{n-1}\mathfrak{I}\subset\dots\subset\Delta_1\mathfrak{I}$ (expand the determinants with respect to the first row).

3) If $f_1,\dots,f_r$ is a set of generators of $\mathfrak{I}$, then $\Delta_s\mathfrak{I}$ is generated by $f_1,\dots,f_r$ and (when $s\le\min(r,n)$) all s×s minors of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$. (we often use "minor" for "determinant of minor")

Let $\mathfrak{I}\subset A_n$ be a proper ideal.

Definition 2.2

$$\operatorname{rank}\mathfrak{I}:=\dim_{\mathbb{C}}\frac{\mathcal{M}_n^2+\mathfrak{I}}{\mathcal{M}_n^2}=\dim_{\mathbb{C}}\frac{\mathfrak{I}}{\mathfrak{I}\cap\mathcal{M}_n^2}$$

$$\operatorname{corank}\mathfrak{I}=n-\operatorname{rank}\mathfrak{I}$$

Note that if $\mathfrak{I}\subset\mathfrak{J}$ then $\operatorname{rank}\mathfrak{I}\le\operatorname{rank}\mathfrak{J}$

Examples $\operatorname{rank}\mathcal{M}_n=n$; $\Delta_i\mathcal{M}_n=\begin{cases}A_n & i\le n\\ \mathcal{M}_n & i>n\end{cases}$

Proposition 2.3

$$\Delta_s\mathfrak{I}=A_n\Leftrightarrow s\le\operatorname{rank}\mathfrak{I}$$

Proof $\Leftarrow$) Let $y_1,\dots,y_s\in\mathfrak{I}$ be elements that are independent when projected in $\frac{\mathfrak{I}}{\mathfrak{I}\cap\mathcal{M}_n^2}$.
Then we may suppose $y_i=x_i+$ terms of degree ≥2, so that $\left|\frac{\partial y_i}{\partial x_j}\right|=\{1+\text{terms of degree}\ge1\}\notin\mathcal{M}_n$. But $\left|\frac{\partial y_i}{\partial x_j}\right|\in\Delta_s\mathfrak{I}$ and $\mathcal{M}_n$ is the only maximal ideal, this implies the thesis.

$\Rightarrow$) Let us suppose $s>\text{rank } \mathfrak{I}$ and let $y_1,\ldots,y_s \in \mathfrak{I}$. We have $y_i=\bar{y}_i+$terms of degree ≥ 2 with $\bar{y}_i$ of degree one. Then $\bar{y}_i$ are dependent and any sxs determinant $\left|\frac{\partial \bar{y}_i}{\partial x_j}\right|$ is zero. This implies that $\left|\frac{\partial y_i}{\partial x_j}\right| \in \mathcal{M}_n$, q.e.d.

The prop. 2.3 is equivalent to the fact that $\Delta_s \mathfrak{I}$ is proper if and only if $s>\text{rank } \mathfrak{I}$.

Definition 2.4 *$\delta\mathfrak{I}:=\Delta_{r+1}\mathfrak{I}$ where $r=\text{rank } \mathfrak{I}$. Hence, $\delta\mathfrak{I}$ is the largest Jacobian extension of $\mathfrak{I}$ which is a proper ideal.*

If $\mathfrak{I}_1 \subset \mathfrak{I}_2$ and rank $\mathfrak{I}_1=$rank $\mathfrak{I}_2$ (essential!) then $\delta\mathfrak{I}_1 \subset \delta\mathfrak{I}_2$.

Definition 2.5 $\beta\mathfrak{I}:=\mathfrak{I}+(\delta\mathfrak{I})^2+(\delta^2\mathfrak{I})^3+\ldots+(\delta^k\mathfrak{I})^{k+1}+\ldots$

Obviously rank $\mathfrak{I}=$rank $\beta\mathfrak{I}$.

2.6 Definition of the Boardman symbol *The Boardman symbol $I(\mathfrak{I})$ of a proper ideal $\mathfrak{I}$ is the sequence of integers $I=(i_1,\ldots,i_k,\ldots)$ where $i_1=\text{corank } \mathfrak{I}$, $i_2=\text{corank } \delta\mathfrak{I},\ldots$, $i_k=\text{corank } \delta^{k-1}\mathfrak{I}$.*

Remark that $\delta^k\mathfrak{I} \supset \delta^{k-1}\mathfrak{I}$, then $i_k \leq i_{k-1}$. Moreover if $\mathfrak{I}_1 \subset \mathfrak{I}_2$ and $I(\mathfrak{I}_1)=I(\mathfrak{I}_2)$ then $\beta(\mathfrak{I}_1) \subset \beta(\mathfrak{I}_2)$.

Definition 2.7 *For $0 \leq k \leq n$, $J_{k,n}$ is the ideal in A_n generated by $x_1,\ldots,x_{n-k}$. If $I=(i_1,i_2,\ldots)$ is a sequence of integers satisfying the condition $n \geq i_1 \geq i_2 \geq \ldots \geq 0$ we set*

$$J_{I,n}:=J_{i_1,n}+J^2_{i_2,n}+\ldots+J^k_{i_k,n}+\ldots$$

In other words $J_{I,n}$ is generated by:

all monomials in the first $n-i_1$ variables of degree one,

all monomials in the first $n-i_2$ variables of degree two,

$\vdots$

and so on.

Lemma 2.8 $I(J_{I,n})=I$

Proof First observe that $\text{corank}(J_{I,n})=\text{corank}(J_{i_1,n})=i_1$. Then we claim that $x_1,\ldots,x_{n-i_2} \in \delta J_{I,n}=\Delta_{n-i_1+1}J_{I,n}$. In fact x_k (for $n-i_1+1 \leq k \leq n-i_2$) is obtained expanding the jacobian of the elements $x_1,\ldots,x_{n-i_1},x_k^2 \in J_{I,n}$ with respect to $x_1,\ldots,x_{n-i_1},x_k$. On the other hand rank $\delta J_{I,n} \leq n-i_2$ (because no other variables appear in the monomials of degree two in $J_{I,n}$), it follows corank $\delta J_{I,n}=i_2$. The proof continues easily in this way.

Definition-example 2.9 *Let $S(i_1,i_2,i_3,\ldots)=(i_2,i_3,\ldots)$ be the <u>shift operator</u>.*

Note that $\delta^k J_{I,n}=J_{S^k I,n}$ (it is obvious from the proof of the lemma 2.8).

Proposition 2.10 *Let $\phi: A_p \to A_n$ be a surjective homomorphism of $\mathbb{C}$-algebras and let $\mathfrak{I}$ be a proper ideal in A_n. Then for every $t \geq 0$ $\Delta_{p-t}\phi^{-1}\mathfrak{I}=\phi^{-1}\Delta_{n-t}\mathfrak{I}$.*

Proof We may suppose $A_p=\mathbb{C}[\![y_1,\ldots,y_n,y_{n+1},\ldots,y_p]\!]$, $A_n=\mathbb{C}[\![x_1,\ldots,x_n]\!]$ and $\phi(y_i)=\begin{cases} x_i & 1 \leq i \leq n \\ 0 & i \geq n+1 \end{cases}$

Let $f_1,\ldots,f_a$ be a set of generators of $\mathfrak{I}$ and $\tilde{f}_i \in \phi^{-1}(f_i)$. Thus $\tilde{f}_1,\ldots,\tilde{f}_a,y_{n+1},\ldots,y_p$ is a set of generators of $\phi^{-1}\mathfrak{J}$. Now $\Delta_{p-t}\phi^{-1}\mathfrak{J}$ is generated by $\widetilde{f}_1,\ldots,\widetilde{f}_a,y_{n+1},\ldots,y_p$ and by all $(p-t)\times(p-t)$ minors of the jacobian of the above functions with respect to $y_1,\ldots,y_p$. It follows that $\Delta_{p-t}\phi^{-1}\mathfrak{J}$ is generated by $\widetilde{f}_1,\ldots,\widetilde{f}_a,y_{n+1},\ldots,y_p$ and by all $(n-t)\times(n-t)$ minors of the jacobian of $\widetilde{f}_1,\ldots,\widetilde{f}_a$ with respect to $y_1,\ldots,y_n$. $\Delta_{n-t}\mathfrak{J}$ is generated by $f_1,\ldots,f_a$ and all $(n-t)\times(n-t)$ determinants of the jacobian of $f_1,\ldots,f_a$ with

respect to $x_1,\ldots,x_n$. As $\phi(\frac{\partial\widetilde{f}_i}{\partial y_j})=\frac{\partial f_i}{\partial x_j}$ this concludes the proof.

Corollary 2.11 *Let $\phi:A_p \to A_n$ be a surjective homomorphism of* C*-algebras and let $\mathfrak{J}$ be a proper ideal in A_n. Then $\delta\phi^{-1}\mathfrak{J}=\phi^{-1}\delta\mathfrak{J}$ and $\beta\phi^{-1}\mathfrak{J}=\phi^{-1}\beta\mathfrak{J}$.*

Proof For δ it follows from the prop. 2.10 because $\text{rank}(\phi^{-1}\mathfrak{J}) = \text{rank}(\mathfrak{J})+p-n$. For β it is a consequence of the result for δ.

Proposition 2.12 $\delta\beta\mathfrak{J}\subset\beta\delta\mathfrak{J}$

Proof It suffices to consider the case when rank $\mathfrak{J}=0$. In fact if rank $\mathfrak{J}=r$ we have a surjective morphism $\phi:A_n \to A_{n-r}$ such that $\phi^{-1}\phi\mathfrak{J}=\mathfrak{J}$ and rank $\phi\mathfrak{J}=0$ (in order to define ϕ construct first $\overline{\phi}:\mathcal{M}_n/\mathcal{M}_n^2 \to \mathcal{M}_{n-r}/\mathcal{M}_{n-r}^2$ and then lift). Then by the corollary 2.11

$\delta\beta\mathfrak{J}=\delta\beta\phi^{-1}(\phi\mathfrak{J})=\phi^{-1}\delta\beta(\phi\mathfrak{J})\subset$ (by the rank zero case)

$\subset\phi^{-1}\beta\delta(\phi\mathfrak{J})=\beta\delta\phi^{-1}(\phi\mathfrak{J})=\beta\delta\mathfrak{J}$.

Thus we may suppose rank $\mathfrak{J}=0$, hence rank $\beta\mathfrak{J}=0$ and

$\delta\beta\mathfrak{J}=\Delta_1[\mathfrak{J}+(\delta\mathfrak{J})^2+(\delta^2\mathfrak{J})^3+\ldots] = \Delta_1\mathfrak{J}+\Delta_1(\delta\mathfrak{J})^2+\Delta_1(\delta^2\mathfrak{J})^3+\ldots\subset$ (by the Leibniz rule)

$\subset\Delta_1\mathfrak{J}+(\delta\mathfrak{J})(\Delta_1\delta\mathfrak{J})+(\delta^2\mathfrak{J})^2(\Delta_1\delta^2\mathfrak{J})+\ldots$

The last expression is contained in

(using $\Delta_1\delta^k\mathfrak{J}=\begin{cases} A_n & \text{for } k\geq k_0 \\ \delta^{k+1}\mathfrak{J} & \text{for } k<k_0\end{cases}$ and $\delta^k\mathfrak{J}\subset\delta^{k+1}\mathfrak{J}$)

$\delta\mathfrak{J}+(\delta^2\mathfrak{J})^2+(\delta^3\mathfrak{J})^3+\ldots=\beta\delta\mathfrak{J}$, q.e.d.

In the prop. 2.12 even the equality holds [8], but we do not need it.

Theorem 2.13 $I\beta(\mathfrak{J})=I(\mathfrak{J})$

Proof It suffices to prove the statement

(1_k) $$\text{rank } \delta^k\mathfrak{J}=\text{rank } \delta^k\beta\mathfrak{J}=\text{rank } \beta\delta^k\mathfrak{J}$$

for all $k\geq 1$. Consider also

(2_k) $$\delta^k\mathfrak{J}\subset\delta^k\beta\mathfrak{J}\subset\beta\delta^k\mathfrak{J}.$$

We will prove (1_k) and (2_k) together by induction on k. Formulas (2_1) and (1_1) are obvious or follow from the prop. 2.12. Moreover, it is clear that $(2_k)\Rightarrow(1_k)$, since rank $\delta^k\mathfrak{J}=$rank $\beta\delta^k\mathfrak{J}$. Finally, one shows $(1_k)+(2_k)\Rightarrow(2_{k+1})$, in fact

$\delta^{k+1}\mathfrak{J}=\delta(\delta^k\mathfrak{J})\subset\delta(\delta^k\beta\mathfrak{J})\subset\delta(\beta\delta^k\mathfrak{J})\subset$ (by the prop. 2.12)

$\subset\beta\delta^{k+1}\mathfrak{J}$, q.e.d.

Lemma 2.14 *Any ideal equivalent to $J_{I,n}$ is maximal among all ideals whose Boardman symbol is* I.

Proof It is sufficient to prove that $J_{I,n}$ is maximal among all ideals whose Boardman symbol is

I. Then suppose $J_{I,n} \subsetneq \mathfrak{I}$ and $I(\mathfrak{I})=I$. Then $J_{I,n}+\mathcal{M}_n^{k+1} \neq \mathfrak{I}+\mathcal{M}_n^{k+1}$ for some $k \geq 0$. Take the smallest such k. Thus there exists $f \in (\mathfrak{I} \cap \mathcal{M}_n^k) \setminus (J_{I,n}+\mathcal{M}_n^{k+1})$. Hence there is a monomial of degree k of the form $x_{j_1}x_{j_2}\cdots x_{j_k}$ with $j_1>n-i_1,\ldots j_k>n-i_k$, $j_1 \leq j_2 \leq \ldots \leq j_k$ whose coefficient in f is nonzero.

Let $D=\dfrac{\partial^{k-1}}{\partial x_{j_1}\ldots\partial x_{j_{k-1}}}$. Then the leading term of Df is a linear form with the coefficient of x_{j_k} different from zero.

Claim: $f \in (\mathfrak{I} \cap \mathcal{M}_n^k) \setminus (J_{I,n}+\mathcal{M}_n^{k+1}) \Rightarrow$
$Df \in \delta^{k-1}\mathfrak{I} = \Delta_{n-i_{k-1}+1} \circ \Delta_{n-i_{k-2}+1} \circ \ldots \circ \Delta_{n-i_1+1}\mathfrak{I}$

From this claim the lemma follows easily. In fact, as $x_1,\ldots,x_{n-i_k} \in \delta^{k-1}J_{I,n} \subset \delta^{k-1}\mathfrak{I}$ (the last inclusion because $I(\mathfrak{I})=I$) and $Df \in \delta^{k-1}\mathfrak{I}$, we have rank $\delta^{k-1}\mathfrak{I} > n-i_k$, a contradiction.

In order to prove the claim, consider first that $\dfrac{\partial f}{\partial x_{j_1}} \in \Delta_{n-i_1+1}\mathfrak{I}$, taking the jacobian of $x_1,\ldots x_{n-i_1}, f$ with respect to $x_1,\ldots x_{n-i_1}, x_{j_1}$. Then $\dfrac{\partial^2 f}{\partial x_{j_2}\partial x_{j_1}} \in \Delta_{n-i_2+1} \circ \Delta_{n-i_1+1}\mathfrak{I} = \delta^2\mathfrak{I}$ taking the jacobian of $x_1,\ldots x_{n-i_2}, \dfrac{\partial f}{\partial x_{j_1}}$ with respect to $x_1,\ldots x_{n-i_2}, x_{j_2}$. We can continue in this way, q.e.d.

Theorem 2.15 *Let $\mathfrak{I}$ be a proper ideal in A_n. $\beta\mathfrak{I}$ is equivalent to $J_{I,n}$ where $I=I(\mathfrak{I})$.*

Proof There exists $f \in Aut(A_n)$ such that $x_1,\ldots,x_{n-i_k} \in f(\delta^{k-1}\mathfrak{I}) = \delta^{k-1}(f\mathfrak{I})$ $\forall k \geq 1$. This implies that $J_{I,n} \subset f(\beta\mathfrak{I}) = \beta f(\mathfrak{I})$. By lemma 2.14 and theorem 2.13 the conclusion follows.

Definition 2.16 *Let $A_n(k)=A_n/\mathcal{M}_n^{k+1}$ be the truncated power series algebra (up to degree k).*

Remark 2.17 The operator $\dfrac{\partial}{\partial x_i}$ is not well defined in $A_n(k)$: in fact $\dfrac{\partial}{\partial x_1}(x_1^{k+1})=(k+1)x_1^k \notin (x_1,\ldots,x_n)^{k+1}$. This fact is important because in order to define Boardman symbols and Jacobian extensions in $A_n(k)$, it is necessary first to work in A_n and then to lift to $A_n(k)$.

Proposition 2.18 *If $\pi_k: A_n \to A_n(k)$ is the natural projection, and $\mathfrak{I}$ is a proper ideal in $A_n(k)$, the Boardman symbol of $\pi_k^{-1}(\mathfrak{I})$ is of the form $(i_1,\ldots,i_k,0,0,\ldots)$*

Proof We have $\mathcal{M}_n^{k+1} \subset \pi_k^{-1}(\mathfrak{I})$, then it is easy to check that $\mathcal{M}_n^k \subset \delta\pi_k^{-1}(\mathfrak{I})$ and so on $\mathcal{M}_n^{k+1-p} \subset \delta^p\pi_k^{-1}(\mathfrak{I})$. In particular $\delta^k\pi_k^{-1}(\mathfrak{I})=\mathcal{M}_n$ has corank zero.

Definitions 2.19 *Let $\mathfrak{I}$ be a proper ideal in $A_n(k)$. By prop. 2.18 $I(\pi_k^{-1}(\mathfrak{I}))=(i_1,\ldots,i_k,0,\ldots,0)$. We define the* <u>*Boardman symbol*</u> *of $\mathfrak{I}$ to be $(i_1,\ldots,i_k)$. There is a unique ideal $\mathfrak{I}_1$ in $A_n(k)$ such that $\beta\pi_k^{-1}\mathfrak{I}=\pi_k^{-1}\mathfrak{I}_1$ and we set*

$$\beta\mathfrak{I} := \mathfrak{I}_1.$$

In fact it suffices to take $\mathfrak{I}_1=\pi_k\beta\pi_k^{-1}\mathfrak{I}$ because $\mathcal{M}_n^{k+1} \subset \pi_k^{-1}\mathfrak{I} \subset \beta\pi_k^{-1}\mathfrak{I}$.

If $I=(i_1,\ldots,i_k)$ we set in $A_n(k)$

$$J_{I,n} := \pi_k J_{I',n} \qquad \text{where } I'=(i_1,\ldots,i_k,0,0,\ldots)$$

Theorem 2.15 holds also in this case, that is:

Theorem 2.20 *Let $\mathfrak{I}$ be a proper ideal in* $A_n(k)$. *$\beta\mathfrak{I}$ is equivalent to* $J_{I(\mathfrak{I}),n}$.

Proof By theorem 2.15 there exists an automorphism f of A_n such that $f(\beta\pi_k^{-1}\mathfrak{I})=J_{I',n}$. f factors to an automorphism f' of $A_n(k)$. Thus $J_{I,n}=\pi_k J_{I',n}=\pi_k f\beta\pi_k^{-1}\mathfrak{I}=f'\pi_k\beta\pi_k^{-1}\mathfrak{I}=f'\beta\mathfrak{I}$, q.e.d.

Lemma 2.21

$$\dim_C A_n(k)/J_{I,n} = \mu_I+1$$

(remember that $n\geq i_k$ and see the definition of μ_I in the introduction)

Proof By the definition of $J_{I,n}$ we get that $\dim_C A_n(k)/J_{I,n}$ is independent of n when $n\geq i_k$. There exists a basis of $J_{I,n}$ that consists of monomials in $x_1,\ldots,x_n$. A monomial of degree$\leq k$ does not belong to $J_{I,n}$ if and only if it is a product of k terms chosen from the following sets:

	sets ↓	
numbered from 0 to i_1 →	$\{1, x_n, x_{n-1},\ldots,x_{n-i_1+1}\}$	choose the 1st term here
numbered from 0 to i_2 →	$\{1, x_n, x_{n-1},\ldots,x_{n-i_2+1}\}$	choose the 2nd term here
⋮	⋮	⋮
numbered from 0 to i_k →	$\{1, x_n, x_{n-1},\ldots,x_{n-i_k+1}\}$	choose the k-th term here

Any choice corresponds to a k-ple $j_1,\ldots,j_k$ with $0\leq j_r\leq i_r$ (the r-th term is the one numbered with j_r). To avoid repetitions we have to suppose $j_1\geq j_2\geq\ldots\geq j_k$. If we suppose also $j_1>0$ we miss only the element $1=1\cdot 1\cdot\ldots\cdot 1$ (k times), q.e.d.

We want now to define the Thom-Boardman singularities in the jet space $J^k(X,Y)$. We will give first a local definition (2.22) on a fiber $J^k(n,m)$ and then we globalize it (2.29).

An element $z\in J^k(n,m)$ is represented by m local functions $(f_1,\ldots,f_m):U \to V$ with $U\subset C^n$, $V\subset C^m$. We may consider $f_i\in A_n$. Let $\mathfrak{I}(z)$ be the ideal in $A_n(k)$ generated by the images of $f_1,\ldots,f_m$ in $A_n(k)$. Obviously $\mathfrak{I}(z)$ does not depend on the representatives f_i.

2.22 Local definition of Σ^I *Let* $I=(i_1,\ldots,i_k)$ *with* $n\geq i_1\geq i_2\geq\ldots\geq i_k\geq 0$. *We set*

$$\Sigma^I:=\{z\in J^k(n,m)|\ I(\mathfrak{I}(z))=I\}$$

Remark 2.23 By the definition of the Boardman symbol, the closure of Σ^I is given by polynomial equations (obtained by expanding determinants of other determinants and so on).

Prop. 2.24

$$\Sigma^I\neq\emptyset \Leftrightarrow \begin{cases} i_1\geq n-m \\ \text{if } i_1=n-m \text{ then } i_1=i_2=\ldots=i_k \end{cases}$$

Proof

$\Rightarrow$) If $z\in\Sigma^I$ then rank $\mathfrak{I}(z)\leq m$, this implies that $i_1\geq n-m$. If $i_1=n-m$ we have rank $\mathfrak{I}(z)=m$, hence the representatives of z $f_1,\ldots,f_m$ generate $\mathcal{M}_m/\mathcal{M}_m^2$ and hence also $\mathcal{M}_m$. Thus $\mathfrak{I}(z)=\mathcal{M}_m$,

$\delta^j \mathfrak{I}(z)=\mathcal{M}_m\ \forall j$ so that rank $\delta^j\mathfrak{I}(z)=m$ as we wanted.

$\Leftarrow$) If $i_1=n-m$, $z=(x_1,\ldots,x_m)\in\Sigma^I$. If $i_1>n-m$, let z be the jet of $f:\mathbf{C}^n \to \mathbf{C}^m$ where

$$f_i=x_i \text{ for } 1\le i\le n-i_1$$
$$f_{n-i_1+1}=x^2_{n-i_1+1}+\ldots+x^2_{n-i_2}+x^3_{n-i_2+1}+\ldots+x^3_{n-i_3}+x^4_{n-i_3+1}+\ldots$$
$$f_i=0 \text{ for } n-i_1+2\le i\le m.$$

We get $z\in\Sigma^I$, q.e.d.

We recall that a subset A of a topological space is said to be locally closed if $\forall a\in A\ \exists$ a neighborhood $U\subset X$ of a such that $A\cap U$ is closed in U

Proposition 2.25 Σ^I *is locally closed (in the Zariski topology of* $J^k(n,m)=\mathbf{C}^N$*).*

Proof The proof is by induction on k and it is obvious for $k=1$ (from the fact that the set of matrices of fixed rank is locally closed in the space of all the matrices). In order to prove the assertion for $k+1$ consider a sequence of integers $(i_1,\ldots,i_k)$ satisfying the conditions of prop. 2.24. Call $p_k:J^{k+1}(n,m)\to J^k(n,m)$. Σ^I is locally closed by induction, hence $p_k^{-1}\Sigma^I=\bigcup_{0\le j\le i_k}\Sigma^{(I,j)}$ is locally closed in J^{k+1}. For every j_0 such that $0\le j_0\le i_k$, the set

$$\bigcup_{j_0\le j\le i_k}\Sigma^{(I,j)}$$

is Zariski closed in $p_k^{-1}\Sigma^I$ (it is given by the vanishing of some determinants for $j_0\neq 0$ and it is equal to $p_k^{-1}\Sigma^I$ for $j_0=0$). Hence, for $j_0=i_k$ we get that Σ^{I,i_k} is locally closed in J^{k+1}, and decreasing j_0 we get the result.

Definition 2.26 $L^k(n)\subset J^k(n,n)$ *is the subset of invertible jets. It is a group in a natural way .*

The group $L^k(n)$ acts on $J^k(n,m)$ in the following way

$$\text{if } z=j^kf\in J^k(n,m) \quad h\cdot z:=j^k(f\circ h^{-1})(hx) \quad \forall h\in L^k(n)$$

Obviously the $\Sigma^I\subset J^k(n,m)$ are invariant under this action.

Let G be the grassmannian of vector spaces in $A_n(k)$ of codim μ_I+1. From lemma 2.21 it follows $J_{I,n}\in G$. Let $U\subset G$ denote the set of ideals in $A_n(k)$ which are equivalent to $J_{I,n}$. $L^k(n)$ acts on $A_n(k)=J^k(n,1)$ and in fact it can be shown that it is isomorphic to its automorphism group. Hence $L^k(n)$ acts also on G. U is an orbit of this action and hence is a complex submanifold. If $z\in\Sigma^I$ we have $\beta\mathfrak{I}(z)\in U$ by theor. 2.20.

So we may define

$$B:\Sigma^I \longrightarrow U$$
$$z\mapsto \beta\mathfrak{I}(z)$$

It is easily verified that B is regular rational and that it commutes with the actions of $L^k(n)$ on U and on Σ^I.

Lemma 2.27 $V=B^{-1}(J_{I,n})$ *is smooth*

Proof Set $W=\{z\in J^k(n,m)|\mathfrak{I}(z)\subset J_{I,n}\}$. W is a vector space, and contains elements of $\Sigma^{I'}$ with I'

bigger or equal than I. If $z \in W \cap \Sigma^I$ then by theor. 2.20 $\beta\mathfrak{J}(z)$ is equivalent to $J_{I,n} = \beta J_{I,n}$(the last equality follows by a direct computation). Now $\mathfrak{J}(z) \subset J_{I,n}$ and they have the same Boardman symbol, hence $\beta\mathfrak{J}(z) \subset \beta J_{I,n}$. Thus, by lemma 2.14 $\beta\mathfrak{J}(z) = J_{I,n}$. Hence $W \cap \Sigma^I = V$ is a Zariski open subset of the vector space W and it is smooth.

Theorem 2.28 $\Sigma^I \subset J^k(n,m)$ *is smooth, hence it is a submanifold, and its codimension is given by* ν_I *(see def. of* ν_I *in the introduction).*

Proof Consider the map $\alpha : L^k(n) \times V \to J^k(n,m)$ induced by the action of $L^k(n)$ on $J^k(n,m)$. By the definition of V, Im $\alpha \subset \Sigma^I$. As B commutes with the action of $L^k(n)$ and U is homogeneous it follows Im $\alpha = \Sigma^I$. So it suffices to prove that α is of constant rank and it is open.

1st claim: $\text{rank}_{h,z}\alpha = \dim U + \dim V \quad \forall h \in L^k(n),\ z \in V$

It is easy to check that in order to prove the claim it is enough to consider the case when $h=1$. Let $\alpha_z : L^k(n) \to J^k(n,m)$ be defined by $\alpha_z(h) = \alpha(h,z) = h \cdot z$ and let $A_z : L^k(n) \to U$ be defined by $A_z(h) = h \cdot Bz = h \cdot J_{I,n}\ \forall z \in V$. For all $\vec{h} \in TL^k(n)_1$ let h(t) be the corresponding path (for small t) in $L^k(n)$ such that $h'(0) = \vec{h}$. Then $\vec{h} \in \text{Ker } d(A_z)_1$ if and only if $h(t) \cdot J_{I,n} = J_{I,n}$ for small t. We have $d(\alpha_z)_1(\vec{h}) = (h(t) \cdot z)'|_{t=0}$; it follows that $d(\alpha_z)_1(\vec{h}) \in TV_z$ if and only if there exists a corresponding path h such that $h(t) \cdot z \in V$ for small t, that means $\beta\mathfrak{J}(h(t) \cdot z) = J_{I,n}$. We have $\beta\mathfrak{J}(h(t) \cdot z) = h(t) \cdot \beta\mathfrak{J}(z) = h(t) \cdot J_{I,n}$, hence $d(\alpha_z)_1(\vec{h}) \in TV_z$ if and only if $\vec{h} \in \text{Ker } d(A_z)_1$. As $d\alpha_{1,z}(\vec{h},\vec{v}) = d(\alpha_z)_1(\vec{h}) + \vec{v}$ we get $\text{Ker}(d\alpha_{1,z}) = \{(\vec{h},\vec{v}) \mid \vec{h} \in \text{Ker } d(A_z)_1,\ \vec{v} = -d(\alpha_z)_1(\vec{h})\}$. Thus

$$\text{rank}_{1,z}\alpha = \dim TL^k(n) + \dim TV_z - \dim \text{Ker } d\alpha_{1,z} = \dim TL^k(n) - \dim \text{Ker } d(A_z)_1 + \dim TV_z =$$

(because A_z is surjective of maximum rank)

$$= \dim U + \dim V.$$

2nd claim: α is open in the metric topology

Consider $z \in V$. It is sufficient to prove that the image of an open ball centered at (1,z) is open. This is equivalent to the statement that $\forall \epsilon > 0\ \exists \delta > 0$ such that if $w \in \Sigma^I$, $|w-z| < \delta$ then $\exists h \in L^k(n)$ satisfying $|h-1| < \epsilon$, $|h^{-1}w - z| < \epsilon$. In fact as α is continuous there exists δ_1 such that if $|w-z| < \delta_1$ and $|h^{-1}-1| < \delta_1$ then $|h^{-1}w - z| < \epsilon$. There exists also δ_2 such that if $|h^{-1}-1| < \delta_2$ then $|h-1| < \epsilon$. Choose $\delta = \min\{\delta_1, \delta_2\}$, q.e.d.

These two claims prove that Σ^I is nonsingular. For the assertion on the codimension we refer to [8] pag. 244-245.

From now on let us suppose that X,Y are complex manifolds with $n = \dim X \le \dim Y = m$.

We recall from 1.b that $J^k(X,Y)$ is a fiber bundle over $X \times Y$ with fibers $J^k(n,m)$. The local coordinates of $j^k f(x) \in J^k(X,Y)$. are $\dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ in a suitable neighborhood.

2.29 Global definition of $\Sigma^I \subset J^k(X,Y)$ (see 2.22, the context should avoid any confusion) $\Sigma^I \subset J^k(n,m)$ *is given locally by equations*

$$G_i\left(\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}\right) = 0 \quad with\ |\alpha| \ge 1$$

where G_i *are suitable polynomials.*

These equations define in a natural way a smooth submanifold $\Sigma^I \subset J^k(X,Y)$ *of codimension* ν_I *which is a fiber bundle on* $X \times Y$. *If* X,Y *are algebraic also* Σ^I *is algebraic.*

2.30 Def. of $\Sigma^I(f)$ *If* $f:X \to Y$ *is holomorphic, we have* $j^k f: X \to J^k(X,Y)$ *and we set* $\Sigma^I(f) := (j^k f)^{-1} \Sigma^I$

The following theorem is the main property of the Thom-Boardman singularities, and justifies its complicated definition.

Theorem 2.31 *Let* $f:X \to Y$ *and* $j^{k-1}f(x) \in \Sigma^I$ (here $I=(i_1,\ldots,i_{k-1})$). *Suppose moreover that* $\Sigma^I(f)$ *is smooth. Then*

$$j^k f(x) \in \Sigma^{I,j} \Leftrightarrow j = \dim[\operatorname{Ker} d(f|_{\Sigma^I(f)})_x]$$

Proof It suffices to prove the "$\Leftarrow$" part because the submanifolds $\Sigma^{I,j}$ form a partition of Σ^I. The statement is local, so let us denote by $(f_1,\ldots,f_m)$ an expression of f in local coord. We first prove the easy case $k=1$. If $\dim[\operatorname{Ker}(df)_x]=j$ this implies that all the $(n-j)\times(n-j)$ minors of the jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ vanish at x (and not lower order determinants), thus the the corank of the ideal generated by the germs at x of $f_1,\ldots,f_m$ is equal to j, hence $j^1 f(x) \in \Sigma^j$, as we wanted. It follows also that:

(T is the holomorphic tangent space,
$<w_i>$ is the vector space generated by the w_i's,
∇g is the complex gradient $(\frac{\partial g}{\partial z_1},\ldots,\frac{\partial g}{\partial z_n})$)

$$T\Sigma^i(f)_x = \{v \mid v\cdot\nabla\left|\frac{\partial f_i}{\partial x_j}\right|(x)=0 \text{ for all determinants of order } n-i\} = \{<\nabla\left|\frac{\partial f_i}{\partial x_j}\right|(x)>\}^{\perp}$$

In the case $k=2$ if $j=\dim[\operatorname{Ker} d(f|_{\Sigma^i(f)})_x]$, set $Z=\Sigma^i(f)$ and we have

$$\dim[\operatorname{Ker}(df)_x \cap TZ]=j \text{ (in particular } j \leq i)$$

We have $\operatorname{Ker}(df)_x = \{v \mid \frac{\partial f_i}{\partial x_j} v_j = 0\} = \{<\frac{\partial f_i}{\partial x_j}>\}^{\perp}$, thus (all determinants are of order $n-i$)

$$j=\dim[\{<\frac{\partial f_i}{\partial x_j}>\}^{\perp} \cap \{<\nabla\left|\frac{\partial f_i}{\partial x_j}\right|>\}^{\perp}] = \dim[<\frac{\partial f_i}{\partial x_j}+\nabla\left|\frac{\partial f_i}{\partial x_j}\right|>]^{\perp}$$

hence:

$$\dim[<\frac{\partial f_i}{\partial x_j}+\nabla\left|\frac{\partial f_i}{\partial x_j}\right|>]=n-j.$$

It follows that the rank of the ideal generated by the germs of $f_1,\ldots,f_m,\left|\frac{\partial f_i}{\partial x_j}\right|$ is $n-j$ and $j^2 f(x) \in \Sigma^{i,j}$.

The reader can easily convince himself that the proof proceeds as well for higher values of k.

3. Proof of theorems 1 and 2

Consider as in the hypothesis of theorem 2 a submanifold $X=X\subset \mathbf{P}^n$ of codimension d. Let T be any linear subspace of $\mathbf{P}^n$ such that $T\cap X=\emptyset$ and $\dim T=t$ and let $L\subset \mathbf{P}^n$ be a fixed linear subspace with dim $L=n-t-1$. Define $f_T:X \to L$ to be the linear projection with center in T, so we have $j^kf_T:X \to J^k(X,L)$. The following proposition should be clear after the theorem 2.31, we state it explicitly because it represents the link between the Thom-Boardman singularities and our original problem.

Proposition 3.1 *Let* $I=(i_1,\dots,i_k)$ *and define (when possible)* X_I *as in the statement of theorem 2. Let* $\Sigma^I \subset J^k(X,L)$. *If* $(j^kf_T)^{-1}\Sigma^I=\Sigma^I(f_T)$ *are smooth for all* I, *then* X_I *are well defined and are equal to* $\Sigma^I(f_T)$.

Proof By induction on k. For $k=1$ there is nothing to prove. For $k\geq 2$:

$$X_I = \{x\in X_{i_1,\dots,i_{k-1}} | \dim[T_xX_{i_1,\dots,i_{k-1}} \cap T]=i_k-1\} =$$

$$= \{x\in X_{i_1,\dots,i_{k-1}} | \dim [\mathrm{Ker}\ d(f_T|_{X_{i_1,\dots,i_{k-1}}})]=i_k\} = \qquad \text{(by the inductive hypothesis)}$$

$$= \{x\in X_{i_1,\dots,i_{k-1}} | \dim[\mathrm{Ker}\ d(f_T|_{\Sigma^{i_1\dots i_{k-1}}(f_T)})]=i_k\} = \qquad \text{(by theorem 2.31)}$$

$$= \{x\in X_{i_1,\dots,i_{k-1}} | j^kf_T(x)\in\Sigma^I\} = (f_T)^{-1}\Sigma^I \quad \text{q.e.d.}$$

Proposition 3.1 implies that in order to prove theorem 2 it suffices to prove that for generic T j^kf_T is transversal to Σ^I (see theorem 1.2). Mather's technique is the following:

step 1) let $W\subset J^k(X,L)$. j^kf_T is transversal to W for generic T if W satisfies a technical additional assumption, precisely that W is *modular* (see 3.9)

step 2) show that Σ^I are modular.

In the sequel we'll do first step 2 (theor. 3.10, proved for the first time in [8]) and then step 1 (theor. 3.11, which is theorem 6 of [9]).

Definition 3.2 *Consider the natural projections* $J^k(X,Y) \xrightarrow{\alpha} X\times Y \xrightarrow{\beta} X$ *and let* $p\in X$, $q\in Y$. *Then* $J^k(X,Y)_{p,q}:=\alpha^{-1}(p,q)$ *is the fiber over* (p,q) *and* $J^k(X,Y)_p:=(\beta\circ\alpha)^{-1}(p)$ *is the fiber over* p. *In the same way if* $W\subset J^k(X,Y)$ *is a submanifold, define* W_p *and* $W_{p,q}$.

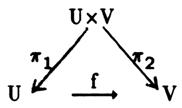
Proposition 3.3 *Let* $f:X \to Y$ *be an holomorphic map.* $\forall p\in X$ *there is a natural isomorphism of* $\mathbb{C}$*-vector spaces (intrinsically defined in the proof):*

$$\lambda:J^k(f^*TY)_p \to T[J^k(X,Y)_p]_{j^kf(p)}.$$

Let $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_m)$ *be local coordinates on* X *and* Y *centered in* p *and* f(p). *If* $\sum g_i(x)f^*(\frac{\partial}{\partial y_i})$ *represents an element* $\sigma\in J^k(f^*TY)_p$ *in a neighborhood of* p, *then* $\lambda(p)$ *is the vector with coordinates*

$\dfrac{\partial^{|\alpha|}g_i(p)}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$ $(|\alpha|\leq k)$ *in the basis of* $T[J^k(X,Y)_p]_{j^kf(p)}$ *corresponding to the chosen coordinates.*

Proof Let $\sigma\in J^k(f^*TY)_p$. In order to define $\lambda(\sigma)$ we first construct a deformation F of f along σ. Let $p\in U\subset X$ be a neighborhood where a section σ' of f^*TY which represents σ is defined and let $V:=f(U)$. We may consider $\sigma':U \to TY$ and the local section $\pi_2^*\sigma'$ of $\pi_2^*TY\subset T(X\times Y)$ as in the following diagram



Now by theorem 1.3 we find (choosing possibly smaller neighb. U,V)

$$H:(-\epsilon,+\epsilon)\times(U\times V)\to X\times Y$$

such that

$$H(0;x,y)=(x,y)$$
$$\frac{\partial H}{\partial t}(t;x,y)=(\pi_2^*\sigma')(H(t;x,y)).$$

Thus in particular

$$\pi_2[H_0(x,f(x))]=\pi_2(x,f(x))=f(x)$$
$$\frac{d}{dt}\pi_2[H_t(x,f(x))]|_{t=0}=\pi_{2*}[(\pi_2^*\sigma')(x,f(x))]=\sigma'(x)$$

hence the function $F_t(x):=\pi_2[H_t(x,f(x))]$ satisfies

$$F_0(x)=f(x)$$
$$\frac{d}{dt}j^kF_t(x)|_{t=0}=\sigma'(x)$$

and it is a deformation of f along σ.

Now consider the path $t\to j^kF_t(p)$ in $J^k(X,Y)_p$ ($F_t(x)$ is holomorphic in x!) and define $\lambda(\sigma):=\frac{d}{dt}j^kF_t(p)|_{t=0}$. The following computation in local coordinates shows that λ does not depend on the representative σ' chosen.

Let $\sigma'=\sum g_i(x)f^*(\frac{\partial}{\partial y_i})$, hence $F_t=(F_t^1,\dots F_t^m)$ with $F_t^i=f_i+tg_i+O(t^2)$. Thus (identifying a jet with the corresponding Taylor expansion)

$$j^kF_t^i(0)=\sum_{|\alpha|\le k}\frac{x^\alpha}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}F_t^i(0)=$$

$$=\sum_{|\alpha|\le k}\frac{x^\alpha}{\alpha!}\Big(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}f_i(0)+t\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}g_i(0)+O(t^2)\Big)$$

and the coordinates of $\frac{d}{dt}[j^kF_t^i(0)]|_{t=0}$ are $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}g_i(0)$ as we wanted. Hence $\lambda(\sigma)$ is completely

determined by $j^kg(0)$, that is the k-jet of σ at p. The local expression of λ shows trivially that it is injective. Hence λ is an isomorphism because $\dim J^k(f^*TY)_p=\dim T[J^k(X,Y)_p]_{j^kf(p)}$, q.e.d.

Corollary 3.4

$$\lambda[\mathcal{M}_xJ^k(f^*TY)_p]=T[J^k(X,Y)_{p,f(p)}]_{j^kf(p)}$$

Proof The only change in the proof above is that we have to consider functions g such that $g_i(0)=0$. Hence $\frac{d}{dt}[j^kF_t^i(0)]|_{t=0}\in T[J^k(X,Y)_{p,f(p)}]_{j^kf(p)}$, q.e.d.

3.5 Definitions of tf, ωf, I_t and I_ω tf:$J^k(TX)\to J^k(p_f{}^*TL)$ *is the* k-jet *map associated to* $d'p_f:TX\to p_f{}^*TL$. *We set* $I_t:=\mathrm{Im}$ tf.

ωf:$J^k(TL)\to J^k(p_f{}^*TL)=p_f{}^*J^k(TL)$ *is the usual pullback. We set* $I_\omega:=\mathrm{Im}\ \omega$f.

Proposition 3.6 *Let* $\Sigma^I \subset J^k(X,Y)_{x,y}$ *and let* $z=j^kf(x)\in\Sigma^I$ *(hence* $y=f(x)$*). With the identifications of* 3.3 *and* 3.4 *we have:*

$$(T\Sigma^I)_z = tf(\mathcal{M}_x J^k TX_x) + \beta(\mathfrak{I}(z))\cdot J^k(f^*TY)_x$$

(*in a suitable coordinate system* $\beta(\mathfrak{I}(z))=J_{I,n}$).

Proof As $z\in\Sigma^I$, $\beta\mathfrak{I}(z)$ is equivalent to $J_{I,n}$ by 2.20. hence, by choosing a suitable coordinate system, we may suppose $\beta\mathfrak{I}(z)=J_{I,n}$, i.e. $z\in V$ (see the proof of 2.28). By the same proof we have the surjective mapping

$$\alpha\colon L^k(n)\times V \to \Sigma^I$$

induced by the action of $L^k(n)$ on $\Sigma^I\supset V$. We have $\alpha(1,z)=z$, hence $(T\Sigma^I)_z=\mathrm{Im}(d\alpha)(1,z)=\mathrm{Im}\,d\alpha_z[T_1L^k(n)]+TV_z$. Now the result follows from the following two claims:

$$\mathrm{Im}\ d\alpha_z[T_1L^k(n)] = tf(\mathcal{M}_x J^k TX_x) \qquad \text{(C1)}$$

$$TV_z = J_{I,n}\cdot J^k(f^*TY)_x \qquad \text{(C2)}$$

For the first one consider that an element belonging to $tf(\mathcal{M}_x J^k TX_x)$ has a local expression $\sum J^k[\sum a_i(x)\frac{\partial f_j}{\partial x_i}]f^*(\frac{\partial}{\partial y_j})$ with $a_i(0)=0$, (i.e. $a(x)\in J^k(n,n)$), so it corresponds to the k-jet of the analytic map $g:X \to Y$ such that $g_j=\sum a_i(x)\frac{\partial f_j}{\partial x_i}$. Now remark that for any vector $a(x)\in T[L^k(n)]_1\simeq J^k(n,n)$ there exists a map $h_t:(-\epsilon,+\epsilon)\to L^k(n)$ such that

$$\begin{cases} h_0=1_{L^k(n)} \\ \frac{dh_t}{dt}|_{t=0}=a(x) \end{cases}$$

The image of this vector $a(x)$ under the linear map $d\alpha_z$ is exactly

$$\frac{d}{dt}J^k[f(x+a(x)t+O(t^2))]|_{t=0} = \sum\frac{\partial^{|\alpha|}}{\partial x_\alpha}[\frac{d}{dt}f(x+a(x)t+O(t^2))|_{t=0}]\frac{x^\alpha}{\alpha!} = \sum\frac{\partial^{|\alpha|}}{\partial x_\alpha}[\frac{\partial f}{\partial x_i}a_i(x)]\frac{x^\alpha}{\alpha!}$$

(with $|\alpha|\le k$) which is the k-jet of the analytic map g. In this way we have defined a natural bijective correspondence between $tf(\mathcal{M}_x J^k TX_x)$ and $\mathrm{Im}\ d\alpha_z[T_1L^k(n)]$.

For (C2) consider that as in the proof of lemma 2.27 V is an open set of the vector space $W=\{z\in J^k(n,m)|\mathfrak{I}(z)\subset J_{I,n}\}$ and so TV_z can be identified with the vector space W. Consider that elements in $J_{I,n}\cdot J^k(f^*TY)_x$ have a local expression $\sum g_i(x)f^*(\frac{\partial}{\partial y_i})$ where $g_i(x)\in J_{I,n}$ (in particular $g_i(0)=0$). Thus in terms of the identifications above the germ of $(g_1,\ldots,g_m)=z$ satisfies $\mathfrak{I}(z)\subset J_{I,n}$ and conversely such germs give elements of $J_{I,n}\cdot J^k(f^*TY)_x$. This concludes the proof.

3.7 Definition of invariant submanifold *A submanifold* $W\subset J^k(X,Y)\xrightarrow{\pi} X\times Y$ *is called* *invariant if for all* $(x,y)\in X\times Y$*, for all local analytic neighborhoods* $U\subset X$, $V\subset Y$, *for every analytic open embeddings* $h:U\to X$, $w:V\to Y$ *and for all* $z=j^kf(x)\in\pi^{-1}(U\times V)\cap W_{x,f(x)}$ (*see* 3.2) *we have that* $z'=j^k(w\circ f\circ h^{-1})(h(x))\in W_{h(x),w(f(x))}$.

This definition in the C^∞ case can be expressed globally by asking that W is invariant under the actions of Diff(X) and Diff(Y) ([7],V).

Theorem 3.8 *Let* $W\subset J^k(X,Y)$ *be an invariant submanifold. Then at* $z=j^kf(x)\in W$ *we have (under the identifications of prop.* 3.3 *and corollary* 3.4*):*

$$T(W_x)_z=T(W_{x,f(x)})_z+(I_\omega)_x$$
$$\pi_f T(W)_z=T(W_x)_z+(I_t)_x$$

where $\pi_f:T[J^k(X,Y)]_z \to T[J^k(X,Y)_x]_z$ *is the natural projection, see* [M1]V, *pag.* 307.

Proof Let O be the orbit of z in the sense that $O=\{z'\in J^k(X,Y)|\exists\ x\in U,\ f(x)\in V$ analytic neighborhoods and $h:U\to X$, $w:V\to Y$ analytic open embeddings as in 3.7 such that $z'=j^k(w\circ f\circ h^{-1})\}$. By the hypothesis it is easy to check that W_x and $W_{x'}$ are isomorphic for all x,x';in fact for all x,x' there are two biholomorphic neighborhoods of x and x'. In the same way we see that $W_{x,y}$ and $W_{x,y'}$ are isomorphic for all x,y,y'. Hence, looking at the fibers of $W\to X$ and $W_x\overset{\gamma}{\to}Y$ we get that W_x and $W_{x,y}$ are submanifolds for all x,y. W is invariant, thus we have $O\subset W$, hence it follows that $(TO_x)_z+T(W_{x,f(x)})_z\subset T(W_x)_z$. As $\gamma(O_x)=Y$, and the projection $W_x\to Y$ is surjective and of maximum rank, we obtain that $T(W_x)_z\subset(TO_x)_z+T(W_{x,f(x)})_z$. But it is easy to check (see [7]V, pag. 307) that under the identifications of prop. 3.3 and corollary 3.4 we have $(TO_x)_z=\mathcal{M}_x(I_t)_x+(I_\omega)_x$ and $T(O_{x,f(x)})_z=\mathcal{M}_x(I_t)_x+\mathcal{M}_{f(x)}(I_\omega)_x\subset T(W_{x,f(x)})_z$. From these identifications we get $T(W_x)_z=T(W_{x,f(x)})_z+(I_\omega)_x$. In order to prove the other equality, note that as above we get $TW_z=(TW_x)_z+TO_z$ and moreover $\pi_f(TW_x)_z=(TW_x)_z$ (obvious) and $\pi_f TO_z=(I_t)_x+(I_\omega)_x$ ([7]V, pag. 308), so that $\pi_f T(W)_z=T(W_x)_z+(I_t)_x$. q.e.d.

In particular if W is invariant then $W_{x,y}$ is invariant under the action of $L^k(n)$ (see 2.26) and of $L^k(m)$. So $T(W_{x,y})$ contains the tangent space to the orbit of the jet z. $L^k(n)$ acts also in a natural way on $J^k(X,Y)_x$, $J^k(X,Y)_{x,y}$, $T[J^k(X,Y)_x]_z$ and $T[J^k(X,Y)_{x,y}]_z$ $\forall(x,y)\in X\times Y$ and $z\in J^k(X,Y)$. By prop. 3.3 and corollary 3.4 $J^k(\mathcal{O}_X)_x=J^k(n,1)$ acts on $T[J^k(X,Y)_x]_z$ and $T[J^k(X,Y)_{x,y}]_z$ as the natural multiplication map.

3.9 Definition of modular submanifold *Let* $W\subset J^k(X,Y)$ *be a smooth submanifold and* $W_{x,y}\subset J^k(X,Y)_{x,y}$ *be the fiber over* $(x,y)\in X\times Y$. W *is said to be* <u>modular</u> *if the following are true* $\forall(x,y)\in X\times Y$:

i) W *is invariant as in* 3.7.

ii) $T(W_{x,y})_z$ *is invariant under the action of* $J^k(\mathcal{O}_X)_x$.

This condition is technical and Mather himself points out that it is just what is needed for the proof. Condition ii) means that $T(W_{x,y})_z$ has to be not only a vector subspace of $T[J^k(X,Y)_{x,y}]_z\simeq\mathcal{M}_x J^k[f^*TY]_x$ ($y=f(x)$ and $z=j^kf_x$ for some local f) but also a $J^k(\mathcal{O})_x$-*submodule.* For example the subspace $\{\sum a_i(f(x))f^*\frac{\partial}{\partial y_i}|a_i(0)=0\}\subset\mathcal{M}_x J^k(f^*TY)_x$ which is the image of $\mathcal{M}_y J^k(TY)_x$ under the natural map $J^k(TY)_x \to J^k(f^*TY)_x$ is not a submodule, although it is invariant under the action of $L^k(n)$ and $L^k(m)$.

Theorem 3.10 *The Thom-Boardman singularities* $\Sigma^I \subset J^k(X,Y)$ *are modular*

Proof i) is obvious by the definition of the Boardman symbol. ii) follows from the formula of the prop. 3.6 because $\mathcal{M}_x$ and $\beta(\mathfrak{I}(z))$ are both ideals in $J^k(\mathcal{O}_X)_x$.

Now we return to our situation. Let $F=\{P^t \in Gr(P^t,P^n) \mid P^t$ does not meet X and L$\}$ and consider $\forall f \in F$ the morphism $p_f:X \to L$.

Theorem 3.11 *Let* W *be a modular algebraic nonsingular subvariety of* $J^k(X,L)$ *(not necessarily closed). Then there exists a Zariski open subset* U *of* F *such that* $j^k p_f:X \to J^k(X,L)$ *is transversal to* W *for* $f \in U$.

We postpone the proof of theorem 3.11 to the end of this section (and of the paper).

Remark 3.12 Covering X by a finite number of open affine subsets, we see that in order to prove the theorem 3.11 we may suppose that $X \subset C^n$ is affine.

Proposition 3.13 *In the hypothesis of prop.* 3.11, *let* $g:X \times F \to J^k(X,L)$ *given by* $g(x,f)=j^k(p_f)_x$ *and let* $W \subset J^k(X,L)$ *a nonsingular subvariety. If* g *is transversal to* W *then there exists a Zariski open subset* U *of* F *such that* $j^k p_f:X \to J^k(X,L)$ *is transversal to* W *for* $f \in U$.

Proof By theorem 1.2 and the hypothesis, $g^{-1}(W)$ is empty or smooth of the expected dimension. We have a natural projection $g^{-1}(W) \to F$ whose fiber over $f \in F$ is isomorphic to $(j^k p_f)^{-1}(W)$. By generic smoothness ([6] III.10.7) there exists $U \subset F$ such that $(j^k p_f)^{-1}(W)$ is smooth of the expected dimension for $f \in U$. Theorem 1.2 implies the result.

Definition 3.14 *Let* $f:X \to Y$ *be a holomorphic map,* $W \subset Y$ *be a submanifold.*

$$\delta(f,W,x):=\begin{cases} 0 & \text{if } f(x) \notin W \\ \dim Y - \dim[TW_{f(x)} + df(TX)_x] & \text{if } f(x) \in W \end{cases}$$

Thus $\delta \geq 0$ and f is transverse to W at x if and only if $\delta(f,W,x)=0$.

Theorem 3.15 *In the notations of the prop.* 3.13, *suppose that* $\forall (x,f) \in X \times F$ *either* $\delta(j^k p_f,W,x)=0$ *or* $\delta(g,W,(x,f)) < \delta(j^k p_f,W,x)$. *Then there exists a Zariski open dense subset* U *of* F *such that* $j^k p_f:X \to J^k(X,L)$ *is transversal to* W *for* $f \in U$.

Proof Let $\Sigma' := \{(x,f) \mid j^k p_f$ is not tranverse to W at x$\} = \{(x,f) \mid \delta(j^k p_f,x,W) \geq 1\} \subset X \times F$. Σ' is a Zariski closed subset of $X \times F$. Let $\pi_1:X \times F \to X$, $\pi_2:X \times F \to F$ be the projections. $\pi_2(\Sigma')$ is closed because π_2 is a proper map. Note that $a \notin \pi_2(\Sigma')$ if and only if $j^k p_f$ is transverse to W, hence the subset $\{f \in F \mid j^k p_f$ is transversal to W$\}$ is a Zariski open subset of F, and it will be sufficient to prove that it is not empty. Let $\delta_g := \sup\{\delta(g,W,(x,f)) \mid (x,f) \in X \times F\}$ and $\Sigma := \{(x,f) \in X \times F \mid \delta(g,W,(x,f)) = \delta_g\}$. Σ and $\pi_2(\Sigma)$ are Zariski closed subsets.

Claim: If $\delta_g > 0$ then $\pi_2(\Sigma)$ is a proper subset of F.

We proceed by contradiction and we suppose that $\pi_2(\Sigma)=F$. Let $(x_0,f_0)\in\Sigma$ and $z_0=j^k g_{(x_0,f_0)}\in W$. We may suppose that $j^k p_{f_0}$ is not transversal to W at x_0. W is nonsingular, hence it is locally a complete intersection ([6] II.8.22.1). In an affine subscheme containing z_0 we have that $W\subset J^k(X,L)$ is given by $\{h_1=\ldots=h_r=0\}$. There exists $\{h_1',\ldots,h_r'\}$ linear combinations of $\{h_1,\ldots,h_r\}$ such that $W=\{h_1'=\ldots=h_r'=0\}$ and $W':=\{h_1'=\ldots=h_{r-\delta_g}'=0\}$ satisfy

$$\dim[dgT(X\times F)_{(x_0,f_0)}+TW'_{z_0}]=\dim[dgT(X\times F)_{(x_0,f_0)}+TW_{z_0}]+\delta_g=\dim J^k(X,L)_{z_0} \quad (*)$$

(this choice is possible by elementary linear algebra because $\delta_g\leq\dim J^k(X,L)-\dim TW=\operatorname{codim} W=r$). Now $\delta(g,W',(x_0,f_0))=0$ by (*), so that g is transversal to W' at (x_0,f_0). Hence there exists Z an open dense subset of $X\times F$ such that if $(x,f)\in Z$ we have that g is transversal to W' at (x,f). We may suppose that Z and W' are both smooth and that $W'\supset W$, possibly replacing them by smaller subsets. The map $g|_Z:Z\to J^k(X,L)$ is transversal to W', hence $(g|_Z)^{-1}(W')=g^{-1}(W')\cap Z$ is smooth. Now consider the projection $g^{-1}(W')\cap Z\xrightarrow{\pi}F$. As $\pi_2(\Sigma)=F$ and $\Sigma\cap Z$ is a Zariski open dense subset of Σ it is easy to check that π is dominant. Then by generic smoothness there exists a dense open subset $U\subset F$ such that if $f\in U$ then $\pi^{-1}(f)=j^k p_f^{-1}(W')\cap Z$ is smooth. $\pi_2(\Sigma\cap Z)$ is dense, hence $\pi_2(\Sigma\cap Z)\cap U\neq\emptyset$. If $f_1\in\pi_2(\Sigma\cap Z)\cap U$ there exists $x_1\in X$ such that

a) $x_1\in j^k p_{f_1}(W')\cap Z,\ f_1\in U$

b) $(x_1,f_1)\in\Sigma$.

a) and 2.1 imply that $j^k p_{f_1}$ is transversal to W' at x_1. b) implies that $\delta(g,W,(x_1,f_1)=\delta_g$.

Thus:

$$0=\delta(j^k p_{f_1},W',x_1)\geq\delta(j^k p_{f_1},W,x_1)-(\dim W'-\dim W)=\delta(j^k p_{f_1},W,x_1)-\delta_g>$$

(by the hypothesis)

$>\delta(g,W,(x_1,f_1))-\delta_g=0$, which is a contradiction. This proves the claim.

Now we can proceed by induction on δ_g. If $\delta_g=0$ the result follows by prop. 3.13. Then, we may suppose $\delta_g>0$. Let $g'=g|_{X\times(F\setminus\pi_2\Sigma)}$. We have $\delta_{g'}<\delta_g$ by definition of Σ and we may use the inductive hypothesis. This concludes the proof.

In order to prove theorem 2 we will verify the hypothesis of theorem 3.15.

Let now $x\in X$, $f\in F$, $y=p_f(x)$ and $z=j^k p_f(x)$. In the following lemmas we use always the identifications of prop. 3.3 and corollary 3.4.

Lemma 3.16 $\delta(j^k p_f,W,x)=\dim\dfrac{J^k(p_f{}^*TL)_x}{(I_t)_x+(I_\omega)_x+T(W_{x,y})_z}$

Proof We have $\delta(j^k p_f,W,x)=\dim\dfrac{(TJ^k(X,L))_z}{(TW)_z+d(J^k p_f)(TX)_x}=$

(by theorem 3.8 and the decomposition $(TJ^k(X,L))_z=(TJ^k(X,L)_x)_z)\oplus d(J^k p_f)(TX)_x$, see [7]V, pag. 307)

$$= \dim \frac{(TJ^k(X,L)_x)_z}{(TW_x)_z+(I_t)_x} = \qquad \text{(by 3.3 and 3.8)}$$

$$= \dim \frac{J^k(p_f{}^*TY)_x}{(I_t)_x+(I_\omega)_x+T(W_{x,y})_z}, \text{ q.e.d.}$$

As in remark 3.12, in the next lemma we substitute X by an affine subset $X \cap C^n$.

Lemma 3.17 $\delta(g,W,(x,f)) = \dim \dfrac{J^k(p_f{}^*TL)_x}{(I_t)_x+B_1(I_\omega)_x+T(W_{x,y})_z}$

where $B_1 \subset J^k(\mathcal{O}_X)_x$ *is the vector subspace generated by* $1,x_1,\ldots,x_n$ *(coord. on* C^n*).*

Proof First of all we consider the analytic map $g_x = g|_{\{x\}\times F}$: $F \to J^k(X,L)_X$. We may identify $L \simeq C^{n-t-1}$. Set $H = \mathrm{Hom}(C^n,C^{n-t-1}) \simeq C^{n(n-t-1)}$ so that $TH \simeq H$. A matrix which belongs to H represents a t-space C^t in the usual description of the grassmannian $Gr(C^t,C^n) \simeq Gr(C^{n-t-1},C^n) \supset F$. If a_{ij} are the enties of a matrix in H, the derivative $d(g_X)_f:TF_f \to T(J^k(X,L)_X)_z$ has the following expression

$$d(g_X)_f(h) = \sum a_{ij}x_jf^*(\frac{\partial}{\partial y_i})$$

with a_{ij} depending on $h \in TF_f$. Now we remark that $dg(T(X\times F)_{(x,f)}) = (dg_x)(TF)_f + d(J^kp_f)(TX)_x$ (note that $d(g_x)(TF)_f \subset (TJ^k(X,L)_x)_z$), so we have

$$\delta(g,W,(x,f)) = \dim \frac{(TJ^k(X,L))_z}{(TW)_z+dg(T(X\times F)_{(x,f)})} =$$

(by using π_f as in the proof of 3.16)

$$= \dim \frac{(TJ^k(X,L)_x)_z}{(TW_x)_z+(dg_x)(TF)_f+(I_t)_x} = \dim \frac{J^k(p_f{}^*TL)_x}{(I_t)_x+(I_\omega)_x+(dg_x)(TF)_f+T(W_{x,y})_z} =$$

$$= \dim \frac{J^k(p_f{}^*TL)_x}{(I_t)_x+B_1(I_\omega)_x+T(W_{x,y})_z}$$

(the last equality follows from the local expression of dg_X), q.e.d.

Definition 3.18 $B_r \subset J^k(\mathcal{O})_X$ is the vector subspace generated by all polynomials in $x_1,\ldots,x_n$ coord. in C^n of degree $\leq r$.

Proof of theorem 3.11

From the lemmas above, it follows that the hypothesis of theorem 3.15 is equivalent to the assertion: either $I_t+I_\omega+T(W_{x,y})_z = J^k(p_f{}^*TL)_X$ or the inclusion $I_t+I_\omega+T(W_{x,y})_z \subset I_t+B_1I_\omega+T(W_{x,y})_z$ is a proper inclusion. Suppose that the first alternative does not hold. We have a chain of vector subspaces $B_1I_\omega \subset \ldots \subset B_rI_\omega \subset B_{r+1}I_\omega \subset \ldots$ such that $\bigcup_{i=1}^{\infty} B_iI_\omega = J^k(p_f{}^*TL)_X$. Hence $B_rI_\omega = J^k(p_f{}^*TL)_X$ for $r \gg 0$. Let q be the largest integer such that $B_qI_\omega \subset I_t+I_\omega+T(W_{x,y})_z$. Thus we have

$$B_{q+1}I_\omega = B_1B_qI_\omega \subset B_1(I_t+I_\omega+T(W_{x,y})_z) = \qquad (I_t \text{ and } T(W_{x,y})_z \text{ are submodules!})$$

$$= I_t+B_1I_\omega+T(W_{x,y})_z.$$

Since $B_{q+1}I_\omega$ is not contained in $I_t+I_\omega+T(W_{x,y})_z$, it follows that $I_t+B_1I_\omega+T(W_{x,y})_z$ is not contained in $I_t+I_\omega+T(W_{x,y})_z$. It follows that the second alternative holds, q.e.d.

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