Università degli Studi di Bologna
Dipartimento di Matematica

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On the irreducible components of the moduli space of instanton bundles on $\mathbf{P}^5$

VINCENZO ANCONA and GIORGIO OTTAVIANI , Dipartimento di Matematica, viale Morgagni 67/A, I-50134 FIRENZE

Abstract We show that the moduli space of mathematical instanton bundles over $\mathbf{P}^5$ with $c_2 = 4$ contains at least two irreducible components.

A (mathematical) instanton bundle on $\mathbf{P}^{2n+1}$ with $c_2 = k$ is a stable bundle of rank $2n$ satisfying one of the following three equivalent conditions:

i) the Chern polynomial of E is $c_t(E) = \frac{1}{(1-t^2)^k}$ and $E(q)$ has natural cohomology in the range $-2n-1 \leq q \leq 0$

ii) E is the cohomology bundle of a monad

$$\mathcal{O}(-1)^k \longrightarrow \mathcal{O}^{2n+2k} \longrightarrow \mathcal{O}(1)^k$$

iii) E is the cohomology bundle of a monad

$$\mathcal{O}(-1)^k \longrightarrow \Omega^1(1)^k \longrightarrow \mathcal{O}^{2n(k-1)}$$

The equivalence of i), ii) and iii) is well-known (see for example [OS] and [AO1]). It seems likely that the stability condition is contained in i), ii) and iii). This is true on $\mathbf{P}^3$ and $\mathbf{P}^5$. Instanton bundles with $c_2 = k$ define a moduli space $MI_{\mathbf{P}^{2n+1}}(k)$ which is an *open subset* of the corresponding Maruyama moduli scheme.

$MI_{\mathbf{P}^3}(k)$ is expected to be irreducible, this is true for $k \leq 4$. $MI_{\mathbf{P}^{2n+1}}(k)$ is irreducible for $k \leq 2$ [AO1].

The aim of this note is to prove the following

Theorem. *$MI_{\mathbf{P}^5}(4)$ contains (at least) two irreducible components of dimension 65 and 68. The points corresponding to the special symplectic instanton bundles lie in their intersection. The generic bundle in the component of dimension 65 is not self-dual, while the generic bundle in the component of dimension 68 is symplectic.*

Remark. *The same technique shows that for $k = 4, 5, 6, 7, 8$ $MI_{\mathbf{P}^5}(k)$ contains components of different dimensions, hence it is reducible.*

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Deformations of bundles with structural group

Let E be a vector bundle of rank r with structural group $G \subset GL(r)$. Let $Ad: G \to GL(Lie\ G)$ be the adjoint representation and let $Ad\ E$ be the corresponding adjoint bundle. According to the Kodaira-Spencer theory, among the small deformations of E preserving the structural group there is a versal deformation. Let X be its base space: then the germ of X at $[E]$ is the zero locus of the Kuranishi map

$$\phi_E: H^1(Ad\ E) \to H^2(Ad\ E)$$

In particular $H^1(Ad\ E)$ identifies with the Zariski tangent space to X at $[E]$; moreover if $H^2(Ad\ E) = 0$ then X is smooth at $[E]$. X is smooth at $[E]$ if and only if $\phi_E \equiv 0$.

When E is stable the germ of the deformation space can be identified with the analytic germ of the corresponding Maruyama moduli scheme.

In particular if $G = Sp(r)$ then E is called a symplectic bundle and we have $Ad\ E = S^2E$. It is equivalent to say that a symplectic bundle E is a bundle E together with an isomorphism $\phi: E \to E^*$ such that $\phi = -\phi^t$.

Cohomology computations for instanton bundles

Let E be an instanton bundle on $\mathbf{P}^5$ with $c_2 = k$. From the Riemann-Roch formula we get

$$h^1(End\ E) - h^2(End\ E) = -3k^2 + 32k - 15 \qquad (1)$$

and if E is symplectic

$$h^1(S^2E) - h^2(S^2E) = -\frac{3}{2}k^2 + \frac{51}{2}k - 10 \qquad (2)$$

A symplectic instanton bundle on $\mathbf{P}^5$ is always the cohomology bundle of a "skew-symmetric" monad

$$\mathcal{O}(-1)^k \xrightarrow{QA^t} \mathcal{O}^{2k+4} \xrightarrow{A} \mathcal{O}(1)^k \qquad (3)$$

where A is represented by a $k \times (2k+4)$ matrix whose entries are homogeneous linear forms and Q is a nondegenerate skew-symmetric matrix of order $2k+4$.

Lemma. *Let E be a symplectic instanton bundle cohomology bundle of a monad (3). Then*

$$H^2(S^2E) \simeq Coker\ \left[H^0(\mathcal{O}^{2k+4} \otimes \mathcal{O}(1)^k) \stackrel{\wedge^2(A\otimes id_k)}{\longrightarrow} H^0(\wedge^2(\mathcal{O}(1)^k))\right]$$

Proof Consider the kernel bundle

$$0 \longrightarrow K \longrightarrow \mathcal{O}^{2k+4} \xrightarrow{A} \mathcal{O}(1)^k \longrightarrow 0$$

It is easy to prove

$$H^1(S^2E) \simeq H^1(S^2K)$$

Then the result follows from the exact sequence

$$0 \longrightarrow S^2 K \longrightarrow S^2 \mathcal{O}^{2k+4} \longrightarrow \mathcal{O}^{2k+4} \otimes \mathcal{O}(1)^k \stackrel{\wedge^2(A \otimes id_k)}{\longrightarrow} \wedge^2 (\mathcal{O}(1)^k) \longrightarrow 0$$

Example. *Let E be a special symplectic instanton bundle on $\mathbf{P}^5$ with $c_2(E) = k$. In [OT] it is computed*

$$h^1(End\ E) = 20k - 3 \qquad h^2(End\ E) = 3(k-2)^2$$

In a similar way C. Dionisi[D] shows

$$h^1(S^2 E) = 18k - 1 \qquad h^2(S^2 E) = \frac{3}{2}(k-2)(k-3)$$

For small values of k the computations can be performed by using [BS] and the above lemma. It follows

Proposition. *The moduli space of symplectic instanton bundles on $\mathbf{P}^5$ with $c_2 = 3$ is smooth of dimension 53 at the points corresponding to special bundles.*

Proof of the theorem. From [AO2] it follows that $MI_{\mathbf{P}^5}(4)$ contains a (reduced) irreducible component of dimension 65. This was proved by constructing $E \in MI_{\mathbf{P}^5}(4)$ such that $h^2(End\ E) = 0$. E is not self-dual. The fact that $MI_{\mathbf{P}^5}(4)$ is reducible follows immediately from (2) because every component of the space of symplectic instanton bundles has dimension $\geq h^1(S^2 E) - h^2 S^2(E) = 68$.

We construct now a class of examples of symplectic instanton bundles on $\mathbf{P}^5$ with $c_2 = 4$. Let (a, b, c, d, e, f) be homogeneous coordinates in $\mathbf{P}^5$. Let

$$A = \begin{bmatrix} a & \alpha b & c & & & & d & \beta e & f & & & \\ & a & b & c & & & & d & e & f & & \\ & & a & b & c & & & & d & e & f & \\ & & & a & b & c & & & & d & e & f \end{bmatrix}$$

Let

$$J = \begin{bmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & J \\ -J^t & 0 \end{bmatrix}$$

It is easy to check that for $\alpha \neq 0$ and $\beta \neq 0$ the rank of A is 4 at every point of $\mathbf{P}^5$ and $AQA^t = 0$. Hence A and Q define a symplectic instanton bundle $E_{\alpha,\beta}$ as in (1).

For $\alpha = \beta = 1$ $E_{\alpha,\beta}$ is special symplectic. For $\alpha = 2$ and $\beta = 3$ one computes by using [BS] that

$$h^2(S^2E_{2,3}) = 0$$

$$h^1(S^2E_{2,3}) = 68$$

It follows that $[E_{2,3}]$ is a smooth point of the moduli space of symplectic bundles.

Remark. *By using [BS] it is possible to compute also the Kuranishi morphism at* $E_{2,3}$ *(see [AABOP]). It turns out that the moduli space of all instanton bundles has still dimension* 68 *at* $E_{2,3}$ *(although we have* $h^1(End\ E_{2,3}) = 72$ *).*

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