G. Ottaviani
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AN INVARIANT REGARDING WARING'S PROBLEM FOR CUBIC POLYNOMIALS

GIORGIO OTTAVIANI

to the memory of Michael Schneider, ten years after

Abstract. We compute the equation of the 7-secant variety to the Veronese variety $(\mathbf{P}^4, \mathcal{O}(3))$, its degree is 15. This is the last missing invariant in the Alexander-Hirschowitz classification. It gives the condition to express a homogeneous cubic polynomial in 5 variables as the sum of 7 cubes (Waring problem). The interesting side in the construction is that it comes from the determinant of a matrix of order 45 with linear entries, which is a cube. The same technique allows to express the classical Aronhold invariant of plane cubics as a pfaffian.

§1. Introduction

We work over an algebraically closed field K of characteristic zero. The Veronese variety, given by $\mathbf{P}^n$ embedded with the linear system $|\mathcal{O}(d)|$, lives in $\mathbf{P}^N$ where $N = \binom{n+d}{d} - 1$. It parametrizes the homogeneous polynomials f of degree d in $n+1$ variables which are the power of a linear form g, that is $f = g^d$.

Let $\sigma_s(\mathbf{P}^n, \mathcal{O}(d))$ be the s-secant variety of the Veronese variety, that is the Zariski closure of the variety of polynomials f which are the sum of the powers of s linear forms g_i, i.e. $f = \sum_{i=1}^s g_i^d$. In particular $\sigma_1(\mathbf{P}^n, \mathcal{O}(d)) = (\mathbf{P}^n, \mathcal{O}(d))$ is the Veronese variety itself and $\sigma_2(\mathbf{P}^n, \mathcal{O}(d))$ is the usual secant variety. For generalities about the Waring's problem for polynomials see [IK] or [RS].

Our starting point is the theorem of Alexander and Hirschowitz (see [AH] or [BO] for a survey, including a self-contained proof) which states that the codimension of $\sigma_s(\mathbf{P}^n, \mathcal{O}(d)) \subseteq \mathbf{P}^N$ is the expected one, that is $\max\{N + 1 - (n+1)s, 0\}$, with the only exceptions

(i) $\sigma_k(\mathbf{P}^n, \mathcal{O}(2))$, $2 \le k \le n$

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(ii) $\sigma_{\frac{1}{2}n(n+3)}(\mathbf{P}^n, \mathcal{O}(4))$, $n = 2, 3, 4$

(iii) $\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$

The case (i) corresponds to the matrices of rank $\leq k$ in the variety of symmetric matrices of order $n+1$. In the cases (ii) and (iii) the expected codimension is zero, while the codimension is one. Hence the equation of the hypersurface $\sigma_s(\mathbf{P}^n, \mathcal{O}(d))$ in these cases is an interesting $SL(n+1)$-invariant. In the cases (ii) it is the catalecticant invariant, that was computed by Clebsch in the 19th century, its degree is $\binom{n+2}{2}$.

The main result of this paper is the computation of the equation of $\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$. This was left as an open problem in [IK, Chap. 2, Rem. 2.4].

We consider a vector space V. For any nonincreasing sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \dots)$ it is defined the Schur module $\Gamma^\alpha V$, which is an irreducible $SL(V)$-module (see [FH]). For $\alpha = (p)$ we get the p-th symmetric power of V and for $\alpha = (1, \dots, 1)$ (p times) we get the p-th alternating power of V. The module $\Gamma^\alpha V$ is visualized as a Young diagram containing α_i boxes in the i-th row. In particular if $\dim V = 5$ then $\Gamma^{2,2,1,1}V$ and its dual $\Gamma^{2,1,1}V$ have both dimension 45.

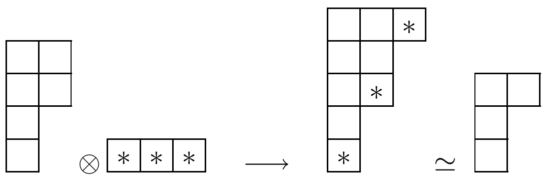
Our main result is the following

Theorem 1.1. *Let V be a vector space of dimension 5. For any $\phi \in S^3V$, let $B_\phi \colon \Gamma^{2,2,1,1}V \to \Gamma^{2,1,1}V$ be the $SL(V)$-invariant contraction operator. Then there is an irreducible homogeneous polynomial P of degree 15 on S^3V such that*

$$2P(\phi)^3 = \det B_\phi$$

The polynomial P is the equation of $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$.

The coefficient 2 is needed because we want the invariant polynomials to be defined over the rational numbers. The picture in terms of Young diagrams is



This picture means that $\Gamma^{2,1,1}V$ is a direct summand of the tensor product $\Gamma^{2,2,1,1}V \otimes S^3V$, according to the Littlewood-Richardson rule ([FH]).

The polynomial P gives the necessary condition to express a cubic homogeneous polynomial in five variables as a sum of seven cubes. We prove in Lemma 3.2 that if ϕ is decomposable then $\mathrm{rk}(B_\phi) = 6$. The geometrical explanation that $\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$ is an exceptional case is related to the fact that given seven points in $\mathbf{P}^4$ there is a unique rational normal curve through them, and it was discovered independently by Richmond and Palatini in 1902, see [CH] for a modern reference. Our approach gives a different (algebraic) proof of the fact that $\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$ is an exceptional case. Another argument, by using syzygies, is in [RS]. B. Reichstein found in [Re] an algorithm to check when a cubic homogeneous polynomial in five variables is the sum of seven cubes, see the Remark 3.4.

The resulting table of the Alexander-Hirschowitz classification is the following

	exp. codim	*codim*	*equation*
$\sigma_k(\mathbf{P}^n, \mathcal{O}(2))$ $2 \le k \le n$	$\max\left(\frac{(n+1)(n+2-2k)}{2}, 0\right)$	$\binom{n-k+2}{2}$	$(k+1) -$ minors
$\sigma_{\frac{1}{2}n(n+3)}(\mathbf{P}^n, \mathcal{O}(4))$ $n = 2, 3, 4$	0	1	catalecticant inv.
$\sigma_7(\mathbf{P}^4, \mathcal{O}(3))$	0	1	see Theorem 1.1

The degree of $\sigma_k(\mathbf{P}^n, \mathcal{O}(2))$ was computed by C. Segre, it is equal to $\prod_{i=0}^{n-k} \binom{n+1+i}{n+1-k-i} / \binom{2i+1}{i}$. We will use in the proof of Theorem 1.1 the fact that $\sigma_{k-1}(\mathbf{P}^n, \mathcal{O}(2))$ is the singular locus of $\sigma_k(\mathbf{P}^n, \mathcal{O}(2))$ for $k \le n$.

A general cubic polynomial in five variables can be expressed as a sum of eight cubes in ∞^5 ways, parametrized by a Fano 5-fold of index one (see [RS]). A cubic polynomial in five variables which can be expressed as a sum of seven cubes was called degenerate in [RS], hence what we have found is the locus of degenerate cubics. A degenerate cubic in five variables can be expressed as a sum of seven cubes in ∞^1 ways, parametrized by $\mathbf{P}^1$ (see [RS, 4.2]).

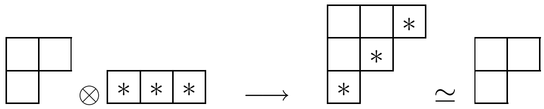
To explain our technique, we consider the Aronhold invariant of plane cubics.

The Aronhold invariant is the degree 4 equation of $\sigma_3(\mathbf{P}^2, \mathcal{O}(3))$, which can be seen as the $SL(3)$-orbit of the Fermat cubic $x_0^3 + x_1^3 + x_2^3$ (sum of three cubes), see [St, Prop. 4.4.7] or [DK, (5.13.1)].

Let W be a vector space of dimension 3. In particular $\Gamma^{2,1}W = \mathrm{ad}\, W$ is self-dual and it has dimension 8. We get

THEOREM 1.2. *For any $\phi \in S^3W$, let $A_\phi\colon \Gamma^{2,1}W \to \Gamma^{2,1}W$ be the $SL(V)$-invariant contraction operator. Then A_ϕ is skew-symmetric and the pfaffian $Pf\, A_\phi$ is the equation of $\sigma_3(\mathbf{P}(W), \mathcal{O}(3))$, i.e. it is the Aronhold invariant.*

The corresponding picture is



The Aronhold invariant gives the necessary condition to express a cubic homogeneous polynomial in three variables as a sum of three cubes. The explicit expression of the Aronhold invariant is known since the 19th century, but we have not found in the literature its representation as a pfaffian. In the Remark 2.3 we apply this representation to the Scorza map between plane quartics.

In Section 2 we give the proof of Theorem 1.2. This is introductory to Theorem 1.1, which is proved in Section 3. In Section 4 we review, for completeness, some known facts about the catalecticant invariant of quartic hypersurfaces.

We are indebted to S. Sullivant, for his beautiful lectures at Nordfjordeid in 2006 about [SS], where a representation of the Aronhold invariant is found with combinatorial techniques.

§2. The Aronhold invariant as a pfaffian

Let e_0, e_1, e_2 be a basis of W and fix the orientation $\bigwedge^3 W \simeq K$ given by $e_0 \wedge e_1 \wedge e_2$. We have $\operatorname{End} W = \operatorname{ad} W \oplus K$. The $SL(W)$-module $\operatorname{ad} W = \Gamma^{2,1}(W)$ consists of the subspace of endomorphisms of W with zero trace. We may interpret the contraction

$$A_\phi\colon \Gamma^{2,1}W \longrightarrow \Gamma^{2,1}W$$

as the restriction of a linear map $A'_\phi\colon \operatorname{End} W \to \operatorname{End} W$, which is defined for $\phi = e_{i_1}e_{i_2}e_{i_3}$ as

$$A'_{e_{i_1}e_{i_2}e_{i_3}}(M)(w) = \sum_\sigma (M(e_{i_{\sigma(1)}}) \wedge e_{i_{\sigma(2)}} \wedge w) e_{i_{\sigma(3)}}$$

where $M \in \operatorname{End} W$, $w \in W$ and σ covers the symmetric group Σ_3.

Then A'_ϕ is defined for a general ϕ by linearity, and it follows from the definition that it is $SL(V)$-invariant.

The Killing scalar product on $\operatorname{End} W$ is defined by $\operatorname{tr}(M \cdot N)$.

LEMMA 2.1. (i) $\operatorname{Im}(A'_\phi) \subseteq \operatorname{ad} W \quad K \subseteq \operatorname{Ker}(A'_\phi)$
(ii) A'_ϕ *is skew-symmetric.*

Proof. (i) follows from

$$\begin{aligned}\operatorname{tr}\Big[A_{e_{i_1}e_{i_2}e_{i_3}}(M)\Big] &= \sum_s A_{e_{i_1}e_{i_2}e_{i_3}}(M)(e_s)e_s^\vee \\ &= \sum_\sigma (M(e_{i_{\sigma(1)}}) \wedge e_{i_{\sigma(2)}} \wedge e_{i_{\sigma(3)}}) = 0\end{aligned}$$

The second inclusion is evident. To prove (ii), we have to check that

$$\operatorname{tr}(A_\phi(M) \cdot N) = -\operatorname{tr}(A_\phi(N) \cdot M)$$

for $M, N \in \operatorname{End} W$. Indeed let $\phi = e_{i_1}e_{i_2}e_{i_3}$. We get

$$\begin{aligned}\operatorname{tr}(A_{e_{i_1}e_{i_2}e_{i_3}}(M) \cdot N) &= \sum_s A_{e_{i_1}e_{i_2}e_{i_3}}(M)(N(e_s))e_s^\vee \\ &= \sum_\sigma M(e_{i_{\sigma(1)}}) \wedge e_{i_{\sigma(2)}} \wedge N(e_{i_{\sigma(3)}})\end{aligned}$$

which is alternating in M and N, where we denoted by $e_i^\vee$ the dual basis. ∎

It follows from Lemma 2.1 that the restriction

$$A'_{\phi|\operatorname{ad} W}\colon \operatorname{ad} W \longrightarrow \operatorname{ad} W$$

coincides, up to scalar multiple, with the contraction operator A_ϕ of Theorem 1.2 and it is skew-symmetric.

LEMMA 2.2. *Let* $\phi = w^3$ *with* $w \in W$. *Then* $\operatorname{rk} A_\phi = 2$. *More precisely*

$$\begin{aligned}\operatorname{Im} A_{w^3} &= \{M \in \operatorname{ad} W \mid \operatorname{Im} M \subseteq \langle w \rangle\} \\ \operatorname{Ker} A_{w^3} &= \{M \in \operatorname{ad} W \mid w \textit{ is an eigenvector of } M\}\end{aligned}$$

Proof. The statement follows from the equality

$$A_{w^3}(M)(v) = 6(M(w) \wedge w \wedge v)w$$

As an example, note that $\operatorname{Im} A_{e_0^3} = \langle e_0 \otimes e_1^\vee, e_0 \otimes e_2^\vee \rangle$ and $\operatorname{Ker} A_{e_0^3}$ is spanned by all the basis monomials, with the exception of $e_0^\vee \otimes e_1$ and $e_0^\vee \otimes e_2$. Due to the $SL(W)$-invariance, this example proves the general case. ∎

Proof of Theorem 1.2. Let $\phi \in \sigma_3(\mathbf{P}(W), \mathcal{O}(3))$. By the definition of higher secant variety, ϕ is in the closure of elements which can be written as $\phi_1 + \phi_2 + \phi_3$ with $\phi_i \in (\mathbf{P}(W), \mathcal{O}(3))$. From Lemma 2.2 it follows that

$$\operatorname{rk} A_\phi \leq \operatorname{rk} A_{\sum_{i=1}^3 \phi_i} = \operatorname{rk} \sum_{i=1}^{3} A_{\phi_i} \leq \sum_{i=1}^{3} \operatorname{rk} A_{\phi_i} = 2 \cdot 3 = 6$$

Hence $Pf(A_\phi)$ has to vanish on $\sigma_3(\mathbf{P}(W), \mathcal{O}(3))$.

Write a cubic polynomial as

$$\begin{aligned}\phi = {} & v_{000}x_0^3 + 3v_{001}x_0^2x_1 + 3v_{002}x_0^2x_2 + 3v_{011}x_0x_1^2 + 6v_{012}x_0x_1x_2 \\ & + 3v_{022}x_0x_2^2 + v_{111}x_1^3 + 3v_{112}x_1^2x_2 + 3v_{122}x_1x_2^2 + v_{222}x_2^3\end{aligned}$$

We order the monomial basis of $\bigwedge^2 W \otimes W$ with the lexicographical order in the following way:

$$\begin{gathered}(w_0 \wedge w_1)w_0,\ (w_0 \wedge w_1)w_1,\ (w_0 \wedge w_1)w_2, \\ (w_0 \wedge w_2)w_0,\ (w_0 \wedge w_2)w_1,\ (w_0 \wedge w_2)w_2, \\ (w_1 \wedge w_2)w_0,\ (w_1 \wedge w_2)w_1,\ (w_1 \wedge w_2)w_2\end{gathered}$$

Call M_i for $i = 1, \ldots, 9$ this basis. The matrix of A'_ϕ, with respect to this basis, has at the entry (i,j) the value $A'_\phi(M_j)(M_i)$ and it is the following

$$\begin{bmatrix}
0 & v_{222} & -v_{122} & 0 & -v_{122} & v_{112} & 0 & v_{022} & -v_{012} \\
-v_{222} & 0 & v_{022} & v_{122} & 0 & -v_{012} & -v_{022} & 0 & v_{002} \\
v_{122} & -v_{022} & 0 & -v_{112} & v_{012} & 0 & v_{012} & -v_{002} & 0 \\
0 & -v_{122} & v_{112} & 0 & v_{112} & -v_{111} & 0 & -v_{012} & v_{011} \\
v_{122} & 0 & -v_{012} & -v_{112} & 0 & v_{011} & v_{012} & 0 & -v_{001} \\
-v_{112} & v_{012} & 0 & v_{111} & -v_{011} & 0 & -v_{011} & v_{001} & 0 \\
0 & v_{022} & -v_{012} & 0 & -v_{012} & v_{011} & 0 & v_{002} & -v_{001} \\
-v_{022} & 0 & v_{002} & v_{012} & 0 & -v_{001} & -v_{002} & 0 & v_{000} \\
v_{012} & -v_{002} & 0 & -v_{011} & v_{001} & 0 & v_{001} & -v_{000} & 0
\end{bmatrix}$$

Deleting one of the columns corresponding to $(w_0 \wedge w_1)w_2$, $(w_0 \wedge w_2)w_1$ or $(w_1 \wedge w_2)w_0$ (respectively the 3rd, the 5th and the 7th, indeed their alternating sum gives the trace), and the corresponding row, we get a skew-symmetric matrix of order 8 which is the matrix of A_ϕ. To conclude the proof, it is enough to check that the pfaffian is nonzero. This can be easily checked on the point corresponding to $\phi = x_0x_1x_2$, that is when $v_{012} = 1$ and all the other coordinates are equal to zero. This means that any triangle is not in the closure of the Fermat curve. we conclude that $Pf(A_\phi)$ is the Aronhold invariant. We verified that it coincides, up to a constant, with the expression given in [St, Prop. 4.4.7] or in [DK, (5.13.1)]. ∎

The vanishing of the Aronhold invariant gives the necessary and sufficient condition to express a cubic polynomial in three variables as the sum of three cubes.

Remark. A'_ϕ can be thought as a map

$$A'_\phi \colon \bigwedge^2 W \otimes W \longrightarrow \bigwedge^2 W^\vee \otimes W^\vee$$

For $\phi = w^3$ we have the formula

$$A'_\phi(\omega \otimes v)(\omega' \otimes v') = (\omega \wedge w) \otimes (v \wedge w \wedge v') \otimes (\omega' \wedge w)$$

This is important for the understanding of the next section.

Remark. We have the decomposition

$$\bigwedge^2(\Gamma^{2,1}W) = S^3W \oplus \Gamma^{2,2,2}W \oplus \operatorname{ad} W$$

and it is a nice exercise to show the behaviour of the three summands. For the first one

$$S^3W \cap \left\{M \in \bigwedge^2(\Gamma^{2,1}W) \mid \operatorname{rk}(M) \leq 2k\right\}$$

is the cone over $\sigma_k(\mathbf{P}(W), \mathcal{O}(3))$, so that we have found the explicit equations for all the higher secant varieties to $(\mathbf{P}(W), \mathcal{O}(3))$. The secant variety $\sigma_2(\mathbf{P}(W), \mathcal{O}(3))$ is the closure of the orbit of plane cubics consisting of three concurrent lines, and its equations are the 6×6 subpfaffians of A_ϕ. It has degree 15. There is a dual description for $\Gamma^{2,2,2}W$.

For the third summand, we have that

$$\operatorname{ad} W \subseteq \left\{M \in \bigwedge^2(\Gamma^{2,1}W) \mid \operatorname{rk}(M) \leq 6\right\}$$

Indeed any $M \in \operatorname{ad} W$ induces the skew-symmetric morphism

$$[M, -]$$

whose kernel contains M. Moreover

$$\operatorname{ad} W \cap \left\{ M \in \bigwedge^2 (\Gamma^{2,1} W) \mid \operatorname{rk}(M) \leq 4 \right\}$$

is the 5-dimensional affine cone consisting of endomorphisms $M \in \operatorname{ad} W$ such that their minimal polynomial has degree ≤ 2.

Remark 2.3. We recall from [DK] the definition of the Scorza map. Let A be the Aronhold invariant. For any plane quartic F and any point $x \in \mathbf{P}(W)$ we consider the polar cubic $P_x(F)$. Then $A(P_x(F))$ is a quartic in the variable x which we denote by $S(F)$. The rational map $S\colon \mathbf{P}(S^4 W) \dashrightarrow \mathbf{P}(S^4 W)$ is called the Scorza map. Our description of the Aronhold invariant shows that $S(F)$ is defined as the degeneracy locus of a skew-symmetric morphism on $\mathbf{P}(W)$

$$\mathcal{O}(-2)^8 \xrightarrow{f} \mathcal{O}(-1)^8$$

It is easy to check (see [Be]) that $\operatorname{Coker} f = E$ is a rank two vector bundle over $S(F)$ such that $c_1(E) = K_{S(F)}$. Likely from E it is possible to recover the even theta-characteristic θ on $S(F)$ defined in [DK, (7.7)]. The natural guess is that

$$h^0(E \otimes (-\theta)) > 0$$

for a unique even θ, but we do not know if this is true.

§3. The invariant for cubic polynomials in five variables

Let now $e_0, \ldots, e_4$ be a basis of V, no confusion will arise with the notations of the previous section. We fix the orientation $\bigwedge^5 V \simeq K$ given by $e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We construct, for $\phi \in S^3 V$, the contraction operator

$$B'_\phi\colon \bigwedge^4 V \otimes \bigwedge^2 V \longrightarrow \bigwedge^4 V^\vee \otimes \bigwedge^2 V^\vee \simeq \bigwedge^3 V \otimes V$$

If $\phi = e_{i_1} e_{i_2} e_{i_3}$, the definition is

$$\begin{aligned} &B'_\phi(v_a \wedge v_b \wedge v_c \wedge v_d) \otimes (v_e \wedge v_f) \\ &\quad = \sum_\sigma \Big(v_a \wedge v_b \wedge v_c \wedge v_d \wedge e_{i_{\sigma(1)}}\Big) \otimes \Big(v_e \wedge v_f \wedge e_{i_{\sigma(2)}}\Big) \otimes e_{i_{\sigma(3)}} \end{aligned}$$

where σ covers the symmetric group Σ_3 and we extend this definition, to a general ϕ, by linearity.

We may interpret B'_ϕ as a morphism

$$B'_\phi\colon\ Hom(V, \textstyle\bigwedge^2 V) \longrightarrow Hom(\textstyle\bigwedge^2 V, V)$$

If $\phi = e_{i_1}e_{i_2}e_{i_3}$ and $M \in Hom(V, \bigwedge^2 V)$ we have

$$B'_{e_{i_1}e_{i_2}e_{i_3}}(M)(v_1 \wedge v_2) = \sum_\sigma (M(e_{i_{\sigma(1)}}) \wedge e_{i_{\sigma(2)}} \wedge v_1 \wedge v_2)e_{i_{\sigma(3)}}$$

We have a $SL(V)$-decomposition

$$\textstyle\bigwedge^4 V \otimes \bigwedge^2 V = \Gamma^{2,2,1,1}V \oplus V$$

Consider the contraction $c\colon \bigwedge^4 V \otimes \bigwedge^2 V \to V$ defined by

$$c(\omega \otimes (v_i \wedge v_j)) = (\omega \wedge v_i)v_j - (\omega \wedge v_j)v_i$$

Then the subspace $\Gamma^{2,2,1,1}V$ can be identified with

$$\left\{M \in \textstyle\bigwedge^4 V \otimes \bigwedge^2 V \mid c(M) = 0\right\}$$

or with

$$\left\{M \in Hom(V, \textstyle\bigwedge^2 V) \mid \sum e_i^\vee M(e_i) = 0\right\}$$

The subspace $V \subset Hom(V, \bigwedge^2 V)$ can be identified with $\{v \wedge - \mid v \in V\}$. At the same time we have a $SL(V)$-decomposition

$$V \otimes \textstyle\bigwedge^3 V = \Gamma^{2,1,1}V \oplus \bigwedge^4 V$$

and the obvious contraction $d\colon V \otimes \bigwedge^3 V \to \bigwedge^4 V$. The subspace $\Gamma^{2,1,1}V$ can be identified with

$$\left\{N \in V \otimes \textstyle\bigwedge^3 V \mid d(N) = 0\right\}$$

LEMMA 3.1. (i) $\mathrm{Im}(B'_\phi) \subseteq \Gamma^{2,1,1}V \quad V \subseteq \mathrm{Ker}(B'_\phi)$
(ii) B'_ϕ *is symmetric.*

Proof. The statement (i) follows from the formula

$$d\Big(B'_{e_{i_1}e_{i_2}e_{i_3}}(v_a\wedge v_b\wedge v_c\wedge v_d)\otimes(v_e\wedge v_f)\Big)$$
$$=\sum_\sigma\Big(v_a\wedge v_b\wedge v_c\wedge v_d\wedge e_{i_{\sigma(1)}}\Big)\otimes\Big(v_e\wedge v_f\wedge e_{i_{\sigma(2)}}\wedge e_{i_{\sigma(3)}}\Big)=0$$

In order to prove the second inclusion, for any $v\in V$ consider the induced morphism $M_v(w)=v\wedge w$. We get

$$B'_{e_{i_1}e_{i_2}e_{i_3}}(M_v)(v_1\wedge v_2)=\sum_\sigma\Big(v\wedge e_{i_{\sigma(1)}}\wedge e_{i_{\sigma(2)}}\wedge v_1\wedge v_2\Big)e_{i_{\sigma(3)}}=0$$

In order to prove (ii) we may assume $\phi=v^3$.

We need to prove that

$$B'_{v^3}(\omega\otimes\xi)(\omega'\otimes\xi')=B'_{v^3}(\omega'\otimes\xi')(\omega\otimes\xi)$$

for every $\omega,\omega'\in\bigwedge^4 V$ and $\xi,\xi'\in\bigwedge^2 V$. Indeed

$$B'_{v^3}(\omega\otimes\xi)(\omega'\otimes\xi')=(\omega\wedge v)\otimes(\xi\wedge v\wedge\xi')\otimes(v\wedge\omega')$$

which is symmetric in the pair (ω,ξ). ∎

It follows from Lemma 3.1 that the restriction $B'_{\phi|\Gamma^{2,2,1,1}}\colon\Gamma^{2,2,1,1}\to\Gamma^{2,1,1}V$ coincides, up to scalar multiple, with the contraction B_ϕ of the Theorem 1.1 and it is symmetric. Note that

$$\operatorname{Ker}(B_\phi)=\operatorname{Ker}(B'_\phi)/V\quad\operatorname{Im}(B_\phi)=\operatorname{Im}(B'_\phi)$$

LEMMA 3.2. *Let $\phi=v^3$ with $v\in V$. Then* $\operatorname{rk}B_\phi=6$. *More precisely*

$$\operatorname{Im}B_{v^3}=\Big\{N\in Hom(\textstyle\bigwedge^2 V,V)\mid\sum e_i^\vee N(e_i\wedge v)=0,\ \forall v\in V,\ \ Im(N)\subseteq\langle v\rangle\Big\}$$
$$\operatorname{Ker}B_{v^3}=\Big\{M\in Hom(V,\textstyle\bigwedge^2 V)\mid\sum e_i^\vee M(e_i)=0,\ \ M(v)\subseteq v\wedge V\Big\}$$

Proof. The statement follows from the equality

$$B_{v^3}(M)(v_1\wedge v_2)=6(M(v)\wedge v\wedge v_1\wedge v_2)v$$

As an example, a basis of $\operatorname{Im}B_{e_0^3}$ is given by $e_0\otimes(e_i^\vee\wedge e_j^\vee)$ for $1\le i<j\le 4$ and a basis of $\operatorname{Ker}B_{e_0^3}$ is given by all the basis monomials with the exceptions of $e_0^\vee\otimes(e_i\wedge e_j)$ for $1\le i<j\le 4$. Due to the $SL(V)$-invariance, this example proves the general case. ∎

We write $\phi \in S^3V$ as $\phi = v_{000}x_0^3 + 3v_{001}x_0^2x_1 + \cdots + v_{444}x_4^3$.

LEMMA 3.3. *Every $SL(V)$-invariant homogeneous polynomial of degree* 15 *on S^3V which contains the monomial*

$$v_{000}^2 v_{012}^3 v_{111} v_{223}^3 v_{334}^3 v_{144}^3$$

is irreducible.

Proof. Let $t_0, \ldots, t_4$ be the canonical basis of $\mathbb{Z}^5$. We denote by $t_i + t_j + t_k$ the weight of the monomial v_{ijk}, according to [St]. For example the weight of v_{000} is $(3, 0, 0, 0, 0)$. We denote the first component of the weight as the x_0-weight, the second component as the x_1-weight, and so on. We recall that every $SL(V)$-invariant polynomial is isobaric, precisely every monomial of a $SL(V)$-invariant polynomial of degree $5k$ has weight $(3k, 3k, 3k, 3k, 3k)$ (see [St, (4.4.14)]), this follows from the invariance with respect to the diagonal torus. We claim that there is no isobaric monomial of weight $(6, 6, 6, 6, 6)$ and degree 10 with variables among v_{000}, v_{012}, v_{111}, v_{223}, v_{334}, v_{144}. We divide into the following cases, by looking at the possibilities for the x_0-weight:

i) The monomial contains v_{000}^2 and does not contain v_{012}. By looking at the x_2-weight, the monomial has to contain v_{223}^3, which gives contribution 3 to the x_3-weight. This gives a contradiction, because from v_{334} the possible values for the x_3-weight are even, and we never make 6.

ii) The monomial contains $v_{000}v_{012}^3$ and not higher powers. This monomial gives contribution 3 to the x_2-weight. From v_{223} the possible values for the x_2-weight are even, and we never make 6, again.

iii) The monomial contains v_{012}^6 and does not contain v_{000}. This monomial gives contribution 6 to the x_0-weight, and the same contribution is given to the x_1-weight and to the x_2-weight. Hence the only other possible monomial that we are allowed to use is v_{334}, which gives a x_3-weight doubled with respect to the x_4-weight, which is a contradiction.

This contradiction proves our claim. Nevertheless, if our polynomial is reducible, also its factors have to be homogeneous and $SL(V)$-invariant, and the monomial in the statement should split into two factors of degree 5 and 10, against the claim. ∎

Proof of Theorem 1.1. Let $\phi \in \sigma_7(\mathbf{P}(V), \mathcal{O}(3))$. By the definition of higher secant variety, ϕ is in the closure of elements which can be written as $\sum_{i=1}^{7} \phi_i$ with $\phi_i \in (\mathbf{P}(V), \mathcal{O}(3))$. From Lemma 3.2 it follows that

$$\operatorname{rk} B_\phi \leq \operatorname{rk} B_{\sum_{i=1}^{7} \phi_i} = \operatorname{rk} \sum_{i=1}^{7} B_{\phi_i} \leq \sum_{i=1}^{7} \operatorname{rk} B_{\phi_i} = 6 \cdot 7 = 42$$

Hence $\det(B_\phi)$ has to vanish on $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$.

We order the monomial basis of S^3V with the lexicographical ordered induced by $x_0 < x_1 < x_2 < x_3 < x_4$. We order also the basis of $\bigwedge^2 V \otimes \bigwedge^4 V$ with the lexicographical order. There are 50 terms, beginning with

$$(e_0 \wedge e_1) \otimes (e_0 \wedge e_1 \wedge e_2 \wedge e_3),\ (e_0 \wedge e_1) \otimes (e_0 \wedge e_1 \wedge e_2 \wedge e_4), \ldots$$

and ending with

$$\ldots, (e_3 \wedge e_4) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4)$$

These 50 terms are divided into 10 blocks, depending on the first factor $e_s \wedge e_t$. The matrix of B'_ϕ, with respect to this basis, is a 50×50 symmetric matrix with linear monomial entries from v_{ijk}.

We describe this matrix in block form. For $i = 0, \ldots, 4$ let A_i be the 5×5 symmetric matrix which at the entry $(5-s, 5-t)$ has $(-1)^{s+t} v_{ist}$, corresponding to the monomial $x_i x_s x_t$. For example

$$A_4 = \begin{bmatrix} v_{444} & -v_{344} & v_{244} & -v_{144} & v_{044} \\ -v_{344} & v_{334} & -v_{234} & v_{134} & -v_{034} \\ v_{244} & -v_{234} & v_{224} & -v_{124} & v_{024} \\ -v_{144} & v_{134} & -v_{124} & v_{114} & -v_{014} \\ v_{044} & -v_{034} & v_{024} & -v_{014} & v_{004} \end{bmatrix}$$

Then the matrix of B'_ϕ has the following block form

$$\begin{bmatrix}
 & & & & & & & A_4 & -A_3 & A_2 \\
 & & & & & -A_4 & A_3 & & & -A_1 \\
 & & & & A_4 & & -A_2 & & A_1 & \\
 & & & & -A_3 & A_2 & & -A_1 & & \\
 & & A_4 & -A_3 & & & & & & A_0 \\
 & -A_4 & & A_2 & & & & & -A_0 & \\
 & A_3 & -A_2 & & & & & A_0 & & \\
A_4 & & & -A_1 & & & A_0 & & & \\
-A_3 & & A_1 & & & -A_0 & & & & \\
A_2 & -A_1 & & & A_0 & & & & &
\end{bmatrix}$$

Among the 50 basis elements, there are 30 tensors $(e_s \wedge e_t) \otimes (e_i \wedge e_j \wedge e_k \wedge e_l)$ such that $\{s,t\} \subseteq \{i,j,k,l\}$. The other 20 elements are divided into 5 groups, depending on the single index $\{s,t\} \cap \{i,j,k,l\}$. The contraction c maps the first group of 30 elements into 30 independent elements of $\Gamma^{2,2,1,1}V$, and each group of 4 elements has the image through c of dimension 3 in $\Gamma^{2,2,1,1}V$, indeed the images of the 4 elements satisfy a linear relation with ± 1 coefficients.

It follows that the matrix of B_ϕ can be obtained from the matrix of B'_ϕ by deleting five rows, one for each of the above groups, and the corresponding five columns. We can delete, for example, the columns and the rows corresponding to

$$\begin{aligned}&(e_0 \wedge e_1) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4),\ (e_0 \wedge e_2) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4),\\ &(e_0 \wedge e_3) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4),\ (e_0 \wedge e_4) \otimes (e_0 \wedge e_1 \wedge e_2 \wedge e_3),\\ &(e_0 \wedge e_4) \otimes (e_1 \wedge e_2 \wedge e_3 \wedge e_4)\end{aligned}$$

which have respectively number 5, 10, 15, 16, 20. Note that in the resulting matrix for B_ϕ, all entries are monomials in v_{ijk} with coefficient ± 1.

In order to show that for general ϕ the morphism B_ϕ is invertible, the simplest way is to look at the monomial $(v_{001}v_{022}v_{113}v_{244}v_{334})^9$ which appears with nonzero coefficient in the expression of $\det B_\phi$. We prefer instead to use the monomial appearing in the statement of Lemma 3.3, which allows to prove the stronger statement that $\det B_\phi$ is the cube of an irreducible polynomial. Indeed, by substituting 0 to all the variables different from v_{000}, v_{012}, v_{111}, v_{223}, v_{334}, v_{144}, we get by an explicit computation that the determinant is equal to

$$-2\left(v_{000}^2 v_{012}^3 v_{111} v_{223}^3 v_{334}^3 v_{144}^3\right)^3$$

Hence for general ϕ we have $\operatorname{rk} B_\phi = 45$. Note that this gives an alternative proof of the fact that $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$ has codimension bigger than zero, and it has to appear in the Alexander-Hirschowitz classification. It follows that on the points of $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$ the rank of $\operatorname{rk} B_\phi$ drops at least by three, so that $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$ is contained in the singular locus of $\det B_\phi$, and in particular $\det B_\phi$ has to vanish with multiplicity ≥ 3 on $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$. It is known that $\sigma_7(\mathbf{P}(V), \mathcal{O}(3))$ is a hypersurface (see [CH]), hence its equation P has to be a factor of multiplicity ≥ 3 of $\det B_\phi$. Since every $SL(V)$-invariant polynomial has degree $5k$, the possible values for the degree of P are 5, 10 or 15. Look at the monomials in P containing some among the variables v_{000}, v_{012}, v_{111}, v_{223}, v_{334}, v_{144}, these monomials have to exist, due to the explicit computation performed before. If the degree of P is ≤ 10, then there exists a $SL(V)$-invariant polynomial of degre 10 with a monomial containing the above variables, but this contradicts the claim proved along the proof of the Lemma 3.3. It follows that $\deg P = \deg \sigma_7(\mathbf{P}(V), \mathcal{O}(3)) = 15$ and P^3 divides $\det B_\phi$, looking again at our explicit computation we see that we can arrange the scalar multiples in order that P is defined over the rational numbers (as all the $SL(V)$-invariants) and the equation $2P(\phi)^3 = \det B_\phi$ holds. The Lemma 3.3 shows that P is irreducible. □

Remark 3.4. The results obtained by Reichstein with his algorithm developed in [Re] can be verified with the Theorem 1.1. For example when w is like in the Example 1 at page 48 of [Re], a computer check shows that $\mathrm{rk}(B_w) = 42$, confirming that $w \in \sigma_7(\mathbf{P}(V), \mathcal{O}(3))$, while when w is like in the Example 2 at page 57 of [Re] then $\mathrm{rk}(B_w) = 45$, so that $w \notin \sigma_7(\mathbf{P}(V), \mathcal{O}(3))$.

The simplest example of a cubic which is not the sum of seven cubes is probably

$$\phi = x_0^2x_1 + x_0x_2^2 + x_1^2x_3 + x_2x_4^2 + x_3^2x_4$$

where $\det(B_\phi) = -2$, which can be checked even without a computer, but with a good amount of patience. The polynomial ϕ defines a smooth cubic 3-fold.

§4. The catalecticant invariant for Clebsch quartics

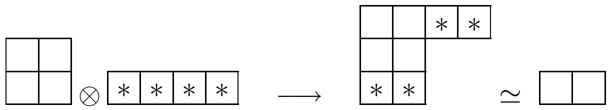
Let U be any vector space of dimension $n+1$.

Every quartic $f \in S^4U$ induces the contraction $C_f\colon S^2U^\vee \to S^2U$. Clebsch realized in 1861 that if $f \in (\mathbf{P}^n, \mathcal{O}(4))$ then $\mathrm{rk}\, A_f = 1$. Indeed, with the notations of the previous sections,

$$C_{v^4}(u_1u_2) = 24u_1(v)u_2(v)v^2$$

is always a scalar multiple of v^2. Clebsch worked in the case $n = 2$ but the same result holds for every n. If $f \in \sigma_k(\mathbf{P}^n, \mathcal{O}(4))$, we get that C_f is the limit of a sum of k matrices of rank one, then $\mathrm{rk}\, C_f \leq k$. The quartic f is called a Clebsch quartic if and only if $\det C_f = 0$, and this equation gives the catalecticant invariant (see [IK] or [DK]). A matrix description is the following. Let D_i for $i = 1, \ldots, \binom{n+2}{2}$ be a basis of differential operators of second order on U. Then $\det(D_iD_jf)$ is the catalecticant invariant.

The picture in terms of Young diagrams for $n = 2$ is



If $n = 2$, we write

$$f = f_{0000}x_0^4 + 4f_{0001}x_0^3x_1 + 6f_{0011}x_0^2x_1^2 + \cdots + 12f_{0012}x_0^2x_1x_2 + \cdots + f_{2222}x_2^4$$

Then the well known expression for the degree 6 equation of $\sigma_5(\mathbf{P}^2, \mathcal{O}(4))$ is the following (we choosed the basis ∂_{00}, ∂_{01}, ∂_{11}, ∂_{02}, ∂_{12}, ∂_{22})

$$\det \begin{bmatrix} f_{0000} & f_{0001} & f_{0011} & f_{0002} & f_{0012} & f_{0022} \\ f_{0001} & f_{0011} & f_{0111} & f_{0012} & f_{0112} & f_{0122} \\ f_{0011} & f_{0111} & f_{1111} & f_{0112} & f_{1112} & f_{1122} \\ f_{0002} & f_{0012} & f_{0112} & f_{0022} & f_{0122} & f_{0222} \\ f_{0012} & f_{0112} & f_{1112} & f_{0122} & f_{1122} & f_{1222} \\ f_{0022} & f_{0122} & f_{1122} & f_{0222} & f_{1222} & f_{2222} \end{bmatrix} = 0$$

The above equation gives the necessary condition to express a quartic homogeneous polynomial in 3 variables as the sum of 5 fourth powers. Mukai proves in [Mu] that a general plane quartic is a sum of 6 fourth powers in ∞^3 ways, parametrized by the Fano 3-fold V_{22}.

The Clebsch quartics give a hypersurface of degree $\binom{n+2}{2}$ in the space of all quartics.

It follows that this hypersurface contains the variety of k-secants to $(\mathbf{P}^n, \mathcal{O}(4))$ for $k = \left[\binom{n+2}{2} - 1\right] = n(n+3)/2$, and it is equal to this secant variety for $1 \leq n \leq 4$, which turns out to be defective for $2 \leq n \leq 4$. Indeed it is a hypersurface while it is expected that it fills the ambient space. This explains why this example appears in the Alexander-Hirschowitz classification.

Added in proof: F. Schreyer communicated to us that B_ϕ of the Theorem 1.1 appears also in the apolar ring of ϕ.

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Dipartimento di Matematica U. Dini
Università di Firenze
viale Morgagni 67/A, 50134 Firenze
Italy

ottavian@math.unifi.it