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AN INTRODUCTION TO THE DERIVED CATEGORIES AND THE THEOREM OF BEILINSON

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In this paper we give a short survey of the theory of *derived categories* with the purpose to state the *Beilinson theorem* [B] in its natural setting. This exposition can be intended as an introduction to [AO].

Derived categories arise for example in the following problem: given the category of complexes of coherent sheaves on some projective variety X, try to identify the complexes with the same cohomology. The natural setting for this construction is a new category (whose objects are still the complexes of coherent sheaves on X), where:

(i) morphisms of objects do not correspond to natural morphisms between complexes of sheaves.

(ii) the kernel of morphisms is not well defined so that the new category is no more abelian.

This new category is called the derived category of coherent sheaves on X: it inherits luckily the structure of a *triangulated* *category*, where *cones of morphisms* partially supply the loss of kernels and cokernels. Despite the technical difficulties which arise in its construction the derived category carries on many informations about the structure of the sheaves on X.

Beilinson showed in 1978 [B] that when $X = \mathbf{P}^n$ any coherent sheaf $\mathcal{F}$ can be constructed by a resolution, defined explicitly from the cohomology of $\mathcal{F}$ itself, where only the generators of the derived category $D^b(X)$ appear. For this reason the generators (in this case the bundles $\Omega^p(p)$ of twisted p-forms or dually the sheaves $O(-i)$ for $i = 0, \dots, n$) are called the *building blocks* for the sheaves on $\mathbf{P}^n$.

Kapranov [K] has extended this result to the quadrics Q_n and to other varieties.

In [AO] we show that when X is a projective space or a quadric, we can detect if a generator of $D^b(X)$ is a direct summand of a given sheaf $\mathcal{F}$ by certain vanishing of cohomology groups. This allows to give simple and unitary proofs of the cohomological characterizations of the building blocks on $\mathbf{P}^n$ and Q_n.

1. Derived categories.

We refer to [Ha] for a complete treatement of this subject.

Let A be an abelian category. A *complex* of objects of A is a collection of objects $(X^n)_{n\in Z}$ of A, together with maps $d^n : X^n \to X^{n+1}$ such that $d^{n+1}d^n = 0$ for all $n \in \mathbf{Z}$. We are interested only in *bounded complexes*, i.e. in complexes X such that $X^n = 0$ for almost all n. In the sequel by the word *complex* we always mean a bounded one.

A *morphism* f of complexes from $X^\cdot$ to $Y^\cdot$ is a collection of

maps $f^n : X^n \to Y^n$ which commute with the maps of complexes:

$$f^{n+1} d_X^n = d_Y^n f^n$$

for all n. Two maps are said to be *homotopic* if there is a collection of maps $k = (k^n)$, $k^n : X^n \to Y^{n-1}$ such that

$$f^n - g^n = d_Y^{n-1} k^n + k^{n+1} d_X^n$$

for all n.

We define the *homotopy category* $K^b(A)$ to be the category whose objects are complexes of objects of A, and whose morphisms are homotopy equivalence classes of morphisms of complexes.

Let us denote by T the operation of shifting one place to the left and changing the sign of the differential, i.e. $T(X^\cdot)^p = X^{p+1}$ and $d_{T(X)} = -d_X$.

Let $u : X^\cdot \to Y^\cdot$ be a morphism. The *mapping cone*, or simply the *cone* of u, denoted $C(u)$ or $C(X^\cdot \to Y^\cdot)$, is defined to be the complex $Z^\cdot = T(X^\cdot) \oplus Y^\cdot$ where the differential operator is given by the matrix

$$\begin{bmatrix} T(d_X) & T(u) \\ 0 & d_Y \end{bmatrix}$$

There are natural morphisms $v : Y^\cdot \to Z^\cdot$ and $w : Z^\cdot \to T(X^\cdot)$. Moreover, there is a long exact cohomology sequence:

$$(1.1) \qquad \dots \to H^i(X^\cdot) \to H^i(Y^\cdot) \to H^i(Z^\cdot) \to H^{i+1}(X^\cdot) \to \dots$$

By a *quasi-isomorphism* we mean a morphism $f : X^\cdot \to Y^\cdot$ in $K^b(A)$ which induces an isomorphism in cohomology (note that two homotopic morphisms induce the same maps in cohomology).

We can now define the *derived category* $D^b(A)$ of A. The objects in $D^b(A)$ are the same as in $K^b(A)$, i.e. $ObD^b(A) = ObK^b(A)$.

A *morphism* in $D^b(A)$ $f : X^\cdot \to Y^\cdot$ is given by definition by a couple of morphisms in $K^b(A), g : Z^\cdot \to X^\cdot$ and $h : Z^\cdot \to Y^\cdot$, where Z is a third complex and g is a quasi isomorphism. A morphism in $D^b(A)$ is not, in general, a real morphism of complexes. Note that by the previous definition it is clear that a quasi-isomorphism is invertible in $D^b(A)$. It follows that a morphism in $D^b(A)$ can be obtained as a composition of the inverse (in $D^b(A)$) of a quasi-isomorphism followed by a real morphism of complexes.

We will say that two complexes X and Y are *equivalent* if they are isomorphic in the category $D^b(A)$ and we will write $X \sim Y$. If two complexes are equivalent, their cohomology groups are isomorphic, but in general there is no quasi-isomorphism between them.

Let $f : X^\cdot \to Y^\cdot$ be a morphism in $D^b(A)$, that is a quasi-isomorphism $g : Z^\cdot \to X^\cdot$ and a morphism $h : Z^\cdot \to Y^\cdot$; the cone $C(h)$ is a complex whose equivalence class in $D^b(A)$ depends only on the morphism f; it is called the *cone* of f, and denoted $C(f)$, or $C(X^\cdot \to Y^\cdot)$. There is a natural morphism in $D^b(A)$ from $Y^\cdot$ to $C(h)$ and from $C(h)$ to $T(X^\cdot)$.

A *triangle* in $D^b(A)$ is a sextuple $(X^\cdot, Y^\cdot, Z^\cdot, u, v, w)$, where $X^\cdot, Y^\cdot, Z^\cdot$ are objects of $D^b(A)$, $u : X^\cdot \to Y^\cdot, v : Y^\cdot \to Z^\cdot$ and $w : Z^\cdot \to T(X^\cdot)$ are morphisms, such that $Z^\cdot$ is isomorphic to the cone $C(u)$ and v and w identify (up to isomorphisms) with the natural maps $Y^\cdot \to C(u)$ and $C(u) \to T(X^\cdot)$. It is not hard to see that there is for the given triangle a long exact sequence (1.1).

The objects of the category A can be considered as objects of $K^b(A)$, or $D^b(A)$, in an obvious way: if $X \in Ob(A)$, it identifies with the complex X such that $X^0 = X$ and $X^n = 0$ for $n \neq 0$. Given an exact sequence $0 \to X \to Y \to Z \to 0$ in A, it is clear that Z is equivalent in $D^b(A)$ to the cone $C(X \to Y)$.

Of course, if we start with not necessarily bounded complexes, the same procedure gives categories $K(A)$ and $D(A)$, containing

$K^b(A)$ and respectively $D^b(A)$ as full subcategories.

2. Derived Functors.

Let A and B two abelian categories. A (covariant) functor $F : K^b(A) \to K(B)$ is called a *∂-functor* if it commutes with the shift operator T and it takes triangles to triangles. Any additive functor $F : A \to B$ can be extended naturally to a ∂-functor from $K^b(A)$ to $K(B)$, which will be denoted with the same letter. A ∂-functor cannot be extended, in general, in a trivial way to $D^b(A)$, because it may not take quasi-isomorphisms to quasi-isomorphisms. The *right derived functor of* F, when it exists, is a ∂-functor $RF : D^b(A) \to D(B)$ close to F. We refer to [Ha] pg. 50 for the definition of RF. Here we limit ourselves to state the next theorem, which gives the existence and the construction of RF in most cases.

By a *triangulated subcategory* of $K^b(A)$ we mean a subcategory S, closed with respect to the shift operator T, such that if $f : X^\cdot \to Y^\cdot$ is a morphism in S, the cone $C(f)$ and the natural morphisms $Y^\cdot \to C(f)$ and $C(f) \to T(X^\cdot)$ belong to S.

THEOREM 2.1. *Let A and B be abelian categories and $F : K^b(A) \to K(B)$ a ∂-functor. Suppose there is a triangulated subcategory S of $K^b(A)$ such that:*

1) Every object of $K^b(A)$ admits a quasi-isomorphism into an object of S;

2) If $I^\cdot \in ObS$ is exact (i.e. $H^i(I^\cdot) = 0$ for all i) then $F(I^\cdot)$ is also exact.

Then F has a right derived functor RF. Moreover if $X^\cdot$ is any object of $D^b(A)$, quasi-isomorphic to an object $I^\cdot$ of $S, RF(X^\cdot)$ is equivalent to $F(I^\cdot)$ in $D(B)$.

When RF exists, we put $R^nF(X^\cdot) = H^n(RF(X^\cdot))$.

An object X of A is said to be *F-acyclic* if $R^nF(X) = 0$ for all $n \neq 0$. We say that F *has finite cohomological dimension* if there is a positive integer n such that $R^iF(X) = 0$ for all $X \in Ob(A)$ and all $i > n$.

THEOREM 2.2. *Under the assumptions of the theorem 2.1, suppose moreover that F has finite cohomological dimension. If $X^\cdot$ is a complex of F-acyclic objects of A (i.e. X^i is F-acyclic for all i), one has $RF(X^\cdot) = F(X^\cdot)$.*

The proof is in [Ha], pg. 58.

EXAMPLE 2.3. The derived functor of Hom.

If $X^\cdot$ and $Y^\cdot$ are complexes of objects of A, we define a complex $\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot)$ by

$$\mathrm{Hom}^n(X^\cdot, Y^\cdot) = \prod_p \mathrm{Hom}_A(X^p, Y^{p+n})$$

and

$$d^n = \prod (d^p_X + (-1)^{n+1} d^{p+n}_Y)$$

Under this definition the n-cycles of the complex $\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot)$ are exactly the morphisms of complexes of $X^\cdot$ to $T^n(Y^\cdot)$, and the n-boundaries are those homotopic to zero, so that:

$$H^n(\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot) = \mathrm{Hom}_{K^b(A)}(X^\cdot, T^n(Y^\cdot))$$

$\mathrm{Hom}(X^\cdot, Y^\cdot)$ is contravariant in $X^\cdot$ and covariant in $Y^\cdot$ so that we obtain a functor

$$\mathrm{Hom}^\cdot : K^b(A)^0 \times K^b(A) \to K(Ab)$$

where $K^b(A)^0$ is the category opposite to $K^b(A)$ and Ab is the category of abelian groups.

Suppose now that A *has enough injectives (projectives)*, i.e. every bounded complex of objects of A admits a quasi-isomorphism to a bounded complex of injective (projectives) objects. (These definition are more restrictive than the usual ones but more useful for our purposes).

Take S to be the subcategory of $K^b(A)$ of bounded complexes of injective objects. For a fixed $X^\cdot \in ObK^b(A)$, S satisfies the assumptions of theorem 2.1 for the functor

$$\mathrm{Hom}^\cdot(X^\cdot, \cdot) : K^b(A) \to K(Ab)$$

by [Ha], Lemma 6.2 pg. 64, hence this functor has a right derived functor, which is, in its turn, functorial in $X^\cdot$, so that we obtain a ∂-functor

$$R_{II}\mathrm{Hom}^\cdot : K^b(A)^0 \times D^b(A) \to K(Ab)$$

Again by the lemma quoted above, this functor is exact in the first variable, giving a trivial right derived functor

$$R_I R_{II}\mathrm{Hom}^\cdot : D^b(A)^0 \times D^b(A) \to D(Ab)$$

On the other hand, as A has enough projectives, we see that there is also a functor

$$R_{II} R_I\mathrm{Hom}^\cdot : D^b(A)^0 \times D^b(A) \to D(Ab)$$

Here the main point is that ***when A has enough injectives and projectives, the above functors agree***, defining a single functor, which is denoted by $R\mathrm{Hom}^\cdot$.

The theorem 6.4 in [Ha] can be formulated, for our purposes, in the following way:

THEOREM 2.4. *Let A be an abelian category having enough injectives. Then for any $X^\cdot, Y^\cdot$ in $D^b(A)$*

$$H^i(R\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot)) = \mathrm{Hom}_{D(A)}(X^\cdot, T^i(Y^\cdot))$$

PROPOSITION 2.5. (lemma 1.6 in [K]) *Let A be an abelian category having enough injectives and projectives. Let $K^\cdot, L^\cdot$ be bounded complexes of objects of A. Suppose that $\mathrm{Ext}^p_A(K^i, L^j) = 0$ for $p > 0$ and all i, j. Then*

$$\mathrm{Hom}_{D^b(A)}(K^\cdot, L^\cdot) = \mathrm{Hom}_{K^b(A)}(K^\cdot, L^\cdot)$$

In order to prove the proposition, consider the following

LEMMA 2.6. *Let $f : X^\cdot \to Y^\cdot$ be a morphism of bounded complexes of objects of A. Assume*

i) $X^\cdot$ is exact

ii) $\mathrm{Ext}^p_A(X^i, Y^j) = 0$ for $p > 0$ and all i, j

Then f is homotopic to zero.

(The proof is well known)

Proof of proposition 2.5.

Now consider the functor $F = \mathrm{Hom}(-, L^\cdot) : (K^b(A))^0 \to K(Ab)$ and denote by $S \subset K(A)$ the triangulated subcategory of $K^b(A)$ consisting of complexes $X^\cdot$ satisfying $\mathrm{Ext}^p(X^i, L^j) = 0$ for $p > 0$ and all i, j. If $X^\cdot \in S$ is exact, by the lemma 2.6 we have for all i $H^i F(X^\cdot) = H^i(\mathrm{Hom}(X^\cdot, L^\cdot) = \mathrm{Hom}_{K^b(A)}(X^\cdot, T^i(L^\cdot)) = 0$ that is $F(X^\cdot)$ is exact too. Moreover S contains all the complexes in $D^b(A)$ formed by projective objects. By he theorem 2.2 $R\mathrm{Hom}(K^\cdot, L^\cdot) = \mathrm{Hom}(K^\cdot, L^\cdot)$.

By the theorem 2.4 $\mathrm{Hom}_{D^b(A)}(K^\cdot, L^\cdot) = H^0(R\mathrm{Hom}(K^\cdot, L^\cdot)) = H^0(\mathrm{Hom}(K^\cdot, L^\cdot)) = \mathrm{Hom}_{K^b(A)}(K^\cdot, L^\cdot)$.

3. The theorem of Beilinson.

Let X be a compact algebraic manifold. We denote by $K^b(X)$ and $D^b(X)$ the bounded homotopy category and the bounded derived category of the category of coherent sheaves on X. $K^b(X)$ and $D^b(X)$ have the same objects; a morphism in $K^b(X)$ is an actual morphism of complexes $F^\cdot \to G^\cdot$, while a morphism in $D^b(X)$ it is not, in general, and it will be denoted with a dotted arrow $F^\cdot \dashrightarrow G^\cdot$. An equivalence of objects or morphisms in $K^b(X)$ will be denoted by $=$, in $D^b(X)$ by $\sim$; thus $F^\cdot = G^\cdot$ means that $F^\cdot$ and $G^\cdot$ are isomorphic in $K^b(X)$, $F^\cdot \sim G^\cdot$ that $F^\cdot$ and $G^\cdot$ are isomorphic in $D^b(X)$.

LEMMA 3.1. [B] *Let $X = \mathbf{P}^n(\mathbf{C})$. Then $\mathrm{Ext}^p_X\ (\Omega^i(i), \Omega^j(j)) = 0$ for $p > 0$ and all i, j.*

Let $X = \mathbf{P}^n(\mathbf{C})$, and denote by $p, q : X \times X \to X$ the two projections. For $F, G \in Coh(X)$ let us put $F \boxtimes G := p^* F \otimes q^* G$. The structure sheaf O_Δ of the diagonal $\Delta \subset X \times X$ admits the following left resolution on $X \times X$

$$(*) \qquad 0 \to \Omega^n(n) \boxtimes O(-n) \to \ldots \to \Omega^1(1) \boxtimes O(-1) \to O_{X \times X} \to O_\Delta \to 0$$

Let $E \in Coh(X)$. Tensoring (*) by $q^* E$ gives a resolution of $q^* E|_\Delta$

$$(**) \qquad \underset{M_n}{\underset{\parallel}{0 \to \Omega^n(n) \boxtimes E(-n)}} \xrightarrow{u_n} \ldots \to \underset{M_1}{\underset{\parallel}{\Omega^1(1) \boxtimes E(-1)}}$$

$$\xrightarrow{u_1} \underset{M_0}{\underset{\parallel}{q^* E}} \xrightarrow{u_0} q^* E|_\Delta \to 0$$

thus we can express $q^* R|_\Delta$ as an iterated cone of morphisms in $D^b(X)$:

$$q^* E|_\Delta = C\ldots(C(CM_n \dashrightarrow M_{n-1}) \dashrightarrow M_{n-2}) \dashrightarrow M_{n-3} \dashrightarrow \ldots)$$

Since Rp_* preserves cones of morphisms (in $D^b(X)$), and $Rp_*(q^* E|_\Delta) = E$, we have

(∗∗∗)

$$E \sim C\ldots(C(C(Rp_* M_n \dashrightarrow Rp_* M_{n-1}) \dashrightarrow Rp_* M_{n-2}) \dashrightarrow Rp_* M_{n-3} \ldots)$$

LEMMA 3.2. *Let X be a compact algebraic manifold, and $F, G \in Coh(X)$ with F locally free. Then $Rp_*(F \boxtimes G) \sim G \otimes H^{\cdot}(X, G)$ where the differentials are the zero maps.*

Proof. Let $\mathcal{U}$ be an affine covering of X, and $C^{\cdot} = C^{\cdot}(\mathcal{U}, G)$ the Čech complex of G with respect to $\mathcal{U}$, which is a right finite resolution of G. Since q^* and the tensorization by F are exact functors $F \boxtimes G \sim F \boxtimes C^{\cdot}$ in $D^b(X \times X)$, hence $Rp_*(F \boxtimes G) \sim Rp_*(F \boxtimes C^{\cdot})$. But each individual C^p is p_*-acyclic by [H, prop. 34.2 pag. 149], by Künneth formula we see that $F \boxtimes C^p$ is p_*-acyclic too, thus by the theorem 2.2 $Rp_*(F \boxtimes C^{\cdot}) = p_*(F \boxtimes C^{\cdot}) = F \otimes H^0(X, C^{\cdot})$ with obvious differentials. It is now easy to see that $F \otimes H^0(X, C^{\cdot}) \sim F \otimes H^{\cdot}(X, G)$ in $D^b(X)$ because each complex of vector spaces is isomorphic in the derived category to the complex given by its cohomology with all zero maps q.e.d.

The lemma above gives $Rp_* M_j = Rp_*(\Omega^j(j) \boxtimes E(-j)) = \Omega^j(j) \otimes H^{\cdot}(E(-j))$ (with zero differentials) and by the proposition 2.5 and the lemma 3.1 the morphisms (in $D^b(X)$) in the right side of (∗∗∗) can be lifted to actual morphisms of complexes in $K^b(X)$ allowing us to compute explicitely E as an iterated cone in $K^b(X)$. More precisely, let us cut (∗∗) into short exact sequences

$$0 \to H_j \xrightarrow{f_j} M_j \xrightarrow{g_j} H_{j-1} \to 0$$

where $H_j = \mathrm{Ker}\, u_j = Im\ u_{j+1}$, so that $H_{j-1} = C(H_j \dashrightarrow M_j)$ and $u_j = f_{j-1} \circ g_j$.

Then $Rp_* H_{j-1} \sim C(Rp_* H_j \xrightarrow{Rp_* f_j} Rp_* M_j)$.

For $s = n, n-1, \ldots, 0$ we define inductively a complex $R_s^{\cdot}$ such that

a) $R_s^k = \bigoplus_{j-i=s-k} X_j^i$ where $X_j^{\cdot} = Rp_* M_j$

b) $R_s^{\cdot} \sim Rp_* H_{s-1}$ in $D^b(X)$

in the following way

$$R_n^{\cdot} = Rp_* M_n = Rp_* H_{n-1}.$$

Suppose $R_{s+1}^{\cdot}$ defined; from $H_{s-1} = C(H_s \to M_s)$ we obtain

$$Rp_* H_{s-1} \sim C(Rp_* H_s \xrightarrow{Rp_* f_s} Rp_* M_s)$$

Since $Rp^* H_s \sim R_{s+1}^{\cdot}$, the map $Rp_* f_s$ gives a map $R_{s+1}^{\cdot} \dashrightarrow Rp_* M_s$ in $D^b(X)$, which is induced by an actual map $v_s : R_{s+1}^{\cdot} \to Rp_* M_s$ (by prop. 2.5 and lemma 3.1); we put $R_s^{\cdot} = C(R_{s+1}^{\cdot} \xrightarrow{v_s} Rp_* M_s)$ (the standard mapping cone in $K^b(X)$), which verifies a) and b) by construction.

Moreover there is a canonical map $Rp_* M_s \xrightarrow{\phi_s} R_s$ and it is clear that $v_s \circ \phi_{s+1} \sim Rp_* u_s : Rp_* M_{s+1} \to Rp_* M_s$. But the complexes $Rp_* M_{s+1}$ and $Rp_* M_s$ coincide with their cohomology complexes, so that actually $v_s \circ \phi_{s+1} = Rp_* u_s$ in $K^b(X)$.

We have $E \sim R_0^{\cdot}$ so at the end we obtain

BEILINSON THEOREM *Let $X = \mathbf{P}^n$, denote by $p, q : X \times X \to X$ the two projections and by Δ the diagonal in $X \times X$. For $\mathcal{F}, \mathcal{G} \in Coh(X)$ let us put $\mathcal{F} \boxtimes \mathcal{G} := p^* \mathcal{F} \otimes q^* \mathcal{G}$. Let $\mathcal{F} \in Coh(X)$, $t \in \mathbf{Z}$. Then there exists a complex of vector bundles $L^{\cdot}(t)$ on X such that:*

1) $H^k(L^\cdot(t)) = \begin{cases} \mathcal{F}(t) & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$

2) $L^k(t) = \bigoplus_{j+k=i} X_j^i(t)$, $X_j^i(t) = \Omega^j(j)^{\oplus h^i(\mathcal{F}(t-j))}$

3) a) the maps $v_j^i(t,s) : X_j^i(t) \to X_{j-s}^{i-s+1}(t)$ $(s \in \mathbf{Z})$ *induced by the differentials* $L^k \to L^{k+1}$ *are zero for* $s \leq 0$

b) the maps $v_j^i(t,1)$ *agree with the natural maps*

$$R^i p_*(\Omega^j(j) \boxtimes \mathcal{F}(t-j)) \to R^i p_*(\Omega^{j-1}(j-1) \boxtimes \mathcal{F}(t-j+1))$$

coming from the exact sequence (tensored by $O(t)$*)*

$$0 \to \Omega^n(n) \boxtimes \mathcal{F}(-n) \xrightarrow{u_n} \ldots \to$$

$$\to [\Omega^1(1) \boxtimes \mathcal{F}(-1)] \xrightarrow{u_1} q^*\mathcal{F} \xrightarrow{u_0} q^*\mathcal{F}|_\Delta \to 0$$

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