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MATHEMATICAL INSTANTON BUNDLES ON PROJECTIVE SPACES: AN ALGORITHMIC APPROACH

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ABSTRACT. We study some computational aspects of the theory of instanton bundles on projective spaces. In particular we give algorithms to compute the tangent space and the local analytic equations for their moduli.

0 Introduction.

Let

$$\pi\colon \mathbf{P}^3(\mathbf{C}) \to \mathbf{P}^1(\mathbf{H}) \simeq S^4$$

be the twistor fibration. Let A be a Sp(1)-connection over the sphere S^4 with Chern class $k \in H^4(S^4, \mathbf{Z}) \simeq \mathbf{Z}$ and let

$$J(A) = \int_{S^4} \|F_A\|^2$$

be the Yang-Mills functional, where we denote by F_A the curvature of A.

The well known ADHM construction ([A.W.]) gives a 1-1 correspondence between the absolute minima A of J and the holomorphic (or anti-holomorphic) vector bundles on $\mathbf{P}^3$ with complex structure induced by π^*A. These holomorphic vector bundles E of rank 2 over $\mathbf{P}^3$ are called real instanton bundles and are precisely the cohomology bundles of a monad:

$$0 \to \mathcal{O}(-1)^k \to \mathcal{O}^{2+2k} \to \mathcal{O}(1)^k \to 0, \tag{0.1}$$

such that their restrictions to the fibers of π are trivial (reality condition).

An instanton bundle on $\mathbf{P}^3$ is the cohomology bundle of a monad (0.1), dropping now the reality condition. The study of the moduli space of instanton bundles on $\mathbf{P}^3$ allowed the remarkable complete classification of the absolute minima of the Yang-Mills functional over S^4 ([A.W.]).

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1 Notations and preliminary definitions.

Let V be a complex vector space of dimension $m+1$, and $\mathbf{P}^m := \mathbf{P}^m(V)$ be the projective space of one-dimensional subspaces of V.

We denote by $S := \bigoplus_{d\geq 0} S_d$ the homogeneous coordinate ring of $\mathbf{P}^m$ where $S_d := S^d(V^*)$. Once we fix homogeneous coordinates $x_0, \dots, x_m$, S will be identified with the polynomial ring $\mathbf{C}[x_0, \dots, x_m]$ and S_d with the vector space of homogeneous polynomials of degree d.

Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^m$; we will denote by:

(1) $\mathcal{F}^* = \mathrm{Hom}(\mathcal{F}, \mathcal{O})$ its dual
(2) $\mathrm{H}^i(\mathcal{F}) = \mathrm{H}^i(\mathbf{P}^m, \mathcal{F})$ its cohomology groups
(3) $\mathrm{h}^i(\mathcal{F}) = \dim_{\mathbf{C}} \mathrm{H}^i(\mathcal{F})$.

Finally, $\mathrm{Mat}(r, s; S_d)$ will be the vector space of the $r \times s$ matrices with entries in S_d.

We recall from [O.S.], [S.T.], [A.O.1], [A.O.2] the definition and properties of istanton bundles on $\mathbf{P}^{2n+1}$.

Definition 1.1. *A mathematical instanton bundle with quantum number k is a holomorphic vector bundle E on $\mathbf{P}^{2n+1}$ of rank $2n$ satisfying the following conditions:*

(1) *the Chern polynomial of E is $c_t(E) = \dfrac{1}{(1-t^2)^{-k}}$,*
(2) *$E(q)$ has natural coomology in the range $-2n-1 \leq q \leq 0$ that is $\mathrm{h}^i(E(q)) \neq 0$ for at most one $i = i(q)$.*

By [O.S.], [A.O.1], the Definition 1.1 is equivalent to: E is the coomology bundle of a monad:

$$0 \longrightarrow \mathcal{O}(-1)^k \xrightarrow{A} \mathcal{O}^{2n+2k} \xrightarrow{B^t} \mathcal{O}(1)^k \longrightarrow 0 \qquad \text{(m)}$$

where A, B are matrices in the space $\mathrm{Mat}(k, 2n+2k, S_1)$; the fact that (m) is a monad is equivalent to the following two conditions on A, B:

(1) A, B have rank k at every point of $\mathbf{P}^{2n+1}$,
(2) $AB^t = 0$.

Example 1.2. *([O.S.]) Let $x_0, \dots, x_n, y_0, \dots, y_n$ the homogeneous coordinates on $\mathbf{P}^{2n+1}$, the following pair $(A, B) \in \mathrm{Mat}(k, 2n+2k, S_1)^{\oplus 2}$:*

$$A = \begin{pmatrix} 0 & \dots & 0 & y_n & \dots & y_0 & 0 & \dots & 0 & -x_n & \dots & -x_0 \\ \dots & 0 & y_n & \dots & y_0 & 0 & \dots & 0 & -x_n & \dots & -x_0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & y_n & \dots & y_0 & 0 & \dots & 0 & -x_n & \dots & -x_0 & 0 & \dots \\ y_n & \dots & y_0 & 0 & \dots & 0 & -x_n & \dots & -x_0 & 0 & \dots & 0 \end{pmatrix}$$

Definition 2.1.1. *A pair* $(A+\epsilon A', B+\epsilon B')$ *satisfying (2.1.1), is called a first-order deformation of* (A, B).

The set of pairs (A', B') corresponding to first-order deformations of (A, B), is a vector space, which we denote as $V_{(A,B)}$.

Let $r = 2n + 2k$.

The Lie group $\mathrm{GL}(k) \times \mathrm{GL}(r) \times \mathrm{GL}(k)$ acts on the pairs $(A, B) \in \mathrm{Mat}(k, r; S_1)^{\oplus 2}$ as

$$\begin{array}{ccc} \mathrm{GL}(k) \times \mathrm{GL}(r) \times \mathrm{GL}(k) & \xrightarrow{\rho} & \mathrm{Aut}(\mathrm{Mat}(k, r; S_1)^{\oplus 2}) \\ (Q, P, R) & \longrightarrow & \rho_{(Q,P,R)} \end{array}$$

where

$$\begin{array}{ccc} \mathrm{Mat}(k, r; S_1)^{\oplus 2} & \xrightarrow{\rho_{(Q,P,R)}} & \mathrm{Mat}(k, r; S_1)^{\oplus 2} \\ (A, B) & \longrightarrow & (QA \cdot P^t, RBP^{-1}). \end{array}$$

We recall ([O.S.S.]) that two instanton bundles E and F on $\mathbf{P}^{2n+1}$ are isomorphic if and only if the corresponding pairs of matrices lie in the same orbit under the action ρ.

The action ρ induces in a natural way an action ρ' of the Lie algebra $\mathrm{gl}(k) \times \mathrm{gl}(r) \times \mathrm{gl}(k)$ on $V_{(A,B)}$, given by $\rho'_{(Q,P,R)}(A', B') = (C', D')$, where

$$\rho_{(I+\epsilon Q', I+\epsilon P, I+\epsilon R)}(A + \epsilon A', B + \epsilon B') = (A + \epsilon C', B + \epsilon D') \qquad \mathrm{mod}\, \epsilon^2$$

which gives:

$$\begin{cases} C' = QA + A' + AP^t \\ D' = RB + B' - BP. \end{cases}$$

Set $\mathcal{U} := \mathrm{Mat}(k; \mathbf{C}) \times \mathrm{Mat}(r; \mathbf{C}) \times \mathrm{Mat}(k; \mathbf{C})$; let us define the following subspace of $V_{(A,B)}$:

$$W_{(A,B)} := \{(M, N) \in V_{(A,B)} \mid \exists (X, Z, Y) \in \mathcal{U} \mid M = XA + AZ;\ N = YB - BZ^t\}.$$

Theorem 2.1.2. $\mathrm{H}^1(E \otimes E^*) \simeq \dfrac{V_{(A,B)}}{W_{(A,B)}}$.

Proof. After suitable tensorizations we obtain from (1.2) and (1.4) the following diagram:

Recalling that $H^0(E \otimes E^*) = \mathbf{C}$ since E is simple, the exact sequence (2.1.6) becomes:

$$0 \to \mathbf{C} \to H^1(M) \xrightarrow{\gamma} H^1(S^* \otimes T^*) \to H^1(E \otimes E^*) \to 0. \tag{2.1.7}$$

Considering the cohomology of (2.1.5), we get:

$$H^1(M) \simeq H^1\{(\mathcal{O}(-1)^k \otimes T^*) \oplus (S^* \otimes \mathcal{O}(-1)^k)\}. \tag{2.1.8}$$

Tensoring (1.3) by $\mathcal{O}(-1)^k$ and taking the cohomology we have:

$$0 \to H^0(\mathcal{O}(-1)^k \otimes T^*) \to H^0(\mathcal{O}(-1)^k \otimes \mathcal{O}^r) \to H^0(\mathcal{O}(-1)^k \otimes \mathcal{O}(1)^k) \to$$
$$\to H^1(\mathcal{O}(-1)^k \otimes T^*) \to H^1(\mathcal{O}(-1)^k \otimes \mathcal{O}^r) \to \dots,$$

hence:

$$H^1(\mathcal{O}(-1)^k \otimes T^*) \simeq H^0(\mathcal{O}(-1)^k \otimes \mathcal{O}(1)^k) \simeq H^0(\mathcal{O}^k \otimes \mathcal{O}^k).$$

In a similar way, tensoring (1.1) by $\mathcal{O}(-1)^k$, we get:

$$H^1(S^* \otimes \mathcal{O}(-1)^k) \simeq H^0(\mathcal{O}(1)^k \otimes \mathcal{O}(-1)^k) \simeq H^0(\mathcal{O}^k \otimes \mathcal{O}^k).$$

Thus (2.1.8) becomes:

$$H^1(M) \simeq H^0(\mathcal{O}^k \otimes \mathcal{O}^k)^{\oplus 2}. \tag{2.1.9}$$

After suitable tensorizations from (1.1) and (1.3) we get the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(1)^k \otimes T^* & \longrightarrow & \mathcal{O}(1)^k \otimes \mathcal{O}^r & \xrightarrow{a_1} & \mathcal{O}(1)^k \otimes \mathcal{O}(1)^k & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle a_2} & & \uparrow{\scriptstyle a_4} & & \\
0 & \longrightarrow & \mathcal{O}^r \otimes T^* & \longrightarrow & \mathcal{O}^r \otimes \mathcal{O}^r & \xrightarrow{a_3} & \mathcal{O}^r \otimes \mathcal{O}(1)^k & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S^* \otimes T^* & \longrightarrow & S^* \otimes \mathcal{O}^r & \longrightarrow & S^* \otimes \mathcal{O}(1)^k & \longrightarrow & 0, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.1.10}$$

with exact rows and columns; the morphisms in diagram (2.1.10) are given by:

$$\begin{cases} a_1 = \mathrm{id}_{\mathcal{O}(1)^k} \otimes A^t \\ a_2 = B^t \otimes \mathrm{id}_{\mathcal{O}^r} \\ a_3 = \mathrm{id}_{\mathcal{O}^r} \otimes A^t \\ a_4 = B^t \otimes \mathrm{id}_{\mathcal{O}(1)^k} . \end{cases}$$

Hence, if $\mathcal{M} = \mathrm{Mat}(r,k;S_1) \oplus \mathrm{Mat}(k,r;S_1) \simeq \mathrm{H}^0(K)$:

$$\begin{cases} \mathrm{Mat}(r,r;\mathbf{C}) \xrightarrow{h} \quad \mathcal{M} \\ \qquad C \qquad \to (CA^t, BC) \\ \\ \qquad \mathcal{M} \qquad \xrightarrow{g} \mathrm{Mat}(k,k;S_2) \\ \quad (P,Q) \quad \to (QA^t - BP). \end{cases}$$

From (2.1.14) we get:

$$\mathrm{H}^0(Z) \simeq \mathrm{Ker}\, g = \{(P,Q) \in \mathcal{M} \mid QA^t - BP = 0\}.$$

Let $f\colon \mathrm{H}^1(M) \to \mathrm{H}^0(K)$ be the morphism defined by

$$\begin{array}{ccc} \mathrm{Mat}(k,k;\mathbf{C}) \oplus \mathrm{Mat}(k,k;\mathbf{C}) & \xrightarrow{f} & \mathcal{M} \\ (R,S) & \to & (A^tR, SB). \end{array}$$

We have $\mathrm{Im}\, f \subset \mathrm{H}^0(Z)$, since $(SB)A^t - B(A^tR) = S(BA^t) - (BA^t)R = 0$.
We can now build the following diagram in which the rows are exact, the second column is exact, and the first column is a complex:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathbf{C} & \xrightarrow{\mathrm{id}} & \mathbf{C} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathrm{H}^1(M) & \xrightarrow{\mathrm{id}} & \mathrm{H}^1(M) & \longrightarrow & 0 \\ & & & & f\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^0(\mathcal{O}^r \otimes \mathcal{O}^r) & \xrightarrow{h} & \mathrm{H}^0(Z) & \longrightarrow & \mathrm{H}^1(S^* \otimes T^*) & \longrightarrow & 0. \\ & & & & \downarrow & & j\downarrow & & \\ & & & & \vdots & & \mathrm{H}^1(E \otimes E^*) & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

As a consequence we obtain an exact sequence:

$$0 \to \mathbf{C} \to \mathrm{H}^1(M) \oplus \mathrm{H}^0(\mathcal{O}^r \otimes \mathcal{O}^r) \xrightarrow{\alpha}$$

and from (2.1.11):

$$H^2(S^* \otimes T^*) \simeq H^1(Z),$$

thus:

$$H^2(E \otimes E^*) \simeq H^1(Z).$$

Extending the cohomology sequence (2.1.14) we get:

$$\ldots \to H^0(K)\ (\simeq \mathcal{M}) \xrightarrow{g} H^0(\mathcal{O}(1)^k \otimes \mathcal{O}(1)^k) \xrightarrow{\sigma} H^1(Z) \to H^1(K) \to 0,$$

and, since $H^1(K) \simeq H^1(\mathcal{O}(1)^{kr})^{\oplus 2} = 0$:

$$H^1(Z) \simeq \frac{H^0(\mathcal{O}(1)^k \otimes \mathcal{O}(1)^k)}{\mathrm{Ker}\sigma} = \frac{H^0(\mathcal{O}(1)^k \otimes \mathcal{O}(1)^k)}{\mathrm{Im}g}.$$

Again the isomorphism ϕ in (2.1.17) identifies $\mathrm{Im}\, g$ to $Z_{(A,B)}$; hence:

$$H^2(E \otimes E^*) \simeq \frac{\mathrm{Mat}(k,k;S_2)}{Z_{(A,B)}}.$$

□

3 Algorithmic methods an experimental results.

We describe here the algorithms we implemented for computing cohomology properties of instanton bundles. The programs that carry out these algorithms have been written as macros for Macaulay, a Computer Algebra system for Algebraic Geometry ([B.S.]), installed on a SUN SPARCstation 10. The main tool used by these macros is the computation of the syzygies of the columns of a given matrix.

3.1 Computation of the bases of $H^1(E \otimes E^*)$ *and* $H^2(E \otimes E^*)$.

Let E be an instanton bundle given by the pair (A,B), we want to compute a basis of $H^1(E \otimes E^*) \simeq \dfrac{V_{(A,B)}}{W_{(A,B)}}$.

The algorithm consists of three steps:

(1) construction of a basis of $V_{(A,B)}$
(2) construction of a set of generators of $W_{(A,B)}$
(3) construction of a basis of a complement $U_{(A,B)}$ of $W_{(A,B)}$ in $V_{(A,B)}$.

Then $U_{(A,B)} \simeq H^1(E \otimes E^*)$.

Step (1). Let us consider the matrix $\boxed{T} \in \mathrm{Mat}(k^2, 2kr; S_1)$:

$$\boxed{T} := \left(\boxed{N_1} \quad \boxed{N_2}\right),$$

Step (2). Let the matrix $\boxed{G} \in \mathrm{Mat}(2kr, 2k^2 + r^2; S_1)$ be defined as:

$$\boxed{G} := \begin{pmatrix} \boxed{M_1} & \boxed{M_2} & \boxed{0} \\ \boxed{0} & \boxed{M_3} & \boxed{M_4} \end{pmatrix},$$

where $\boxed{M_1}, \boxed{M_4} \in \mathrm{Mat}(kr, k^2; S_1)$ are:

$$\boxed{M_1} := \left.\begin{pmatrix} A^t & 0 & \dots & 0 \\ 0 & A^t & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A^t \end{pmatrix}\right\} k$$

$$\boxed{M_4} = \boxed{N_1}^t := \left.\begin{pmatrix} B^t & 0 & \dots & 0 \\ 0 & B^t & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & B^t \end{pmatrix}\right\} k$$

$\boxed{M_2} \in \mathrm{Mat}(kr, r^2; S_1)$ is given by:

$$\boxed{M_2} := \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_1 \\ \dots & \dots & \dots & \dots \\ A_k & 0 & \dots & 0 \\ 0 & A_k & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_k \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}A\\A\\A\\A\end{matrix}}\right\} r \\ \\ \\ \\ \\ \end{matrix}$$

and $\boxed{M_3} := (M_{ij}) \in \mathrm{Mat}(kr, r^2; S_1)$ where $M_{ij} = -b_{ij} I_r$, b_{ij} being the entries of the matrix B and I_r is the $r \times r$ unit matrix.

Proposition 3.1.2. *The columns of* $\boxed{G}$ *are a set of generators of the space* $W_{(A,B)}$.

Proof. According with the notation of §2.1, we define the following maps:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\omega} & V_{(A,B)} \\ (X, Z, Y) & \to & (XA + AZ, YB - BZ^t). \end{array}$$

(i) is a consequence of Theorem 2.1.2, Remark 2.1.5 and Proposition 3.1.1.
(ii) is a consequence of (1.1) and Riemann-Roch formula.

3.2 The Kuranishi map.

Let E be an instanton bundle represented by the pair (A, B), and K the Kuranishi map

$$\mathrm{H}^1(\mathrm{End}(E)) \xrightarrow{K} \mathrm{H}^2(\mathrm{End}(E)).$$

By Theorems 2.1.2 and 2.1.6 K can be seen as a map

$$\frac{V_{(A,B)}}{W_{(A,B)}} \xrightarrow{K} \frac{\mathrm{Mat}(k, k; S_2)}{Z_{(A,B)}}.$$

In [A.O.3] it is proved that the Kuranishi map, in the situation we are concerned with, coincides with its quadratic part K_2, and this latter one is given by:

$$K_2((A', B') + W_{(A,B)}) = A'B'^t + Z_{(A,B)}.$$

Let $U_{(A,B)}$ and $U'_{(A,B)}$ be the subspaces of $V_{(A,B)}$ and $\mathrm{Mat}(k, k; S_2)$ respectively isomorphic to $\mathrm{H}^1(E \otimes E^*)$ and $\mathrm{H}^2(E \otimes E^*)$, as in §3.1. Let $\{(A_i, B_i)\}_{i=1,\dots,s}$ be a basis of $U_{(A,B)}$, and $\{X_i\}_{i=1,\dots,s}$ the components in this basis of an element (A', B') in $U_{(A,B)}$. Then:

$$A'B'^t = \left(\sum_{i=1}^{s} X_i A_i\right)\left(\sum_{j=1}^{s} X_j B_j^t\right) = \sum_{i=1}^{s}\sum_{j=1}^{s} X_i X_j A_i B_j^t.$$

Let now $\{C_i\}_{i=1,\dots,N}$ be a basis of $\mathrm{Mat}(k, k; S_2)$ such that $\{C_1, \dots, C_t\}$ is a basis of $U'_{(A,B)}$. Then if

$$A_i B_j^t = \sum_{l=1}^{N} Y_l^{ij} C_l,$$

we have

$$A'B'^t = \sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{l=1}^{N} X_i X_j Y_l^{ij} C_l,$$

hence the projection $(A'B'^t)_{U'_{(A,B)}}$ of $A'B'^t$ onto $U'_{(A,B)}$ is:

$$(A'B'^t)_{U'_{(A,B)}} = \sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{l=1}^{t} X_i X_j Y_l^{ij} C_l.$$

The Kuranishi map is thus known once the coefficients Y_l^{ij} of the t quadratic forms:

$$\sum_{i=1}^{s}\sum_{j=1}^{s} Y_l^{ij} X_i X_j \qquad l = 1, \dots, t.$$

holds. Since for $|\epsilon|$ small the matrices $(A+\epsilon A')$ and $(B+\epsilon B')$ are still everywhere non-singular in $\mathbf{P}^{2n+1}$, the pair $(A+\epsilon A', B+\epsilon B')$ represents the monad of an instanton bundle. Though the condition (3.3.1) is very strong, "in practice" it is satisfied much more often than it might be expected.

Starting with a pair (A,B) representing an already known instanton bundle E, it is possible to implement an easy algorithm which computes, for a basis $\{(A'_j, B'_j)\}$ of $U_{(A,B)} \simeq \mathrm{H}^1(E \otimes E^*)$ all the products $A'_j B'^t_j$; each pair (A'_j, B'_j) such that $A'_j B'^t_j = 0$ produces a new instanton, represented by the pair $(A+\epsilon A'_j, B+\epsilon B'_j)$.

Example 3.3.1. *Let x_0, x_1, y_0, y_1 be homogeneous coordinates on $\mathbf{P}^3$. Starting from the istanton bundle of Example 1.2, with $k=3$, iterating twice the process stated before, we get the following pair of matrices:*

$$A = \begin{pmatrix} 0 & -x_0 & y_1 & y_0 & 0 & 0 & -x_1 & -x_0 \\ -x_0 & y_1 & -x_0+y_0 & 0 & 0 & -x_1 & -x_0 & 0 \\ y_1 & y_0 & 0 & 0 & -x_1 & -x_0 & 0 & x_0 \end{pmatrix}$$

$$B = \begin{pmatrix} x_0 & -x_1 & 0 & 0 & y_0 & y_1 & -x_0 & 0 \\ 0 & x_0 & x_1 & 0 & 0 & -x_0+y_0 & y_1 & -x_0 \\ -x_0 & 0 & x_0 & x_1 & 0 & 0 & y_0 & y_1 \end{pmatrix},$$

which represents an instanton bundle E satisfying $\mathrm{H}^0(E(1)) = 0$.

In the same way we find several examples of instanton bundles E on $\mathbf{P}^3, \mathbf{P}^5, \mathbf{P}^7$ with quantum numbers $k = 4, 5, 6\ldots$, and $\mathrm{H}^0(E(1)) = 0$.

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