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The Horrocks bundles of rank three on $\mathbb{P}^5$

In memoria di Franco Tricerri e della sua famiglia

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There are only few examples of indecomposable vector bundles of small rank on the complex projective space $\mathbb{P}^n$. The only known indecomposable rank 2 bundles on $\mathbb{P}^4$ are the Horrocks-Mumford bundle and its pullbacks under a finite morphism $\pi : \mathbb{P}^4 \to \mathbb{P}^4$. Moreover these 2-bundles on $\mathbb{P}^4$ are stable and the families obtained by pulling back the Horrocks-Mumford bundle under all the finite morphisms of fixed degree are invariant under small deformations [DS]. No indecomposable 2-bundle is known on $\mathbb{P}^5$.

Horrocks defined [Hor 2] a stable 3-bundle E on $\mathbb{P}^5$, called the *parent bundle*. Decker, Manolache and Schreyer proved that every small deformation of the parent bundle can be obtained by the action of an automorphism of $\mathbb{P}^5$ [DMS].

Horrocks also showed how to modify the parent bundle in order to obtain some 3-bundles $E_{\alpha,\beta,\gamma}$ (which are a particular case of the *relation bundles* defined in this paper) depending on nonnegative integers $\alpha \leqq \beta \leqq \gamma$ satisfying $\alpha + \beta < \gamma$.

The main goal of this paper is the study of the small deformations of $E_{\alpha,\beta,\gamma}$.

More precisely let us take into account the following diagram

$$\begin{array}{ccc} \mathbb{C}^6\backslash 0 & \xrightarrow{\omega} & \mathbb{C}^6\backslash 0 \\ \downarrow\eta & & \downarrow\eta \\ \mathbb{P}^5 & & \mathbb{P}^5 \end{array} \tag{0.1}$$

where in a suitable system of coordinates ω is given by six homogeneous polynomials $f_1, \ldots, f_6$ without common zeroes of degree

$$\gamma - \alpha,\quad \gamma - \beta,\quad \gamma + \alpha + \beta,\quad \gamma + \alpha,\quad \gamma + \beta,\quad \gamma - \alpha - \beta\,.$$

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Horrocks proved that $\omega^*\eta^*E$ descends to a vector bundle $E_{\alpha,\beta,\gamma}$ on $\mathbb{P}^5$, so that

$$\eta^*E_{\alpha,\beta,\gamma} \simeq \omega^*\eta^*E.$$

Of course $E_{\alpha,\beta,\gamma}$ depends on ω but for simplicity we omit this fact in the notations. The *parent bundle* correspond to $\alpha = \beta = 0$, $\gamma = 1$ and its pullbacks, under a finite morphism $\omega' : \mathbb{P}^5 \to \mathbb{P}^5$ of degree d^5, correspond to $\alpha = \beta = 0$, $\gamma = d$. We refer to bundles obtained from the diagram (0.1) with the construction we mentioned above as *bundles coming as pullback over* $\mathbb{C}^6 \setminus 0$. The reader should bare in mind that only in the case $\alpha = \beta = 0$ the map ω descends to $\omega' : \mathbb{P}^5 \to \mathbb{P}^5$.

We prove that the family of bundles $E_{\alpha,\beta,\gamma}$, which Horrocks constructed by pulling back the parent bundle over $\mathbb{C}^6 \setminus 0$, is invariant under small deformations if and only if $\alpha = \beta = 0$ (see corollary 3.11). In order to construct all the small deformations of the bundles $E_{\alpha,\beta,\gamma}$ obtained by pulling back the parent bundle over $\mathbb{C}^6 \setminus 0$, we proceed as follows. Let $Q_{\alpha,\beta,\gamma}$ be obtained by pulling back over $\mathbb{C}^6 \setminus 0$ the quotient bundle Q on $\mathbb{P}^5$ (we call $Q_{\alpha,\beta,\gamma}$ a weighted quotient bundle). Let $\mathscr{W} = \mathcal{O}(\alpha) \oplus \mathcal{O}(\beta) \oplus \mathcal{O}(-\alpha-\beta)$, $\mathscr{H} = \mathscr{W} \oplus \mathscr{W}^*$. A *relation bundle* $E_{\alpha,\beta,\gamma}$ is defined as the cohomology bundle of a monad

$$\mathcal{O}(-\gamma) \to B_{\alpha,\beta,\gamma} \to \mathcal{O}(\gamma)$$

where $B_{\alpha,\beta,\gamma}$ is, in its turn, the cohomology bundle of a monad

$$Q_{\alpha,\beta,\gamma}(-1) \to \wedge^2\mathscr{H} \otimes \mathcal{O} \to Q^*_{\alpha,\beta,\gamma}(1).$$

The $E_{\alpha,\beta,\gamma}$ coming as pullbacks are relation bundles.

Our main results are as follows:

Theorem A. *Let $E^0_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6 \setminus 0$. Every small deformation of $E^0_{\alpha,\beta,\gamma}$ is a relation bundle $E_{\alpha,\beta,\gamma}$. Moreover, the Kuranishi space of $E^0_{\alpha,\beta,\gamma}$ is smooth at $E^0_{\alpha,\beta,\gamma}$.*

Theorem B. *Let $E_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6 \setminus 0$. What follows is equivalent:*

(i) *$E_{\alpha,\beta,\gamma}$ is stable.*

(ii) *$E_{\alpha,\beta,\gamma}$ is simple.*

(iii) $3\gamma - 2\alpha - 4\beta > 0$.

An immediate consequence of theorem A and theorem B is the following

Theorem C. *Let $3\gamma - 2\alpha - 4\beta > 0$. The moduli space of stable 3-bundles with Chern classes $c_1 = c_3 = 0$, $c_2 = 3\gamma^2 + 4\alpha\beta - 4(\alpha+\beta)^2$ is smooth in those points which correspond to relation bundles $E_{\alpha,\beta,\gamma}$ coming as pullback over $\mathbb{C}^6 \setminus 0$; its dimension in those points has been computed in* [AO]. *If $\gamma > 3\alpha + 3\beta$ this dimension is $h^0(\mathscr{H}(\gamma)) - h^0(\operatorname{End}\mathscr{W})$.*

The Chern classes of $E_{\alpha,\beta,\gamma}$ are $c_1 = c_3 = 0$, $c_2 = 3\gamma^2 + 4\alpha\beta - 4(\alpha+\beta)^2$. Schwarzenberger conditions imply that 3-bundles on $\mathbb{P}^5$ with Chern classes $c_1 = c_3 = 0$ can exist only if $c_2 \equiv 0, 3, 8$ or 11 (mod 12) [Hor2].

Theorem D. *Let* $t > 0$, $t \equiv 0, 3, 8$ *or* 11 (mod 12). *There exists a semistable* $E_{\alpha,\beta,\gamma}$ *coming as pullback such that* $c_2(E_{\alpha,\beta,\gamma}) = t$.

Let $t > 0$, $t \equiv 3, 8$ *or* 11 (mod 12). *There exists a stable* $E_{\alpha,\beta,\gamma}$ *coming as pullback such that* $c_2(E_{\alpha,\beta,\gamma}) = t$.

As a consequence of our results we prove the following two theorems about 3-bundles on $\mathbb{P}^5$ by using some elementary number-theoretical arguments.

Theorem E. $\forall N \in \mathbb{N}$, $\forall t \in \mathbb{Z}$, $t \equiv 0, 3, 8$ *or* 11 (mod 12) *there exists a family of non-isomorphic* 3-*bundles on* $\mathbb{P}^5$ *with Chern classes* $c_1 = c_3 = 0$, $c_2 = t$ *of dimension* $\geqq N$.

Theorem F. *Let* $M_{\mathbb{P}^5}(0, t, 0) = X_1 \cup X_2 \cup \ldots \cup X_{n(t)}$ *be the decomposition into irreducible components of the moduli space of stable* 3-*bundles on* $\mathbb{P}^5$ *with Chern classes* $c_1 = c_3 = 0$, $c_2 = t$. *Then* $\limsup_t n(t) = +\infty$.

Theorem E generalizes the analogous result for 2-bundles on $\mathbb{P}^3$ obtained by Hartshorne [Har] and it shows that there is plenty of 3-bundles on $\mathbb{P}^5$. Theorem F generalizes the analogous result for 2-bundles on $\mathbb{P}^3$ obtained by Ein [Ein].

The first draft of this paper [AO] contains more results about weighted lambda-three bundles and weighted nullcorrelation bundles.

The authors benefited from many helpful conversations with W. Decker, N. Manolache and F.O. Schreyer. In particular N. Manolache communicated to us the minimal resolution of the parent bundle (theorem 1.9).

0. Notations and conventions

Let H be a 6-dimensional complex vector space and $\mathbb{P}^5 = \mathbb{P}(H)$ the projective space of lines in H with homogeneous coordinate ring $S = \mathbb{C}[x_1, \ldots, x_6]$.

Let V be an m-dimensional complex vector space, and denote by $\mu_1, \ldots, \mu_{m-1}$ the fundamental weights of $SL(V) \simeq SL(m) = SL(m, \mathbb{C})$. We remind the reader that the irreducible representation of $SL(V)$ with highest weight $\sum a_i \mu_i$ ($a_i \in \mathbb{Z}_{\geq 0}$) can be represented by the Young diagram consisting of $n_1 = a_1 + \cdots + a_{m-1}$ boxes in the first row, and of $n_2 = a_2 + \cdots + a_{m-1}$ boxes in the second row, up to $n_{m-1} = a_{m-1}$ boxes in the $(m-1)$-th row. We denote such representation either by the symbol $V_{\sum a_i \mu_i}$ or $\Gamma^{n_1, \ldots, n_{m-1}} V$. In particular

$$V_{\mu_i} = \Gamma^{\overbrace{1,\ldots,1}^{i\ \text{times}},0,\ldots,0} V \simeq \wedge^i V.$$

If $\mathscr{F}$ is a bundle on $\mathbb{P}^{m-1} = \mathbb{P}(V)$, the bundle $\mathscr{F}_{\sum a_i \mu_i} = \Gamma^{n_1, \ldots, n_{m-1}} \mathscr{F}$ is defined.

If $\mathscr{F} = \bigoplus_{i=1}^{m} \mathcal{O}(d_i)$, then

$$\Gamma^{n_1, \ldots, n_{m-1}} \bigoplus_{i=1}^{m} \mathcal{O}(d_i) \simeq \bigoplus_{j \in J} \mathcal{O}(b_j) \tag{0.2}$$

where J is the set of all the combinations of the d_i's which fill the boxes of the Young diagram with n_i boxes in the i-th row. The indices of the d_i's are strictly increasing in the columns and non decreasing in the rows, and b_j is the sum of all the d_i appearing in the combination j.

We are mainly interested to the two cases:

(1) $V = H$ a 6-dimensional vector space.

(2) $V = W$ a 3-dimensional vector space. Note that in this case

$$\Gamma^{i,0} W = S^i W, \quad \Gamma^{1,1} W = \wedge^2 W = W^*,$$

moreover $\Gamma^{p,q} W = \Gamma^{p,p-q} W^*$ and $\dim \Gamma^{p,q} W = \dfrac{(p+2)(q+1)(p-q+1)}{2}$.

Let $\lambda_1, \lambda_2, \lambda_3$ be the fundamental weights of $Sp(6) = Sp(6, \mathbb{C})$. We denote by H_λ the irreducible representation of $Sp(6)$ with highest weight $\lambda = \sum_{i=1}^{3} a_i \lambda_i$. For example $H_{\lambda_1} \simeq H$, $H_{\lambda_2} \simeq \wedge^2 H / \mathbb{C}$, $H_{\lambda_3} \simeq \wedge^3 H / H$. If $\mathscr{F}$ is a symplectic 6-bundle the bundle $\mathscr{F}_{\sum a_i \lambda_i}$ is naturally defined, for example $\mathscr{F}_{\lambda_2} = \wedge^2 \mathscr{F} / \mathcal{O}$. We will use this notation throughout the paper many times when $\mathscr{F}$ is the bundle $\mathscr{H}$ defined in (2.1).

If W is a complex vector space of dimension 3, then $W \oplus W^*$ has a natural symplectic structure.

We will use Mumford-Takemoto definition of stability.

If $\mathscr{F}$ is a coherent sheaf over a complex space X and $f: X \to S$ is a morphism, we denote by $\mathrm{Quot}_{\mathscr{F}/X/S}$ the *Grothendieck space* which parametrizes the coherent quotient sheaves of $\mathscr{F}$ which are flat over S. We have a projection $\mathrm{Quot}_{\mathscr{F}/X/S} = Z \to S$ and for $s \in S$ we have $Z_s \simeq \mathrm{Quot}_{\mathscr{F}_s/X_s}$.

If E is a vector bundle on a compact complex space X, a *Kuranishi space* Z exists which is the base for the *versal deformation* of E. Let $z_0 \in Z$ be the point corresponding to E. Z is equipped with a universal family and the germ (Z, z_0) is unique up to automorphisms. The same bundle can appear many times in the versal deformation but only once does the E itself appear in a neighborhood of z_0.

1. Some known results about bundles on $\mathbb{P}^5$

Let G be a connected, simply connected, semisimple complex Lie group and let ϕ be the set of the roots of G. Let $\Delta = \{\alpha_1, \ldots, \alpha_k\}$ be a fundamental system of roots. We have the Cartan decomposition

$$\operatorname{Lie} G = \mathscr{G}_0 \oplus \sum_{\alpha \in \phi^-} \mathscr{G}_\alpha \oplus \sum_{\alpha \in \phi^+} \mathscr{G}_\alpha .$$

Let $\phi^+(i) = \{\alpha \in \phi^+ \mid \alpha = \sum n_j \alpha_j \text{ with } n_i = 0\}$ and let $P(\alpha_i) \subset G$ be the parabolic subgroup such that $\operatorname{Lie} P(\alpha_i) = \mathscr{G}_0 \oplus \sum_{\alpha \in \phi^-} \mathscr{G}_\alpha \oplus \sum_{\alpha \in \phi^+(i)} \mathscr{G}_\alpha$. Then $G/P(\alpha_i)$ is a rational homogeneous manifold with $\operatorname{Pic} = \mathbb{Z}$.

Let $\{\nu_1, \ldots, \nu_k\}$ be the fundamental weights with respect to Δ.

We will apply this construction to the cases

(i) $G = SL(6)$, $\Delta = \{\beta_1, \ldots, \beta_5\}$, $SL(6)/P(\beta_1) \simeq \mathbb{P}^5$; the reductive factor in the Levi decomposition of $P(\beta_1)$ is isomorphic to $SL(5) \cdot \mathbb{C}^*$. We denote in this case by $\{\mu_1, \ldots, \mu_5\}$ the fundamental weights.

(ii) $G = Sp(6)$, $\Delta = \{\sigma_1, \sigma_2, \sigma_3\}$, $Sp(6)/P(\sigma_1) \simeq \mathbb{P}^5$; the reductive factor in the Levi decomposition of $P(\sigma_1)$ is isomorphic to $Sp(4) \cdot \mathbb{C}^*$. We denote in this case by $\{\lambda_1, \lambda_2, \lambda_3\}$ the fundamental weights.

Let $\varrho(\nu)$ be the irreducible representation of $P(\alpha_i)$ whose restriction to the reductive factor has highest weight $\nu = \sum n_j \nu_j$ with $n_j \geqq 0$ for $j \neq i$. Let E^ν be the homogeneous vector bundle over $G/P(\alpha_i)$ associated to $\varrho(\nu)$.

The *quotient bundle* Q on $\mathbb{P}^5 = \mathbb{P}(H)$, is defined by the Euler sequence

$$0 \to \mathcal{O}(-1) \to H \otimes \mathcal{O} \to Q \to 0 .$$

The bundle Q, as well as Q^*, is stable and $SL(6)$-invariant, precisely

$$Q \simeq E^{\mu_5}, \quad Q^* \simeq E^{\mu_2 - \mu_1} .$$

Remember also $\mathcal{O}(t) \simeq E^{t\mu_1} \forall t \in \mathbb{Z}$.

We list now some cohomological lemmas that are applications of Bott theorem [Bo] which will be used in the rest of the paper. For more details see [AO].

Lemma 1.1. $H^1(\operatorname{End} \wedge^2 Q(t)) = H^1(\operatorname{End} Q(t)) = \begin{cases} 0 & \textit{for } t \neq -1, \\ H & \textit{for } t = -1 . \end{cases}$

Lemma 1.2. $H^1(\wedge^2 Q \otimes \wedge^4 Q^*(t)) = 0 \ \forall t \in \mathbb{Z}$.

Lemma 1.3.

$$H^0(\wedge^4 Q \otimes \wedge^2 Q^*) = H_{\mu_4},$$

$$H^0(\wedge^4 Q \otimes \wedge^2 Q^*(t)) = H_{(t-1)\mu_1+\mu_2+\mu_3} \oplus H_{t\mu_1+\mu_4} \quad \textit{for } t \geqq 1$$

(*all the previous groups are zero for* $t < 0$),

$$H^1(\wedge^4 Q \otimes \wedge^2 Q^*(t)) = \begin{cases} 0 & \textit{for } t \neq -1, \\ \wedge^3 H & \textit{for } t = -1, \end{cases}$$

$$H^0(\wedge^4 Q(t)) = H_{t\mu_1+\mu_2} \quad \textit{for } t \geqq 0.$$

Let N be a nullcorrelation bundle on $\mathbb{P}^5$ corresponding to a nondegenerate symplectic form $\psi \in \wedge^2 H^*$ (see [DMS], §1 for details). The bundle N is symplectic, hence $\wedge^2 N$ has a distinguished nowhere vanishing section $\mathcal{O} \xrightarrow{s} \wedge^2 N$ whose cokernel is a rank 5 bundle.

Definition 1.4. A *lambda-three bundle* is the bundle $\wedge^2 N/\mathcal{O}$ for some nullcorrelation bundle N.

A lambda-three bundle B is stable and orthogonal, and has a resolution

$$0 \to \wedge^4 Q^* \xrightarrow{f} \wedge^2 Q^* \to B \to 0 \tag{1.1}$$

where f is defined by contraction with the symplectic form $\psi \in \wedge^2 H^*$.

The moduli space of lambda-three bundles is naturally isomorphic to the moduli space of nullcorrelation bundles, which is the space of nondegenerate symplectic forms $\psi \in \wedge^2 H^*$ (up to a scalar multiple). Moreover, any small deformation of a lambda-three bundle is again a lambda-three bundle. As $\wedge^4 Q^* = Q(-1)$, any lambda-three bundle is the cohomology bundle of a monad

$$Q(-1) \to \wedge^2 H \otimes \mathcal{O} \to Q^*(1). \tag{1.2}$$

The symplectic form $\psi \in \wedge^2 H^*$ induces a natural action of $Sp(6)$ on $\mathbb{P}^5$, such that

$$\mathbb{P}^5 \simeq Sp(6)/P(\sigma_1)$$

and N and B are naturally isomorphic to $E^{\lambda_2-\lambda_1}$ and $E^{\lambda_3-\lambda_1}$ respectively. Hence from Bott theorem for $Sp(6)$ [Bo] we obtain easily

Lemma 1.5.

$$H^0(B(t)) = H_{(t-1)\lambda_1+\lambda_3} \quad \textit{for } t \geqq 1,$$

$$H^i(B(t)) = 0 \quad \textit{for} \quad 1 \leqq i \leqq 4$$

$\forall t \in \mathbb{Z}$ *with the only exceptions* $H^2(B(-2)) = H^3(B(-4)) = \mathbb{C}$,

$$H^1(\operatorname{End} B(t)) = 0$$

$\forall t \in \mathbb{Z}$ *with the only exceptions* $H^1(\mathrm{End}\,B) = H^1(\wedge^2 B) = H_{\lambda_2}$,

$$H^1(\mathrm{End}\,B(-1)) = H^1(\wedge^2 B(-1)) = H.$$

The minimal resolution of B is by [DMS]

$$(1.3)\quad 0 \to \mathcal{O}(-4) \to H \otimes \mathcal{O}(-3) \to H_{\lambda_2} \otimes \mathcal{O}(-2) \to H_{\lambda_3} \otimes \mathcal{O}(-1) \to B \to 0$$

and is $Sp(6)$-invariant.

Lemma 1.6.

$$H^1(B \otimes \wedge^2 Q^*(t)) = 0 \quad \text{for } t \neq 0, \quad H^1(B \otimes \wedge^2 Q^*) = H_{\lambda_2},$$
$$H^2(B \otimes \wedge^2 Q^*(t)) = 0 \quad \text{for } t \neq -1, \quad H^2(B \otimes \wedge^2 Q^*(-1)) = H_{\lambda_3}.$$

Proof. Straightforward computation from (1.3).

Definition 1.7. Let B be a lambda-three bundle. A *parent bundle E* is defined as the cohomology of a monad

$$\mathcal{O}(-1) \to B \to \mathcal{O}(1).$$

The existence of parent bundles was shown by Horrocks in [Hor2] (see also remark 2.13). All parent bundles are isomorphic under the action of $SL(6)$.

It is easy to check that $c_1(E) = c_3(E) = 0$, $c_2(E) = 3$ and that E is stable [DMS]. As in [Hor2] we can split $H = W \oplus W^*$ so that the symmetry group of E is $SL(W) \rtimes \mathbb{Z}_2$, in particular E is $SL(W)$-invariant. We remark that in Horrocks notations [Hor2]:

$$[m] \simeq [m]' \simeq \Gamma^{2m,m} W \quad \text{for } m \geq 0,$$
$$[p, -q] \simeq \Gamma^{p+q,q} W \oplus \Gamma^{p+q,q} W^* \quad \text{for } p \neq q,\ p, q \geq 0.$$

Lemma 1.8.

$$H^2(E(-2)) = H^3(E(-4)) = \mathbb{C},$$
$$H^1(E(-1)) = H^4(E(-5)) = \mathbb{C},$$
$$H^1(E) = H^4(E(-6)) = W \oplus W^*,$$
$$H^1(E(1)) = H^4(E(-7)) = \Gamma^{2,1} W = \Gamma^{2,1} W^* = W \otimes W^*/\mathbb{C}.$$

All other intermediate $H^i(E(t))$ *for* $1 \leq i \leq 4$, $t \in \mathbb{Z}$ *are zero.*

$$H^1(\mathrm{End}\,E) = (W \otimes W) \oplus (W \otimes W^*) \oplus (W^* \otimes W^*),$$
$$H^1(\mathrm{End}\,E(-1)) = W \oplus W^*,$$
$$H^1(\mathrm{End}\,E(t)) = 0 \quad \text{for } t \leq -2.$$

Proof. [Hor2], [DMS].

We point out that no parent bundle is self-dual, in particular $h^0(E \otimes E) = 0$.

N. Manolache kindly communicated to us the minimal resolution of a parent bundle E in terms of $SL(W) \rtimes \mathbb{Z}_2$ representations. Later W. Decker showed us that this resolution can be obtained by using the same techniques of [DMS], prop. 2.1. We are only referring to the case of $SL(W)$-representations because it will be sufficient for our purposes (e.g. for the theorem 2.15).

Theorem 1.9 ([DMS]). *Let $\mathbb{P}^5 = \mathbb{P}(W \oplus W^*)$. The minimal resolution (which is $SL(W)$-invariant) of a parent bundle E on $\mathbb{P}^5$ is*

$$0 \to L_4 \otimes \mathcal{O}(-7) \to L_3 \otimes \mathcal{O}(-6) \to L_{21} \otimes \mathcal{O}(-5) \oplus \mathcal{O}(-4)$$
$$\to L_{11} \otimes \mathcal{O}(-4) \oplus L_{12} \otimes \mathcal{O}(-3) \to L_{01} \otimes (-3) \oplus L_{02} \otimes \mathcal{O}(-2) \to E \to 0$$

where

$$L_4 = \Gamma^{2,1} W, \quad L_3 = [S^2 W \oplus S^2 W^* \oplus \Gamma^{3,1} W \oplus \Gamma^{3,1} W^*],$$
$$L_{21} = [S^3 W \oplus S^3 W^* \oplus \Gamma^{3,1} W \oplus \Gamma^{3,1} W^* \oplus \Gamma^{4,2} W \oplus \Gamma^{2,1} W],$$
$$L_{11} = [\Gamma^{4,1} W \oplus \Gamma^{4,1} W^* \oplus \Gamma^{3,1} W \oplus \Gamma^{3,1} W^*],$$
$$L_{12} = [W \oplus W^*], \quad L_{01} = \Gamma^{4,2} W, \quad L_{02} = \Gamma^{2,1} W.$$

Proof (Compare with [DMS], prop. 2.1). Remind that $S = \bigoplus_t S^t(W \oplus W^*)$ is the coordinate ring. The display of the monad of definition 1.7 gives the two exact sequences

$$0 \to R \to B \to \mathcal{O}(1) \to 0,$$
$$0 \to \mathcal{O}(-1) \to R \to E \to 0.$$

First we claim that the minimal ($SL(W)$-invariant) resolution of the artinian module $\bigoplus_t H^1(\mathbb{P}^5, E(t)) = \bigoplus_t H^1(\mathbb{P}^5, R(t))$ is

$$0 \to L_4 \otimes S(-7) \to L_3 \otimes S(-6) \to L_{21} \otimes S(-5) \oplus S(-4)^{\oplus 2}$$
$$\to L_{11} \otimes S(-4) \oplus L_{12}^{\oplus 2} \otimes S(-3)$$
$$\to L_{01} \otimes S(-3) \oplus [\Gamma^{2,1} W \oplus \Gamma^{2,1} W \oplus W \oplus W^*] \otimes S(-2)$$
$$\to [S^2 W \oplus S^2 W^* \oplus \mathbb{C}] \otimes S(-1) \to S(1) \to \bigoplus_t H^1(\mathbb{P}^5, R(t)) \to 0.$$

This claim can be straightforwardly verified. For example the 22-dimensional term $[\Gamma^{2,1} W \oplus \Gamma^{2,1} W \oplus W \oplus W^*]$ is obtained as the complement of

$$(W \oplus W^*) \otimes [S^2 W \oplus S^2 W^* \oplus \mathbb{C}]$$

in $S^3(W \oplus W^*)$, inspecting the degree 3 terms of the resolution of $\bigoplus_t H^1(\mathbb{P}^5, R(t))$. Then we prove the theorem comparing the resolutions of each term of the two exact sequences of S-modules

$$0 \to S(-1) \to \bigoplus_t H^0(\mathbb{P}^5, R(t)) \to \bigoplus_t H^0(\mathbb{P}^5, E(t)) \to 0\,,$$

$$0 \to \bigoplus_t H^0(\mathbb{P}^5, R(t)) \to \bigoplus_t H^0(\mathbb{P}^5, B(t)) \to S(1) \to \bigoplus_t H^1(\mathbb{P}^5, R(t)) \to 0\,.$$

2. Pulling back bundles over $\mathbb{C}^6 \setminus 0$

In this section we explore the construction of pulling back bundles over $\mathbb{C}^6 \setminus 0$ introduced by Horrocks in the last section of [Hor2]. This construction can be applied to any bundle on $\mathbb{P}^n$ whose symmetry group contains a copy of $\mathbb{C}^*$.

Let $\alpha \leqq \beta$ be two nonnegative integers, we define

$$\mathcal{W} := \mathcal{O}(\alpha) \oplus \mathcal{O}(\beta) \oplus \mathcal{O}(-\alpha-\beta)\,, \qquad (2.1)$$

$$\mathcal{H} := \mathcal{W} \oplus \mathcal{W}^*\,.$$

Let moreover $f_1, \ldots, f_6$ be homogeneous polynomials of degree

$$\gamma-\alpha, \quad \gamma-\beta, \quad \gamma+\alpha+\beta, \quad \gamma+\alpha, \quad \gamma+\beta, \quad \gamma-\alpha-\beta\,,$$

defining a surjective morphism $\mathcal{H} \xrightarrow{f} \mathcal{O}(\gamma)$ (this happens if and only if they have no common zeroes).

Lemma 2.1. *The dimension of the degree t summand A_t of the artinian algebra $S/(f_1, \ldots, f_6)$ is equal to*

$$\sum_{j=0}^{6} (-1)^j h^0[\wedge^j \mathcal{H} \otimes \mathcal{O}(t-j\gamma)]\,.$$

In particular it is nonzero if and only if $0 \leqq t \leqq 6\gamma - 6$.

Proof. Immediate from the twisted Koszul complex of the map $\mathcal{H} \to \mathcal{O}(\gamma)$

$$0 \to \mathcal{O}(t-6\gamma) \to \mathcal{H}(t-5\gamma) \to \wedge^2\mathcal{H}(t-4\gamma) \to \wedge^3\mathcal{H}(t-3\gamma) \to \wedge^2\mathcal{H}(t-2\gamma)$$
$$\to \mathcal{H}(t-\gamma) \to \mathcal{O}(t) \to 0$$

and the fact that A_t is isomorphic to the cokernel of the map $H^0[\mathcal{H}(t-\gamma)] \to H^0[\mathcal{O}(t)]$.

In order to study the deformations of Horrocks 3-bundles defined in [Hor2] we need to pull back over $\mathbb{C}^6 \setminus 0$ not only the parent bundle but also some other bundles on $\mathbb{P}^5$. We begin with the easy example of weighted quotient bundles.

The polynomials $f_1, \ldots, f_6$ define a map $\omega: \mathbb{C}^6 \backslash 0 \to \mathbb{C}^6 \backslash 0$. Look at the diagram (0.1). On the domain of ω consider the standard multiplicative action of $\mathbb{C}^*$ and on the codomain of ω the action $\tau_{\alpha,\beta,\gamma}: \mathbb{C}^* \times \mathbb{C}^6 \backslash 0 \to \mathbb{C}^6 \backslash 0$ defined by

$$(2.2) \qquad \tau_{\alpha,\beta,\tau}(t, v_1, \ldots, v_6) = (t^{\gamma-\alpha} v_1, t^{\gamma-\beta} v_2, t^{\gamma+\alpha+\beta} v_3, t^{\gamma+\alpha} v_4, t^{\gamma+\beta} v_5, t^{\gamma-\alpha-\beta} v_6)$$

so that

$$\omega \textit{ is } \mathbb{C}^*\textit{-equivariant}.$$

Observe that the action of $\mathbb{C}^*$ given by $\tau_{\alpha,\beta,\gamma}$ descends to $\mathbb{P}^5$. The quotient bundle Q is $SL(H)$-invariant, hence $\eta^* Q$ is $\mathbb{C}^*$-invariant under the action of $\tau_{\alpha,\beta,\gamma}$. It follows that $\omega^*\eta^* Q$ is $\mathbb{C}^*$-invariant under the multiplicative action and then *it descends to a bundle* $\tilde{Q}$ on $\mathbb{P}^5$, that is

$$\omega^*\eta^* Q \simeq \eta^* \tilde{Q}.$$

We say that $\tilde{Q}$ is obtained *by pulling back* Q *over* $\mathbb{C}^6 \backslash 0$.

Definition 2.2. Let $\gamma > \alpha + \beta$. A *weighted quotient bundle* $Q_{\alpha,\beta,\gamma}$ is a bundle obtained pulling back the quotient bundle over $\mathbb{C}^6 \backslash 0$ or equivalently, a bundle defined by an exact sequence:

$$0 \to \mathcal{O}(-\gamma) \to \mathscr{H} \to Q_{\alpha,\beta,\gamma} \to 0.$$

We often drop the indices α, β, γ and we use $\tilde{Q}$ for $Q_{\alpha,\beta,\gamma}$.

We get also $\omega^*\eta^*(H \otimes \mathcal{O}) = \eta^*\mathscr{H}$ and if T is any representation of $SL(H)$ then (with obvious notations) $\omega^*\eta^*(T(H) \otimes \mathcal{O}) = \eta^* T(\mathscr{H})$. The functor η^* gives an equivalence of categories between bundles over $\mathbb{P}^5$ and bundles over $\mathbb{C}^6 \backslash 0$ endowed with the standard multiplicative $\mathbb{C}^*$-action. Hence the minimal resolution of $\tilde{Q}^*$ can be obtained by pulling back over $\mathbb{C}^6 \backslash 0$ the minimal ($SL(6)$-invariant) resolution of Q^* and indeed it is

$$(2.3) \qquad \begin{aligned} 0 &\to \mathcal{O}(-5\gamma) \to \wedge^5 \mathscr{H}(-4\gamma) \to \wedge^4 \mathscr{H}(-3\gamma) \\ &\to \wedge^3 \mathscr{H}(-2\gamma) \to \wedge^2 \mathscr{H}(-\gamma) \to \tilde{Q}^* \to 0. \end{aligned}$$

Lemma 2.3 (Bohnhorst-Spindler). *A weighted quotient bundle* $Q_{\alpha,\beta,\gamma}$ *is stable if and only if* $\gamma > 5\alpha + 5\beta$.

Proof. [BoS].

This construction can be generalized in the following way (we refer to [Hor2] for more details). Fix a decomposition $H = W \oplus W^*$ and choose the coordinates $(x_1, \ldots, x_6)$ so that (x_1, x_2, x_3) are coordinates on W and (x_4, x_5, x_6) are coordinates on W^* (see [DMS]). Then $SL(W)$ embeds into $SL(H)$.

Definition 2.4. Let G be any bundle on $\mathbb{P}^5$ which is $SL(W)$-invariant. Then $\omega^*\eta^* G$ is $\mathbb{C}^*$-invariant under the standard multiplicative action and it descends to a bundle $\tilde{G}$ on $\mathbb{P}^5$, so that

$$\omega^*\eta^* G \simeq \eta^* \tilde{G}.$$

We say that $\tilde{G}$ is obtained *by pulling back* G *over* $\mathbb{C}^6 \backslash 0$.

Our basic examples of $SL(W)$-invariant bundles will be B, E, $\operatorname{End} B$, $\operatorname{End} E$ where B is a lambda-three bundle and E is a parent bundle.

Remark 2.5. The minimal resolution of $\tilde{G}$ can be obtained by pulling back over $\mathbb{C}^6\setminus 0$ the minimal resolution of G, which is $SL(W)$-invariant. We would point out that if T is any representation of $SL(W)$ then $\omega^*\eta^*(T(W)\otimes\mathcal{O})\simeq\eta^*T(\mathcal{W})$.

The cohomology module $\bigoplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^5, G(t))$ has a natural structure of $SL(W)$-module. We remind that $H^i(\mathbb{C}^6\setminus 0, \eta^*G)$ is isomorphic to $\bigoplus_{t\in\mathbb{Z}} H^i(P^5, G(t))$ and is endowed with the $\mathbb{C}^*$-action induced by the standard multiplicative action (that we notice by $v\mapsto\sigma(t)v$). The graded summand $H^i(\mathbb{P}^5, G(m))$ is isomorphic to the *weight space*

$$\{v\in H^i(\mathbb{C}^6\setminus 0, \eta^*G)\,|\,\sigma(t)v = t^m v \;\; \forall t\in\mathbb{C}^*\}.$$

For $t\in\mathbb{C}^*$ let $\tau'_{\alpha,\beta,\gamma}(t)\in GL\left(H^i(\mathbb{C}^6\setminus 0, \eta^*G)\right)$ denote the $\mathbb{C}^*$-action induced by $\tau_{\alpha,\beta,\gamma}$. Then on $H^i(\mathbb{C}^6\setminus 0, \eta^*G)$ we have the new graduation

$$V^m_{\alpha,\beta,\gamma} = \{v\in H^i(\mathbb{C}^6\setminus 0, \eta^*G)\,|\,\tau'_{\alpha,\beta,\gamma}(t)v = t^m v \;\; \forall t\in\mathbb{C}^*\}$$

so that $H^i(\mathbb{C}^6\setminus 0, \eta^*G) = \bigoplus_{m\in\mathbb{Z}} V^m_{\alpha,\beta,\gamma}$. We have the isomorphism

$$(2.4)\qquad H^i(\mathbb{C}^6\setminus 0, \omega^*\eta^*G)\simeq H^i(\mathbb{C}^6\setminus 0, \eta^*G)\otimes_{\mathbb{C}} S/(f_1,\dots,f_6).$$

This isomorphism is $\mathbb{C}^*$-*equivariant* if we consider on the left side the standard multiplicative action and on the right side the action $\tau'_{\alpha,\beta,\gamma}$ on $H^i(\mathbb{C}^6\setminus 0, \eta^*G)$. Let A_m be the degree m summand of $S/(f_1,\dots,f_6)$ (see lemma 2.1). Then our main technical tool is the formula (obtained by (2.4))

$$(2.5)\qquad H^i(\mathbb{P}^5, \tilde{G}(m))\simeq\bigoplus_{j=0}^{6\gamma-6} A_j\otimes V^{m-j}_{\alpha,\beta,\gamma}.$$

In the special case $\alpha=\beta=0$ we have $V^{m\gamma}_{0,0,\gamma} = H^i(\mathbb{P}^5, G(m))$ and we have the well known formula

$$\bigoplus_{m\in\mathbb{Z}} H^i(\mathbb{P}^5, \tilde{G}(m))\simeq S/(f_1,\dots,f_6)\otimes_{\mathbb{C}}[\bigoplus_{m\in\mathbb{Z}} H^i(\mathbb{P}^5, G(m))]$$

with the natural graduation. In this case the dimensions $h^i(\mathbb{P}^5, G(m))$, $m\in\mathbb{Z}$ suffice to compute the dimensions $h^i(\mathbb{P}^5, \tilde{G}(q))$, $q\in\mathbb{Z}$ and the module structure of $\bigoplus_m H^i(\mathbb{P}^5, G(m))$ is not involved. More precisely $h^i(\mathbb{P}^5, \tilde{G}(m))$ depends only on $h^i(\mathbb{P}^5, G(m'))$ with $\gamma m'\leqq m$. In particular if $h^0(\mathbb{P}^5, G)=0$ then also $h^0(\mathbb{P}^5, \tilde{G})=0$. This fails in the general case, in fact we will see (theorem B) that $\tilde{G}$ can be unstable even if G is stable.

In order to compute the eigenspaces of $\tau'_{\alpha,\beta,\gamma}$ let us write $\tau_{\alpha,\beta,\gamma}=\tau_1\circ\tau_2$ where $\tau_1(t, v_1,\dots,v_6) = (t^{-\alpha}v_1, t^{-\beta}v_2, t^{+\alpha+\beta}v_3, t^{+\alpha}v_4, t^{+\beta}v_5, t^{-\alpha-\beta}v_6)$ and τ_2 is the multiplication by t^γ. Now τ_1 factors through $SL(W)$ and the corresponding $\mathbb{C}^*$-action on the cohomology can be easily computed by looking at the cohomology groups as $SL(W)$-representations. For example if

$$H^1(G(t)) = \begin{cases} 0 & t \neq -2, \\ W \oplus W^* & t = -2 \end{cases}$$

then $\bigoplus_t H^1(G(t)) = \bigoplus_m V^m_{\alpha,\beta,\gamma}$ and $\dim V^m_{\alpha,\beta,\gamma}$ is equal to the number of times m occurs in the sequence $\{-2\gamma+\alpha, -2\gamma+\beta, -2\gamma-\alpha-\beta, -2\gamma-\alpha, -2\gamma-\beta, -2\gamma+\alpha+\beta\}$. This remark together with lemma 2.1 allows us to write (2.5) more explicitly, as follows:

Theorem 2.6. *Let G be an $SL(W)$-invariant bundle and let $H^i(G(t)) = T^t(W)$ where T^t is a representation of $SL(W)$. Let $\tilde{G}$ be obtained pulling back G over $\mathbb{C}^6 \setminus 0$. Then*

$$h^i(\tilde{G}(t)) = \sum_{h\in\mathbb{Z}} \sum_{j=0}^{6} (-1)^j h^0[\wedge^j(\mathscr{H}(-\gamma)) \otimes T^h(\mathscr{W}^*) \otimes \mathcal{O}(t-h\gamma)] .$$

Remark 2.7. An analogous result holds with other linear groups in place of $SL(W)$. Observe that $SL(W) \subset Sp(6)$, so that a lambda-three bundle is $SL(W)$-invariant.

Proposition 2.8. *Let $Q_{\alpha,\beta,\gamma}$ be a weighted quotient bundle. Then*

$$(2.6) \qquad h^1(\operatorname{End} Q_{\alpha,\beta,\gamma})$$
$$= h^0(\mathscr{H}(\gamma)) - h^0(\mathscr{H} \otimes \mathscr{H}) + h^0(\wedge^2\mathscr{H} \otimes \mathscr{H}(-\gamma)) - h^0(\wedge^3\mathscr{H} \otimes \mathscr{H}(-2\gamma)) .$$

Proof. By lemma 1.1 and by theorem 2.6 we have as follows:

$$h^1(\operatorname{End} Q_{\alpha,\beta,\gamma}) = \sum_{j=0}^{6} (-1)^j h^0[\wedge^j[\mathscr{H}(-\gamma)] \otimes \mathscr{H}(\gamma)] .$$

We have proved the proposition because the summands with $j \geqq 4$ are zero.

By pulling back over $\mathbb{C}^6 \setminus 0$ a lambda-three bundle we have a bundle $B_{\alpha,\beta,\gamma}$. It is easy to check that it fits into an exact sequence

$$(2.7) \qquad 0 \to \wedge^4 Q^*_{\alpha,\beta,\gamma} \to \wedge^2 Q^*_{\alpha,\beta,\gamma} \to B_{\alpha,\beta,\gamma} \to 0$$

hence $B_{\alpha,\beta,\gamma}$ is the cohomology of a monad

$$(2.8) \qquad Q_{\alpha,\beta,\gamma}(-\gamma) \to \wedge^2\mathscr{H} \to Q^*_{\alpha,\beta,\gamma}(\gamma) .$$

Definition 2.9. A *weighted lambda-three bundle* $B_{\alpha,\beta,\gamma}$ is the cohomology of a monad (2.8) where $Q_{\alpha,\beta,\gamma}$ is a weighted quotient bundle.

Remark 2.10. It follows from remark 2.5 that if $B_{\alpha,\beta,\gamma}$ is obtained as pullback over $\mathbb{C}^6 \setminus 0$ from a lambda-three bundle, its minimal resolution is

$$(2.9) \qquad 0 \to \mathcal{O}(-4\gamma) \to \mathscr{H}(-3\gamma) \to \mathscr{H}_{\lambda_2}(-2\gamma) \to \mathscr{H}_{\lambda_3}(-\gamma) \to B_{\alpha,\beta,\gamma} \to 0 .$$

We often use $\tilde{B}$ for $B_{\alpha,\beta,\gamma}$.

Note that the dual of a lambda-three bundle is again a lambda-three bundle.

Proposition 2.11. *Let $B_{\alpha,\beta,\gamma}$ be a weighted lambda-three bundle obtained by pulling back over $\mathbb{C}^6\setminus 0$ a lambda-three bundle. Then*

$$\begin{aligned} h^1(\operatorname{End} B_{\alpha,\beta,\gamma}) &= \sum_{j=0}^{6} (-1)^j h^0[\wedge^j(\mathscr{H}(-\gamma)) \otimes (\mathscr{H}(\gamma) \oplus \mathscr{H}_{\lambda_2})] \\ &= h^0(\mathscr{H}(\gamma)) - h^0(S^2\mathscr{H}) - h^0(\wedge^3\mathscr{H} \otimes \mathscr{H}(-2\gamma)) \\ &\quad + h^0(\wedge^2\mathscr{H} \otimes \wedge^2\mathscr{H}(-2\gamma)) - h^0(\wedge^3\mathscr{H} \otimes \wedge^2\mathscr{H}(-3\gamma)) - 1\,. \end{aligned}$$

Proof. By lemma 1.5 and by theorem 2.6.

Now we can define the 3-bundles which are the main subject of this paper.

Definition 2.12. A *relation bundle* $E_{\alpha,\beta,\gamma}$ is the cohomology of a monad

$$\mathcal{O}(-\gamma) \to B_{\alpha,\beta,\gamma} \to \mathcal{O}(\gamma)$$

where $B_{\alpha,\beta,\gamma}$ is a weighted lambda-three bundle.

We remark that $E_{0,0,1}$ is a *parent bundle*. The bundles obtained by pulling back over $\mathbb{C}^6\setminus 0$ the parent bundle (which is $SL(W)$-invariant), are particular cases of the relation bundles and were constructed by Horrocks in [Hor2]. In the notations of the last section of [Hor2] we have $m_1 = \alpha$, $m_2 = \beta$, $m_3 = -\alpha - \beta$, $r = \gamma$.

Sometimes we use $\tilde{E}$ for $E_{\alpha,\beta,\gamma}$. It follows from the definition that the dual of a relation bundle is again a relation bundle.

Remark 2.13. It is interesting to explicitly construct some relation bundles. Let B a lambda-three bundle. The 14×14 matrix of the composition

$$H_{\lambda_3} \otimes \mathcal{O}(-1) \to B \simeq B^* \to H_{\lambda_3} \otimes \mathcal{O}(1)$$

(where the morphisms are defined by (1.3)) has been computed in [DMS], proof of prop. 1.3. Denote by M this matrix. The space generated by the rows of M identifies naturally with the space $H^0(B(1)) \subset H_{\lambda_3} \otimes H^0(\mathcal{O}(2))$. Decker, Manolache and Schreyer pointed out that the section σ (resp. τ) given by the sum (resp. difference) of 1st and 2nd rows does not vanish anywhere and that $\tau \circ \sigma = 0$. Hence this pair of sections defines a monad whose cohomology is a parent bundle. Now, consider a morphism $\omega : \mathbb{C}^6\setminus 0 \to \mathbb{C}^6\setminus 0$ as in (0.1) given by the polynomials $f_1, \ldots, f_6$. ω defines a bundle $B_{\alpha,\beta,\gamma}$ obtained by pulling back over $\mathbb{C}^6\setminus 0$ a lambda-three bundle B. The 14×14 matrix $\tilde{M}$ of the composition

$$\mathscr{H}_{\lambda_3}(-\gamma) \to B_{\alpha,\beta,\gamma} \simeq B^*_{\alpha,\beta,\gamma} \to \mathscr{H}_{\lambda_3}(\gamma)$$

(where the morphisms are defined by (2.9)) is obtained from M replacing x_i by f_i. The space $H^0(B_{\alpha,\beta,\gamma}(\gamma))$ can be interpreted as the space of linear combinations of the rows of $\tilde{M}$ with coefficients homogeneous polynomials of degree 0, 0, 2α, $\alpha+\beta$, $-\beta$, 2β, $-\alpha$, $-2\alpha-2\beta$, -2α, $-\alpha-\beta$, β, -2β, α, $2\alpha+2\beta$ (the coefficients are zero if the corresponding degrees are negative). Let σ (resp. $\tau) \in H^0(B_{\alpha,\beta,\gamma}(\gamma))$ be given by the coefficients $(1,1,0,0,0,0,0,0,0,0,0,0,0,0)$ (resp. $(1,-1,0,0,0,0,0,0,0,0,0,0,0,0)$). Then the cohomology of the monad

$$\mathcal{O}(-\gamma) \xrightarrow{\sigma} B_{\alpha,\beta,\gamma} \xrightarrow{\tau} \mathcal{O}(\gamma)$$

is a 3-bundle $E_{\alpha,\beta,\gamma}$. $E_{\alpha,\beta,\gamma}$ comes as pullback over $\mathbb{C}^6\setminus 0$ from the parent bundle constructed above. Note that for $0<\alpha<\beta$ only the first two coefficients, out of the fourteen we mentioned before, are allowed to be constant, hence *only the relation bundles defined by sections of $B_{\alpha,\beta,\gamma}(\gamma)$, which are suitable linear combinations of σ, τ, come as pullbacks over* $\mathbb{C}^6\setminus 0$. This family fibers over the family of their corresponding weighted lambda-three bundles coming as pullbacks over $\mathbb{C}^6\setminus 0$, with 1-dimensional fibers.

We summarize in the following theorem the intermediate cohomology of a relation bundle $E_{\alpha,\beta,\gamma}$ coming as pullback over $\mathbb{C}^6\setminus 0$. It follows from lemma 1.8 and theorem 2.6. H^3 and H^4 can be found by Serre duality because $E^*_{\alpha,\beta,\gamma}$ is again a relation bundle coming as pullback over $\mathbb{C}^6\setminus 0$. H^0 can be computed from the minimal resolution (see theorem 2.15).

Theorem 2.14. *Let $E_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6\setminus 0$. What follows next holds:*

$$H^1(E_{\alpha,\beta,\gamma}(t)) = \sum_j (-1)^j h^0\{\wedge^j\mathcal{H}\otimes\mathcal{O}(t-j\gamma)\otimes[\mathcal{O}(+\gamma)\oplus\mathcal{H}\oplus\Gamma^{2,1}\mathcal{W}(-\gamma)]\},$$

$$H^2(E_{\alpha,\beta,\gamma}(t)) = \sum_j (-1)^j h^0\{\wedge^j(\mathcal{H}(-\gamma))\otimes\mathcal{O}(t+2\gamma)\}.$$

Theorem 2.15. *Let $E_{\alpha,\beta,\gamma}$ be a relation bundle on $\mathbb{P}^5$ coming as pullback over $\mathbb{C}^6\setminus 0$. Let $\mathcal{W}=\mathcal{O}(\alpha)\oplus\mathcal{O}(\beta)\oplus\mathcal{O}(-\alpha-\beta)$. The minimal resolution of $E_{\alpha,\beta,\gamma}$ is*

$$\begin{aligned}0 &\to \Gamma^{2,1}\mathcal{W}(-7\gamma)\to[S^2\mathcal{W}\oplus S^2\mathcal{W}^*\oplus\Gamma^{3,1}\mathcal{W}\oplus\Gamma^{3,1}\mathcal{W}^*](-6\gamma)\\ &\to[S^3\mathcal{W}\oplus S^3\mathcal{W}^*\oplus\Gamma^{3,1}\mathcal{W}\oplus\Gamma^{3,1}\mathcal{W}^*\oplus\Gamma^{4,2}\mathcal{W}\oplus\Gamma^{2,1}\mathcal{W}](-5\gamma)\oplus\mathcal{O}(-4\gamma)\\ &\to[\Gamma^{4,1}\mathcal{W}\oplus\Gamma^{4,1}\mathcal{W}^*\oplus\Gamma^{3,1}\mathcal{W}\oplus\Gamma^{3,1}\mathcal{W}^*](-4\gamma)\oplus[\mathcal{W}\oplus\mathcal{W}^*](-3\gamma)\\ &\to\Gamma^{4,2}\mathcal{W}(-3\gamma)\oplus\Gamma^{2,1}\mathcal{W}(-2\gamma)\to E_{\alpha,\beta,\gamma}\to 0.\end{aligned}$$

Proof. By theorem 1.9 and remark 2.5.

3. Proof of theorem A

Proposition 3.1. *Let $Q^0_{\alpha,\beta,\gamma}$ be a weighted quotient bundle. Every small deformation of $Q^0_{\alpha,\beta,\gamma}$ is again a weighted quotient bundle $Q_{\alpha,\beta,\gamma}$. Moreover the Kuranishi space of $Q^0_{\alpha,\beta,\gamma}$ is smooth at the point corresponding to $Q^0_{\alpha,\beta,\gamma}$.*

Proof. Let $\tilde{Q}$, $\tilde{Q}'$ be two weighted quotient bundles. Every morphism from $\tilde{Q}$ to $\tilde{Q}'$ lifts to a morphism of sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(-\gamma) & \to & \mathcal{H} & \to & \tilde{Q} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}(-\gamma) & \to & \mathcal{H} & \to & \tilde{Q}' & \to & 0 \end{array}$$

(by the vanishing of $H^1(\mathcal{H}(-\gamma))$). Moreover two elements $f, f' \in \mathrm{Hom}(\mathcal{O}(-\gamma), \mathcal{H})$ give the same element of $\mathrm{Quot}_{\mathcal{H}/\mathbb{P}^5}$ if and only if $g \in \mathrm{Aut}(\mathcal{O}(-\gamma))$ exists such that $f = f' \circ g$. Let $\tilde{Q}_0 := Q^0_{\alpha,\beta,\gamma}$ be the cokernel of $f_0 \in \mathrm{Hom}(\mathcal{O}(-\gamma), \mathcal{H})$. Let Y be the Kuranishi space of $\tilde{Q}_0$ and $y_0 \in Y$ be the point corresponding to $\tilde{Q}_0$. Let $x_0 \in \mathrm{Quot}_{\mathcal{H}/\mathbb{P}^5}$ be the point corresponding to $\tilde{Q}_0$ and let X be the irreducible component of $\mathrm{Quot}_{\mathcal{H}/\mathbb{P}^5}$ containing x_0. We have a natural morphism of germs $\pi : (X, x_0) \to (Y, y_0)$, then

$$\dim_{y_0} Y \geqq \dim_{x_0} X - \dim_{x_0} \pi^{-1}(y_0).$$

If $Z = \{x \in X : \tilde{Q}_x \simeq \tilde{Q}_0\}$ we have $(\pi^{-1}(y_0), x_0) \subset (Z, x_0)$, hence

$$\dim_{y_0} Y \geqq \dim_{x_0} X - \dim_{x_0} Z .$$

We have $\dim_{x_0} X = h^0(\mathcal{H}(\gamma)) - 1 = h^0(\tilde{Q}_0(\gamma))$. We obtain the formula

$$\dim_{x_0} Z = h^0(\mathrm{End}\,\mathcal{H}) - \{\text{dimension of endomorphisms of } \mathcal{H} \text{ which fix } f_0\} - 1 .$$

The sequence

$$(3.1) \qquad 0 \to \tilde{Q}_0^* \otimes \mathcal{H} \to \mathrm{End}\,\mathcal{H} \to \mathcal{H}(\gamma) \to 0$$

shows that the number in braces in the last formula is equal to $h^0(\tilde{Q}_0^* \otimes \mathcal{H})$.

It follows $\dim_{y_0} Y \geqq h^0(\mathcal{H}(\gamma)) - h^0(\mathrm{End}\,\mathcal{H}) + h^0(\tilde{Q}_0^* \otimes \mathcal{H}) = h^1(\tilde{Q}_0^* \otimes \mathcal{H})$ where the last equality follows again from the sequence (3.1). Now the exact sequence

$$0 \to \tilde{Q}_0^*(-\gamma) \to \tilde{Q}_0^* \otimes \mathcal{H} \to \mathrm{End}\,\tilde{Q}_0 \to 0$$

shows $h^1(\tilde{Q}_0^* \otimes \mathcal{H}) = h^1(\mathrm{End}\,\tilde{Q}_0)$, hence $\dim_{y_0} Y \geqq h^1(\mathrm{End}\,\tilde{Q}_0)$ and the equality holds because the right-hand side is the dimension of the Zariski tangent space to Y at y_0. In particular $\dim_{y_0} Y = \dim_{x_0} X - \dim_{x_0} \pi^{-1}(y_0)$ and π is surjective between germs, q.e.d.

Corollary 3.2. *The dimension of the Kuranishi space of $Q_{\alpha,\beta,\gamma}$ is*

$$h^0(\mathcal{H}(\gamma)) - h^0(\mathcal{H} \otimes \mathcal{H}) + h^0(\wedge^2 \mathcal{H} \otimes \mathcal{H}(-\gamma)) - h^0(\wedge^3 \mathcal{H} \otimes \mathcal{H}(-2\gamma)) .$$

Proof. By proposition 2.8 and proposition 3.1.

Corollary 3.3 (Bohnhorst-Spindler). *Let $\gamma > 5\alpha + 5\beta$ (see lemma 2.3). The weighted quotient bundles $Q_{\alpha,\beta,\gamma}$ form a smooth open irreducible subset of dimension*

$$h^0(\mathscr{H}(\gamma)) - h^0(\mathscr{H} \otimes \mathscr{H})$$

of the moduli space of stable bundles with the same rank and Chern classes.

Lemma 3.4. *Let $\tilde{B}$, $\tilde{B}'$ be two isomorphic weighted lambda-three bundles defined from the sequences*

$$0 \to \wedge^4 \tilde{Q}^* \to \wedge^2 \tilde{Q}^* \to \tilde{B} \to 0,$$

$$0 \to \wedge^4 \tilde{Q}'^* \to \wedge^2 \tilde{Q}'^* \to \tilde{B}' \to 0$$

where $\tilde{Q}$, $\tilde{Q}'$ are weighted quotient bundles (as in (2.7)). Then $\tilde{Q} \simeq \tilde{Q}'$.

Proof. By putting together the resolutions of $\wedge^4 \tilde{Q}^*$ and $\wedge^2 \tilde{Q}^*$ we obtain the resolution

$$(3.2) \qquad 0 \to \mathcal{O}(-4\gamma) \to \mathscr{H}(-3\gamma) \oplus \mathcal{O}(-2\gamma) \to \wedge^2 \mathscr{H}(-2\gamma) \oplus \mathscr{H}(-\gamma) \to \wedge^3 \mathscr{H}(-\gamma) \to \tilde{B} \to 0.$$

The corresponding sequence of S-modules is exact. The piece $\mathcal{O}(-4\gamma) \xrightarrow{k} \mathscr{H}(-3\gamma)$ does not contain any summand that cancels in the minimal resolution because $\gamma > \alpha + \beta$. Hence $\tilde{Q}(-3\gamma) \simeq \operatorname{Coker} k$ is defined directly from the minimal resolution of $\tilde{B}$.

Lemma 3.5. *Let $Q^0_{\alpha,\beta,\gamma}$ be a weighted quotient bundle. Any small deformation of $\wedge^2 Q^0_{\alpha,\beta,\gamma}$ is isomorphic to $\wedge^2 Q_{\alpha,\beta,\gamma}$ where $Q_{\alpha,\beta,\gamma}$ is again a weighted quotient bundle. Moreover the map $Q_{\alpha,\beta,\gamma} \mapsto \wedge^2 Q_{\alpha,\beta,\gamma}$ induces an isomorphism between the germs of the corresponding Kuranishi spaces.*

Proof. We remark that lemma 1.1 combined with theorem 2.6 implies that

$$H^1(\operatorname{End} \wedge^2 \tilde{Q}) \simeq H^1(\operatorname{End} \tilde{Q})$$

for any weighted quotient $\tilde{Q}$. Now, it is sufficient to verify that if $\wedge^2 \tilde{Q}' \simeq \wedge^2 \tilde{Q}''$ then $\tilde{Q}' \simeq \tilde{Q}''$ and this comes from the fact that in the minimal resolution of $\wedge^2 \tilde{Q}$

$$0 \to \mathcal{O}(-2\gamma) \to \mathscr{H}(-\gamma) \to \wedge^2 \mathscr{H} \to \wedge^2 \tilde{Q} \to 0$$

the first cokernel on the left is isomorphic to $\tilde{Q}(-\gamma)$.

Lemma 3.6. *Let $\tilde{Q}$ be a weighted quotient bundle. Then $H^1(\wedge^2 \tilde{Q} \otimes \wedge^4 \tilde{Q}^*) = 0$.*

Proof. From lemma 1.2 and theorem 2.6.

Theorem 3.7. *Let $B^0_{\alpha,\beta,\gamma}$ be a lambda-three bundle coming as pullback over $\mathbb{C}^6 \setminus 0$. Every small deformation of $B^0_{\alpha,\beta,\gamma}$ is again a lambda-three bundle $B_{\alpha,\beta,\gamma}$ defined by a sequence* (2.7). *Moreover the Kuranishi space of $B^0_{\alpha,\beta,\gamma}$ is smooth at $B^0_{\alpha,\beta,\gamma}$.*

Proof. From lemmas 3.4 and 3.6 it follows that every isomorphism between two lambda-three bundles $\tilde{B}$, $\tilde{B}'$, as in the statement of lemma 3.4, is induced by a morphism of sequences

$$\begin{array}{ccccccccc} 0 & \to & \wedge^4\tilde{Q}^* & \to & \wedge^2\tilde{Q}^* & \to & \tilde{B} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \wedge^4\tilde{Q}'^* & \to & \wedge^2\tilde{Q}'^* & \to & \tilde{B}' & \to & 0\,. \end{array}$$

$\tilde{B}_0$ stands for $B^0_{\alpha,\beta\gamma,\gamma}$ and we denote $\tilde{Q}_0$ the weighted quotient corresponding to $\tilde{B}_0$ uniquely defined by lemma 3.4. Let now $f_0 \in \mathrm{Hom}(\wedge^4\tilde{Q}_0^*, \wedge^2\tilde{Q}_0^*)$ be a morphism which defines $\tilde{B}_0$. $f, f' \in \mathrm{Hom}(\wedge^4\tilde{Q}_0^*, \wedge^2\tilde{Q}_0^*)$ give the same element of $\mathrm{Quot}_{\wedge^2\tilde{Q}_0^*/\mathbb{P}^5}$ if and only if there is an invertible $h \in \mathrm{End}(\wedge^4\tilde{Q}_0)$ such that $f = f' \circ h$. Let (Y, y_0) be the Kuranishi space of $\wedge^2\tilde{Q}_0$ and the (T, t_0) be the Kuranishi space of $\tilde{B}_0$. Let $\mathscr{F}$ be the universal family over $Y \times \mathbb{P}^5$ and let $Z = \mathrm{Quot}_{\mathscr{F}/Y\times\mathbb{P}^5/Y}$. Let $z_0 \in Z$ be the element corresponding to $\tilde{B}_0$. We have two natural morphisms $\phi: Z \to Y$ and $\pi: (Z, z_0) \to (T, t_0)$.

Let Z' be the subvariety of the component of Z, containing z_0, which consists of quotients $\wedge^2\tilde{Q}''^* \xrightarrow{g''} \mathscr{G}$ for some weighted quotient $\tilde{Q}''$ such that $\mathrm{Ker}\, g'' \simeq \wedge^4\tilde{Q}''^*$. Hence we have $\dim_{t_0} T \geqq \dim_{z_0} Z - \dim_{z_0}\pi^{-1}(t_0) \geqq \dim_{z_0} Z' - \dim_{z_0}\pi^{-1}(t_0)$. Moreover from lemma 3.4 we have $(\pi^{-1}(t_0), z_0) \subset (\phi^{-1}(y_0), z_0) = (\mathrm{Quot}_{\wedge^2\tilde{Q}_0^*/\mathbb{P}^5}, z_0)$. If

$$P := \{x \in \mathrm{Quot}_{\wedge^2\tilde{Q}_0^*/\mathbb{P}^5} : \tilde{B}_x \simeq \tilde{B}_0\}$$

we check $(\pi^{-1}(t_0), z_0) \subset (P, z_0)$. We have $\dim_{z_0} P = h^0(\mathrm{End}\wedge^2\tilde{Q}_0)$ – {dimension of endomorphisms of $\wedge^2\tilde{Q}_0$ that fix f_0} $- h^0(\mathrm{End}\wedge^4\tilde{Q}_0)$. The exact sequence

$$(3.3) \qquad 0 \to \tilde{B}_0 \otimes \wedge^2\tilde{Q}_0^* \to \mathrm{End}\wedge^2\tilde{Q}_0 \to \wedge^4\tilde{Q}_0 \otimes \wedge^2\tilde{Q}_0^* \to 0\,.$$

shows that the term in braces of the last formula is equal to $h^0(\tilde{B}_0 \otimes \wedge^2\tilde{Q}_0^*)$, hence $\dim_{z_0} P = h^0(\mathrm{End}\wedge^2\tilde{Q}_0) - h^0(\tilde{B}_0 \otimes \wedge^2\tilde{Q}_0^*) - h^0(\mathrm{End}\wedge^4\tilde{Q}_0)$. Now consider that all the fibers of $\phi|_{Z'}: Z' \to Y$ have the same dimension $h^0(\wedge^4\tilde{Q}_0 \otimes \wedge^2\tilde{Q}_0^*) - h^0(\mathrm{End}\wedge^4\tilde{Q}_0)$ (depending only on α, β, γ). By lemma 3.5 and prop. 3.1,

$$\dim_{y_0} Y = h^1(\mathrm{End}\wedge^2\tilde{Q}_0) = h^1(\mathrm{End}\,\tilde{Q}_0) = h^1(\mathrm{End}\wedge^4\tilde{Q}_0)\,,$$

hence $\dim Z' = h^0(\wedge^4\tilde{Q}_0 \otimes \wedge^2\tilde{Q}_0^*) - h^0(\mathrm{End}\wedge^4\tilde{Q}_0) + h^1(\mathrm{End}\wedge^4\tilde{Q}_0)$. It follows

$$(3.4) \qquad \dim_{t_0} T \geqq \dim_{z_0} Z' - \dim_{z_0} P$$
$$= h^0(\wedge^4\tilde{Q}_0 \otimes \wedge^2\tilde{Q}_0^*) - h^0(\mathrm{End}\wedge^2\tilde{Q}_0) + h^0(\tilde{B}_0 \otimes \wedge^2\tilde{Q}_0^*) + h^1(\mathrm{End}\wedge^2\tilde{Q}_0)\,.$$

We claim that the image of the morphism $H^1(\mathrm{End}\wedge^2\tilde{Q}_0) \to H^1(\wedge^4\tilde{Q}_0 \otimes \wedge^2\tilde{Q}_0^*)$ defined by the sequence (3.3) has the following dimension $\sum_{j=0}^{6} h^0[\wedge^j\mathscr{H} \otimes \mathscr{H}((1-j)\gamma)]$.

In fact from the hypothesis the morphism $\operatorname{End}\wedge^2\tilde{Q}_0 \to \wedge^4\tilde{Q}_0 \otimes \wedge^2\tilde{Q}_0^*$ is obtained by pulling back over $\mathbb{C}^6\setminus 0$ a morphism $\operatorname{End}\wedge^2 Q \to \wedge^4 Q \otimes \wedge^2 Q^*$. $H^1(\wedge^4 Q \otimes \wedge^2 Q^*(t))$ is zero for $t \neq -1$ and it is isomorphic to $\wedge^3 H = H \oplus H_{\lambda_3}$ for $t = -1$ (lemma 1.3). We have also $H^1(\operatorname{End}\wedge^2 Q(-1)) = H$ (lemma 1.1), and we see from lemma 1.6 and the cohomology sequence associated to (3.3) that the morphism

$$\bigoplus H^1(\operatorname{End}\wedge^2 Q(t)) \to \bigoplus H^1(\wedge^4 Q \otimes \wedge^2 Q^*(t))$$

is an isomorphism on the subspace H just considered in degree -1 thus proving our claim from theorem 2.6. Then from (3.4) and the cohomology sequence associated to (3.3) it follows:

$$\dim_{t_0} T \geqq h^1(\tilde{B}_0 \otimes \wedge^2\tilde{Q}_0^*) + \sum_{j=0}^{6} h^0[\wedge^j\mathscr{H} \otimes \mathscr{H}((1-j)\gamma)]\,.$$

By lemma 1.5 we have $H^1(\operatorname{End} B(t)) = 0$ for $t \neq -1, 0$ and by lemma 1.6 we have

$$H^1(B \otimes \wedge^2 Q^*(t)) = 0$$

for $t \neq 0$, $H^1(B \otimes \wedge^2 Q^*) = H_{\lambda_2} = H^1(\operatorname{End} B)$. Hence, as $H^1(\operatorname{End} B(-1)) = H$ gives a contribution to $H^1(\operatorname{End}\tilde{B}_0)$ in the formula of theorem 2.6 exactly equal to

$$\sum_{j=0}^{6} h^0[\wedge^j\mathscr{H} \otimes \mathscr{H}((1-j)\gamma)]$$

we get $\dim_{t_0} T \geqq h^1(\operatorname{End}\tilde{B}_0)$, thus the equality holds and π is surjective, q.e.d.

Corollary 3.8. *The dimension of the Kuranishi space of $B^0_{\alpha,\beta,\gamma}$ is*

$$\sum_{j=0}^{6} (-1)^j h^0[\wedge^j(\mathscr{H}(-\gamma)) \otimes (\mathscr{H}(\gamma) \oplus \mathscr{H}_{\lambda_2})]\,.$$

Proof. By proposition 2.11.

The display of the monad defining $\tilde{E}$ gives the two exact sequences

(3.5) $$0 \to R_{\alpha,\beta,\gamma} \to B_{\alpha,\beta,\gamma} \to \mathcal{O}(\gamma) \to 0\,,$$

(3.6) $$0 \to \mathcal{O}(-\gamma) \to R_{\alpha,\beta,\gamma} \to E_{\alpha,\beta,\gamma} \to 0\,.$$

Lemma 3.9. *Let $R^0_{\alpha,\beta,\gamma}$ has a bundle appearing as a kernel in a sequence*

$$0 \to R^0_{\alpha,\beta,\gamma} \to B^0_{\alpha,\beta,\gamma} \to \mathcal{O}(\gamma) \to 0$$

where $B^0_{\alpha,\beta,\gamma}$ is a weighted lambda-three bundle coming as pullback over $\mathbb{C}^6\setminus 0$ and such that also $R^0_{\alpha,\beta,\gamma}$ comes as pullback over $\mathbb{C}^6\setminus 0$. Every small deformation of $R^0_{\alpha,\beta,\gamma}$ appears again as a kernel in a sequence as (3.5) *where $B_{\alpha,\beta,\gamma}$ is a weighted lambda-three bundle defined by a sequence* (2.7). *Moreover the Kuranishi space of $R^0_{\alpha,\beta,\gamma}$ is smooth at $R^0_{\alpha,\beta,\gamma}$.*

Proof. The proof is analogous to the proof of the theorem 3.7 and it is up to the reader to check it out.

Theorem A follows now from the following

Theorem 3.10. *Let $E^0_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6 \setminus 0$. Every small deformation of $E^0_{\alpha,\beta,\gamma}$ is a relation bundle $E_{\alpha,\beta,\gamma}$ corresponding to a weighted lambda-three bundle $B_{\alpha,\beta,\gamma}$ defined from a sequence (2.7). Moreover the Kuranishi space of $E^0_{\alpha,\beta,\gamma}$ is smooth at $E^0_{\alpha,\beta,\gamma}$.*

Proof. We use $\tilde{E}_0$ for $E^0_{\alpha,\beta,\gamma}$. Let $\tilde{R}_0$ be corresponding to $\tilde{E}_0$, that is the unique non-splitting extension

$$0 \to \mathcal{O}(-\gamma) \to ? \to \tilde{E}_0 \to 0 .$$

Consider the exact sequence

$$0 \to \tilde{E}_0^* \otimes \tilde{R}_0 \to \operatorname{End} \tilde{R}_0 \to \tilde{R}_0(\gamma) \to 0 . \tag{3.7}$$

We begin to prove that the induced morphism $g: H^1(\tilde{E}_0^* \otimes \tilde{R}_0) \to H^1(\operatorname{End} \tilde{R}_0)$ is surjective. We set

$$g_t: H^1(E^* \otimes R(t)) \to H^1(\operatorname{End} R(t)) .$$

As the tensor product is right exact we have that Coker g is the degree 0 summand in $[\bigoplus_t \operatorname{Coker} g_t] \otimes_{\mathbb{C}} S/(f_1, \ldots, f_6)$ (see (2.4) and (2.5)).

Consider the two exact sequences

$$0 \to E^* \otimes R(t) \to \operatorname{End} R(t) \to R(t+1) \to 0 ,$$

$$0 \to E^*(t-1) \to R \otimes E^*(t) \to \operatorname{End} E(t) \to 0 .$$

$\operatorname{Coker} g_t = 0$ for $t \geqq 1$ and for $t \leqq -3$ from the first sequence. It is easy to check

$$H^1(\operatorname{End} R(-2)) = 0 .$$

From the second sequence $H^1(\operatorname{End} E) = H^1(E^* \otimes R)$, from the first

$$H^1(E^* \otimes R) \subset H^1(\operatorname{End} R)$$

and the last inclusion is the identity because $H^1(\operatorname{End} E)$ surjects naturally over $H^1(\operatorname{End} R)$ (by lemma 3.9 in the case $\alpha = \beta = 0$). In the case $t = -1$ we have that

$$\operatorname{Coker} g_{-1} \subset H^1(R) = H^1(E) = W \oplus W^*$$

and hence it cannot contribute to the degree zero summand of the tensor product. Let us also observe that from the second sequence it is easy to prove in the same way that $h^1(\tilde{E}_0^* \otimes \tilde{R}_0) = h^1(\operatorname{End} \tilde{E}_0)$. Hence we have proved g to be surjective.

Let (T, t_0) be the Kuranishi space of E_0. As in the proofs of prop. 3.1 and theorem 3.7 we can check that

$$\dim_{t_0} T \geqq h^0(\tilde{R}_0(\gamma)) - h^0(\operatorname{End} \tilde{R}_0) + h^0(\tilde{E}_0^* \otimes \tilde{R}_0) + \dim\{\text{Kuranishi space of } \tilde{R}_0\}$$
$$= (\text{by lemma 3.9})\ h^0(\tilde{R}_0(\gamma)) - h^0(\operatorname{End} \tilde{R}_0) + h^0(\tilde{E}_0^* \otimes \tilde{R}_0) + h^1(\operatorname{End} \tilde{R}_0)$$

where we used the sequence (3.7). Again from sequence (3.7) and from the fact that g is surjective we have

$$\dim_{t_0} T \geqq h^1(\tilde{E}_0^* \otimes \tilde{R}_0) = h^1(\operatorname{End} \tilde{E}_0)$$

as we wanted.

Corollary 3.11. *The family of bundles obtained by pulling back a parent bundle over* $\mathbb{C}^6 \setminus 0$ *is invariant under small deformations if and only if* $\alpha = \beta = 0$.

Proof. If $\beta \neq 0$ a monad exists $\mathcal{O}(-\gamma) \xrightarrow{\sigma} B_{\alpha,\beta,\gamma} \xrightarrow{\tau} \mathcal{O}(\gamma)$ where σ, τ correspond to some linear combinations of the rows of the matrix $\tilde{M}$ (see remark 2.13) with non-constant coefficients. If $\alpha = \beta = 0$ let $\omega' : \mathbb{P}^5 \to \mathbb{P}^5$ be a finite morphism of degree t^5. By theorem 3.7 every small deformation of $\omega'^* B = B_{0,0,t}$ is isomorphic to $\omega''^* B$ for some $\omega'' : \mathbb{P}^5 \to \mathbb{P}^5$ because all the morphisms $\wedge^4 Q_{0,0,t}^* \to \wedge^2 Q_{0,0,t}^*$ are obtained as pull-backs from a morphism $\wedge^4 Q^* \to \wedge^2 Q^*$. In the same way every small deformation of $\omega'^* E = E_{0,0,t}$ is isomorphic to $\omega''^* E$ for some ω''.

Remark 3.12. There is a simpler proof of corollary 3.11 without using theorem 3.10, following the lines of [DS].

4. Proof of theorems B, C, D, E, F

Theorem 4.1. *Let* $\tilde{E} = E_{\alpha,\beta,\gamma}$ *be a relation bundle coming as pullback over* $\mathbb{C}^6 \setminus 0$. *Then* $h^0(\tilde{E}(t)) \neq 0$ *if and only if* $\min\{2\gamma - \alpha - 2\beta, 3\gamma - 2\alpha - 4\beta\} \leqq t$.

Proof. By theorem 2.15 it is easy to check that $h^0(\tilde{E}(t)) \neq 0$ if and only if

$$h^0(\Gamma^{4,2}\mathscr{W}(-3\gamma + t) \oplus \Gamma^{2,1}\mathscr{W}(-2\gamma + t)) \neq 0 .$$

Now consider that the sum of the degree in the Young diagram (according to (0.2))

$\alpha + \beta$	$\alpha + \beta$
$-\alpha$	

is $\alpha + 2\beta$, while the sum of the degrees in

$\alpha+\beta$	$\alpha+\beta$	$\alpha+\beta$	$\alpha+\beta$
$-\alpha$	$-\alpha$		

is $2\alpha+4\beta$.

Proof of theorem B. (i) $\Rightarrow$ (ii) is well known.

(ii) $\Rightarrow$ (iii). If $3\gamma-2\alpha-4\beta\leqq 0$ then $h^0(E_{\alpha,\beta,\gamma})\neq 0$ and $h^0(E^*_{\alpha,\beta,\gamma})\neq 0$ from the theorem 4.1.

(iii) $\Rightarrow$ (i). If $3\gamma-2\alpha-4\beta>0$ then $h^0(E_{\alpha,\beta,\gamma})=0$ and

$$h^0(\wedge^2 E_{\alpha,\beta,\gamma})=h^0(E^*_{\alpha,\beta,\gamma})=0$$

from the theorem 4.1 (recall that we have always $2\gamma-\alpha-2\beta>0$).

Corollary 4.2. *Let $\tilde{E}$ be a relation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. The following are equivalent*:

(i) *$\tilde{E}$ is semistable,*

(ii) $3\gamma-2\alpha-4\beta\geqq 0$.

Remark 4.3. In particular $h^0(E_{0,t-1,t}(-t+4))\neq 0$ for $t\geqq 1$, hence the bundles $E_{0,t-1,t}$ are "strongly unstable" for $t\gg 0$. On the other side the pullback bundles $E_{0,0,t}$ satisfy $h^0(E_{0,0,t}(t))=0$, so they are "strongly stable" for $t\gg 0$.

Theorem C is immediate from theorem A and from theorem B.

Proof of theorem D. We apply theorem B and corollary 4.2. Choosing $\alpha=n-3$, $\beta=n$, $\gamma=2n-2$ for $n\geqq 3$ we have

$$3\gamma-2\alpha-4\beta=0\,,$$
$$c_2=3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=12(n-2)\,.$$

Choosing $\alpha=n-2$, $\beta=n$, $\gamma=2n-1$ for $n\geqq 2$ we have

$$3\gamma-2\alpha-4\beta=1\,,\quad c_2=12(n-1)-1\,.$$

Choosing $\alpha=n-1$, $\beta=n$, $\gamma=2n$ for $n\geqq 1$ we have

$$3\gamma-2\alpha-4\beta=2\,,\quad c_2=12n-4\,.$$

Choosing $\alpha=\beta=n$, $\gamma=2n+1$ for $n\geqq 0$ we have

$$3\gamma-2\alpha-4\beta=3\,,\quad c_2=12n+3\,.$$

Remark 4.4. For $c_2 = 24$ there is no stable $E_{\alpha,\beta,\gamma}$ while for $c_2 = 12$ $E_{0,0,2}$ is stable. By using a computer to check the values of k such that a stable $E_{\alpha,\beta,\gamma}$ exists with $c_2 = 12k$ we can see that for $k \leqq 100$ the only gaps are $k = 2, 10, 14, 26, 34, 70$.

Remark 4.5. There is no semistable $E_{\alpha,\beta,\gamma}$ with $c_2 = 0$.

Proof of theorem E. Every relation bundle coming as pullback over $\mathbb{C}^6 \setminus 0$ determines a unique lambda-three bundle, hence, by lemma 3.4, it determines a unique weighted quotient bundle. Then it is sufficient so show that for a fixed second Chern class $3\gamma^2 + 4\alpha\beta - 4(\alpha+\beta)^2$ we can find α, β, γ satisfying $\alpha + \beta < \gamma$ such that

$$f(\alpha,\beta,\gamma) := h^0(\mathscr{H}(\gamma)) - h^0(\mathscr{H}\otimes\mathscr{H}) + h^0(\wedge^2\mathscr{H}\otimes\mathscr{H}(-\gamma)) - h^0(\wedge^3\mathscr{H}\otimes\mathscr{H}(-2\gamma))$$

(see corollary 3.2) is arbitrarily big.

Starting from an integral solution $(\alpha_0, \beta_0, \gamma_0)$ of the equation

$$3\gamma^2 + 4\alpha\beta - 4(\alpha+\beta)^2 = t$$

(it exists by [Hor2], prop. 4.4, see also the proof of theorem D) we can check with easy computations that $(\alpha_n, \beta_n, \gamma_n)$ is an integral solution for every even n, where

$$\alpha_n = \alpha_0,$$

$$\beta_n = \left(\frac{\alpha_0}{4} + \frac{\beta_0}{2} + \frac{\sqrt{3}}{4}\gamma_0\right)(2+\sqrt{3})^n + \left(\frac{\alpha_0}{4} + \frac{\beta_0}{2} - \frac{\sqrt{3}}{4}\gamma_0\right)(2-\sqrt{3})^n,$$

$$\gamma_n = \left(\frac{\sqrt{3}}{6}\alpha_0 + \frac{\sqrt{3}}{3}\beta_0 + \frac{\gamma_0}{2}\right)(2+\sqrt{3})^n + \left(-\frac{\sqrt{3}}{6}\alpha_0 - \frac{\sqrt{3}}{3}\beta_0 + \frac{\gamma_0}{2}\right)(2-\sqrt{3})^n.$$

In order to see whether this solution is integral, we remind that

$$\sqrt{3}\,[(2+\sqrt{3})^n - (2-\sqrt{3})^n] \equiv 0 \pmod 6 \quad \forall n \in \mathbb{N},$$

$$(2+\sqrt{3})^n + (2-\sqrt{3})^n \equiv 0 \pmod 4 \quad \text{for every even } n.$$

It's straightforwardly verifiable: if $n \gg 0$ then $\alpha_n + \beta_n < \gamma_n$ and

$$\lim_{n\to+\infty} f(\alpha_n, \beta_n, \gamma_n) = +\infty,$$

q.e.d.

Proposition 4.6. *If two relation bundles* $E_{\alpha,\beta,\gamma}$ *and* $E_{\alpha',\beta',\gamma'}$ *defined by two lambda-three bundles* $B_{\alpha,\beta,\gamma}$ *and* $B_{\alpha',\beta',\gamma'}$, *as in the statement of lemma* 3.4, *are isomorphic then* $(\alpha,\beta,\gamma) = (\alpha',\beta',\gamma')$.

Proof. Every relation bundle determines a unique lambda-three bundle, hence, by lemma 3.4 and by our assumption, it determines a unique weighted quotient bundle. The result follows because from the minimal resolution (2.3) of $Q^*_{\alpha,\beta,\gamma}$ we can recover the integers α, β, γ.

Proposition 4.7. *Let $E_{\alpha,\beta,\gamma}$ and $E_{\alpha',\beta',\gamma'}$ be two stable relation bundles coming as pullbacks from $\mathbb{C}^6 \backslash 0$ with $(\alpha, \beta, \gamma) \neq (\alpha', \beta', \gamma')$. They determine two different irreducible components in the moduli space.*

Proof. By the theorem 3.10 and the proposition 4.6.

Proof of theorem F. We can apply the prop. 4.7. We will prove a little bit more, that is, that the number of components goes to infinity even in the range where $\tilde{Q}$ is stable. We will prove that the number

$$N(t) := \#\{(a, \beta, \gamma) \mid 3\gamma^2 + 4\alpha\beta - 4(\alpha+\beta)^2 = t, \gamma > 5\alpha + 5\beta\}$$

satisfies $\limsup N(t) = +\infty$. Let ε, x_0 be such that

$$8e^{1+2\varepsilon} \leqq 27 \quad \text{and} \quad x \cdot \ln\left(1 + \frac{1}{x}\right) \geqq 1 - \varepsilon \quad \text{for} \quad x \geqq x_0 .$$

It is sufficient to check that if $x \geqq x_0$ then

$$\#\{(\alpha, \beta, \gamma) \mid 3\gamma^2 + 4\alpha\beta - 4(\alpha+\beta)^2 = 12x^x(x+1)^x, \gamma > 5\alpha + 5\beta\} \geqq \frac{x}{6} - 2 .$$

For every integer a such that $\frac{x}{3} \leqq a \leqq \frac{x}{2}$ we set

$$A := (x+1)^{x-a}x^a, \quad B := x^{x-a}(x+1)^a, \quad \alpha = \beta = \frac{A-B}{2}, \quad \gamma = A + B .$$

These choices of a are at least $\frac{x}{6} - 2$. Now, we observe that in order to have α, β nonnegative we need $A \geqq B$ which is equivalent to $(x+1)^{x-2a} \geqq x^{x-2a}$ which is satisfied because $a \leqq \frac{x}{2}$. We have

$$3\gamma^2 + 4\alpha\beta - 4(\alpha+\beta)^2 = 3(A+B)^2 + (A-B)^2 - 4(A-B)^2 = 12AB = 12(x+1)^x x^x$$

as we wanted.

The inequality $5\alpha + 5\beta < \gamma$ remains to be checked. It is equivalent to $2A < 3B$, that is

$$\left(\frac{x+1}{x}\right)^{x-a} < \frac{3}{2}\left(\frac{x+1}{x}\right)^a \quad \text{or} \quad \left(1 + \frac{1}{x}\right)^x < \frac{3}{2}\left(1 + \frac{1}{x}\right)^{2a} .$$

It is sufficient to verify $e \leqq \frac{3}{2}\left(1+\frac{1}{x}\right)^{2a}$ that is

$$\ln\frac{2e}{3} \leqq 2a\ln\left(1+\frac{1}{x}\right), \quad a \geqq \frac{\ln(2e/3)}{2\ln(1+1/x)}.$$

It is sufficient to check $\frac{x}{3} \geqq \frac{\ln(2e/3)}{2\ln(1+1/x)}$ and this is true by the choices of ε and x_0.

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