Canonical resolutions of sheaves on Schubert and Brieskorn varieties

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Introduction

Let Ω^j be the bundle of j-forms on the complex projective space $\mathbf{P}^n$ and let $\Omega^j(j)=\Omega^j\otimes_{\mathcal{O}}\mathcal{O}(j)$. In 1978 Beilinson [B] showed that every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$ has a canonical two sided finite *resolution* of the form

$$\ldots \xrightarrow{d_{-2}} C^{-1}_{\mathcal{F}} \xrightarrow{d_{-1}} C^{0}_{\mathcal{F}} \xrightarrow{d_{0}} C^{1}_{\mathcal{F}} \xrightarrow{d_{1}} \ldots$$

where

$$C^p_{\mathcal{F}}=\bigoplus_{s-j=p}\Omega^j(j)\otimes_{\mathbf{C}}H^s(\mathbf{P}^n,\mathcal{F}(-j)).$$

This means that $\frac{\mathrm{Ker}\, d_i}{\mathrm{Im}\, d_{i-1}}=\begin{cases}\mathcal{F} & \text{for } i=0\\ 0 & \text{for } i\neq 0\end{cases}$

Note that the bundles $\Omega^j(j)$ are the same for each sheaf $\mathcal{F}$: for this reason they are called the building blocks for the sheaves on $\mathbf{P}^n$. In particular the canonical resolution of a building block is the trivial resolution with only nonzero element the building block itself. This fact is known as "orthogonality relations".

There are many applications of this theorem of Beilinson: see for example [OSS],[E],[D],[AO].

In general on a manifold X we call *building blocks* some sheaves $\{A_1,\ldots A_k\}$ such that every coherent sheaf on X has a two sided finite resolution whose terms are direct sums of the A_i's.

The Beilinson theorem holds also for complexes of sheaves $\mathcal{F}^\cdot$.

Explicit resolutions of this type are known for sheaves on grassmann manifolds [K1], flag manifolds and quadric smooth hypersurfaces [K2]. All these manifolds are homogeneous.

In this note we find explicit resolutions for complexes of sheaves on every smooth Schubert variety in a flag manifold and on some other non homogeneous varieties. Moreover the orthogonality relations hold also in these cases. The smooth Schubert varieties were classified in [R], among them there are the desingularizations of the classical Schubert cycles in a grassmannian [S].

As in [B] and [K2], we need to use the *derived categories* of bounded complexes of sheaves on X [H]. We underline that the "natural" candidates to be building blocks work on the desingularization of the classical Schubert varieties but fail on general Schubert varieties (see theor. 6 and rem. 7), where we have to twist the natural polarization.

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General remarks

Let E be a holomorphic vector bundle of rank r+1 on a complex manifold X. We denote by E^* its dual. Let $G=Gr(k,E)\subset\mathbf{P}(\overset{k+1}{\wedge}E)$ the *grassmann bundle* of subspaces of projective dimension k.

Let U,Q be the relative universal and quotient bundles on G. Consider the fiber product

$$\begin{array}{ccc} & G\times_X G & \\ {}^{p}\swarrow & & \searrow^{q} \\ G & & G\end{array}$$

We have by the Leray sequence $H^0(G\times_X G,p^*U^*\otimes q^*Q)\simeq H^0(X,E\otimes E^*)\simeq \mathrm{End}(E)$.

The section of $p^*U^*\otimes q^*Q$ corresponding to the identity endomorphism of E in the isomorphism above vanishes exactly on the relative diagonal $\Delta_G\subset G\times_X G$.

Let $\alpha=(\alpha_1,\ldots,\alpha_m)$ be a non increasing sequence of integers. It corresponds to a Young diagram, and let α^* be the sequence corresponding to the transpose diagram. Let $|\alpha|=\sum_{i=1}^m\alpha_i$ be the length of α. We denote by $\Gamma^\alpha E$ the vector bundle corresponding to the irreducible representation of $GL(C^r)$ with highest weight α.

Let $\mathcal{F}^\cdot$ be a complex of sheaves on X. We denote by $\mathbf{H}^\cdot(X,\mathcal{F}^\cdot)$ the hypercohomology complex of $\mathcal{F}^\cdot$. In the derived category it is equal to $R\Gamma(\mathcal{F}^\cdot)$ where $R\Gamma$ is the derived functor of $\Gamma=H^0$.

Theorem 1(Beilinson[B]) *Let $A^j_i,B^j_i(i=1,\ldots,n;j=1,\ldots,k_i)$ be bundles on X and denote by α,β the two projections of $X\times_X X$ on X. Suppose that we have*

(*i*) *a resolution of the diagonal $\Delta_X\subset X\times X$ given by*

$$\ldots\to\bigoplus_{j=1}^{k_2}\alpha^*A^j_2\otimes\beta^*B^j_2\to\bigoplus_{j=1}^{k_1}\alpha^*A^j_1\otimes\beta^*B^j_1\to\mathcal{O}_{X\times X}\to\mathcal{O}_{\Delta_X}\to 0$$

(*ii*) $\mathrm{Ext}^p(X,B^j_i,B^s_t)=0$ for $p>0$ $\forall i,j,t,s$ (1)

Then each complex $\mathcal{F}^\cdot$ on X is obtained as the cohomology of a complex $C_{\mathcal{F}}$ with

$$C^p_{\mathcal{F}}=\bigoplus_{s-i=p}\bigoplus_{j=1}^{k_i}H^s(X,\mathcal{F}^\cdot\otimes A^j_i)\otimes_{\mathbf{C}}B^j_i \qquad (2)$$

so that B^i_j are building blocks for the sheaves on X.

Proof Tensor the resolution in (i) by $\alpha^*\mathcal{F}^\cdot$ and obtain in the derived category

$$[\ldots\to\bigoplus_{j=1}^{k_2}\alpha^*(A^j_2\otimes\mathcal{F}^\cdot)\otimes\beta^*B^j_2\to\bigoplus_{j=1}^{k_1}\alpha^*(A^j_1\otimes\mathcal{F}^\cdot)\otimes\beta^*B^j_1\to\alpha^*\mathcal{F}^\cdot]\sim\alpha^*\mathcal{F}^\cdot|_{\Delta_X}$$

Now apply the derived functor $R\beta_*$ to both sides and obtain $C'^\cdot_{\mathcal{F}}\sim\mathcal{F}$ where $C'^\cdot_{\mathcal{F}}$ is a complex with the same terms of $C^\cdot_{\mathcal{F}}$ but with morphisms defined only in the derived category. By (ii) these morphisms arise from true morphisms of sheaves.

Example 2 (Kapranov[K1]) The hypothesis of theorem 1 are satisfied if $X=Gr(k,n)$ (grassmannian of k-planes in $\mathbf{P}^n$) and $A^j_i=\Gamma^{\alpha_i^*}U$, $B^j_i=\Gamma^{\alpha_i}Q^*$ where $\{\alpha_1,\ldots\alpha_{k_i}\}$ are the Young diagrams of length i,number of rows$\leq n-k$ and number of columns$\leq k+1$ for $i=1,\ldots,(k+1)(n-k)$.

Proposition 3 (relative case of example 2) *Let* $G=Gr(k,E)\xrightarrow{\pi}X$. *Let* A_i^j, B_i^j $(i=1,\dots,n; j=1,\dots,k_i)$ *be bundles on X such that each complex* $\mathcal{F}^\cdot$ *on X is obtained as the cohomology of a complex* $C_{\mathcal{F}}$ *with*

$$C^p_{\mathcal{F}}=\bigoplus_{s-i=p}\bigoplus_{j=1}^{k_i} H^s(X,\mathcal{F}^\cdot\otimes A_i^j)\otimes_{\mathbb{C}} B_i^j$$

If

$$Ext^p(G,\pi^*B_i^j\otimes\Gamma^\alpha Q^*,\pi^*B_t^s\otimes\Gamma^\beta Q^*)=0 \text{ for } p>0, \text{ n. of col. of } \alpha,\beta\le r-k,\ \forall i,j,t,s \quad (3)$$

then each complex of sheaves $\mathcal{G}^\cdot$ *on G is obtained as the cohomology of a complex* $\tilde{C}_{\mathcal{G}}$ *with*

$$\tilde{C}^p_{\mathcal{G}}=\bigoplus_{s-i-h=p}\bigoplus_{|\alpha|=h}\bigoplus_{j=1}^{k_i} H^s(G,\mathcal{G}^\cdot\otimes\pi^*A_i^j\otimes\Gamma^{\alpha^*}U)\otimes_{\mathbb{C}}\pi^*B_i^j\otimes\Gamma^\alpha Q^* \quad (4)$$

so that $\pi^*B_j^i\otimes\Gamma^\alpha Q^*$ *(number of columns of* $\alpha\le r-k$*) are building blocks for the sheaves on G.*

Remark (3) implies (1) taking $\alpha=\beta=0$

Theorem 4 (3) *is always satisfied if* (1) *is satisfied and we substitute E with* $E\otimes L$, *L a sufficiently ample line bundle on X.*

Sketch of the proof Remember that $Gr(k,E)=Gr(k,E\otimes L)$. Apply the Leray spectral sequence and use the fact that $\pi_*(\Gamma^\alpha Q\otimes\Gamma^\beta Q^*)=\Gamma^{\alpha|\beta}E$ (see[L]).

Theorem 5 (orthogonality relations down $\Rightarrow$ orthogonality relations up)
Let $C_{\mathcal{F}}$ *as in* (2) *and* $C'_{\mathcal{F}}$ *as in* (4). *If* $C^p_{\mathcal{F}}=\begin{cases}\mathcal{F} & \text{for } p=0\\ 0 & \text{for } p\neq 0\end{cases}$ *for any* $\mathcal{F}=B_t^s$ *building block*

on X then $C'^p_{\mathcal{F}}=\begin{cases}\mathcal{F} & \text{for } p=0\\ 0 & \text{for } p\neq 0\end{cases}$ *for any* $\mathcal{F}=\pi^*B_t^s\otimes\Gamma^\alpha Q^*$ *(number of columns of* $\alpha\le r-k$*) building block on G.*

The proof of theorem 5 is an application of the generalized Bott theorem as given in [L].

The case of Schubert varieties

Let $m_0\le\dots\le m_s\le j_0\le\dots\le j_k\le n$ be any sequence of integers. Let $\mathbf{F}=\mathbf{F}(m_0,\dots,m_s,j_0,\dots,j_k,n)$ be the flag manifold which parametrizes the flags of subspaces of projective dimension $m_0,\dots,m_s,j_0,\dots,j_k$ in P^n(here the two sets of indexes are inessential). $\mathbf{F}$ is a quotient of the simple Lie group $SL(n+1,\mathbb{C})$ by a parabolic subgroup P. A *Schubert variety* is by definition the closure of a P-orbit in $\mathbf{F}$. Fix now a flag of subspaces in $\mathbf{P}^n$ $B_{t_0}\subset\dots\subset B_{t_s}\subset A_{i_0}\subset\dots\subset A_{i_k}\subset\mathbf{P}^n$ (dim $A_t=t$, dim $B_i=i$). Ryan in [R] proves that the smooth Schubert varieties X in $\mathbf{F}$ are exactly those for which there exists a flag $B_{t_0}\subset\dots\subset B_{t_s}\subset A_{i_0}\subset\dots\subset A_{i_k}$ of subspaces such that $X=X_{B_{t_0}\dots B_{t_s}A_{i_0}\dots A_{i_k}}$ $=\{(X_{m_0},\dots,X_{m_s},X_{j_0},\dots,X_{j_k})\in F(m_0,\dots,m_s,j_0,\dots,j_k,n)\mid B_{t_h}\subset X_{m_h}, X_{j_s}\subset A_{i_s}\ \forall h,s\}$.
$X_{B_{t_0}\dots B_{t_s}}$ $(k=-1)$ and $X_{A_{i_0}\dots A_{i_k}}$ $(s=-1)$ are exactly the desingularizations of the Schubert varieties in a grassmannian [S].

With a slight abuse of notation, on $X_{B_{t_0}\dots B_{t_s}A_{i_0}\dots A_{i_k}}$ we denote by X_{m_h}, X_{j_q} the universal bundles of rank resp. m_h+1, j_q+1 and by X_{m_h}/B_{t_h} and A_{i_q}/X_{j_q} the obvious universal and quotient bundles of rank resp. m_h-t_h and i_q-j_q. $X_{B_{t_0}\dots B_{t_s}A_{i_0}\dots A_{i_k}}$ is for $k\ge1$ the grassmann bundle of subspaces of dimension j_k-j_{k-1} in the bundle $A_{i_k}/X_{j_{k-1}}$ on $X_{B_{t_0}\dots B_{t_s}A_{i_0}\dots A_{i_{k-1}}}$. In this situation the relative quotient and universal bundles are resp. A_{i_k}/X_{j_k} and $X_{j_k}/X_{j_{k-1}}$. In the same way $X_{B_{t_0}\dots B_{t_s}A_{i_0}\dots A_{i_k}}$ is a grassmann bundle on $X_{B_{t_1}\dots B_{t_s}A_{i_0}\dots A_{i_k}}$, so that every smooth Schubert variety can be obtained as a repeated fibration in grassmannians.

Let $\overline{\alpha}=\{\alpha_0,\dots,\alpha_s\}$, $\overline{\beta}=\{\beta_0,\dots,\beta_k\}$ be sequences of Young diagrams. Denote now $\psi^{\overline{\alpha}\overline{\beta}}=\{\bigotimes_{h=0}^{s}\Gamma^{\alpha_h}(X_{m_h}/B_{t_h})\}\otimes\{\bigotimes_{q=0}^{k}\Gamma^{\beta_q}[(A_{i_q}/X_{j_q})\otimes L]^*\}$ for #col. $\alpha_h\le m_{h+1}-m_h$ $(h\le s-1)$, #col. $\alpha_s\le i_0-m_s$, #col. $\beta_q\le j_q-j_{q-1}$ $(q\ge1)$, #col. $\beta_0\le j_0-m_s$, where $L=\det(X^*_{m_s})^{\otimes i_0-m_s-1}$

In the same way denote $\phi^{\overline{\alpha}^*\overline{\beta}^*}=\{\bigotimes_{h=0}^{s}\Gamma^{\alpha_h^*}(X_{m_{h+1}}/X_{m_h})\}\otimes\{\bigotimes_{q=0}^{k}\Gamma^{\beta_q^*}[(X_{j_q}/X_{j_{q-1}})\otimes L]^*\}$ where $X_{m_{s+1}}=A_{i_0}$ and $X_{j_{-1}}=X_{m_s}$.

Let $|\overline{\alpha}|=\sum|\alpha_i|$, $|\overline{\beta}|=\sum|\beta_i|$. Our main theorem is:

Theorem 6 *Every complex of sheaves* $\mathcal{F}^\cdot$ *on the Schubert variety* $X=X_{B_{t_0}\dots B_{t_s}A_{i_0}\dots A_{i_k}}$ *is obtained as the cohomology of a complex* $C_{\mathcal{F}}$ *with*

$$C^p_{\mathcal{F}}=\bigoplus_{s-i=p}\bigoplus_{|\overline{\alpha}|+|\overline{\beta}|=i} H^s(X,\mathcal{F}^\cdot\otimes\phi^{\overline{\alpha}^*\overline{\beta}^*})\otimes_{\mathbb{C}}\psi^{\overline{\alpha}\overline{\beta}}$$

so that $\psi^{\overline{\alpha}\overline{\beta}}$ *are building blocks for the sheaves on X. Moreover, the orthogonality relations hold on X, that is*

$$C^p_{\mathcal{F}}=\begin{cases}\mathcal{F} & \text{for } p=0\\ 0 & \text{for } p\neq 0\end{cases} \text{ for any } \mathcal{F}=\psi^{\overline{\alpha}\overline{\beta}}.$$

Sketch of the proof Consider X as a repeated fibration in grassmannians and apply prop. 3 and theor. 5. The "strange" term L appear in the definition of $\psi^{\overline{\alpha}\overline{\beta}}$ to get the vanishing (3) by Bott theorem at the first step $X_{B_{t_s}A_{i_0}}\to X_{B_{t_s}}=Gr(m_s-t_s,i_0-t_s)$, so that we have a concrete application of the theorem 4.

Remark 7 We may substitute L with $\det(X^*_{m_s})^{\otimes z}$ for $z\ge i_0-m_s-1$.
If $k=-1$ we get the desingularizations of Schubert varieties in the grassmannians and we do not need the theorem 4.
If $k=-1$ and $B_{t_i}=0$ we get the ordinary flag manifolds as in [K2].

Other manifolds

Consider now the (generalized) Brieskorn varieties given by the projective bundle $\mathbf{P}(E)$ where E is a direct sum of line bundles on $\mathbf{P}^n$.

Theorem 8 *Let* $E=\oplus\mathcal{O}(a_i)$ *on* $\mathbf{P}^n$ *with* $a_i\geq 0$ *and* $X=\mathbf{P}(E)\xrightarrow{\pi}\mathbf{P}^n$. *Let* $\mathcal{O}_{rel}(-1)$ *and* Q *be the relative universal and quotient bundle. Then any complex* $\mathcal{F}^\cdot$ *of coherent sheaves on* X *is obtained as the cohomology of the complex* $C^\cdot_{\mathcal{F}}$ *where*

$$C^p_{\mathcal{F}}=\bigoplus_{s-i=p}\bigoplus_{q+h=i}H^s(\pi^*\mathcal{O}_{\mathbf{P}^n}(-q)\otimes\mathcal{O}_{rel}(-h)\otimes\mathcal{F}^\cdot)\otimes\pi^*\Omega^q_{\mathbf{P}^n}(q)\otimes\stackrel{h}{\wedge}Q^*$$

The orthogonality relations hold on X.

Proof The vanishing (3) are satisfied by the Leray spectral sequence and by $H^s(\mathbf{P}^n,\Omega^i(i)\otimes\Omega^t(t)^*\otimes\mathcal{O}(a))=0\ \forall s>0,\forall a\geq 0,\ \forall\ i,t$. Then the theorem is a standard application of the proposition 3 and the theorem 5.

Remark The theorem 8 is easily generalized to the case of grassmann bundles $Gr(k,E)$ with $E=\oplus\mathcal{O}(a_i)$ $(a_i\geq 0)$ splitting bundle on the grassmannian $Gr(m,n)$.

We underline that a necessary condition on a variety X to have a finite number of building blocks is that $h^1(X,\mathcal{O}_X)=0$. In fact the set of the line bundles that have a two sided resolution by a finite number of other sheaves is easily seen to be countable.

It seems to be an open problem to find explicit building blocks for the sheaves on the rational homogeneous varieties G/P. We are able to find for every G/P a finite number of (homogeneous) bundles $\{A_1,\ldots,A_k\}$ such that for each sheaf $\mathcal{F}$ on G/P there exists a spectral sequence whose terms are direct sums of the A_i's abutting to $\mathcal{F}$.

References

[AO] *V.Ancona, G.Ottaviani*, Some applications of Beilinson theorem to projective spaces and quadrics, preprint

[B] *A.A.Beilinson*, Coherent sheaves on $\mathbf{P}^n$ and problems of linear algebra, Funkt. Analiz Prilozheniya,12 n.3,68-69(1978),

[D] *W.Decker*, Stable rank 2 vector bundles with Chern classes $c_1=-1$, $c_2=4$, Math. Ann. 275, 481-500(1986)

[E] *L.Ein*, Some stable vector bundles on $\mathbf{P}^4$ and $\mathbf{P}^5$, Journal reine angew. Math. 337, 142-153(1982)

[H] *R.Hartshorne*, Residues and duality, Springer LNM 20, New York Heidelberg Berlin 1966

[K1] *M.M.Kapranov*, On the derived category of coherent sheaves on Grassmann varieties, USSR Math. Izvestija 48,192-202(1984)

[K2] *M.M.Kapranov*, On the derived categories of coherent sheaves on some homogeneous spaces, Inv.Math. 92,479-508(1988)

[L] *A.Lascoux*, Syzygies des variétés déterminantales, Adv. in Math. 30 n.3,202-237(1978)

[OSS] *C.Okonek, M.Schneider, H.Spindler*, Vector bundles on complex projective spaces, Progress in Math. 3, Birkhauser Boston 1980

[R] *K.Ryan*, On Schubert varieties in the flag manifold of Sl(n,C), Math. Ann. 276,205-224(1987)

[S] *T.Svanes*, Coherent cohomology on Schubert subschemes of flag schemes and applications, Adv. in Math. 14,369-453(1974)

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