ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

R. BRAUN
G. OTTAVIANI
M. SCHNEIDER
F.-O. SCHREYER

Classification of conic bundles in $\mathbb{P}^5$

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 23, nº 1 (1996), p. 69-97.

<http://www.numdam.org/item?id=ASNSP_1996_4_23_1_69_0>

Classification of Conic Bundles in $\mathbb{P}_5$

R. BRAUN - G. OTTAVIANI - M. SCHNEIDER - F.-O. SCHREYER

1. - Introduction

Smooth threefolds $X \subset \mathbb{P}_5$ which are not of general type have bounded degree d_0 (cf. [BOSS]). At the moment it is not clear what the best bound d_0 should be.

The adjunction-theoretic approach leads in a very natural way to the class of log general type 3-folds (cf. [BBS]).

DEFINITION. *A smooth 3-fold $X \subset \mathbb{P}_5$ is said to be of log general type if $K_X + H$ is big and nef. Here H denotes the hyperplane class. We say that X is log special if X is not of log general type.*

The purpose of this paper is to give a complete classification of all log special 3-folds in $\mathbb{P}_5$. It turns out that their degree is bounded by 12.

Note that 3-folds in $\mathbb{P}_5$ are classified up to degree 11 ([BSS1, 2]), uniqueness is known only up to degree 10. For partial results on the classification in degree 12 see also [E].

By general adjunction theory due to Sommese [S3] the structure of log special manifolds is well understood and the problem is to decide whether they live in $\mathbb{P}_5$ and up to what degree.

In most cases this is either well known or easy to decide. The two difficult cases are:

a) X is ruled in lines;

b) X is a conic bundle over a surface.

Case a) has been dealt with in [O]: there are four types of scrolls, all known classically.

Case b) is more delicate and its analysis is the heart of this paper.

The only classically known example is the Castelnuovo conic bundle X of degree 9 which is the determinantal variety defined by the maximal minors

Pervenuto alla Redazione il 25 Agosto 1994 e in forma definitiva il 28 Agosto 1995.

of a 3×4-matrix with linear and quadratic entries:

$$0 \to \mathcal{O}^3 \to \mathcal{O}(1)^3 \oplus \mathcal{O}(2) \to \mathcal{I}_X(5) \to 0.$$

The morphism given by $K + H$ makes X into a conic bundle over $\mathbb{P}_2$.

One of the points of this paper is the construction of a classically unknown example of a conic bundle of degree 12:

Let E be *the* rank 5 vector bundle on $\mathbb{P}_5$, irreducible for the action of Sp(6), with $c_1(E) = 5$, c.f. [H]. The bundle E is generated by global sections and four general sections of E define a smooth 3-fold $X \subset \mathbb{P}_5$

$$0 \to \mathcal{O}^4 \to E \to \mathcal{I}_X(5) \to 0.$$

The morphism $X \to \mathbb{P}_3$ given by $K + H$ makes X into a conic bundle over a quartic surface $B \subset \mathbb{P}_3$.

A resolution of its ideal sheaf is of the form

$$0 \to \mathcal{O}^4 \oplus \Omega^4(5) \to \Omega^2(3) \to \mathcal{I}_X(5) \to 0.$$

THEOREM. *Let $X \subset \mathbb{P}_5$ be a smooth conic bundle over a surface. Then* $\deg X = 9$ *or* $\deg X = 12$ *and X is as in the above examples.*

Putting things together we get the following picture.

THEOREM. *Let $X \subset \mathbb{P}_5$ be a non degenerate and log special smooth 3-fold. Then* $\deg(X) \leq 12$ *and X is one of the varieties of the following table:*

d	log-special 3-folds in $\mathbb{P}_5$	resolution of $\mathcal{I}_X(5)$
3	cubic Segre scroll	$\mathcal{O}(2)^2 \to \mathcal{O}(3)^3$
4	complete intersection of 2 quadrics	$\mathcal{O}(1) \to \mathcal{O}(3)^2$
5	quadric fibration over $\mathbb{P}_1$	$\mathcal{O}(1)^2 \to \mathcal{O}(2)^2 \oplus \mathcal{O}(3)$
6	complete intersection of a quadric and a cubic	$\mathcal{O} \to \mathcal{O}(2) \oplus \mathcal{O}(3)$
6	Bordiga scroll over $\mathbb{P}_2$	$\mathcal{O}(1)^3 \to \mathcal{O}(2)^4$
7	Palatini scroll over cubic in $\mathbb{P}_3$	$\mathcal{O}(1)^4 \to \Omega^1(3)$
7	$S_{(2,2,2)}(x_0)$, the blow up of a complete intersection of 3 quadrics in $\mathbb{P}_6$	$\mathcal{O} \oplus \mathcal{O}(1) \to \mathcal{O}(2)^3$
7	Del Pezzo fibration over $\mathbb{P}_1$	$\mathcal{O}^2 \to \mathcal{O}(1)^2 \oplus \mathcal{O}(3)$
8	Del Pezzo fibration over $\mathbb{P}_1$	$\mathcal{O}^2 \to \mathcal{O}(1) \oplus \mathcal{O}(2)^2$
9	conic bundle over $\mathbb{P}_2$	$\mathcal{O}^3 \to \mathcal{O}(1)^3 \oplus \mathcal{O}(2)$
9	scroll over a $K3$ surface	$\mathcal{O}(1)^9 \to \Omega^3(5)$
12	conic bundle over quartic in $\mathbb{P}_3$	$\mathcal{O}^4 \oplus \Omega^4(5) \to \Omega^2(3)$

($\mathcal{F} \to \mathcal{G}$ in the right hand column means that we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{I}_X(5) \to 0).$$

Acknowledgement: We would like to thank the DFG-Schwerpunktprogramm "Komplexe Mannigfaltigkeiten" for support.

We thank the referee for his detailed suggestions for improvement and S. Mukai for pointing out the note added in proof.

2. - Notations and Preliminaries

2.1. In the following X is a (smooth) 3-fold of degree d in $\mathbb{P}_5$, H is the hyperplane divisor and K_X is the canonical divisor. We denote by Y a generic hyperplane section of X; C is a generic hyperplane section of Y and g the genus of C.

2.2. From the exact sequence

$$0 \to T_X \to T_{\mathbb{P}_5}|_X \to N_{X|\mathbb{P}_5} \to 0$$

one gets

$$c_2(X) = (15-d)H^2 - 6Hc_1(X) + c_1^2(X)$$

$$c_3(X) = c_2(X)(c_1(X) - 6H) - c_1(X)dH^2 + 20d.$$

2.3. Selfintersection formula:

$$c_2(N_{X|\mathbb{P}_5}) = dH^2.$$

2.4. Double point formula for smooth surfaces in $\mathbb{P}_4$:

$$2K_Y^2 = d(d-5) - 10(g-1) + 12\chi(\mathcal{O}_Y).$$

2.5. Castelnuovo bound [GP]: If $C \subset \mathbb{P}_3$ is not contained in a hypersurface of degree $s-1$, then

$$g - 1 \leq \frac{d^2}{2s} + \frac{d}{2}(s-4).$$

2.6. Roth theorem: If $Y \subset \mathbb{P}_4$ is of degree $d > s^2$ and C is contained in a hypersurface of degree s, then Y is contained in a hypersurface of degree s.

2.7. Ellingsrud-Peskine bound [EP, lemme 1]:

If the minimal degree of a hypersurface containing $Y \subset \mathbb{P}_4$ is s, then

$$g - 1 \geq \frac{d^2}{2s} + \frac{d}{2}(s-4) - \frac{(s-1)^2}{2s}d.$$

3. - Reduction to conic bundles

In this section we show via adjunction theory that the degree of log special 3-folds in $\mathbb{P}_5$, which are not conic bundles, is bounded by 9. Basically this is a summary of results contained in [BSS1, 2], [O], [Sch]. For the convenience of the reader we have also included some proofs.

Let X be a 3-fold of degree d in $\mathbb{P}_5$.

PROPOSITION 3.1 [BSS1]. *$K_X + 2H$ is generated by sections, if $d \geq 4$.*

PROOF. By [S1], [SV] $K_X + 2H$ is generated, unless (X, H) is one of the following:

1. $(\mathbb{P}_3, \mathcal{O}(1))$
2. $(\mathcal{Q}, \mathcal{O}_{\mathbb{P}_4}(1)|_{\mathcal{Q}})$, $\mathcal{Q}$ a smooth quadric in $\mathbb{P}_4$
3. (X, H) is a $\mathbb{P}_2$-bundle over a curve and $H|_F = \mathcal{O}(1)$, where F is a fibre.

In (1) and (2) X is degenerate. In (3) the exact sequence of normal bundles of $F \subset X \subset \mathbb{P}_5$ reads

$$0 \to \mathcal{O}_{\mathbb{P}_2} \to \mathcal{O}_{\mathbb{P}_2}(1)^{\oplus 3} \to N_{X|\mathbb{P}_5}|_F \to 0.$$

This implies

$$c_2(N_{X|\mathbb{P}_5}|_F) = 3$$

which gives $d = 3$ by 2.3.

From now on assume $K_X + 2H$ to be generated by sections.

PROPOSITION 3.2. *If $K_X + 2H$ is not nef and big, then $d \leq 9$.*

PROOF. By [S2], [SV], [S3], $K_X + 2H$ is nef and big, unless:

1. (X, H) is a del Pezzo variety, i.e. $K_X = -2H$
2. (X, H) is a scroll over a curve
3. (X, H) is a quadric fibration over a smooth curve
4. (X, H) is a scroll over a surface.

(1) implies that Y is a del Pezzo surface in $\mathbb{P}_4$; hence $d = 4$ and X is a complete intersection of 2 hyperquadrics.

For (2) the same argument as in Proposition 3.1, case (3), applies, i.e. $d = 3$.

In case (3) let F be a smooth fiber, i.e. F is a smooth quadric in $\mathbb{P}_3$, and consider the sequence of normal bundles

$$0 \to \mathcal{O}_F \to \mathcal{O}_F(2) \oplus \mathcal{O}_F(1)^{\oplus 2} \to N_{X|\mathbb{P}_5}|_F \to 0.$$

This gives

$$c_2(N_{X|\mathbb{P}_5}|_F) = 10$$

and thus $d = 5$ by 2.3. (4) is studied in [O], where it turns out that $d \leq 9$.

Assume now that K_X+2H is nef and big. Therefore by general adjunction theory (X,H) has a unique (first) reduction (Z,L), i.e.

$$\pi : X \to Z$$

is a blow up of a finite number γ of smooth points of the smooth 3-fold Z and

$$K_X + 2H = \pi^*(K_Z + 2L),$$

L being ample on Z.

PROPOSITION 3.3 [BSS2]. *π is an isomorphism, if $d \neq 7$.*

PROOF. If there exists an exceptional linear $\mathbb{P}_2 =: P$ in X, the normal bundle sequence of $P \subset X \subset \mathbb{P}_5$ is (see [Sch])

$$0 \to \mathcal{O}_{\mathbb{P}_2}(-1) \to \mathcal{O}_{\mathbb{P}_2}(1)^{\oplus 3} \to N_{X|\mathbb{P}_5}|_P \to 0.$$

Hence $c_2(N_{X|\mathbb{P}_5}|_P) = 7$ which implies $d = 7$ by 2.3.

Next we suppose that X coincides with its first reduction. Note that among the three types of 3-folds of degree 7 there is only $S_{(2,2,2)}(x_0)$ for which X is different from the first reduction.

PROPOSITION 3.4. *If $K_X + H$ is not nef and big and X is not a conic bundle, then $d \leq 8$.*

PROOF. Under the assumptions above, by [S3], $K_X + H$ is nef and big, unless:

1. $(X,H) = (\mathbb{P}_3, \mathcal{O}(3))$
2. $(X,H) = (\mathcal{Q}, \mathcal{O}_{\mathbb{P}_4}(2)|_{\mathcal{Q}})$, $\mathcal{Q}$ a smooth quadric in $\mathbb{P}_4$
3. (X,H) is a Fano variety, i.e. $K_X = -H$
4. (X,H) is a del Pezzo fibration over a smooth curve, i.e. the general fiber F is a del Pezzo surface and $H|_F = -K_F$
5. (X,H) is a Veronese fibration over a smooth curve, i.e. the general fiber F is a $\mathbb{P}_2$ and $H|_F = \mathcal{O}(2)$
6. (X,H) is a conic bundle over a surface.

(1) and (2) do not occur since X has to be linearly normal.

(3) implies that $K_Y = \mathcal{O}_Y$; hence $\chi(\mathcal{O}_Y) = 2$ and by the adjunction formula $d = 2g - 2$. Therefore the double point formula 2.4 gives

$$0 = d(d-5) - 5d + 24 = (d-4)(d-6).$$

Hence $d = 4$ or 6, from which we deduce that X is a complete intersection of type $(1,4)$ or $(2,3)$.

In case (4) notice first that for a general fiber F we have

$$3 \leq \deg F = H^2 F =: \sigma \leq 9$$

$$c_1^2(F) = \sigma; \quad c_2(F) = 12 - \sigma.$$

From the exact sequence

$$0 \to T_F \to T_{\mathbb{P}_5}|_F \to N_{F|\mathbb{P}_5} \to 0$$

we get

$$c_1(N_{F|\mathbb{P}_5}) = 5H|_F$$

$$c_2(N_{F|\mathbb{P}_5}) = 11\sigma - 12.$$

2.3 and the exact sequence of normal bundles

$$0 \to \mathcal{O}_F \to N_{F|\mathbb{P}_5} \to N_{X|\mathbb{P}_5}|_F \to 0$$

yield

$$d \cdot \sigma = dH^2 F = c_2(N_{X|\mathbb{P}_5}|_F) = c_2(N_{F|\mathbb{P}_5}) = 11\sigma - 12.$$

Hence

$$0 = (d - 11)\sigma + 12,$$

which implies $\sigma = 3$, 4 or 6 and $d = 7$, 8 or 9 respectively, furthermore $d = 9$ is ruled out by the classification [BSS1].

Case (5) is excluded by calculating as in case (4) which gives:

$$4d = c_2(N_{X|\mathbb{P}_5}|F) = c_2(N_{X|\mathbb{P}_5}) = 30.$$

The rest of the paper is devoted to case (6), i.e. conic bundles over a surface.

4. - Conic bundles I

4.1. Let $X \subset \mathbb{P}_5$ be a conic bundle in the adjunction theoretic sense, i.e. there exists a morphism $p: X \to B$ onto a normal surface B and an ample Cartier divisor L on B such that $p^* L = K_X + H$.

The main result of this section is that the degree of X has to be 9 or 12 (see Theorem 4.18).

First we shall show that B is necessarily smooth and that all the fibres of p are one-dimensional.

Beltrametti and Sommese have classified the possible fibers of a conic bundle.

THEOREM 4.2 ([BS], Theorem 2.3). *If $p: X \to B$ is a conic bundle in the adjunction theoretic sense, then there are the following possibilities for the fiber over $b \in B$ ($\mathbb{F}_r$ is the r-th Hirzebruch surface):*

$$p^{-1}(b) \simeq \begin{cases} \text{a conic} \\ \mathbb{F}_0 \simeq \mathbb{P}_1 \times \mathbb{P}_1 & \text{with} \quad H_{\mathbb{F}_0} \simeq \mathcal{O}_{\mathbb{F}_0}(1,2) \\ \mathbb{F}_0 \cup \mathbb{F}_1 & \text{with} \quad H_{\mathbb{F}_0} \simeq \mathcal{O}_{\mathbb{F}_0}(1,1), \ H_{\mathbb{F}_1} \simeq E + 2f. \end{cases}$$

DEFINITION. *A 3-fold X is called a geometric conic bundle, if there exists a morphism $p: X \to B$ onto a normal surface B such that every fiber $p^{-1}(b)$ is isomorphic to a conic.*

PROPOSITION 4.3. *Let $X \subset \mathbb{P}_5$ be a conic bundle in the adjunction theoretic sense. Then X is a geometric conic bundle.*

PROOF. Assume first that there is a divisorial fibre $\mathbb{F}_0$ with $H_{\mathbb{F}_0} \simeq \mathcal{O}_{\mathbb{F}_0}(1,2)$. From the sequence

$$0 \to T_{\mathbb{F}_0} \to T_{\mathbb{P}_5}|_{\mathbb{F}_0} \to N_{\mathbb{F}_0|\mathbb{P}_5} \to 0$$

we compute

$$4 + c_2(N_{\mathbb{F}_0|\mathbb{P}_5}) - K_{\mathbb{F}_0}(6H + K_{\mathbb{F}_0}) = 15H^2 \cdot \mathbb{F}_0.$$

Substitute $K_{\mathbb{F}_0} \cdot H = \mathcal{O}(-2,-2) \cdot \mathcal{O}(1,2) = -6$, $K_{\mathbb{F}_0}^2 = 8$ and get

$$c_2(N_{\mathbb{F}_0|\mathbb{P}_5}) = 28.$$

Since $K_X + H$ is trivial on the fibers, we have $N_{\mathbb{F}_0|X} = K_{\mathbb{F}_0} - K_X|_{\mathbb{F}_0} = K_{\mathbb{F}_0} + H|_{\mathbb{F}_0}$ and $c_1(N_{X|\mathbb{P}_5}|_{\mathbb{F}_0}) = (6H + K_X)|_{\mathbb{F}_0} = 5H|_{\mathbb{F}_0}$. Now from the sequence

$$0 \to N_{\mathbb{F}_0|X} \to N_{\mathbb{F}_0|\mathbb{P}_5} \to N_{X|\mathbb{P}_5}|_{\mathbb{F}_0} \to 0$$

and the self intersection formula $c_2(N_{X|\mathbb{P}_5}|_{\mathbb{F}_0}) = dH^2 \cdot \mathbb{F}_0 = 4d$ (cf. 2.3) we have

$$28 = (K_{\mathbb{F}_0} + H) \cdot 5H + 4d$$

$$28 = -30 + 20 + 4d, \quad 4d = 38$$

which is a contradiction.

In the same way we exclude $\mathbb{F}_0$ with $H|_{\mathbb{F}_0} \simeq \mathcal{O}_{\mathbb{F}_0}(1,1)$. Here $K_{\mathbb{F}_0} \cdot H = \mathcal{O}(-2,-2) \cdot \mathcal{O}(1,1) = -4$ which yields

$$c_2(N_{\mathbb{F}_0|\mathbb{P}_5}) = 10.$$

As above we find $d = 10$. But for $d = 10$ we know from [BSS1] that there are no conic bundles.

NOTATIONS 4.4. Let $p: X \to B$ be a geometric conic bundle in $\mathbb{P}_5$. We have a natural morphism $f: B \to \mathrm{Gr}(\mathbb{P}_2, \mathbb{P}_5)$. We set $p_* \mathcal{O}_X(1) =: E$, a rank 3 vector bundle on B. We have $E = f^*(U^\vee)$, where U is the universal bundle of $\mathrm{Gr}(\mathbb{P}_2, \mathbb{P}_5)$, in particular $\det E = f^*(\wedge^2 U^\vee)$ is ample. $W := \mathbb{P}(E)$ is a $\mathbb{P}_2$-bundle in the natural incidence variety in $\mathbb{P}_5 \times \mathrm{Gr}(\mathbb{P}_2, \mathbb{P}_5)$ whose projection π into $\mathbb{P}_5$ is the hypersurface V given by the union of all the planes containing the conics of X.

Furthermore $\tilde{X}$, the strict transform of X under π, is smooth and isomorphic to X. Since the natural projection of $\tilde{X}$ onto B is $p \circ \pi$, we denote the natural projection of W onto B also by p and by H the divisor on W corresponding to $\mathcal{O}_W(1)$. Then $\tilde{X} = 2H - p^*L$ for some divisor L on B.

The divisor $D \subset B$ corresponding to points whose fiber is a singular conic, is called the *discriminant divisor*. More precisely, $\tilde{X}$ determines a section of $S^2E \otimes L^\vee$, hence a morphism $\phi: L \otimes E^\vee \to E$. D is given by the equation $\det \phi = 0$. Consequently $D = c_1(E) - c_1(L \otimes E^\vee) = 2c_1(E) - 3L$.

PROPOSITON 4.5. *Let X be a geometric conic bundle in $\mathbb{P}_5$. Then B is smooth.*

PROOF. The assertion is local in the base B. Therefore we may assume that X is embedded into $B \times \mathbb{P}_2 =: Y$ as a *Cartier divisor*, given by the composition

$$\mathcal{O}_Y \to p^*(S^2(E) \otimes L^{-1}) \to \mathcal{O}_Y(2) \otimes p^*(L^{-1}).$$

Let $p \in X$ be an arbitrary point. We obtain

$$\dim(m_{Y,p}/m_{Y,p}^2) \leq \dim(m_{X,p}/m_{X,p}^2) + 1 = \dim X + 1 = \dim_p Y.$$

Hence Y is smooth in p and therefore also B is smooth.

REMARK. Besana proves in [Bes], using Mori theory, that the base surface of a smooth adjunction theoretic conic bundle is smooth. He observes that a geometric conic bundle is also an adjunction theoretic one.

From now on let $p: X \to B$ in $\mathbb{P}_5$ be a geometric conic bundle over the smooth surface B and let $f: B \to \mathrm{Gr}(\mathbb{P}_2, \mathbb{P}_5)$ be the natural morphism.

LEMMA 4.6. *Let $Y = X \cap \mathbb{P}_4$ be a generic hyperplane section. Then the restriction $p: Y \to B$ is finite* $2:1$.

PROOF. $\mathrm{Gr}(\mathbb{P}_2, \mathbb{P}_4)$ is embedded in $\mathrm{Gr}(\mathbb{P}_2, \mathbb{P}_5)$ as a Schubert cycle of codimension 3. Hence if $\mathbb{P}_4$ is generic $f(B) \cap \mathrm{Gr}(\mathbb{P}_2, \mathbb{P}_4) = \emptyset$.

Let us introduce some more notation.

Let $2R \subset B$ be the branch divisor of $p: Y \to B$.

Recall that we want to show $d = 9$ or 12. As a first approximation we prove $d \leq 72$ (see Proposition 4.17). This follows basically from the three inequalities $D \cdot R \geq 0$, $c_2(E) \geq 0$ and $c_1(E) \cdot D \geq 0$. To express these in terms of $x = K_B^2$, $y = D \cdot R$ and d we need several preliminary computations.

PROPOSITION 4.7.

$$c_1(W) = 3H - p^*c_1(E) - p^*K_B$$

$$c_2(W) = 3H^2 + H(-3p^*K_B - 2p^*c_1(E)) + p^*c_2(B) + p^*K_B p^*c_1(E) + p^*c_2(E)$$

$$c_3(W) = H^2(-3p^*K_B) + H(3p^*c_2(B) + 2p^*K_B p^*c_1(E))$$

$$H^3 - H^2p^*c_1(E) + Hp^*c_2(E) = 0.$$

PROOF. Consider the sequence

$$0 \to \mathcal{O} \to p^*E^\vee \otimes \mathcal{O}_W(1) \to T_W \to p^*T_B \to 0.$$

The Chern polynomial of $p^*E^\vee \otimes \mathcal{O}_W(1)$ is

$$1 + [3H - p^*c_1(E)]t + [3H^2 - 2H \cdot p^*c_1(E) + p^*c_2(E)]t^2.$$

Hence we have

$$1 + c_1(W)t + c_2(W)t^2 + c_3(W)t^3 = \{1 + [3H - p^*c_1(E)]t + [3H^2 - 2H \cdot p^*c_1(E) + p^*c_2(E)]t^2\} \cdot \{1 - p^*K_Bt + c_2(B)t^2\}.$$

Expanding the right-hand side we get the first three equations.

The last equation is the Wu-Chern equation on $W = \mathbb{P}(E)$, which is equivalent to $c_3(p^*E^\vee \otimes \mathcal{O}_W(1)) = 0$.

LEMMA 4.8. *$c_1(E) = 3R - D$, $L = 2R - D$ (L is defined in 4.6).*

PROOF. We have $K_Y = p^*(K_B + R)$ (see 4.4), hence by the adjunction formula $K_X = -H + p^*K_B + p^*R$. From Proposition 4.7 $K_W = -3H + p^*c_1(E) + p^*K_B$. Putting together with the adjunction formula $K_X = K_W|_X + (2H - p^*L)|_X$ we get $-p^*L + p^*c_1(E) + p^*K_B = p^*K_B + p^*R$, that is $c_1(E) = L + R$.

Substituting into $c_1(E) = (D + 3L)/2$ (4.6) we obtain $D + 3L = 2L + 2R$, that is $L = 2R - D$. Hence $c_1(E) = (2R - D) + R = 3R - D$.

PROPOSITION 4.9.

$$c_1(X) = H - p^*K_B - p^*R$$

$$c_2(X) = H^2 + H[-p^*K_B - p^*c_1(E) + p^*R] + p^*R^2 + p^*R \cdot p^*K_B - p^*c_1(E) \cdot p^*R + p^*c_2(B) + p^*c_2(E)$$

$$c_3(X) = -2H^3 + H^2(3p^*c_1(E) - p^*K_B - 3p^*R) + H[p^*c_2(B) + p^*K_B \cdot p^*c_1(E) - 3p^*R^2 - p^*R \cdot p^*K_B + 4p^*c_1(E) \cdot p^*R - 2p^*c_2(E) - p^*c_1^2(E)].$$

PROOF. Straightforward computation from the sequence

$$0 \to T_X \to T_W|_X \to \mathcal{O}(2H + R - c_1(E)) \to 0$$

using Proposition 4.7.

LEMMA 4.10. *Let Z, Z' be arbitrary divisors on B.*

i) $H \cdot p^*Z \cdot p^*Z' = 2Z \cdot Z'$

ii) $H^2 \cdot p^*Z = [c_1(E) + R] \cdot Z = [4R - D] \cdot Z.$

PROOF. (i) is obvious because the fibers are conics. For (ii) note that $H^2 \cdot p^*Z$ is equal to the intersection product on W

$$H^2 \cdot p^*Z \cdot (2H + p^*R - p^*c_1(E)).$$

Intersect the Wu-Chern equation with p^*Z to obtain

$$H^3 \cdot p^*Z = H^2 \cdot p^*c_1(E) \cdot p^*Z = c_1(E) \cdot Z.$$

Hence

$$\begin{aligned} H^2 \cdot p^*Z \cdot (2H + p^*R - p^*c_1(E)) &= 2H^3 \cdot p^*Z + R \cdot Z - c_1(E) \cdot Z \\ &= 2c_1(E) \cdot Z + R \cdot Z - c_1(E) \cdot Z \\ &= c_1(E) \cdot Z + R \cdot Z. \end{aligned}$$

PROPOSITION 4.11. *The surface $f(B)$ in* $\mathrm{Gr}(\mathbb{P}_2, \mathbb{P}_5)$ *has bidegree* $(\delta, c_2(E))$ *where*

$$\delta = \deg V = \frac{d - R \cdot c_1(E) + c_1^2(E)}{2}, \quad c_2(E) = \frac{-d + R \cdot c_1(E) + c_1^2(E)}{2}.$$

More precisely, if $\mathcal{L}$ and $\mathcal{P}$ are respectively a generic line and a generic $\mathbb{P}_3$, $\delta = \#\{\mathbb{P}_2 | \mathbb{P}_2 \cap \mathcal{L} \neq \emptyset, \mathbb{P}_2 \in f(B)\}$, $c_2(E) = \#\{\mathbb{P}_2 | \dim \mathbb{P}_2 \cap \mathcal{P} \geq 1, \mathbb{P}_2 \in f(B)\}$. Moreover $c_1^2(E) = \delta + c_2(E)$ is the degree of $f(B)$ in the Plücker embedding.

PROOF. We intersect the Wu-Chern equation in 4.7 with H and we get $H^4 - H^3 \cdot p^*c_1(E) + H^2 \cdot p^*c_2(E) = 0$, that is $\delta - c_1^2(E) + c_2(E) = 0$. Now cut the equation $\tilde{X} = 2H + p^*R - p^*c_1(E)$ with H^3 and obtain $d = 2\delta + R \cdot c_1(E) - c_1^2(E)$. From these equalities the expressions of δ and $c_2(E)$ follow. The geometrical interpretation of δ is clear, because $\delta = \#\{V \cap \mathcal{L}\}$. The other statements are applications of Schubert calculus.

PROPOSITION 4.12.

$$\begin{aligned}
c_1(X) &= H - p^*K_B - p^*R \\
c_2(X) &= H^2 + H(-p^*K_B - 2p^*R + p^*D) \\
&\qquad + (-2p^*R^2 + p^*R \cdot p^*K_B + p^*D \cdot p^*R + p^*c_2(B) + p^*c_2(E)) \\
c_3(X) &= 2c_2(B) - D^2 - D \cdot K_B.
\end{aligned}$$

PROOF. Easy computations, using 4.8-4.11, give the formulas for $c_2(X)$ and $c_3(X)$.

PROPOSITION 4.13. *The following hold:*

$$\begin{aligned}
&-10K_B \cdot R + 4(d-12)R^2 - (d-11)D \cdot R = 0 \\
&10K_B^2 - 4(d-12)R + (d-11)K_B \cdot D = 0 \\
&-10K_B \cdot D + 4(d-12)D \cdot R - (d-11)D^2 = 0 \\
&d(d-10) - 22K_B \cdot R - 18R^2 - 2K_B^2 + 2c_2(B) + 5K_B \cdot D + 5D \cdot R = 0 \\
&d(d-20) - 4(d+3)K_B \cdot R - 4dR^2 + 12c_2(B) - 2K_B^2 + (d+5)K_B \cdot D \\
&\qquad + (d+6)D \cdot R - D^2 = 0.
\end{aligned}$$

PROOF. From (2.2) we have $c_2(X) = (15-d)H^2 - 6H \cdot c_1(X) + c_1^2(X)$. Substituting the values of $c_1(X)$, $c_2(X)$ of Proposition 4.12 we get

$$\begin{aligned}
&H^2 + H(-p^*K_B - 2p^*R + p^*D) + (-2p^*R^2 + p^*D \cdot p^*R + p^*R \cdot p^*K_B \\
&\quad + p^*c_2(B) + p^*c_2(E)) = H^2(10-d) + H(4p^*K_B + 4p^*R) \\
&\quad + (p^*K_B^2 + p^*R^2 + 2p^*K_B \cdot p^*R).
\end{aligned}$$

Now cut respectively with p^*R, p^*K_B, p^*D, H and obtain the first four equations. For example cutting with p^*R we have

$$\begin{aligned}
&H^2 \cdot p^*R + H(-p^*K_B \cdot p^*R - 2p^*R^2 + p^*D \cdot p^*R) \\
&\qquad = H^2 \cdot p^*R(10-d) + H(4p^*K_B \cdot p^*R + 4p^*R^2)
\end{aligned}$$

so that by lemmas 4.8 and 4.10

$$R \cdot (4R - D) - 2K_B \cdot R - 4R^2 + 2D \cdot R = (10-d)R \cdot (4R - D) + 8K_B \cdot R + 8R^2$$

and simplifying

$$-10K_B \cdot R + 4(d-12)R^2 - (d-11)D \cdot R = 0.$$

For the last equation consider the second formula of 2.2:

$$c_3(X) = c_2(X) \cdot (c_1(X) - 6H) - c_1(X)dH^2 + 20d.$$

Substituting the values of Proposition 4.12 we get the assertion by a straightforward calculation.

PROPOSITION 4.14. *Set*

$$P(d) = \frac{1}{3d^3 + 21d^2 - 1619d + 11295}, \quad x = K_B^2, \quad y = D \cdot R.$$

The following hold:

$$\begin{aligned}
R^2 = & -\frac{1}{4}P(d)[100x(7d-82) + (11-d)[y(3d^2+67d-1140) \\
& + 25d(d^2-19d+88)]] \\
K_B \cdot R = & -\frac{1}{2}P(d)[20x(d-12)(7d-82) + (11-d)[y(2d^2-65d+477) \\
& + 5d(d^3-31d^2+316d-1056)]] \\
c_2(B) = & \frac{1}{4}P(d)[4x(3d^3+26d^2-884d+3180) + y(d^3-38d^2+381d-1044) \\
& + d(4d^4-77d^3+378d^2-1865d+18660)] \\
K_B \cdot D = & -P(d)[10x(31d^2-638d+3267) - 2(d-12)[y(2d^2-65d+477) \\
& + 5d(d^3-31d^2+316d-1056)]] \\
D^2 = & 2P(d)[50x(31d-297) + 2(d-12)[y(3d^2+44d-810) \\
& - 25d(d^2-20d+96)]].
\end{aligned}$$

PROOF. Solve the system of Proposition 4.13 which is linear with coefficients rational functions of d. A computer is helpful, although not indispensable.

LEMMA 4.15. $y = D \cdot R \geq 0$.

PROOF. D and R are both effective divisors. At the generic point q of D the fibre is a reducible conic. For hyperplane sections which do not meet the intersection of the two lines, q is not contained in R. Hence D and R have no common component.

PROPOSITION 4.16.

$$\begin{aligned} c_2(E) &= \frac{1}{2} P(d)[100x(10d-51) - y(13d^2 - 350d + 2565) \\ &\quad - d(28d^3 - 929d^2 + 9706d - 31305)] \\ g - 1 &= \frac{1}{2} P(d)[10x(3d^2 - 44d + 151) + y(17d^2 - 388d + 2199) \\ &\quad + d(38d^3 - 1039d^2 + 8966d - 23465)]. \end{aligned}$$

PROOF. We have (using 4.11 and 4.8)

$$c_2(E) = \frac{R \cdot c_1(E) + c_1^2(E) - d}{2} = \frac{12R^2 + D^2 - 7R \cdot D - d}{2}.$$

Moreover from the adjunction formula $c_1(X)$. $H^2 = 2d + 2 - 2g$ and 4.10ii) we obtain

$$\begin{aligned} g - 1 &= d - \frac{1}{2}(H - p^*K_B - p^*R)H^2 = d - \frac{1}{2}d + \frac{1}{2}(K_B + R)(4R - D) \\ &= \frac{1}{2}d + 2K_B \cdot R - \frac{1}{2}K_B \cdot D + 2R^2 - \frac{1}{2}R \cdot D. \end{aligned}$$

Substituting the values of Proposition 4.14 and simplifying gives the assertion.

Now we are in the position to give the first upper bound on the degree of a conic bundle in $\mathbb{P}_5$.

PROPOSITION 4.17. *Let X be a conic bundle in $\mathbb{P}_5$. Then* $d = \deg(X) \leq 72$.

PROOF. We consider the possible values of x, y compatible with the three inequalities:

$$\begin{aligned} &D \cdot R \geq 0, \quad \text{(Lemma 4.15)} \\ &c_2(E) \geq 0, \quad \text{(Proposition 4.11)} \\ &c_1(E) \cdot D \geq 0, \quad (c_1(E) \text{ is ample, see 4.6}). \end{aligned}$$

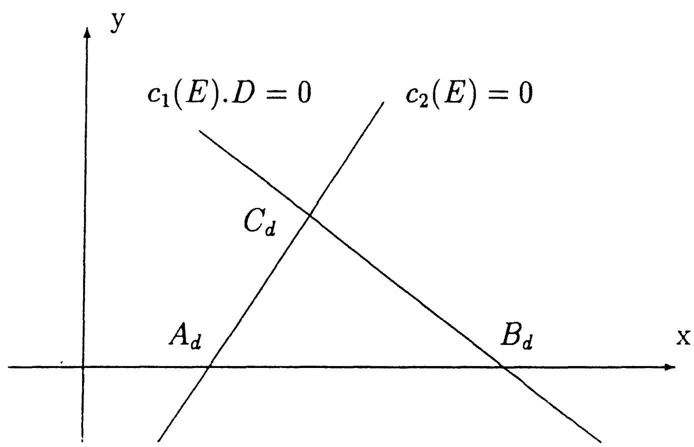
We have respectively (using 4.10, 4.11, 4.14 and 4.16) after easy computations:

$$y \geq 0$$

$$100x(10d - 51) - y(13d^2 - 350d + 2565) - d(28d^3 - 929d^2 + 9706d - 31305) \geq 0$$

$$-100x(31d - 297) - y(3d^3 - 31d^2 - 495d + 4995) + 100d(d - 12)^2(d - 8) \geq 0.$$

These inequalities bound the inside of a triangle as in the following picture (the triangle is acutangle for $d \geq 14$):



A_d, B_d, C_d are the three vertices of the triangle. We have coordinates

$$A_d = \left(\frac{d(28d^3 - 929d^2 + 9706d - 31305)}{100(10d - 51)}, 0\right),$$

$$B_d = \left(\frac{d(d - 12)^2(d - 8)}{31d - 297}, 0\right)$$

$$C_d = \left(\frac{d(28d^3 - 981d^2 + 11130d - 40005)}{1000(d - 9)}, \frac{d(44d - 303)}{10(d - 9)}\right).$$

The minimum and the maximum of $g - 1$ considered as a function in x and y with d fixed (see Proposition 4.16) have to be attained in one of the vertices. Substituting the coordinates of A_d, B_d, C_d in the expression of $g - 1$ we get respectively

$$g - 1 = \frac{d(28d^2 - 269d + 641)}{20(10d - 51)}, \frac{d(10d^2 - 144d + 463)}{2(31d - 297)}, \frac{d(28d^2 - 321d + 745)}{200(d - 9)}.$$

Hence we have in our case

$$(*) \qquad \frac{d(28d^2 - 269d + 641)}{20(10d - 51)} \leq g - 1 \leq \frac{d(10d^2 - 144d + 463)}{2(31d - 297)}.$$

Now we distinguish two cases.

Suppose first that the curve section C of X is not contained in a cubic. Then from 2.5 and (*) we have

$$\frac{d(28d^2 - 269d + 641)}{20(10d - 51)} \leq \frac{d^2}{8},$$

that is $6d^2 - 283d + 1282 \leq 0$, which gives $d \leq 42$.

If otherwise C is contained in a cubic, from 2.6 we get that also Y is contained in a cubic. Hence 2.7 and (*) give

$$\frac{d^2-7d}{6} \leq \frac{d(10d^2-144d+463)}{2(31d-297)},$$

that is $d^2-82d+690 \leq 0$, which implies $d \leq 72$.

THEOREM 4.18. *Let X be a conic bundle in $\mathbb{P}_5$. Then* $d = \deg(X) = 9$ *or* 12.

PROOF. From [BSS1] we know that for $d \leq 10$ there exists precisely one conic bundle of degree 9. For $11 \leq d \leq 72$ there are only finitely many integer pairs (x, y) inside the triangle considered in the proof of Proposition 4.17. Moreover we impose the Hodge inequality

$$(K_B \cdot R)^2 \geq K_B^2 \cdot R^2 \tag{*.1}$$

and the integrality conditions

$$R^2 \in \mathbb{Z} \tag{*.2}$$

$$K_B \cdot R \in \mathbb{Z} \tag{*.3}$$

$$c_2(B) \in \mathbb{Z} \tag{*.4}$$

$$K_B \cdot D \in \mathbb{Z} \tag{*.5}$$

$$D^2 \in \mathbb{Z}. \tag{*.6}$$

Using the formulas of Theorem 4.14 all the conditions (*.1),...,(*.6) involve only x, y, d. We have checked conditions (*.1),...,(*.6) with a Pascal program. It turns out that only the following values are possible:

$$\begin{aligned} &d = 12, \quad x = 0, \quad &&y = 0 \\ &d = 29, \quad x = 212, \quad &&y = 1 \\ &d = 45, \quad x = 1233, \quad &&y = 60. \end{aligned}$$

From these values one can compute K^3, HK^2, H^2K using the formulas of Propositions 4.12 and 4.14. But from Riemann-Roch

$$\chi(\mathcal{O}_X(1)) = \chi(\mathcal{O}_X) + \frac{1}{12}d(15-d) + \frac{1}{6}HK^2 + \frac{1}{4}H^2K + \frac{d}{6},$$

hence $\frac{1}{12}d(15-d) + \frac{1}{6}HK^2 + \frac{1}{4}H^2K + \frac{1}{6}d \in \mathbb{Z}$.

For $d = 29$ we have $HK^2 = 413$, $H^2K = 148$ and the above value is $\frac{461}{6} \notin \mathbb{Z}$.

For $d = 45$ we have $HK^2 = 2667$, $H^2K = 428$ and the above value is $\frac{893}{2} \notin \mathbb{Z}$.

Therefore $d = 12$ is the only possible value (for $d \geq 11$).

5. - Conic bundles II

5.1. We proceed by showing that there exist conic bundles X in $\mathbb{P}_5$ of degree 9 resp. 12 and that they are all obtained by the constructions given in Example 1 and 2 below. In degree 9 this is contained in [BSS1]. The proof of "uniqueness" in degree 12 requires some work. In particular we obtain that the Hilbert scheme of smooth 3-dimensional conic bundles in $\mathbb{P}_5$ has 2 components. Both are unirational of expected dimension.

5.2. A quick way to construct 3-folds in $\mathbb{P}_5$ is to use the following result of Kleiman:

THEOREM (Kleiman). *Let E be a globally generated rank r vector bundle on $\mathbb{P}_n$, $n \leq 5$. Then the morphism determined by $r-1$ generic sections of E degenerates on a smooth codimension two subvariety X, and we have the exact sequence*

$$0 \to \mathcal{O}^{r-1} \to E \to \mathcal{I}_X(c_1(E)) \to 0.$$

EXAMPLE 1. 3 generic sections of $E := \mathcal{O}_{\mathbb{P}_5}(1)^3 \oplus \mathcal{O}_{\mathbb{P}_5}(2)$ degenerate on the Castelnuovo conic bundle X of degree 9; hence a resolution of its ideal sheaf is

$$0 \to \mathcal{O}^3 \to \mathcal{O}(1)^3 \oplus \mathcal{O}(2) \to \mathcal{I}_X(5) \to 0.$$

The morphism associated to the line bundle $K_X + H$ is $\phi_{K_X+H}: X \to \mathbb{P}_2$ which gives the structure of a conic bundle. Note that X is also obtained by linkage with a cubic and a quartic from the Segre variety $\mathbb{P}_1 \times \mathbb{P}_2$.

EXAMPLE 2. Consider the rank 5 vector bundle E on $\mathbb{P}_5$ irreducible for the action of Sp(6) and unique up to PGL-action with $c_1(E) = 5$, c.f. [H]. E is globally generated and by Kleiman's theorem for generic f we obtain a smooth 3-fold X of degree 12 with resolution

$$0 \to \mathcal{O}^4 \xrightarrow{f} E \to \mathcal{I}_X(5) \to 0.$$

The morphism $\phi_{K_X+H}: X \to B \subset \mathbb{P}_3$ (where B is a smooth quartic surface) gives the structure of a conic bundle.

Before proving uniqueness in degree 12 we describe these two examples in more detail.

5.3. In the Example 1 one computes $h^0(K_X + H) = 3$, $(K_X + H)^3 = 0$, hence $\phi_{K_X+H}: X \to \mathbb{P}_2$. Explicitly the matrix of the homomorphism $F: \mathcal{O}^3 \to$

$\mathcal{O}(1)^3 \oplus \mathcal{O}(2)$ is given by

$$F = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ c_1 & c_2 & c_3 \end{pmatrix}$$

with $a_{ij} \in H^0(\mathbb{P}_5, \mathcal{O}(1))$, $c_i \in H^0(\mathbb{P}_5, \mathcal{O}(2))$. The four 3×3 minors of F give the equations of X (in fact I_X is generated by a cubic and three quartic hypersurfaces). If $x' \in X$ then rank $(F) \leq 2$ and there exist $(\lambda_1, \lambda_2, \lambda_3)$ such that

$$(*) \qquad \begin{gathered} \sum_{j=1}^{3} \lambda_j a_{ij}(x') = 0 \text{ for } i = 1, 2, 3 \\ \sum_{j=1}^{3} \lambda_j c_j(x') = 0. \end{gathered}$$

The morphism $p: X \to \mathbb{P}_2$ defined by $p(x') = (\lambda_1, \lambda_2, \lambda_3)$ gives the structure of a conic bundle and (*) are the equations of the fiber over $(\lambda_1, \lambda_2, \lambda_3)$.

LEMMA 5.4. *Let X be the Castelnuovo conic bundle as in the Example 1. We have $2R = \mathcal{O}_{\mathbb{P}_2}(8)$ and $D = \mathcal{O}_{\mathbb{P}_2}(9)$. (Notations as in Section 4).*

PROOF. Let $Y = X \cap H$ be a generic hyperplane section. By Lemma 4.5 $p: Y \to \mathbb{P}_2$ is finite 2:1 and we have $p_*\mathcal{O} = \mathcal{O} \oplus \mathcal{O}(-R)$. We have

$$0 \to \mathcal{O}_H^3 \to \mathcal{O}_H(1)^3 \oplus \mathcal{O}_H(2) \to \mathcal{O}_H(5) \to \mathcal{O}_Y(5) \to 0$$

hence

$$0 \to \mathcal{O}_H(-5)^3 \to \mathcal{O}_H(-4)^3 \oplus \mathcal{O}_H(-3) \to \mathcal{O}_H \to \mathcal{O}_Y \to 0.$$

It follows $\chi(\mathcal{O}_Y) = 1 + 3 = 4$. Now $\chi(\mathcal{O}_Y) = \chi(p_*\mathcal{O}_Y) = \chi(\mathbb{P}_2, \mathcal{O}) + \chi(\mathbb{P}_2, \mathcal{O}(-R))$. We obtain $\chi(\mathbb{P}_2, \mathcal{O}(-R)) = 3$ and $R = \mathcal{O}_{\mathbb{P}_2}(4)$. The Hilbert polynomial is $\chi(\mathcal{O}_X(t)) = \frac{3}{2}t^3 + \frac{1}{2}t^2 + 3t + 1$ and it is easy to check that $H^2K = -2$, $HK^2 = -3$, $K^3 = 6$. Let S be a smooth surface of the system $|K_X + H|$, thus $K_S = (2K_X + H)|_S$. S is the blow up in $\deg(D)$ points of a ruled rational surface. K^2 decreases by one at each blow up, hence $K_S^2 = 8 - \deg(D)$. But we compute $K_S^2 = (2K + H)^2(K + H) = 4K^3 + 8HK^2 + 5H^2K + H^3 = 24 - 24 - 10 + 9 = -1$, then $\deg(D) = 9$, as we wanted.

5.5. In Example 2 the bundle E can be constructed directly from the nullcorrelation bundle N. As N carries a nondegenerate symplectic form, $\mathcal{O}$ is a direct summand of $\wedge^2 N$ and we may define $\wedge^2 N = \mathcal{O} \oplus E(-1)$. One computes (e.g. from Bott's theorem) the following table of cohomology:

$$h^0(E(t)) = -h^5(E(-8-t)) = \frac{(t+1)(t+2)(t+4)(t+6)(t+7)}{24}$$

for $t \geq 0$, $h^2(E(-3)) = h^3(E(-5)) = 1$ and all other $h^i(E(j))$ are zero. Hence from Beilinson's theorem we have also

$$0 \to \Omega^4(5) \to \Omega^2(3) \to E \to 0$$

or dually ($E^\vee = E(-1)$ by [DMS], Proposition 1.2))

$$0 \to E \to \Omega^3(5) \to \Omega^1(3) \to 0$$

and the ideal sheaf of X has the following resolution

$$0 \to \mathcal{O}^4 \oplus \Omega^4(5) \to \Omega^2(3) \to \mathcal{I}_X(5) \to 0.$$

The dual sequence shows that $K_X + H$ is globally generated by 4 sections and an easy calculation gives $(K_X+H)^3 = 0$, $(K_X+H)^2 H = 4$. Hence the morphism associated to $K_X + H$ maps X onto a quartic surface of $\mathbb{P}_3$ and exhibits X as a conic bundle.

LEMMA 5.6. *Let X be the conic bundle of Example* 2. *Then* $R = \mathcal{O}_{\mathbb{P}_3}(1)|_B$ *and* $D = 0$.

PROOF. Exactly as in the proof of Lemma 5.4 we get $\chi(\mathcal{O}_Y) = 6$, hence $\chi(\mathcal{O}(-R)) = 6 - \chi(\mathcal{O}_B) = 4$, $K^3 = 12$, $HK^2 = -12$, $H^2K = 4$, $K_S^2 = (2K_X + H)^2(K+H) = -16$. S is the blow up of a ruled surface over the hyperplane section of B which has genus 3. As $-16 = 8(1-3)$ there are no blow ups to perform, that is there are no reducible conics. Moreover $p^*\mathcal{O}_{\mathbb{P}_3}(1)|_B = (K_X+H)|_Y = K_Y = p^*(K_B + R) = p^*(R)$.

REMARK. The bundle E appearing in the resolution of the Example 2 is explicitly described in [DMS]. From the description given there one can see directly that the fibers of the projection $p: X \to B$ are smooth conics. B is the intersection of a generic linear $\mathbb{P}_3$ with the quartic hypersurface $T \subset \mathbb{P}_{13}$ in the notations of [DMS].

5.7. The invariants of the conic bundles of the Examples 1 and 2 are summarized in the following table:

	$d=9$	$d=12$
$c_1^2(E)$	9	36
$c_2(E)$	6	18
$\delta=\deg(V)$	3	18
B	$\mathbb{P}_2$	$S_4 \subset \mathbb{P}_3$
$\deg(D)$	9	0
R^2	16	4
K^3	6	12
HK^2	-3	-12
H^2K	-2	4
$\chi(\mathcal{O}_X(t))$	$\frac{3}{2}t^3+\frac{1}{2}t^2+3t+1$	$2t^3-t^2+3t+2$
g	9	15

where $p\colon X \to B$, $E=p_*\mathcal{O}(1)$, $(\delta, c_2(E))$ is the bidegree of the embedding of B in $\mathrm{Gr}(2,5)$ and V is the hypersurface spanned by the planes of the conics.

It remains to show uniqueness in degree 12. We achieve this in two steps: first, given a conic bundle X of degree 12 in $\mathbb{P}_5$, we prove that the dimensions of certain cohomology groups $h^q(\mathcal{I}_{X|\mathbb{P}_5}(m))$ are the same as in Example 2; secondly the morphisms in the corresponding Beilinson spectral sequences are also the same.

PROPOSITION 5.8. *Let X be a conic bundle of degree* 12 *in* $\mathbb{P}_5$. *Then* $h^q(\mathcal{I}_{X|\mathbb{P}_5}(m))$ *is as in the following table:*

q							
5	0	0	0	0	0	0	
4	24	4	0	**0**	**0**	**0**	
3	0	0	1	**0**	**0**	**0**	
2	0	0	0	**0**	**1**	**0**	
1	0	0	0	0	**0**	**0**	
0	0	0	0	0	0	0	
	-2	-1	0	1	2	3	m

PROOF. By Theorem 4.18 and Propositions 4.12 and 4.14 the numerical invariants of X are as in Example 2, especially (see 5.7): $\chi(\mathcal{O}_X(t))=2t^3-t^2+3t+2$.

Consider the exact sequence

$$0 \to \mathcal{I}_{X|\mathbb{P}_5}(m) \to \mathcal{O}_{\mathbb{P}_5}(m) \to \mathcal{O}_X(m) \to 0.$$

The associated cohomology sequence gives the zeros in the top row of the diagram. By Riemann-Roch, Kodaira-vanishing and $h^1(\mathcal{O}_X)=h^3(\mathcal{O}_X)=0$ we obtain the -2, -1 and 0-columns. Since X is linearly normal we have $h^1(\mathcal{I}_X(1)) = 0$. From [K] we know that Y (hence also X) cannot be contained in a cubic. Hence we are left with the bold face numbers.

Now $h^0(K_X) = h^3(\mathcal{O}_X) = 0$ yields for $m > 0$:

$$0 = h^0(K_X(-m)) = h^3(\mathcal{O}_X(m)) = h^4(\mathcal{I}_{X|\mathbb{P}_5}(m)).$$

CLAIM 1. $h^2(\mathcal{I}_{X|\mathbb{P}_5}(1)) = h^3(\mathcal{I}_{X|\mathbb{P}_5}(1)) = 0$.

PROOF. Riemann-Roch and the previous results give

$$0 = \chi(\mathcal{I}_{X|\mathbb{P}_5}(1)) = h^2(\mathcal{I}_{X|\mathbb{P}_5}(1)) - h^3(\mathcal{I}_{X|\mathbb{P}_5}(1)).$$

$K_Y = (K_X + H)|_Y$ is nef and $(K_Y - H) \cdot K_Y = K_X \cdot (K_X + H) \cdot H = -12$ which implies $0 = h^0(K_Y - H) = h^2(\mathcal{O}_Y(1))$.

Let h be a hyperplane in $\mathbb{P}_5$ and consider the commutative diagram

$$\begin{array}{ccccccc} \mathbb{C} & \simeq & H^2(\mathcal{O}_X) & \to & H^2(\mathcal{O}_X(1)) & \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & H^3(\mathcal{I}_{X|\mathbb{P}_5}) & \xrightarrow{\alpha_h} & H^3(\mathcal{I}_{X|\mathbb{P}_5}(1)). & & \end{array}$$

Let V be a $\mathbb{C}$-vector space with $\mathbb{P}_5 = \mathbb{P}(V)$. In Beilinson's spectral sequence

$$E_1^{p,q} = H^q(\mathcal{I}_{X|\mathbb{P}_5}(p+3)) \otimes \Omega_{\mathbb{P}_5}^{-p}(-p)$$

the d_1-morphism $H^3(\mathcal{I}_{X|\mathbb{P}_5}) \otimes \Omega^3_{\mathbb{P}_5}(3) \to H^3(\mathcal{I}_{X|\mathbb{P}_5}(1)) \otimes \Omega^2_{\mathbb{P}_5}(2)$ is an element of $\mathrm{Hom}(\Omega^3_{\mathbb{P}_5}(3), \Omega^2_{\mathbb{P}_5}(2)) \simeq V$. Consequently α_h can be identified with a point $P \in \mathbb{P}(V)$ and if we choose h, such that $P \in h$, we see that α_h is in fact the zero map. This implies Claim 1.

CLAIM 2. $h^3(\mathcal{I}_{X|\mathbb{P}_5}(2)) = h^3(\mathcal{I}_{X|\mathbb{P}_5}(3)) = 0$.

PROOF. Immediately from Claim 1.

To control the remaining four groups we need some information on Y. We have the following table for $h^q(\mathcal{I}_{Y|\mathbb{P}_4}(m))$:

q					
3	5	0	0	0	
2	0	1	$1+a$	b	
1	0	0	a	$1+b$	
0	0	0	0	0	
	0	1	2	3	m

where $a := h^1(\mathcal{I}_{Y|\mathbb{P}_4}(2))$ and $b := h^2(\mathcal{I}_{Y|\mathbb{P}_4}(3))$.

CLAIM 3. $a \leq 1$.

PROOF. By the cohomology of the exact sequence

$$0 \to \mathcal{O}_Y(1) \to \mathcal{O}_Y(2) \to \mathcal{O}_C(2) \to 0$$

and $h^1(\mathcal{O}_Y(1)) = h^2(\mathcal{I}_{Y|\mathbb{P}_4}(1)) = 1$ it is enough to show that

$$1 \geq h^1(\mathcal{O}_C(2)) = h^0(\mathcal{O}_C(K_C - 2H)).$$

Assume the contrary and let B be the base locus of two independent sections of $\mathcal{O}_C(K_C - 2H)$ and let W be the subspace of $H^0(\mathcal{O}_C(K_C - 2H - B))$ generated by the induced sections. The Base-Point-Free Pencil Trick [ACGH, p. 126] gives an exact sequence

$$0 \to H^0(\mathcal{O}_C(3H - K_C + B)) \to W \otimes H^0(\mathcal{O}_C(H)) \to H^0(\mathcal{O}_C(K_C - H - B)).$$

Now $h^0(\mathcal{O}_C(H)) = 4$ and $h^0(\mathcal{O}_C(K_C - H - B)) \leq h^0(\mathcal{O}_C(K_C - H)) = h^1(\mathcal{O}(H)) = 6$, where the last equality follows from Riemann-Roch.

Hence $h^0(\mathcal{O}_C(3H - K_C + B)) \geq 2$.

Take $s_1, s_2 \in H^0(\mathcal{O}_C(3H - K_C + B))$ and $t_1, t_2 \in H^0(\mathcal{O}_C(K - 2H - B))$. The images of $s_i \otimes t_j$ in $H^0(\mathcal{O}_C(H))$ satisfy the relation

$$s_1 \otimes t_1 \cdot s_2 \otimes t_2 - s_1 \otimes t_2 \cdot s_2 \otimes t_1 = 0$$

i.e. C is contained in a quadric, which is a contradiction.

CLAIM 4. $b = 0$.

PROOF. Since $h^1(\mathcal{O}_C(3)) = h^0(\mathcal{O}_C(K_C - 3H)) = 0$ we obtain from the exact sequence

$$0 \to \mathcal{O}_Y(2) \to \mathcal{O}_Y(3) \to \mathcal{O}_C(3) \to 0$$

that $b = h^1(\mathcal{O}_Y(3)) \leq h^1(\mathcal{O}_Y(2)) = a + 1$, hence by Claim 3: $b \leq 2$.

The same argument shows that $h^2(\mathcal{I}_{Y|\mathbb{P}_4}(r)) = h^1(\mathcal{O}_Y(r)) \leq 2$ for all $r \geq 2$.

Define $k := \max\{t \in \mathbb{N} : h^2(\mathcal{I}_{Y|\mathbb{P}_4}(t)) \neq 0\}$ and consider the Beilinson spectral sequence

$$E_1^{p,q} = H^q(\mathcal{I}_{Y|\mathbb{P}_4}(p + k + 1)) \otimes \Omega_{\mathbb{P}_5}^{-p}(-p).$$

By construction the d_1-morphism

$$H^2(\mathcal{I}_{Y|\mathbb{P}_4}(k - 1)) \otimes \Omega^2_{\mathbb{P}_5}(2) \to H^2(\mathcal{I}_{Y|\mathbb{P}_4}(k)) \otimes \Omega^1_{\mathbb{P}_5}(1)$$

has to be surjective. Now $h^2(\mathcal{I}_{Y|\mathbb{P}_4}(k)) > 0$ and if $k \geq 3$, we have $h^2(\mathcal{I}_{Y|\mathbb{P}_4}(k - 1)) \leq 2$. Hence we get a morphism of vector bundles $\Omega^2_{\mathbb{P}_5}(2)^{\oplus 2} \to \Omega^1_{\mathbb{P}_5}(1)$.

Taking the dual twisted by $\mathcal{O}(-1)$ we obtain an injective map in each fiber over a point $\langle v\rangle \in \mathbb{P}(V) = \mathbb{P}_5$, i.e. for all $v \in V\backslash\{0\}$

$$\langle v\rangle \wedge V \hookrightarrow 2\left(\langle v\rangle \wedge \bigwedge^2 V\right)$$

given by wedging with $\begin{pmatrix} w \\ u \end{pmatrix}$ for certain w and u.

This is a contradiction (take e.g. $w \wedge u \in \langle w\rangle \wedge V$).

We conclude $k = 2$ which means $b = 0$.

CLAIM 5. $a = 0$.

PROOF. Assume $a = 1$.

We argue as in the proof of Claim 1 and find a point $P \in \mathbb{P}_4$ such that for every hyperplane h in $\mathbb{P}_4$ with $P \in h$ the multiplication map

$$H^1(\mathcal{I}_{Y|\mathbb{P}_4}(2)) \xrightarrow{\alpha_h} H^1(\mathcal{I}_{Y|\mathbb{P}_4}(3))$$

is zero. Consequently the corresponding hyperplane sections of Y are contained in a cubic. By Aure's proof [A] of Roth's theorem in the case $s = 3$, we find that Y is contained in a cubic – a contradiction.

CLAIM 6. $h^2(\mathcal{I}_{X|\mathbb{P}_5}(2)) = 1$ and $h^1(\mathcal{I}_{X|\mathbb{P}_5}(2)) = 0$.

PROOF. We have $h^1(\mathcal{I}_{X|\mathbb{P}_5}(1)) = 0$ and by Claim 5 also $h^2(\mathcal{I}_{Y|\mathbb{P}_4}(2)) = 0$. Riemann-Roch and the cohomology of the exact sequence

$$0 \to \mathcal{I}_{X|\mathbb{P}_5}(1) \to \mathcal{I}_{X|\mathbb{P}_5}(2) \to \mathcal{I}_{Y|\mathbb{P}_4}(2) \to 0$$

give the assertion.

CLAIM 7. $h^1(\mathcal{I}_{X|\mathbb{P}_5}(3)) = h^2(\mathcal{I}_{X|\mathbb{P}_5}(3)) = 0$.

PROOF. From Riemann-Roch we obtain $0 = \chi(\mathcal{I}_{X|\mathbb{P}_5}(3)) = h^2(\mathcal{I}_{X|\mathbb{P}_5}(3)) - h^1(\mathcal{I}_{X|\mathbb{P}_5}(3))$. The cohomology of the exact sequence

$$0 \to \mathcal{I}_{X|\mathbb{P}_5}(2) \to \mathcal{I}_{X|\mathbb{P}_5}(3) \to \mathcal{I}_{Y|\mathbb{P}_4}(3) \to 0$$

yields $h^2(\mathcal{I}_{X|\mathbb{P}_5}(3)) \leq 1$. Arguing again as in Claim 1, we find a hyperplane h such that the induced multiplication map

$$H^2(\mathcal{I}_{X|\mathbb{P}_5}(2)) \xrightarrow{\alpha_h} H^2(\mathcal{I}_{X|\mathbb{P}_5}(3))$$

is zero. From $h^2(\mathcal{I}_{Y|\mathbb{P}_4}(3)) = 0$ follows Claim 7 and this finishes the proof of Proposition 5.8.

To show that our table in the introduction is complete we now only need:

PROPOSITION 5.9. *Every conic bundle X of degree* 12 *in $\mathbb{P}_5$ is as in Example* 2.

PROOF. From the cohomology table of Proposition 5.8 we obtain via Beilinson's spectral sequence that $\mathcal{I}_X(3)$ is the cohomology of a monad

$$\mathcal{O}_{\mathbb{P}_5}(-1)^{\oplus 24} \to \Omega^3_{\mathbb{P}_5}(3) \oplus \Omega^4_{\mathbb{P}_5}(4)^{\oplus 4} \to \Omega^1_{\mathbb{P}_5}(1)$$

i.e. we have the commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
0 & \to & A & \to & \mathcal{O}_{\mathbb{P}_5}(-1)^{\oplus 24} & \xrightarrow{\beta} & \Omega^4_{\mathbb{P}_5}(4)^{\oplus 4} & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & \Omega^3_{\mathbb{P}_5}(3) & \xrightarrow{\alpha} & \Omega^1_{\mathbb{P}_5}(1) & \to & D & \to & 0 \\
 & & \downarrow & & & & & & \downarrow & & \\
 & & \mathcal{I}_X(3) & & & & & & 0 & & \\
 & & \downarrow & & & & & & & & \\
 & & \operatorname{Ker}(C \to D) & & & * & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

where α is given by contraction with a 2-form ω.

We can choose a basis $\{e_1, \dots, e_6\}$ of $V(\mathbb{P}_5 = \mathbb{P}(V))$ such that ω is one of the following:

(1) $e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6$

(2) $e_1 \wedge e_2 + e_3 \wedge e_4$

(3) $e_1 \wedge e_2$

(4) 0.

CASE (1). In this case α is surjective and the induced map $H^0(\Omega^3_{\mathbb{P}_5}(4)) \to H^0(\Omega^1_{\mathbb{P}_5}(2))$ is an isomorphism, hence $h^0(B(1)) = 0$. We deduce $B(2) \simeq E$ (compare 5.5). Furthermore A cannot contain a summand $\mathcal{O}_{\mathbb{P}_5}(-1)$, which implies that β is a direct sum of four copies of the Euler sequence map $\mathcal{O}_{\mathbb{P}_5}(-1)^{\oplus 6} \to \Omega^4_{\mathbb{P}_5}(4)$. Consequently we have $C = 0$ and $A = \mathcal{O}_{\mathbb{P}_5}(-2)^{\oplus 4}$. Hence the ideal sheaf of X has a resolution as in Example 2.

It remains to exclude the cases (2), (3) and (4).

CASE (2). By restricting to the $\mathbb{P}_3$ which is spanned by $e_1, \dots, e_4$ we obtain from the diagram $*$

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}_3}(-1)^{\oplus 24} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}_3}(-1)^{\oplus 8} \oplus \Omega^2_{\mathbb{P}_3}(2)^{\oplus 4} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{P}_3}(-1) \oplus \Omega^2_{\mathbb{P}_3}(2)^{\oplus 2} \oplus \Omega^1_{\mathbb{P}_3}(1) & \xrightarrow{\alpha} & \Omega^1_{\mathbb{P}_3}(1) \oplus \Omega^{\oplus 2}_{\mathbb{P}_3}. \end{array}$$

Note that the homology of the restriction monad is $\mathcal{I}_{X\cap\mathbb{P}_3|\mathbb{P}_3}(3)$. α splits in the following way:

$$\begin{array}{ccccccccc} & & 0 & \to & \mathcal{O}_{\mathbb{P}_3}(-1) & \to & \Omega^1_{\mathbb{P}_3}(1) & \to & N & \to & 0 \\ & & \oplus & & \oplus & & \oplus & & & & \\ 0 & \to & M^{\oplus 2} & \to & \Omega^2_{\mathbb{P}_3}(2)^{\oplus 2} & \to & \mathcal{O}^{\oplus 2}_{\mathbb{P}_3} & \to & 0 & & \\ & & \oplus & & \oplus & & \oplus & & & & \\ 0 & \to & \Omega^1_{\mathbb{P}_3}(1) & \to & \Omega^1_{\mathbb{P}_3}(1) & \to & 0 & & & & \end{array}$$

where N is a null correlation bundle and $M \simeq N(-1)$.

Via the Euler sequence we replace β by

$$\mathcal{O}_{\mathbb{P}_3}(-2)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}_3}(-1)^{\oplus 24} \xrightarrow{\tilde{\beta}} \mathcal{O}_{P_3}(-1)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}_3}(-1)^{\oplus 16}$$

and obtain a surjection

$$\mathcal{O}_{\mathbb{P}_3}(-1)^{\oplus 8} \oplus \mathcal{O}_{\mathbb{P}_3}(-1)^{\oplus 16} \to N \to 0.$$

Since the second Segre class of $N(1)$ is not zero, $N(1)$ cannot be generated by less than four sections, hence the kernel of $\tilde{\beta}$ contains at least four copies of $\mathcal{O}_{\mathbb{P}_3}(-1)$. This means that we have an injective map

$$0 \to \mathcal{O}_{\mathbb{P}_3}(-1)^{\oplus 4} \to M^{\oplus 2} \oplus \Omega^1_{\mathbb{P}_3}(1).$$

But $M(1) \simeq N$ has no sections and $\mathrm{rank}(\Omega^1_{\mathbb{P}_3}(1)) = 3$, which is a contradiction.

CASE (3). Assume $\omega = e_1 \wedge e_2$ and consider the map α of diagram $*$. After dualizing and twisting by $\mathcal{O}(-1)$ we obtain in each fiber over a point $\langle v\rangle \in \mathbb{P}(V) = \mathbb{P}_5$:

$$\alpha^{\vee}\colon \langle v\rangle \wedge V \xrightarrow{\wedge(e_1\wedge e_2)} \langle v\rangle \wedge \bigwedge^3 V.$$

For $v \notin \langle e_1, e_2\rangle$ the kernel is spanned by $v\wedge e_1$ and $v \wedge e_2$, which means that the rank of D is 2 and consequently B has rank 7. Furthermore

$$
\begin{aligned}
h^0(B(1)) &= \dim(\ker(H^0(\Omega^3_{\mathbb{P}_5}(4)) \xrightarrow{\alpha} H^0(\Omega^1_{\mathbb{P}_5}(2)))) \\
&= \dim\left(\ker\left(\bigwedge^4 V^\vee \xrightarrow{\neg(e_1\wedge e_2)} \bigwedge^2 V^\vee\right)\right) \\
&= \dim\left(\ker\left(\bigwedge^2 V \xrightarrow{\wedge(e_1\wedge e_2)} \bigwedge^4 V\right)\right) \\
&= \dim(e_1 \wedge V + e_2 \wedge V) = 9.
\end{aligned}
$$

And from diagram $*$: $h^0(A(1)) \le \operatorname{rank}(A) \le \operatorname{rank}(B) = 7$.

In fact $\operatorname{rank}(A) < \operatorname{rank}(B)$, since otherwise $A \simeq B$ which leads to

$$7 \ge h^0(A(1)) = h^0(B(1)) = 9.$$

Hence $\gamma := h^0(\mathcal{I}_X(4)) \ge 3$. Riemann-Roch and the previous computations give the following table for $h^q(\mathcal{I}_X(m))$:

q							
5	0	0	0	0	0	0	
4	4	0	0	0	0	0	
3	0	1	0	0	0	0	
2	0	0	0	1	0	0	
1	0	0	0	0	0	γ	
0	0	0	0	0	0	γ	
	-1	0	1	2	3	4	m

From Beilinson's spectral sequence we obtain a complex

$$\Omega^5_{\mathbb{P}_5}(5)^{\oplus 4} \oplus \Omega^4_{\mathbb{P}_5}(4) \xrightarrow{(\varphi,\phi)} \Omega^2_{\mathbb{P}_5}(2) \xrightarrow{\psi} H^1(\mathcal{I}_X(4)) \otimes \mathcal{O}_{\mathbb{P}_5}.$$

Notice first that ψ has to be surjective and ϕ is given by contraction with $\omega = e_1 \wedge e_2$. Since composition is wedge product we have $e_1 \wedge e_2 \wedge \psi = 0$. So we write ψ as a $1 \times \gamma$-matrix $\psi = (e_1 \wedge a_i + e_2 \wedge b_i)_{i=1,\dots,\gamma}$ with certain $a_i,\ b_i \in V$. Let $P = \lambda e_1 + \mu e_2$ be any point on the line spanned by e_1 and e_2. Dualizing ψ and twisting by $\mathcal{O}(1)$ induces an injective map in the fibers over P:

$$\psi^\vee : \langle \lambda e_1 + \mu e_2 \rangle^{\oplus\gamma} \hookrightarrow \langle \lambda e_1 + \mu e_2 \rangle \wedge \bigwedge^2 V.$$

This means that the elements of the $\gamma \times 1$-matrix

$$(\lambda e_1 + \mu e_2) \wedge \psi = (e_1 \wedge e_2 \wedge (\lambda b_i - \mu a_i))_{i=1,\dots,\gamma}$$

are linearly independent in $\bigwedge^3 V$. Consider the projection map $p: V \to V/\langle e_1, e_2\rangle =: U$ and let $\tilde{a}_i := p(a_i)$ and $\tilde{b}_i := p(b_i)$. Now $(\lambda \tilde{b}_i - \mu \tilde{a}_i)_{i=1,\dots,\gamma}$ are linearly independent in U, which implies $\gamma \le 4$. γ can not be 4, since in this case the determinantal locus of the matrix

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \tilde{a}_1 & \dots & \tilde{a}_\gamma \\ \tilde{b}_1 & \dots & \tilde{b}_\gamma \end{pmatrix}$$

in $\mathbb{P}(U^\vee)$ would be non-empty and any such point would give λ, μ such that $(\lambda \tilde{b}_i - \mu \tilde{a}_i)_{i=1,\dots,4}$ are linearly dependent. Hence $\gamma = 3$.

Moreover we have a well-defined morphism

$$\mathbb{P}_1 \to \mathbb{P}(U^\vee) = \mathbb{P}_3 \quad (\lambda : \mu) \mapsto \ker(\lambda B - \mu A)$$

and the image is a rational normal curve. Therefore there exists a basis $e_1, \dots, e_6$ of V such that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \tilde{e}_3 & \tilde{e}_4 & \tilde{e}_5 \\ \tilde{e}_4 & \tilde{e}_5 & \tilde{e}_6 \end{pmatrix}$$

where $\tilde{e}_i := p(e_i)$. Thus

$$\psi = \begin{pmatrix} e_1 \wedge e_3 + e_2 \wedge e_4 \\ e_1 \wedge e_4 + e_2 \wedge e_5 \\ e_1 \wedge e_5 + e_2 \wedge e_6 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \cdot e_1 \wedge e_2$$

with $c_i \in \mathbb{C}$. Consider now $\varphi = (\varphi_i)_{i=1,\dots,4}: \Omega^5_{\mathbb{P}_5}(5)^{\oplus 4} \to \Omega^2_{\mathbb{P}_5}(2)$.

φ_l is given by contraction with a 3-form and after subtracting $v_l \wedge e_1 \wedge e_2$ (suitable $v_l \in V$) we can assume

$$\varphi_l = \sum \alpha^l_{ijk} e_i \wedge e_j \wedge e_k$$

where the sum runs over all indices $1 \le i < j < k \le 6$ and $\alpha^l_{12k} = 0$ for all k.

From $\psi \wedge \varphi_l = 0$ we get by an easy computation $\alpha^l_{ijk} = 0$ for all i, j, k, l, i.e. $\varphi = 0$. We conclude that there exists an injective map $\mathcal{O}_{\mathbb{P}_5}(-1)^{\oplus 4} = \Omega^5_{\mathbb{P}_5}(5)^{\oplus 4} \hookrightarrow H^0(\mathcal{I}_X(4)) \otimes \mathcal{O}_{\mathbb{P}_5}$ which is a contradiction since $\gamma = h^0(\mathcal{I}_X(4)) = 3$.

CASE (4). We have $B \simeq \Omega^3_{\mathbb{P}_5}(3)$ and $h^0(B(1)) = 15$. Arguing as in case 3 we find $\operatorname{rank}(A) = 9$, $h^0(A(1)) \le 9$, $\gamma = h^0(\mathcal{I}_X(4)) \ge 6$ and the map

$$\psi: \Omega^2_{\mathbb{P}_5}(2) \to H^1(\mathcal{I}_X(4)) \otimes \mathcal{O}_{\mathbb{P}_5}$$

is again surjective. Equivalently we get an injection $\mathcal{O}_{\mathbb{P}_5} \hookrightarrow \wedge^2(T_{\mathbb{P}_5}(-1))$. Since $c_4(\wedge^2(T_{\mathbb{P}_5}(-1))) \neq 0$ and $\wedge^2(T_{\mathbb{P}_5}(-1))$ is generated by global sections it follows $\gamma \leq 6$, i.e. $\gamma = 6$.

From diagram $*$ we deduce $h^0(A(1)) = 9 = \text{rank}(A)$ which implies $A \simeq \mathcal{O}_{\mathbb{P}_5}(-1)^{\oplus 9}$. We twist with $\mathcal{O}_{\mathbb{P}_5}(1)$ and use $\Omega^3_{\mathbb{P}_5}(4) \simeq \bigwedge^2(T_{\mathbb{P}_5}(-1))$ to obtain from the first column of diagram $*$:

$$0 \to \mathcal{O}_{\mathbb{P}_5}^{\oplus 9} \to \bigwedge^2(T_{\mathbb{P}_5}(-1)) \to \mathcal{I}_Z(4) \to 0,$$

where the codimension of Z in $\mathbb{P}_5$ is 2 and $\deg(Z) = c_2(\wedge^2(T_{\mathbb{P}_5}(-1))) = 9$. But $\mathcal{I}_Z \subset \mathcal{I}_X$, i.e. $X \subset Z$ which contradicts $\deg(X) = 12$.

Thus the proof of Proposition 5.9 is complete.

Putting everything together we get the theorem in the introduction.

Note added in proof

It is natural to ask whether the conic bundle X of degree 12 in $\mathbb{P}_5$ is a $\mathbb{P}_1$-bundle, all fibres being smooth. It is fairly easy to see that X cannot be a $\mathbb{P}_1$-bundle in case $\text{Pic}(B) \simeq \mathbb{Z}$. Indeed, assuming $X = \mathbb{P}(E)$, and using [O], p. 469, equations (C) and (D), we get

$$c_1^2(E) - 4c_2(E) = 4\xi^3 - 3\xi^2 \cdot c_1(E) = -6.$$

Here $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. If $\text{Pic}(B) \simeq \mathbb{Z}$, we have $c_1(E) = ac_1(\mathcal{O}(B(1))$, leading to

$$4a^2 - 4c_2(E) = -6.$$

But this is clearly impossible.

For the general conic bundle $X \to B$ in $\mathbb{P}_5$, the isomorphism $\text{Pic}(B) \simeq \mathbb{Z}$ follows from a beautiful idea of Mukai:

Let $U^{14} = H^0(\mathbb{P}_5, E)$ be the 14-dimensional space of sections of the $\text{Sp}(3, \mathbb{C})$-bundle E. We have seen that the Hilbert scheme of degree 12 conic bundles in $\mathbb{P}_5$ is birational to $G(4, U^{14})$, the Grassmannian of 4-dimensional subspaces of U^{14}. The moduli space of $K3$ surfaces B, arising as base spaces of our conic bundles, is therefore birational to $G(4, U^{14})/\text{Sp}(3, \mathbb{C})$.

On the other hand Mukai [M1], [M2] has shown that $G(10, U^{14})/\text{Sp}(3, \mathbb{C})$ is birational to the moduli space of polarized $K3$ surfaces of genus 9 and Clifford index 4. The symplectic form on $\mathbb{P}_5$ induces a skew symmetric non degenerate form on U^{14}. With the induced isomorphism $U^{14} \simeq (U^{14})^\vee$ the Grassmannians $G(4, U^{14})$ and $G(10, U^{14})$ are identified and this gives rise to a birational correspondence of the moduli space of $K3$ surfaces of genus 9 and the moduli space of $K3$ surfaces occuring as base spaces of our conic bundles.

Hence both spaces are of dimension 19, and by Hodge theory the general base space B has Picard group $\mathbb{Z}$. In particular we see that the general quartic in $\mathbb{P}_3$ arises as a base B.

REFERENCES

[ACGH] E. ARBARELLO - M. CORNALBA - P. GRIFFITHS - J. HARRIS, Geometry of algebraic curves I, Grundlehren der math. Wissenschaften 267, Springer, Berlin, 1985.

[A] A. AURE, *On surfaces in projective 4-space*, Thesis, Oslo, 1987.

[BBS] M. BELTRAMETTI - A. BIANCOFIORE - A.J. SOMMESE, *Projective N-folds of log-general type I.* Trans. Am. Math. Soc. **314** (1989), 825-849.

[Be] A. BEAUVILLE, *Variétés de Prym et Jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup., **10** (1977), 309-391.

[BeSo] M. BELTRAMETTI - A.J. SOMMESE, *New properties of special varieties arising from adjunction theory*, J. Math. Soc. Japan **43** (1991), 381-412.

[Bes] G.M. BESANA, *On the geometry of conic bundles arising in adjunction theory*, Math. Nachr. **160** (1993), 223-251.

[BOSS] R. BRAUN - G. OTTAVIANI - F.O. SCHREYER - M. SCHNEIDER, *Boundedness for non-general type 3-folds in $\mathbb{P}^5$*. In "Complex Analysis and Geometry", V. Ancona - A. Silva Eds., Plenum Press, 1993, pp. 311-338.

[BSS1] M. BELTRAMETTI - M. SCHNEIDER - A.J. SOMMESE, *Threefolds of degree 9 and 10 in $\mathbb{P}^5$*, Math. Ann. **288** (1990), 613-644.

[BSS2] M. BELTRAMETTI - M. SCHNEIDER - A.J. SOMMESE, *Threefolds of degree 11 in $\mathbb{P}_5$*. In: "Complex projective geometry", London Math. Soc. Lecture Note Series 179, 1992, pp. 59-80.

[DMS] W. DECKER - N. MANOLACHE - F.O. SCHREYER, *Geometry of the Horrocks bundle on $\mathbb{P}_5$*. In: "Complex projective geometry", London Math. Soc. Lecture Note Series 179, 1992, pp. 128-148.

[E] G. EDELMANN, *3-folds in $\mathbb{P}_5$ of degree 12*, Manuscripta math. **82** (1994), 393-406.

[EP] G. ELLINGSRUD - C. PESKINE, *Sur les surfaces lisses de $\mathbb{P}^4$*, Invent. Math. **95** (1989), 1-11.

[GP] L. GRUSON - C. PESKINE, *Genre des courbes de l'espace projectif.* In: "Algebraic Geometry", Lecture Notes in Math. 687, Springer, Berlin, 1977, pp. 31-59.

[H] G. HORROCKS, *Examples of rank three vector bundles on five-dimensional projective space*, J. London Math. Soc. **18** (1978), 15-27.

[K] C. KOELBLEN, *Surfaces de $\mathbb{P}_4$ tracées sur une hypersurface cubique*, J. reine angew. Math. **433** (1992), 113-141.

[M1] S. MUKAI, *Curves and symmetric spaces*, Proc. Japan Acad. Ser. A Math. Sci. **68** (1992), 7-10.

[M2] S. MUKAI, *Fano 3-folds*. In: "Complex Projective Geometry", London Math. Soc. Lecture Notes Series 179, 1992, pp. 255-263.

[O] G. OTTAVIANI, *3-folds in $\mathbb{P}_5$ which are scrolls*, Annali Scuola Norm. Sup. Pisa Cl. Sci., **19** (1992), 451-471.

[Sch] M. SCHNEIDER, *3-folds in $\mathbb{P}_5$: Classification in low degree and finiteness results*. In: "Geometry of complex projective varieties", Cetraro 1990, A. Lanteri - M. Palleschi - D. Struppa Eds., Mediterranean Press, Rende, 1993, pp. 275-288.

[S1] A.J. SOMMESE, *Hyperplane sections of projective surfaces I. The adjunction mapping*. Duke Math. J. **46** (1979), 377-401.

[S2] A.J. SOMMESE, *Hyperplane sections. Proc. of the Algebraic Geometry conference*, Univ. of Illinois at Chicago Circle, 1980, Lecture Notes in Math. 862, Springer-Verlag, Berlin, 1981, pp. 232-271.

[S3] A.J. SOMMESE, *On the adjunction theoretic structure of projective varieties*, Complex Analysis and Algebraic Geometry, Proceedings Göttingen 1985, Lecture Notes in Math. 1194, Springer, Berlin, 1986, pp. 175-213.

[SV] A.J. SOMMESE - A. VAN DE VEN, *On the adjunction mapping*, Math. Ann. **278** (1987), 593-603.

Mathematisches Institut
Universität Bayreuth
Universitätsstrasse
D-95440 Bayreuth
Germany

Dipartimento di Matematica
Università dell'Aquila
Via Vetoio, Coppito
I-67010 L'Aquila
Italy