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On Cayley Bundles on the Five-Dimensional Quadric.

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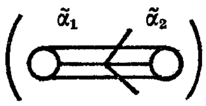
Sunto. – *Studiamo una famiglia di fibrati (olomorfi) stabili di rango 2 sulla quadrica Q_5 di dimensione 5. Questi fibrati si definiscono in modo naturale descrivendo Q_5 come la sottovarietà degli ottetti proiettivi di Cayley complessificati data dagli elementi con quadrato nullo, sulla quale agisce il gruppo di Lie eccezionale G_2. Lo spazio dei moduli è fine ed è isomorfo a $\boldsymbol{P}^7 \setminus Q_6 = \mathrm{Spin}(7)/\boldsymbol{Z}_2 \cdot G_2$, le rette di salto sono le rette dove il prodotto di Cayley vale zero e la sezione generica si annulla sulla varietà di bandiera $\boldsymbol{F}(0, 1, 2)$ che risulta così immersa in modo liscio in Q_5.*

Introduction.

Let $\boldsymbol{P}^{n+1}$ be the $(n+1)$-dimensional complex projective space and Q_n be the quadric hypersurface.

There are no known indecomposable rank 2 vector bundles on $\boldsymbol{P}_Q$ for $n \geqslant 5$ and only few are known on $\boldsymbol{P}^4$ (essentially one: the Horrocks-Mumford bundle).

We want now to consider the same situation with regard to Q_n. There are no known indecomposable 2-bundles on Q_n for $n \geqslant 6$. The aim of this paper is to describe a family of *indecomposable 2-bundles on Q_5*.

These bundles on Q_5 have been implicitly known for a long time. In fact the five-dimensional quadric is a quotient of the simple *exceptional Lie group G_2*  by the parabolic subgroup $P(\tilde{\alpha}_1)$ [*T*i1]. The semisimple part of $P(\tilde{\alpha}_1)$ is $SL(2)$, and the standard representation of $SL(2)$ defines a stable 2-bundle on Q_5 with $c_1 = c_2 = 3$. We call the normalized of such a bundle (with $c_1 = -1$, $c_2 = 1$) a *Cayley bundle*. There are three more under-

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standable descriptions of this bundle (up to twost by a line bundle $\mathcal{O}(t)$):

(i) as the extension of the 2-bundle defined on a hyperplane section Q_4 from the codimension two subvariety $\pi \cup \pi'$ (π, π' *disjoint 2-planes*) via the Hartshorne-Serre construction;

(ii) as the rank 2 irreducible component associated to the filtration of the rank 4 *spinor bundle* on Q_5 (or similarly of the rank 5 tangent bundle), induced by the action of G_2;

(iii) as the bundle whose fibers of the projectivization on the point p are given by all the *special lines* through p. Q_5 can be described as the variety of projectivized *complexified Cayley octonions* with null square, and the special lines through p are the lines concontained in the special 2-plane

$$\pi_p = \{x \in Q_5 : x \cdot p = 0 \text{ (Cayley product)}\} .$$

This third description is the geometrical interpretation of the definition via representation.

We will examine closely the second description: we can make explicit it in the following way. Let S^* be the dual of the spinor bundle on Q_5. The generic section of S^* does not vanish and defines a bundle G from the sequence

$$0 \to \mathcal{O} \to S^* \to G \to 0$$

The 2-bundle G has been studied in [Ot2]. It is stable with Chern classes $c_1 = c_2 = c_3 = 2$ and all stable 3-bundles with these Chern classes arise in this way and are parametrized by a fine moduli space which is $\boldsymbol{P}^7 \setminus Q_6$. We will see that G is homogeneous under the action of G_2.

We will see also that $G^*(1)$ has a nowhere vanishing section which defines a bundle C from the sequence

$$0 \to \mathcal{O} \to G^*(1) \to C(1) \to 0 .$$

C is normalized with $c_1 = -1$, $c_2 = 1$. Observe that $c_1(C(2)) = c_2(C(2)) = 3$. A bundle C arising in this way is a *Cayley bundle*. Our main result (proved in section 2) is the following:

MAIN THEOREM. – (i) *each stable 2-bundle on Q_5 with Chen classes $c_1 = -1$, $c_2 = 1$ is a Cayley bundle (i.e. is homogeneous and irreducible under the action of G_2)*;

(ii) *bundles in* (i) *are parametrized by a fine moduli space which is* $\boldsymbol{P}^7 \setminus Q_6$;

(iii) *the natural action of* Aut (Q_5) *on the moduli space is transitive.*

It follows from the theorem that the three description given at the beginning are equivalent (up to twist).

We obtain also that on Q_4 all stable 2-bundles F with Chern classes $c_1 = -1$, $c_2 = (1, 1)$ have a section of $F(1)$ vanishing on two disjoint 2-planes and their moduli space is always $\boldsymbol{P}^7 \setminus Q_6$ which is a $\boldsymbol{P}^1$-fibration over $\boldsymbol{P}^3 \times \boldsymbol{P}^3 \setminus H$. This last result was obtained in a different way by Arrondo and Sols [*SA*].

In section 3 we compute the cohomology of Cayley bundles and we prove that Cayley bundles do not extend on Q_6. We prove also that the special lines are exactly the jumping lines and that the generic section vanishes on the *homogeneous* 3-*fold* $\boldsymbol{F}(0, 1, 2)$.

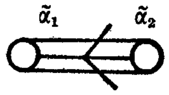
I thank V. Ancona and A. Huckleberry for many helpful talks on this subject.

1. – Preliminaries.

1.1. – We use the Mumford-Takemoto definition of stability. We have natural isomorphisms $H^{2i}(Q_4, \boldsymbol{Z}) = H^{2j}(Q_5, \boldsymbol{Z}) = \boldsymbol{Z}$ except $H^4(Q_4, \boldsymbol{Z}) = \boldsymbol{Z} \oplus \boldsymbol{Z}$. Thus we denote the Chern classes by inters and the second Chern class on Q_4 by a pair (a, b) of integers. E^* is the dual of the bundle E. $E(t)$ means $E \otimes \mathcal{O}(t)$.

We refer to [St] for Bott theorem and basic facts about homogeneous rational manifolds.

1.2. – Let $\boldsymbol{O}$ be the algebra of *complexified Cayley octonions*, which are defined (either over $\boldsymbol{R}$ or over $\boldsymbol{C}$) by the generators $1, e_1, e_2, \ldots, e_7$ subjected to the relations $e_i^2 = -1$, $e_i \cdot e_j = -e_j \cdot e_i$ $(i \neq j)$ and $e_1 \cdot e_2 = e_3$, $e_1 \cdot e_4 = e_5$, $e_1 \cdot e_6 = e_7$, $e_2 \cdot e_5 = e_7$, $e_2 \cdot e_4 = -e_6$, $e_3 \cdot e_4 = e_7$, $e_3 \cdot e_5 = e_6$ (more cyclic permutations) [Di]. Other definitions in literature are equivalent to this one. Over $\boldsymbol{R}$ the left-hand (or right-hand) division except by zero is always possible and unique. Over $\boldsymbol{C}$ division is no more always possible. In fact if we define the norm of an element $a_0 + \sum_{i=1}^{7} a_i e_i$ to be $N = \sum_{i=1}^{8} a_i^2$ we get that Cayley product preserves the norms, so that an element of norm zero cannot be inverted.

G_2 is the automorphism group of $\boldsymbol{O}$ (in the sense that the action of G_2 preserves sums and Cayley products) and it is easy to see that G_2 acts on the variety of projectivized elements of $\boldsymbol{O}$ with null square which is in a natural way isomorphic to the variety given bu the homogeneous relations $a_0 = \sum_{i=1}^{7} a_i^2 = 0$, that is to Q_5. The isotropy subgroup in G_2 of a point in Q_5 is a parabolic subgroup which is $P(\tilde{\alpha}_1)$ [Ti1], [St]. The Dynkin diagram of G_2 in 

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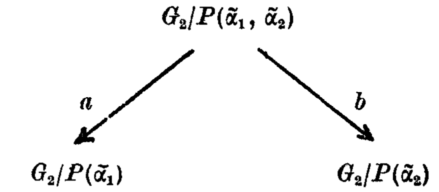
Here $|\tilde{\alpha}_2| = \sqrt{3}\,|\tilde{\alpha}_1|$ so that $\tilde{\alpha}_1$ is the shortest root. Let $\tilde{\lambda}_1, \tilde{\lambda}_2$ be the fundamental weights associated to the roots $\tilde{\alpha}_1, \tilde{\alpha}_2$.

1.3. – Tits describes also $G_2/P(\tilde{\alpha}_2)$ as the variety of special lines in Q_5. A line l is called special if $x, y \in l \Rightarrow x\cdot y = 0$ (Cayley product). All special lines through p are exactly the lines contained in the special plane $\pi_p = \{x \in Q_5 : x\cdot p = 0\}$. $G_2/P(\tilde{\alpha}_1)$ is a 5-dimensional variety in $\boldsymbol{P}^{13}$. Consider the diagram



$G_2/P(\tilde{\alpha}_1, \tilde{\alpha}_2)$ is the flag variety $\{(p, l) : p \in l,\ l$ special line in $Q_5\}$. We have the Levi decomposition $P(\tilde{\alpha}_1) = \tilde{U} \bowtie [SL(2)\cdot \boldsymbol{C}^*]$ where $\tilde{U}$ is the unipotent radical, $SL(2)\cdot \boldsymbol{C}^*$ is reductive and $SL(2)$ is the semisimple part. We set $B = P(\tilde{\alpha}_1, \tilde{\alpha}_2)$ (Borel subgroup). The standard representation of $SL(2)$ (with maximal weight $\tilde{\lambda}_2$) is by the Borel-Weil theorem the natural action of $SL(2)$ on

$$H^0(SL(2)/SL(2) \cap B, L_{\tilde{\lambda}_2})$$

where $L_{\tilde{\lambda}_2}$ is the line bundle with maximal weight $\tilde{\lambda}_2$. As $SL(2)/SL(2) \cap B \simeq P(\tilde{\alpha}_1)/B \simeq \{$fiber of $a\}$ it follows that the representation extends to $P(\tilde{\alpha}_1)$ and that the bundle defined on $Q_5 \simeq G_2/P(\tilde{\alpha}_1)$ by this representation is $a_* b^* \mathcal{O}(1)$. We have called such a bundle (twisted by $\mathcal{O}(-2)$ to normalize) a Cayley bundle and we denote it by C. Then the fibres over p of the projectivized of C are given by all the special lines through p. Note also that by the Borel-Weil theorem the natural action of G_2 on $H^0(Q_5, C(2))$ is the adjoint representation.

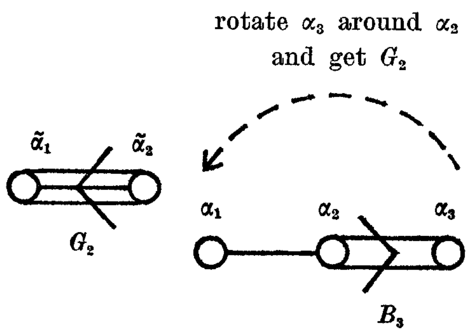
1.4. – The inclusion of the subgroup Spin (7) $\subset Cl_7 = \mathbf{C}(8) \oplus \mathbf{C}(8)$ gives the spin representation σ: Spin (7) $\to \mathbf{C}(8) = GL(H^0(Q_5, S^*))$ (the last equality by Borel-Weil thorem, Spin (7) acts in a natural way on $Q_5 =$ Spin $(7)/P(\alpha_1)$ where $P(\alpha_1) = U \bowtie$ [Spin $(5) \cdot \mathbf{C}^*$] and S^* is defined from the spin representation of Spin (5) [Ot2]).

Let $\mathbf{1} = (1, 0, ..., 0) \in \mathbf{O} = \mathbf{C}^8 = H^0(Q_5, S^*)$. We have obviously $G_2 \subset \{v \in$ Spin $(7): \sigma(v)\mathbf{1} = \mathbf{1}\}$, that is the action of G_2 on Q_5 lifts to an action on S^* which fixes exactly one section (for this see also the remark after the proof of the main theorem).

As $Q_6 \simeq \{\mathbf{P}^2 : \mathbf{P}^2 \subset Q_5\} \simeq$ Spin $(7)/P(\alpha_3)$ (Spin (7) acts transitively on Q_6 and the spinor bundle on Q_5 lifts to any of the two spinor bundles on Q_6) we can consider also the action of G_2 on Q_6. It is easy to check that G_2 preserves the form $a_0^2 - \sum_{i=1}^{7} a_i^2$ and the element $\mathbf{1}$ so that we can verify from the fact that G_2 is transitive on Q_5 that the action of G_2 in Q_6 has exactly two orbits: an hyperplane section Q_5 (given by $a_0 = 0$, consisting of imaginary numbers) and the complement $Q_6 \setminus Q_5$. Note that G_2 acts on the subset of $\mathbf{O}$ given by imaginary numbers: in fact $\mathbf{C}\cdot\mathbf{1}$ is an invariant subspace and $x \in \mathbf{O}$ is imaginary or $x \in \mathbf{C}\cdot\mathbf{1}$ if and only if $x^2 \in \mathbf{C}\cdot\mathbf{1}$. The orbit $Q_5 \to Q_6$ can be realized as the variety of special planes into the variety of all planes in Q_5.

1.5. – At the level of Lie algebras, we can choose fundamental systems of roots $\{\alpha_1, \alpha_2, \alpha_3\}$ for Spin (7) and $\{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ for G_2 with respective maximal weights $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{\tilde{\lambda}_1, \tilde{\lambda}_2\}$ such that the restriction of α_1 or α_3 (resp. λ_1 and λ_3) is $\tilde{\alpha}_1$ (resp. $\tilde{\lambda}_1$) and the restriction of α_2 (resp. λ_2) is $\tilde{\alpha}_2$ (resp. $\tilde{\lambda}_2$).

The following picture explains this restriction at the level of Dynkin diagrams



A quick proof of this fact is as follows. The tangent bundle is defined by the adjoint representation [St]. The weights of the adjoint representation which defines the tangent bundle on $Q_5 \simeq \simeq \mathrm{Spin}\,(7)/P(\alpha_1)$ are exactly the roots which contain α_1, that are in order

$$\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3,\ \alpha_1+2\alpha_2+2\alpha_3\ \text{[Ti2]}\,.$$

In a analogous way on $Q_5 \simeq G_2/P(\tilde{\alpha}_1)$ the weights of the adjoint representation which defines the tangent bundle are the roots which contain $\tilde{\alpha}_1$, that are in order $\tilde{\alpha}_1$, $\tilde{\alpha}_1+\tilde{\alpha}_2$, $2\tilde{\alpha}_1+\tilde{\alpha}_2$, $3\tilde{\alpha}_1+\tilde{\alpha}_2$, $3\tilde{\alpha}_1+ + 2\tilde{\alpha}_2$. This proves our claim. The tangent bundle is irreducible under the action of Spin (7). When we restrict to the action of G_2 the tangent bundle is reducible and we have the ordering

$$\underbrace{\tilde{\alpha}_1 = 2\tilde{\lambda}_1-\tilde{\lambda}_2 \leqslant \tilde{\alpha}_1+\tilde{\alpha}_2 = \tilde{\lambda}_2-\tilde{\lambda}_1}_{\text{weights of } C(1)} \leqslant \underbrace{2\tilde{\alpha}_1+\tilde{\alpha}_2 = \tilde{\lambda}_1}_{\text{weight of } \mathcal{O}(1)} \leqslant$$
$$\leqslant \underbrace{3\tilde{\alpha}_1+\tilde{\alpha}_2 = 3\tilde{\lambda}_1-\tilde{\lambda}_2 \leqslant 3\tilde{\alpha}_1+2\tilde{\alpha}_2 = \tilde{\lambda}_2}_{\text{weights of } C(2)}$$

2. – Proof of the main theorem.

LEMMA 2.1. – *Let F be a stable 2-bundle on Q_4 with $c_1 = -1$, $c_2 = (1, 1)$, We have $H^2(F(t)) = 0\ \forall t \in \mathbf{Z}$.*

PROOF. – By [ES] F remains stable on the generic hyperplane section Q_3 and we have $h^1(F|_{Q_3}(m)) = 0$ for $m \leqslant -1$, then $h^1(F(m)) = 0$ for $m \leqslant -1$ (this argument works on Q_n). By hypothesis we have $h^0(F) = 0$, $h^4(F) = h^0(F^*(-4)) = h^0(F(-3)) = 0$. Moreover we have $h^3(F) = h^1(F(-3)) = 0$. By the Hirzebruch-Riemann-Roch theorem we compute $\chi(F) = -1$ and then it follows $h^1(F) - h^2(F) = 1$, so that

$$H^1(F) = H^1(F^*(-1)) = \mathrm{Ext}^1\,(F(1), \mathcal{O}) \neq 0\,.$$

We obtain a nonsplitting extension $0 \to \mathcal{O} \to E^*(1) \to F(1) \to 0$ with $c_1(E) = 2$, $E_{\mathrm{norm}} = E(-1)$, $(E^*)_{\mathrm{norm}} = E^*$.

From the sequence

$$0 \to \mathcal{O}(-1) \to E^* \to F \to 0$$

it follows $h^0(E^*) = 0$. Now we want to prove that $h^0(E(-1)) = 0$. If on the contrary $E(-1)$ has one nonzero section we consider the diagram

$$\begin{array}{ccccccccc} & & & 0 & & & & & \\ & & & \downarrow & & & & & \\ & & & \mathcal{O} & \overset{\psi}{\searrow} & & & & \\ & & & \downarrow & & & & & \\ 0 & \to & F & \to & E(-1) & \to & \mathcal{O} & \to & 0 \end{array}$$

We must have $\psi = 0$, otherwise the above row splits. Thus we obtain an injective morphism from $\mathcal{O}$ to F, contradicting $h^0(F) = 0$. It follows that E is stable. The Chern classes of E are $c_1 = 2$, $c_2 = (2, 2)$, $c_3 = 2$. In [HS] the moduli of E are computed and it follows in particular that $h^2(E^*(t)) = 0\ \forall t$ and then $h^2(F(t)) = 0\ \forall t$.

Lemma 2.2. – *Let C be a stable 2-bundle on Q_5 with $c_1 = -1$, $c_2 = 1$. Then we have an exact sequence $0 \to \mathcal{O} \to G^*(1) \to C(1) \to 0$ with G stable 3-bundle with $c_1 = c_2 = c_3 = 2$.*

Proof. – By [ES] for the generic hyperplane section Q_4 $C|_{Q_4}$ is stable and from the proof of the lemma 2.1 we have the sequence on Q_4 $0 \to \mathcal{O} \to E^*(1) \to C(1)|_{Q_4} \to 0$, with E stable 3-bundle with $c_1 = 2$, $c_2 = (2, 2)$, $c_3 = 2$. From the lemma 2.1 $h^2(C(t)|_{Q_4}) = 0\ \forall t$, then from the sequence $0 \to C(t-1) \to C(t) \to C(t)|_{Q_4} \to 0$ it follows $h^2(C(t)) = h^3(C(t)) = 0\ \forall t$. Then the morphism $H^1(C) \to H^1(C|_{Q_4})$ is surjective, and when we interpret $H^1(C)$ in terms of extensions we obtain a 3-bundle G on Q_5 and a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O} & \to & G^*(1) & \to & C(1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_{Q_4} & \to & E^*(1) & \to & C(1)|_{Q_4} & \to & 0 \end{array}$$

As in the proof above we have $h^0(G_{\text{norm}}) = h^0(G^*_{\text{norm}}) = 0$, so that G is stable and also in this case the moduli of G are known from [Ot2].

Proof of the main theorem. – Our aim is to show that the moduli space of C is the same of G of the previous lemma, that is $\mathbf{P}^7 \setminus Q_6$ as in [Ot2], where the cohomology of the bundle $G(t)$ of the previous lemma is computed. In particular $h^0(G^*(1)) = 1$, $h^0(G^*(1)|_{Q_4}) = 2$ and we have also $h^1(C|_{Q_4}) = h^1(C) = h^1(G^*) = 1$.

Then we have a natural biunivoc map

$$\left\{\begin{matrix}\text{stable 3-bundle on } Q_5 \\ \text{with } c_1 = c_2 = c_3 = 2\end{matrix}\right\} \underset{\text{nonsplitting extension of } C(1) \text{ by } \mathcal{O}}{\overset{\text{quotient by the section of } G^*(1) \text{ and tensor by } \mathcal{O}(-1)}{\rightleftarrows}} \left\{\begin{matrix}\text{stable 2-bundle on } Q_5 \\ \text{with } c_1 = -1,\ c_2 = 1\end{matrix}\right\}$$

$$G \longmapsto C .$$

We remark that both the extension and the section considered in the map just defined are unique. This proves point (i) and (ii). Spin (7) acts on the space $\boldsymbol{P}(H^0(Q_5, S^*)) = \boldsymbol{P}^7$ where Q_6 is a compact orbit of codimension one, so that $P^7 \setminus Q_6$ must be another orbit because the action has no fixed points and stabilizes the closure of every orbit. Of course one can also compute directly $h^0(C \otimes C^*) = 1$ $h^1(C \otimes C^*) = 7$ $h^2(C \otimes C^*) = 0$ from Bott theorem, but this is not needed. This concludes the proof.

Remark on the proof. – In this remark we explicit the relation between Cayley bundles and embeddings on G_2 in Spin (7). This gives a more algebraic description of the moduli space $\boldsymbol{P}^7 \setminus Q_6$. The isotropy subgroup of Spin (7) acting on $\boldsymbol{P}^7 \setminus Q_6$ contains G_2 because Cayley bundles are G_2-homogeneous and has dimension $14 = \dim G_2$. The affine quadric $Q_{(7)}$ is the universal covering 2:1 of $\boldsymbol{P}^7 \setminus Q_6$ so that $\pi_1(\boldsymbol{P}^7 \setminus Q_6) = \boldsymbol{Z}_2$ and the homotopy sequence of the fibration Spin (7) $\to \boldsymbol{P}^7 \setminus Q_6$ shows that

$$\boldsymbol{P}^7 \setminus Q_6 \simeq \text{Spin}(7)/\boldsymbol{Z}_2 \cdot G_2 .$$

Now the fact that G_2 fixes the element $\mathbf{1} \in \boldsymbol{O} \simeq H^0(Q_5, S^*)$ can be interpreted in the following way. G_2 is the connected group of the automorphisms of the bundle S^* which fixes exactly one section, i.e. in the sequence $0 \to \mathcal{O} \to S^* \to G \to 0$ we have that G is homogeneous under the action of G_2. The action is reducible and each bundle G defines a Cayley bundle as the rank 2 component of the irreducible filtration. We emphasize that any embedding $G_2 \to$ Spin (7) defines an action of G_2 on S^* an then defines a Cayley bundle considering the filtration as above and conversely any Cayley bundle defines a 3-bundle G as nonsplitting extension and then an embedding

$$G_2 = \{g \in \text{Spin}(7) \colon g^* G \simeq G\}^0 = \{g \in \text{Spin}(7) \colon g^* C \simeq C\}^0 \to \text{Spin}(7)$$

(the superscript ° means « connected component of the identity »).

In the same way Spin (8) acts transitively on the moduli space of all 3-bundles on Q_5 with $c_1 = 2$ $c_2 = 2$ $c_3 = (2, 0)$ or $(0, 2)$ which are Spin (7)-homogeneous [Ot2]. This gives a geometric interpretation of the isomorphism

$$\boldsymbol{P}^7 \setminus Q_6 \simeq \text{Spin}\,(8)/\boldsymbol{Z}_2 \cdot \text{Spin}\,(7)\,.$$

Note that we have also

$$\boldsymbol{P}^7 \setminus Q_6 \simeq SO(7)/G_2 \simeq SO(8)/\text{Spin}\,(7)$$

and

$$Q_{(7)} \simeq \text{Spin}\,(7)/G_2 \simeq \text{Spin}\,(8)/\text{Spin}\,(7)\,.$$

3. – Further properties of Cayley bundles.

THEOREM 3.1. – *Let C be a Cayley bundle (normalized), and let*

$$A(t) = \frac{1}{30}(t-1)(t+1)(t+2)(t+3)(t+5\,.$$

Then $h^0(C(t)) = A(t)$ for $t \geqslant 2$, $h^5(C(t)) = -A(t)$ for $t \leqslant -6$, all other values of $h^i(C(t))$ vanish except $h^1(C) = h^4(C(-4)) = 1$.

PROOF. – It follows from Bott theorem. The weights of $C(2)$ are $\tilde{\lambda}_2$ and $3\tilde{\lambda}_1 - \tilde{\lambda}_2$ and we may suppose that the Killing form is

$$(\tilde{\lambda}_i, \tilde{\alpha}_j) = \begin{cases} \delta_{ij} & \text{if } (i, j) \neq (2, 2) \\ 3 & \text{if } (i, j) = (2, 2)\,. \end{cases}$$

Then for example $\tilde{\lambda}_1 + 2\tilde{\lambda}_2$ is regular of index 0, $2\tilde{\lambda}_2$ is singular, $-\lambda_1 + 2\tilde{\lambda}_2$ is regular of index 1 and so on. For $t > 0$ from Weyl formula

$$\prod_{\alpha \in \Phi^+} \frac{(t\tilde{\lambda}_1 + 2\tilde{\lambda}_2, \alpha)}{(\tilde{\lambda}_1 + \tilde{\lambda}_2, \alpha)} = A(t+1)$$

$$\text{as } \Phi^+ = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_1 + \tilde{\alpha}_2, 2\tilde{\alpha}_1 + \tilde{\alpha}_2, 3\tilde{\alpha}_1 + \tilde{\alpha}_2, 3\tilde{\alpha}_1 + 2\tilde{\alpha}_2\}$$

THEOREM 3.2. – *Any Cayley bundle C does not extend on Q_6 as a vector bundle.*

PROOF. – Suppose that C extends to Q_6 to the bundle $\tilde{C}$ (in the sense that $\tilde{C}|_{Q_5} \simeq C$). Then we have a sequence $0 \to \tilde{C}(t-1) \to \to \tilde{C}(t) \to C(t) \to 0$ and from that we obtain $h^0(\tilde{C}(t)) = 0$ for $t \leqslant 1$, $h^1(\tilde{C}(t)) = 0$ for $t \leqslant -1$, $h^2(\tilde{C}(t)) = 0$ $\forall t$, $h^1(\tilde{C}) = h^1(\tilde{Q}(1)) = 1$. $\tilde{C}$ restricts to a Cayley bundle at any smooth hyperplane section by the main theorem. We get an extension on Q_6

$$0 \to \mathcal{O} \to \tilde{G}^*(1) \to \tilde{C}(1) \to 0$$

with $c_1(\tilde{G}) = c_2(\tilde{G}) = 2$ $c_3(\tilde{G}) = (1, 1)$. The restriction $H^1(\tilde{C}(1)) \to \to H^1(C(1))$ is a isomorphism of one dimensional vector spaces and then $\tilde{G}|_{Q_5} \simeq G$. As $h^1(\tilde{G}^*(-1)) = h^2(\tilde{G}^*(-1)) = 0$ we get that the morphism $H^1(\tilde{G}^*) \to H^1(G^*)$ is again a isomorphism of one dimensional vector spaces: this means that there is a sequence on Q_6 $0 \to \mathcal{O} \to A \to \tilde{G} \to 0$ where A is a bundle which restricts to the spinor bundle at any smooth hyperplane section Q_5. By [Ot2], th. 2.11 it follows that A is itself a spinor bundle and looking at Chern classes we get $c_3(\tilde{G}) = (2, 0)$ or $(0, 2)$ in contradiction with $c_3(\tilde{G}) = (1, 1)$. This concludes the proof.

The theorem above implies that there are no stable 2-bundles on Q_6 with $c_1 = -1$, $c_2 = 1$. In [Ot2] a class of stable 3-bundles E on Q_6 with $c_1 = 2$, $c_2 = 2$, $c_3 = (2, 0)$ or $(0, 2)$ are constructed (they are Spin (7)-homogeneous) but $E^*(1)$ has no sections.

REMARK 3.3. – From 1.5 we obtain the following filtration $0 \subset C(1) \subset G \subset TQ_5$ with the sequences

$$0 \to C(1) \to G \to \mathcal{O}(1) \to 0 \qquad 0 \to G \to TQ_5 \to C(2) \to 0 .$$

The dual of the surjective morphism $TQ_5 \to C(2)$ is given on each fiber by the inclusion of the 2-plane given by all special lines through p into the space of all tangent lines through p.

REMARK 3.4. – On Q_4 we have again a unique nonsplitting extension E of $C(1)|_{Q_4} = F(1)$ by $\mathcal{O}$ but the section of $E^*(1)$ is not unique. It follows that a stable 2-bundle on Q_4 with $c_1 = -1$, $c_2 = (1, 1)$ extends in a unique way on Q_5. The moduli spaces of these bundles on Q_4 is fine and is again $\mathbf{P}^7 \setminus Q_6$. These bundles can be constructed directly from the union of two disjoint 2-planes via the Hartshorne-Serre construction. The computation on the moduli shows that all stable 2-bundles F with $c_1 = -1$, $c_2 = (1, 1)$ can be constructed in this way. Then each of these bundles on Q_4, twisted so that $c_1 = 1$ $c_2 = (1, 1)$ has a unique (up to scalar multiple) section which vanishes on the union of two disjoint 2-planes.

The theorem 3.1 shows that

$$h^0(F(t)) = h^4(F(-t-3)) = \frac{(t+1)(t+2)(t^2+3t-3)}{6} \text{ for } t \geqslant 1$$

and all other values of $h^i(F(t))$ vanish except $h^1(F) = h^3(F(-3)) = 1$.

THEOREM 3.5. – *Let C be a Cayley bundle. If r is a special line then $C|_r \simeq \mathcal{O}(-2) \oplus \mathcal{O}(1)$. If r is not a special line then $C|_r \simeq \mathcal{O}(-1) \oplus \mathcal{O}$. Then the special lines are exactly the jumping lines of C.*

PROOF. – In [Ot1] th. 3.2 the jumping lines of G restricted to $Q_4 \simeq \mathrm{Gr}(1,3)$ are computed. It follows that on the generic 2-plane in Q_4 there are not jumping lines for G and the generic splitting is $\mathcal{O} \oplus \mathcal{O}(1)^2$ while on some 2-planes there is a pencil of jumping lines (all the lines through a fixed point) where the splitting is $\mathcal{O}^2 \oplus \mathcal{O}(2)$. Then the two possible splitting for $G^*(1)$ are $\mathcal{O}^2 \oplus \mathcal{O}(1)$ and $\mathcal{O}(-1) \oplus \mathcal{O}(1)^2$. This is important because looking at the cohomology sequence associated to

$$0 \to \mathcal{O}_r \to G^*(1)|_r \to C(1)|_r \to 0$$

we get

$$G^*(1)|_r \simeq \mathcal{O}^2 \oplus \mathcal{O}(1) \Leftrightarrow C(1)|_r \simeq \mathcal{O} \oplus \mathcal{O}(1)$$

$$G^*(1)|_r \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)^2 \Leftrightarrow C(1)|_r \simeq \mathcal{O}(-1) \oplus \mathcal{O}(2)\,.$$

Consider now that from 1.4 G_2 acts on $\{\boldsymbol{P}^2 \colon \boldsymbol{P}^2 \subset Q_5\} \simeq Q_6$ with two orbits: the hyperplane Q_5 given by the special planes (i.e. the planes that $\forall p \in Q_5$ contain all special lines through p) and the complement $Q_6 \setminus Q_5$. Both G and C are homogneneous by the action of G_2. Then when π is not a special plane $C|_\pi$ has to be uniform (precisely $C|_\pi \simeq T\boldsymbol{P}^2(-2)$ by cohomology reasons) and when π is special $C|_\pi$ has a pencil of jumping lines that has to be the pencil of special lines because on the special lines the splitting has to be the same by the homogeneity. This concludes the proof.

REMARK 3.6. – The theorem 3.5 shows that the special lines determines the isomorphism class of the bundle C. In fact the set of special lines determines an embedding $Q_5 \to \{\boldsymbol{P}^2 \colon \boldsymbol{P}^2 \subset Q_5\} \simeq Q_6$ given by the hyperplane section and these are parametrized by $\boldsymbol{P}^7 \setminus Q_6$ (we have to exclude the tangent hyperplanes).

The special lines form a 5-dimensional submanifold in the 7-dimensional manifold of all lines in Q_5.

The behaviour of G and C when restricted to linear subspaces in Q_5 is summarized in the table 1.

Table 1

	restricted to lines	restricted to planes
C	nonspecial: $\mathcal{O}\oplus\mathcal{O}(-1)$ special: $\mathcal{O}(-2)\oplus\mathcal{O}(1)$	nonspecial: $TP^2(-2)$ special: not semistable bundle with $c_1=-1$ $c_2=1$
G	nonspecial: $\mathcal{O}\oplus\mathcal{O}(1)^2$ special: $\mathcal{O}^2\oplus\mathcal{O}(2)$	nonspecial: $\Omega^1(2)\oplus\mathcal{O}(1)$ special: $N|_{P^2}\oplus\mathcal{O}$ N = nullcorrelation on P^3

THEOREM 3.7. – *Let C be a Cayley bundle. $C(2)$ is the first twist which has sections and it is globally generated. Each section of $C(2)$ with smooth zero loci (and hence the generic one) vanishes on a subvariety isomorphic to the complete flag manifold $\boldsymbol{F}(0,1,2)$ of linear elements of $\boldsymbol{P}^2$.*

PROOF. – $C(2)$ is generated by global sections because is a quotient of TQ_5. By the theorem 3.1 $h^0(C(2))=14$. Let V be the smooth zero loci of a section. By the adjunction formula $K_V\simeq \simeq K_{Q_5}\otimes\det C(2)|_V\simeq\mathcal{O}(-2)|_V$. We have $V\sim c_2(C(2))$ as cycles in Q_5 and $c_2(C(2))$ is equivalent to three times a section Q_3, so that $\deg V=6$. V is then a Fano threefold of index 2 and degree 6. By the classification given in [Mu] it follows that there are only two possibilities: $V\simeq L\cdot S(\boldsymbol{P}^2\times\boldsymbol{P}^2)$ or $V\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$. The first case corresponds to a hyperplane section of the Segre variety $\boldsymbol{P}^2\times\boldsymbol{P}^2$ and it is easy to check that it is isomorphic to $\boldsymbol{F}(0,1,2)$. As $\boldsymbol{P}^2\times\boldsymbol{P}^2$ can be smoothly projected from $\boldsymbol{P}^8$ in $\boldsymbol{P}^7$ then $\boldsymbol{F}(0,1,2)$ can be smoothly embedded in $\boldsymbol{P}^6$. The second case $V\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ cannot occur and is excluded by the following computation. We have:

$$h^0(T_{P^1\times P^1\times P^1})-h^1(T_{P^1\times P^1\times P^1})=9-0=9$$

while on the other hand we consider the sequences:

(3.1) $\quad 0\to TV\to TQ_5|_V\to C(2)|_V\to 0$

(3.2) $\quad 0\to \mathcal{J}_V(1)\to\mathcal{O}_{Q_5}(1)\to\mathcal{O}_V(1)\to \qquad (\mathcal{J}_V \text{ means } \mathcal{J}_{V,Q_5})$

(3.3) $\quad 0\to\mathcal{J}_V\otimes C(2)\to C(2)\to C(2)|_V\to 0$

(3.4) $\quad 0\to\mathcal{O}(-3)\to C(-1)\to\mathcal{J}_V\to 0$

(Koszul complex of the section defining V)

(3.5) $0 \to C(-1) \to C \otimes C^* \to \mathfrak{J}_V \otimes C(2) \to 0$

(above tensored by $C(2)$)

(3.6) $$0 \to TQ_5|_V \to TP^6|_V \to \mathcal{O}_V(2) \to 0$$

(3.7) $$0 \to \mathcal{O}_V \to \mathcal{O}(1)^7|_V \to TP^6|_V \to 0$$

By (3.5) one computes $h^0(\mathfrak{J}_V \otimes C(2)) = 1$, $h^1(\mathfrak{J}_V \otimes C(2)) = 7$, $h^2(\mathfrak{J}_V \otimes C(2)) = 0$, then by (3.3) it follows $h^0(C(2)|_V) = 14 + 7 - 1 = 20, h^1(C(2)|_V) = 0$. By (3.4) twisted by $\mathcal{O}(1)$ we have $h^0(\mathfrak{J}_V(1)) = 0$, $h^1(\mathfrak{J}_V(1)) = 1$, $h^2(\mathfrak{J}_V(1)) = 0$, then by (3.2) $h^0(\mathcal{O}_V(1)) = 8$, $h^1(\mathcal{O}_V(1)) = 0$ and also in the same way $h^0(\mathcal{O}_V) = 1$, $h^1(\mathcal{O}_V) = h^2(\mathcal{O}_V) = 0$, $h^0(\mathcal{O}_V(2)) = 27$. By (3.7) $h^0(TP^6|_V) = 7 \cdot 8 - 1 = 55$, $h^1(TP^6|_V) = 0$, then by (3.6) $h^0(TQ_5|_V) - h^1(TQ_5|_V) = 55 - 27 = 28$. At last by (3.2)

$$h^0(TV) - h^1(TV) =$$

$$= h^0(TQ_5|_V) - h^1(TQ_5|_V) - h^0(C(2)|_V) = 28 - 20 = 8 \quad \text{(instead of 9)}$$

This gives the contradiction. Of course one can computes directly $h^0(TF) - h^1(TF) = 8$ (TF is the tangent bundle of $F(0,1,2)$).

REMARK 3.8. – We observe that the hyperplane sections of $F(0,1,2)$ are *Del Pezzo surfaces* isomorphic to P^2 blown up at three points and imbedded with the anticanonical system. These surfaces are the zero loci of sections of $C(2)|_{Q_4}$. From the proof of the theorem 3.7 it follows also that, if we set $F = F(0,1,2)$ $h^0(\mathfrak{J}_{F,Q_5}(2)) = h^0(C(1)) = 0$. Then $h^0(\mathfrak{J}_{F,P^6}(2)) = 1$, that means F is contained in a unique quadric in P^6, which is smooth by our construction. The generic section of $C(2)$ on F is a smooth curve of degree 18 and genus 10.

REMARK 3.9. – The restriction $C(2)|_F$ is the normal bundle N_{F,Q_5} but the restriction on F of the sequence $0 \to G \to TQ_5 \to C(2) \to 0$ of remark 3.3 is *not* the sequence $0 \to TF \to TQ_5|_F \to N_{F,Q_5} \to 0$. In fact from the embedding $P^1 = SL(2)/B \to SL(3)/B = B = F$ it is easy to see that TF restricted to any line is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$ that is not a possible splitting for G (see table 1). Then we can construct *deformations of TF* as kernels of the surjective morphisms $TQ_5|_F \to N_{F\,Q_5}$. Note also the filtration on F $0 \to \mathcal{O} \to TF^*(1) \to \mathcal{O}(-1,2) \oplus \mathcal{O}(2,-1) \to 0$ which is analog on Q_5 at $0 \to \mathcal{O} \to G^*(1) \to C(1) \to 0$.

We can also show that F contains *at least one* special line. We sketch only the reasoning. Considering the fibres of one of the two projections $p: F \to P^2$ we may assume that C is uniform on this family. Then by a well known argument we have that (up to twist) $p^* p_* C$ is a line subbundle of C. Then C is an extensions of two line bundles and looking at Chern classes the only possibility is $0 \to \mathcal{O}(-2, 1) \to C \to \mathcal{O}(1, -2) \to 0$ so that the splitting of C on one family is $\mathcal{O}(-2) \oplus \mathcal{O}(1)$ which corresponds to special lines.

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Note added in proof. – After this paper was written, B. Fantechi pointed out to me using simple linear algebra that the chordal variety of $P^1 \times P^1 \times P^1$ is P^7: this allows to avoid the cohomological computations in the proof of the theorem 3.7.

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