3-bundles on $\mathbb{P}^5$

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There are only few examples of indecomposable vector bundles of small rank on the complex projective space $\mathbb{P}^n$. The only known indecomposable rank 2 bundles on $\mathbb{P}^4$ are the Horrocks-Mumford bundle and its pullbacks under a finite morphism $\pi:\mathbb{P}^4 \to \mathbb{P}^4$. Moreover these 2-bundles on $\mathbb{P}^4$ are stable and any family obtained pulling back the Horrocks-Mumford bundle under a finite morphism is invariant by small deformations [DS]. No indecomposable 2-bundle is known on $\mathbb{P}^5$.

Horrocks defined on [Hor2] a stable 3-bundle on $\mathbb{P}^5$ (called the *parent bundle*). He also showed how to modify this example in order to obtain some weighted 3-bundles $E_{\alpha,\beta,\gamma}$ (called in this paper relation bundles, see definition 6.1) depending on nonnegative integers $\alpha\leq\beta\leq\gamma$ satisfying $\alpha+\beta<\gamma$. The parent bundle corresponds to $\alpha=\beta=0$, $\gamma=1$. More precisely consider the diagram

$$\begin{array}{ccc} \mathbb{C}^6\backslash 0 & \xrightarrow{\omega} & \mathbb{C}^6\backslash 0 \\ \downarrow\eta & & \downarrow\eta \\ \mathbb{P}^5 & & \mathbb{P}^5 \end{array} \tag{0.1}$$

where ω is given by six homogeneous polynomials $f_1,\ldots, f_6$ without common zeroes of degree $\gamma-\alpha$, $\gamma-\beta$, $\gamma+\alpha+\beta$, $\gamma+\alpha$, $\gamma+\beta$, $\gamma-\alpha-\beta$. $\omega^*\eta^*E$ descends to a vector bundle $E_{\alpha,\beta,\gamma}$ on $\mathbb{P}^5$, so that we have $\eta^*E_{\alpha,\beta,\gamma}\simeq\omega^*\eta^*E_{0,0,1}$. Of course $E_{\alpha,\beta,\gamma}$ depends on ω but for simplicity we forget this fact in the notations. We refer to bundles obtained from the diagram (0.1) with the construction above as *bundles coming as pullback over* $\mathbb{C}^6\backslash 0$, warning the reader that only in the case $\alpha=\beta=0$ the map ω descends to $\omega':\mathbb{P}^5 \to \mathbb{P}^5$.

Decker, Manolache and Schreyer studied the moduli space and the geometry of sections of the parent bundle [DMS]. In particular they proved that every small deformation of the parent bundle can be obtained by the action of a automorphism of $\mathbb{P}^5$. In section 6 we remark that the family of bundles $E_{\alpha,\beta,\gamma}$ constructed by Horrocks pulling back the parent bundle over $\mathbb{C}^6\backslash 0$ is invariant by small deformations if and only if $\alpha=\beta=0$ (the case of finite morphisms $\mathbb{P}^5 \to \mathbb{P}^5$). We give a more general construction of 3-bundles in terms of monads (the relation bundles), that includes all the small deformations of Horrocks bundles $E_{\alpha,\beta,\gamma}$(see theorem 6.5). In corollary 6.10 and theorem 6.12 we prove that the generic $E_{\alpha,\beta,\gamma}$ is stable if and only if $3\gamma-2\alpha-4\beta>0$.

In prop. 7.2 it is computed the dimension of the Kuranishi space of $E_{\alpha,\beta,\gamma}$, which turns out to be $h^1(\mathrm{End}\ E_{\alpha,\beta,\gamma})$, in terms of α,β,γ. The formula has a "principal part" that has a clear meaning plus some "correction terms" that vanish when $\gamma\gg 0$. The Chern classes of $E_{\alpha,\beta,\gamma}$ are $c_1=c_3=0$,

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$c_2=3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2$. Schwarzenberger conditions imply that 3-bundles on $\mathbb{P}^5$ with $c_1=c_3=0$ can exist only if $c_2\equiv 0,3,8$ or 11 (mod 12). A consequence of our computations is the following:

Theorem 1 $\forall N\in\mathbb{N}$, $\forall t\in\mathbb{Z}$, $t\equiv 0,3,8$ or 11 (mod 12) there exists a family of nonisomorphic 3-bundles on $\mathbb{P}^5$ with $c_1=c_3=0$, $c_2=t$ of dimension $\geq N$.

This generalizes the analogous result for 2-bundles on $\mathbb{P}^3$ obtained by Hartshorne[Har] and shows that there are plenty of 3-bundles on $\mathbb{P}^5$. Let $M_{\mathbb{P}^5}(0,t,0)$ be the moduli space of stable 3-bundles on $\mathbb{P}^5$ with $c_1=c_3=0$, $c_2=t$, then we get:

Theorem 2 Let $M_{\mathbb{P}^5}(0,t,0)=X_1\cup X_2\cup\ldots\cup X_{n(t)}$ be the decomposition into irreducible components. Then $\limsup_t n(t)=+\infty$.

This generalizes the analogous result for 2-bundles on $\mathbb{P}^3$ obtained by Ein [Ein]. Our approach gives also an alternative proof of Ein result, using representation theory instead of Cech cohomology computations.

For some computations in theorem 6.7 and in lemma 7.10 we used the program Macaulay [BaS], running on a personal computer. Anyway, the help of a computer is not necessary in order to prove theorems 1 and 2.
This is the content of the sections:

0. Notations and conventions
1. Some known results about bundles on $\mathbb{P}^5$
2. The parent bundle
3. Weighted quotient bundles
4. Weighted nullcorrelation bundles
5. Weighted lambda-three bundles
6. Relation bundles
7. The computation of $h^1(\mathrm{End}\ E_{\alpha,\beta,\gamma})$

The main technique used in this paper, that is the computation of cohomology of bundles coming as pullback over $\mathbb{C}^6\backslash 0$ using representation theory, is explained with full details in sections 3 and 4, and then used throughout the paper. The proofs of the theorems 1 and 2 will be given in section 6. Sections 3 and 5 contain results that are used in section 6. In the appendix we have collected some numerical information on the moduli of the relation bundles.

The authors benefited from many helpful conversations with W.Decker, N.Manolache and F.Schreyer. In particular N.Manolache communicated to us the minimal resolution of the parent bundle (theorem 6.7).

0. Notations and conventions

Let H be a complex vector space of dimension 6, we consider $\mathbb{P}^5 = \mathbb{P}(H^*)$ with homogeneous coordinates (a,b,c,d,e,f).

Let $\mu_1,\ldots,\mu_5$ be the fundamental weights of $SL(6)=SL(6,\mathbb{C})$. Let us recall that the irreducible representation of SL(6) corresponding to the weight $\sum a_i\mu_i$ is represented by the Young diagram consisting of $a_1+\ldots+a_5$ boxes in the first row, $a_2+\ldots+a_5$ boxes in the second row, up to a_5 boxes in the fifth row. We will denote such representation with both the symbols $H_{\sum a_i\mu_i}$ or $\Gamma^{a_1+\ldots+a_5,\ldots,a_5}H$. In particular

$$H_{\mu_i} \simeq \Gamma^{\overbrace{1,\ldots,1}^{i\ \text{times}}} H \simeq \wedge^i H.$$

If $\mathcal{F}$ is any 6-bundle, the bundle $\mathcal{F}_{\sum a_i\mu_i} \simeq \Gamma^{a_1+\ldots+a_5,\ldots,a_1}\mathcal{F}$ is naturally defined.

In particular if $\mathcal{F} = \bigoplus_{i=1}^{6} \mathcal{O}(d_i)$, then

$$\Gamma^{n_1,\ldots,n_5} \bigoplus_{i=1}^{6} \mathcal{O}(d_i) \simeq \bigoplus_{j\in J} \mathcal{O}(b_j) \tag{0.2}$$

where J is the set of all the combinations of the d_i's filling the boxes of the Young diagram with n_i boxes in the i-th row in such a way that the indexes are strictly increasing in the columns and increasing in the rows. and b_j is the sum of all the d_i appearing in the combination j.

Let ν_1, ν_2, ν_3 be the fundamental weights of Sp(6). Let ν be a weight in the fundamental chamber of Sp(6), we denote by H_ν the corresponding representation. For example $H_{\nu_1} \simeq H$, $H_{\nu_2} \simeq \wedge^2 H/\mathbb{C}$, $H_{\nu_3} \simeq \wedge^3 H/H$. If $\mathcal{F}$ is a symplectic 6-bundle it is naturally defined the bundle $\mathcal{F}_\nu$, for example $\mathcal{F}_{\nu_2} = \wedge^2\mathcal{F}/\mathcal{O}$. We will use throughout the paper this notation many times when $\mathcal{F}$ is the bundle $\mathcal{H}$ defined in (3.1).

If $\mathcal{F}$ is a bundle, the *adjoint bundle* ad $\mathcal{F}$ is End $\mathcal{F}/\mathcal{O}$.

If W is a complex vector space of dimension 3, then $W\oplus W^*$ has a natural symplectic structure. We denote by $\Gamma^{p,q}W$ the representation of W corresponding to the Young diagram with p boxes in the first row and q boxes in the second row. In particular $\Gamma^{i,0}W = S^iW$, $\Gamma^{1,1}W = \wedge^2 W = W^*$, moreover $\Gamma^{p,q}W = \Gamma^{p,p-q}W^*$ and $\dim \Gamma^{p,q}W = \frac{(p+2)(q+1)(p-q+1)}{2}$.

If $\mathcal{F}$ is any 3-bundle, the bundle $\Gamma^{p,q}\mathcal{F}$ is naturally defined .

We will use Mumford-Takemoto definition of stability.

If $\mathcal{F}$ is a coherent sheaf over a complex space X and $f: X \to S$ is a morphism, we denote by $\mathrm{Quot}_{\mathcal{F}/X/S}$ the *Grothendieck space* parametrizing the coherent quotient sheaves of $\mathcal{F}$ which are flat on S. We have a projection $Z \to S$ and for $s\in S$ we have $Z_s \simeq \mathrm{Quot}_{\mathcal{F}_s/X_s}$.

If E is a vector bundle on a compact complex space X there exists a *Kuranishi space* Z which is the base for the *versal deformation* of E. Let $z_0 \in Z$ be the point corresponding to E. Z is equipped with a universal family and the germ (Z,z_0) is unique up to automorphisms. The same bundle can appear many times in the versal deformation but E itself appears in a neighborhood of z_0 only once.

1. Some known results about bundles on $\mathbb{P}^5$

Let G be a semisimple complex Lie group and let ϕ be the set of the roots of G. Let $\Delta=\{\alpha_1,\dots,\alpha_k\}$ be a fundamental system of roots . We have the Cartan decomposition

$$\text{Lie } G=\mathfrak{g}_0\oplus\sum_{\alpha\in\phi^-}\mathfrak{g}_\alpha\oplus\sum_{\alpha\in\phi^+}\mathfrak{g}_\alpha$$

Let $\phi^+(i)=\{\alpha\in\phi^+ \mid \alpha=\sum n_j\alpha_j \text{ with } n_i=0\}$ and let $P(\alpha_i)\subset G$ be the parabolic subgroup such that $\text{Lie } P(\alpha_i)=\mathfrak{g}_0\oplus\sum_{\alpha\in\phi^-}\mathfrak{g}_\alpha\oplus\sum_{\alpha\in\phi^+(i)}\mathfrak{g}_\alpha$. Then $G/P(\alpha_i)$ is a rational homogeneous manifold with $\text{Pic}=\mathbb{Z}$.

Let $\{\lambda_1,\dots,\lambda_k\}$ be the fundamental weights with respect to Δ.

We will apply this construction to the cases

i) $G=SL(6)$, $\Delta=\{\beta_1,\dots,\beta_5\}$, $SL(6)/P(\beta_1)\simeq\mathbb{P}^5$; the reductive part in the Levi decomposition of $P(\beta_1)$ is isomorphic to $SL(5)\cdot\mathbb{C}^*$. We denote in this case $\{\mu_1,\dots,\mu_5\}$ the fundamental weights.

ii) $G=Sp(6)$, $\Delta=\{\sigma_1,\sigma_2,\sigma_3\}$, $Sp(6)/P(\sigma_1)\simeq\mathbb{P}^5$; the reductive part in the Levi decomposition of $P(\sigma_1)$ is isomorphic to $Sp(4)\cdot\mathbb{C}^*$. We denote in this case $\{\nu_1,\nu_2,\nu_3\}$ the fundamental weights.

Let (,) be the Killing form over Lie G. We recall that a weight λ is called singular if there exists $\alpha\in\phi^+$ such that $(\lambda,\alpha)=0$ and is called regular of index p if it is not singular and if there exist exactly p roots $\alpha\in\phi^+$ such that $(\lambda,\alpha)<0$.

Let $\rho(\lambda)$ be the irreducible representation of $P(\alpha_i)$ whose restriction to the reductive part has maximal weight $\lambda=\sum n_j\lambda_j$ with $n_j\geq 0$ for $j\neq i$. Let E^λ be the homogeneous vector bundle over $G/P(\alpha_i)$ associated to $\rho(\lambda)$. Let $\delta=\sum\lambda_i$. The main result about the cohomology of E^λ is:

Bott theorem [Bo]

i)If $\lambda+\delta$ is singular then $H^k(G/P(\alpha_i),E^\lambda)=0 \ \forall k$

ii) If $\lambda+\delta$ is regular of index p then

$H^k(G/P(\alpha_i),E^\lambda)=0 \ \forall k\neq p$

$H^p(G/P(\alpha_i),E^\lambda)=V_{s(\lambda+\delta)-\delta}$ where $s(\lambda+\delta)$ is the unique element of the fundamental Weyl chamber of G congruent to $\lambda+\delta$ under the action of the Weyl group.

The *quotient bundle* Q on $\mathbb{P}^5=\mathbb{P}(H^*)$, is defined by the Euler sequence

$$0\to\mathcal{O}(-1)\to H^*\otimes\mathcal{O}\to Q\to 0$$

The bundle Q, as well as Q^*, is stable and SL(6)-homogeneous, precisely $Q=E^{\mu_5}$, $Q^*=E^{\mu_2-\mu_1}$. Recall also $\mathcal{O}(t)=E^{t\mu_1}$ $\forall t\in\mathbb{Z}$. By Bott theorem $H^0(Q^*(t))\simeq H_{(t-1)\mu_1+\mu_2}$ for $t\geq 1$, the intermediate $H^i(Q^*(t))$ for $1\leq i\leq 4$ are zero for every t with the only exception $H^1(Q^*(-1))=C$. Moreover $H^1(\mathrm{End}\ Q(t))=0$ for $t\neq -1$ and $H^1(\mathrm{End}\ Q(-1))=H$, so that every small deformation of Q is isomorphic to Q. The minimal resolution of Q^* is

$$0\to\mathcal{O}(-5)\to\wedge^5H\otimes\mathcal{O}(-4)\to\wedge^4H\otimes\mathcal{O}(-3)\to\wedge^3H\otimes\mathcal{O}(-2)\to\wedge^2H\otimes\mathcal{O}(-1)\to Q^*\to 0$$

We list now some cohomological lemmas that are applications of Bott theorem and will be used in the rest of the paper.

Lemma 1.1 $H^1(\mathrm{End}\ \wedge^2Q(t)) = H^1(\mathrm{End}\ Q(t))$ $\forall t\in\mathbb{Z}$

Proof From the Littlewood-Richardson rule $\mathrm{End}\ \wedge^2Q(t) = \mathrm{End}\ Q(t)\oplus E_{\mu_3+\mu_4}(t-1)$. The weight $t\mu_1+\mu_3+\mu_4+\delta$ is regular of index 0 for $t\geq 0$, of index 2 for $t=-3$, of index 3 for $t=-5$, of index 5 for $t\leq -8$ and it is singular for $t=-1, -2, -4, -6, -7$. In particular $t\mu_1+\mu_3+\mu_4+\delta$ is never regular of index one, then from Bott theorem $H^1(\mathrm{End}\ \wedge^2Q(t)) = H^1(\mathrm{End}\ Q(t))$ $\forall t\in\mathbb{Z}$, hence our claim follows.

Lemma 1.2 $H^1(\wedge^2Q\otimes\wedge^4Q^*(t))=0$ $\forall t\in\mathbb{Z}$

Proof Note that $\wedge^2Q\otimes\wedge^4Q^*(t)=\wedge^2Q\otimes Q(t-1)$ $\forall t\in\mathbb{Z}$. We have $\wedge^2Q\otimes Q=\Gamma^{2,1}Q\oplus\wedge^3Q$. $\Gamma^{2,1}Q$ is homogeneous corresponding to the maximal weight $\mu_4+\mu_5$. It is easy to check that the weight $t\mu_1+\mu_2+\mu_3+2\mu_4+2\mu_5$ is never regular of index one, hence $H^1(\Gamma^{2,1}Q(t-1))=0$ $\forall t\in\mathbb{Z}$ by Bott theorem. The analogous vanishing for $\wedge^3Q$ is well known.

Lemma 1.3

$H^0(\wedge^4Q\otimes\wedge^2Q^*)=H_{\mu_4}$, $H^0(\wedge^4Q\otimes\wedge^2Q^*(t))=H_{(t-1)\mu_1+\mu_2+\mu_3}\oplus H_{t\mu_1+\mu_4}$ for $t\geq 1$ (all the previous groups are zero for $t<0$)

$H^1(\wedge^4Q\otimes\wedge^2Q^*(t))$ is zero for $t\neq -1$ and it is isomorphic to $\wedge^3H$ for $t=-1$.

$H^0(\wedge^4Q(t))=H_{t\mu_1+\mu_2}$ for $t\geq 0$

Proof $\wedge^4Q\otimes\wedge^2Q^*(t)=\Gamma^{2,2,2,1}Q(t-1)\oplus\wedge^2Q(t)$. The cohomology of the second summand is well known. The first summand corresponds to the maximal weight $(t-1)\mu_1+\mu_2+\mu_3$; $t\mu_1+2\mu_2+2\mu_3+\mu_4+\mu_5$ is regular of index 0 for $t\geq 1$ and it is regular of index one for $t=-1$. $-\mu_1+2\mu_2+2\mu_3+\mu_4+\mu_5$ is congruent to $\mu_1+\mu_2+2\mu_3+\mu_4+\mu_5$ under the action of the Weyl group. The result follows from Bott theorem.

Lemma 1.4 $H^1(\Gamma^{2,2,2}Q(t)) = 0$ $\forall t\in\mathbb{Z}$

$H^0(\Gamma^{2,2,2}Q(t-2))=\Gamma^{t,t,t,t-2,t-2}H = H_{(t-2)\mu_1+2\mu_3}$ for $t\geq 2$ and $H^0(\Gamma^{2,2,2}Q(t-2))=0$ for $t<2$.

$H^1(\Gamma^{2,1,1}Q(t))=0$ $\forall t\in\mathbb{Z}$

Proof From Bott theorem

Lemma 1.5 $H^0(\Gamma^{2,2}Q(t-2))=\begin{cases}\Gamma^{t,t,t-2,t-2,t-2}H & \text{for } t\geq 2\\ 0 & \text{for } t<2\end{cases}$

$$H^0(\Gamma^{3,2,2,2}Q(t-3))=\begin{cases}\Gamma^{t,t-1,t-1,t-1,t-3}H & \text{for } t\geq 3\\ 0 & \text{for } t<3\end{cases}$$

$$H^1(\Gamma^{3,2,1,1}Q(-3+t))=\begin{cases}\Gamma^{2,1}H & \text{for } t=1\\ 0 & \text{for } t\neq 1\end{cases}$$

$$H^1(\Gamma^{2,2,2,1}Q(-3+t))=\begin{cases}\wedge^3 H & \text{for } t=1\\ 0 & \text{for } t\neq 1\end{cases}$$

$H^1(\wedge^2 Q(t))=\ \forall t\in\mathbb{Z}$

$H^2(\Gamma^{3,1,1}Q(t))=H^2(\Gamma^{2,1,1,1}Q(t))=0\ \forall t\in\mathbb{Z}$

Proof From Bott theorem

Definition 1.6 A *nullcorrelation bundle* N is the cohomology bundle of a monad

$$\mathcal{O}(-1)\xrightarrow{a} H\otimes\mathcal{O}\xrightarrow{b}\mathcal{O}(1)$$

Every nullcorrelation bundle fits into an exact sequence

$$0\to\mathcal{O}(-1)\to Q^*\to N\to 0$$

The following lemma is well known (for a more general fact see lemma 4.2).

Lemma 1.7 Every nullcorrelation bundle N is symplectic

Proof Let $x_0,\ldots,x_5$ be homogeneous coordinates on $\mathbb{P}^5$. We can identify $a=\sum x_i a_i$, $b=\sum x_i b_i$ with a_i, b_i^t 1×6 matrices. Let A be the square matrix whose i-th row is a_i and B be the square matrix whose i-th column is b_i. The monad condition is equivalent to AB nondegenerate and skew-symmetric. We set $Q:=(B^{-1})^tA$. We get Q skew-symmetric and $A=B^tQ$, $B=-Q^{-1}A^t$, that is the dual monad is isomorphic to the monad itself.

The above proof shows that the moduli space of nullcorrelation bundles is isomorphic to the space of nondegenerate skew-symmetric 6×6 matrices (up to scalar multiple), that is to $\mathbb{P}^{14}\setminus V_3$ where V_3 is the cubic hypersurface given by the pfaffian. Given a nullcorrelation bundle N, we can write a suitable isomorphism $\mathbb{P}^5\simeq Sp(6)/P(\sigma_1)$ in such a way that $N=E^{\nu_2-\nu_1}$.

By Bott theorem $H^0(N(t))\simeq H_{(t-1)\nu_1+\nu_2}$ for $t\geq 1$, the intermediate $H^i(N(t))$ for $1\leq i\leq 4$ are zero with the only exceptions $H^1(N(-1))=H^4(N(-5))=\mathbb{C}$. Moreover $H^1(\mathrm{End}\ N(t))=0$ for every t with the only exceptions $H^1(\mathrm{End}\ N)=H^1(S^2N)=H_{\nu_2}$, $H^1(\mathrm{End}\ N(-1))=H^1(S^2N(-1))=H$. In particular every small deformation of a nullcorrelation is again a nullcorrelation. The minimal resolution of N is

$$0 \to \mathcal{O}(-5) \to H\otimes\mathcal{O}(-4) \to \wedge^2 H\otimes\mathcal{O}(-3) \to \wedge^3 H\otimes\mathcal{O}(-2) \to H_{\nu_2}\otimes\mathcal{O}(-1) \to N \to 0$$

Definition 1.8 A *lambda-three bundle* B is the bundle $\wedge^2 N/\mathcal{O}$ for some nullcorrelation bundle N.

We have $B=E^{\nu_3-\nu_1}$ (as N, also B depends on the choice of a nondegenerate skew-symmetric matrix)

B is stable and orthogonal. By Bott theorem $H^0(B(t))=H_{(t-1)\nu_1+\nu_3}$ for $t\geq 1$, the intermediate $H^i(B(t))$ for $1\leq i\leq 4$ are zero with the only exceptions $H^2(B(-2))=H^3(B(-4))=C$. Moreover $S^2N\simeq\wedge^2B$, and we have $H^1(\mathrm{End}\ B(t))=0$ for every t with the only exceptions $H^1(\mathrm{End}\ B)=H^1(\wedge^2B)=H_{\nu_2}$, $H^1(\mathrm{End}\ B(-1))=H^1(\wedge^2B(-1))=H$. From Beilinson theorem we get the resolution

$$0 \to \wedge^4 Q^* \xrightarrow{f} \wedge^2 Q^* \to B \to 0 \tag{1.1}$$

where f is defined by contraction with the same element of $\wedge^2 H$ defining N. Hence the moduli space of lambda-three bundles B is naturally isomorphic to the moduli space of nullcorrelation bundles. As $\wedge^4Q^*=Q(-1)$ we have the

Alternative definition of lambda-three bundle, 1.9 A *lambda-three bundle* B is the cohomology bundle of a monad

$$Q(-1) \to \wedge^2 H\otimes\mathcal{O} \to Q^*(1)$$

Every small deformation of a lambda-three bundle B is again a lambda-three bundle. The minimal resolution of B is [DMS]

$$0 \to \mathcal{O}(-4) \to H\otimes\mathcal{O}(-3) \to H_{\nu_2}\otimes\mathcal{O}(-2) \to H_{\nu_3}\otimes\mathcal{O}(-1) \to B \to 0$$

Lemma 1.10 $H^1(B\otimes\wedge^2Q^*(t))=0$ for $t\neq -1$, $H^2(B\otimes\wedge^2Q^*(-1))=H_{\nu_3}$
$H^1(B\otimes\wedge^2Q^*(t))=0$ for $t\neq 0$, $H^1(B\otimes\wedge^2Q^*)=H_{\nu_2}$

Proof Straightforward computation from the minimal resolution of B.

Remark 1.11 The 14×14 matrix of the composition $H_{\nu_3}\otimes\mathcal{O}(-1) \to B\simeq B^* \to H_{\nu_3}\otimes\mathcal{O}(1)$ has been computed in [DMS]. In suitable coordinates a,b,c,d,e,f it is:

$$
\begin{bmatrix}
 & q_1 & d^2 & bd & cd & b^2 & bc & c^2 & & & & & & \\
q_1 & & & & & & & & a^2 & ae & af & e^2 & ef & f^2 \\
d^2 & & & & & f^2 & -ef & e^2 & q_2 & de & df & & & \\
bd & & & -\frac{1}{2}f^2 & \frac{1}{2}ef & & \frac{1}{2}af & -ae & ab & -\frac{1}{2}cf & \frac{1}{2}bf & de & \frac{1}{2}df & \\
cd & & & \frac{1}{2}ef & -\frac{1}{2}e^2 & -af & \frac{1}{2}ae & & ac & \frac{1}{2}ce & -\frac{1}{2}be & & \frac{1}{2}de & df \\
b^2 & & f^2 & & -af & & & a^2 & & ab & & q_3 & bf & \\
bc & & -ef & \frac{1}{2}af & \frac{1}{2}ac & & -\frac{1}{2}a^2 & & & \frac{1}{2}ac & \frac{1}{2}ab & ce & -\frac{1}{2}ad & bf \\
c^2 & & e^2 & -ae & & a^2 & & & & & ac & & ce & q_4 \\
 & a^2 & q_2 & ab & ac & & & & & & & c^2 & -bc & b^2 \\
 & ae & de & -\frac{1}{2}cf & \frac{1}{2}ce & ab & \frac{1}{2}ac & & & -\frac{1}{2}c^2 & \frac{1}{2}bc & & \frac{1}{2}cd & -bd \\
 & af & df & \frac{1}{2}bf & -\frac{1}{2}be & & \frac{1}{2}ab & ac & & \frac{1}{2}bc & -\frac{1}{2}b^2 & -cd & \frac{1}{2}bd & \\
 & e^2 & & de & & q_3 & ce & & c^2 & & -cd & & & d^2 \\
 & ef & & \frac{1}{2}df & \frac{1}{2}de & bf & -\frac{1}{2}ad & ce & -bc & \frac{1}{2}cd & \frac{1}{2}bd & & -\frac{1}{2}a^2 & \\
 & f^2 & & & df & & bf & q_4 & b^2 & -bd & & d^2 & &
\end{bmatrix}
\tag{1.2}
$$

where $q_1 = ad+be+cf$. $q_2 = ad-be-cf$, $q_3 = -ad+be-cf$, $q_4 = -ad-be+cf$

2. The parent bundle

Definition 2.1 Let B be a lambda-three bundle. A *parent bundle* E is defined as the cohomology of a monad

$$\mathcal{O}(-1) \to B \to \mathcal{O}(1)$$

The display of the monad defines the two exact sequences

$$0 \to R \to B \to \mathcal{O}(1) \to 0$$

$$0 \to \mathcal{O}(-1) \to R \to E \to 0$$

The existence of a parent bundle is due to Horrocks and can be verified explicitly, as in [DMS], in the following way.

The space $H^0(B(1)) \subset H_{\nu_3} \otimes H^0(\mathcal{O}(2))$ identifies naturally to the space generated by the rows of (1.2). In particular the section σ (resp. τ) given by the sum (resp. difference) of 3^{rd} and 9^{th} rows does not vanish anywhere and $\tau \circ \sigma = 0$. Hence this pair of sections defines a monad whose cohomology is a parent bundle.

It is easy to check that $c_1(E) = c_3(E) = 0$, $c_2(E) = 3$ and that E is stable [DMS]. As in [Hor2] one can split $H = W \oplus W^*$ and the symmetry group of E is $SL(W) \times| \mathbb{Z}_2$, in particular E is SL(W)-homogeneous. With the coordinates of remark 1.11 we have that a,b,c are coordinates of the subspace W and d,e,f are coordinates of the subspace W^*. Restricting representations of $SL(W) \rtimes \mathbb{Z}_2$ to representations of SL(W) we obtain, in the Littlewood notation used by Horrocks in [Hor2]:

$$[m]\simeq[m]'\simeq\Gamma^{2m,m}W \text{ for } m\geq 0 \qquad [p,-q]\simeq\Gamma^{p+q,q}W\oplus\Gamma^{p+q,q}W^* \text{ for } p\neq q,\ p,q\geq 0$$

Lemma 2.2

$H^2(E(-2))=H^3(E(-4))=\mathbb{C}$

$H^1(E(-1))=H^4(E(-5))=\mathbb{C}$

$H^1(E)=H^4(E(-6))=W\oplus W^*$

$H^1(E(1))=H^4(E(-7))=\Gamma^{2,1}W=\Gamma^{2,1}W^*=W\otimes W^*/\mathbb{C}$

All other intermediate $H^i(E(t))$ for $1\leq i\leq 4$, $t\in\mathbb{Z}$ are zero.

$H^1(\text{End}E)=W\otimes W\oplus W\otimes W^*\oplus W^*\otimes W^*$, $H^1(\text{End } E(-1))=W\oplus W^*$, $H^1(\text{End } E(t))=0$ for $t\leq -2$

Proof The statements about the cohomology of E follow from [Hor2] or [DMS]. The statements about the cohomology of End E follow from the cohomology of End B computed in section 1 and from the following four exact sequences of SL(W)-homogeneous bundles

$$0\to R\otimes B(t)\to \text{End } B(t)\to B(t+1)\to 0 \tag{2.1}$$

$$0\to R(t-1)\to B\otimes R(t)\to \text{End } R(t)\to 0 \tag{2.2}$$

$$0\to E^*\otimes R(t)\to \text{End } R(t)\to R(t+1)\to 0 \tag{2.3}$$

$$0\to E^*(t-1)\to E^*\otimes R(t)\to \text{End } E(t)\to 0 \tag{2.4}$$

with $t\leq 0$, considering the following restrictions to SL(W) of Sp(6)-representations.

$H=W\oplus W^*$, $H_{\nu_2}=\Gamma^{2,1}W\oplus W\oplus W^*$, $H_{\nu_3}=\mathbb{C}\oplus\mathbb{C}\oplus S^2W\oplus S^2W^*$

and the fact that $h^1(\text{End } E)=27$ [DMS]. Moreover we remark that given a parent bundle E_0 (that determines B_0) $H^1(\text{End } E_0(-1))$ is the tangent space at E_0 of the moduli space of bundles E such that $E|_{\mathbb{P}^4}\simeq E_0|_{\mathbb{P}^4}$ on a fixed hyperplane $\mathbb{P}^4$. From the exact sequence

$$0\to B\to B(1)\to B(1)|_{\mathbb{P}^4}\to 0$$

one gets $H^0(B(1))=H^0(B(1)|_{\mathbb{P}^4})$. Hence every B such that $B|_{\mathbb{P}^4}=B_0|_{\mathbb{P}^4}$ determines uniquely E such that $E|_{\mathbb{P}^4}\simeq E_0|_{\mathbb{P}^4}$. This explains that the natural morphism $H^1(\text{End } E(-1))\to H^1(\text{End } B(-1))=W\oplus W^*$ is an isomorphism.

Remark 2.3 In [DMS] the following formula is proved

$$\bigoplus_t H^1(E(t))=\frac{\mathbb{C}[a,b,c,d,e,f]}{(ad+be+cf,(a,b,c)^2,(d,e,f)^2)}(1).$$

The interpretation of this module structure in terms of SL(W)-representations is the following.

We have $W=(a,b,c)$, $W^*=(d,e,f)$, $S=\mathbb{C}[a,b,c,d,e,f]=\bigoplus_i S^i(W\oplus W^*)$, $H^1(E(-1))=\mathbb{C}=S_0$, $H^1(E)=W\oplus W^*=S_1$. $H^1(E(1))=\Gamma^{2,1}W$ is a quotient of $S^2(W\oplus W^*)$. In the decomposition $S^2(W\oplus W^*)=S^2W\oplus S^2W^*\oplus\mathbb{C}\oplus\Gamma^{2,1}W$, the killing of S^2W, S^2W^*, $\mathbb{C}$ is given respectively by $(a,b,c)^2$, $(d,e,f)^2$, $ad+be+cf$, where the last quadratic polynomial corresponds to the identity endomorphism of

W.

It is proved in [DMS] that a small deformation of a parent bundle is again a parent bundle and the 27-dimensional moduli space M_0 of the parent bundles has a natural fibration $M_0 \to \mathbb{P}^{14}\setminus V_3$ (corresponding to $E \mapsto B$) whose fibers are isomorphic to $\mathbb{P}^{13}\setminus V_4$ where V_4 is the hypersurface given by the tangent variety to the isotropic grassmannian $\mathrm{Grn}(\mathbb{P}^2,\mathbb{P}^5)\subset\mathbb{P}^{13}$. $\mathrm{Aut}(\mathbb{P}^5)$ acts transitively on M_0. We remark also that no parent bundle is self-dual, in particular $h^0(E\otimes E)=0$.

3. Weighted quotient bundles

Now we modify the constructions of the bundles defined in the previous section.

Let $\alpha\leq\beta$ be two nonnegative integers, we define

$$\mathcal{W}:=\mathcal{O}(\alpha)\oplus\mathcal{O}(\beta)\oplus\mathcal{O}(-\alpha-\beta),$$
$$\mathcal{H}:=\mathcal{W}\oplus\mathcal{W}^*. \tag{3.1}$$

Definition 3.1 Let $\gamma>\alpha+\beta$. A *weighted quotient bundle* $Q_{\alpha,\beta,\gamma}$ is a bundle defined by an exact sequence:

$$0\to\mathcal{O}(-\gamma)\to\mathcal{H}\to Q_{\alpha,\beta,\gamma}\to 0$$

We often drop the indexes α,β,γ and we use $\widetilde{Q}$ for $Q_{\alpha,\beta,\gamma}$.

Lemma 3.2 (Bohnhorst-Spindler) A weighted quotient bundle $Q_{\alpha,\beta,\gamma}$ is stable if and only if $\gamma>5\alpha+5\beta$

Proof [BoS]

Proposition 3.3 Let $Q^0_{\alpha,\beta,\gamma}$ be a weighted quotient bundle. Every small deformation of $Q^0_{\alpha,\beta,\gamma}$ is again a weighted quotient bundle $Q_{\alpha,\beta,\gamma}$. Moreover the Kuranishi space of $Q^0_{\alpha,\beta,\gamma}$ is *smooth* at the point corresponding to $Q^0_{\alpha,\beta,\gamma}$.

Proof Let $\widetilde{Q}$, $\widetilde{Q}'$ be two weighted quotient bundles. Every morphism from $\widetilde{Q}$ to $\widetilde{Q}'$ lifts to a morphism of sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(-\gamma) & \to & \mathcal{H} & \to & \widetilde{Q} & \to & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}(-\gamma) & \to & \mathcal{H} & \to & \widetilde{Q}' & \to & 0\end{array}$$

(by the vanishing of $H^1(\mathcal{H}(-\gamma))$). Moreover two elements f, $f'\in\mathrm{Hom}(\mathcal{O}(-\gamma),\mathcal{H})$ give the same element of $\mathrm{Quot}_{\mathcal{H}/\mathbb{P}^5}$ if and only if there exists $g\in\mathrm{Aut}(\mathcal{O}(-\gamma))$ such that $f=f'\circ g$. Let $\widetilde{Q}_0:=Q^0_{\alpha,\beta,\gamma}$ be the cokernel of $f_0\in\mathrm{Hom}(\mathcal{O}(-\gamma),\mathcal{H})$. Let Y be the Kuranishi space of $\widetilde{Q}_0$ and $y_0\in Y$ be the point corresponding to $\widetilde{Q}_0$. Let $x_0\in\mathrm{Quot}_{\mathcal{H}/\mathbb{P}^5}$ be the point corresponding to $\widetilde{Q}_0$ and let X be the irreducible component of $\mathrm{Quot}_{\mathcal{H}/\mathbb{P}^5}$ containing x_0. We have a natural morphisms of germs $\pi:(X,x_0)\to(Y,y_0)$, then $\dim_{y_0} Y \geq \dim_{x_0} X - \dim_{x_0} \pi^{-1}(y_0)$. If $Z=\{x\in X: \widetilde{Q}_x \simeq \widetilde{Q}_0\}$ we get $(\pi^{-1}(y_0),x_0)\subset(Z,x_0)$, hence $\dim_{y_0} Y \geq \dim_{x_0} X - \dim_{x_0} Z$. We have $\dim_{x_0} X = h^0(\mathcal{H}(\gamma))-1 = h^0(\widetilde{Q}_0(\gamma))$. We obtain the formula

$$\dim_{x_0} Z = h^0(\text{End } \mathcal{H}) - \{\text{dimension of endomorphisms of } \mathcal{H} \text{ which fix } f_0\} - 1.$$

The sequence

$$0 \to \widetilde{Q}^* \otimes \mathcal{H} \to \text{End } \mathcal{H} \to \mathcal{H}(\gamma) \to 0 \tag{3.2}$$

shows that the number in braces in the last formula is equal to $h^0(\widetilde{Q}^* \otimes \mathcal{H})$.

It follows $\dim_{y_0} Y \geq h^0(\mathcal{H}(\gamma)) - h^0(\text{End } \mathcal{H}) + h^0(\widetilde{Q}^* \otimes \mathcal{H}) = h^1(\widetilde{Q}^* \otimes \mathcal{H})$ where the last equality follows again from the sequence (3.2). Now the exact sequence

$$0 \to \widetilde{Q}_0^*(-\gamma) \to \widetilde{Q}_0^* \otimes \mathcal{H} \to \text{End } \widetilde{Q}_0 \to 0$$

shows $h^1(\widetilde{Q}^* \otimes \mathcal{H}) = h^1(\text{End } \widetilde{Q}_0)$, hence $\dim_{y_0} Y \geq h^1(\text{End } \widetilde{Q}_0)$ and the equality holds because the right-hand side is the dimension of the Zariski tangent space to Y at y_0. In particular $\dim_{y_0} Y = \dim_{x_0} X - \dim_{x_0} \pi^{-1}(y_0)$ and π is surjective between germs, q.e.d.

Now consider a bundle $\widetilde{Q}^*$ given as kernel of a surjective morphism $\mathcal{H} \xrightarrow{f} \mathcal{O}(\gamma)$. f is given by six homogeneous polynomials f_1, f_2, f_3, f_4, f_5, f_6 (of degree $\gamma-\alpha$, $\gamma-\beta$, $\gamma+\alpha+\beta$, $\gamma+\alpha$, $\gamma+\beta$, $\gamma-\alpha-\beta$) which define a map $\omega: \mathbb{C}^6\backslash 0 \to \mathbb{C}^6\backslash 0$. Look at the diagram (0.1). On the domain of ω consider the multiplicative action of $\mathbb{C}^*$ and on the codomain of ω the action $\tau_{\alpha,\beta,\gamma}: \mathbb{C}^* \times \mathbb{C}^6\backslash 0 \to \mathbb{C}^6\backslash 0$ given by

$$\tau_{\alpha,\beta,\gamma}(t,v_1,\ldots,v_6) = (t^{\gamma-\alpha}v_1, t^{\gamma-\beta}v_2, t^{\gamma+\alpha+\beta}v_3, t^{\gamma+\alpha}v_4, t^{\gamma+\beta}v_5, t^{\gamma-\alpha-\beta}v_6) \tag{3.3}$$

so that ω is $\mathbb{C}^*$-equivariant. The quotient bundle Q is SL(H)-invariant, in particular η^*Q is $\mathbb{C}^*$-invariant under the action of $\tau_{\alpha,\beta,\gamma}$. It follows that $\omega^*\eta^*Q$ is $\mathbb{C}^*$-invariant under the multiplicative action and then *it descends to a bundle* $\widetilde{Q}$ on $\mathbb{P}^5$, that is

$$\omega^*\eta^*Q \simeq \eta^*\widetilde{Q}.$$

We say that $\widetilde{Q}$ is obtained *pulling back* Q *over* $\mathbb{C}^6\backslash 0$ (we refer to [Hor2] for more details). It is easy to check that any weighted quotient bundle is obtained pulling back suitably over $\mathbb{C}^6\backslash 0$ the quotient bundle Q. We get also $\omega^*\eta^*(H\otimes\mathcal{O}) = \eta^*\mathcal{H}$ and if T is any representation of SL(H) then (with obvious notations) $\omega^*\eta^*(T(H)\otimes\mathcal{O}) = \eta^*T(\mathcal{H})$. The functor η^* gives an equivalence of categories between bundles over $\mathbb{P}^5$ and bundles over $\mathbb{C}^6\backslash 0$ endowed with the multiplicative $\mathbb{C}^*$-action. Hence the minimal resolution of $\widetilde{Q}^*$ can be obtained pulling back over $\mathbb{C}^6\backslash 0$ the minimal resolution of Q and indeed it is

$$0 \to \mathcal{O}(-5\gamma) \to \wedge^5\mathcal{H}(-4\gamma) \to \wedge^4\mathcal{H}(-3\gamma) \to \wedge^3\mathcal{H}(-2\gamma) \to \wedge^2\mathcal{H}(-\gamma) \to \widetilde{Q}^* \to 0 \tag{3.4}$$

When the six homogeneous have the same degree (this happens if and only if $\alpha=\beta=0$) then ω

induces a finite morphism $\omega':\mathbb{P}^5 \to \mathbb{P}^5$ and $\widetilde{Q}$ is the pullback of Q under ω'. When ω is a Galois covering (e.g. ω given by six monomials which are powers of the six indeterminates) then $\widetilde{Q}$ is invariant under the action of the finite group $\mathbb{Z}_{\gamma-\alpha}\times\mathbb{Z}_{\gamma-\beta}\times\mathbb{Z}_{\gamma+\alpha+\beta}\times\mathbb{Z}_{\gamma+\alpha}\times\mathbb{Z}_{\gamma+\beta}\times\mathbb{Z}_{\gamma-\alpha-\beta}$ and then it descends to the corresponding weighted projective space $\mathbb{P}(\gamma-\alpha,\ \gamma-\beta,\ \gamma+\alpha+\beta,\ \gamma+\alpha,\ \gamma+\beta,\ \gamma-\alpha-\beta)$. In this case $\widetilde{Q}$ is obtained as a pullback $\bar{\eta}^*Q'$ where $\bar{\eta}:\mathbb{P}^5 \to \mathbb{P}(\gamma-\alpha,\ \gamma-\beta,\ \gamma+\alpha+\beta,\ \gamma+\alpha,\ \gamma+\beta,\ \gamma-\alpha-\beta)$.

We recall that the cohomology group $H^i(\mathbb{C}^6\backslash 0,\ \eta^*Q)$ is isomorphic to its local cohomology $\oplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^5,Q(t))$ [Hor1]. In particular

$$\oplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^5,\widetilde{Q}(t)) \simeq \oplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^5,Q(t))\otimes_{\mathbb{C}[f_1,f_2,f_3,f_4,f_5,f_6]}\mathbb{C}[a,b,c,d,e,f] \simeq$$

$$\simeq \oplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^5,Q(t))\otimes_{\mathbb{C}} \frac{\mathbb{C}[a,b,c,d,e,f]}{(f_1,f_2,f_3,f_4,f_5,f_6)} \tag{3.5}$$

In these isomorphisms the graded summands correspond if we consider $\oplus_{t\in\mathbb{Z}} H^i(\mathbb{P}^5,Q(t))$ as $\mathbb{C}[f_1,f_2,f_3,f_4,f_5,f_6]$-module and then perform the graded tensor product over $\mathbb{C}$. We recall that if S is a graded ring, the degree t summand of the graded tensor product between two S-modules $\oplus A_k$ and $\oplus B_k$ is $\oplus_{j=0}^{t}(A_j\otimes B_{t-j})$.

The formula (3.5) holds with End Q in the place of Q. In general if $\widetilde{G}$ is a bundle obtained pulling back over $\mathbb{C}^6\backslash 0$ a bundle G then the formula (3.5) holds with G in the place of Q. The 2nd proof of prop. 3.7, where we consider cohomology groups as SL(H)-representations, should clarify this formula. First we need

Lemma 3.4 Let $(f_1,f_2,f_3,f_4,f_5,f_6)$ be homogeneous polynomials defining a surjective morphism $\mathcal{H} \to \mathcal{O}(\gamma)$ (this happens if and only if they have no common zeroes). Then the dimension of the degree t summand of the artinian algebra $\frac{\mathbb{C}[a,b,c,d,e,f]}{(f_1,f_2,f_3,f_4,f_5,f_6)}$ is equal to

$$\sum_{j=0}^{6}(-1)^j h^0[\wedge^j\mathcal{H}\otimes\mathcal{O}(t-j\gamma)].$$

In particular it is nonzero only if and only if $0\le t\le 6\gamma-6$.

Proof Immediate from the twisted Koszul complex of the map $\mathcal{H} \to \mathcal{O}(\gamma)$

$$0 \to \mathcal{O}(t-6\gamma) \to \mathcal{H}(t-5\gamma) \to \wedge^2\mathcal{H}(t-4\gamma) \to \wedge^3\mathcal{H}(t-3\gamma) \to \wedge^2\mathcal{H}(t-2\gamma) \to$$
$$\to \mathcal{H}(t-\gamma) \to \mathcal{O}(t) \to 0$$

and the fact that the needed degree t summand is isomorphic to the cokernel of the map $H^0[\mathcal{H}(t-\gamma)] \to H^0[\mathcal{O}(t)]$.

Remark 3.5 The dimension computed in the previous lemma is equal to $h^1(\widetilde{Q}^*(t-\gamma))$, in fact

$\bigoplus_t H^1(\widetilde{Q}^*(t)) = \dfrac{\mathbb{C}[a,b,c,d,e,f]}{(f_1,f_2,f_3,f_4,f_5,f_6)}(\gamma)$. Note also $h^0(\widetilde{Q}^*(t)) = \sum_{j=2}^{6}(-1)^j h^0[\wedge^j \mathcal{H}\otimes \mathcal{O}(t+(1-j)\gamma)]$

Example 3.6 In the case $\alpha=0$, $\beta=1$, $\gamma=2$ the dimension of degree t summands of $\dfrac{\mathbb{C}[a,b,c,d,e,f]}{(a^2,b,c^3,d^2,e^3,f)}$ for $t=0,\dots,6$ are 1,4,8,10,8,4,1.

Proposition 3.7

$$h^1(\text{End } Q_{\alpha,\beta,\gamma}) = h^0(\mathcal{H}(\gamma)) - h^0(\mathcal{H}\otimes\mathcal{H}) + h^0(\wedge^2\mathcal{H}\otimes\mathcal{H}(-\gamma)) - h^0(\wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma)) \qquad (3.6)$$

(By prop. 3.3 this number is the dimension of the Kuranishi space of $Q_{\alpha,\beta,\gamma}$; observe that this formula depends only on α,β,γ)

1st *proof* We use $\widetilde{Q}$ for $Q_{\alpha,\beta,\gamma}$. From the proof of 3.3 it follows $h^1(\text{End } \widetilde{Q}) = h^1(\widetilde{Q}^*\otimes\mathcal{H})$. Tensoring (3.4) by $\mathcal{H}$ we obtain the exact sequence

$$0 \to \mathcal{H}(-5\gamma) \to \wedge^5\mathcal{H}\otimes\mathcal{H}(-4\gamma) \to \wedge^4\mathcal{H}\otimes\mathcal{H}(-3\gamma) \to$$
$$\to \wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma) \to \wedge^2\mathcal{H}\otimes\mathcal{H}(-\gamma) \to \widetilde{Q}^*\otimes\mathcal{H} \to 0.$$

From this we get $h^0(\widetilde{Q}^*\otimes\mathcal{H}) = h^0(\wedge^2\mathcal{H}\otimes\mathcal{H}(-\gamma)) - h^0(\wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma))$ because $h^0(\wedge^2\mathcal{H}\otimes\mathcal{H}(-3\gamma))=0$ due to $\gamma>\alpha+\beta$. From (3.2) it follows $h^0(\widetilde{Q}^*\otimes\mathcal{H}) - h^1(\widetilde{Q}^*\otimes\mathcal{H}) = h^0(\mathcal{H}\otimes\mathcal{H}) - h^0(\mathcal{H}(\gamma))$, hence the result.

2nd *proof* We computed in section 1 $\bigoplus_{t\in\mathbb{Z}} H^1(\text{End } Q(t)) = H$ at degree -1. As $\mathcal{O}(-1)=\det Q^*$ and $\det\widetilde{Q}^* = \mathcal{O}(-\gamma)$ it follows that $\bigoplus_{t\in\mathbb{Z}} H^1(\text{End } Q(t))$ as $\mathbb{C}[f_1,f_2,f_3,f_4,f_5,f_6]$-module has graded summands of dimension 1 exactly in the degrees $-\gamma-\alpha$, $-\gamma-\beta$, $-\gamma+\alpha+\beta$, $-\gamma+\alpha$, $-\gamma+\beta$, $-\gamma-\alpha-\beta$ (with possible overlappings!). In fact the degrees of the generators a,b,...,f become $\gamma-\alpha$, $\gamma-\beta$, $\gamma+\alpha+\beta$, $\gamma+\alpha$, $\gamma+\beta$, $\gamma-\alpha-\beta$. Then in the graded tensor product with $\dfrac{\mathbb{C}[a,b,c,d,e,f]}{(f_1,f_2,f_3,f_4,f_5,f_6)}$ (see (3.5)) the contribution to $H^1(\text{End } \widetilde{Q})$ is equal to the direct sum of the summands of $\dfrac{\mathbb{C}[a,b,c,d,e,f]}{(f_1,f_2,f_3,f_4,f_5,f_6)}$ of degree resp. $+\gamma+\alpha$, $+\gamma+\beta$, $+\gamma-\alpha-\beta$, $+\gamma-\alpha$, $+\gamma-\beta$, $+\gamma+\alpha+\beta$ that is by lemma 3.4 to $\sum_{j=0}^{6}(-1)^j h^0[\wedge^j[\mathcal{H}(-\gamma)]\otimes\mathcal{H}(\gamma)]$ which is the thesis because the summands with $j\geq 4$ are zero.

Corollary 3.8 (Bohnhorst-Spindler) If $\gamma>5\alpha+5\beta$ then the bundles $\widetilde{Q}$ fill up a *smooth open irreducible* subset of dimension $h^0(\mathcal{H}(\gamma)) - h^0(\mathcal{H}\otimes\mathcal{H})$ of the moduli space of stable bundles with the same rank and Chern classes.

Corollary 3.9 $h^0(\text{End } Q_{\alpha,\beta,\gamma}) = h^0(\wedge^2\mathcal{H}\otimes\mathcal{H}(-\gamma)) - h^0(\wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma)) + 1$. In particular $Q_{\alpha,\beta,\gamma}$ is *simple* if and only if $\gamma>2\alpha+3\beta$.

Proof From the exact sequence

$$0 \to \text{End } \widetilde{Q} \to \widetilde{Q}\otimes\mathcal{H} \to \widetilde{Q}(\gamma) \to 0$$

it follows $h^0(\text{End } \widetilde{Q}) - h^1(\text{End } \widetilde{Q}) = h^0(\widetilde{Q}\otimes\mathcal{H}) - h^0(\widetilde{Q}(\gamma)) = h^0(\mathcal{H}\otimes\mathcal{H}) - h^0(\mathcal{H}(\gamma)) + 1$. Then use

(3.6). For the last assertion observe that $h^0(\wedge^2\mathcal{H}\otimes\mathcal{H}(-\gamma))=0$ if and only if $\gamma>2\alpha+3\beta$ and when it is nonzero it is always bigger than $h^0(\wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma))$.

Remark 3.10 The assertion about the simplicity of $\widetilde{Q}$ can be verified also from the formula $H^0(\mathrm{End}\ Q(t)) = S^tH\oplus H_{(t-1)\mu_1+\mu_2+\mu_5}$ for $t\geq 1$. Indeed S^tH gives no contribution to $h^0(\mathrm{End}\ \widetilde{Q})$ in the graded tensor product (3.5) with End Q in the place of Q, while $H_{(t-1)\mu_1+\mu_2+\mu_5}$ contributes to $h^0(\mathrm{End}\ \widetilde{Q})$ only if $t\gamma\leq(t+1)\alpha+(t+2)\beta$. Indeed the maximum degree in the corresponding Young diagram $t+1,2,1,1,1$ is given by ($t=4$ in the picture)

$\alpha+\beta$	$\alpha+\beta$	$\alpha+\beta$	$\alpha+\beta$	$\alpha+\beta$
β	β			
α				
$-\alpha$				
$-\beta$				

and the sum is $(t+1)\alpha+(t+2)\beta$ (we used (0.2)). Note that $\gamma>2\alpha+3\beta$ implies $t\gamma>(t+1)\alpha+(t+2)\beta$ $\forall t\geq 1$.

Remark 3.11 In general one can define $\mathcal{W}=\mathcal{O}(\alpha)\oplus\mathcal{O}(\beta)\oplus\mathcal{O}(-\delta)$ with $0\leq\alpha\leq\beta\leq\delta$ and $\widetilde{Q}$ from

$$0\to\widetilde{Q}\to\mathcal{W}\oplus\mathcal{W}^*\to\mathcal{O}(\gamma)\to 0$$

By [BoS] $\widetilde{Q}$ is stable if and only if $\gamma>5\delta$. Prop. 3.3, prop. 3.7 and cor. 3.9 hold also in this case, while $\widetilde{Q}$ turns out to be simple if and only if $\gamma>2\delta+\beta$. We can study in this way even more general bundles with $\mathcal{H}=\bigoplus_{i=1}^{6}\mathcal{O}(a_i)$, but this brings us far from the subject of this paper.

4. Weighted nullcorrelation bundles

Definition 4.1 Let $\gamma>\alpha+\beta$. A *weighted nullcorrelation bundle* $N_{\alpha,\beta,\gamma}$ is defined as the cohomology of a monad

$$\mathcal{O}(-\gamma)\to\mathcal{H}\to\mathcal{O}(\gamma)$$

($\mathcal{H}$ is defined in (3.1)).

$N_{\alpha,\beta,\gamma}$ fits into an exact sequence

$$0\to\mathcal{O}(-\gamma)\to Q^*_{\alpha,\beta,\gamma}\to N_{\alpha,\beta,\gamma}\to 0$$

where $Q_{\alpha,\beta,\gamma}$ is a weighted quotient bundle.

We often drop the indexes α,β,γ and we use $\widetilde{N}$ for $N_{\alpha,\beta,\gamma}$.

Weighted nullcorrelation bundles were studied by Ein [Ein] on $\mathbb{P}^3$ by different techniques. Our approach can easily be extended to $\mathbb{P}^{2n+1}$.

Lemma 4.2 Every weighted nullcorrelation bundle is *symplectic*

Proof Let $\widetilde{N}$ be the cohomology of the monad $\mathcal{O}(-\gamma) \xrightarrow{b} \mathcal{H} \xrightarrow{a} \mathcal{O}(\gamma)$. We have to prove that there exists a symplectic automorphism $t:\mathcal{H} \to \mathcal{H}$ such that $b=t\circ a^t$ (that is the monad is self-dual). We have the exact sequence

$$0 \to \widetilde{Q}^*(\gamma) \to \mathcal{H}(\gamma) \xrightarrow{p} \mathcal{O}(2\gamma) \to 0$$

where $H^0(p):\mathrm{Hom}(\mathcal{O}(-\gamma),\mathcal{H}) \to \mathrm{Hom}(\mathcal{O}(-\gamma),\mathcal{O}(\gamma))$ is the composition with a, that is $H^0(p)(s)=a\circ s$. Then the space of $b':\mathcal{O}(-\gamma) \to \mathcal{H}$ such that $a\circ b=0$ identifies with $H^0(\widetilde{Q}^*(\gamma))$. Moreover we have the resolution

$$0 \to \mathcal{O}(-4\gamma) \to \mathcal{H}(-3\gamma) \to \wedge^2\mathcal{H}(-2\gamma) \to \wedge^3\mathcal{H}(-\gamma) \to \wedge^2\mathcal{H} \xrightarrow{q} \widetilde{Q}^*(\gamma) \to 0$$

where $H^0(q)(t)=t\circ a^t$. From the resolution it follows that $H^0(q)$ is surjective. It remains to check that $H^0(q)^{-1}\{$injective bundle morphisms in $\mathrm{Hom}(\mathcal{O}(-g),\widetilde{Q}^*)\}$ is equal to the set of symplectic automorphisms. If t is invertible then $t\circ a^t$ is injective. Conversely let us suppose that $\forall x\in\mathbb{P}^5\ t(x)\circ a^t(x)\neq 0$. If t is not invertible then there exists a vector c with entries homogeneous polynomials such that $t(x)c^t(x)=0\ \forall x\in\mathbb{P}^5$ (det t is a constant). There exists a point x_0 where $a(x_0)$ and $c(x_0)$ are proportional, hence we get the desired contradiction.

Corollary 4.3 The minimal resolution of a weighted nullcorrelation bundle $\widetilde{N}$ is

$$0 \to \mathcal{O}(-4\gamma) \to \mathcal{H}(-3\gamma) \to \wedge^2\mathcal{H}(-2\gamma) \to \wedge^3\mathcal{H}(-\gamma) \to \mathcal{H}_{\nu_2} \to \widetilde{N} \to 0$$

Proof If $t:\mathcal{O} \to \wedge^2\mathcal{H}$ is any symplectic automorphism of $\mathcal{H}$ that corresponds to $\widetilde{N}$ as in the above proof, then there exists $s:\wedge^2\mathcal{H} \to \mathcal{O}$ such that $s\circ t=\mathrm{id}$. It follows Coker $t\simeq\mathcal{H}_{\nu_2}$ and the statement follows from (3.4).

Our main result about $\widetilde{N}$ is the following

Theorem 4.4 Let $N^0_{\alpha,\beta,\gamma}$ be a weighted nullcorrelation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. Every small deformation of $N^0_{\alpha,\beta,\gamma}$ is again a nullcorrelation bundle $N_{\alpha,\beta,\gamma}$. Moreover the Kuranishi space of deformations of $N^0_{\alpha,\beta,\gamma}$ is *smooth* at $N^0_{\alpha,\beta,\gamma}$.

Proof First we remark that as in the proof of prop. 3.3 every morphism between two weighted nullcorrelation bundles $\widetilde{N}$, $\widetilde{N}'$ is induced by a morphism of sequences

$$\begin{array}{ccccccc} 0 \to & \mathcal{O}(-\gamma) & \to & \widetilde{Q}^* & \to & \widetilde{N} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathcal{O}(-\gamma) & \to & \widetilde{Q}'^* & \to & \widetilde{N}' & \to 0 \end{array}$$

because of the vanishing of $H^1(Q(-\gamma))$. We use $\widetilde{N}_0$ for $N^0_{\alpha,\beta,\gamma}$ and we denote $\widetilde{Q}_0$ the weighted quotient corresponding to $\widetilde{N}_0$. Let Y be the Kuranishi space of $\widetilde{Q}_0$ and let T be the Kuranishi space of $\widetilde{N}_0$ with $t_0\in T$ being the point corresponding to $\widetilde{N}_0$. Let $\mathcal{F}$ be the universal family over $Y\times\mathbb{P}^5$. We set $Z:=\mathrm{Quot}_{\mathcal{F}/Y\times\mathbb{P}^5/Y}$ and we denote by $\phi:Z \to Y$ the natural projection. Let $z_0\in Z$ be the point corresponding to $\widetilde{N}_0$. We have a morphism of germs $\pi:(Z,z_0) \to (T,t_0)$, hence

$\dim_{t_0}T \geq \dim_{z_0}Z - \dim_{z_0}\pi^{-1}(t_0)$. Every $\widetilde{N}$ arises from a unique $\widetilde{Q}$, in fact because of $H^1(\widetilde{N}(-\gamma)) = \mathbb{C}$ the corresponding $\widetilde{Q}$ can be constructed as the unique nonsplitting extension $0 \to \mathcal{O}(-\gamma) \to ? \to \widetilde{N} \to 0$. It follows that we have $(\pi^{-1}(t_0), z_0) \subset (\phi^{-1}(y_0), z_0) = (\mathrm{Quot}_{\widetilde{Q}_0^*/\mathbb{P}^5}, z_0)$. Let $f_0 \in \mathrm{Hom}(\mathcal{O}(-\gamma), \widetilde{Q}_0)$ be the morphism defining $\widetilde{N}_0$. We set $P = \{x \in \mathrm{Quot}_{\widetilde{Q}_0^*/\mathbb{P}^5} : \widetilde{N}_x \simeq \widetilde{N}_0\}$, hence $(\pi^{-1}(t_0), z_0) \subset (P, z_0)$ and we get $\dim_{t_0}T \geq \dim_{z_0}Z - \dim_{z_0}P$. We have

$\dim_{z_0}P = h^0(\mathrm{End}\ \widetilde{Q}_0) - \{$dimension of endomorphisms of $\widetilde{Q}_0$ that fix $f_0\} - 1$. Considering the exact sequence

$$0 \to \widetilde{Q}^* \otimes \widetilde{N} \to \mathrm{End}\ \widetilde{Q} \to \widetilde{Q}(\gamma) \to 0$$

we get that the term in braces in the last formula is equal to $h^0(\widetilde{Q}^* \otimes \widetilde{N})$. The fibers of ϕ are isomorphic to a component of $\mathrm{Quot}_{\widetilde{Q}_0^*/\mathbb{P}^5}$ containing a weighted nullcorrelation bundle and they have all the same dimension equal to

$$h^0(\widetilde{Q}_0^*(\gamma)) - 1.$$

Hence $\dim_{z_0}Z = \dim_{y_0}Y + h^0(\widetilde{Q}_0^*(\gamma)) - 1$ =(by prop. 3.3) $h^1(\mathrm{End}\ \widetilde{Q}_0) + h^0(\widetilde{Q}_0^*(\gamma)) - 1$. It follows $\dim_{t_0}T \geq h^0(\widetilde{Q}^*(\gamma)) - h^0(\mathrm{End}\ \widetilde{Q}) + h^0(\widetilde{Q}^* \otimes \widetilde{N}) + h^1(\mathrm{End}\ \widetilde{Q})$. Now consider the exact sequence

$$0 \to \widetilde{N} \otimes \widetilde{Q}^* \to \mathrm{End}\ \widetilde{Q} \to \widetilde{Q}^*(\gamma) \to 0 \tag{4.1}$$

From the cohomology sequence associated to (4.1) it follows that $\dim_{t_0}T \geq h^1(N \otimes \widetilde{Q}^*)$ because the morphism $H^1[\mathrm{End}\ \widetilde{Q}] \to H^1[\widetilde{Q}^*(\gamma)]$ is zero. In order to prove this last claim consider that the natural morphism $\oplus H^1[\mathrm{End}\ Q(t)] \to \oplus H^1[Q^*(t)]$ (that shifts one degree) is obviously zero and the morphism we are considering is obtained by tensoring this one. Now we consider the cohomology sequence associated to the sequence

$$0 \to \widetilde{N}(-\gamma) \to \widetilde{Q}^* \otimes \widetilde{N} \to \mathrm{End}\ \widetilde{N} \to 0$$

The map $H^0(\mathrm{End}\ \widetilde{N}) \to H^1(\widetilde{N}(-\gamma)) = \mathbb{C}$ is surjective and $h^2(\widetilde{N}(-\gamma)) = 0$, hence $h^1(\widetilde{Q}^* \otimes \widetilde{N}) = h^1(\mathrm{End}\ \widetilde{N}_0)$. Then $\dim_{t_0}T = h^1(\mathrm{End}\ \widetilde{N}_0)$ and π is surjective, concluding the proof.

Consider the nullcorrelation bundle N on $\mathbb{P}^5$ which is invariant by the action of the symplectic group $Sp(6) = \{A \in SL(6) | AJA^t = J\}$ where

$$J = \begin{bmatrix} & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ -1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \end{bmatrix}.$$

The map

$$t \mapsto \begin{bmatrix} t^{-\alpha} & & & & & \\ & t^{-\beta} & & & & \\ & & t^{\alpha+\beta} & & & \\ & & & t^{\alpha} & & \\ & & & & t^{\beta} & \\ & & & & & t^{-\alpha-\beta} \end{bmatrix}$$

gives an embedding of $\mathbb{C}^*$ in $Sp(6)$, hence η^*N is $\mathbb{C}^*$-invariant under the action of $\tau_{\alpha,\beta,\gamma}$ (see (3.3)). It follows also in this case, with the notations of (0.1), that there exists a weighted nullcorrelation $N_{\alpha,\beta,\gamma}$ such that

$$\omega^*\eta^*N=\eta^*N_{\alpha,\beta,\gamma}.$$

Differently from the case of weighted quotient bundles, a general weighted nullcorrelation bundle $N_{\alpha,\beta,\gamma}$ *does not come pulling back over* $\mathbb{C}^6\backslash 0$ *from a diagram as in* (0.1). Indeed consider the piece of the Koszul complex

$$\wedge^2\mathcal{H}(-\gamma)\xrightarrow{g}\mathcal{H}\to\mathcal{O}(\gamma)\to 0$$

where in a convenient basis the morphism g is represented by the 15×6 matrix

$$\begin{bmatrix}
-f_2 & f_1 & & & & \\
-f_3 & & f_1 & & & \\
-f_4 & & & f_1 & & \\
-f_5 & & & & f_1 & \\
-f_6 & & & & & f_1 \\
 & -f_3 & f_2 & & & \\
 & -f_4 & & f_2 & & \\
 & -f_5 & & & f_2 & \\
 & -f_6 & & & & f_2 \\
 & & -f_4 & f_3 & & \\
 & & -f_5 & & f_3 & \\
 & & -f_6 & & & f_3 \\
 & & & -f_5 & f_4 & \\
 & & & -f_6 & & f_4 \\
 & & & & -f_6 & f_5
\end{bmatrix}$$

The space of section of $\widetilde{Q}^*(\gamma)$ can be interpreted as the space of linear combinations of the rows of the previous matrix, the coefficients being homogeneous polynomials of degree resp. $\alpha+\beta$, $-\beta$, 0, $\alpha-\beta$, $2\alpha+\beta$, α, $-\alpha+\beta$, 0, $\alpha+2\beta$, $-2\alpha-\beta$, $-\alpha-2\beta$, 0, $-\alpha-\beta$, β, α (of course any coefficient is zero when the corresponding degree is negative). When $\alpha=\beta=0$ we have the case of pullback over $\mathbb{P}^5$, all these coefficients are constants and every $\widetilde{N}$ is a pullback of a classical nullcorrelation bundle. On the contrary, when $0<\alpha<\beta$ only bundles given by linear combinations with nonzero coefficients of rows number 3, 8 and 12 come as pullback over $\mathbb{C}^6\backslash 0$.

Exactly in the same way of 3.7 and 3.9 one can compute the cohomology groups of $\widetilde{N}$ coming as pullback over $\mathbb{C}^6\backslash 0$. The results are the following

Proposition 4.5 Let $\widetilde{N}$ be a weighted nullcorrelation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. The following hold

$h^2(\widetilde{N}(t))=h^3(\widetilde{N}(t))=0\ \forall t\in\mathbb{Z}$

$h^1(\widetilde{N}(t))=h^4(\widetilde{N}(-t-6))=\sum_{j=0}^{6}(-1)^j h^0[\wedge^j\mathcal{H}\otimes\mathcal{O}(t+(1-j)\gamma)]$

$$h^0(\widetilde{N}(t)) = \sum_{j=2}^{6}(-1)^j h^0[\wedge^j\mathcal{H}\otimes\mathcal{O}(t+(1-j)\gamma)]-h^0(\mathcal{O}(t-\gamma))]$$

$$h^1(\mathrm{End}\,\widetilde{N}) = \sum_{j=0}^{6}(-1)^j h^0[\wedge^j(\mathcal{H}(-\gamma))\otimes(\mathcal{H}(\gamma)\oplus\mathcal{H}_{\nu_2})] =$$

$=h^0(\mathcal{H}(\gamma))-h^0(S^2\mathcal{H})-h^0(\wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma))+h^0(\wedge^2\mathcal{H}\otimes\wedge^2\mathcal{H}(-2\gamma))-h^0(\wedge^3\mathcal{H}\otimes\wedge^2\mathcal{H}(-3\gamma))-1=$
$=h^0(\mathcal{H}(\gamma))-h^0(S^2\mathcal{H})+h^0(\mathcal{O}(4\alpha+2\beta-2\gamma))+h^0(\mathcal{O}(2\alpha+4\beta-2\gamma))+2h^0(\mathcal{O}(3\alpha+\beta-2\gamma))+$
$2h^0(\mathcal{O}(\alpha+3\beta-2\gamma))+2h^0(\mathcal{O}(3\alpha+2\beta-2\gamma))+2h^0(\mathcal{O}(2\alpha+3\beta-2\gamma))-h^0(\mathcal{O}(4\alpha+3\beta-3\gamma))-$
$h^0(\mathcal{O}(3\alpha+4\beta-3\gamma))-h^0(\mathcal{O}(4\alpha+\beta-3\gamma))-h^0(\mathcal{O}(\alpha+4\beta-3\gamma))$

$h^0(\mathrm{End}\,\widetilde{N}) =$
$1+h^0(\wedge^3\mathcal{H}(-\gamma))-h^0(\wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma))+h^0(\wedge^2\mathcal{H}\otimes\wedge^2\mathcal{H}(-2\gamma))-h^0(\wedge^3\mathcal{H}\otimes\wedge^2\mathcal{H}(-3\gamma))$

Remark 4.6 In particular a bundle $\widetilde{N}$ coming as pullback is simple if and only if $\gamma>2\alpha+2\beta$ (see theorem 4.8 for a more general statement)

In order to study the stability of $\widetilde{N}$ we need the following

Lemma 4.7 Let F be a symplectic rank 4 bundle on $\mathbb{P}^5$ such that $h^0(F)=0$, $h^0(\wedge^2F)=1$. Then F is stable.

Proof The hypothesis $h^0(F)=0$ implies the nonexistence of destabilizing subsheaves of rank 1 or 3. We have $\wedge^2F=\mathcal{O}\oplus F'$ with $h^0(F')=0$. If $\mathcal{G}$ is a destabilizing torsion-free subsheaf of F of rank 2 we can suppose $c_1(\mathcal{G})=0$, hence $\mathcal{O}=\wedge^2\mathcal{G}^{**}\subset\wedge^2F$ gives the only section of $\wedge^2F$. On an open subset the fiber $\mathcal{G}_x$ is a subspace of F_x of constant dimension 2. If $\mathcal{G}_x$ is spanned by v_1 and v_2, then $\wedge^2\mathcal{G}_x$ is spanned by $v_1\wedge v_2$ in $\wedge^2F_x$, in particular it corresponds to a 2-vector of rank 2, while the only section of $\wedge^2F$ has rank 6 everywhere. This is the desired contradiction.

Theorem 4.8 Let $\widetilde{N}$ be a weighted nullcorrelation bundle. The following are equivalent

i) $\gamma>2\alpha+2\beta$

ii) $\widetilde{N}$ is *stable*

iii) $\widetilde{N}$ is *simple*

Proof i)$\Rightarrow$ii) If $\gamma>2\alpha+2\beta$ then $h^0(\widetilde{Q}^*)=0$ from (3.4), hence $h^0(\widetilde{N})=0$. Consider the exact sequence

$$0\to\widetilde{N}(-\gamma)\to\wedge^2\widetilde{Q}^*\to\wedge^2\widetilde{N}\to 0$$

From the resolution

$$0\to\mathcal{O}(-4\gamma)\to\wedge^5\mathcal{H}(-3\gamma)\to\wedge^4\mathcal{H}(-2\gamma)\to\wedge^3\mathcal{H}(-\gamma)\to\wedge^2\widetilde{Q}^*\to 0$$

we have $h^0(\wedge^2\widetilde{Q}^*)=h^1(\wedge^2\widetilde{Q}^*)=0$, hence $h^0(\wedge^2\widetilde{N})=h^1(\widetilde{N}(-\gamma))=1$ and the thesis follows from lemma 4.7.

ii)$\Rightarrow$iii) is well known

iii) $\Rightarrow$ i) If $\gamma\leq 2\alpha+2\beta$ then $h^0(\wedge^3\mathcal{H}(-\gamma))\neq 0$, then $h^0(\wedge^2\widetilde{Q}^*)\neq 0$ from the above resolution of $\wedge^2\widetilde{Q}^*$. In particular we get $h^0(\wedge^2\widetilde{N})\neq 0$, then $\widetilde{N}$ is not simple.

Corollary 4.9 If $\gamma>2\alpha+2\beta$ the weighted nullcorrelation bundles fill up an *open reduced irreducible* subset of dimension $h^0(\mathcal{H}(\gamma))-h^0(S^2\mathcal{H})-1$ of the moduli space of stable bundles with the same rank and Chern classes. Bundles coming as pullback over $\mathbb{C}^6\setminus 0$ are *smooth* points.

Remark 4.10 The proof of corollary 4.9 translates with slights modifications to $\mathbb{P}^3$ giving another proof of theorem 3.1 b of [Ein]

Remark 4.11 As in remark 3.11 one can define in the obvious way $N_{\alpha,\beta,\gamma,\delta}$. In this case the necessary and sufficient condition for the stability and semplicity of $N_{\alpha,\beta,\gamma,\delta}$ is $\gamma>\alpha+\beta+\delta$.

5. Weighted lambda-three bundles

Let $N_{\alpha,\beta,\gamma}$ be a weighted nullcorrelation bundle appearing in the sequence

$$0\to\mathcal{O}(-\gamma)\to Q^*_{\alpha,\beta,\gamma}\to N_{\alpha,\beta,\gamma}\to 0$$

The second exterior power of this sequence gives

$$0\to N_{\alpha,\beta,\gamma}(-\gamma)\to\wedge^2Q^*_{\alpha,\beta,\gamma}\to\wedge^2N_{\alpha,\beta,\gamma}\to 0$$

Lemma 4.2 provides a splitting $\wedge^2N_{\alpha,\beta,\gamma}\simeq B_{\alpha,\beta,\gamma}\oplus\mathcal{O}$. The bundle $B_{\alpha,\beta,\gamma}$ is orthogonal and it is easy to check that it fits into an exact sequence

$$0\to\wedge^4Q^*_{\alpha,\beta,\gamma}\to\wedge^2Q^*_{\alpha,\beta,\gamma}\to B_{\alpha,\beta,\gamma}\to 0 \qquad (5.1)$$

Let $w\in\mathrm{Hom}(\mathcal{O}(-\gamma),\wedge^2\mathcal{H}(-\gamma))=H^0(\wedge^2\mathcal{H})$ be a lifting as in the following diagram

$$\begin{array}{ccccccccc} & & & w & \wedge^2\mathcal{H}(-\gamma) & & & & \\ & & & \nearrow & \downarrow & & & & \\ 0 & \to & \mathcal{O}(-\gamma) & \to & Q^*_{\alpha,\beta,\gamma} & \to & N_{\alpha,\beta,\gamma} & \to & 0 \end{array}$$

where the vertical arrow comes from (3.4). The morphism w defines naturally $w'\colon\mathcal{H}(-\gamma)\to\wedge^3\mathcal{H}(-\gamma)$ and one can check that the following diagram is commutative

$$
\begin{array}{ccccccccc}
 & & 0 \to & \mathcal{O}(-2\gamma) & \to & \mathcal{H}(-\gamma) & \to & \wedge^4 Q^*_{\alpha,\beta,\gamma} & \to 0 \\
 & & & \downarrow w & & \downarrow w' & & \downarrow & \\
0 \to \mathcal{O}(-4\gamma) & \to & \mathcal{H}(-3\gamma) & \to \wedge^2\mathcal{H}(-2\gamma) & \to & \wedge^3\mathcal{H}(-\gamma) & \to & \wedge^2 Q^*_{\alpha,\beta,\gamma} & \to 0 \\
 & & & & & & & \downarrow & \\
 & & & & & & & B_{\alpha,\beta,\gamma} &
\end{array}
$$

The proof of lemma 4.2 shows that w corresponds to a symplectic automorphism of $\mathcal{H}$, hence Coker $w \simeq \mathcal{H}_{\nu_2}(-2\gamma)$, Coker $w' \simeq \mathcal{H}_{\nu_3}(-\gamma)$, and the minimal resolution of $B_{\alpha,\beta,\gamma}$ is

$$0 \to \mathcal{O}(-4\gamma) \to \mathcal{H}(-3\gamma) \to \mathcal{H}_{\nu_2}(-2\gamma) \to \mathcal{H}_{\nu_3}(-\gamma) \to B_{\alpha,\beta,\gamma} \to 0 \tag{5.2}$$

From the natural isomorphism $\wedge^4 Q^*_{\alpha,\beta,\gamma} \simeq Q_{\alpha,\beta,\gamma}(-\gamma)$, it is easy to check that $B_{\alpha,\beta,\gamma}$ is the cohomology of a monad

$$Q_{\alpha,\beta,\gamma}(-\gamma) \to \wedge^2\mathcal{H} \to Q_{\alpha,\beta,\gamma}{}^*(\gamma) \tag{5.3}$$

Definition 5.1 A *weighted lambda-three bundle* $B_{\alpha,\beta,\gamma}$ is the cohomology of a monad (5.3) where $Q_{\alpha,\beta,\gamma}$ is a weighted quotient bundle.

We often use $\widetilde{B}$ for $B_{\alpha,\beta,\gamma}$.

Note that the dual of a lambda-three bundle is again a lambda-three bundle.

Remark 5.2 As a particular case we have the weighted lambda-three bundles obtained pulling back over $\mathbb{C}^6\backslash 0$ a lambda-three bundle B. If the morphism ω: $\mathbb{C}^6\backslash 0 \to \mathbb{C}^6\backslash 0$ as in (0.1) is given by $f_1,\ldots,f_6$ then the composition $\mathcal{H}_{\nu_3}(-\gamma) \to B_{\alpha,\beta,\gamma} \simeq B_{\alpha,\beta,\gamma}{}^* \to \mathcal{H}_{\nu_3}(\gamma)$ is described by the 14×14 matrix (1.2) where we replace f_1 by a, f_2 by b and so on. Let us call M this new matrix. The sections of $H^0(B_{\alpha,\beta,\gamma}(\gamma))$ can be interpreted as the space of linear combination of the rows of M with coefficients homogeneous polynomials of degree -2α, 2α, 0, $\alpha+\beta$, $-\beta$, $2\alpha+2\beta$, α, -2β, 0, $-\alpha-\beta$, β, $-2\alpha-2\beta$, $-\alpha$, 2β.

Lemma 5.3 Two isomorphic lambda-three bundles are defined by the same weighted quotient $\widetilde{Q}$.

Proof In the case $\widetilde{B} = \wedge^2\widetilde{N}/\mathcal{O}$ then the statement is obvious from the minimal resolution (5.2) because $\widetilde{Q}(-3\gamma)$ is the first cokernel on the left. In the general case the minimal resolution could be a priori different. Putting together the resolutions of $\wedge^4\widetilde{Q}^*$ and $\wedge^2\widetilde{Q}^*$ one gets the resolution

$$0 \to \mathcal{O}(-4\gamma) \to \mathcal{H}(-3\gamma)\oplus\mathcal{O}(-2\gamma) \to \wedge^2\mathcal{H}(-2\gamma)\oplus\mathcal{H}(-\gamma) \to \wedge^3\mathcal{H}(-\gamma) \to \widetilde{B} \to 0 \tag{5.4}$$

Because $H^1(\tilde{B}(*))=0$ the corresponding sequence of $\mathbb{C}[a,\ldots,f]$-modules is exact. In particular the piece $\mathcal{O}(-4\gamma) \xrightarrow{k} \mathcal{H}(-3\gamma)$ does not contain any summands that cancel in the minimal resolution.

Hence $\widetilde{Q}(-3\gamma)$=Coker k is defined directly from the minimal resolution of $\widetilde{B}$.

Lemma 5.4 Let $Q^0_{\alpha,\beta,\gamma}$ be a weighted quotient bundle. Any small deformation of $\wedge^2 Q^0_{\alpha,\beta,\gamma}$ has the form $\wedge^2 Q_{\alpha,\beta,\gamma}$ where $Q_{\alpha,\beta,\gamma}$ is again a weighted quotient bundle. Moreover the map $Q_{\alpha,\beta,\gamma} \mapsto \wedge^2 Q_{\alpha,\beta,\gamma}$ induces an isomorphism between the germs of the corresponding Kuranishi spaces.

Proof We remark that lemma 1.1 combined with (3.5) (replacing Q by $\wedge^2$Q) implies that $H^1(\mathrm{End}\ \wedge^2\widetilde{Q}) \simeq H^1(\mathrm{End}\ \widetilde{Q})$ for any weighted quotient $\widetilde{Q}$. Now it is sufficient to verify that if $\wedge^2\widetilde{Q}' \simeq \wedge^2\widetilde{Q}''$ then $\widetilde{Q}' \simeq \widetilde{Q}''$ and this follows from the fact that in the minimal resolution of $\wedge^2\widetilde{Q}$

$$0 \to \mathcal{O}(-2\gamma) \to \mathcal{H}(-\gamma) \to \wedge^2\mathcal{H} \to \wedge^2\widetilde{Q} \to 0$$

the first cokernel on the left is $\widetilde{Q}(-\gamma)$.

As in the case of weighted nullcorrelation $\widetilde{N}$, *not all weighted lambda-three* $\widetilde{B}$ *come as pullback from* $\mathbb{C}^6\backslash 0$. From the fact that $H^1(\mathrm{End}\ N(*)) \simeq H^1(\mathrm{End}\ B(*))$ (see section 1) it follows that if $\widetilde{N}$ comes as pullback over $\mathbb{C}^6\backslash 0$ then also $\widetilde{B} = \wedge^2\widetilde{N}/\mathcal{O}$ comes as pullback over $\mathbb{C}^6\backslash 0$ and $H^1(\mathrm{End}\ \widetilde{B}) = H^1(\mathrm{End}\ \widetilde{N})$, already computed in prop. 4.5.

Lemma 5.5 Let $\widetilde{Q}$ be a weighted quotient bundle. Then $H^1(\wedge^2\widetilde{Q} \otimes \wedge^4\widetilde{Q}^*) = 0$

Proof From lemma 1.2.

Theorem 5.6 Let $B^0_{\alpha,\beta,\gamma}$ be a lambda-three bundle coming as pullback over $\mathbb{C}^6\backslash 0$. Every small deformation of $B^0_{\alpha,\beta,\gamma}$ is again a lambda-three bundle $B_{\alpha,\beta,\gamma}$. Moreover the Kuranishi space of $B^0_{\alpha,\beta,\gamma}$ is *smooth* at $B^0_{\alpha,\beta,\gamma}$.

Proof From lemmas 5.3 and 5.5 it follows that every isomorphism between two lambda-three bundles $\widetilde{B}$, $\widetilde{B}'$ is induced by a morphism of sequences

$$\begin{array}{ccccccccc} 0 & \to & \wedge^4\widetilde{Q}^* & \to & \wedge^2\widetilde{Q}^* & \to & \widetilde{B} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \wedge^4\widetilde{Q}^* & \to & \wedge^2\widetilde{Q}^* & \to & \widetilde{B}' & \to & 0 \end{array}$$

We use $\widetilde{B}_0$ for $B^0_{\alpha,\beta,\gamma}$ and we denote $\widetilde{Q}_0$ the weighted quotient corresponding to $\widetilde{B}_0$ uniquely defined by lemma 5.3. Let now $f_0 \in \mathrm{Hom}(\wedge^4\widetilde{Q}_0^*, \wedge^2\widetilde{Q}_0^*)$ be a morphism defining $\widetilde{B}_0$. $f, f' \in \mathrm{Hom}(\wedge^4\widetilde{Q}_0^*, \wedge^2\widetilde{Q}_0^*)$ give the same element of $\mathrm{Quot}_{\wedge^2\widetilde{Q}_0^*/\mathbb{P}^5}$ if and only if there is an invertible $h \in \mathrm{End}(\wedge^4\widetilde{Q}_0)$ such that $f = f' \circ h$. Let (Y, y_0) be the Kuranishi space of $\wedge^2\widetilde{Q}_0$ and let (T, t_0) be the Kuranishi space of $\widetilde{B}_0$. Let $\mathcal{F}$ be the universal family over $Y \times \mathbb{P}^5$ and let $Z = \mathrm{Quot}_{\mathcal{F}/Y\times\mathbb{P}^5/Y}$. We have two natural morphisms $\phi: Z \to Y$ and $\pi: (Z, z_0) \to (T, t_0)$.

Let Z' be the subvariety of the component of Z containing z_0 consisting of quotients

$\wedge^2 Q''^* \xrightarrow{g''} \mathcal{G}$ for some weighted quotient Q'' (we are using lemma 5.4) such that Ker $g'' \simeq \wedge^4 \widetilde{Q}''^*$. Hence we have $\dim_{t_0} T \geq \dim_{z_0} Z - \dim_{z_0} \pi^{-1}(t_0) \geq \dim_{z_0} Z' - \dim_{z_0} \pi^{-1}(t_0)$. Moreover from lemma 5.3 we get

$(\pi^{-1}(t_0), z_0) \subset (\phi^{-1}(y_0), z_0) = (\mathrm{Quot}_{\wedge^2 \widetilde{Q}_0^*/\mathbb{P}^5}, z_0)$. If $P := \{x \in \mathrm{Quot}_{\wedge^2 \widetilde{Q}_0^*/\mathbb{P}^5} : \widetilde{B}_x \simeq \widetilde{B}_0\}$ we check

$(\pi^{-1}(t_0), z_0) \subset (P, z_0)$. We have $\dim_{z_0} P = h^0(\mathrm{End} \wedge^2 \widetilde{Q}_0) - \{$dimension of endomorphisms of $\wedge^2 \widetilde{Q}_0$ that fix $f_0\} - h^0(\mathrm{End} \wedge^4 \widetilde{Q}_0)$. The exact sequence

$$0 \to \widetilde{B}_0 \otimes \wedge^2 \widetilde{Q}_0^* \to \mathrm{End}\ \wedge^2 \widetilde{Q}_0 \to \wedge^4 \widetilde{Q}_0 \otimes \wedge^2 \widetilde{Q}_0^* \to 0 \tag{5.5}$$

shows that the term in braces of the last formula is equal to $h^0(\widetilde{B}_0 \otimes \wedge^2 \widetilde{Q}_0^*)$. Now consider that all the fibers of $\phi|_{Z'}: Z' \to Y$ have the same dimension $h^0(\wedge^4 \widetilde{Q}_0 \otimes \wedge^2 \widetilde{Q}_0^*) - h^0(\mathrm{End} \wedge^4 \widetilde{Q}_0)$, (depending only on α, β, γ). By lemmas 5.4 and 3.3 $\dim_{y_0} Y = h^1(\mathrm{End}\ \widetilde{Q}_0) = h^1(\mathrm{End}\ \wedge^4 \widetilde{Q}_0) = h^1(\mathrm{End}\ \wedge^2 \widetilde{Q}_0)$, hence $\dim Z' = h^0(\wedge^4 \widetilde{Q}_0 \otimes \wedge^2 \widetilde{Q}_0^*) - h^0(\mathrm{End} \wedge^4 \widetilde{Q}_0) + h^1(\mathrm{End} \wedge^4 \widetilde{Q}_0)$. It follows

$\dim_{t_0} T \geq \dim_{z_0} Z' - \dim_{z_0} P$, that is

$$\dim_{t_0} T \geq h^0(\wedge^4 \widetilde{Q}_0 \otimes \wedge^2 \widetilde{Q}_0^*) - h^0(\mathrm{End}\ \wedge^2 \widetilde{Q}_0) + h^0(\widetilde{B}_0 \otimes \wedge^2 \widetilde{Q}_0^*) + h^1(\mathrm{End}\ \wedge^2 \widetilde{Q}_0). \tag{5.6}$$

We claim that the image of the morphism $H^1(\mathrm{End}\ \wedge^2 \widetilde{Q}_0) \to H^1(\wedge^4 \widetilde{Q}_0 \otimes \wedge^2 \widetilde{Q}_0^*)$ defined by the sequence (5.5) has dimension $\sum_{j=0}^{6} h^0[\wedge^j \mathcal{H} \otimes \mathcal{H}((1-j)\gamma)]$. In fact from the hypothesis the morphism End $\wedge^2 \widetilde{Q}_0 \to \wedge^4 \widetilde{Q}_0 \otimes \wedge^2 \widetilde{Q}_0^*$ comes as pullback over $\mathbb{C}^6 \setminus 0$ from a morphism $B \otimes \wedge^2 Q^* \to \mathrm{End}\ \wedge^2 Q$ and it can be computed by a graded tensor product. $H^1(\wedge^4 Q \otimes \wedge^2 Q^*(t))$ is zero for $t \neq -1$ and it is isomorphic to $\wedge^3 H = H \oplus H_{\nu_3}$ for $t = -1$ (lemma 1.3). We have also $H^1(\mathrm{End}\ \wedge^2 Q(-1) = H$ (lemma 1.1), and we check from lemma 1.10 and the cohomology sequence associated to (5.5) that the morphism $\oplus H^1(\mathrm{End}\ \wedge^2 Q(t)) \to \oplus H^1(\wedge^4 Q \otimes \wedge^2 Q^*(t))$ is an isomorphism on the subspace H just considered in degree -1 and this proves our claim. Then from (5.6) and the cohomology sequence associated to (5.5) it follows:

$\dim_{t_0} T \ \geq h^1(\widetilde{B}_0 \otimes \wedge^2 \widetilde{Q}_0^*) + \sum_{j=0}^{6} h^0[\wedge^j \mathcal{H} \otimes \mathcal{H}((1-j)\gamma)]$

We recall also from section 1 that $H^1(\mathrm{End}\ B(t)) = 0$ for $t \neq -1, 0$ and from lemma 1.10 that $H^1(B \otimes \wedge^2 Q^*(t)) = 0$ for $t \neq 0$, $H^1(B \otimes \wedge^2 Q^*) = H_{\nu_2} = H^1(\mathrm{End}\ B)$. Hence, as $H^1(\mathrm{End}\ B(-1)) = H$ gives a contribution to $H^1(\mathrm{End}\ \widetilde{B}_0)$ in the graded tensor product exactly equal to $\sum_{j=0}^{6} h^0[\wedge^j \mathcal{H} \otimes \mathcal{H}((1-j)\gamma))$ we get $\dim_{t_0} T \geq h^1(\mathrm{End}\ \widetilde{B}_0)$, thus the equality holds and π is surjective, q.e.d.

Lemma 5.7 Let $\widetilde{B}$ be a weighted lambda-three bundle. Then $H^0(\widetilde{B})$ is zero if and only if $\gamma > 2\alpha + 2\beta$.

Proof From (5.4) we get $H^0(\widetilde{B}) = H^0(\mathcal{H}_{\nu_3}(-\gamma))$.

Lemma 5.8 Let $\widetilde{B}$ be any weighted lambda-three bundle. If $\gamma > 2\alpha + 2\beta$ then $h^0(S^2 \widetilde{B}) = 1$.

Proof Taking the second symmetric power from (5.1) we get

$$0 \to \wedge^2(\wedge^4\widetilde{Q}^*) \to \wedge^4\widetilde{Q}^*\otimes\wedge^2\widetilde{Q}^* \to S^2(\wedge^2\widetilde{Q}^*) \to S^2\widetilde{B} \to 0 \tag{5.7}$$

We have $S^2(\wedge^2\widetilde{Q}^*)\simeq\Gamma^{2,2,2}\widetilde{Q}(-2\gamma)\oplus\widetilde{Q}(-\gamma)$. From lemma 1.4 we have $H^1(\Gamma^{2,2,2}Q(*)) = H^1(Q(*)) = 0$ and $H^0(\Gamma^{2,2,2}Q(t-2)) = H_{(t-2)\mu_1+2\mu_3}$ for $t\geq 2$. The maximum degree appearing in $H_{(t-2)\mu_1+2\mu_3}$ is $(t+2)\alpha+(t+2)\beta$ hence if $\gamma>2\alpha+2\beta$ we have $H^0(\Gamma^{2,2,2}\widetilde{Q}(-2\gamma))=0$. Moreover $H^0(\widetilde{Q}(-\gamma))=0$. Summarizing we get for $\gamma>2\alpha+2\beta$ that $H^0(S^2(\wedge^2\widetilde{Q}^*)) = H^1(S^2(\wedge^2\widetilde{Q}^*)) = 0$. From the isomorphism $\wedge^2(\wedge^4\widetilde{Q}^*)=\wedge^3\widetilde{Q}^*(-\gamma)$ it follows $H^1(\wedge^2(\wedge^4\widetilde{Q}^*))=H^2(\wedge^2(\wedge^4\widetilde{Q}^*))=0$. From these vanishings and the sequence (5.7) we get $H^0(S^2\widetilde{B})=H^1(\wedge^4\widetilde{Q}^*\otimes\wedge^2\widetilde{Q}^*)$. The thesis is now a consequence of the equality $\wedge^4\widetilde{Q}^*\otimes\wedge^2\widetilde{Q}^*=\Gamma^{2,1,1}\widetilde{Q}(-2\gamma)\oplus\widetilde{Q}^*(-\gamma)$ and lemma 1.4.

Proposition 5.9 Let $\widetilde{B}$ a weighted lambda-three bundle. If $\gamma>2\alpha+2\beta$ then $h^0(\wedge^2\widetilde{B})=h^0(\wedge^3\widetilde{B})=0$

Proof Taking the third exterior power from (5.1) we get

$$0 \to S^3(\wedge^4\widetilde{Q}^*) \to S^2(\wedge^4\widetilde{Q}^*)\otimes\wedge^2\widetilde{Q}^* \to \wedge^4\widetilde{Q}^*\otimes\wedge^2(\wedge^2\widetilde{Q}^*) \to \wedge^3(\wedge^2\widetilde{Q}^*) \to \wedge^3\widetilde{B} \to 0 \tag{5.8}$$

Using Littlewood-Richardson rule for $(\wedge^2\widetilde{Q}^*)^{\otimes 3}$ and checking the dimension of the summands we find that $\wedge^3(\wedge^2\widetilde{Q}^*)=\Gamma^{2,2}\widetilde{Q}(-2\gamma)\oplus\Gamma^{3,2,2,2}\widetilde{Q}(-3\gamma)$. If $\gamma>\alpha+2\beta$ it follows from lemma 1.5 that

$$H^0(\wedge^3(\wedge^2\widetilde{Q}^*))=0. \tag{5.9}$$

Consider now the decomposition $\wedge^4\widetilde{Q}^*\otimes\wedge^2(\wedge^2\widetilde{Q}^*) = \Gamma^{3,2,1,1}\widetilde{Q}(-3\gamma)\oplus\Gamma^{2,2,2,1}\widetilde{Q}(-3\gamma)\oplus\wedge^2\widetilde{Q}(-2\gamma)$. Again from lemma 1.5 we have for $\gamma>2\alpha+2\beta$

$$H^1(\wedge^4\widetilde{Q}^*\otimes\wedge^2(\wedge^2\widetilde{Q}^*))=0. \tag{5.10}$$

Going on, we look at the decomposition $S^2(\wedge^4\widetilde{Q}^*)\otimes\wedge^2\widetilde{Q}^*=\Gamma^{3,1,1}\widetilde{Q}(-3\gamma)\oplus\Gamma^{2,1,1,1}\widetilde{Q}(-3\gamma)$.

From lemma 1.5 we have as above

$$H^2(S^2(\wedge^4\widetilde{Q}^*)\otimes\wedge^2\widetilde{Q}^*)=0 \tag{5.11}$$

Moreover $H^3(S^3Q(t))=0\ \forall t\in\mathbb{Z}$, thus

$$H^3(S^3(\wedge^4\widetilde{Q}^*))=0 \tag{5.12}$$

From (5.9), (5.10), (5.11), (5.12) and the cohomology sequence associated to (5.8) it follows that $h^0(\wedge^3\widetilde{B})=0$ for $\gamma>2\alpha+2\beta$.

On the other hand $\wedge^2\widetilde{B}=\wedge^3\widetilde{B}^*$ hence $h^0(\wedge^2\widetilde{B})=0$ for $\gamma>2\alpha+2\beta$ because $\widetilde{B}^*$ too is a weighted lambda-three bundle. This concludes the proof.

Theorem 5.10 Let $\widetilde{B}$ a weighted lambda-three bundle. The following are equivalent

i) $\gamma>2\alpha+2\beta$

ii) $\widetilde{B}$ is *stable*

iii) $\widetilde{B}$ is *simple*

Proof i)$\Rightarrow$ii) If $\gamma>2\alpha+2\beta$, then $h^0(\widetilde{B})=h^0(\wedge^4\widetilde{B})=0$ from lemma 5.7 and $h^0(\wedge^2\widetilde{B})=h^0(\wedge^3\widetilde{B})=0$

from prop. 5.8. If $\mathcal{F}\subset\widetilde{B}$ is a proper subsheaf of rank f with torsion-free quotient, we get $(\wedge^f\mathcal{F})^{**}\subset\wedge^f\widetilde{B}$, hence $c_1(\mathcal{F})<0$ and $\widetilde{B}$ is stable.

ii)$\Rightarrow$iii) is well known

iii)$\Rightarrow$i) If $\gamma\leq 2\alpha+2\beta$ then from lemma 5.7 we have $h^0(\widetilde{B})\neq 0$, $h^0(\widetilde{B}^*)\neq 0$, then we can construct a section of $\widetilde{B}\otimes\widetilde{B}^* = \text{End}\,\widetilde{B}$ which is not a homothety.

Proposition 5.11 Let $\widetilde{B}$ a lambda-three bundle. If $\gamma>2\alpha+2\beta$ then $\widetilde{B}$ is *orthogonal*, in particular $\widetilde{B}\simeq\widetilde{B}^*$.

Proof In the hypothesis $\widetilde{B}$ is stable by theorem 5.10. By lemma 5.8 there exists a nonzero symmetric morphism $\phi:B\to B^*$, that has to be a isomorphism because both $\widetilde{B}$, $\widetilde{B}^*$ are stable.

We do not know if for $\gamma\leq 2\alpha+2\beta$ any lambda-three bundle is orthogonal. This is true in the case of bundles coming as pullback over $\mathbb{C}^6\backslash 0$.

Corollary 5.12 If $\gamma>2\alpha+2\beta$ then the bundles $\widetilde{B}$ fill up a *open reduced irreducible* subset of dimension $h^0(\mathcal{H}(\gamma))-h^0(S^2\mathcal{H})-1$ of the moduli space of stable bundles with the same rank and Chern classes. Bundles coming as pullback over $\mathbb{C}^6\backslash 0$ are *smooth* points.

Proof From theorem 5.6 and theorem 5.10.

Proposition 5.13 Let $\widetilde{B}$ be a weighted lambda-three bundle coming as pullback over $\mathbb{C}^6\backslash 0$. Then

$$H^0(\text{End}\,\widetilde{B})=1+h^0(\mathcal{H}_{\nu_3}\otimes\mathcal{H}_{\nu_3}(-2\gamma))-2h^0(\mathcal{H}_{\nu_2}\otimes\mathcal{H}_{\nu_3}(-3\gamma))$$

Proof Tensoring by $\widetilde{B}^*$ the minimal resolution of $\widetilde{B}$.

Proposition 5.14 Let $\widetilde{Q}$ be a weighted quotient bundle. $\wedge^2\widetilde{Q}$ is *simple* if and only if $\gamma>3\alpha+4\beta$

Proof Starting from the sequence

$$0\to\wedge^2\widetilde{Q}(-2\gamma)\to\wedge^2\widetilde{Q}^*\otimes H(-\gamma)\to\wedge^2\widetilde{Q}^*\otimes\wedge^2 H\to\text{End}\,\wedge^2\widetilde{Q}^*\to 0$$

one computes

$$h^0(\text{End}\,\wedge^2\widetilde{Q}^*) = 1+h^0(\wedge^3\mathcal{H}\otimes\wedge^2\mathcal{H}(-\gamma))-h^0(\wedge^2\mathcal{H}\otimes\wedge^2\mathcal{H}(-2\gamma))-h^0(\wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma))$$

Now consider that $\wedge^3\mathcal{H}\otimes\wedge^2\mathcal{H}=\bigoplus_i\mathcal{O}(a_i)$ with $\max_i\{a_i\}=3\alpha+4\beta$

Theorem 5.15 When $\gamma>3\alpha+4\beta$ then the morphism $\widetilde{N}\mapsto\wedge^2\widetilde{N}/\mathcal{O}$ from the moduli space of weighted nullcorrelation bundles into the moduli space of weighted lambda-three bundles is bijective.

Proof We have $\text{Hom}(\mathcal{O}(-\gamma),\widetilde{Q}^*)=\wedge^4\widetilde{Q}$, $\text{Hom}(\wedge^4\widetilde{Q}^*,\wedge^2\widetilde{Q}^*)=\wedge^4\widetilde{Q}\otimes\wedge^2\widetilde{Q}^*$. Looking at lemma 1.3 and considering the corresponding Young diagrams, one can check that the maximum degree appearing in $H_{t\mu_1+\mu_2}$ is $(t+1)\alpha+(t+2)\beta$, in $H_{(t-1)\mu_1+\mu_2+\mu_3}$ is $(t+2)\alpha+(t+3)\beta$ and in $H_{t\mu_1+\mu_4}$ is $(t+1)\alpha+(t+2)\beta$. Hence if $\gamma>3\alpha+4\beta$ both $h^0(\wedge^4\widetilde{Q})$ and $h^0(\wedge^4\widetilde{Q}\otimes\wedge^2\widetilde{Q}^*)$ are equal to $\sum_{j=0}^{6}(-1)^j h^0[\wedge^j\mathcal{H}(-\gamma)\otimes\wedge^2\mathcal{H}]$ (only the summands with $t=0$ give contribution). We have then a natural isomorphism

$Hom(\mathcal{O}(-\gamma),\widetilde{Q}^*) \to Hom(\wedge^4\widetilde{Q}^*,\wedge^2\widetilde{Q}^*)$ corresponding to $\widetilde{N}\mapsto\wedge^2\widetilde{N}/\mathcal{O}$. By corollary 3.9 and prop. 5.14 with our assumptions both $\widetilde{Q}^*$ and $\wedge^2\widetilde{Q}^*$ are simple, hence weighted nullcorrelation bundles (resp. weighted lambda-three bundles) defined by the weighted quotient bundle $\widetilde{Q}$ (see lemma 5.3) correspond to a unique element of $Hom(\mathcal{O}(-\gamma),,\widetilde{Q}^*)$ (resp. $Hom(\wedge^4\widetilde{Q}^*,\wedge^2\widetilde{Q}^*)$), q.e.d.

6. The relation bundles

Now we construct the 3-bundles which are the main subject of this paper. We saw in the remark 5.2 that if $B_{\alpha,\beta,\gamma}$ is a weighted lambda-three bundle coming as pullback over $\mathbb{C}^6\backslash 0$ then $H^0(B_{\alpha,\beta,\gamma}(\gamma))$ can be interpreted as the space of linear combinations of the rows of the matrix M defined there with coefficients being homogeneous polynomials of degree -2α, 2α, 0, $\alpha+\beta$, $-\beta$, $2\alpha+2\beta$, α, -2β, 0, $-\alpha-\beta$, β, $-2\alpha-2\beta$, $-\alpha$, 2β. Let σ, $\tau\in H^0(B_{\alpha,\beta,\gamma}(\gamma))$ be given by the coefficients (0,0,1,0,0,0,0,0,1,0,0,0,0,0) and (0,0,1,0,0,0,0,0,−1,0,0,0,0,0). Then by the explicit form of the matrix M, σ and τ do not vanish anywhere (because f_1, ...,f_6 have no common zeroes). Moreover $\sigma^t\circ\tau=(0,0,1,0,0,0,0,0,1,0,0,0,0,0)M(0,0,1,0,0,0,0,0,-1,0,0,0,0,0)^t=0$ because the matrix M is symmetric. Hence the cohomology of the monad

$$\mathcal{O}(-\gamma) \overset{\sigma}{\to} B_{\alpha,\beta,\gamma} \overset{\tau}{\to} \mathcal{O}(\gamma)$$

is a 3-bundle $E_{\alpha,\beta,\gamma}$. The action (3.3) of $\tau_{\alpha,\beta,\gamma}$ gives an embedding of $\mathbb{C}^*$ in SL(W) and as in the discussion after (3.3) it is clear that $\eta^*E_{\alpha,\beta,\gamma}=\omega^*\eta^*E$, that is $E_{\alpha,\beta,\gamma}$ comes as pullback over $\mathbb{C}^6\backslash 0$ from a parent bundle. We remark that $\eta^*\mathcal{W}=\omega^*\eta^*(W\otimes\mathcal{O})$. This construction was performed by Horrocks in [Hor2]. In the notations of the last section of [Hor2] we have $m_1=\alpha$, $m_2=\beta$, $m_3=-\alpha-\beta$, $r=\gamma$.

Definition 6.1 A *relation* (=weighted parent) *bundle* $E_{\alpha,\beta,\gamma}$ is the cohomology of a monad

$$\mathcal{O}(-\gamma) \to B_{\alpha,\beta,\gamma} \to \mathcal{O}(\gamma)$$

where $B_{\alpha,\beta,\gamma}$ is a weighted lambda-three bundle.

Sometimes we use $\widetilde{E}$ for $E_{\alpha,\beta,\gamma}$. It follows from the definition that the dual of a relation bundle is again a relation bundle. Let $\widetilde{B}$ be a weighted lambda-three bundle coming as pullback over $\mathbb{C}^6\backslash 0$. for $0<\alpha<\beta$ *only the relation bundles defined by sections of* $B_{\alpha,\beta,\gamma}(\gamma)$ *which are suitable linear combination of* σ, τ *above come as pullback over* $\mathbb{C}^6\backslash 0$. This family fibers over the family of corresponding weighted lambda-three bundles coming as pullback over $\mathbb{C}^6\backslash 0$, with 1-dimensional fibers (see the discussion after theorem 4.4).

This construction explains why we restricted the definitions of $\widetilde{Q}$, $\widetilde{N}$, $\widetilde{B}$ to the case $\mathcal{W}=\mathcal{O}(\alpha)\oplus\mathcal{O}(\beta)\oplus\mathcal{O}(-\alpha-\beta)$. In the general case where $\mathcal{W}=\mathcal{O}(\alpha)\oplus\mathcal{O}(\beta)\oplus\mathcal{O}(-\delta)$, suppose that $N_{\alpha,\beta,\gamma,\delta}$ is the cohomology bundle of a monad $\mathcal{O}(-\gamma) \to \mathcal{H} \to \mathcal{O}(\gamma)$ (see remark 4.11),

$B_{\alpha,\beta,\gamma,\delta} = \wedge^2 N_{\alpha,\beta,\gamma,\delta}/\mathcal{O}$ and $\mathcal{E}$ is the cohomology sheaf of a monad

$$\mathcal{O}(-\gamma) \to B_{\alpha,\beta,\gamma,\delta} \to \mathcal{O}(\gamma)$$

then $c_4(\mathcal{E})=(\alpha+\beta+\delta)(\alpha-\beta-\delta)(\alpha-\beta+\delta)(\alpha+\beta-\delta)$ so that $c_4(\mathcal{E})$ is zero if and only if $\delta=\alpha+\beta$.

Example 6.2 The cohomology of relation bundles coming as pullback over $\mathbb{C}^6\backslash 0$ is completely determined by the cohomology of the parent bundle. Even in the simplest cases the cohomology table of $E_{\alpha,\beta,\gamma}$ is quite complicate. In the case $\alpha=0$, $\beta=1$, $\gamma=2$ we have $c_2(E_{0,1,2})=8$ and

$$\bigoplus_t H^1(E_{0,1,2}(t)) = \frac{\mathbb{C}[a,b,c,d,e,f]}{(ad+be+cf,(a,b,c)^2,(d,e,f)^2)}(1)\otimes_{\mathbb{C}[a^2,b,c^3,d^2,e^3,f]}\mathbb{C}[a,b,c,d,e,f] =$$

$$= \frac{\mathbb{C}[a,b,c,d,e,f]}{(a^2d^2+be^3+c^3f,(a^2,b,c^3)^2,(d^2,e^3,f)^2)}(2).$$

Th only nonzero values of $h^1(E_{0,1,2}(t))$ are 1,6,19,42,70,92,98,86,63,38,18,6,1 corresponding respectively to $t=-2,-1,\dots,10$. It is interesting to remark that in this case $h^0(E_{0,1,2}(2))=2$.

For the convenience of the reader we summarize in the following theorem the intermediate cohomology of a relation bundle $E_{\alpha,\beta,\gamma}$ coming as pullback over $\mathbb{C}^6\backslash 0$. H^3 and H^4 can be found by Serre duality because $E^*_{\alpha,\beta,\gamma}$ is again a relation bundle. H^0 can be computed from the minimal resolution (see corollary 6.8).

Theorem 6.3 Let $E_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. The following hold

$$H^1(E_{\alpha,\beta,\gamma}(t))=\sum_j(-1)^j h^0\{\wedge^j\mathcal{H}\otimes\mathcal{O}(t-j\gamma)\otimes[\mathcal{O}(-\gamma)\oplus\mathcal{H}\oplus\Gamma^{2,1}\mathcal{W}(\gamma)]\}$$

$$H^2(E_{\alpha,\beta,\gamma}(t))=\sum_j(-1)^j h^0\{\wedge^j[\mathcal{H}(-\gamma)]\otimes\mathcal{O}(t-2\gamma)\}$$

The display of the monad defining $\widetilde{E}$ gives the two exact sequences

$$0 \to R_{\alpha,\beta,\gamma} \to B_{\alpha,\beta,\gamma} \to \mathcal{O}(\gamma) \to 0 \qquad (6.1)$$

$$0 \to \mathcal{O}(-\gamma) \to R_{\alpha,\beta,\gamma} \to E_{\alpha,\beta,\gamma} \to 0 \qquad (6.2)$$

Lemma 6.4 Let $R^0_{\alpha,\beta,\gamma}$ be a bundle appearing as a kernel in a sequence

$$0 \to R^0_{\alpha,\beta,\gamma} \to B^0_{\alpha,\beta,\gamma} \to \mathcal{O}(\gamma) \to 0$$

where $B^0_{\alpha,\beta,\gamma}$ is a weighted lambda-three bundle coming as pullback over $\mathbb{C}^6\backslash 0$ and such that also $R^0_{\alpha,\beta,\gamma}$ comes as pullback over $\mathbb{C}^6\backslash 0$. Every small deformation of $R^0_{\alpha,\beta,\gamma}$ appears again as a kernel in a sequence as (6.1) where $B_{\alpha,\beta,\gamma}$ is a weighted lambda-three bundle. Moreover the Kuranishi space of $R^0_{\alpha,\beta,\gamma}$ is *smooth* at $R^0_{\alpha,\beta,\gamma}$.

Proof We use $\widetilde{R}_0$ (resp. $\widetilde{B}_0$) for $R^0_{\alpha,\beta,\gamma}$ (resp. $B^0_{\alpha,\beta,\gamma}$). We replace $\widetilde{R}_0$ by the dual bundle $\widetilde{R}^*_0$ that appears as a quotient in the sequence

$$0 \to \mathcal{O}(-\gamma) \to \tilde{B}_0^* \to \tilde{R}_0^* \to 0$$

Let (T,t_0) be the Kuranishi space for $\widetilde{R}_0^*$ and let (Y,y_0) be the Kuranishi space for $\widetilde{B}_0^*$. Let $\mathcal{F}$ be the universal family over $Y\times\mathbb{P}^5$ and let Z be the component of $\mathrm{Quot}_{\mathcal{F}/Y\times\mathbb{P}^5/Y}$ containing $\widetilde{R}_0^*$. As in the proof of theorem 4.4 we have a natural map $\pi:(Z,z_0) \to (T,t_0)$ so that $\dim_{t_0}T\geq\dim_{z_0}Z-\dim_{z_0}\pi^{-1}(t_0)\geq h^1(\mathrm{End}\ \widetilde{B}_0)+h^0(\widetilde{B}_0^*(\gamma))-h^0(\mathrm{End}\ \widetilde{B}_0)+h^0(\widetilde{R}_0\otimes\widetilde{B}_0^*)$, the last inequality is a consequence of the exact sequence

$$0 \to \widetilde{R}_0\otimes\widetilde{B}_0^* \to \widetilde{B}_0\otimes\widetilde{B}_0^* \to B_0^*(\gamma) \to 0,$$

the fact that $h^0(\widetilde{B}_0(\gamma))$ depends only on α,β,γ, and the theorem 5.6.
Again from the above sequence we have

$$\dim_{t_0}T\geq h^1(\widetilde{R}_0\otimes\widetilde{B}_0^*)$$

(because $h^1(B_0^*(\gamma))=0$).

Now consider the sequence

$$0 \to \widetilde{R}_0(-\gamma) \to \widetilde{R}_0\otimes\widetilde{B}_0 \to \widetilde{R}_0\otimes\widetilde{R}_0{}^* \to 0$$

The cohomology sequence associated to

$$0 \to R(t-1) \to R\otimes B(t) \to \mathrm{End}\ R(t) \to 0$$

gives

$h^1(R\otimes B(t))=H^1(\mathrm{End}\ R(t))=0$ if $t\leq-1$

$H^0(\mathrm{End}\ R) \to \underset{\overset{\parallel}{\mathbb{C}}}{H^1(R(-1))} \to 0 \to H^1(R\otimes B) \to H^1(\mathrm{End}\ R) \to 0$ for $t=0$, and

$$H^1(R)=W\oplus W^* \to H^1(R\otimes B(1)) \to H^1(\mathrm{End}\ R(1)) \to 0 \quad \text{for } t=1$$

The mophisms in this last sequence are SL(W)-invariant. $W\oplus W^*$ cannot contribute to $H^1(\widetilde{R}_0\otimes\widetilde{B}_0)$ in the graded tensor product $H^1(\widetilde{R}_0\otimes\widetilde{B}_0)=[\underset{t}{\oplus}H^1(R\otimes B(t))]\otimes_{\mathbb{C}}\frac{\mathbb{C}[a,b,c,d,e,f]}{(f_1,f_2,f_3,f_4,f_5,f_6)}$, and the same reasoning holds for $t\geq 2$. It follows

$$H^1(\widetilde{R}_0\otimes\widetilde{B}_0)=H^1(\mathrm{End}\ \widetilde{R}_0)$$

as we wanted.

Theorem 6.5 Let $E^0_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. Every small deformation of $E^0_{\alpha,\beta,\gamma}$ is a relation bundle $E_{\alpha,\beta,\gamma}$. Moreover the Kuranishi space of $E^0_{\alpha,\beta,\gamma}$ is *smooth* at $E^0_{\alpha,\beta,\gamma}$.

Proof We use $\widetilde{E}_0$ for $E^0_{\alpha,\beta,\gamma}$. Let $\widetilde{R}_0$ be corresponding to $\widetilde{E}_0$, that is the unique nonsplitting extension

$$0 \to \mathcal{O}(-\gamma) \to ? \to \widetilde{E}_0 \to 0$$

Consider the exact sequence

$$0 \to \widetilde{E}_0^*\otimes\widetilde{R}_0 \to \mathrm{End}\ \widetilde{R}_0 \to \widetilde{R}_0(\gamma) \to 0 \tag{6.3}$$

We begin to prove that the induced morphism
$g: H^1(\widetilde{E}_0{}^*\otimes\widetilde{R}_0) \to H^1(\mathrm{End}\ \widetilde{R}_0)$ is surjective. We set

$$g_t: H^1(E^*\otimes R(t)) \to H^1(\mathrm{End}\ R(t))$$

As tensor product is right exact we have that Coker g is the degree 0 summand in $[\bigoplus_t \mathrm{Coker}\ g_t]\otimes_{\mathbb{C}} \frac{\mathbb{C}[a,b,c,d,e,f]}{(f_1,f_2,f_3,f_4,f_5,f_6)}$.

Consider the two exact sequences

$$0 \to E^*\otimes R(t) \to \mathrm{End}\ R(t) \to R(t+1) \to 0$$
$$0 \to E^*(t-1) \to R\otimes E^*(t) \to \mathrm{End}\ E(t) \to 0$$

Coker $g_t=0$ for $t\geq1$ and for $t\leq-3$ from the first sequence. It is easy to check $H^1(\mathrm{End}\ R(-2))=0$. From the second sequence $H^1(\mathrm{End}\ E)=H^1(E^*\otimes R)$, from the first $H^1(E^*\otimes R)\subset H^1(\mathrm{End}\ R)$ and the last inclusion is the identity because $H^1(\mathrm{End}\ E)$ surjects naturally over $H^1(\mathrm{End}\ R)$ (by the previous lemma in the case $\alpha=\beta=0$). In the case $t=-1$ we have that Coker $g_{-1}\subset H^1(R)=H^1(E)=W\oplus W^*$ and hence cannot contribute to the degree zero summand of the tensor product. Let us observe also that from the second sequence it is easy to prove in the same way that $h^1(\widetilde{E}_0{}^*\otimes\widetilde{R}_0)=h^1(\mathrm{End}\ \widetilde{E}_0)$.

Let (T,t_0) be the Kuranishi space of E_0. As in the proof of theorem 4.4 we can check that

$$\dim_{t_0}T\geq h^0(\widetilde{R}_0(\gamma))-h^0(\mathrm{End}\ \widetilde{R}_0)+h^0(\widetilde{E}_0{}^*\otimes\widetilde{R}_0)+\dim\ \{\text{Kuranishi space of } \widetilde{R}_0\} =$$
$$= (\text{by theor. 6.4})\ h^0(\widetilde{R}_0(\gamma))-h^0(\mathrm{End}\ \widetilde{R}_0)+h^0(\widetilde{E}_0{}^*\otimes\widetilde{R}_0)+h^1(\mathrm{End}\ \widetilde{R}_0)$$

where we used sequence (6.3) Again from sequence (6.3) and from the fact that g is surjective we have

$$\dim_{t_0}T\geq h^1(\widetilde{E}_0{}^*\otimes\widetilde{R}_0)=h^1(\mathrm{End}\ \widetilde{E}_0)$$

as we wanted.

Remark 6.6 In the case of pullback bundles $E_{0,0,t}$ there is a simpler proof of the above theorem following the lines of [DS]. In fact in this case considering any finite morphism $\pi:\mathbb{P}^5 \to \mathbb{P}^5$ of degree d^5 we have $H^1(\mathrm{End}\ E_{0,0,t})=H^1(\pi^*\mathrm{End}\ E_{0,0,1})=H^1(\mathrm{End}\ E_{0,0,1}\otimes\pi_*\mathcal{O})$ and from the formulas given in [DS] it follows

$$h^1(\mathrm{End}\ E_{0,0,t})=h^1(\mathrm{End}\ E_{0,0,1})+\left[\binom{d+5}{5}-6\right]h^1(\mathrm{End}\ E_{0,0,1}(-1))=27+\left[\binom{d+5}{5}-6\right]6=6\binom{d+5}{5}-9$$

From the fact that the cohomology module $H^2(E_{0,0,1}(*))$ is concentrated in one degree it is easy to prove as in [DS] that given two finite morphisms π, π' as above and given a parent bundle E then $\pi^*E\simeq\pi'^*E$ if and only if there exists $\sigma\in\mathrm{Aut}(\mathbb{P}^4)$ such that $\pi=\sigma\circ\pi'$ and $\sigma^*E\simeq E$. It follows that the family of bundles obtained pulling back a parent bundle by any finite morphism of degree d^5 has dimension equal to {dimension variety of morphisms of degree d^5}$-${dimension symmetry group of the parent bundle} $=$ $\{6\binom{d+5}{5}-1\}-\{8\}=6\binom{d+5}{5}-9$ (same number as before!), and hence this family fills up a smooth open subset of a irreducible component of the moduli space of stable bundles containing $E_{0,0,d}$. We will repeat these computations in the more general setting of theorem 7.1.

Proof of theorem 1 From the proofs of 6.4, 6.5 and from lemma 5.3 it follows that any relation bundle $\widetilde{E}$ determines a unique weighted quotient bundle $\widetilde{Q}$. Then it is sufficient to show that when the second Chern class $3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2$ is fixed one can find α, β, γ satisfying $\alpha+\beta<\gamma$ such that $f(\alpha,\beta,\gamma):= h^0(\mathcal{H}(\gamma))-h^0(\mathcal{H}\otimes\mathcal{H})+h^0(\wedge^2\mathcal{H}\otimes\mathcal{H}(-\gamma))-h^0(\wedge^3\mathcal{H}\otimes\mathcal{H}(-2\gamma))$ is arbitrarily big.

Starting from an integral solution $(\alpha_0,\beta_0,\gamma_0)$ of the equation $3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=t$ (it exists by [Hor2], see also the proof of corollary 6.14) one can check with easy computations that $(\alpha_n,\beta_n,\gamma_n)$ is a integral solution for every even n, where

$$\alpha_n=\alpha_0$$

$$\beta_n=(\frac{\alpha_0}{4}+\frac{\beta_0}{2}+\frac{\sqrt{3}}{4}\gamma_0)(2+\sqrt{3})^n+(\frac{\alpha_0}{4}+\frac{\beta_0}{2}-\frac{\sqrt{3}}{4}\gamma_0)(2-\sqrt{3})^n$$

$$\gamma_n=(\frac{\sqrt{3}}{6}\alpha_0+\frac{\sqrt{3}}{3}\beta_0+\frac{\gamma_0}{2})(2+\sqrt{3})^n+(-\frac{\sqrt{3}}{6}\alpha_0-\frac{\sqrt{3}}{3}\beta_0+\frac{\gamma_0}{2})(2-\sqrt{3})^n$$

In order to check that this solution is integer, we recall that

$$\sqrt{3}[(2+\sqrt{3})^n-(2-\sqrt{3})^n]\equiv 0 \pmod 6 \quad \forall n\in\mathbb{N}$$
$$(2+\sqrt{3})^n+(2-\sqrt{3})^n\equiv 0 \pmod 4 \text{ for every even } n$$

It is straightforward to verify that if $n\gg 0$ then $\alpha_n+\beta_n<\gamma_n$ and $\lim_{n\to+\infty} f(\alpha_n,\beta_n,\gamma_n) = +\infty$, q.e.d.

N.Manolache kindly communicated to us the minimal resolution of a parent bundle E in terms of $SL(W)\times|\mathbb{Z}_2$ representations. In the next theorem we show how to compute the minimal resolution of E in terms of SL(W)-representations using [BaS]. This weaker statement will be sufficient for our purposes (e.g. for the computation of $h^1(\mathrm{End}\,\widetilde{E})$, see section 7).

Theorem 6.7 (Manolache)
Let $\mathbb{P}^5=\mathbb{P}(W\oplus W^*)$. The minimal SL(W)-invariant resolution of a parent bundle E on $\mathbb{P}^5$ is
$0\to L_4\otimes\mathcal{O}(-7)\to L_3\otimes\mathcal{O}(-6)\to L_{21}\otimes\mathcal{O}(-5)\oplus\mathcal{O}(-4)\to L_{11}\otimes\mathcal{O}(-4)\oplus L_{12}\otimes\mathcal{O}(-3)\to$
$\to L_{01}\otimes(-3)\oplus L_{02}\otimes\mathcal{O}(-2)\to E\to 0$
where $L_4=\Gamma^{2,1}W$, $L_3=[S^2W\oplus S^2W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*]$,
$L_{21}=[S^3W\oplus S^3W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus\Gamma^{4,2}W\oplus\Gamma^{2,1}W]$, $L_{11}=[\Gamma^{4,1}W\oplus\Gamma^{4,1}W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*]$,
$L_{12}=[W\oplus W^*]$, $L_{01}=\Gamma^{4,2}W$, $L_{02}=\Gamma^{2,1}W$

Proof Let E be the cohomology of the monad

$$\mathcal{O}(-1)\to B\to\mathcal{O}(1)$$

We have the following presentation of B:

$$H_{\nu_3}\otimes\mathcal{O}(-1)$$
$$\downarrow u$$
$$0 \to B \xrightarrow{s} H_{\nu_3}\otimes\mathcal{O}(1) \xrightarrow{t} H_{\nu_2}\otimes\mathcal{O}(2)$$

The 14×14 matrix of $s\circ u$ is (1.2). Computing its first syzygies with [BaS] and transposing we obtain the matrix of t which is

$$\left[\begin{array}{cccccccccccccc}
 & & b & -d & & & & & & & & & -f & e \\
-f & & & & & & & & & & d & & b & c \\
 & & -c & & d & & & & & & & -f & e & \\
-e & & & & & & & & & d & & b & c & \\
 & & & & -c & & & d & & -e & & a & & \\
 & & & c & & & -d & & & -f & & & a & \\
-a & & & & & & & & d & b & c & & & \\
 & & & & & & c & -b & -e & a & & & & \\
 & & & & & -c & b & & -f & & a & & & \\
 & -c & & & a & & e & f & & & & & & \\
 & -b & & a & & e & f & & & & & & & \\
 & & & c & -b & & & & & -f & e & & & \\
 & & & -b & & d & & & & & -f & & & a \\
 & -d & a & e & f & & & & & & & & &
\end{array}\right]$$

We use now the fact that given $\omega:\mathbb{C}^6\backslash 0 \to \mathbb{C}^6\backslash 0$ as in (0.1) then the pullback $\omega^*\eta^*$ of the minimal resolution of $E_{0,0,1}$ descends on $\mathbb{P}^5$ to the minimal resolution of $E_{\alpha,\beta,\gamma}$. In particular every summand $\Gamma^{a,b}W\otimes(\det Q)^{\otimes t}$ in the minimal resolution of $E_{0,0,1}$ becomes a summand $\Gamma^{a,b}\mathcal{W}\otimes(\det \tilde{Q})^{\otimes t}$ in the minimal resolution of $E_{\alpha,\beta,\gamma}$.

Replacing a,b,c,d,e,f by polynomials $f_1, \ldots, f_6$ of degrees $\gamma-\alpha$, $\gamma-\beta$, $\gamma+\alpha+\beta$, $\gamma+\alpha$, $\gamma+\beta$, $\gamma-\alpha-\beta$ the degrees of the rows of the above matrix are $2\gamma+\beta$, $2\gamma-\alpha+\beta$, $2\gamma-\alpha-\beta$, $2\gamma-2\alpha-\beta$, $2\gamma-\alpha-2\beta$, 2γ , $2\gamma-\alpha$, $2\gamma-\beta$, $2\gamma+\alpha+\beta$, $2\gamma+\alpha-\beta$, $2\gamma+2\alpha+\beta$, 2γ, $2\gamma+\alpha+2\beta$, $2\gamma+\alpha$ while the degrees of the columns are $\gamma-2\alpha$, $\gamma+2\alpha$, γ, $\gamma+\alpha+\beta$, $\gamma-\beta$, $\gamma+2\alpha+2\beta$, $\gamma+\alpha$, $\gamma-2\beta$, γ, $\gamma-\alpha-\beta$, $\gamma+\beta$, $\gamma-2\alpha-2\beta$, $\gamma-\alpha$, $\gamma+2\beta$. Let R be defined by

$$0 \to R \to B \xrightarrow{h} \mathcal{O}(1) \to 0$$

where h is given by the sum of the rows number 3 and 9 of the matrix (1.2). Then we have the presentation

$$0 \to R \to [\mathbb{C}\oplus\mathbb{C}\oplus S^2W\oplus S^2W^*]\otimes\mathcal{O}(1) \xrightarrow{t'} [W\oplus W^*\oplus\Gamma^{2,1}W]\otimes\mathcal{O}(2)\oplus\mathcal{O}(1)$$

where the matrix of t' is obtained adding at the bottom of the matrix of t the row (0,0,1,0,0,0,0,0,1,0,0,0,0,0). As in [DMS], the resolution of t' gives

$$\mathcal{O}(-1)\oplus\mathcal{O}(-2)^8\oplus\mathcal{O}(-3)^{27}\to R\to 0$$

where the 14×8 matrix of the composition

$$\mathcal{O}(-2)^8\to\mathcal{O}(-1)\oplus\mathcal{O}(-2)^8\oplus\mathcal{O}(-3)^{27}\to[C\oplus C\oplus S^2W\oplus S^2W^*]\otimes\mathcal{O}(1)$$

is

$$\begin{bmatrix}
-bce-c^2f & b^2e+bcf & -c^2d & 0 & 2b^2d & -2bcd & 0 & 0\\
0 & 0 & ae^2 & 0 & -2af^2 & -2aef & -2bef-2cf^2 & be^2+cef\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac12aef & \frac12af^2 & ade-\frac12cef & \frac12bef+\frac12cf^2 & -bf^2 & -adf+cf^2 & -bdf & bde+\frac12cdf\\
-\frac12ae^2 & -\frac12aef & \frac12ce^2 & -\frac12be^2-\frac12cef & bef-2adf & -ade-cef & -bde-2cdf & \frac12cde\\
-a^2f & 0 & abe-a^2d & -abf & 0 & -2abf & -2b^2f & b^2e-abd\\
\frac12a^2e & -\frac12a^2f & \frac12ace & \frac12abe-\frac12acf & -abf & a^2d-acf & abd\text{-}2bcf & bce-\frac12acd\\
0 & a^2e & 0 & ace & 2a^2d-2acf & 0 & 2acd-2c^2f & c^2e\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac12ace & abe+\frac12acf & -\frac12c^2e & \frac12bce+\frac12c^2f & 2abd-bcf & c^2f-acd & bcd & \frac12c^2d\\
-\frac12abe-acf & \frac12abf & \frac12bce-acd & -\frac12b^2e-\frac12bcf & b^2f & -abd-bcf & -b^2d & -\frac12bcd\\
-ce^2 & be^2-ade & 0 & -cde & 2bde-2ad^2 & -2cde & -2cd^2 & 0\\
\frac12ade-cef & bef-\frac12adf & -\frac12cde & \frac12bde-\frac12cdf & bdf & ad^2-cdf & bd^2 & -\frac12cd^2\\
adf-cf^2 & bf^2 & ad^2-cdf & bdf & 0 & 0 & 0 & bd^2
\end{bmatrix}$$

In this matrix the degrees of the rows are $\gamma-2\alpha$, $\gamma+2\alpha$, γ, $\gamma+\alpha+\beta$, $\gamma-\beta$, $\gamma+2\alpha+2\beta$, $\gamma+\alpha$, $\gamma-2\beta$, γ, $\gamma-\alpha-\beta$, $\gamma+\beta$, $\gamma-2\alpha-2\beta$, $\gamma-\alpha$, $\gamma+2\beta$ hence we can compute the degrees of the columns which are $-2\gamma-\alpha+\beta$, $-2\gamma-2\alpha-\beta$, $-2\gamma+\alpha+2\beta$, -2γ, $-2\gamma-\alpha-2\beta$, -2γ, $-2\gamma+\alpha-\beta$, $-2\gamma+2\alpha+\beta$, i.e. exactly the integers appearing in $\Gamma^{2,1}\mathcal{W}\otimes\mathcal{O}(-2\gamma)$ (and $\Gamma^{2,1}W$ is the only representation $\Gamma^{a,b}W$ which gives $\Gamma^{a,b}\mathcal{W}$ with the required splitting). From the columns of the 14×27 matrix obtained by the composition

$$\mathcal{O}(-3)^{27}\to\mathcal{O}(-1)\oplus\mathcal{O}(-2)^8\oplus\mathcal{O}(-3)^{27}\to[C\oplus C\oplus S^2W\oplus S^2W^*]\otimes\mathcal{O}(1)$$

we can find the degrees appearing in $\Gamma^{4,2}W$. Continuing in this way we can find all the resolution.

Corollary 6.8 Let $E_{\alpha,\beta,\gamma}$ be a relation bundle on $\mathbb{P}^5$ coming as pullback over $\mathbb{C}^6\backslash 0$. Let $\mathcal{W}=\mathcal{O}(\alpha)\oplus\mathcal{O}(\beta)\oplus\mathcal{O}(-\alpha-\beta)$. The minimal resolution of $E_{\alpha,\beta,\gamma}$ is

$$0\rightarrow\Gamma^{2,1}\mathcal{W}(-7\gamma)\rightarrow[S^2\mathcal{W}\oplus S^2\mathcal{W}^*\oplus\Gamma^{3,1}\mathcal{W}\oplus\Gamma^{3,1}\mathcal{W}^*](-6\gamma)\rightarrow$$
$$\rightarrow[S^3\mathcal{W}\oplus S^3\mathcal{W}^*\oplus\Gamma^{3,1}\mathcal{W}\oplus\Gamma^{3,1}\mathcal{W}^*\oplus\Gamma^{4,2}\mathcal{W}\oplus\Gamma^{2,1}\mathcal{W}](-5\gamma)\oplus\mathcal{O}(-4\gamma)\rightarrow$$
$$\rightarrow[\Gamma^{4,1}\mathcal{W}\oplus\Gamma^{4,1}\mathcal{W}^*\oplus\Gamma^{3,1}\mathcal{W}\oplus\Gamma^{3,1}\mathcal{W}^*](-4\gamma)\oplus[\mathcal{W}\oplus\mathcal{W}^*](-3\gamma)\rightarrow$$
$$\rightarrow\Gamma^{4,2}\mathcal{W}(-3\gamma)\oplus\Gamma^{2,1}\mathcal{W}(-2\gamma)\rightarrow E_{\alpha,\beta,\gamma}\rightarrow 0$$

Theorem 6.9 Let $\widetilde{E}=E_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. Then $h^0(\widetilde{E}(t))\neq 0$ if and only if $\min\{2\gamma-\alpha-2\beta,\ 3\gamma-2\alpha-4\beta\}\leq t$

Proof From corollary 6.8 it is easy to check that $h^0(\widetilde{E}(t))\neq 0$ if and only if $h^0(\Gamma^{4,2}\mathcal{W}(-3\gamma+t)\oplus\Gamma^{2,1}\mathcal{W}(-2\gamma+t))\neq 0$. Now consider that the sum of the degrees in the Young diagram, according to (0.2)

$\alpha+\beta$	$\alpha+\beta$
$-\alpha$	

is $\alpha+2\beta$, while the sum of the degrees in

$\alpha+\beta$	$\alpha+\beta$	$\alpha+\beta$	$\alpha+\beta$
$-\alpha$	$-\alpha$		

is $2\alpha+4\beta$.

Corollary 6.10 Let $\widetilde{E}$ be a relation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. The following are equivalent

i) $\widetilde{E}$ is *stable*

ii) $\widetilde{E}$ is *simple*

iii) $3\gamma-2\alpha-4\beta>0$

Proof i) $\Rightarrow$ ii) is well known

ii)$\Rightarrow$iii) if $3\gamma-2\alpha-4\beta\leq 0$ then $h^0(E_{\alpha,\beta,\gamma})\neq 0$ and $h^0(E^*_{\alpha,\beta,\gamma})\neq 0$ from the theorem.

iii) $\Rightarrow$ i) if $3\gamma-2\alpha-4\beta>0$ then $h^0(E_{\alpha,\beta,\gamma})=0$ and $h^0(\wedge^2E_{\alpha,\beta,\gamma})=h^0(E^*_{\alpha,\beta,\gamma})=0$ from the theorem.

Corollary 6.11 Let $\widetilde{E}$ be a relation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. The following are equivalent

i) $\widetilde{E}$ is semistable

ii) $3\gamma-2\alpha-4\beta\geq 0$

Theorem 6.12 Let $3\gamma-2\alpha-4\beta\leq 0$. Then any relation bundle $E_{\alpha,\beta,\gamma}$ is ***unstable.***

Proof Let $E_{\alpha,\beta,\gamma}$ be the cohomology of the monad

$$\mathcal{O}(-\gamma) \to B_{\alpha,\beta,\gamma} \xrightarrow{\tau} \mathcal{O}(\gamma)$$

Let Ker $\tau = R_{\alpha,\beta,\gamma}$. It is sufficient to prove that $H^0(R_{\alpha,\beta,\gamma}) \neq 0$, or equivalently that the composition

$$H^0(\wedge^3\mathcal{H}(-\gamma)) \to H^0(B_{\alpha,\beta,\gamma}) \xrightarrow{H^0(\tau)} H^0(\mathcal{O}(\gamma))$$

is nonzero. The corresponding morphism $\wedge^3\mathcal{H}(-\gamma) \to \mathcal{O}(\gamma)$ is given by 20 homogeneous polynomials $g_1, \ldots g_{20}$, and up to permutations we may suppose that g_1 has degree $-2\alpha-2\beta+2\gamma$ and g_2 has degree $-2\beta+2\gamma$. The map $H^0(\wedge^3\mathcal{H}(-\gamma)) \to H^0(\mathcal{O}(\gamma))$ is given by $(f_1,\ldots,f_{20}) \mapsto \sum f_i g_i$ where deg $f_1=2\alpha+2\beta-\gamma$, deg $f_2=2\beta-\gamma$. If $g_1=g_2=0$ it is clear that the morphism is nonzero. Otherwise we can take $f_1=g_2 a^{2\alpha+4\beta-3\gamma}$, $f_2=-g_1 a^{2\alpha+4\beta-3\gamma}$, $f_3=\ldots=f_{20}=0$.

Remark 6.13 The proof of theorem 6.12 shows in the same way that if $t<\gamma$ and $2\alpha+4\beta-3\gamma+t\leq 0$ then any relation bundle $E_{\alpha,\beta,\gamma}$ satisfies $h^0(E_{\alpha,\beta,\gamma}(t))\neq 0$. In particular $h^0(E_{0,t-1,t}(-t))\neq 0$ for $t\geq 1$, hence all the bundles $E_{0,t-1,t}$ are "strongly unstable". On the other side the pullback bundles $E_{0,0,t}$ satisfy $h^0(E_{0,0,t}(t))=0$, so they are "strongly stable".

Corollary 6.14 Let $t>0$, $t\equiv 0,3,8$ or 11 (mod 12). There exists a *semistable* $E_{\alpha,\beta,\gamma}$ such that $c_2(E_{\alpha,\beta,\gamma})=t$.

Let $t>0$, $t\equiv 3,8$ or 11 (mod 12). There exists a *stable* $E_{\alpha,\beta,\gamma}$ such that $c_2(E_{\alpha,\beta,\gamma})=t$.

Proof Choosing $\alpha=n-3$, $\beta=n$, $\gamma=2n-2$ for $n\geq 3$ we have

$$3\gamma-2\alpha-4\beta=0$$
$$c_2=3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=12(n-2)$$

Choosing $\alpha=n-2$, $\beta=n$, $\gamma=2n-1$ for $n\geq 2$ we have

$$3\gamma-2\alpha-4\beta=1$$
$$c_2=3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=12(n-1)-1$$

Choosing $\alpha=n-1$, $\beta=n$, $\gamma=2n$ for $n\geq 1$ we have

$$3\gamma-2\alpha-4\beta=2$$
$$c_2=3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=12n-4$$

Choosing $\alpha=\beta=n$, $\gamma=2n+1$ for $n\geq 0$ we have

$$3\gamma-2\alpha-4\beta=3$$
$$c_2=3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=12n+3$$

Remark 6.15 For $c_2=24$ do not exist any stable $E_{\alpha,\beta,\gamma}$ while for $c_2=12$ the pullback with $\alpha=\beta=0$, $\gamma=2$ is stable. A computer checking of values of k such that there exists a stable $E_{\alpha,\beta,\gamma}$ with $c_2=12k$ shows that for $k\leq 100$ the only gaps are k=2, 10, 14, 26, 34, 70.

Remark 6.16 There are no semistable $E_{\alpha,\beta,\gamma}$ with $c_2=0$.

Proof of Theorem 2 We will prove a little bit more, that is that the number of components

goes to infinity even in the range where $\widetilde{Q}$ is stable, that is we will prove that the number $N(t):=\#\{(\alpha,\beta,\gamma)|3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=t,\ \gamma>5\alpha+5\beta\}$ satisfies $\limsup N(t)=+\infty$. Let ϵ, x_0 be such that $8e^{1+2\epsilon}\leq 27$ and $x\cdot\ln(1+\frac{1}{x})\geq 1-\epsilon$ for $x\geq x_0$. It is sufficient to check that if $x\geq x_0$ then $\#\{(\alpha,\beta,\gamma)|3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=12x^x(x+1)^x,\ \gamma>5\alpha+5\beta\}\geq\frac{x}{6}-2$. For every integer a such that $\frac{x}{3}\leq a\leq\frac{x}{2}$ we set $A:=(x+1)^{x-a}x^a$, $B:=x^{x-a}(x+1)^a$, $\alpha=\beta=\frac{A-B}{2}$, $\gamma=A+B$. These choices of a are at least $\frac{x}{6}-2$. Now we observe that in order to have α, β nonnegative we need $A\geq B$ which is equivalent to $(x+1)^{x-2a}\geq x^{x-2a}$ which is satisfied because $a\leq\frac{x}{2}$. We get
$3\gamma^2+4\alpha\beta-4(\alpha+\beta)^2=3(A+B)^2+(A-B)^2-4(A-B)^2=12AB=12(x+1)^x x^x$ as we wanted.

The inequality $5\alpha+5\beta<\gamma$ remains to be checked. It is equivalent to $2A<3B$, that is

$$\left(\frac{x+1}{x}\right)^{x-a}<\frac{3}{2}\left(\frac{x+1}{x}\right)^a$$

or $\left(1+\frac{1}{x}\right)^x<\frac{3}{2}\left(1+\frac{1}{x}\right)^{2a}$

It is sufficient to verify $e\leq\frac{3}{2}\left(1+\frac{1}{x}\right)^{2a}$ that is $\ln\frac{2e}{3}\leq 2a\ln(1+\frac{1}{x})$

$$a\geq\frac{\ln(2e/3)}{2\ln(1+1/x)}$$

It is sufficient to check $\frac{x}{3}\geq\frac{\ln(2e/3)}{2\ln(1+1/x)}$ and this is true by the choices of ϵ and x_0.

7. The computation of $h^1(\mathrm{End}\ E_{\alpha,\beta,\gamma})$

Theorem 7.1 Let $E_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. Then

$$h^1(\mathrm{End}\ E_{\alpha,\beta,\gamma}) = \sum_j(-1)^j h^0[\Lambda^j\mathcal{H}(-\gamma)\otimes\mathcal{H}(\gamma)] + \sum_j(-1)^j h^0\{\Lambda^j\mathcal{H}(-\gamma)\otimes[\mathcal{W}\otimes\mathcal{W}\oplus\mathcal{W}\otimes\mathcal{W}^*\oplus\mathcal{W}^*\otimes\mathcal{W}^*]\} + \sum_j(-1)^j h^0\{\Lambda^j\mathcal{H}(-\gamma)\otimes[S^3\mathcal{W}(-\gamma)\oplus S^3\mathcal{W}^*(-\gamma)\oplus\Gamma^{3,1}\mathcal{W}(-\gamma)\oplus\Gamma^{3,1}\mathcal{W}^*(-\gamma)\oplus(\Gamma^{2,1}\mathcal{W}(-\gamma))^2\oplus \Gamma^{4,2}\mathcal{W}(-2\gamma)\oplus\Gamma^{4,1}\mathcal{W}(-2\gamma)\oplus\Gamma^{4,1}\mathcal{W}^*(-2\gamma)\oplus\Gamma^{3,1}\mathcal{W}(-2\gamma)\oplus\Gamma^{3,1}\mathcal{W}^*(-2\gamma)\oplus \Gamma^{5,2}\mathcal{W}(-3\gamma)\oplus\Gamma^{5,2}\mathcal{W}^*(-3\gamma)\oplus\Gamma^{6,3}\mathcal{W}(-4\gamma)]\}$$

We postpone to the end the proof of theorem 7.1.

For practical purposes, the following formula is more useful

Corollary 7.2 Let $E_{\alpha,\beta,\gamma}$ be a relation bundle coming as pullback over $\mathbb{C}^6\backslash 0$. Then

$$h^1(\mathrm{End}\ E_{\alpha,\beta,\gamma}) = h^0(\mathcal{H}(\gamma))-h^0(\mathrm{End}\mathcal{W})-h^0(\mathcal{O}(4\alpha+4\beta-2\gamma))-h^0(\mathcal{O}(4\alpha-2\gamma))-h^0(\mathcal{O}(4\beta-2\gamma))- h^0(\mathcal{O}(3\alpha+\beta-2\gamma))-h^0(\mathcal{O}(\alpha+3\beta-2\gamma))-h^0(\mathcal{O}(3\alpha+2\beta-2\gamma))-h^0(\mathcal{O}(2\alpha+3\beta-2\gamma))+h^0(3\alpha-2\gamma))+ h^0(\mathcal{O}(3\beta-2\gamma))+h^0(\mathcal{O}(3\alpha+3\beta-2\gamma))+h^0(\mathcal{O}(4\alpha+\beta-2\gamma))+h^0(\mathcal{O}(\alpha+4\beta-2\gamma))+h^0(\mathcal{O}(3\beta-\alpha-2\gamma))+ h^0(\mathcal{O}(4\alpha+3\beta-2\gamma))+h^0(\mathcal{O}(3\alpha+4\beta-2\gamma))+h^0(\mathcal{O}(4\alpha+4\beta-3\gamma))+h^0(\mathcal{O}(4\alpha-3\gamma))+h^0(\mathcal{O}(4\beta-3\gamma))+ h^0(\mathcal{O}(5\alpha+2\beta-3\gamma))+h^0(\mathcal{O}(2\alpha+5\beta-3\gamma))+h^0(\mathcal{O}(5\alpha+3\beta-3\gamma))+h^0(\mathcal{O}(3\alpha+5\beta-3\gamma))- h^0(\mathcal{O}(4\alpha+2\beta-3\gamma))-h^0(\mathcal{O}(2\alpha+4\beta-3\gamma))+h^0(\mathcal{O}(\alpha+4\beta-3\gamma))+h^0(\mathcal{O}(4\alpha+\beta-3\gamma))+ h^0(\mathcal{O}(3\alpha+6\beta-5\gamma)+h^0(\mathcal{O}(6\alpha+3\beta-5\gamma))+h^0(\mathcal{O}(4\alpha+8\beta-6\gamma))+h^0(\mathcal{O}(8\alpha+4\beta-6\gamma))- h^0(\mathcal{O}(8\alpha+3\beta-7\gamma))-h^0(\mathcal{O}(3\alpha+8\beta-7\gamma))-h^0(\mathcal{O}(8\alpha+5\beta-7\gamma))-h^0(\mathcal{O}(5\alpha+8\beta-7\gamma))$$

Proof Substituting $\mathcal{H}=\mathcal{W}\oplus\mathcal{W}^*$ in the formula of the theorem 7.1 and applying the Littlewood-Richardson rule (see the lemma 7.5) we get $h^1(\text{End } E_{\alpha,\beta,\gamma}) = h^0(\mathcal{H}(\gamma))-h^0(\text{End } \mathcal{W})+h^0[(\Gamma^{4,1}\mathcal{W}\oplus\Gamma^{4,1}\mathcal{W}^*\oplus S^3\mathcal{W}\oplus S^3\mathcal{W}^*)(-2\gamma)]-h^0[(S^4\mathcal{W}\oplus S^4\mathcal{W}^*)(-2\gamma)]+ h^0[(S^4\mathcal{W}\oplus S^4\mathcal{W}^*\oplus\Gamma^{5,2}\mathcal{W}\oplus\Gamma^{5,2}\mathcal{W}^*)(-3\gamma)]-h^0[(\Gamma^{4,2}\mathcal{W})(-3\gamma)]+$
$+h^0(\Gamma^{6,3}\mathcal{W}(-5\gamma))+h^0(\Gamma^{8,4}\mathcal{W}(-6\gamma))- h^0[(\Gamma^{8,3}\mathcal{W}\oplus\Gamma^{8,3}\mathcal{W}^*)(-7\gamma)]$. Note that after the "principal part", which is $h^0(\mathcal{H}(\gamma))-h^0(\text{End } \mathcal{W})$, both the correction terms with $-\gamma$ and -4γ vanish. If we expand all the terms and simplify, we find that many summands are zero because of the inequality $\alpha+\beta<\gamma$.

Corollary 7.3 The component of the moduli space of stable 3-bundles with Chern classes $c_1=c_3=0$, $c_2=3\gamma^2-4\alpha\beta-4(\alpha+\beta)^2$ containing a relation bundle $E_{\alpha,\beta,\gamma}$ is smooth at points corresponding to bundles coming as pullback over $\mathbb{C}^6\backslash 0$; its dimension is
$h^0(\mathcal{H}(\gamma))-h^0(\text{End } \mathcal{W})- h^0(\mathcal{O}(4\alpha+4\beta-2\gamma))-h^0(\mathcal{O}(4\beta-2\gamma))-h^0(\mathcal{O}(3\alpha+2\beta-2\gamma))-$
$h^0(\mathcal{O}(2\alpha+3\beta-2\gamma))+h^0(\mathcal{O}(3\alpha+3\beta-2\gamma))+h^0(\mathcal{O}(4\alpha+\beta-2\gamma))+h^0(\mathcal{O}(\alpha+4\beta-2\gamma))+$
$h^0(\mathcal{O}(4\alpha+3\beta-2\gamma))+h^0(\mathcal{O}(3\alpha+4\beta-2\gamma))+h^0(\mathcal{O}(4\alpha+4\beta-3\gamma))+h^0(\mathcal{O}(5\alpha+3\beta-3\gamma))+$
$h^0(\mathcal{O}(3\alpha+5\beta-3\gamma))+ h^0(\mathcal{O}(5\alpha+2\beta-3\gamma))+h^0(\mathcal{O}(2\alpha+5\beta-3\gamma))$

Proof Apply theorem 6.5 and consider that $3\gamma-2\alpha-4\beta>0$ by corollary 6.10. Then many summands in the formula of corollary 7.2 are zero.

Corollary 7.4 Let $\gamma>2\alpha+2\beta$. Then the component of the moduli space of stable 3-bundles with Chern classes $c_1=c_3=0$, $c_2=3\gamma^2-4\alpha\beta-4(\alpha+\beta)^2$ containing a relation bundle $E_{\alpha,\beta,\gamma}$ is smooth at points corresponding to bundles coming as pullback over $\mathbb{C}^6\backslash 0$ of dimension

$$h^0(\mathcal{H}(\gamma))-h^0(\text{End } \mathcal{W})$$

In particular the fibres of the map {moduli space of $E_{\alpha,\beta,\gamma}$} → {moduli space of $B_{\alpha,\beta,\gamma}$} have dimension $h^0(S^2\mathcal{W})+h^0(S^2\mathcal{W}^*)+1$.

Before proving the theorem we need some lemmas. For the convenience of the reader we recall first some formulas that are obtained as a straightforward application of the Littlewood-Richardson rule.

Lemma 7.5 Let W be a 3-dimensional vector space. The following decompositions of SL(W)-representations are true

$S^nW\otimes W=S^{n+1}W\oplus\Gamma^{n,1}W \qquad S^nW\otimes W^*=\Gamma^{n+1,1}W\oplus S^{n-1}W \quad \forall n\geq 1$
$\Gamma^{a,b}W\otimes W=\Gamma^{a+1,b}W\oplus\Gamma^{a,b+1}W\oplus\Gamma^{a-1,b-1}W$ for $0<b<a$
$\Gamma^{a,b}W\otimes W^*=\Gamma^{a+1,b+1}W\oplus\Gamma^{a,b-1}W\oplus\Gamma^{a-1,b}W$ for $0<b<a$
$S^nW\otimes S^2W=S^{n+2}W\oplus\Gamma^{n+1,1}W\oplus\Gamma^{n,2}W \qquad S^nW\otimes S^2W^*=\Gamma^{n+2,2}W\oplus\Gamma^{n,1}W\oplus S^{n-2}W \quad \forall n\geq 1$
$\Gamma^{2,1}W\otimes S^2W=\Gamma^{4,1}W\oplus S^2W\oplus\Gamma^{3,2}W$

$\Gamma^{2,1}W\otimes S^3W=\Gamma^{5,1}W\oplus\Gamma^{4,2}W\oplus\Gamma^{2,1}W\oplus S^3W$
$\Gamma^{2,1}W\otimes\Gamma^{2,1}W=\Gamma^{4,2}W\oplus(\Gamma^{2,1}W)^2\oplus S^3W\oplus S^3W^*\oplus C$
$\Gamma^{3,1}W\otimes\Gamma^{2,1}W=\Gamma^{5,2}W\oplus S^4W\oplus(\Gamma^{3,1}W)^2\oplus\Gamma^{4,1}W^*\oplus S^2W^*\oplus W$
$\Gamma^{4,2}W\otimes\Gamma^{2,1}W=\Gamma^{6,3}W\oplus\Gamma^{5,4}W\oplus\Gamma^{5,1}W\oplus S^3W\oplus S^3W^*\oplus(\Gamma^{4,2}W)^2\oplus\Gamma^{2,1}W$
$\Gamma^{4,1}W\otimes\Gamma^{2,1}W=\Gamma^{6,2}W\oplus S^5W\oplus\Gamma^{4,1}W\oplus\Gamma^{5,3}W\oplus\Gamma^{3,2}W\oplus\Gamma^{4,1}W\oplus S^2W$
$\Gamma^{5,2}W\otimes\Gamma^{2,1}W=\Gamma^{7,3}W\oplus\Gamma^{6,1}W^*\oplus\Gamma^{6,2}W^*\oplus\Gamma^{5,2}W\oplus\Gamma^{4,1}W^*\oplus\Gamma^{5,2}W\oplus S^4W\oplus\Gamma^{3,1}W$

Lemma 7.6 The following decompositions of tensor products between Sp(6)-representations are true

$H_{\nu_1}\otimes H_{\nu_1}=H_{2\nu_1}\oplus H_{\nu_2}\oplus C$ $\quad$ $H_{\nu_1}\otimes H_{\nu_2}=H_{\nu_1+\nu_2}\oplus H_{\nu_1}\oplus H_{\nu_3}$
$H_{\nu_1}\otimes H_{\nu_3}=H_{\nu_1+\nu_3}\oplus H_{\nu_2}$ $\quad$ $H_{\nu_2}\otimes H_{\nu_3}=H_{\nu_1+\nu_2}\oplus H_{\nu_2+\nu_3}\oplus H_{\nu_1}$
$H_{\nu_1}\otimes H_{\nu_1+\nu_3}=H_{2\nu_1+\nu_3}\oplus H_{\nu_2+\nu_3}\oplus H_{\nu_1+\nu_2}\oplus H_{\nu_3}$
$H_{\nu_2}\otimes H_{\nu_2}=H_{2\nu_2}\oplus H_{2\nu_1}\oplus H_{\nu_1+\nu_3}\oplus H_{\nu_2}\oplus C$
$H_{\nu_3}\otimes H_{\nu_3}=H_{2\nu_3}\oplus H_{2\nu_2}\oplus H_{2\nu_1}$
Proof [Lit]

Lemma 7.7 Let B a lambda-three bundle
$H^0(B(1))=H_{\nu_3}=S^2W\oplus S^2W^*\oplus C\oplus C$
$H^0(B(2))=H_{\nu_1+\nu_3}= S^3W\oplus S^3W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus\Gamma^{2,1}W\oplus W^2\oplus W^{*2}$
$H^0(B(3))=H_{2\nu_1+\nu_3}=S^4W\oplus S^4W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus\Gamma^{4,1}W\oplus\Gamma^{4,1}W^*\oplus(S^2W\oplus S^2W^*)^3\oplus(\Gamma^{4,2}W\oplus \Gamma^{2,1}W)^2\oplus C^2$
where the right-hand sides are the restrictions to SL(W) of the Sp(6)-representations on the left.
Proof Bott theorem gives all the left equalities. A quick way to obtain the decompositions in terms of SL(W)-representations is to start from $H_{\nu_1}=W\oplus W^*$, $H_{\nu_2}=W\oplus W^*\oplus\Gamma^{2,1}W$, $H_{\nu_3}=S^2W\oplus S^2W^*\oplus C\oplus C$ and then apply the previous lemma, the Littlewood-Richardson rule for SL(W)-representations and cancel the extra terms.

Lemma 7.8
Let B a lambda-three bundle.
$H^0(\text{ad }B(1))=0$
$H^0(\text{ad }B(2))=H_{2\nu_2}\oplus H_{2\nu_3}=[(S^2W\oplus S^2W^*)^2\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus\Gamma^{4,2}W\oplus\Gamma^{2,1}W\oplus C]\oplus [S^4W\oplus S^4W^*\oplus(S^2W\oplus S^2W^*)^2\oplus\Gamma^{4,2}W\oplus C^3]$

Proof We have End $B=\mathcal{O}\oplus$ad B where ad $B=E^{2\nu_2}(-2)\oplus E^{2\nu_3}(-2)$. Then argue as in lemma 7.7.

Lemma 7.9 Let E be a parent bundle.

$H^0(\mathrm{End}\ E(2))=H^0(\mathcal{O}(2))=S^2W\oplus S^2W^*\oplus\Gamma^{2,1}W\oplus C$

$H^1(\mathrm{End}\ E(2))=\Gamma^{4,2}W\oplus\Gamma^{4,1}W\oplus\Gamma^{4,1}W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus S^2W\oplus S^2W^*$

Proof We consider the exact sequence

$$0\to R\otimes B(2)\to \mathrm{End}\ B(2)\to B(3)\to 0 \tag{7.1}$$

We have from lemmas 7.7 and 7.8

$H^0(\mathrm{End}\ B(2)) = H_{2\nu_1}\oplus H_{2\nu_2}\oplus H_{2\nu_3} = [S^2W\oplus S^2W^*\oplus\Gamma^{2,1}W\oplus C]\oplus$
$[(S^2W\oplus S^2W^*)^2\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus\Gamma^{4,2}W\oplus\Gamma^{2,1}W\oplus C]\oplus$
$[S^4W\oplus S^4W^*\oplus(S^2W\oplus S^2W^*)^2\oplus\Gamma^{4,2}W\oplus C^3]$

$H^0(\mathrm{End}\ B(2))\subset H^0(B(1))\otimes H^0(B(1))= H_{\nu_3}\otimes H_{\nu_3}$ = (by lemma 7.6) $H_{2\nu_1}\oplus H_{2\nu_2}\oplus H_{2\nu_3}\oplus C =$
$H^0(\mathrm{End}\ B(2))\oplus C$

$H^0(B(3)) = H_{2\nu_1+\nu_3} =$
$S^4W\oplus S^4W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus\Gamma^{4,1}W\oplus\Gamma^{4,1}W^*\oplus(S^2W\oplus S^2W^*)^3\oplus(\Gamma^{4,2}W\oplus\Gamma^{2,1}W)^2\oplus C^2$

The main point is that the restriction of the morphism $H^0(\mathrm{End}\ B(2))\to H^0(B(3))$ to the summand $\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*$ is an isomorphism, and then $\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*$ does not appear as summand in $H^0(R\otimes B(2))$. We have $H_{\nu_3}\otimes H_{\nu_3} = [S^2W\oplus S^2W^*\oplus C\oplus C]\otimes[S^2W\oplus S^2W^*\oplus C\oplus C]$. The summand $\Gamma^{3,1}W$ appears in the above tensor product as a summand of $S^2W\oplus S^2W$, indeed $S^2W\otimes S^2W=S^4W\oplus\Gamma^{3,1}W\oplus S^2W^*$ (more precisely $\Gamma^{3,1}W=\wedge^2(S^2W)$). If v_1, $v_2\in W$ then we have $(v_1\otimes v_2+v_2\otimes v_1)\in S^2W$, $v_1\otimes v_1\in S^2W$ and

$$(v_1\otimes v_2+v_2\otimes v_1)\otimes(v_1\otimes v_1)-(v_1\otimes v_1)\otimes(v_1\otimes v_2+v_2\otimes v_1)\in\Gamma^{3,1}W\subset W\otimes W\otimes W\otimes W$$

Looking at the matrix (1.2) we choose v_1 of degree α, v_2 of degree β. The morphism $B(1)\to \mathcal{O}(2)$ is given by the sum of the rows number 3 and 9.

Then $(v_1\otimes v_2+v_2\otimes v_1)\in H^0(B(1))$ corresponds to the 4^{th} row
$(bd,0,0,-\frac{1}{2}f^2,0,\frac{1}{2}ef,\frac{1}{2}af,-ae,ab,-\frac{1}{2}cf,\frac{1}{2}bd,de,\frac{1}{2}df,0)$ and $(v_1\otimes v_2+v_2\otimes v_1)\in H^0(B(1))\subset H^0(\mathcal{O}(2))$ corresponds to the quadratic polynomial ab.

$v_1\otimes v_1\in H^0(B(1))$ corresponds to the 2^{nd} row
$(ad+be+cf,0,0,0,0,0,0,0,a^2,ae,af,e^2,ef,f^2)$ and $v_1\otimes v_1\in H^0(B(1))\subset H^0(\mathcal{O}(2))$ corresponds to the quadratic polynomial a^2. Putting together
$(v_1\otimes v_2+v_2\otimes v_1)\otimes(v_1\otimes v_1)-(v_1\otimes v_1)\otimes(v_1\otimes v_2+v_2\otimes v_1)\in\Gamma^{3,1}W\subset H^0(B(3))$ corresponds to the row
$ab(ad+be+cf,0,0,0,0,0,0,0,a^2,ae,af,e^2,ef,f^2)-a^2(bd,0,0,-\frac{1}{2}f^2,0,\frac{1}{2}ef,\frac{1}{2}af,-ae,ab,-\frac{1}{2}cf,\frac{1}{2}bd,de,\frac{1}{2}df,0)\neq 0$
It follows that $H^0(R\otimes B(2))$ does not contain the summand $\Gamma^{3,1}W$ and in the same way we can prove that it does not contain the summand $\Gamma^{3,1}W^*$. Then from the sequences (2.2), (2.3), (2.4) for t=2 we get that $H^0(\mathrm{End}\ R(2))$, $H^0(E^*\otimes R(2))$ and $H^0(\mathrm{End}\ E(2))$ do not contain $\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*$ either. From (7.1) it follows easily that $H^0(R\otimes B(2))$ does not contain the summand $W\oplus W^*$ and again from the

sequences (2.2), (2.3), (2.4) for $t=2$, $H^0(\mathrm{End}\ E(2))$ does not contain $W\oplus W^*$either.

Now we tensor by $E^*(2)$ the minimal resolution of E of theorem 6.7 and we get

$$0\to L_4\otimes E^*(-5)\to L_3\otimes E^*(-4)\to L_{21}\otimes E^*(-3)\oplus E^*(-2)\to$$
$$\to L_{11}\otimes E^*(-2)\oplus L_{12}\otimes E^*(-1)\to L_{01}\otimes E^*(-1)\oplus L_{02}\otimes E^*\to \mathrm{End}\ E(2)\to 0$$

Set $\mathcal{A}:=\mathrm{Ker}\ L_{01}\otimes E^*(-1)\oplus L_{02}\otimes E^*\to \mathrm{End}\ E(2)$.

$\mathfrak{B}:=\mathrm{Ker}\ L_{11}\otimes E^*(-2)\oplus L_{12}\otimes E^*(-1)\to L_{01}\otimes E^*(-1)\oplus L_{02}\otimes E^*$

Then we have

$H^0(\mathfrak{B})=H^1(\mathfrak{B})=H^3(\mathfrak{B})=0,\ H^2(\mathfrak{B})=\mathbb{C}\oplus L_3\oplus L_4$

$$0\to L_{12}\to H^1(\mathcal{A})\to \mathbb{C}\oplus L_3\oplus L_4\to L_{11}\to H^2(\mathcal{A})\to 0 \tag{7.2}$$

$$0\to H^0(\mathrm{End}\ E(2))\to H^1(\mathcal{A})\to L_{01}\oplus L_{02}\otimes(W\oplus W^*)\to H^1(\mathrm{End}\ E(2))\to H^2(\mathcal{A})\to 0 \tag{7.3}$$

From these two sequences we obtain $H^0(\mathrm{End}\ E(2))\subset\mathbb{C}\oplus L_3\oplus L_4\oplus L_{12} = \mathbb{C}\oplus S^2W\oplus S^2W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus\Gamma^{2,1}W\oplus W\oplus W^*$.

Moreover $H^0(\mathrm{End}\ E(2))\supset H^0(\mathcal{O}(2))=S^2(W\oplus W^*)=S^2W\oplus S^2W^*\oplus\Gamma^{2,1}W\oplus\mathbb{C}$. Since we proved that $H^0(\mathrm{End}\ E(2))$ does not contain $\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus W\oplus W^*$ it follows $H^0(\mathrm{End}\ E(2))=S^2W\oplus S^2W^*\oplus\Gamma^{2,1}W\oplus\mathbb{C}$. Finally $H^1(\mathrm{End}\ E(2))$ can be found by the sequences (7.2) and (7.3).

It is useful to recall the Riemann-Roch formula

$$\chi(\mathrm{End}\ E(t)) = \frac{3}{40}\,(t^5+15t^4+45t^3-135t^2-566t-240) \tag{7.4}$$

Lemma 7.10 Let E be a parent bundle.

$H^0(\mathrm{End}\ E(3))=H^0(\mathcal{O}(3))\oplus S^2W\oplus S^2W^*$

$H^1(\mathrm{End}\ E(3))=S^3W\oplus S^3W^*\oplus\Gamma^{5,2}W\oplus\Gamma^{5,2}W^*$

Proof From Beilinson theorem it follows as in [DMS] that any parent bundle is the cohomology of a monad

$$\mathcal{O}(-1)\oplus\wedge^4Q^*\to Q^*\oplus\wedge^2Q^*\to\mathcal{O}^6$$

Let K (resp. K′) be the kernel bundle of the monad corresponding to E (resp. E^*). In particular we have the two exact sequences

$$0\to\mathcal{O}(-1)\oplus\wedge^4Q^*\to K\to E\to 0 \tag{7.5}$$

$$0\to\mathcal{O}(-1)\oplus\wedge^4Q^*\to K'\to E^*\to 0 \tag{7.6}$$

After easy computations we have:

$h^0(K(1))=h^0(K'(1))=7,\ h^0(K(2))=h^0(K'(2))=49$

$h^1(K(1))=h^1(K'(1))=8,\ h^1(K(2))=h^1(K'(2))=0$

Tensoring (7.6) by $E^*(3)$ and (7.5) by $K'(3)$ and using the Euler sequence

$$0 \to \mathcal{O}(1) \to \mathcal{O}(2)^6 \to \wedge^4 Q^*(3) \to 0$$

we get

$h^0(\mathrm{End}\ E(3)) = h^0(K \otimes K'(3)) - h^0(K'(2)) - h^0(K' \otimes \wedge^4 Q^*(3)) - h^0(E(2)) - h^0(E \otimes \wedge^4 Q^*(3)) =$
$h^0(K \otimes K'(3)) - 49 - 295 - 8 - 56$

that is

$$h^0(\mathrm{End}\ E(3)) = h^0(K \otimes K'(3)) - 408 \tag{7.7}$$

From the exact sequence

$$0 \to K \to Q^* \oplus \wedge^2 Q^* \to \mathcal{O}^6 \to 0$$

and the minimal resolutions of Q^* and $\wedge^2 Q^*$ we get the exact sequence

$$0 \to K \to \mathcal{O}^6 \oplus \mathcal{O}^{15} \overset{\psi}{\to} \mathcal{O}(1) \oplus \mathcal{O}(1)^6 \oplus \mathcal{O}^6$$

where the matrix of ψ is given by (see [DMS]):

a	b	c	d	e	f															
										−f				−e			−d		−c	−b
									−f				−e			−d		−c		a
								−f				−e			−d			b	a	
							−f				−e				c	b	a			
						−f					d	c	b	a						
						e	d	c	b	a										
1						1														
	1						−1													
		1									1									
			1															1		
				1															−1	
					1															1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

In the same way we have

$$0 \to K' \to \mathcal{O}^6 \oplus \mathcal{O}^{15} \overset{\psi'}{\to} \mathcal{O}(1) \oplus \mathcal{O}(1)^6 \oplus \mathcal{O}^6$$

where the matrix of ψ' is given by:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
a	b	c	d	e	f															
										−f				−e			−d		−c	−b
									−f				−e			−d		−c		a
								−f				−e			−d			b	a	
							−f				−e				c	b	a			
						−f					d	c	b	a						
						e	d	c	b	a										
−1						1														
	−1						−1													
		−1									1									
			1															1		
				1															−1	
					1															1

Tensoring we get the exact sequence

$$0 \to K\otimes K' \to \mathcal{O}^{441} \xrightarrow{(\psi\otimes \mathrm{id},\mathrm{id}\otimes\psi')} (\mathcal{O}^{126}\oplus\mathcal{O}(1)^{147})^{\oplus 2}$$

Using [BaS] we can compute the syzygies of $(\psi\otimes\mathrm{id},\mathrm{id}\otimes\psi')$ and we obtain

$$h^0(K\otimes K'(1))=0$$

$$h^0(K\otimes K'(2))=50$$

$$h^0(K\otimes K'(3))=476$$

hence by (7.7)

$$h^0(\mathrm{End}\ E(3))=476-408=68$$

$$h^0(\mathrm{ad}\ E(3)) = 68-56 = 12$$

By (7.4) $\chi(\mathrm{End}\ E(3))=h^0(\mathrm{End}\ E(3))-h^1(\mathrm{End}\ E(3))=-36$,
this implies

$$h^1(\mathrm{End}\ E(3))=104$$

Now we tensor by $E^*(3)$ the minimal resolution of E in the theorem 6.7 and we set

$$\mathcal{C}:=\mathrm{Ker}\ L_{01}\otimes E^*\oplus L_{02}\otimes E^*(1) \to \mathrm{End}\ E(3).$$

$$\mathcal{D}:=\mathrm{Ker}\ L_{11}\otimes E^*(-1)\oplus L_{12}\otimes E^* \to L_{01}\otimes E^*\oplus L_{02}\otimes E^*(1)$$

We get the following exact sequences:

$$0 \to H^0(\mathrm{End}\ E(3)) \to H^1(\mathcal{C}) \to [L_{01}\otimes(W\oplus W^*)]\oplus[L_{02}\otimes\Gamma^{21}W] \to H^1(\mathrm{End}\ E(3)) \to 0$$

$$0 \to H^1(\mathcal{D}) \to L_{11}\oplus L_{12}\otimes(W\oplus W^*) \to H^1(\mathcal{C}) \to H^2(\mathcal{D}) \to 0$$

$$0 \to \mathbb{C} \to H^1(\mathcal{D}) \to L_4 \to L_{21} \to H^2(\mathcal{D}) \to 0$$

Using lemma 7.5 we have

$L_{01}\otimes(W\oplus W^*)=\Gamma^{5,2}W\oplus\Gamma^{5,2}W^*\oplus\Gamma^{4,1}W\oplus\Gamma^{4,1}W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*$
$L_{02}\otimes\Gamma^{2,1}W=\Gamma^{4,2}W\oplus S^3W\oplus S^3W^*\oplus(\Gamma^{2,1}W)^2\oplus\mathbb{C}$

It is easy to verify that $\Gamma^{5,2}W\oplus\Gamma^{5,2}W^*$ does not appear in $H^1(\mathcal{C})$, then we obtain $H^1(\text{End } E(3))=\Gamma^{5,2}W\oplus\Gamma^{5,2}W^*\oplus J$ where $\dim J=h^1(\text{End } E(3))-\dim\Gamma^{5,2}W\oplus\Gamma^{5,2}W^* = 104-84 = 20$. From the sequences above the only possibility is $J=S^3W\oplus S^3W^*$. The result for $H^0(\text{End } E(3))$ follows in the same way.

Remark 7.11 In principle it is possible to determine $h^0(\text{End } E(3))$ computing syzygies with [BaS] directly from the presentation obtained by the minimal resolution of E, that is

$$0\to E\otimes E^*\to(\mathcal{O}(2)^8\oplus\mathcal{O}(3)^{27})^{\otimes 2}\to\ldots$$

This computation requires much more computer-memory than the one performed in lemma 7.10.

Remark 7.12 As in [Hor2] if E is a parent bundle on $\mathbb{P}(W\oplus W^*)$ which is SL(W)-invariant we have $E|_{\mathbb{P}(W)}\simeq \text{ad } T\mathbb{P}(W)$.

Lemma 7.13 $H^0(\text{End } E|_{\mathbb{P}(W)})=\mathbb{C}$ $\quad H^1(\text{End } E|_{\mathbb{P}(W)})=S^3W$
$H^0(\text{End } E(1)|_{\mathbb{P}(W)})=W^*\oplus S^2W$ $\quad H^1(\text{End } E(t)|_{\mathbb{P}(W)})=0$ for $t\geq 1$
$H^0(\text{End } E(2)|_{\mathbb{P}(W)})=S^2W^*\oplus\Gamma^{3,1}W\oplus S^4W$
$H^0(\text{End } E(3)|_{\mathbb{P}(W)})=S^3W^*\oplus\Gamma^{4,2}W\oplus\Gamma^{5,1}W$
$H^0(\text{End } E(4)|_{\mathbb{P}(W)})=S^4W^*\oplus\Gamma^{5,3}W\oplus\Gamma^{6,2}W$

Proof Tensoring the exact sequence

$$0\to\mathcal{O}(-3)\to W^*\otimes(-2)\to W\otimes\mathcal{O}(-1)\to\mathcal{O}\to\mathcal{O}|_{\mathbb{P}(W)}\to 0$$

by End E(t) and using Bott theorem over $\mathbb{P}(W)$

Lemma 7.14 The only summand $\Gamma^{a,b}W$ of $H^1(\text{End } E(4))$ with $a\geq 5$ is $\Gamma^{6,3}W$

Proof Tensoring by $E^*(4)$ the minimal resolution of E and cutting into short exact sequences one obtains:
(set $\mathcal{I}:=\text{Ker }[\Gamma^{4,2}W\otimes E^*(1)\oplus\Gamma^{2,1}W\otimes E^*(2)\to\text{End } E(4)]$)

$$0\to H^0(\mathcal{I})\to L_{21}\oplus W\oplus W^*\oplus L_3\to L_{11}\otimes(W\oplus W^*)\oplus L_{12}\otimes\Gamma^{2,1}W\to H^1(\mathcal{I})\to 0$$

$$H^1(\mathcal{I})\to L_{01}\otimes\Gamma^{2,1}W\to H^1(\text{End } E(4))\to 0$$

The only summands $\Gamma^{a,b}W$ of $L_{01}\otimes\Gamma^{2,1}W$ with $a\geq 5$ are $\Gamma^{6,3}W$, $\Gamma^{5,1}W$, $\Gamma^{5,1}W^*$. From the first sequence one sees that $\Gamma^{5,1}W$, $\Gamma^{5,1}W^*$ appear both in $H^1(\mathcal{I})$ while $\Gamma^{6,3}W$ does not appear in $H^1(\mathcal{I})$ and then it must be a summand of $H^1(\text{End } E(4))$.

In order to exclude the summand $\Gamma^{5,1}W$ we consider the exact sequence (see remark 7.12 and lemma 7.13)

$$0\to\text{End } E(1)\to\text{End } E(2)\otimes W^*\to\text{End } E(3)\otimes W\to\text{End } E(4)\to\text{End } E(4)|_{\mathbb{P}(W)}\to 0$$

From lemmas 7.11 and 7.5 we find that $\Gamma^{5,1}W$ does not appear in $H^1(\text{End } E(3))\otimes W$, hence does not appear in $H^1(\text{End } E(4)\otimes \mathcal{I}_{\mathbb{P}(W),\mathbb{P}^5})$ either. Consider the exact sequence (see lemma 7.13)

$$H^1(\text{End } E(4)\otimes \mathcal{I}_{\mathbb{P}(W),\mathbb{P}^5}) \to H^1(\text{End } E(4)) \to S^4W\oplus\Gamma^{5,3}W\oplus\Gamma^{6,2}W$$

it follows that $H^1(\text{End } E(4))$ does not contain the summand $\Gamma^{5,1}W$, as we claimed. The same argument, restricting to $\mathbb{P}(W^*)$, shows that $H^1(\text{End } E(4))$ does not contain the summand $\Gamma^{5,1}W^*$either.

Proof of theorem 7.1

We will apply the formula

$$\bigoplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^5, \text{End } E_{\alpha,\beta,\gamma}(t)) \simeq \bigoplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^5,\text{End } E_{0,0,1}(t))\otimes_{\mathbb{C}} \frac{\mathbb{C}[a,b,c,d,e,f]}{(f_1,f_2,f_3,f_4,f_5,f_6)}$$

and then we need the module structure of $\bigoplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^5,\text{End } E_{0,0,1}(t))$ in terms of SL(W)-representations. We denote by E a parent bundle.

In the lemma 2.2 we already computed $H^1(\text{End } E(t))$ for $t\leq 0$. In the same way one can show $H^1(\text{End } E(1))=S^3W\oplus S^3W^*\oplus\Gamma^{3,1}W\oplus\Gamma^{3,1}W^*\oplus(\Gamma^{2,1}W)^2\oplus W\oplus W^*$, $H^2(\text{End } E(1))=0$. From the minimal resolution of the theorem 6.7 one can check that $H^1(\text{End } E(t))=0$ for $t\geq 5$. The necessary computations of $h^1(\text{End } E(t))$ for $t=2,3,4$ are done respectively in lemmas 7.9, 7.10 and 7.14.

Appendix

In the following *table* 1 we collect some numerical informations on the components $M_{\alpha,\beta,\gamma}$ of the moduli space of stable 3-bundles on $\mathbb{P}^5$ containing a relation bundle $E_{\alpha,\beta,\gamma}$ with second Chern class ≤ 50. In the *table* 2 we list some informations about the interesting case $c_1=c_2=c_3=0$. All the values are obtained by the formulas of theorems 6.9 and 7.2.

Table 1

c_2	(α,β,γ)	dim $M_{\alpha,\beta,\gamma}$	maximum t such that $h^0(E_{\alpha,\beta,\gamma}(t))=0$ for $E_{\alpha,\beta,\gamma}$ coming as pullback over $\mathbf{C}^6\backslash 0$
3	(0,0,1)	27	1
8	(0,1,2)	130	1
11	(0,2,3)	471	0
12	(0,0,2)	117	3
15	(1,1,3)	427	2
20	(1,2,4)	1171	1
23	(0,1,3)	370	3
	(1,3,5)	2814	0
24	——		
27	(0,0,3)	327	5
	(2,2,5)	2604	2
32	(0,2,4)	1047	3
	(2,3,6)	5342	1
36	(1,1,4)	981	4
39	(0,3.5)	2545	2
	(3,3,7)	9700	2
44	(0,1,4)	832	5
	(0,4,6)	5474	1
	(3,4,8)	16,901	1
47	(0,5,7)	10,756	0
	(1,2,5)	2343	4
	(3,5,9)	28,382	0
48	(0,0,4)	747	7

$c_2=71$ is the first value where there are four components: (α,β,γ) is respectively (0,1,5), (1,6,9), (2,3,7). (5,7,13) .

Table 2 (α,β,γ)	$c_2=0$ dimension of the base of the versal deformation at $E_{\alpha,\beta,\gamma}$ pullback over $\mathbb{C}^6\backslash 0$	$\max\{t \mid h^0(E_{\alpha,\beta,\gamma}(t))=0\}$ (for any $E_{\alpha,\beta,\gamma}$)
(1,22,26)	5,444,021	−13
(2,11,14)	297,555	−7
(4,22,28)	7,076,165	−13
(11,26,38)	23,100,774	−13
(2,44,52)	148,201,315	−25
...	...	...

(the list is infinite)

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