Polynomial Interpolation and Sums of Powers

SIAM Conference on Applied Algebraic Geometry, Fort Collins, August 2013
Colorado State University MS76, Formulas in Interpolation

Chiara Brambilla, Giorgio Ottaviani[1]

Università di Firenze[1]

Interpolation in one variable

The first result on polynomial interpolation goes back to Newton and Lagrange.

Given distinct points $x_1, \ldots, x_{d+1} \in \mathbf{A}^1$ there is a unique polynomial $p(x)$ of degree $\leq d$ which has assigned values y_i on these points, namely

$$p(x_i) = y_i$$

Hermite interpolation

The Lagrange interpolation can be generalized by considering derivatives.

Given distinct points $x_1, \ldots, x_k \in \mathbf{A}^1$, one looks for a polynomial p of degree $\leq d$ satisfying

$$p^{(j)}(x_i) = y_{i,j} \quad \text{for } j = 0, \ldots d_i - 1$$

where $y_{i,j}$ are given scalars.
The assigned $\sum_{i=1}^{k} d_i$ conditions are independent, so that there is a unique solution of deg $\leq d$ if $d + 1 = \sum_{i=1}^{k} d_i$.

Many variables

The extension to many variables is more difficult. The space $R_{d,n}$ of polynomials of degree $\leq d$ in n variables have dimension $\binom{n+d}{d}$. First of all, the conditions of assigned values at distinct points may be dependent.

Example If $d+2$ among the points x_i are on a line then in general there is no solution, unless the assigned values y_i satisfy special conditions.

Luckily, if the points x_i are general and their number is $\leq \binom{n+d}{d}$, then there is a polynomial p of degree $\leq d$ satisfying $p(x_i) = y_i$ for every assigned values y_i.

Classical geometers of XIX century expressed this fact by saying that *"general points impose independent conditions"*.

Many variables and derivatives

If we assign the values of partial derivatives, up to some order, at some points, again we need that the points are general; indeed for points on a line we meet the same phenomenon as in previous slide. Something new here happens. *Even if the points are general, there may be situations where there is no solution to the interpolation problem.*

Alexander and Hirschowitz solved the case where the values of *all* the first partial derivatives at some general points are assigned.

The Alexander-Hirschowitz Theorem

Theorem (Alexander-Hirschowitz)

Let $d \neq 2$ and $char(K) = 0$. For a general choice of points p_i $i = 1, \ldots, k$ the space of polynomials having assigned values of all the first partial derivatives at p_i has codimension in $R_{d,n}$ equal to

$$\min\{(n+1)k, \dim R_{d,n}\}$$

with the following list of exceptions

$$\begin{aligned} a)\ & n = 2,\ d = 4,\ k = 5, \\ b)\ & n = 3,\ d = 4,\ k = 9, \\ c)\ & n = 4,\ d = 3,\ k = 7, \\ d)\ & n = 4,\ d = 4,\ k = 14. \end{aligned}$$

The first exceptional case: two points on the plane

The first nontrivial exception is the following. Consider $k = 2$ and $(a_1, a_2) = (n, n)$. Then the interpolating affine space is given by quadratic polynomials with assigned tangent spaces A_1, A_2 at two points p_1, p_2. The interpolating space is not empty if and only if the intersection space $A_1 \cap A_2$ is not empty and its projection on K^n contains the midpoint of $p_1 p_2$, which is a codimension one condition. In order to prove this fact restrict to the line through p_1 and p_2 and use a well known property of the tangent lines to the parabola.

The orthogonal space to the space of polynomials of degree d, having a double point at $(a_0, \ldots, a_n)$, is the degree d part of the ideal generated by $(a_0x_0 + \ldots + a_nx_n)^{d-1}$. This remark and the Terracini Lemma allow to reformulate the Alexander-Hirschowitz theorem, in terms of sum of powers, which is the version more widely known (see [Iarrobino-Kanev]).

Originally Alexander and Hirschowitz worked on the interpolation problem!

The Alexander-Hirschowitz Theorem in terms of sums of powers

Theorem (Alexander-Hirschowitz)

Let $d \neq 2$ and $char(K) = 0$. The variety of polynomials in $R_{d,n}$ which are sum of k powers of linear forms has dimension equal to

$$\min\{(n+1)k, \dim R_{d,n}\}$$

with the following list of exceptions

$$\begin{aligned} a)\ & n=2,\ d=4,\ k=5,\\ b)\ & n=3,\ d=4,\ k=9,\\ c)\ & n=4,\ d=3,\ k=7,\\ d)\ & n=4,\ d=4,\ k=14, \end{aligned}$$

where the corresponding varieties are hypersurfaces.

Beyond Alexander-Hirschowitz

With Chiara Brambilla we studied the case of any general linear combination of first partial derivatives. (Asking just some partial derivatives is "coordinate dependent")

So we assign a general subspace of dimension d_i inside $\langle \frac{\partial}{\partial x_1} \ldots \frac{\partial}{\partial x_n} \rangle$

General setting

Let $p_1, \ldots, p_k \in K^n$ be k general points and assume that over each of these points a general affine proper subspace $A_i \subset K^n \times K$ is given. Assume that $\dim A_1 \geq \ldots \geq \dim A_k$. Let $\Gamma_f \subseteq K^n \times K$ be the graph of $f \in R_{d,n}$ and $T_{p_i}\Gamma_f$ be its tangent space at the point $(p_i, f(p_i))$. Consider the conditions

$$A_i \subseteq T_{p_i}\Gamma_f, \text{ for } i = 1, \ldots, k$$

Explanation of the conditions.

- When $\dim A_i = 0$, the assumption (11) means that the value of f at p_i is assigned.
- When $\dim A_i = n$, (11) means that the value of f at p_i and the values of all first partial derivatives of f at p_i are assigned.
- In the intermediate cases, (11) means that the value of f at p_i and the values of some linear combinations of first partial derivatives of f at p_i are assigned.

The interpolation space.

Consider now the affine space

$$V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k) = \{f \in R_{d,n} | A_i \subseteq T_{p_i}\Gamma_f, \quad i = 1, \ldots, k\} \tag{1}$$

we may call the *interpolation space* with data p_i, A_i, because its elements correspond to interpolating polynomials.

The Main Theorem

Theorem (Brambilla-O)

Let $d \neq 2$ and $char(K) = 0$. For a general choice of points p_i and subspaces A_i, the interpolation space $V_{d,n}(p_1, \ldots, p_k, A_1, \ldots, A_k)$ has codimension in $R_{d,n}$ equal to

$$\min\{\sum_{i=1}^{k}(\dim A_i + 1), \dim R_{d,n}\}$$

with the following list of exceptions

$$
\begin{array}{ll}
a) & n = 2,\ d = 4,\ k = 5,\ \dim A_i = 2 \ for\ i = 1, \ldots 5 \\
b) & n = 3,\ d = 4,\ k = 9,\ \dim A_i = 3 \ for\ i = 1, \ldots 9 \\
b') & n = 3,\ d = 4,\ k = 9,\ \dim A_i = 3 \ for\ i = 1, \ldots 8, \dim A_9 = 2 \\
c) & n = 4,\ d = 3,\ k = 7,\ \dim A_i = 4 \ for\ i = 1, \ldots 7 \\
d) & n = 4,\ d = 4,\ k = 14,\ \dim A_i = 4 \ for\ i = 1, \ldots 14
\end{array}
$$

The perfect case

In particular, when $\sum_{i=1}^{k}(a_i+1)=\binom{n+d}{d}$ there is a unique polynomial f in $V_{d,n}(p_1,\ldots,p_k,A_1,\ldots A_k)$, with the above exceptions a), b'), c), d).

In the exceptional cases the space $V_{d,n}(p_1,\ldots,p_k,A_1,\ldots A_k)$ is empty.

The case of quadrics

Assume now $d = 2$. We set $a_i = -1$ for $i > k$. For any $1 \leq i \leq n$ we denote

$$\delta_{a_1,\ldots,a_k}(i) = \max\{0, \sum_{j=1}^{i} a_j - \sum_{j=1}^{i}(n+1-j)\}$$

Theorem (Brambilla-O)

Let K be an infinite field. For a general choice of points p_i and subspaces A_i of dimension a_i, the interpolation space $V_{2,n}(p_1,\ldots,p_k,A_1,\ldots A_k)$ has codimension in $R_{2,n}$ equal to $\min\{\sum_{i=1}^{k}(a_i+1), \dim R_{2,n}\}$ if and only if one of the following conditions takes place:

1. *either $\delta_{a_1,\ldots,a_k}(i) = 0$ for all $1 \leq i \leq n$;*
2. *or $\sum_i (a_i+1) \geq \binom{n+2}{2} + \max\{\delta_{a_1,\ldots,a_k}(i) : 1 \leq i \leq n\}$.*

The perfect case for quadrics

In particular when $\sum_{i=1}^{k}(a_i+1)=\binom{n+2}{2}$ there is a unique polynomial f in $V_{2,n}(p_1,\ldots,p_k,A_1,\ldots A_k)$ if and only if, for any $1\leq i\leq n$, we have

$$\sum_{j=1}^{i} a_j \leq \sum_{j=1}^{i}(n+1-j).$$

Table for three variables

With two variables, the only exceptional case for quadrics is when X is given by two double points. In the 3dimensional space, we have the following lists of subschemes which do not impose independent conditions on quadrics.

X	$\deg X$	$\max\{\delta_X(i)\}$	$(m_1, \ldots, m_4)$	$\dim I_X(2)$
4,4,4	12	3	$(0,0,0,3)$	1
4,4,3	11	2	$(0,0,1,2)$	1
4,4,2	10	1	$(0,1,0,2)$	1
4,4,1,1	10	1	$(2,0,0,2)$	1
4,4,1	9	1	$(1,0,0,2)$	2
4,4	8	1	$(0,0,0,2)$	3
4,3,3	10	1	$(0,0,2,1)$	1

Open problem

Study the dimension of the interpolation spaces with assigned values of partial derivatives of higher order.

Many partial results are known (Ciliberto, Miranda, Harbourne, Hirschowitz, Gimigliano, Bocci, Dumitrescu, Carlini, Ballico, Brambilla, Laface, Ugaglia, Orecchia, Cioffi, Chiantini, ...)

Thanks !!