INTRODUCTION TO THE EUCLIDEAN DISTANCE DEGREE

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Abstract. The points at a fixed distance t from an affine variety X in a Euclidean space make an "offset variety", whose algebraic closure is a hypersurface with equation a polynomial in t^2 called E$Dpoly_X(t^2)$ (Euclidean Distance polynomial). Its degree is the Euclidean distance degree, and measures the complexity to compute the closest point lying in X from a given general one. We see the main properties of the Euclidean Distance degree (EDdegree) of an affine variety X. As an application, we describe the d-singular d-ples of tensors.

1. Introduction

Let V be finite dimensional spaces over $\mathbb{R}$ equipped with a nonsingular positive definite quadratic form q. The pair (V, q) is called a Euclidean space.

Let $X \subset V$ be a closed subset. The (Euclidean) distance function from $u \in V$ to X is

$$d_X(u) := \min_{x \in X} \sqrt{q(u-x)}.$$

We are interested mainly in critical points of the distance function d_X. They do not change if we square it as d_X^2. This is an advantage because the square root disappears and we may work with a polynomial function, allowing the tools of Algebraic Geometry.

1.1. The critical points of the distance function from u on X.

- If X is a smooth subvariety, the minimum of the distance from u is attained among the points x such that $T_xX \perp (u-x)$. Checking the distance of all of the critical points guarantees to compute the global distance from u to X.

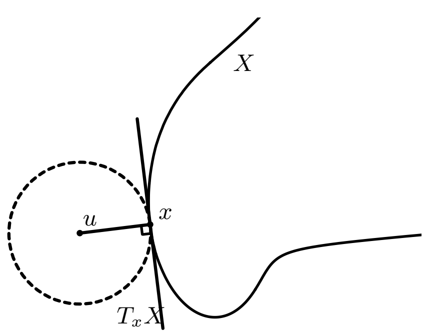

Figure 1. A critical point $x \in X$ for the distance function from u on X.

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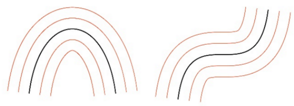
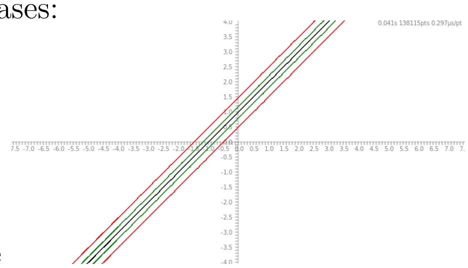
Let X = black curve. The red curves are *offset curves of* X, i.e. loci of points at a given distance from X.

All the curves of the same family have the same evolute.

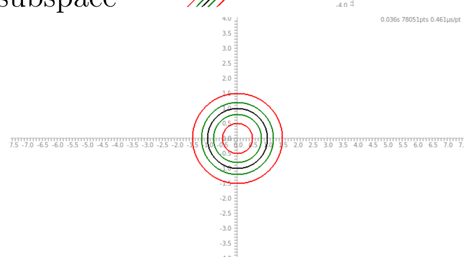


Offset varieties are obtained as envelopes of spheres, centered on the variety itself, they have striking engineering applications, in CAD/CAM manifacturing tools.

Given X, how to compute d_X ?
There are two trivial cases:



1) X = affine subspace



2) X = sphere

Definition 1.1. The Euclidean Distance polynomial $EDpoly_{X,u}(t^2)$ defines the offset hypersurface of all points u with squared distance from X being equal to t^2 and, more generally, its algebraic closure contains all points u having a critical point for the squared distance function from u, with value t^2. If X is algebraic, it can be explicitly computed by elimination, see [OS20] and the following 1.4.

In the next sections 1.2, 1.4 we make explicit the case when X is an affine conic.

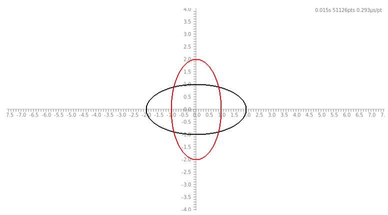
1.2. **The ED polynomial of a conic, a classical story.** For an ellipse E, the distance function was found in XIX century with the help of invariant theory (see [20]).

The following Proposition gives the idea how to attack algebraically the problem. It is an elementary case of a more general result regarding pencil of quadrics. It is instructive to give an elementary direct proof, which needs the reader to revisit the concept of discriminant of a polynomial.

Proposition 1.2. *Let C, D be two smooth conics in the projective plane. The following are equivalent*

- *(i) C and D are not tangent*
- *(ii) the intersection $C \cap D$ is given by 4 distinct points*

- *(iii) the pencil of conics $\alpha C + \beta D$ contains three distinct singular conics.*



Proof. The equivalence of (i) and (ii) follows from Bezout Theorem. Let $C \cap D = \{P_1, P_2, P_3, P_4\}$, note no three of them are collinear by the smoothness of C. (ii) implies (iii) since the three distinct singular conics through P_1, P_2, P_3, P_4 may be constructed directly as

$$\overline{P_1P_2} \cup \overline{P_3P_4}, \quad \overline{P_1P_3} \cup \overline{P_2P_4}, \quad \overline{P_1P_4} \cup \overline{P_2P_3}.$$

In order to prove that (iii) implies (ii) we may assume that C and D meet in $(1, 0, 0)$ with common tangent $x_1 = 0$, then their matrices are

$$A_i = \begin{pmatrix} 0 & a_i & 0 \\ a_i & * & * \\ 0 & * & * \end{pmatrix}$$

for $i = 0, 1$, so that the determinant $\det(\alpha A_0 + \beta A_1)$ contains the factor $(\alpha a_0 + \beta a_1)^2$ against the assumption. □

Cayley computation, in Salmon "Treatise on Conics" (1879 edition)

Let E be the matrix of the ellipse and B the matrix of the circle $(x - u_1)^2 + (y - u_2)^2 - t^2$.

Proposition 1.2 implies that E and B are tangent if and only if the determinant $\det(E + \lambda B)$ has a double root in λ. The discriminant of the polynomial in λ

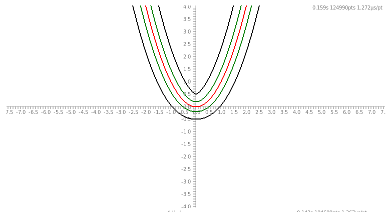
$$\det(E + \lambda B)$$

is $\mathrm{EDpoly}_{E,u}(t^2)$, whose roots correspond to the distances of the critical points considered above, EDpoly stands for Euclidean Distance polynomial.

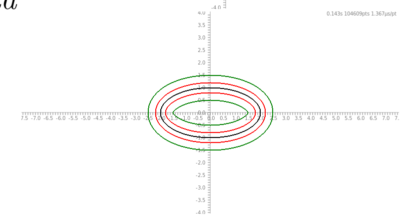
$\mathrm{EDpoly}_{X,u}(t^2)$ allows to compute the distance from X.

Theorem 1.3 (Cayley).
- $\deg \mathrm{EDpoly}_{E,u}(t^2) = 2 \iff$ *E is a circle.*
- $\deg \mathrm{EDpoly}_{E,u}(t^2) = 3 \iff$ *E is a parabola.*
- $\deg \mathrm{EDpoly}_{E,u}(t^2) = 4$ *for all other smooth conics.*

In projective case, EDdegree(conic)=4 unless A has multiple eigenvalues, where EDdegree=2. To be precise, unless $\det(A + \lambda E)$ has multiple eigenvalues, that is unless the conic is tangent to the isotropic quadric. It gives a orthogonally invariant family of projective conics, which has affine forms given by circles and equilateral hyperbolas.



Parabola



Ellipse

Remark 1.4. *Note that, posing* $X_{t_0} = \mathrm{EDpoly}_{X,u}(t_0^2)$, *since distances can be added, then*

$$\mathrm{EDpoly}_{X_{t_0},u}(t^2) = \mathrm{EDpoly}_{X,u}((t+t_0)^2)\mathrm{EDpoly}_{X,u}((t-t_0)^2)$$

The formula is true for small values of t *and, beibg a polynomial identity, it is true for any* t.

1.3. **The dual variety.** In a metric setting, the dual variety $X^\vee$ lives in the same space where X lives, and it can be defined by orthogonality.

Let X be a cone, as given by a projective variety.

$X^\vee := \overline{\bigcup_{y\in X_{sm}} (T_yX)^\perp}$ "union of normal spaces".

A conic with 3×3 matrix A has dual conic with matrix A^{-1}.

The main result on dual varieties is the *Biduality Theorem*

$$(X^\vee)^\vee = X$$

Example 1.5. The dual variety $(\mathbb{P}^n\times\mathbb{P}^m)^\vee$ is the locus of matrices of corank ≥ 1, it is a hypersurface when $n=m$ (square case), namely the *determinant* hypersurface.

For tensors, the dual variety of the Segre variety is a hypersurface when the triangle inequality (1.2) is satisfied, it defines the hyperdeterminant.

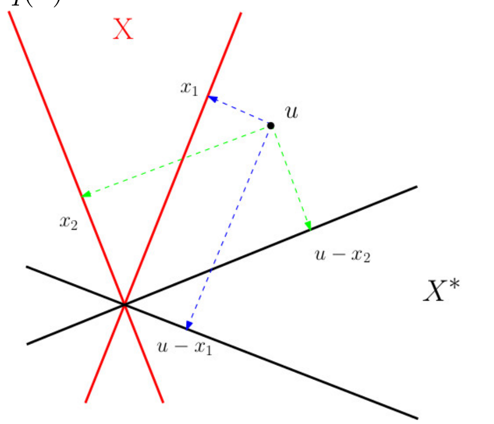
Example 1.6. The dual variety of the Veronese variety $v_d(\mathbb{P}^n)$ is the discriminant hypersurface, whose members correspond to singular hypersurfaces of degree d in $\mathbb{P}^n$. It has degree given by $(n+1)(d-1)^n$, it is classically called the Boole formula.

1.4. **Duality property of ED polynomial and first computations.**

Theorem 1.7 (Draisma-Horobeţ-O-Sturmfels-Thomas, O-Sodomaco)**.** *Let* X *be a projective variety and* $X^\vee$ *its dual. Let* $q(u)$ *be the Euclidean quadratic form. Then for any data point* $u\in V$

$$\mathrm{EDpoly}_{X,u}(t^2) = \mathrm{EDpoly}_{X^\vee,u}(q(u)-t^2).$$

Proof. $x \in X$ is a critical point for d_X at u if and only if $u - x \in X^\vee$ is a critical point for $d_{X^\vee}$ at u, as it is clear from the figure. The squared length of $u - x$ is t^2, then by the Pythagorean Theorem it follows that the squared length of $x = u - (u - x)$ is $q(u) - t^2$.



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The Theorem means that projective duality corresponds to variable reflection for the ED polynomial.

Reformulation of EDdegree Given an affine real variety $X \subset \mathbb{R}^N$, the number of *complex critical regular points* for the distance function from $y \in \mathbb{R}^N$ to X is constant for general $y \in \mathbb{R}^N$.

Definition 1.8 (Draisma, Horobeţ, O., Sturmfels, Thomas)**.** [3] The number of critical points for the distance function from a general $y \in \mathbb{R}^N$ to X is called the Euclidean distance degree of X, denoted by $\mathrm{EDdegree}(X)$.

Indeed the degree of $\mathrm{EDpoly}_{X,u}(t^2)$ is $\mathrm{EDdegree}(X)$ (Horobeţ-Weinstein, [HW]). The reason is clear. Fixing a point u, the positive roots $\{t_1, \ldots, t_k\}$ of $\mathrm{EDpoly}_{X,u}(t^2)$ correspond to the critical points of the squared distance function from u to X. This means that u belongs to the "offset varieties" at distance t_i for $i = 1, \ldots, k$ form X. Here $k = \mathrm{EDdegree}(X)$. By the duality property seen, we get

$$\mathrm{EDdegree}(X) = \mathrm{EDdegree}(X^\vee)$$

Steps to compute the ED polynomial with a *Computer Algebra System* like Macaulay2, Singular, CoCoA, Sage,...

(1) Pick the ring $\mathbb{Q}[u_0, \ldots, u_n, x_0, \ldots, x_n, t]$
(2) Input is the ideal I_X with generators $f = f_1, \ldots, f_m$
(3) Let $c = \mathrm{codim}\ X$
(4) Compute $I_{X_{sing}}$ singular locus, by c-minors of $Jac(f)$.
(5) Compute the critical ideal as $I_u := \left(I_X + (c+1)\text{-minors of } \begin{pmatrix} u - x \\ Jac(f) \end{pmatrix} \right) : \left(I_{X_{sing}}\right)^\infty$
(6) Eliminate $x_0, \ldots, x_n$ in $I_u + (t^2 - q(x - u))$, get $\mathrm{EDpoly}_{X,u}(t^2)$.

The ED polynomial of an ellipse Let X be the projective ellipse with equation $4x^2 + y^2 - z^2 = 0$.

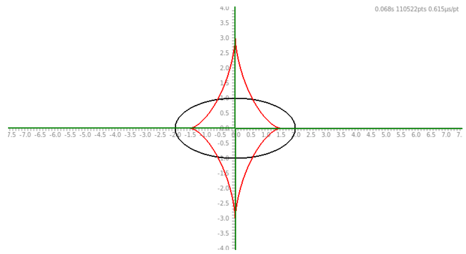

FIGURE 2. The Lamé sextic, evolute of the ellipse, sometimes called an astroid.

Then $\mathrm{EDpoly}_{X,(x,y,z)}(t^2) = (4x^2+y^2-z^2)^2(64x^4+80x^2y^2+25y^4+48x^2z^2-30y^2z^2+9z^4)+$

$(2048x^6+2208x^4y^2+540x^2y^4-25y^6+1184x^4z^2-224x^2y^2z^2+185y^4z^2+324x^2z^4-207y^2z^4+63z^6)(-2t^2)+$

$(6016x^4+2960x^2y^2-275y^4+3312x^2z^2-810y^2z^2+621z^4)t^4+$

$(64x^2+5y^2+21z^2)(-60)t^6+$

$900t^8$.

Given (x,y,z), get 4 roots in t^2, the minimum positive value of t is the distance from X.

We may substitute $t^2 \to -t^2+(x^2+y^2+z^2)$ in the above polynomial $\mathrm{EDpoly}_{X,(x,y,z)}(t^2)$, get

$\mathrm{EDpoly}_{X^\vee,(x,y,z)}(t^2) = (x^2+4y^2-4z^2)^2(4x^4+20x^2y^2+25y^4-12x^2z^2+30y^2z^2+9z^4)+$

$\ldots+$

$900t^8$.

Note the equation in red of the ellipse $(x^2+4y^2-4z^2)$ which is dual to $(4x^2+y^2-z^2)$.

Compute the discriminant of EDpoly.

The discriminant of ED polynomial of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ is

$$L^3x^2y^2,$$

where $c^2=a^2-b^2$, L is the evolute with equation the Lamé sextic (see Figure 1.4)

$$L=(a^2x^2+b^2y^2-c^4)^3+27a^2b^2c^4x^2y^2$$

Note the two symmetry axis x, y appear in the discriminant.

In [OS20, Prop. 2.5] it is proved that the evolute of a curve C always divides the discriminant of $\mathrm{EDpoly}_{C,u}(t^2)$. The EDdiscriminant [3] generalizes the evolute to any variety X.

This is another general phenomenon, the ED polynomial contains informations on the symmetry axis.

1.5. **Catanese-Trifogli formula.** Let X smooth projective, $\dim X = m$

Theorem 1.9 (Catanese-Trifogli). *If X is transversal to Q then*

$$\mathrm{EDdegree}(X) = \sum_{i=0}^{m} (-1)^i (2^{m+1-i} - 1) \deg c_i(X)$$

where c_i are Chern classes.

If X is affine, transversality is needed with both the hyperplane at infinity and the quadric at infinity. This explains the different behaviour proved by Cayley concerning circle, parabola and general conic.

1.6. **Matrices, SVD, Spectral Theorem.** There is a significant case where the ED polynomial has a nice form.

In the space of $n \times m$ matrices equipped with the Bombieri-Weyl norm (according to **) $q(A) = \mathrm{tr}(AA^t)$, let X = variety of corank one matrices.

Proposition 1.10.

$$\mathrm{EDpoly}_{X,A}(t^2) = \det(AA^t - t^2 I)$$

For general matrices A of size $n \times m$, with $n \leq m$, there are n critical points $\sigma_i v_i \otimes w_i$ (singular pairs) of the distance function to the variety of rank one matrices.

$A = \sum_i \sigma_i v_i \otimes w_i^t$ is the *Singular Value Decomposition* (SVD) of A.

If A is a symmetric matrix, we get a splitting

$$\det(AA^t - t^2 I) = \det(A - tI)\det(A + tI), \tag{1.1}$$

the critical points are $v_i v_i^t$ where v_i are eigenvectors of A.

This allows to define the spectrum of u with respect to X the set of roots of $\mathrm{EDpoly}_{X,u}(t^2)$.

We get the spectral decomposition

$$A = \sum_i \lambda_i v_i \otimes v_i^t$$

where λ_i are the eigenvalues of A.

Remark 1.11. *A general symmetric matrix of size n has n critical points of the distance functions to the variety of rank one symmetric matrices, which is a cone over $v_2(\mathbb{P}^{n-1})$. This means that $\mathrm{EDdegree} v_2(\mathbb{P}^{n-1}) = n$ with respect to the Bombieri-Weyl metric. We emphasize that the quadric $Q \subset \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^n)$ of isotropic vectors is not transversal to $v_2(\mathbb{P}^{n-1})$. Indeed Q cuts the Veronese variety in the quartic hypersurface corresponding to q^2 (here q is the quadric in $\mathbb{P}^{n-1}$), which is not reduced. In particular the formula by Catanese-Trifogli in Theorem 1.9 cannot be applied. This formula predicts the value $\frac{3^n-1}{2}$ in this example, which is the correct value with respect to a general metric (note the value n obtained with respect to the Bombieri-Weyl metric is much smaller).*

1.7. **The Eckart-Young Theorem. Best rank k approximation for matrices.**

Theorem 1.12 (Eckart-Young, 1936). *Let* $A = \sum_{i=1}^n \sigma_i u_i \otimes v_i$ *be the SVD of* A, *with* $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$.

Then for any $k = 1, \ldots n-1$, $\sum_{i=1}^k \sigma_i u_i \otimes v_i$ *is the best rank* k *approximation of* A.

Let $X_r = \{M \in M_{n\times t} | \mathrm{rk}(M) \leq r\}$. It is closed, so for matrices, border rank = rank.

Theorem 1.13 (All critical points). *Let* $A = \sum_{i=1}^n \sigma_i u_i \otimes v_i$ *be the SVD of* A. *All critical points of the distance function from* A *to* X_k *are* $\sum_{i\in I_k} \sigma_i u_i \otimes v_i$, *for any* $I_k \subset \{1, \ldots, n\}$ *of cardinality* k. *It follows* $\mathrm{EDdegree}(X_k) = \binom{n}{k}$.

Since $(X_k)^\vee = X_{n-k}$, the duality statement for EDdegree is confirmed.

1.8. **The critical space.** Let $V = \mathrm{Sym}^{d_1}V_1 \otimes \ldots \otimes \mathrm{Sym}^{d_k}V_k$, let $G = SO(V_1) \times \ldots \times SO(V_k)$.

The *critical space* of $f \in V$ is

$$H_f := [(Lie\ G) \cdot f]^\perp$$

For a matrix A, $H_A = \{B | A^tB, AB^t \text{ are symmetric}\}$

Theorem 1.14. [DOT18, O22] *All the critical points for* f *lie in* H_f.

The critical points span the critical space if triangle inequality

$$\dim V_i \leq \sum_{j\neq i} \dim V_j \tag{1.2}$$

is satisfied for all i such that $d_i = 1$. It is the condition such that the dual variety of the Segre-Veronese variety is a hypersurface, called the *hyperdeterminant*. For square matrices it is the classical determinant.

Theorem 1.15 (Banach 1938). *Let* t *be a symmetric tensor. The closest rank one tensor to* t *may be chosen symmetric.*

There are other critical points for the distance function, beyond the symmetric ones.

1.9. **Decomposable (rank one) tensors.** Multidimensional version of matrices are *tensors*.

$V = V_1 \otimes \ldots \otimes V_k$ is a tensor space.

The set of decomposable tensors is closed , it is the cone over the *Segre variety*. For $k = 2$ it is the variety of matrices of rank ≤ 1, $a_{ij} = x_i y_j$

The symmetric setting In the symmetric setting $Sym^d W \subset W \otimes \ldots \otimes W$, *symmetric tensors* are identified with homogeneous polynomials of degree d.

The set of decomposable symmetric tensors is the cone over the Veronese variety. For $d = 2$ it is the variety of symmetric matrices of rank ≤ 1, $a_{ij} = x_i x_j$.

1.10. **The singular k-ples.** Any tensor $t \in \mathbb{R}^{m_1+1} \otimes \ldots \otimes \mathbb{R}^{m_k+1}$ defines a distance function $f_t \colon X = \mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_k} \to \mathbb{R}$ over the Segre variety X of decomposable tensors.

Theorem 1.16 (Lim, Qi)**.** *The critical points of f_t corresponds to tensors $(x_1, \ldots, x_k) \in X$ such that*

$$t(x_1, \ldots, \hat{x_i}, \ldots, x_k) = \lambda_i x_i \qquad \forall i = 1, \ldots, k$$

which are called singular d-ples.

Reference book: [18] Qi, Luo, "Tensor Analysis and Spectral Theory", SIAM, 2017.

Theorem 1.17 (Friedland-O, EDdegree of Segre variety)**.** *The number of singular k-ples of a general tensor t over $\mathbb{C}$ of format $(m_1+1) \times \ldots \times (m_k+1)$ is the coefficient of $\prod_{i=1}^k t_i^{m_i}$ in the polynomial*

$$\prod_{i=1}^k \frac{\hat{t_i}^{m_i+1} - t_i^{m_i+1}}{\hat{t_i} - t_i}$$

where $\hat{t_i} = \sum_{j \neq i} t_j$. This number is EDdegree$(\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_k})$ *with respect to Bombieri-Weyl product.*

Theorem 1.18 (Special case of binary tensors)**.**

$$\text{EDdegree}(\underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{k}) = k!$$

Theorem 1.19 (Friedland-O, EDdegree of Segre-Veronese variety)**.** *The number of singular k-ples of a general tensor $t \in Sym^{d_1}\mathbb{C}^{m_1+1} \otimes \ldots \otimes Sym^{d_k}\mathbb{C}^{m_k+1}$ is the coefficient of $\prod_{i=1}^k t_i^{m_i}$ in the polynomial*

$$\prod_{i=1}^k \frac{\hat{t_i}^{m_i+1} - t_i^{m_i+1}}{\hat{t_i} - t_i}$$

where $\hat{t_i} = \sum_{j=1}^k d_j t_j - t_i$. This number is EDdegree *of the variety $(\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_k})$ embedded with $\mathcal{O}(d_1, \ldots, d_k)$, with respect to Bombieri-Weyl product.*

Theorem 1.17 is the special case of Theorem 1.19 when all $d_i = 1$.

The proof of Theorem 1.19 is geometrical. It can be shown that the singular t-ples correspond to the zero loci of a general section of the bundle E defined in the following way. Consider the projections $\pi_i \colon X \to \mathbb{P}^{n_i}$, then $E = \oplus_{i=1}^k \pi_i^* Q\,(d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_k)$. The rank of E coincides with the dimension of X. The formula in Theorem 1.19 computes the top Chern class of E.

The following result by Aluffi and Harris makes transparent the dependency of EDdegree on the quadratic form chosen.

Theorem 1.20. *[Aluffi-Harris] Let $X \subset \mathbb{P}^n$ be smooth of dimension m. Let Q be the quadric which defines the distance function. Then*

$$\mathrm{EDdegree}_Q(X) = (-1)^m \left(\chi(X) - \chi(Q \cap X) - \chi(H \cap X) + \chi(Q \cap H \cap X)\right)$$

Corollary 1.21. *The formula in Theorem 1.17 works for any SO-invariant symmetric bilinear form that does not contain the variety of decomposable tensors.*

Proof. On each factor, the epression $q(v,w)^{d-2i}q(v,v)^i q(w,w)^i$ collapses, when $v = w$, to $q(v,v)^d$. So $Q \cap X$ is the same and we may apply Theorem 1.20. □

1.11. The generating function and a first asymptotics.

Theorem 1.22 (Zeilberger)**.** *Let $a_k(m_1, \ldots, m_k)$ be the number of critical points of format $\prod_{i=1}^k (m_i + 1)$, then*

$$\sum_{m \in \mathbb{N}^k} a_k(m_1, \ldots, m_k)\mathbf{x}^m = \frac{1}{\left(1 - \sum_{i=2}^k (i-1)e_i(\mathbf{x})\right)} \prod_{i=1}^k \frac{x_i}{1 - x_i}$$

where e_i is the i-th elementary symmetric function.

Theorem 1.23 (Zeilberger, Pantone)**.**

$$a_3(n,n,n) \sim \frac{2}{\sqrt{3}\pi}\frac{8^n}{n} \qquad \text{for } n \to \infty$$

Pantone finds similar asymptotical formulas for any multidimensional format.

1.12. **Tensor Eigenvectors in the symmetric case.** The critical points of the distance function from a symmetric tensors $A \in \mathrm{Sym}^d V$ to the Veronese variety of decomposable tensors have the form λv^d, v is eigenvector with eigenvalue λ.

Theorem 1.24. *)Fornaess-Sibony, Cartwright-Sturmfels,* [7, 1]*) The number of eigenvectors of a symmetric tensor $A \in \mathrm{Sym}^d \mathbb{C}^{m+1}$ is (for $d \geq 2$)*

$$\frac{(d-1)^{m+1} - 1}{d - 2}$$

This number is EDdegree *of d-Veronese embedding of $\mathbb{P}^{m-1}$ with respect to Bombieri-Weyl product.*

As in Theorem 1.19, the proof of Theorem 1.24 follows by the computation of the top Chern class of the bundle $Q(d-1)$ [13].

The last result we present generalizes (1.1) from matrices to tensors.

Theorem 1.25. *(Qi,* [17]*) If X = discriminant hypersurface, and d is even,*

$$\mathrm{EDpoly}_{X,f}(t^2) = \Delta_d\left(f(x) - tq(x)^{d/2}\right)\Delta_d\left(f(x) + tq(x)^{d/2}\right).$$

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