Introduction to Tensor Spaces
Doctoral course in Mathematics, 2024/25
University of Florence

Choose and solve 3 exercises among the following.

1. Let $f = x^3 \pm 12xy^2$. Compute the apolar ideal $f^\perp$ and decompose f as the sum of two cubes over $\mathbb{C}$. Is the construction possible over $\mathbb{R}$ or modifications are needed ?

2. Let $V_{n-1} \subset \mathbb{C}^n$ be the standard representation of the symmetric group Σ_n. Find a basis of V_{n-1} in terms of the canonical basis of $\mathbb{C}^n$. Compute the character $\chi_{V_{n-1}} \colon \Sigma_n \to \mathbb{C}$ of the standard representation.

3. Let $\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$. Compute the symmetrizer c_λ and its square c_λ^2 as elements of the group algebra $\mathbb{C}\Sigma_3$.

4. Let V be a complex vector space of dimension 5 with basis e_i, $i = 0, \ldots, 4$. Consider the variety
$$Gr(\mathbb{P}^1, \mathbb{P}(V)) \subset \mathbb{P}(\wedge^2 V)$$
consisting of decomposable tensors. For any $\omega \in \wedge^2 V$ consider the map $C_\omega \colon V^\vee \to V$ given by the natural contraction. Prove that C_ω is skew-symmetric and that $[\omega] \in Gr(\mathbb{P}^1, \mathbb{P}(V))$ if and only if the rank of C_ω is ≤ 2. For any Young diagram λ describe the Schubert cells $X_\lambda \subset Gr(\mathbb{P}^1, \mathbb{P}(V))$, that is describe its representatives as 2×5 matrices and the dimension of intersection $[\omega] \cap \langle e_i, \ldots, e_4 \rangle$ for $i = 1, \ldots, 4$.