Tensor decomposition and tensor rank from the point of view of Classical Algebraic Geometry

RTG Workshop

Tensors and their Geometry in High Dimensions

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Content of the three talks

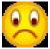
- **Wednesday** Rank and symmetric rank. Tensor decomposition. Classical apolarity and Sylvester algorithm. Secant varieties. Clebsch quartics. Sum of squares, sum of k-th powers.
- **Thursday** Cases where classical apolarity fails. Vector bundles and non abelian apolarity. Equations for secant varieties, infinitesimal criterion for smoothness. Scorza map and Lüroth quartics. Identifiability.
- **Friday** Actions of $SL(2)$. The complexity of Matrix Multiplication Algorithm.

The first case when classic apolarity fails

Consider $X = \mathbb{P}^2$.
Plane cubics $f \in S^3\mathbb{C}^3$ of rank ≤ 2 may be detected by the minors of the morphism

$$C_f^2 \colon S^2{\mathbb{C}^3}^\vee \to \mathbb{C}^3$$

cubics of rank 1 are defined by 2×2 minors of A_f
cubics of rank ≤ 2 are defined by 3×3 minors of A_f
cubics of rank ≤ 3 are defined by ...



$\sigma_3(v_3(\mathbb{P}^2))$ is a hypersurface of degree 4 in $\mathbb{P}^9$ *(Aronhold invariant)*. It cannot be defined by the minors of the contraction C_f^2

Non abelian apolarity, I

Let E be a vector bundle on X embedded, by the very ample line bundle L, in $\mathbb{P}(H^0(L)^\vee)$.
Consider the natural morphism

$$H^0(E) \otimes H^0(L)^\vee \xrightarrow{A} H^0(E^\vee \otimes L)^\vee$$

which induces $\forall f \in H^0(L)^\vee$ the linear map

$$A_f \colon H^0(E) \to H^0(E^\vee \otimes L)^\vee$$

Non abelian apolarity, II

Theorem (Landsberg-O)

Let $Z = \{x_1, \ldots x_k\} \subset X$ *be a set of points such that* $H^0(E^\vee \otimes L) \to H^0(E^\vee \otimes L_{|Z})$ *is surjective.*
Let $f = \sum_{i=1}^k x_i \in H^0(L)^\vee$. *Then*

$$\ker A_f = H^0(I_Z \otimes E)$$

$$Z \subseteq \textit{ base locus of } \ker A_f$$

In particular

$$\mathrm{rk}\ A_f = \mathrm{rk}\ E \cdot rk\ f$$

Steps of the proof

1. The inclusion
$$\ker A_f \supseteq H^0(I_Z \otimes E)$$
always holds. Indeed assume $f = [x]$ with
$x \in \mathrm{Cone}(X) \subset H^0(L)^\vee$
We have
$H^0(L)^\vee \otimes H^0(I_Z \otimes E) \otimes H^0(E^\vee \otimes L) \to H^0(L)^\vee \otimes H^0(I_Z \otimes L)$
the latter pairing associates to (x, s) the values of $s(x)$, which vanishes.
By linearity $A_{\sum_i [x_i]} = \sum A_{[x_i]}$ and the same argument works in general.
2. The surjectivity assumption allows a local computation to check the dimension of the kernel.

Non abelian apolarity for plane cubics

Consider again $X = \mathbb{P}^2$, $L = \mathcal{O}(3)$, $f \in S^3\mathbb{C}^3$.
Let $E = Q(1)$ where Q is the quotient bundle of rank two on $\mathbb{P}^2$.
Consider

$$A_f \colon H^0(E) = \mathrm{ad}\ \mathbb{C}^3 \to \mathrm{ad}\ \mathbb{C}^3 = H^0(E^\vee \otimes L)^\vee$$

cubics of rank $\leq k$ are defined by the condition $\mathrm{rk}\ A_f \leq 2k$ for $k \leq 3$.

The equation of the hypersurface $\sigma_3(v_3(\mathbb{P}^2))$ is given by the *Pfaffian* of A_f, *(Aronhold invariant).*

Explicit construction of the minors of A_f from a presentation of E

On $\mathbb{P}^n$ we have the presentation

$$\begin{tikzcd}[column sep=small]
\dots \arrow[r] & L_2 \arrow[r] & L_1 \arrow[rr, "p_E"] \arrow[dr] & & L_0 \arrow[r] & L_{-1} \arrow[r] & \dots \\
& & & E \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

the matrix A_f can be obtained by differentiating with respect to p_E the catalecticant matrix. The presentation of $E = Q(1)$ on $\mathbb{P}^2$ is

$$\begin{bmatrix} & x_2 & -x_1 \\ -x_2 & & x_0 \\ x_1 & -x_0 & \end{bmatrix}$$

The corresponding picture is

$$\ydiagram{2,1} \otimes \ytableaushort{***} \longrightarrow \ytableaushort{{}{}*,{}*,*} \simeq \ydiagram{2,1}$$

Explicit form of Aronhold invariant

Let $f = f_{000}x_0^3 + f_{111}x_1^3 + f_{222}x_2^3 + 3f_{001}x_0^2x_1 + 3f_{011}x_0x_1^2 + 3f_{002}x_0^2x_2 + 3f_{022}x_0x_2^2 + 3f_{112}x_1^2x_2 + 3f_{122}x_1x_2^2 + 6f_{012}x_0x_1x_2 \in S^3\mathbb{C}^3$

The minors of A_f coincide with the minors of the following map $P_f\colon End(V) \to End(V)$ (regardless the trace):

$$\begin{bmatrix} & & & f_{002} & f_{012} & f_{022} & -f_{010} & -f_{011} & -f_{012} \\ & & & f_{012} & f_{112} & f_{122} & -f_{011} & -f_{111} & -f_{112} \\ & & & f_{012} & f_{112} & f_{222} & -f_{012} & -f_{112} & -f_{122} \\ -f_{002} & -f_{012} & -f_{022} & & & & f_{000} & f_{001} & f_{002} \\ -f_{012} & -f_{112} & -f_{122} & & & & f_{001} & f_{011} & f_{012} \\ -f_{012} & -f_{112} & -f_{222} & & & & f_{001} & f_{011} & f_{022} \\ f_{010} & f_{011} & f_{012} & -f_{000} & -f_{001} & -f_{002} & & & \\ f_{011} & f_{111} & f_{112} & -f_{001} & -f_{011} & -f_{012} & & & \\ f_{012} & f_{112} & f_{122} & -f_{001} & -f_{011} & -f_{022} & & & \end{bmatrix}$$

All the subPfaffians of size 8 extracted by P_f coincide, up to scalar multiple, with the *Aronhold invariant*.

Expression as blocks of catalecticant

The resulting matrix 9×9 of A'_ϕ, with respect to this basis, has a block structure with the following 3×3 blocks

$$\begin{bmatrix} 0 & C(f_2) & -C(f_1) \\ -C(f_2) & 0 & C(f_0) \\ C(f_1) & -C(f_0) & 0 \end{bmatrix}$$

All the pfaffians of this matrix are proportional to the Aronhold invariant.

The Aronhold invariant in classical notations

The Aronhold invariant for plane cubics was defined classically by the "symbolic representation"
as a multilinear symmetric function
$S\colon S^3\mathbb{C}^3 \times S^3\mathbb{C}^3 \times S^3\mathbb{C}^3 \times S^3\mathbb{C}^3 \to \mathbb{C}$

$$S(x^3, y^3, z^3, w^3) := (x \wedge y \wedge z)(x \wedge y \wedge w)(x \wedge z \wedge w)(y \wedge z \wedge w)$$

If a plane cubic f is sum of three cubes (namely, it is $SL(3)$-equivalent to the Fermat cubic $x_0^3 + x_1^3 + x_2^3$) we have $S(f) = 0$. Indeed $S(f, f, f, f)$ splits as a sum of $S(x_{i_0}^3, x_{i_1}^3, x_{i_2}^3, x_{i_3}^3$ where $i_k \in \{0, 1, 2\}$, so that $\{i_0, i_1, i_2, i_3\}$ contains at least a repetition, in such a way that all summands contributing to $S(f, f, f, f)$ vanish.
The Aronhold invariant is the degree 4 equation of $\sigma_3(\mathbb{P}^2, \mathcal{O}(3))$, which can be seen as the $SL(3)$-orbit of the Fermat cubic $x_0^3 + x_1^3 + x_2^3$ (sum of three cubes),

The Aronhold invariant as a pfaffian, from scratch, I

Let W be a vector space of dimension 3. In particular $End_0 W \subset End W$ the space of traceless endomorphisms is self-dual and it has dimension 8.
For any $\phi \in S^3 W$, let $A_\phi \colon End_0 W \to End_0 W$ be the $SL(V)$-invariant contraction operator defined in the following way
If $\phi = u^3$ and $M \in End_0 W$ then $A_{u^3}(M)(w) := (M(u) \wedge u \wedge w)\, u$.
A_f is defined by linearity for general $f \in S^3 W$

Theorem

The operator A_ϕ is skew-symmetric and the pfaffian $Pf\ A_\phi$ is the equation of $\sigma_3(\mathbb{P}(W), \mathcal{O}(3))$, i.e. it is the Aronhold invariant.

The Aronhold invariant as a pfaffian, from scratch, II

Lemma

Let $\phi = w^3$ with $w \in W$. Then $rkA_\phi = 2$. More precisely

$$ImA_{w^3} = \{M \in ad\ W | Im\ M \subseteq < w >\}$$

$$KerA_{w^3} = \{M \in ad\ W | w\ is\ an\ eigenvector\ of\ M\}$$

Proof The statement follows from the equality

$$A_{w^3}(M)(v) = 6(M(w) \wedge w \wedge v)w$$

Direct proof that invariant defined by Pfaffian vanishes on $\sigma_3(v_3(\mathbb{P}^2))$

Let $\phi \in \sigma_3(\mathbb{P}(W), \mathcal{O}(3))$. By the definition of higher secant variety, ϕ is in the closure of elements which can be written as $\phi_1 + \phi_2 + \phi_3$ with $\phi_i \in (\mathbb{P}(W), \mathcal{O}(3))$. From the previous Lemma it follows that

$$\mathrm{rk}\ A_\phi \leq \mathrm{rk}\ A_{\sum_{i=1}^3 \phi_i} = \mathrm{rk}\ \sum_{i=1}^{3} A_{\phi_i} \leq \sum_{i=1}^{3} \mathrm{rk}\ A_{\phi_i} = 2 \cdot 3 = 6$$

Hence $Pf(A_\phi)$ has to vanish on $\sigma_3(\mathbb{P}(W), \mathcal{O}(3))$.

finer description

Remark We have the decomposition

$$\wedge^2(S^{2,1}W) = S^3W \oplus S^{2,2,2}W \oplus S^{2,1}W$$

what we found corresponds to the embedding
$S^3W \subset \wedge^2(S^{2,1}W)$

Koszul flattening algorithm [Oeding-O.]

Input: $f \in S^d V$

Construct $P_f \colon \mathrm{Hom}(S^m V, \wedge^a V) \to \mathrm{Hom}(\wedge^{n-a} V, S^{d-m-1} V)$, with suitable m, a.

Compute $\ker P_f$. Note that $\mathrm{rk}(f) \geq \frac{\mathrm{rk}(P_f)}{\binom{n}{a}} = s$.

Find Z the common eigenvectors of elements in the kernel of P_f. If Z is infinite then stop, this method fails.

Otherwise let $Z = \{v_1, \ldots, v_s\}$. Solve the linear system defined by $f = \sum_{i=1}^{s} c_i v_i^d$ in the unknowns c_i.

Output: The unique Waring decomposition of f.

Feasibility of the algorithm, general case

Theorem

Suppose $n \geq 3$ and set $d = 2m+1$. Let $f = \sum_{i=1}^{r} v_i^d$ be a form in $S^d V$ of rank r such that f is general among the forms in $S^d V$ of rank r, let $z_i = [v_i] \in \mathbb{P}(V)$ be the corresponding points and let $Z = \{z_1, \ldots, z_r\}$.

1. *Let Z' be the set of common eigenvectors (up to scalars) of $\ker P_f$. If n is even and $r \leq \binom{m+n}{n}$, then Z' agrees with Z. Moreover the Koszul Algorithm produces the unique Waring decomposition of f.*
2. *If n is odd and $r \leq \binom{m+n}{n}$ let Z' be the set of common eigenvectors (up to scalars) of $\ker P_f$ and $(\mathcal{I}m\ P_f)^{\perp}$. Then $Z' = Z$ and Koszul Algorithm, with this modification, produces the unique Waring decomposition of f.*

Feasibility of the algorithm, $n = 3$

If $n = 3$ and $r \leq \frac{1}{3}\left(\frac{1}{2}(m+4)(m+3)(m+1) - \frac{m^2}{2} - \frac{m}{2} - 8\right)$ let Z' be the set of common eigenvectors (up to scalars) of $\ker P_f$. Then Koszul Algorithm produces the unique Waring decomposition of f.

Feasibility of the algorithm, $n = 2$, d odd

Theorem

Suppose $n = 2$ and set $d = 2m + 1$. Let $f = \sum_{i=1}^{r} v_i^d$ be a general form of rank r in $S^d V$, let $z_i = [v_i] \in \mathbb{P}(V)$ be the corresponding points and let $Z = \{z_1, \ldots, z_r\}$. Let Z' be the set of common eigenvectors (up to scalars) of $\ker P_f$.

1. *If $2r \leq m^2 + 3m + 4$ then $Z' = Z$. Moreover Koszul Algorithm produces the unique Waring decomposition of f.*
2. *If $2r \leq m^2 + 4m + 2$, then it is possible that $Z \subsetneq Z'$. Even in this case, Koszul Algorithm will produce the unique minimal Waring decomposition.*

Explicit form of Pentahedral Sylvester Theorem [Oeding-O]

Theorem (Pentahedral)

For a general $f \in S^3\mathbb{C}^4$ there exist unique $l_1, \dots l_5$ such that $f = \sum_{i=1}^5 l_i^3$

We set $E = \wedge^2 Q(1)$ We can merge $A_f \colon H^0(\wedge^2 Q(1)) \to H^0(Q(1))^\vee$ as a block in the 16×24 matrix constructed by the Koszul matrix 4×6

$$\begin{pmatrix} -x_1 & -x_2 & 0 & -x_3 & 0 & 0 \\ x_0 & 0 & -x_2 & 0 & -x_3 & 0 \\ 0 & x_0 & x_1 & 0 & 0 & -x_3 \\ 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}$$

The sections belonging to $\ker A_f$ vanish on $\{l_1, \dots l_5\}$. Note that $c_3(\wedge^2 Q(1)) = 5$, meaning that the general section of $\wedge^2 Q(1)$ vanish on five points.

Infinitesimal criterion

Let Z be a general collection of points and $f = \sum_{x_i \in Z} x_i$.

If

$$H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^\vee \otimes L) \longrightarrow H^0(I_{Z^2} \otimes L)$$

is surjective, then the locus where $\mathrm{rk}\ A_f \leq k \cdot \mathrm{rk}\ E$ contains $\sigma_k(X)$ as an irreducible component.

Theorem (Landsberg-O., [2010])

Let $d = 2\delta + 1$*,* $a = \lfloor \frac{n}{2} \rfloor$*. Let* $Z \subset \mathbb{P}^n$ *of length* $k \le \binom{\delta+n}{n}$*. Then the map*

$$H^0(I_Z \otimes \wedge^a Q(\delta)) \otimes H^0(I_Z \otimes \wedge^{n-a} Q(\delta)) \to H^0(I_{Z^2}(d))$$

is surjective. Hence we may apply the infinitesimal criterion and get local equations for $\sigma_k(v_d(\mathbb{P}^n))$*.*

Equations of secants of d-Veronese surface

All equations of $\sigma_k(v_3(\mathbb{P}^2))$ can be found with $E = Q(1)$ All equations of $\sigma_k(v_4(\mathbb{P}^2))$ can be found with $E = \mathcal{O}(2)$ In case $\sigma_k(v_5(\mathbb{P}^2))$, with $E = Q(2)$, we find equations cutting the secant variety as irreducible component. In case $\sigma_k(v_6(\mathbb{P}^2))$, with $E = \mathcal{O}(3)$ and $E = S^2Q(2)$ together, we find equations cutting the secant variety as irreducible component. In case $\sigma_k(v_7(\mathbb{P}^2))$, with $E = Q(3)$, we find equations cutting the secant variety as irreducible component, only for $k \leq 10$. We do not know any equations for the last one $\sigma_{11}(v_7(\mathbb{P}^2))$, which has codimension 3. Challenging question: $\sigma_{18}(v_9(\mathbb{P}^2))$ is a hypersurface. Even the degree is unknown.

Interpretation of catalecticant algorithm in terms of vector bundles (abelian apolarity)

The classical apolarity fits in this setting. If we have $f \in S^d\mathbb{C}^{n+1}$, we set $L = \mathcal{O}(d)$, $E = \mathcal{O}(e)$, so that $C_f^a \colon S^e{\mathbb{C}^{n+1}}^\vee \to S^{d-e}\mathbb{C}^{n+1}$ Since E is a line bundle, with abelian structural group, it is natural to call it "abelian apolarity" and "not abelian apolarity" refers to the case $\mathrm{rk}\ E \geq 2$. Note that $H^0(I_Z \otimes E^\vee \otimes L)$ coincides with $H^0(I_Z(d-e))$ and so the infinitesimal criterion reads as the surjectivity of the map

$$H^0(I_Z(e)) \otimes H^0(I_Z(d-e)) \to H^0(I_{Z^2}(d))$$

A numerical example, Waring decomposition of a quintic in three variables.

Hilbert, 1888, letter to Hermite

The general f of order 5 in three variables has a unique decomposition as a sum of seven powers of linear forms.

As an example we pick

$$f = 19x_0^5 + 25x_0^4x_1 + 44x_0^3x_1^2 + 35x_0^2x_1^3 + 30x_0x_1^4 + 36x_1^5 + 38x_0^4x_2 + 50x_0^3x_1x_2 - 20x_0^2x_1^2x_2 + 27x_0x_1^3x_2 + 14x_1^4x_2 - 23x_0^3x_2^2 + 10x_0^2x_1x_2^2 + 45x_0x_1^2x_2^2 - 13x_1^3x_2^2 + 11x_0^2x_2^3 - 29x_0x_1x_2^3 + 29x_1^2x_2^3 + 13x_0x_2^4 - 28x_1x_2^4 + 34x_2^5$$

Question

How to construct explicitly $f = \sum_{i=1}^{7} c_i l_i^5$, with $c_i \in \mathbb{C}$
$l_i = a_i x_0 + b_i x_1 + c_i x_2$?

We apply non abelian apolarity and Koszul algorithm with
$E = Q(2)$, $L = \mathcal{O}(5)$, so that $E^\vee \otimes L = Q(2)$ again.
For any $f \in S^5\mathbb{C}^3$ we have
$A_f \colon H^0(Q(2)) \to H^0(Q(2))^\vee$.
To construct it we take a presentation of $Q(2)$.

The contraction map P_f

$Hom(S^2\mathbb{C}^3, \mathbb{C}^3)$ represents tensors of order 3 partially symmetric in two indices.
We construct the map

$$\begin{array}{ccc} H^0(Q(2) & \xrightarrow{A_f} & H^0(Q(2))^\vee \\ \downarrow & & \downarrow \\ Hom(S^2\mathbb{C}^3, \mathbb{C}^3) & \xrightarrow{P_f} & Hom(\mathbb{C}^3, S^2\mathbb{C}^3) \end{array}$$

if $f = v^5$, $g \in Hom(S^2\mathbb{C}^3, \mathbb{C}^3)$

$$P_{v^5}(g)(w) := \left(g(v^2) \wedge v \wedge w\right) v^2 \tag{1}$$

and then extended by linearity.
This means $P_{\sum_i c_i v_i^5} = \sum_i c_i P_{v_i^5}$
The formula (1) is the key to understand the connection between tensor decomposition and eigenvectors.

Lemma

$P_{v^5}(M) = 0$ *if and only if there exists* λ *such that* $M(v^2) = \lambda v$.

If all v_i are eigenvectors of g then $g \in \ker P_{\sum_i c_i v_i^5}$.
So we have candidates to decompose f: compute the eigenvectors of $\ker P_f$.
P_f is given by a 18×18 block matrix with three classical catalecticants.
We compute the three partials

$\frac{\partial f}{\partial x_0} = 95x_0^4 + 100x_0^3x_1 + 132x_0^2x_1^2 + 70x_0x_1^3 + 30x_1^4 + 152x_0^3x_2 + 150x_0^2x_1x_2 - 40x_0x_1^2x_2 + 27x_1^3x_2 - 69x_0^2x_2^2 + 20x_0x_1x_2^2 + 45x_1^2x_2^2 + 22x_0x_2^3 - 29x_1x_2^3 + 13x_2^4$

$\frac{\partial f}{\partial x_1} = 25x_0^4 + 88x_0^3x_1 + 105x_0^2x_1^2 + 120x_0x_1^3 + 180x_1^4 + 50x_0^3x_2 - 40x_0^2x_1x_2 + 81x_0x_1^2x_2 + 56x_1^3x_2 + 10x_0^2x_2^2 + 90x_0x_1x_2^2 - 39x_1^2x_2^2 - 29x_0x_2^3 + 58x_1x_2^3 - 28x_2^4$

$\frac{\partial f}{\partial x_2} = 38x_0^4 + 50x_0^3x_1 - 20x_0^2x_1^2 + 27x_0x_1^3 + 14x_1^4 - 46x_0^3x_2 + 20x_0^2x_1x_2 + 90x_0x_1^2x_2 - 26x_1^3x_2 + 33x_0^2x_2^2 - 87x_0x_1x_2^2 + 87x_1^2x_2^2 + 52x_0x_2^3 - 112x_1x_2^3 + 170x_2^4$

The three catalecticant matrices corresponding to the three partial derivatives $\frac{\partial f}{\partial x_0}$, $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$ are $\begin{pmatrix} 2280 & 600 & 912 & 528 & 300 & -276 \\ 600 & 528 & 300 & 420 & -80 & 40 \\ 912 & 300 & -276 & -80 & 40 & 132 \\ 528 & 420 & -80 & 720 & 162 & 180 \\ 300 & -80 & 40 & 162 & 180 & -174 \\ -276 & 40 & 132 & 180 & -174 & 312 \end{pmatrix}$

$$\begin{pmatrix} 600 & 528 & 300 & 420 & -80 & 40 \\ 528 & 420 & -80 & 720 & 162 & 180 \\ 300 & -80 & 40 & 162 & 180 & -174 \\ 420 & 720 & 162 & 4320 & 336 & -156 \\ -80 & 162 & 180 & 336 & -156 & 348 \\ 40 & 180 & -174 & -156 & 348 & -672 \end{pmatrix}$$

$$\begin{pmatrix} 912 & 300 & -276 & -80 & 40 & 132 \\ 300 & -80 & 40 & 162 & 180 & -174 \\ -276 & 40 & 132 & 180 & -174 & 312 \\ -80 & 162 & 180 & 336 & -156 & 348 \\ 40 & 180 & -174 & -156 & 348 & -672 \\ 132 & -174 & 312 & 348 & -672 & 4080 \end{pmatrix}$$

The construction of P_f

We get, in exact arithmetic, that the 18×18 matrix of P_f is the following block matrix

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 912 & 300 & -276 & -80 & 40 & 132 & -600 & -528 & -300 & -420 & 80 & -40 \\
0 & 0 & 0 & 0 & 0 & 0 & 300 & -80 & 40 & 162 & 180 & -174 & -528 & -420 & 80 & -720 & -162 & -180 \\
0 & 0 & 0 & 0 & 0 & 0 & -276 & 40 & 132 & 180 & -174 & 312 & -300 & 80 & -40 & -162 & -180 & 174 \\
0 & 0 & 0 & 0 & 0 & 0 & -80 & 162 & 180 & 336 & -156 & 348 & -420 & -720 & -162 & -4320 & -336 & 156 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 180 & -174 & -156 & 348 & -672 & 80 & -162 & -180 & -336 & 156 & -348 \\
0 & 0 & 0 & 0 & 0 & 0 & 132 & -174 & 312 & 348 & -672 & 4080 & -40 & -180 & 174 & 156 & -348 & 672 \\
-912 & -300 & 276 & 80 & -40 & -132 & 0 & 0 & 0 & 0 & 0 & 0 & 2280 & 600 & 912 & 528 & 300 & -276 \\
-300 & 80 & -40 & -162 & -180 & 174 & 0 & 0 & 0 & 0 & 0 & 0 & 600 & 528 & 300 & 420 & -80 & 40 \\
276 & -40 & -132 & -180 & 174 & -312 & 0 & 0 & 0 & 0 & 0 & 0 & 912 & 300 & -276 & -80 & 40 & 132 \\
80 & -162 & -180 & -336 & 156 & -348 & 0 & 0 & 0 & 0 & 0 & 0 & 528 & 420 & -80 & 720 & 162 & 180 \\
-40 & -180 & 174 & 156 & -348 & 672 & 0 & 0 & 0 & 0 & 0 & 0 & 300 & -80 & 40 & 162 & 180 & -174 \\
-132 & 174 & -312 & -348 & 672 & -4080 & 0 & 0 & 0 & 0 & 0 & 0 & -276 & 40 & 132 & 180 & -174 & 312 \\
600 & 528 & 300 & 420 & -80 & 40 & -2280 & -600 & -912 & -528 & -300 & 276 & 0 & 0 & 0 & 0 & 0 & 0 \\
528 & 420 & -80 & 720 & 162 & 180 & -600 & -528 & -300 & -420 & 80 & -40 & 0 & 0 & 0 & 0 & 0 & 0 \\
300 & -80 & 40 & 162 & 180 & -174 & -912 & -300 & 276 & 80 & -40 & -132 & 0 & 0 & 0 & 0 & 0 & 0 \\
420 & 720 & 162 & 4320 & 336 & -156 & -528 & -420 & 80 & -720 & -162 & -180 & 0 & 0 & 0 & 0 & 0 & 0 \\
-80 & 162 & 180 & 336 & -156 & 348 & -300 & 80 & -40 & -162 & -180 & 174 & 0 & 0 & 0 & 0 & 0 & 0 \\
40 & 180 & -174 & -156 & 348 & -672 & 276 & -40 & -132 & -180 & 174 & -312 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Note that $\mathrm{rank}\,(P_f) = 14$

The kernel of P_f

The general elemment of the kernel of P_f is an element of $Hom(S^2\mathbb{C}^3, \mathbb{C}^3)$ and it is described by $3 \times 3 \times 3$ tensor with three symmetric slices $A = [A_0, A_1, A_2]$.

$$A_0 = \begin{bmatrix} -70889812761849564 & 37854909005395720 & -15413901438761293 3 \\ 37854909005395720 & 14380829313586918 & -17754154469945324 \\ -15413901438761293 3 & -17754154469945324 & 17745274511945518 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -51678516968532528 & 20166198255841392 & -4194619430531880 \\ 20166198255841392 & 75444945154321110 & -53236221702271281 \\ -4194619430531880 & -53236221702271281 & -28743768948593570 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 59045630248326600 & -91504273685786376 & 0 \\ -91504273685786376 & 34278108136451138 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The first E-eigenvector

We need to compute the E-eigenvectirs of $A = [A_0, A_1, A_2]$
We remain in Macaulay2. We have

$$\begin{bmatrix} 1 & 7.97577 & 1.82513 \end{bmatrix} \cdot A_i \cdot \begin{bmatrix} 1 \\ 7.97577 \\ 1.82513 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 7.97577 \\ 1.82513 \end{bmatrix}$$

where $\lambda = 4.2733 \cdot 10^{17}$. After normalization, $5.1952 \cdot 10^{16}$ is the corresponding E-eigenvalue, but we will do not need it.
The vector computed is an approximate real Z-eigenvector (not normalized) of A, computed with (classical) eigenvector method for zero-dimensional ideals, well explained in the book [Elkadi-Mourrain], applied to the 2-minors of

$$\begin{bmatrix} x^t A_0 x & x^t A_1 x & x^t A_2 x \\ x_0 & x_1 & x_2 \end{bmatrix}$$

setting $x_0 = 1$.

All the seven E-eigenvector of the tensor A

There are three real eigenvectors and two pairs of conjugate eigenvectors, which are

$$\begin{bmatrix} 1 \\ 7.97577 \\ 1.82513 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ -6.7325 + 2.91924\sqrt{-1} \\ -3.49842 - 3.27128\sqrt{-1} \end{bmatrix}, \begin{bmatrix} 1 \\ -6.7325 - 2.91924\sqrt{-1} \\ -3.49842 + 3.27128\sqrt{-1} \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ .39844 \\ .112957 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ .122478 + .537715\sqrt{-1} \\ -.436832 - .342586\sqrt{-1} \end{bmatrix}, \begin{bmatrix} 1 \\ .122478 - .537715\sqrt{-1} \\ -.436832 + .342586\sqrt{-1} \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ -2.94762 \\ 12.5538 \end{bmatrix}$$

Why seven eigenvectors ?

The formula

$$\#\ \text{eigenvectors of a tensor in}\ (\mathbb{C}^n)^{\otimes m} = \frac{(m-1)^n - 1}{m-2}$$

seen yesterday in Shenglong Hu talk,
proved in some cases by [Ni-Qi-Wang-Wang] and in general by [Cartwright, Sturmfels]
for $m = 3$, $n = 3$ gives 7.

The decomposition of f

We substitute the seven eigenvectors computed

$$f = c_0(x_0 + 7.97577x_1 + 1.82513x_2)^5 +$$
$$c_1(x_0 + x_1(-6.7325 + 2.91924\sqrt{-1}) + x_2(-3.49842 - 3.27128\sqrt{-1}))^5 +$$
$$c_2(x_0 + x_1(-6.7325 - 2.91924\sqrt{-1}) + x_2(-3.49842 + 3.27128\sqrt{-1}))^5 +$$
$$c_3(x_0 + (.39844)x_1 + (.112957)x_2)^5 +$$
$$c_4(x_0 + x_1(.122478 + .537715\sqrt{-1}) + x_2(-.436832 - .342586\sqrt{-1}))^5 +$$
$$c_5(x_0 + x_1(.122478 - .537715\sqrt{-1}) + x_2(-.436832 + .342586\sqrt{-1}))^5 +$$
$$c_6(x_0 + (-2.94762)x_1 + (12.5538)x_2)^5$$

We need just to solve a square system in the seven unknowns $c_0 \ldots c_6$.

This is the Waring decomposition of f

$$f = .0011311(x_0 + 7.97577x_1 + 1.82513x_2)^5 +$$
$$(.000199669 + .000111056\sqrt{-1})(x_0 + x_1(-6.7325 + 2.91924\sqrt{-1}) + x_2(-3.49842 - 3.27128\sqrt{-1}))^5 +$$
$$(+.000199669 - .000111056\sqrt{-1})(x_0 + x_1(-6.7325 - 2.91924\sqrt{-1}) + x_2(-3.49842 + 3.27128\sqrt{-1}))^5 +$$
$$(24.25)(x_0 + (.39844)x_1 + (.112957)x_2)^5 +$$
$$(-2.62582 + 3.74206\sqrt{-1})(x_0 + x_1(.122478 + .537715\sqrt{-1}) + x_2(-.436832 - .342586\sqrt{-1}))^5 +$$
$$(-2.62582 - 3.74206\sqrt{-1})(x_0 + x_1(.122478 - .537715\sqrt{-1}) + x_2(-.436832 + .342586\sqrt{-1}))^5 +$$
$$(.000108482)(x_0 + (-2.94762)x_1 + (12.5538)x_2)^5$$

The Scorza map between plane quartics

Let A be the Aronhold invariant. For any plane quartic F and any point $x \in \mathbf{P}(W)$ we consider the polar cubic $P_x(F)$. Then $A(P_x(F))$ is a quartic in the variable x which we denote by $S(F)$. The rational map $S\colon \mathbf{P}(S^4W) \dashrightarrow \mathbf{P}(S^4W)$ is called the Scorza map. Our description of the Aronhold invariant shows that $S(F)$ is defined as the degeneracy locus of a skew-symmetric morphism on $\mathbf{P}(W)$

$$\mathcal{O}(-2)^8 \xrightarrow{f} \mathcal{O}(-1)^8$$

It is possible to check (see [Beauville]) that $\mathrm{Coker}\, f = E$ is a rank two vector bundle over $S(F)$ such that $c_1(E) = K_{S(F)}$.

Theta characteristic

We recall that a *theta characteristic* on a general plane quartic f is a line bundle θ on f such that θ^2 is the canonical bundle. Hence $\deg \theta = 2$. There are 64 theta characteristic on f. If the curve is general, every bitangent is tangent in two distinct points P_1 and P_2, and the divisor $P_1 + P_2$ defines a theta characteristic θ such that $h^0(\theta) = 1$, these are called odd theta characteristic and there are 28 of them. The remaining 36 theta characteristic θ are called even and they satisfy $h^0(\theta) = 0$.

The Scorza map is 36 : 1. Indeed the quartic image comes equipped with a theta characteristic θ. I owe to A. Buckley the claim that from E it is possible to recover the even theta-characteristic θ on $S(F)$, with a construction performed by Pauly . By [Lange-Narasimhan] there are exactly eight maximal line subbundles P_i of E of maximal degree equal to 1 such that $h^0(E \otimes P_i^\vee) > 0$. These eight line bundles are related by the equality (Lemma 4.2 in [Pauly])

$$\otimes_{i=1}^{8} P_i = K_{S(F)}^2$$

Pauly construction in §4.2 [Pauly] gives a net of quadrics in the following way.

For the general stable E as ours, there exists a unique stable bundle E' with $c_1(E') = \mathcal{O}$ such that $h^0(E \otimes E')$ has the maximal value 4. The extensions

$$0 \longrightarrow P_i \longrightarrow E \longrightarrow K_{S(F)} \otimes P_i^{-1} \longrightarrow 0$$

$$0 \longrightarrow P_i^{-1} \longrightarrow E' \longrightarrow P_i \longrightarrow 0$$

define eight sections in $Hom(E', E) \simeq E \otimes E'$ as

$$E' \longrightarrow P_i \longrightarrow E$$

which give eight points in $\mathbb{P}H^0(E \otimes E')$. These eight points are base locus for a net of quadrics, which gives a symmetric representation of the original quartic curve $S(F)$ and then defines a theta characteristic.

Clebsch quartics

A plane quartic $f \in S^4V$ is called *Clebsch* if it has an apolar conic, that is if there exists a nonzero $q \in S^2V^\vee$ such that $q \cdot f = 0$.
One defines, for any $f \in S^4V$, the catalecticant map $C_f \colon S^2V^\vee \to S^2V$ which is the contraction by f. If

$$f = a_{00}x^4+4a_{10}x^3y+4a_{01}x^3z+6a_{20}x^2y^2+12a_{11}x^2yz+6a_{02}x^2z^2+4a_{30}xy^3+$$

$$4a_{03}xz^3 + a_{40}y^4 + 4a_{31}y^3z + 6a_{22}y^2z^2 + 4a_{13}yz^3 + a_{04}z^4$$

then the matrix of C_f is

$$C_f = \begin{bmatrix} a_{00} & a_{10} & a_{01} & a_{20} & a_{11} & a_{02} \\ a_{10} & a_{20} & a_{11} & a_{30} & a_{21} & a_{12} \\ a_{01} & a_{11} & a_{02} & a_{21} & a_{12} & a_{03} \\ a_{20} & a_{30} & a_{21} & a_{40} & a_{31} & a_{22} \\ a_{11} & a_{21} & a_{12} & a_{31} & a_{22} & a_{13} \\ a_{02} & a_{12} & a_{03} & a_{22} & a_{13} & a_{04} \end{bmatrix}$$

Clebsch quartics have border rank five

We get that a plane quartic f is Clebsch if and only if $\det C_f = 0$. The basic property is that if $f = l^4$ is the 4-th power of a linear form, then C_f has rank 1. It follows that if $f = \sum_{i=1}^{5} l_i^4$ is the sum of five 4-th powers of linear forms, then

$$\mathrm{rk}\ C_f = \mathrm{rk}\ \sum_{i=1}^{5} C_{l_i^4} \leq \sum_{i=1}^{5} \mathrm{rk}\ C_{l_i^4} = \sum_{i=1}^{5} 1 = 5$$

Theorem (Clebsch)

A plane quartic f is Clebsch if and only if there is an expression $f = \sum_{i=0}^{4} l_i^4$ (or a limit of such an expression)

In conclusion, $\det C_f = 0$ is the equation (of degree six) of $\sigma_5(v_4(\mathbb{P}^2))$, which is called the Clebsch hypersurface.

Defectivity of plane quartics

It is expected by naive dimensional count that the general rank for a plane quartic is five.
On the contrary, the general rank is six. Five summands are not sufficient, and describe Clebsch quartics.
A general Clebsch quartic f can be expressed as a sum of five 4-th powers in ∞^1 many ways. Precisely the 5 lines l_i belong to a unique smooth conic Q in the dual plane, which is apolar to f and it is found as the generator of $\ker\ C_f$.

Scorza(Clebsch) is inscribed in a pentalateral

Theorem

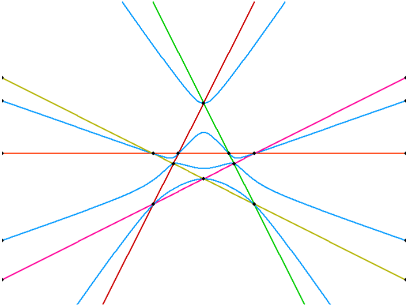
Let f be a Clebsch quartic with apolar conic Q, then $S(f)$ is a Lüroth quartic equipped with the pentalateral theta corresponding to Q.

Proof Let $f = \sum_{i=1}^{5} l_i^4$. Let l_p and l_q be two lines in the pentalateral and let $x_{pq} = l_p \cap l_q$. Then

$$P_{x_{pq}}(f) = \sum_{i=1}^{5} P_{x_{pq}}(l_i^4) = \sum_{i=1}^{5} 4 l_i^3 P_{x_{pq}}(l_i)$$

In the above sum at most three summands survive, because the ones with $i = p, q$ are killed. Then $P_{x_{pq}}(f)$ is a Fermat cubic and $Ar(P_{x_{pq}}(f)) = 0$, hence $x_{pq} \in S(f)$. It follows that $S(f)$ is inscribed in the pentalateral and it is Lueroth. $\square$

A *Lüroth quartic* is a plane quartic containing the ten vertices of a complete *pentalateral*, or the limit of such curves.



The pentalateral theta

If l_i for $i = 0, \ldots, 4$ are the lines of the pentalateral, we may consider as divisor (of degree 4) over the curve. Then $l_0 + \ldots + l_4$ consists of 10 double points, the meeting points of the 5 lines. Let $P_1 + \ldots + P_{10}$ the corresponding reduced divisor of degree 10. So

$$2(P_1 + \ldots + P_{10}) = 5H$$

Then $P_1 + \ldots + P_{10} = 2H + \theta$ where H is the hyperplane divisor and θ is a even theta characteristic, which is called the pentalateral theta.

Explicitly, if f is Clebsch with equation

$$l_0^4 + \ldots + l_4^4$$

then $S(f)$ has equation

$$\sum_{i=0}^{4} k_i \prod_{j \neq i} l_j$$

where $k_i = \prod_{p<q<r, i \notin \{p,q,r\}} |l_p l_q l_r|$ so that $l_0, \ldots, l_4$ is a pentalateral inscribed in $S(f)$. Note that the conic where the five lines which are the summands of f are tangent, is the same conic where the pentalateral inscribed in $S(f)$ is tangent.

Lüroth quartics are jumping curves for stable bundles

Theorem (Barth 1977)

Let E be a stable rank 2 bundle on $\mathbb{P}^2$ with $c_1 = 0$, $c_2 = 4$. Then $\{l \in \mathbb{P}^{2\vee} | E_{|l} \textit{ is not trivial}\}$ is a Lüroth quartic curve.

Moduli space of these bundles correspond to pairs (Lüroth, theta).

The symmetric case: uniqueness in the subgeneric case

Theorem (Sylvester[1851], Chiantini-Ciliberto, Mella, Ballico, [2002-2005])

The general $f \in S^d\mathbb{C}^{n+1}$ of rank s smaller than the generic one has a unique Waring decomposition, with the only exceptions

- *rank $s = \binom{n+2}{2} - 1$ in $S^4\mathbb{C}^{n+1}$, $2 \leq n \leq 4$, when there are infinitely many decompositions*
- *rank 7 in $S^3\mathbb{C}^5$, when there are infinitely many decompositions*
- *rank 9 in $S^6\mathbb{C}^3$, where there are exactly two decompositions*
- *rank 8 in $S^4\mathbb{C}^4$, where there are exactly two decompositions*

The cases listed in red are called the *defective cases.*
The cases listed in blue are called the *weakly defective cases.*

Defectivity and weak defectivity

Weak defectivity goes back to classical papers by Terracini and it has been studied extensively by Chiantini and Ciliberto.
We realized recently that it allows computer experiments regarding the tensor decomposition.

Weakly defective examples

Assume for simplicity $k = 3$. Only known examples where the general $f \in V_1 \otimes V_2 \otimes V_3$ ($\dim V_i = n_i + 1$) of subgeneric rank s has a NOT UNIQUE decomposition, besides the defective ones, are

- unbalanced case, rank $s = n_1 n_2 + 1$, $n_3 \geq n_1 n_2 + 1$
- rank 6 $(n_1, n_2, n_3) = (3, 3, 3)$ where there are two decompositions
- rank 8 $(n_1, n_2, n_3) = (2, 5, 5)$, sporadic case [CO], maybe six decompositions

Theorem

- *The unbalanced case is understood [Chiantini-O. [2011]].*
- *There is a unique decomposition for general tensor of rank* s *in* $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$
  if $s \leq \frac{3n+1}{2}$ *[Kruskal[1977]]*
  if $s \leq \frac{(n+2)^2}{16}$ *[Chiantini-O. [2011]]*
- *The exceptions to uniqueness listed in the previous slide are the only ones in the cases*
  (i) $n_i \leq 6$
  (ii) $s \leq 6$ *[Chiantini-O. [2011]]*

Proof uses a generalization of the inductive technique in [AOP] plus the weak defectivity.

The contact locus

It is given by the subvariety of X where the general hyperplane tangent at k general points is tangent.
In the case $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ and $k = 6$, the contact locus is an ellipitic normal curve.
In the case $\mathbb{P}^2 \times \mathbb{P}^5 \times \mathbb{P}^5$ and $k = 8$, the contact locus is $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ embedded with $\mathcal{O}(3, 1, 1)$.