Special Session on Secant Varieties and Related Topics

Secant Varieties of Grassmann and Segre Varieties

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Content

- 1) Historical perspective, the Alexander-Hirschowitz Theorem for the Veronese Varieties

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- 2) Segre Varieties

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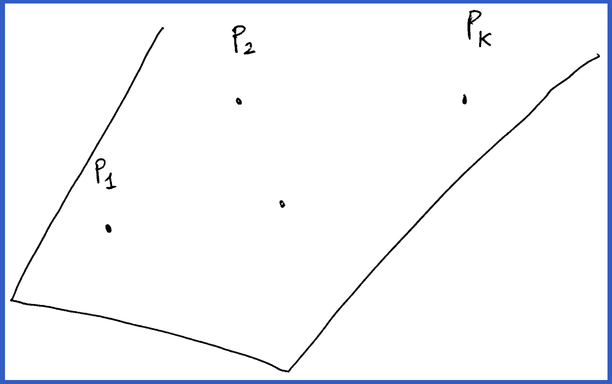
- 2) Segre Varieties
- 3) Grassmann Varieties

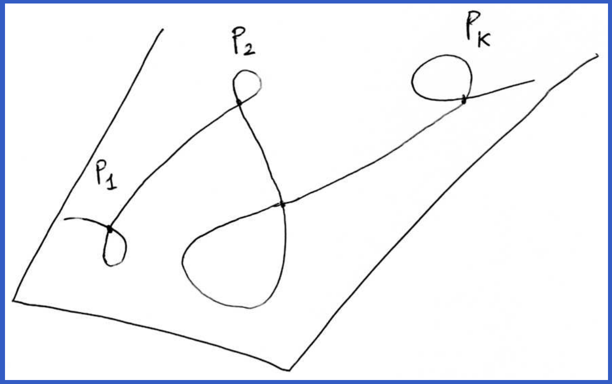
Interpolation problem

- Fix k general points $P_1, \ldots, P_k$ in the plane



Interpolation problem

- Fix k general points $P_1, \ldots, P_k$ in the plane
- How many plane curves of degree d exist which are singular at $P_1, \ldots, P_k$?



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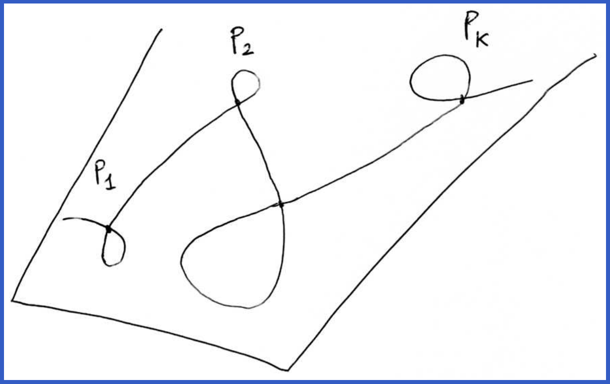

The expected dimension of the linear system is $\max(\binom{d+2}{2} - 3k, 0)$

Campbell Theorem, 1892

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  The dimension of the system of plane curves of degree d, singular at k general points, is $\max(\binom{d+2}{2} - 3k, 0)$ with the only exceptions
  - $(d, k) = (2, 2)$ conics through two points
  - $(d, k) = (4, 5)$ quartics through five points

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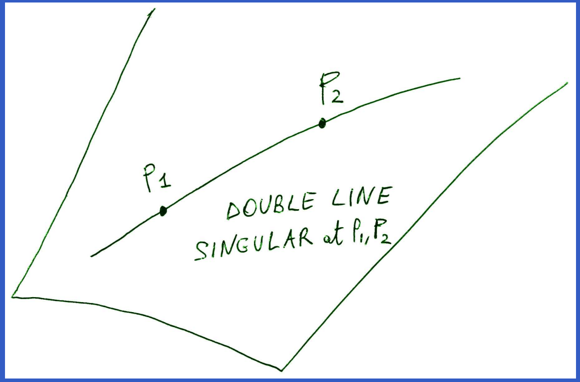

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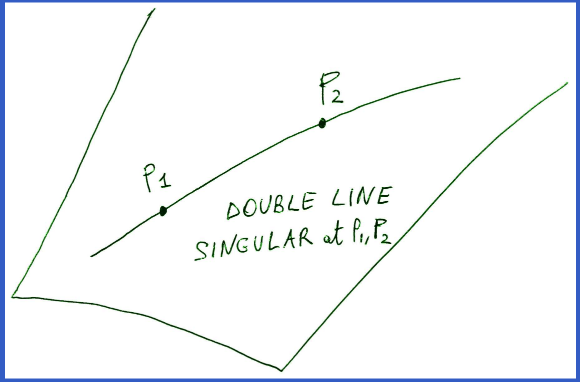

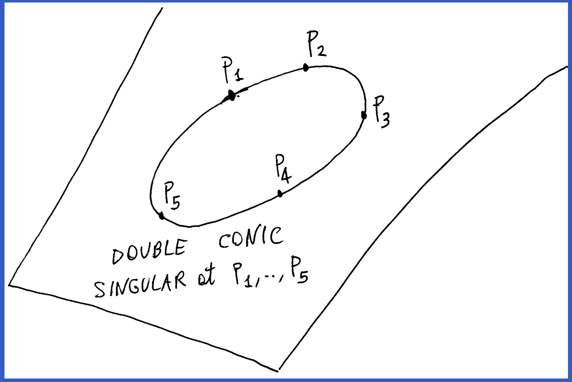

Terracini's work

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- Let $X_1, \ldots, X_s \subset \mathbf{P}^N$ be irreducible varieties. The join of $X_1, \ldots, X_s$ is

$$J(X_1, \ldots, X_s) := \overline{\bigcup_{x_i \in X_i} < x_1, \ldots, x_s >}$$

where the overbar means Zariski closure.

Dimension of the join

- Its virtual dimension is

$$\text{Virt dim } J(X_1, \dots, X_s) = \sum_{i=1}^{s} \dim X_i + (s-1)$$

and its expected dimension is

$$\text{Exp dim } J(X_1, \dots, X_s) = \min\{\sum_{i=1}^{s} \dim X_i + (s-1), N\}$$

The higher secant variety

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- The minimal s such that $\sigma_s(X)$ fills the ambient space is called the typical rank and it is denoted by $\underline{R}(X)$.

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- The minimal s such that $\sigma_s(X)$ fills the ambient space is called the typical rank and it is denoted by $\underline{R}(X)$.
- X is called defective if there exists a p such that $\dim \sigma_p(X) < \text{Exp} \dim(\sigma_p(X))$.

Terracini lemma

- Terracini Lemma Let $P_i \in X_i$ and $z \in < P_1, \ldots, P_k >$ be general. Then

$$T_z J(X_1, \ldots, X_k) = < T_{x_1} X_1, \ldots, T_{x_k} X_k >$$

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- Corollary Exceptional cases in polynomial interpolation correspond to defective Veronese varieties.

Alexander-Hirschowitz Theorem, (1995

Theorem[AH], Classification of defective Veronese varieties Let $d \geq 3$. $\sigma_s(v_d(\mathbf{P}^n))$ has the expected dimension with the only exceptions:

		codim	exp. codim
1)	$\sigma_5(v_4(\mathbf{P}^2))$	1	0
2)	$\sigma_9(v_4(\mathbf{P}^3))$	1	0
3)	$\sigma_{14}(v_4(\mathbf{P}^4))$	1	0
4)	$\sigma_7(v_3(\mathbf{P}^4))$	1	0

Equations of the exceptional cases, I

- In the cases 1), 2), 3), the equation of the 'last' secant variety is the catalecticant invariant (Clebsch). For $\phi \in S^4V$ let $A_\phi \colon S^2V^\vee \to S^2V$ be the contraction operator. Then $\det A_\phi$ is the catalecticant invariant.

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- For $n = 2$ it has degree 6 and it gives the condition to express a homogeneous quartic polynomial in 3 variables as the sum of 5 fourth powers (Waring problem).
- *Sketch of proof:* If $\phi \in v_4(\mathbf{P}^2)$ then $rk(A_\phi) = 1$. If $\phi \in \sigma_5(v_4(\mathbf{P}^2))$ it follows that $rk(A_\phi) \leq 5$.

Equations of the exceptional cases, II

- In the case 4) let $\phi \in S^3V$, where $\dim V = 5$. Let $B_\phi\colon \Gamma^{2,2,1,1}V \longrightarrow \Gamma^{2,1,1}V$ be the $SL(V)$-invariant contraction operator.

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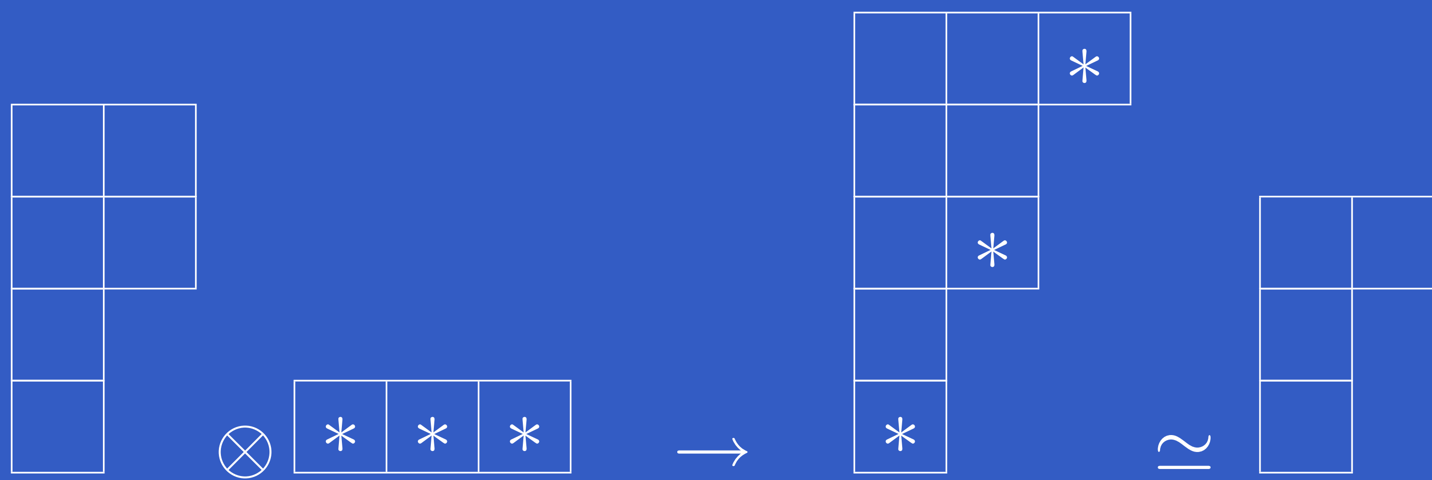
- 

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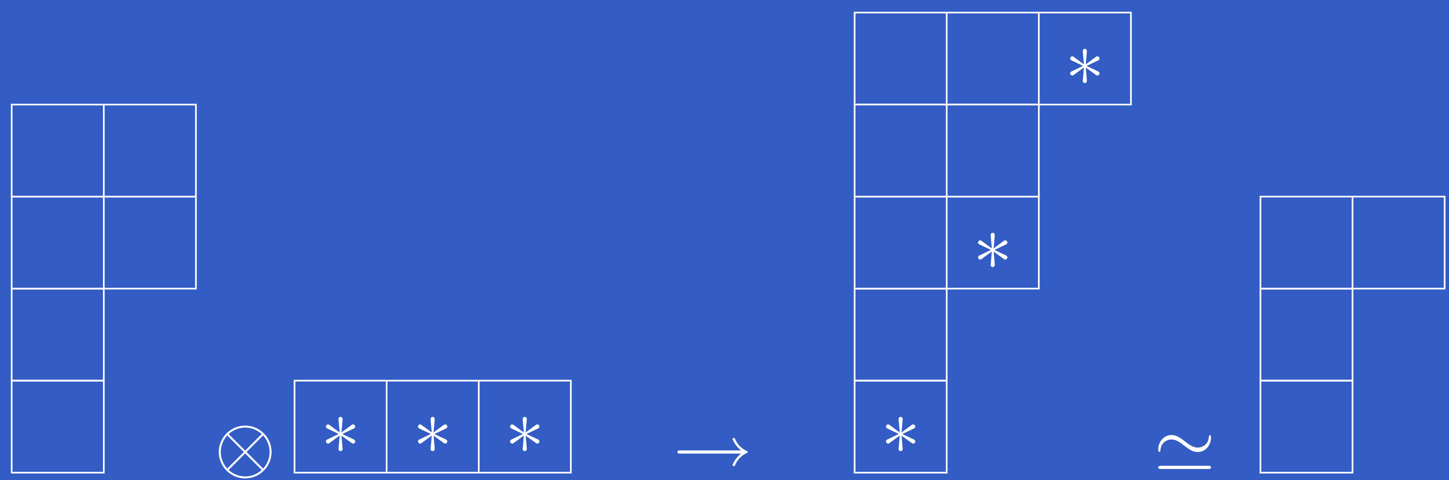
- 
- Theorem, arXiv:0712.2527 $\det B_\phi = 2P(\phi)^3$ where P is the equation of $\sigma_7(v_3(\mathbf{P}^4))$, it has degree 15.

Waring problem for cubics

- *Sketch of proof:* If $\phi \in v_3(\mathbf{P}^4)$ then $rk(B_\phi) = 6$. If $\phi \in \sigma_7(v_3(\mathbf{P}^4))$ it follows that $rk(B_\phi) \leq 42$, while $\dim \Gamma^{2,2,1,1} V = 45$.

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- $P(\phi) = 0$ gives the condition to express the homogeneous cubic polynomial ϕ in 5 variables as the sum of 7 cubes.

Sketch of proof of AH-Theorem,I

Let $s = s' + s''$. Specialize s' points on a hyperplane $X' = v_d(\mathbf{P}^{n-1}) \subset X = v_d(\mathbf{P}^n)$.

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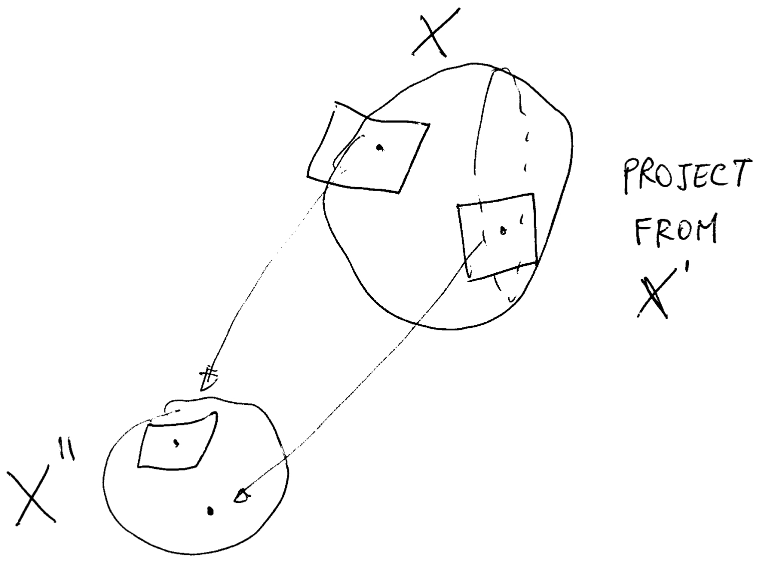

Sketch of proof of AH-Theorem,II

- Splitting Theorem for Veronese varieties
  If Dim$J(s'X')$ = Virt Dim$J(s'X')$ AND
  Dim$J(s''X'', s'P)$ = Virt Dim$J(s''X'', s'P)$
  it follows that
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- May apply induction!

Sketch of proof of AH-Theorem,III

- First difficulty: what is the starting case of the induction? Quadrics are defective. They correspond to the matrices of rank $\leq k$ inside the symmetric matrices of order $n+1$. See next talk by C. Brambilla for a generalization to partial polynomial interpolation. Cubics become the starting case.

Sketch of proof of AH-Theorem,III

- First difficulty: what is the starting case of the induction? Quadrics are defective. They correspond to the matrices of rank $\leq k$ inside the symmetric matrices of order $n+1$. See next talk by C. Brambilla for a generalization to partial polynomial interpolation. Cubics become the starting case.
- If we show that cubics are not defective for $n \geq 5$, then the inductive procedure shows that $\underline{\mathrm{R}}(v_d(\mathbf{P}^n)) \sim \binom{n+d}{d}/(n+1)$ if $n \to \infty$ or $d \to \infty$. (weak asympt. version of AH-theor.)

The tropical approach

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- As far as I know, at present the proof works for $n \leq 3$.

The case of Segre Varieties

- $\sigma_4(\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2)$ and $\sigma_3(\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1)$ (four qubits) are the first defective cases.

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- In 1985, Lickteig completes previous work by Strassen and shows that

$$\mathbf{P}^n \times \mathbf{P}^n \times \mathbf{P}^n$$

is never defective for $n \geq 3$.

Some cases where equations are known

- Landsberg and Manivel (2003) show that $\sigma_2(\mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \times \ldots \times \mathbf{P}^{n_k})$ is defined by the cubics of the various flattening, algebraically for $k = 3$ and set-theoretically for $k \geq 4$.

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- Allman and Rhodes (2004) extend the algebraic statement to $k \leq 5$. Garcia, Sturmfels and Sullivant conjecture that this is true $\forall k$.

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- Allman and Rhodes (2004) extend the algebraic statement to $k \leq 5$. Garcia, Sturmfels and Sullivant conjecture that this is true $\forall k$.
- Landsberg and Weyman (2006) have found equations for $\sigma_k(\mathbf{P}^1 \times \mathbf{P}^{n_2} \times \mathbf{P}^{n_3})$, $\sigma_2(\mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \times \mathbf{P}^{n_3} \times \mathbf{P}^{n_4})$ and $\sigma_3(\mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \times \mathbf{P}^{n_3})$

The unbalanced case

- **Definition** Let $n_1 \leq n_2 \leq \ldots \leq n_k$. $\mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \times \ldots \times \mathbf{P}^{n_k}$ is called *balanced* if $\sum_{i=1}^{k} n_i \leq \prod_{i=1}^{k-1}(n_i + 1)$. Otherwise is called *unbalanced*.

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- Unbalanced means $n_i \ll n_k$
- Catalisano-Geramita-Gimigliano (2006) find, in the unbalanced case, equations for $\sigma_s(\mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \times \ldots \times \mathbf{P}^{n_k})$ when s is sufficiently large and describe exactly which σ_s are defective. Unbalanced implies defective.

Many copies of $\mathbb{P}^1$

- Catalisano-Geramita-Gimigliano in 2005 prove that $\sigma_s(\mathbf{P}^1 \times \mathbf{P}^1 \times \ldots \times \mathbf{P}^1)$ is never defective, with at most one exception for any such variety.

Many copies of $\mathbb{P}^1$

- Catalisano-Geramita-Gimigliano in 2005 prove that $\sigma_s(\mathbf{P}^1 \times \mathbf{P}^1 \times \ldots \times \mathbf{P}^1)$ is never defective, with at most one exception for any such variety.
- The only known defective case is $\sigma_3(\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1)$ which has codimension 2, while the expected codimension is 1.

Inductive technique for Segre varieties

- Let $X = \mathbf{P}^{n_1} \times \mathbf{P}^{n_2} \times \ldots \times \mathbf{P}^{n_k}$. Fix a linear subspace $\mathbf{P}^{n_1'} \subset \mathbf{P}^{n_1}$. Specialize s' points on $X' = \mathbf{P}^{n_1'} \times \mathbf{P}^{n_2} \times \ldots \times \mathbf{P}^{n_k}$ and project from X' on $X'' = \mathbf{P}^{n_1''} \times \mathbf{P}^{n_2} \times \ldots \times \mathbf{P}^{n_k}$, where $(n_1' + 1) + (n_1'' + 1) = (n_1 + 1)$.

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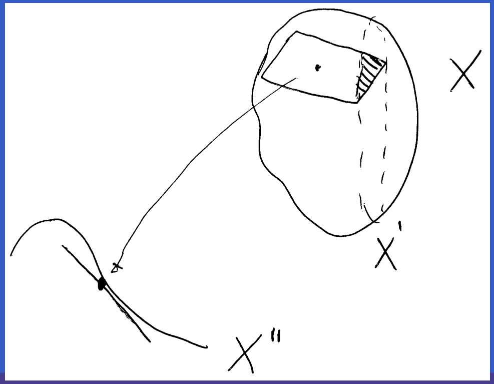
- The difficulty is that, even if a point is not specialized, its tangent space meets $< X' >$.



The tangent spaces of Segre varieties

- The tangent space at $v_1 \otimes v_2 \otimes v_3$ is $V_1 \otimes v_2 \otimes v_3 + v_1 \otimes V_2 \otimes v_3 + v_1 \otimes v_2 \otimes V_3$. The three summands are $E(Q_1)$, $E(Q_2)$, $E(Q_3)$, where Q_i is the i-th quotient bundle and $E(Q_i)$ is the Poincaré dual of the Euler class of Q_i.

The splitting Theorem[AOP]

- Splitting Theorem for Segre Varieties [AOP]
  Let $s = s' + s''$, $a_i = a_i' + a_i''$. If

$$\text{Dim } J(s'X', (a_1 + s'')E(Q_1'), a_2'E(Q_2), a_3'E(Q_3)) = \text{Virt Dim } J(\ldots)$$

AND

$$\text{Dim } J(s''X'', (a_1 + s')E(Q_1''), a_2''E(Q_2), a_3''E(Q_3)) = \text{Virt Dim } J(\ldots)$$

then

$$\text{Dim } J(sX, a_1E(Q_1), a_2E(Q_2), a_3E(Q_3)) = \text{Virt Dim } J(\ldots)$$

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  $$\text{Dim } J(sX, a_1E(Q_1), a_2E(Q_2), a_3E(Q_3)) = \text{Virt Dim } J(\ldots)$$

- The same is true if the joins fill their ambient space.

Notation for statement T

The statement

$$Dim\ J(sX, a_1E(Q_1), a_2E(Q_2), a_3E(Q_3)) = ExpDim\ J(\dots)$$

is denoted by

$$T(n_1, n_2, n_3; s; a_1, a_2, a_3)$$

The goal is to prove $T(n_1, n_2, n_3; s; 0, 0, 0)$ for as many s as possible.

The inductive procedure at work

Example: $\sigma_6(\mathbf{P}^3 \times \mathbf{P}^3 \times \mathbf{P}^3)$

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$$T(3,3,3;6;0,0,0)$$

$$\uparrow$$

$$2T(1,3,3;3;3,0,0)$$

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Example: $\sigma_6(\mathbf{P}^3 \times \mathbf{P}^3 \times \mathbf{P}^3)$

$$
\begin{array}{ccc}
T(3,3,3;6;0,0,0) & & \\
\uparrow & & \\
2T(1,3,3;3;3,0,0) & \leftarrow & 2T(1,1,3;1;2,2,0) \\
\uparrow & & \\
2T(1,1,3;2;1,1,0) & &
\end{array}
$$

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starting cases, I

- $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ Up to permutation of the three factors the list of defective cases is $(0; 0, 1, 3), (1; 0, 0, 2)$

starting cases, I

- $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ Up to permutation of the three factors the list of defective cases is $(0;0,1,3)$, $(1;0,0,2)$
- Why $(1;0,0,2)$ is defective ? Consider X as a pencil of smooth quadrics parametrized by the third factor. A point of X is a point of one of these quadrics, say Q. The two lines meet Q in two disjoint points, and the line spanned by these points meets every tangent plane of Q.

starting cases, II

- $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ Up to permutation of the first two factors the list of minimal defective cases is $(0;0,1,3)$, $(0;0,4,1)$, $(0;5,1,0)$, $(1;0,3,0)$, $(1;0,0,2)$

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$(0;0,1,3)$, $(0;0,4,1)$, $(0;5,1,0)$, $(1;0,3,0)$, $(1;0,0,2)$
- We have also the lists for $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ and $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$

starting cases, II

- $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^2$ Up to permutation of the first two factors the list of minimal defective cases is $(0;0,1,3)$, $(0;0,4,1)$, $(0;5,1,0)$, $(1;0,3,0)$, $(1;0,0,2)$
- We have also the lists for $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ and $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$
- Examples of applications of the inductive technique are $\underline{R}(2^3) = 5$, $\underline{R}(2^4) = 9$, $\underline{R}(2^5) = 23$, $\underline{R}(3^3) = 7$, $\underline{R}(3^4) = 20$, $\underline{R}(3^5) = 64$, $\underline{R}(3^6) = 215$,

Asymptotic behaviour is non defective

- Theorem [Abo, O., Peterson] Let $X = (\mathbf{P}^n)^k$, $k \geq 3$. Let $s_k := \lfloor \frac{(n+1)^k}{nk+1} \rfloor$ and $\delta_k := s_k$ mod $(n+1)$.
(i) If $s \leq s_k - \delta_k$ then $\sigma_s(X)$ has the expected dimension.
(ii) If $s \geq s_k - \delta_k + n + 1$ then $\sigma_s(X)$ fills the ambient space.

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- Corollary on typical rank $\underline{\mathrm{R}}(n^k) \sim \frac{(n+1)^k}{nk+1}$ if $n \to \infty$ or $k \to \infty$.

A Conjecture on Segre varieties

Conjecture Let $d \geq 3$. $\sigma_s(\mathbf{P}^{n_1} \times \ldots \times \mathbf{P}^{n_d})$ has the expected dimension with the only exceptions:

		codim	exp. codim
1)	unbalanced	$\ldots$	$\ldots$
2)	$\sigma_{\frac{3n}{2}+1}(\mathbf{P}^2 \times \mathbf{P}^n \times \mathbf{P}^n)$, n even	1	0
3)	$\sigma_{2n+1}(\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^n)$	2	1
4)	$\sigma_5(\mathbf{P}^2 \times \mathbf{P}^3 \times \mathbf{P}^3)$	4	3

Conjecture true for $k \leq 6$

Theorem [Abo,O., Peterson] The conjecture for $\sigma_k(\text{Segre})$ is true if $k \leq 6$.

Equations in the defective cases, I

In the unbalanced case([CGG]), and in the case $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^n \times \mathbf{P}^n$ ([CGG], Carlini), the flattening technique works.

Equations in the defective cases, II

- Consider the case $\mathbf{P}^2 \times \mathbf{P}^n \times \mathbf{P}^n = \mathbf{P}(U) \times \mathbf{P}(V) \times \mathbf{P}(V')$. For every $\phi \in U \otimes V \otimes V'$ define the contraction $A_\phi : U \otimes V^\vee \to \wedge^2 U \otimes V'$

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- If P, Q, R are the three $(n+1) \times (n+1)$ slices of ϕ, the matrix representing A_ϕ is

$$\begin{bmatrix} 0 & P & -Q \\ -P & 0 & R \\ Q & -R & 0 \end{bmatrix}$$

Equations in the defective cases, III

- Theorem(Strassen, 1983, but in a different form) For n even, $\det(A_\phi)$ is the equation of $\sigma_{\frac{3n}{2}+1}(\mathbf{P}^2 \times \mathbf{P}^n \times \mathbf{P}^n)$, which has degree $3(n+1)$.

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- *Sketch of proof:* If $\phi \in \mathbf{P}^2 \times \mathbf{P}^n \times \mathbf{P}^n$ then $rk(A_\phi) = 2$. If $\phi \in \sigma_{\frac{3n}{2}+1}(\mathbf{P}^2 \times \mathbf{P}^n \times \mathbf{P}^n)$ it follows that $rk(A_\phi) \leq 3n+2$, while $\dim U \otimes V^\vee = 3(n+1)$.

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- When n is odd, the above determinant vanishes on $\sigma_{\frac{3n+1}{2}}(\mathbf{P}^2 \times \mathbf{P}^n \times \mathbf{P}^n)$, which has bigger codimension.

A Conjecture for Grassmannians

- **Conjecture I** Let $k \geq 2$. $\sigma_s(Gr(k,n))$ has the expected dimension with the only exceptions:

		codim	exp. codim
$1)$	$\sigma_3(Gr(2,6))$	1	0
$2)$	$\sigma_3(Gr(3,7))$	20	19
$2')$	$\sigma_4(Gr(3,7))$	6	2
$3)$	$\sigma_4(Gr(2,8))$	10	8

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- All the examples have been written by Catalisano, Geramita, Gimigliano (2002), with the help of Catalano-Johnson.

Evidence for the conjecture

- Theorem The conjecture is true by Montecarlo computations for $n \leq 14$ (McGillivray 2005)
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The inductive step for Grassmannians

- Let $X = Gr(k, n)$. Specialize some points on $X' = Gr(k, n-1)$ and project to $X'' = Gr(k-1, n-1)$.

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- Let $X = Gr(k,n)$. Specialize some points on $X' = Gr(k, n-1)$ and project to $X'' = Gr(k-1, n-1)$.
- Let U and Q be the universal and the quotient bundle on $Gr(k,n)$. Let $E(Q)$ be the Poincaré dual of the Euler class of Q, namely $E(Q) = \{\mathbf{P}^k | \mathbf{P}_0^{k-1} \subset \mathbf{P}^k\} \simeq \mathbf{P}^{n-k}$ for a fixed $\mathbf{P}_0^{k-1}$.

Splitting Theorem for Grassmannians

- Splitting Theorem for Grassmann varieties[AOP] Let $s = s' + s''$, $a = a' + a''$, $b = b' + b''$. Let P be the class of a point. If
  Dim $J(s'Gr(k, n-1), (s''+a')E(Q), b'E(U^\vee), b''P) =$
  Virt Dim $J(\ldots)$
  AND
  Dim $J(s''Gr(k-1, n-1), a''E(Q), (s'+b'')E(U^\vee), a'P) =$
  Virt Dim $J(\ldots)$
  then
  Dim $J(sGr(k,n), aE(Q), bE(U^\vee)) =$ Virt Dim $J(\ldots)$

A stronger Conj. for Grassmannians

Conjecture II Let $k \geq 2$. $J(sGr(k,n), a\mathbf{P}^{n-k}, b\mathbf{P}^{k+1})$ has the expected dimension with the only exceptions for (s,a,b,k,n), up to duality:

- $(2,0,1,2,6)$ $(2,0,2,2,6)$ $(2,1,1,2,6)$ $(2,2,0,2,6)$ <u>(3,0,0,2,6)</u>
- $(3,1,0,2,7)$
- <u>(3, 0, 0,3,7)</u> $(3,0,1,3,7)$ $(3,0,2,3,7)$ $(3,0,3,3,7)$ $(3,1,1,3,7)$ $(3,1,2,3,7)$ <u>(4, 0, 0,3,7)</u> $(4,0,1,3,7)$
- <u>(4, 0, 0,2, 8)</u> $(4,0,1,2,8)$ $(4,0,2,2,8)$ $(4,1,0,2,8)$

Starting case for Grassmannians

We still need to manage with the starting case of the induction. It is $Gr(2, n)$, Grassmannians of planes. It turns out that the technique given in [Brambilla-O.], to prove the cubic case in AH-theorem, works also for $Gr(2, n)$.

Proof of cubic case in AH-Theor. [BO],

- Cubics in $\mathbf{P}^n$ have dim $f(n) = \frac{(n+3)(n+2)(n+1)}{6}$
  Consider $\frac{(n+3)(n+2)}{6}$ points P_i (it is an integer if $n \neq 2$ mod 3)

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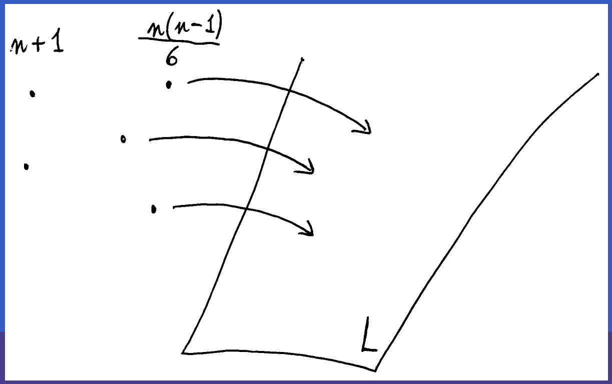

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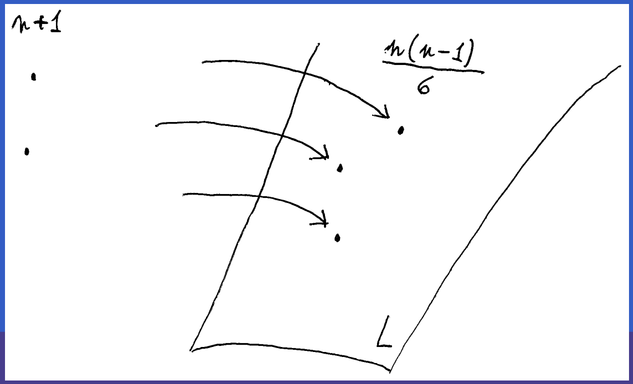

Proof of cubic case in AH-Theor., II

Applying induction we reduce to cubics in $\mathbf{P}^n$ containing L. They have dim
$\Delta_3 f(n) = f(n) - f(n-3) = \frac{3n^2}{2} + \frac{3n}{2} + 1$

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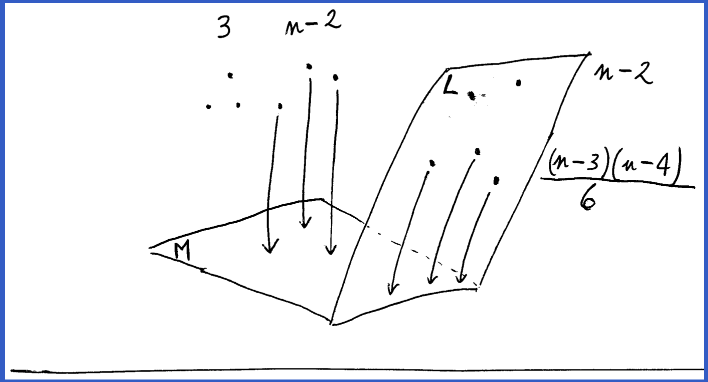

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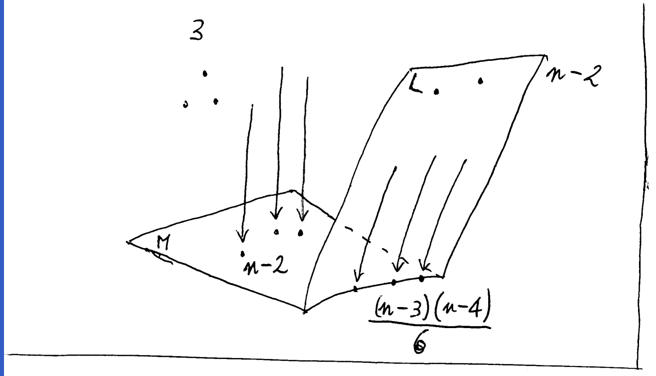

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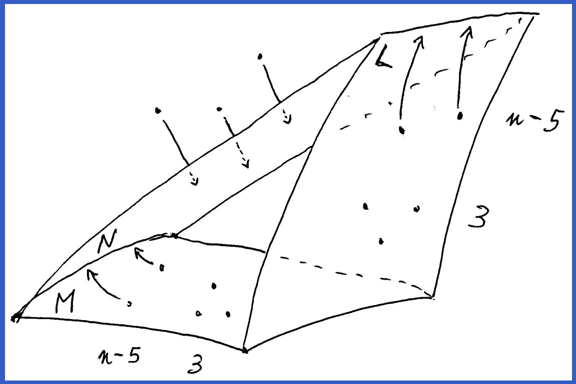

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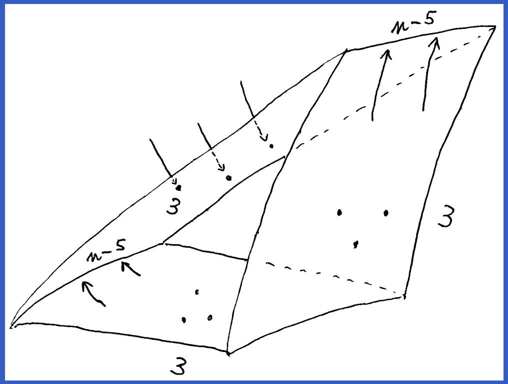

Proof of cubic case in AH-Theor., IV

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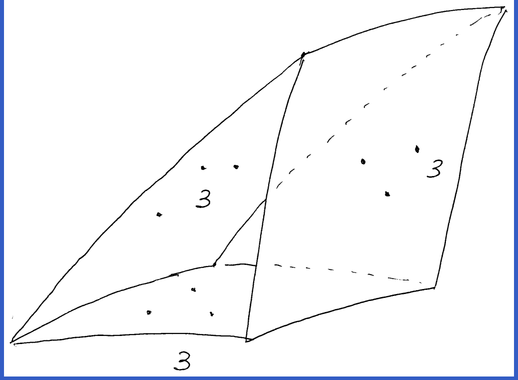

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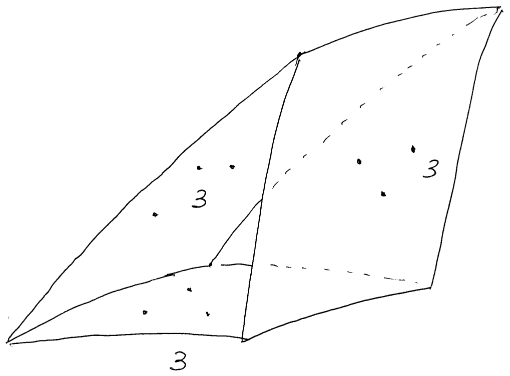

It is enough to compute the rank of a 27×27 matrix. It is 27 and the cubic case is proved.

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- Ehrenborg proved (1999) that $\underline{R}(Gr(2,n)) \leq \frac{n^2}{12} + O(n)$
- Application of the technique: Theorem[AOP] $\underline{R}(Gr(2,n)) \sim \frac{n^2}{18}$ (sharp asymptotical value)