J. Math. Anal. Appl. ••• (••••) ••••••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Growth conditions and regularity for weak solutions to nonlinear elliptic pdes

Paolo Marcellini

Università di Firenze, Italy

A R T I C L E I N F O

Article history:
Received 30 April 2020
Available online xxxx
Submitted by G. Mingione

Keywords:
Elliptic equations
Calculus of variations
Regularity of solutions
Local Lipschitz continuity
$p, q-$growth conditions
General growth conditions

A B S T R A C T

We describe some aspects of the process/approach to interior regularity of *weak solutions* to a class of nonlinear elliptic equations in divergence form, as well as of *minimizers* of integrals of the calculus of variations. In particular with respect to the *growth conditions* we emphasize some differences between solving equations and minimizing energy integrals, and also between $x-$dependence or $x-$independence.

Contents

1. Introduction . . . 2
2. The process/approach to regularity . . . 6
   2.1. The second regularity step (2nd) . . . 7
   2.2. The first regularity step (1st) . . . 9
   2.3. Conclusion: from $W^{1,p}$ to $C^{1,\alpha}$ or C^{∞} . . . 9
3. Some examples with general growth . . . 9
   3.1. General $p, q-$growth . . . 9
   3.2. Double phase integrals . . . 11
4. Proofs of Theorems 1.1, 1.2, 1.3, and 1.4 . . . 12
   4.1. First step: a-priori estimates . . . 13
      4.1.1. Difference quotient . . . 14
      4.1.2. Weak solutions with second derivatives . . . 17
      4.1.3. Moser's iteration scheme for the gradient . . . 17
   4.2. Second step: interpolation . . . 22
      4.2.1. Description and statement of the interpolation . . . 22
      4.2.2. Proof of the interpolation Theorem 4.4 . . . 24
   4.3. Third step: approximation . . . 26
      4.3.1. Proof of Theorem 1.1 . . . 26
      4.3.2. Proof of Theorem 1.2 . . . 28
      4.3.3. Proof of Theorem 1.3 . . . 28
      4.3.4. Proof of Theorem 1.4 . . . 29
References . . . 30

E-mail address: paolo.marcellini@unifi.it.

https://doi.org/10.1016/j.jmaa.2020.124408

1. Introduction

We propose a full self-contained proof of the interior regularity of *weak solutions* to a class of nonlinear elliptic partial differential equations and of local minimizers of some integrals of the calculus of variations. We focus on the *growth conditions*, since we deal with *natural growth conditions*, as well as with *general growth conditions*, for instance the so-called *double phase integrals*, which are a special case of the $p,q-$*growth*, considered in details in this paper. The simplest *nonlinear* model under *natural* growth conditions is the $p-$*Dirichlet integral*, or the *non-degenerate* $p-$*Dirichlet integral*, respectively given by

$$\int_{\Omega} |Du(x)|^p \, dx \,, \quad \int_{\Omega} \left(1+|Du|^2\right)^{\frac{p}{2}} dx \,, \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^n$ for some $n \geq 2$, $u = u(x)$ is a map from $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}$ and Du is its *gradient*. Any local minimizer to the $p-$Dirichlet integral in (1.1) is a weak solution for $x \in \Omega$ to the $p-$*Laplace equation*, or respectively to the *non-degenerate* $p-$*Laplace equation*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ |Du|^{p-2} u_{x_i} \right\} = 0 \,; \quad \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left\{ \left(1+|Du|^2\right)^{\frac{p-2}{2}} u_{x_i} \right\} = 0 \,. \tag{1.2}$$

These equations enter in the regularity theory presented here, but more general cases are considered, some examples being described in Section 3. For instance we consider weak solutions to a class of variational elliptic equations of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du(x)) = b(x) \,, \qquad x \in \Omega \,, \tag{1.3}$$

where the vector field $\left(a^i(x,\xi)\right)_{i=1,\ldots,n}$ is locally Lipschitz continuous in $\Omega \times \mathbb{R}^n$ and it satisfies the so called $p,q-$*growth conditions* as in (1.10), (1.11) below. In this general context, i.e. under $p,q-$growth conditions with $q \neq p$, we notice that even the existence of weak solutions to the elliptic equation (1.3) in general is an open problem; in particular in a naive way, in analogy when (1.3) is derived through the first variation of a minimization process, we could expect weak solutions to be in the Sobolev class $W^{1,p}(\Omega)$, however the pairing, i.e. the distributional *weak form* of the equation, is well defined only if $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$; see (4.2), (4.3) for details.

One of our aims is to emphasize that, in the context of *general growth conditions*, there is a difference both in the proofs but also in the growth assumptions, between the regularity of minimizers of integral of the calculus of variations and the regularity of weak solutions to elliptic equations. This is clear if we compare the different growth assumptions (1.5), (1.17) respectively in the Theorems 1.1 and 1.4 below: we have $\frac{q}{p} < 1 + \frac{2}{n}$ versus $\frac{q}{p} < 1 + \frac{1}{n}$. A second similar difference appears, in the minimization context, when we compare energy integrands explicitly depending on x or not; see condition (1.5) in Theorem 1.1 versus (1.9) of Theorem 1.2 with x dependence: again $\frac{q}{p} < 1 + \frac{2}{n}$ versus $\frac{q}{p} < 1 + \frac{1}{n}$.

An other aim in this paper is to emphasize an aspect of the process/approach to regularity of weak solutions to nonlinear elliptic equations and systems under general growth conditions. This will be described more precisely in Section 2, where we divide this process of interior regularity into two steps: 1st$-$ from either a *minimizer*, or a *weak solution*, $u \in W^{1,p}$ to $W^{1,\infty}_{\mathrm{loc}}$; 2nd$-$ from a weak solution $u \in W^{1,p} \cap W^{1,\infty}_{\mathrm{loc}}$, under some smoothness of the data, to *more regularity* of the type $C^{1,\alpha}$, or C^k, or C^∞.

As explained in Section 2, we concentrate here to the local boundedness of the gradient; i.e. to the local Lipschitz regularity of local minimizers and of weak solutions. We start to consider an energy integral of the type $\int_\Omega f(Du(x))\,dx$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function of class $C^2(\mathbb{R}^n)$ satisfying the $p,q-$growth conditions

$$m\,|\xi|^{p-2}\,|\lambda|^2 \le \sum_{i,j=1}^{n} \frac{\partial^2 f(\xi)}{\partial \xi_i \partial \xi_j}\lambda_i\lambda_j \le M\left(1+|\xi|^2\right)^{\frac{q-2}{2}}|\lambda|^2\,, \tag{1.4}$$

for some exponents $q \ge p \ge 2$, some positive constants m, M and for every $\lambda, \xi \in \mathbb{R}^n$. We denote by B_R a generic ball of radius R compactly contained in Ω and by B_ϱ a ball of radius $\varrho < R$ concentric with B_R.

Theorem 1.1 *(Local minimizers). Let u be a local minimizer in $W^{1,p}(\Omega)$ of the energy integral $\int_\Omega f(Du(x))\,dx$ with $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the $p, q-$growth conditions in (1.4) and*

$$\frac{q}{p} < 1 + \frac{2}{n}\,. \tag{1.5}$$

Then u is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and there exist constants c, c', δ such that

$$\begin{aligned}\|Du(x)\|_{L^\infty(B_\varrho;\mathbb{R}^n)} &\le \left(\frac{c}{(R-\varrho)^\delta}\int_{B_R}\left(1+|Du|^2\right)^{\frac{p}{2}}dx\right)^{\frac{2}{(n+2)p-nq}} \\ &\le \left(\frac{c'}{(R-\varrho)^\delta}\int_{B_R}\{1+f(Du)\}\,dx\right)^{\frac{2}{(n+2)p-nq}}\,,\end{aligned} \tag{1.6}$$

for all ϱ, R with $0 < \varrho < R \le \varrho + 1$. The constants c, c', δ depend only on p, q, n.

Note that $(n+2)\,p - nq > 0$ and thus the exponent in the right hand side of the above gradient estimates is well defined. Moreover, since $\frac{2}{(n+2)p-nq} \ge \frac{1}{p}$, this exponent has the form $\frac{\vartheta}{p}$, i.e. the right hand side can be compared with $\|Du(x)\|^{\vartheta}_{L^p(B_R;\mathbb{R}^n)}$, where ϑ is greater than or equal to 1 and it is equal to 1 if and only if $q = p$. In the statement of Theorem 1.1 we mainly refer to the specific notation for the case $n > 2$; while details when $n = 2$ can be found below, see specifically the Remarks 4.2, 4.5 and 4.7. In particular for $n = 2$ the proper exponent in the right hand side of (1.6), derived from the application of Theorem 4.4, is $\frac{\alpha}{p}$ with α given in (4.33) which is the correct exponent in any dimension $n \ge 2$ and it reduces to $\frac{2}{(n+2)p-nq}$ when $n > 2$.

A similar gradient bound as in Theorem 1.1 can be proved for energy integral of the type

$$\int_\Omega \{f(Du) + b(x)\,u\}\,dx \tag{1.7}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function of class $C^2(\mathbb{R}^n)$ satisfying the $p, q-$growth conditions (1.4). In fact, by mean of the Euler's first variation, this case reduces to an example in the class of pde's in divergence form as in (1.3). Then the growth conditions (1.10), (1.11) below are satisfied with $a_i(\xi) = \frac{\partial f(\xi)}{\partial \xi_i}$, since the left hand side of (1.11) is identically equal to zero. With the same proof of Theorem 1.1 and Theorem 1.3 below we can obtain the gradient bound (1.6) for the energy integral (1.7), in the case $b \in L^\infty_{\mathrm{loc}}(\Omega)$, with the $p, q-$growth conditions (1.4), $q \ge p \ge 2$ and $\frac{q}{p} < 1 + \frac{2}{n}$.

In this context we quote the recent article [5] by Beck-Mingione where, among other results, the authors extend Theorem 1.1 to functionals (1.7) with a forcing term $b(x)$, under sharp assumptions on the regularity of b. Precisely, under the same bound $q \ge p \ge 2$ and $\frac{q}{p} < 1+\frac{2}{n}$ (and also for some cases with $q \ge p > 1$; and for the vector-valued case with $f(\xi) = g(|\xi|)$ too) Beck-Mingione [5, Theorem 1.2] considered sharp assumptions on the function $b(x)$ of the type $b \in L(n, 1)$ in dimension $n > 2$; i.e., $\int_0^{+\infty} \operatorname{meas}\{x \in \Omega : |b(x)| > \lambda\}^{1/n}\,d\lambda < +\infty$ (note that $L^{n+\varepsilon} \subset L(n, 1) \subset L^n$) or $b \in L^2(\log L)^\alpha$ for some $\alpha > 2$ when $n = 2$.

We can also consider more general energy integrands $f = f(x,\xi)$ convex in $\xi \in \mathbb{R}^n$, with $f_{\xi\xi}(x,\xi)$, $f_{\xi x}(x,\xi)$ Carathéodory functions satisfying the $p,q-$growth conditions $(1 < p \le q)$

$$\begin{cases} m\,|\xi|^{p-2}\,|\lambda|^2 \le \sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j \le M\left(1+|\xi|^2\right)^{\frac{q-2}{2}}|\lambda|^2 \\ |f_{\xi x}(x,\xi)| \le\; M\left(1+|\xi|^2\right)^{\frac{q-1}{2}} \end{cases} \tag{1.8}$$

for constants $m, M > 0$, for a.e. $x \in \Omega$ and for all $\lambda, \xi \in \mathbb{R}^n$ with $|\xi| \ge 1$. The following is an a-priori estimate obtained in [44]; see also Theorem 3.4 in Section 3.2.

Theorem 1.2 *(Local minimizers with $x-$dependence [44]). Under the condition (1.8) with exponents p, q satisfying $1 < p \le q$ and*

$$\frac{q}{p} < 1 + \frac{1}{n}, \tag{1.9}$$

any smooth local minimizer in $W^{1,p}(\Omega)$ of the energy integral $\int_\Omega f(x, Du(x))\,dx$ is locally Lipschitz continuous in Ω and an estimate of the type (1.6) holds, with exponent $\frac{1}{(n+1)p-nq}$ in the right hand side.

We go back to the class of pde's in divergence form as in (1.3). We assume that the vector field $\left(a^i(x,\xi)\right)_{i=1,\dots,n}$ is locally Lipschitz continuous in $\Omega \times \mathbb{R}^n$ with $\left|a^i(x,0)\right|$ bounded in Ω, satisfying the so called *$p,q-$growth conditions:* for every $\lambda, \xi \in \mathbb{R}^n$, $x \in \Omega$ and for some exponents $p \le q$ and positive constants m, M

$$\sum_{i,j=1}^n \frac{\partial a_i}{\partial \xi_j}\lambda_i\lambda_j \ge m\left(1+|\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2, \quad \sum_{i,j=1}^n \left|\frac{\partial a_i}{\partial \xi_j}\right| \le M\left(1+|\xi|^2\right)^{\frac{q-2}{2}}, \tag{1.10}$$

$$\sum_{i,j=1}^n \left|\frac{\partial a_i}{\partial \xi_j} - \frac{\partial a_j}{\partial \xi_i}\right| \le M\left(1+|\xi|^2\right)^{\frac{p+q-4}{4}}, \quad \sum_{i,s=1}^n \left|\frac{\partial a_i}{\partial x_s}\right| \le M\left(1+|\xi|^2\right)^{\frac{p+q-2}{4}}. \tag{1.11}$$

Finally, for simplicity, we assume $b \in L^\infty_{\mathrm{loc}}(\Omega) \cap L^{\frac{p}{p-1}}(\Omega)$. We notice that the *Lax-Milgram existence theory* does not apply when the exponents p, q are different each other. A weak solution to the elliptic equation (1.3) needs to be a function u in the class $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$; in fact, the $p-$ellipticity condition alone in general is not sufficient for the existence of weak solutions in the Sobolev class $W^{1,p}(\Omega)$. See the details in Section 4, in particular the weak form of the equation in (4.2).

Theorem 1.3 *(Weak solutions with growth (1.10), (1.11)). Let Ω be an open and bounded set in $\mathbb{R}^n$. Under the $p,q-$growth conditions (1.10), (1.11), if $q \ge p \ge 2$ and*

$$\frac{q}{p} < 1 + \frac{2}{n}, \tag{1.12}$$

there exists a weak solution $u \in W^{1,p}_{u_0}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ to the Dirichlet problem

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(x, Du(x)) = b(x) \quad \text{in } \Omega; \quad u = u_0 \in W^{1,p\frac{q-1}{p-1}} \quad \text{on } \partial\Omega. \tag{1.13}$$

Moreover u is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and there exist positive constants c, δ such that, for every ϱ and R such that $0 < \rho < R \le \varrho + 1$,

$$\|Du(x)\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \leq \left(\frac{c}{(R-\varrho)^\delta} \int_{B_R} \left(1+|Du(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{2}{(n+2)p-nq}} . \tag{1.14}$$

Less strict growth conditions than (1.11) can be requested; for instance

$$\sum_{i,s=1}^{n} \left| \frac{\partial a_i}{\partial x_s} \right| \leq M \left(1+|\xi|^2\right)^{\frac{q-1}{2}}, \tag{1.15}$$

for every $\lambda, \xi \in \mathbb{R}^n$, $x \in \Omega$ and for some positive constants m, M. We notice that automatically from (1.10) we get

$$\sum_{i,j=1}^{n} \left| \frac{\partial a_i}{\partial \xi_j} - \frac{\partial a_j}{\partial \xi_i} \right| \leq 2M \left(1+|\xi|^2\right)^{\frac{q-2}{2}}; \tag{1.16}$$

therefore (1.10), (1.15) really are less strict than the growth conditions (1.10), (1.11).

Theorem 1.4 *(Weak solutions with growth (1.10), (1.15)). Let Ω be open and bounded in $\mathbb{R}^n$. Under the $p,q-$growth conditions (1.10), (1.15), if $q \geq p \geq 2$ and*

$$\frac{q}{p} < 1 + \frac{1}{n}, \tag{1.17}$$

there exists a weak solution $u \in W^{1,p}_{u_0}(\Omega) \cap W^{1,2q-p}_{\mathrm{loc}}(\Omega) \subset W^{1,p}_{u_0}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ to the Dirichlet problem

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du(x)) = b(x) \quad in\ \Omega; \quad u = u_0 \in W^{1,p\frac{2q-p-1}{p-1}} \ on\ \partial\Omega. \tag{1.18}$$

Moreover u is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and there exist positive constants c, δ such that, for every ϱ and R such that $0 < \rho < R \leq \varrho + 1$,

$$\|Du(x)\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \leq \left(\frac{c}{(R-\varrho)^\delta} \int_{B_R} \left(1+|Du(x)|^2\right)^{\frac{p}{2}} dx \right)^{\frac{1}{(n+1)p-nq}} . \tag{1.19}$$

We emphasize the interest to compare the growth conditions on the exponents p, q in (1.5), (1.12): $\frac{q}{p} < 1 + \frac{2}{n}$ and in (1.9), (1.17): $\frac{q}{p} < 1 + \frac{1}{n}$ in the context of local Lipschitz continuity of $f(x,\xi)$ and $a^i(x,\xi)$ with respect to the $x-$variable, versus the growth conditions in (3.9): $\frac{q}{p} < 1 + \frac{q}{n}$ for local boundedness, and in (3.10): $\frac{q}{p} < 1 + \frac{\alpha}{n}$, with $\alpha-$Hölder continuity of $f(x,\xi)$ and in (3.14): $\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}$ under Sobolev $r-$summability of the derivative $f_{\xi x}(x,\xi)$ for some $r > n$. See also the recent results by Bella-Schäffner [6], [7] and Remark 4.6 in Section 4.3.1. Finally, as in (3.6), we recall that for some energy integrals under $p, q-$growth conditions the regularity results hold under less restrictive conditions on the ratio $\frac{q}{p}$, or even without restrictions such as for instance $f(\xi) = |\xi|^{a+b\sin(\log\log|\xi|)}$, with $a, b > 0$ and $a > 1 + b\sqrt{2}$, which is a convex function satisfying the $p, q-$*growth conditions* with $p = a - b$ and $q = a + b$ and the Δ_2-condition (see (2.10) in [58] and Remark 3.3 in [14]).

Theorems 1.1-1.4 give local Lipschitz continuity of local minimizers and of weak solutions. From $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ further regularity applies; see details in Section 2. Theorem 1.2 has been proved in [44]; see also Theorem 3.4 below. The proofs of Theorems 1.1, 1.3 and 1.4 are in Section 4.

Of course these results are also valid when $p = q$, i.e. for interior Lipschitz continuity of weak solutions to the $p-$*Laplace equation* in (1.2). Since their proofs are presented in Section 4 in a complete form and are self contained, they could be used in a mini pde's course, as already done by the author in a PhD mini-course held in Germany, at the *Department Mathematik, Friedrich-Alexander Universität Erlangen-Nürnberg* on December 2019. The author thanks Frank Duzaar for the invitation and for his warm hospitality at the Math. Dept.

2. The process/approach to regularity

We briefly describe some aspects of the process/approach to regularity of weak solutions to nonlinear elliptic equations and systems (in this section more generally we treat *systems* too). Let $n \geq 2$, $m \geq 1$, let Ω be an open set of $\mathbb{R}^n$ and let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a *weak solution* of a *nonlinear elliptic system* of the form

$$\operatorname{div} A(Du) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^{\alpha}(Du) = 0, \qquad \alpha = 1, 2 \dots m, \tag{2.1}$$

where $Du : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{m\times n}$ denotes the *gradient* of the map u, by components $x = (x_i)_{i=1,2,\dots,n}$, $u = (u^\alpha)^{\alpha=1,2,\dots,m}$ and $Du = (\partial u^\alpha / \partial x_i) = (u^\alpha_{x_i})^{\alpha=1,2,\dots,m}_{i=1,2,\dots,n}$.

Then $A(\xi) = (a_i^\alpha(\xi))^{\alpha=1,2,\dots,m}_{i=1,2,\dots,n}$ is a given *vector field* $A : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ of class C^1, satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{m} \frac{\partial a_i^\alpha(\xi)}{\partial \xi_j^\beta} \lambda_i^\alpha \lambda_j^\beta > 0, \qquad \forall\, \lambda, \xi \in \mathbb{R}^{m\times n} : \lambda \neq 0. \tag{2.2}$$

In the general context of nonlinear elliptic systems the vector field $A(\xi)$ is more general than in the specific context of the calculus of variations, where the vector field $A(\xi)$ is the *gradient* of a function $f(\xi)$; i.e., when there exists a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ of class $C^2(\mathbb{R}^{m\times n})$ such that $A(\xi) = D_\xi f(\xi)$; in terms of components

$$a_i^\alpha = \frac{\partial f}{\partial \xi_i^\alpha} = f_{\xi_i^\alpha}, \qquad \forall\, \alpha = 1, 2, \dots, m;\ \forall\, i = 1, 2, \dots, n.$$

Under this variational condition, the ellipticity condition (2.2) can be equivalently written in the form

$$\sum_{i,j,\alpha,\beta} \frac{\partial^2 f(\xi)}{\partial \xi_i^\alpha \partial \xi_j^\beta} \lambda_i^\alpha \lambda_j^\beta > 0, \qquad \forall\, \lambda, \xi \in \mathbb{R}^{m\times n} : \lambda \neq 0. \tag{2.3}$$

Thus the ellipticity condition of the system is equivalent to the positivity on $\mathbb{R}^{m\times n}$ of the quadratic form $D^2_\xi f(\xi)$; i.e., for every $\xi \in \mathbb{R}^{m\times n}$, $\left(D^2_\xi f(\xi)\lambda, \lambda\right) > 0$, for all $\lambda, \xi \in \mathbb{R}^{m\times n}$, $\lambda \neq 0$, which implies the *(strict) convexity* of the function f. In this case any *weak solution* (in a class of maps u to be defined) to the differential elliptic system is a *minimizer* (also here, we need to define that class of maps which compete with u in the minimization process) of the energy functional $F(u) = \int_\Omega f(Du)\, dx$ and, in general, the vice-versa does not hold.

In the general *vectorial setting* of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ which are *weak solution* of the *nonlinear elliptic system* (2.1) it is well known that, in general, we can look for the so called *partial regularity*, since the pioneering work of Morrey and De Giorgi on this subject. If some additional structure conditions are assumed, then some studies can be found in the mathematical literature on the subject for *everywhere regularity*. For instance, the celebrated everywhere regularity results on minimizers of the $p-$*Laplace* energy-integral obtained by Uhlenbeck [67] in 1977 specifically with $f(\xi) = |\xi|^p$ and $p \geq 2$; that is $F(u) = \int_\Omega |Du(x)|^p \, dx$. The *regularity problem* for the previous elliptic system consists in asking if the solution $u = u(x) = (u^\alpha(x))^{\alpha=1,2,\ldots,m}$, *a-priori* only *measurable map* in a given Sobolev class, in fact is *continuous* or more regular; i.e., if u is of class either $C^{0,\alpha}$, C^1, $C^{1,\alpha}$, or C^k for some k, or even C^∞, under suitable assumption of smoothness of the data. With the aim to explain the situation, we split the regularity process into two main parts (other points of view of smoothness are possible too), both relevant steps by themselves:

1st− from either a *minimizer*, or a *weak solution*, $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ (either in $W^{1,p}$ or in some other Sobolev or Orlicz classes) to $W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$;
2nd− from a weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^m) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, under some smoothness of the data, to *more regularity* of the type $C^{1,\alpha}$, or C^k, or C^∞.

2.1. The second regularity step (2nd)

Let us start to briefly discussing the second regularity step: from $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ to $C^{1,\alpha}(\Omega, \mathbb{R}^m)$ and to $C^\infty(\Omega, \mathbb{R}^m)$. I.e., we consider the case when the weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ also belongs to $W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Let us first treat equations; i.e. the scalar case $m = 1$, that is the case when the nonlinear system reduces to a nonlinear elliptic equation. Under (the so called *"natural"*) *ellipticity* and *p−growth conditions* ($p \geq 2$) on the function $f \in C^2(\mathbb{R}^{m\times n})$, of the type

$$\begin{cases} \left(D^2_\xi f(\xi)\lambda, \lambda\right) \geq m\left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \\ \left|D^2_\xi f(\xi)\right| \leq M\left(1 + |\xi|^2\right)^{\frac{p-2}{2}} \end{cases}, \quad \forall\, \lambda, \xi \in \mathbb{R}^n,$$

it is possible to see that u admits second derivatives in weak form, i.e., $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ (see Section 4.1.1). Then, fixed $k \in \{1, 2, \ldots, n\}$, we can take the $k-$derivative in both sides to the equation $\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(Du) = 0$ and we obtain

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n \frac{\partial a_i(Du(x))}{\partial \xi_j}\,(u_{x_j})_{x_k}\right) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du(x))}{\partial \xi_j}\,(u_{x_k})_{x_j}\right) = 0\,.$$

Therefore the partial derivative u_{x_k} satisfies an elliptic differential equation

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du(x))}{\partial \xi_j}\,(u_{x_k})_{x_j}\right) = 0\,.$$

Recall that the map u is given (u is fixed); then we can *"forget"* the explicit dependence of $\partial a_i/\partial \xi_j$ on $Du(x)$. We define the element a_{ij} of an $n \times n$ matrix

$$a_{ij}(x) = \frac{\partial a_i(Du(x))}{\partial \xi_j} = f_{\xi_i \xi_j}(Du(x))\,, \qquad i, j = 1, 2, \ldots, n.$$

Recalling that

$$m\left(1 + |Du|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \leq \left(D^2_\xi f(Du)\lambda, \lambda\right) \leq M\left(1 + |Du|^2\right)^{\frac{p-2}{2}}$$

and since *the gradient* Du *is locally bounded* in Ω ($u \in W^{1,\infty}_{\text{loc}}(\Omega)$) then the $n \times n$ square matrix $(a_{ij}(x))_{n\times n}$ is uniformly elliptic, with *measurable locally bounded coefficients*. Thus - as well known - we can apply the celebrated De Giorgi's Hölder continuity result [37], dated 1957, for the *linear* elliptic equation with bounded measurable coefficients,

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u_{x_k}}{\partial x_j}\right) = 0\,;$$

then, for every $k \in \{1,2,\ldots,n\}$, the partial derivative u_{x_k} is Hölder continuous for some exponent $\alpha \in (0,1)$. Thus $u \in C^{1,\alpha}(\Omega,\mathbb{R}^m)$.

In the vector-valued case $m \geq 1$ we need to assume a structure condition of the type $f(\xi) = g(|\xi|)$ for all $\xi \in \mathbb{R}^{m\times n}$. Then, again it is possible to show (in some cases) that

$$\begin{gathered} u \in W^{1,\infty}_{\text{loc}}(\Omega,\mathbb{R}^m),\; A \in C^{1,\gamma} \text{ for some } \gamma \in (0,1) \\ \Downarrow \\ u \in C^{1,\alpha} \text{ for some } \gamma \in (0,1)\,. \end{gathered}$$

See for instance the $p-$*Laplace* energy-integral, studied by Uhlenbeck [67], with $f(\xi) = |\xi|^p$ and $p \geq 2$. Moreover under natural assumptions more regularity applies (although we also refer to some recent examples by Mooney-Savin [60] and Mooney [59] when further regularity of solutions may fail). In fact, if the function f is smooth, say $f \in C^{2,\gamma}(\mathbb{R}^{m\times n})$, similarly to the scalar case, u admits second derivatives in weak form and, fixed $k \in \{1,2,\ldots,n\}$, we can take the $k-$derivative in the system

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i^\alpha(Du) = 0, \qquad \alpha = 1,2\ldots m.$$

Thus the partial derivative $u_{x_k} = \left(u^\beta_{x_k}\right)^{\beta=1,2,\ldots,m}$ satisfies (u_{x_k} is a vector-valued map, a vector-valued partial derivative)

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}\sum_{\beta=1}^{m} \frac{\partial a_i^\alpha(Du(x))}{\partial \xi_j^\beta}\left(u^\beta_{x_j}\right)_{x_k}\right) \\ &= \sum_{i,j,\beta} \frac{\partial}{\partial x_i}\left(\frac{\partial a_i^\alpha(Du(x))}{\partial \xi_j^\beta}\left(u^\beta_{x_k}\right)_{x_j}\right) = 0\,,\;\; \alpha = 1,2\ldots m. \end{aligned}$$

That is, for every $k \in \{1,2,\ldots,n\}$, the (vector-valued) map $u_{x_k} = \left(u^\beta_{x_k}\right)^{\beta=1,2,\ldots,m}$ is a weak solution to the elliptic differential system

$$\sum_{i,j,\beta} \frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}(x)\frac{\partial u^\beta_{x_k}}{\partial x_j}\right) = 0\,, \qquad \alpha = 1,2\ldots m,$$

where $a_{ij}^{\alpha\beta}(x) \overset{\text{def}}{:=} \partial a_i^\alpha/\partial \xi_j^\beta(Du(x)) = f_{\xi_i^\alpha \xi_j^\beta}(Du(x))$ are now Hölder continuous coefficients, since $u \in C^{1,\alpha}$. Thus we can apply the classical regularity results in the literature for *linear* elliptic systems with *smooth coefficients* (see for instance Section 3 of Chapter 3 of the book by Giaquinta [46]) to infer

$$u \in C^{1,\alpha},\;\; A \in C^{k,\gamma} \qquad \Longrightarrow \qquad u \in C^{k,\alpha}, \quad \forall\, k = 2,3,\ldots$$

In particular, if $A \in C^\infty$ (or equivalently $f \in C^\infty$) then $u \in C^\infty(\Omega,\mathbb{R}^m)$.

2.2. The first regularity step (1st)

We discuss the process from $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ to $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$. The problem to be considered is: *under which conditions on the vector field* $A : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$, $A(\xi) = (a_i^\alpha(\xi))_{i=1,2,\dots,n}^{\alpha=1,2,\dots,m}$, *the gradient* Du *is in fact locally bounded? I.e., we look for sufficient conditions for* $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$.

Why the local boundedness of the gradient Du is a so relevant condition for regularity? Because the differential system heavily depends on Du in a nonlinear way, in particular through $a_i^\alpha(Du)$ and, if $Du(x)$ is bounded, then $a_i^\alpha(Du(x))$ (here "$p \geq 2$") is bounded too and far away from zero. Thus the behavior of $A(\xi) = (a_i^\alpha(\xi))$ for $|\xi| \to +\infty$ becomes irrelevant.

On the contrary, the local boundedness of the gradient is a property related to the behavior of $A(\xi)$ as $|\xi| \to +\infty$. Growth conditions play a relevant role in the $W^{1,\infty}_{\text{loc}}$ estimates. In the general context of $p, q-$*growth conditions* the first $L^\infty-$*gradient estimates* have been obtained in 1989-1991 in [50], [51], [52].

2.3. Conclusion: from $W^{1,p}$ to $C^{1,\alpha}$ or C^∞

We summarize with a scheme the regularity process from $W^{1,p}(\Omega, \mathbb{R}^m)$ to $C^{1,\alpha}$ or C^∞:

1st$-$ from $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ to $W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$; growth conditions, either of the vector field $A(\xi) = (a_i^\alpha(\xi))$ or of the integrand $f(\xi)$, play a central role;
2nd$-$ from $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^m)$ to $C^{1,\alpha}$ or C^∞; essentially growth conditions are not needed; however, of course, in this step some uniform ellipticity must be considered too.

The $C^{1,\alpha}$ and C^∞ regularity is explicitly stated for instance in the Corollary 2.2 of [51]. The article [51], together with [50], was the first paper dealing with the interior Lipschitz regularity under $p, q-$growth conditions. The statement in the Corollary 2.2 is related to weak solutions a-priori in the class $W^{1,q}_{\text{loc}}$, to handle at the same time equations, as in Theorems 1.3 and 1.4, and minimizers as in Theorems 1.1 and 1.2. But just following one of these quoted Theorems 1.1-1.4, or applying the interpolation Theorem 3.1 in [51], we see that in the non-degenerate case the $C^{1,\alpha}$ or C^∞ regularity can be stated also for weak solutions in the larger class $W^{1,p}_{\text{loc}}$, of course assuming the corresponding bound on the ratio $\frac{q}{p}$ in (1.5), (1.9), (1.12), (1.17).

3. Some examples with general growth

3.1. General $p, q-$growth

For some $p > 1$ and $q > p$ let us mention some examples of energy-integrals with p, q and/or general growth conditions:

$$\int_\Omega |Du(x)|^{p(x)} \, dx \,, \tag{3.1}$$

$$\int_\Omega |Du(x)|^p \log(1 + |Du(x)|) \, dx \,, \tag{3.2}$$

$$\int_\Omega \exp\left(a(x) |Du(x)|^2\right) dx \,, \tag{3.3}$$

$$\int_\Omega \{a(x) |Du(x)|^p + b(x) |Du(x)|^q\} \, dx \,. \tag{3.4}$$

The first example (3.1) enters in the $p,q-$growth context with $p = \inf\{p(x) : x \in \Omega\}$, $q = \sup\{p(x) : x \in \Omega\}$; more precisely, following the mathematical literature where the exponent $p(x)$ is continuous in Ω, for the regularity purpose we can fix a ball $B_R \subset \Omega$ of small radius R in such a way that $p = \inf\{p(x) : x \in B_R\}$, $q = \sup\{p(x) : x \in B_R\}$ can be chosen arbitrarily close each other so that any bound of the type $1 \le \frac{q}{p} < 1 + O\left(\frac{1}{n}\right)$ can be satisfied and the regularity results stated in Theorems 1.2, 1.3, 1.4 apply. Similarly the example (3.2), where we can fix $q > p$ arbitrarily close to p. In the example (3.3) the integrand has exponential growth and we refer to [53], [58] and more recently to [38]. The last example in (3.4) is known - in the terminology introduced in 2015 by Colombo–Mingione - as a *double phase energy integral*, with possibly zero coefficients, by assuming however that the two exponents $a(x), b(x)$ are not equal to zero at the same time; i.e.,

$$a(x), b(x) \ge 0, \quad a(x) + b(x) > 0, \quad \text{a.e. } x \in \Omega. \tag{3.5}$$

The $p,q-$growth appear also with f independent of x, of the form $f(\xi) = g(|\xi|)$ when g does not behave like a power when $|\xi| \to \infty$; for instance

$$f(\xi) = g(|\xi|) = |\xi|^{a+b\sin\log\log|\xi|}, \qquad g(t) = t^{a+b\sin\log\log t}, \tag{3.6}$$

for $|\xi|$ large, precisely for $t \ge e$, and properly extended for $t \in [0,e)$. In fact a computation shows that g is a convex function for $t \ge e$ and the function $g(|\xi|) = |\xi|^{a+b\sin\log\log|\xi|}$, a-priori defined for $|\xi| \ge e$, can be extended to all $\xi \in \mathbb{R}^n$ as a convex function on $\mathbb{R}^n$ if a, b are positive real numbers with $a > 1 + b\sqrt{2}$. In this case our integrand satisfies the $p,q-$growth conditions

$$m|\xi|^p \le f(\xi) \le M(1 + |\xi|^q), \quad \forall\, \xi \in \mathbb{R}^n$$

with $p = a - b > 1$ and $q = a + b$. We notice that the *"Δ_2-condition"* (well known in the mathematical literature; see for instance [20]) is considered to be a generalization of the *uniformly elliptic case*. The function $f(\xi)$ in (3.6) satisfies the Δ_2-condition; while we can construct (details by Bögelein-Duzaar-Marcellini-Scheven [13], [14]; see also Chlebicka [20]) a convex function $f(\xi) = g(|\xi|)$ satisfying the $p,q-$growth conditions with $q > p$ and q arbitrarily close to p, which does not satisfy the Δ_2-condition and which enters in the regularity theory presented here.

The class of energy functionals modelized by the example (3.4) enters in the context of general $p,q-$growth conditions; it is also named *double phase integrals* and has been recently (starting from 2015) explored in a series of interesting papers by M. Colombo-Mingione [23], [24], Baroni-M. Colombo-Mingione [2], [3], [4]. From different points of view Eleuteri-Marcellini-Mascolo [41], [42], [43], [44] and DeFilippis [33], DeFilippis-Ho [35]. For related recent references we also quote Duzgun-Marcellini-Vespri [39], [40], Rădulescu-Zhang [63], [64], Cencelja-Rădulescu-Repovš [19], Papageorgiou-Rădulescu-Repovš-Dušan [62], Ambrosio-Rădulescu [1], Ragusa-Tachikawa [65], Chlebicka [20], Chlebicka-DeFilippis [21], [22], Cupini-Giannetti-Giova-Passarelli [25], Carozza-Giannetti-Leonetti-Passarelli [17], Carozza-Kristensen-Passarelli [18], Cupini-Marcellini-Mascolo [26–32], Harjulehto-Hästö-Toivanen [47], Hästö-Ok [48], Bousquet-Brasco [16], DeFilippis-Palatucci [36], Bildhauer-Fuchs [8], [9], TN Nguyen-MP Tran [61], Sin [66]. A special mention to [34] by DeFilippis-Mingione with some interesting considerations about the so-called *Lavrentiev phenomenon*. General growth conditions even for the one-dimensional case $n = 1$ have been studied in [15], [45]. For the general case $n > 1$ and $m > 1$ under *quasiconvexity conditions* see [49] and the *integral convexity condition* [11] by Bögelein-Dacorogna-Duzaar-Marcellini-Scheven. Further references can be found in [38], [55], [56], [57]. We devote the next section to describe some results related to the class of energy integrals as in the example (3.4).

3.2. Double phase integrals

In the terminology introduced by Colombo-Mingione [23], but already studied from the regularity point of view, in fact this is a particular case of the p, q-growth (see also the model examples in Section 1.3 of [12]), we consider the *double phase* integral (3.4)

$$\int_{\Omega} \left\{ a\,(x)\,|Du\,(x)|^{p} + b\,(x)\,|Du\,(x)|^{q} \right\}\,dx\;, \tag{3.7}$$

with $q > p > 1$ and $a\,(x)$,$b\,(x) \geq 0$, $a\,(x) + b\,(x) > 0$ a.e. $x \in \Omega$. Independently of the continuity of the coefficients $a\,(x)$, $b\,(x)$, we first state a local boundedness result for minimizers of the energy integral as in (3.4), obtained by Cupini-Marcellini-Mascolo [32] in the spirit of previous related results by Boccardo-Marcellini-Sbordone [10].

Theorem 3.1 *(Cupini-Marcellini-Mascolo). Let $q \geq p > 1$, $a^{-1} \in L^{r}_{\mathrm{loc}}\,(\Omega)$ and $b \in L^{s}_{\mathrm{loc}}\,(\Omega)$ for some exponents $r \in \left(\frac{1}{p-1}, +\infty\right]$, $s \in (1, +\infty]$, with*

$$\frac{1}{pr} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{n}\,. \tag{3.8}$$

Then every local minimizer of the energy integral (3.7) is locally bounded in Ω.

Note that in the special relevant case $r = s = +\infty$ the above condition reduces to $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, that is

$$\frac{q}{p} < 1 + \frac{q}{n}\,. \tag{3.9}$$

More regularity of minimizers, in fact the local Hölder continuity of their gradients, has been proved by M. Colombo-Mingione [23], [24], Baroni-M. Colombo-Mingione [2], [3], [4]. For the local Lipschitz continuity without structure conditions see Eleuteri-Marcellini-Mascolo [41], [42], [43], [44]; see also De Filippis [33]. The following results have been obtained by M. Colombo-Mingione; of course in the first one we need a more strict assumption than either (3.8) or (3.9).

Theorem 3.2 *(Colombo-Mingione). Let $q \geq p > 1$, $a^{-1} \in L^{\infty}_{\mathrm{loc}}\,(\Omega)$ and $a, b \in C^{\alpha}_{\mathrm{loc}}\,(\Omega)$ for some $\alpha \in (0, 1]$, with*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}\,. \tag{3.10}$$

Then every local minimizer of (3.7) is of class $C^{1,\beta}_{\mathrm{loc}}\,(\Omega)$ for some $\beta \in (0, 1)$.

Theorem 3.3 *(Colombo-Mingione). Let $q \geq p$ and $1 < p \leq n$, $a^{-1} \in L^{\infty}_{\mathrm{loc}}\,(\Omega)$ and $a, b \in C^{\alpha}_{\mathrm{loc}}\,(\Omega)$ for some $\alpha \in (0, 1]$, with*

$$\frac{q}{p} < 1 + \frac{\alpha}{p}\,. \tag{3.11}$$

Then any locally bounded minimizer of (3.7) is in $C^{1,\beta}_{\mathrm{loc}}\,(\Omega)$ for some $\beta \in (0, 1)$.

The following is a related regularity result by Eleuteri-Marcellini-Mascolo [44], valid for a generalized class of double (or *multi*) phase energy integrands, whose prototype is given by

Please cite this article in press as: P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124408

$$f(x,\xi)=a(x)\,|\xi|^{p}+b(x)\,|\xi|^{s}+|\xi_{n}|^{q}\ , \tag{3.12}$$

ξ_n being the last (or any other) component of the vector $\xi=(\xi_1,\xi_2,\ldots,\xi_n)\in\mathbb{R}^n$ and $s\le\frac{p+q}{2}$. Note however that we do not assume a structure representation of the integrand, for instance of the type (3.12), which is only a model example. In fact we can also consider more general energy integrands $f=f(x,\xi)$ without a structure, i.e. not necessarily depending on the modulus of ξ. We assume that $f:\Omega\times\mathbb{R}^n\to[0,+\infty)$ is a convex function with respect to the gradient variable ξ and it is strictly convex only at infinity; more precisely, $f_{\xi\xi}$, $f_{\xi x}$ are Carathéodory functions satisfying

$$\begin{cases} m\,|\xi|^{p-2}\,|\lambda|^{2}\le\sum_{i,j}f_{\xi_i\xi_j}\,(x,\xi)\,\lambda_i\lambda_j\le M\left(1+|\xi|^{2}\right)^{\frac{q-2}{2}}|\lambda|^{2}\\ \textit{either}\quad |f_{\xi x}\,(x,\xi)|\le\ h(x)\left(1+|\xi|^{2}\right)^{\frac{p+q-2}{4}}, \\ \textit{or, respectively}\quad |f_{\xi x}\,(x,\xi)|\le\ h(x)\left(1+|\xi|^{2}\right)^{\frac{q-1}{2}} \end{cases} \tag{3.13}$$

for some constants $m,M>0$, for almost every $x\in\Omega$ and for all $\lambda,\xi\in\mathbb{R}^n$ with $|\xi|\ge1$. Here $1<p\le q$ and $h\in L^r(\Omega)$ for some $r>n$. The following a-priori estimate has been obtained in [44].

Theorem 3.4 *(Eleuteri-Marcellini-Mascolo). Under the growth assumptions (3.13) with exponents p,q satisfying*

$$\frac{q}{p}<1+2\left(\frac{1}{n}-\frac{1}{r}\right)\quad \textit{or, respectively}\quad \frac{q}{p}<1+\frac{1}{n}-\frac{1}{r}\,, \tag{3.14}$$

any smooth local minimizer of the energy integral $\int_\Omega f\,(x,Du\,(x))\,dx$ is locally Lipschitz continuous in Ω.

If we specialize the above theorem with integrand $f(x,\xi)$ as in (3.12), with

$$a\,(x)=1\,,\quad b\,(x)=|x|^{\alpha}\,,$$

for some $\alpha\in(0,1)$ and $0\in\Omega$, then $b\in C^{0,\alpha}\cap W^{1,r}$ with $\frac{1}{r}=\frac{1-\alpha}{n}$. The function h belongs to L^r for the same $r=\frac{n}{1-\alpha}$ and the assumption on the exponents p,q can be written in terms of the parameter α in the equivalent form

$$\frac{q}{p}<1+\frac{2\alpha}{n}\,. \tag{3.15}$$

Differently, if we take under consideration the double phase integral (3.7) with the same coefficients $a\,(x)=1$ and $b\,(x)=|x|^{\alpha}$, then a computation gives $\frac{q}{p}<1+\frac{\alpha}{n}$, as in the Colombo-Mingione Theorem 3.2.

4. Proofs of Theorems 1.1, 1.2, 1.3, and 1.4

The proofs of Theorems 1.1, 1.3 and 1.4 have a unique roof and rely on the a-priori gradient estimate of Theorem 4.1. We divide the proofs of these results into several steps, which are detailed and self contained. As references, a reader can find related details in some of the author's papers, in particular in [51] and [54].

4.1. First step: a-priori estimates

Let us consider again the elliptic equation

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}a^i\left(x,Du\left(x\right)\right)=b\left(x\right),\qquad x\in\Omega\,. \tag{4.1}$$

Under $p,q-$growth conditions, with $q\geq p$, as in (1.10), (1.11), a *weak solution* to (4.1) is a function $u\in W^{1,q}_{\mathrm{loc}}\left(\Omega\right)$ such that, for every $\Omega'\subset\subset\Omega$,

$$\int_{\Omega}\sum_{i=1}^{n}\underbrace{a^i\left(x,Du\left(x\right)\right)}_{\in L^{\frac{q}{q-1}}_{\mathrm{loc}}}\underbrace{\varphi_{x_i}}_{\in L^q}\,dx+\int_{\Omega}b\left(x\right)\varphi\,dx=0,\qquad\forall\varphi\in W^{1,q}_0\left(\Omega'\right), \tag{4.2}$$

where $\frac{q}{q-1}$ and q are conjugate exponents (i.e., the sum of the reciprocals is equal to 1). In fact by (1.10)

$$\left|a^i\left(x,Du\left(x\right)\right)-a^i\left(x,0\right)\right|=\left|\int_0^1\frac{d}{dt}a^i\left(x,tDu\left(x\right)\right)\,dt\right| \tag{4.3}$$

$$=\left|\int_0^1\sum_{j=1}^{n}a^i_{\xi_j}\left(x,tDu\left(x\right)\right)u_{x_j}\,dt\right|\leq M\left(1+\left|Du\left(x\right)\right|^2\right)^{\frac{q-2}{2}}\left|Du\left(x\right)\right|$$

$$\leq M\left(1+\left|Du\left(x\right)\right|^2\right)^{\frac{q-1}{2}}\in L^{\frac{q}{q-1}}_{\mathrm{loc}}\quad\text{if}\quad\left|Du\left(x\right)\right|\in L^q_{\mathrm{loc}}\,.$$

Thus $a^i\left(x,Du\right)\in L^{\frac{q}{q-1}}_{\mathrm{loc}}$, as indicated in (4.2), if $\left|Du\left(x\right)\right|\in L^q_{\mathrm{loc}}$ and if $\left|a^i\left(x,0\right)\right|$ has the right summability (i.e., $\left|a^i\left(x,0\right)\right|\in L^{\frac{q}{q-1}}_{\mathrm{loc}}$), in particular if it is locally bounded in Ω. The definition of weak solution is consistent if a-priori $u\in W^{1,q}_{\mathrm{loc}}\left(\Omega\right)$ and in general (if $q\neq p$) it is not sufficient $u\in W^{1,p}_{\mathrm{loc}}\left(\Omega\right)$.

Theorem 4.1 *(A-priori estimate). Under the $p,q-$growth conditions (1.10), (1.11), if $q\geq p\geq 2$ and*

$$\frac{q}{p}<\frac{n}{n-2} \tag{4.4}$$

(without restrictions on the exponents p,q if $n=2$), then every weak solution $u\in W^{1,q}_{\mathrm{loc}}\left(\Omega\right)$ to the pde (4.1) is of class $W^{1,\infty}_{\mathrm{loc}}\left(\Omega\right)$; i.e., the gradient Du is locally bounded in Ω and the following estimate holds: there exist $c,\beta>0$ and $\vartheta\geq 1$ such that, for every ϱ,R $(0<\rho<R\leq\varrho+1$ with ratio $\frac{R}{\varrho}$ bounded),

$$\sup_{x\in B_\varrho}\left(1+\left|Du\left(x\right)\right|^2\right)^{\frac{1}{2}}\leq\frac{c}{\left(R-\rho\right)^{\vartheta\beta}}\left(\left\|\left(1+\left|Du\right|^2\right)^{\frac{1}{2}}\right\|_{L^q(B_R)}\right)^{\vartheta}$$

$$\underset{\textit{for } n>2}{=}c\left(\frac{1}{R^n}\int_{B_R}\left(1+\left|Du\right|^2\right)^{\frac{q}{2}}dx\right)^{\frac{2}{np-(n-2)q}}. \tag{4.5}$$

The proof of Theorem 4.1 follows below; it is divided into several steps.

Remark 4.2. We emphasize that in the a-priori gradient bound (4.5) the exponents β in (4.21) and ϑ in (4.17) have explicit expression; in particular $\vartheta:=\frac{(2^*-2)q}{2^*p-2q}$ which is equal to $\frac{2q}{np-(n-2)q}$ when $n\geq 3$. With

Please cite this article in press as: P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124408

abuse of notation sometime we denote with the same expression the value of ϑ also when $n = 2$; however, more precisely, for every $n \geq 2$ in fact ϑ means $\vartheta := \frac{2^*-2}{2^*\frac{p}{q}-2}$, where 2^* for $n = 2$ is equal to any fixed real number greater than $\frac{2q}{p}$. We also observe for $n = 2$ that $\vartheta \to \frac{q}{p}$ as $2^* \to +\infty$ so that ϑ can be any number close to (and greater than) $\frac{q}{p}$. In any case $\vartheta \geq 1$; i.e. for every $n \geq 2$ and every $q \geq p$.

The exponent $\vartheta = \frac{2^*-2}{2^*\frac{p}{q}-2}$ is equal to 1 if and only if $q = p$. Therefore in the classical case of the *non-degenerate $p-$Laplace equation* (1.2) the a-priori estimate in Theorem 4.1 is a final result for the local Lipschitz continuity of the weak solutions. In fact if $q = p$ then condition (4.4) holds; Theorem 4.1 applies to every weak solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and further steps are not needed. In this case with $q = p$ then $\vartheta = 1$ and (4.5) takes the form, for every ϱ, R $(0 < \rho < R \leq \varrho + 1)$,

$$\sup_{x \in B_\varrho} \left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \leq \left(\frac{c}{(R-\rho)^n} \int_{B_R} \left(1 + |Du|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}. \tag{4.6}$$

4.1.1. Difference quotient

Fixed $s \in \{1, 2, \ldots n\}$ and $h \in \mathbb{R}$, $h \neq 0$, as usual the *difference quotient* is defined by $\Delta_h \psi = \frac{\psi(x+he_s)-\psi(x)}{h}$. Given $g : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, with $0 \leq g' \leq L$, we consider a test function of the form $\varphi = \Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)$, $\eta \in C^1_0(\Omega')$, $\eta \geq 0$, $\Omega' \subset\subset \Omega$. We get

$$\int_\Omega \left\{\sum_{i=1}^n \Delta_h a^i(x, Du(x)) \left(\eta^2 g(\Delta_h u)\right)_{x_i} + b(x)\, \Delta_{-h}\left(\eta^2 g(\Delta_h u)\right)\right\} dx = 0.$$

We compute $\left(\eta^2 g(\Delta_h u)\right)_{x_i} = 2\eta\eta_{x_i} g(\Delta_h u) + \eta^2 g'(\Delta_h u)\, \Delta_h u_{x_i}$ and

$$\begin{aligned}
\Delta_h a^i(x, Du(x)) &= \frac{1}{h}\int_0^1 \frac{d}{dt} a^i(x + the_s, Du(x) + th\Delta_h Du)\, dt\,, \\
&= \int_0^1 \left(a^i_{x_s} + \sum_{j=1}^n a^i_{\xi_j} \Delta_h u_{x_j}\right) dt\,.
\end{aligned}$$

We obtain

$$\begin{aligned}
\boxed{\mathbf{0}} \quad & \int_0^1 dt \int_\Omega \eta^2 g'(\Delta_h u) \sum_{i,j=1}^n a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j} dx \\
\boxed{\mathbf{1}} \quad & = -\int_0^1 dt \int_\Omega \eta^2 g'(\Delta_h u) \sum_{i=1}^n a^i_{x_s} \Delta_h u_{x_i} dx \\
\boxed{\mathbf{2}} + \boxed{\mathbf{3}} \quad & -\int_0^1 dt \int_\Omega 2\eta g(\Delta_h u) \sum_{i=1}^n \left(a^i_{x_s} + \sum_{j=1}^n a^i_{\xi_j} \Delta_h u_{x_j}\right) \eta_{x_i} dx \\
\boxed{\mathbf{4}} \quad & + \int_\Omega b(x)\, \Delta_{-h}\left(\eta^2 g(\Delta_h u)\right) dx\,.
\end{aligned}$$

We discuss all these terms, separately each other. Let us start with $\boxed{\mathbf{0}}$:

$$\boxed{0} \quad \int_0^1 dt \int_\Omega \eta^2 g'\left(\Delta_h u\right) \sum_{i,j=1}^n a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j}\, dx$$

$$\geq m \int_0^1 dt \int_\Omega \eta^2 g' \cdot \left(1 + |Du + th\Delta_h u|^2\right)^{\frac{p-2}{2}} |\Delta_h Du|^2\, dx\,.$$

However we do not immediately apply this estimate; before we treat the other terms, since - also in the other terms - the quadratic form $\sum_{i,j=1}^n a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j}$ will appear. With $\boxed{1}$, since

$$a^i_{x_s} = a^i_{x_s}\left(Du + th\Delta_h u\right) \in L^{\frac{q}{q-1}}_{\mathrm{loc}}, \qquad a^i_{\xi_j} = a^i_{\xi_j}\left(Du + th\Delta_h u\right) \in L^{\frac{q}{q-2}}_{\mathrm{loc}},$$

as in Lemma 2.5 of [51] we obtain the estimate

$$\left| \int_0^1 dt \int_\Omega \eta^2 g'\left(\Delta_h u\right) \sum_{i=1}^n \underbrace{a^i_{x_s}}_{\frac{q}{q-1}} \underbrace{\Delta_h u_{x_i}}_{q}\, dx \right| \qquad \boxed{\tfrac{q-1}{q} + \tfrac{1}{q} = 1}$$

$$\leq c \int_0^1 dt \int_\Omega \eta^2 g'\left(\Delta_h u\right) \underbrace{\left(\sum_{i,j=1}^n \underbrace{a^i_{\xi_j}}_{\frac{q}{q-2}} \underbrace{\Delta_h u_{x_i}}_{q} \underbrace{\Delta_h u_{x_j}}_{q} \right)^{\frac{1}{2}}}_{\in L^2_{\mathrm{loc}}} \underbrace{\left(1 + |Du + th\Delta_h u|^2\right)^{\frac{q}{4}}}_{\in L^2_{\mathrm{loc}}} dx\,.$$

As a scruple, we also make a dimensional control of the previous estimates by separating the first and the second derivatives:

$$\left| \int_0^1 dt \int_\Omega \eta^2 g'\left(\Delta_h u\right) \sum_{i=1}^n \underbrace{a^i_{x_s}}_{\frac{q}{q-1}} \underbrace{\Delta_h u_{x_i}}_{q}\, dx \right|$$

$$\text{dimensional control } \; |Du|^{q-1} \cdot |\Delta_h Du|$$

$$\leq c \int_0^1 dt \int_\Omega \eta^2 g'\left(\Delta_h u\right) \underbrace{\left(\sum_{i,j=1}^n \underbrace{a^i_{\xi_j}}_{\frac{q}{q-2}} \underbrace{\Delta_h u_{x_i}}_{q} \underbrace{\Delta_h u_{x_j}}_{q} \right)^{\frac{1}{2}}}_{\in L^2_{\mathrm{loc}}} \underbrace{\left(1 + |Du + th\Delta_h u|^2\right)^{\frac{q}{4}}}_{\in L^2_{\mathrm{loc}}} dx\,.$$

$$\text{dimensional control } \; |Du|^{\frac{q-2}{2}} \cdot |\Delta_h Du| \cdot |Du|^{\frac{q}{2}} = |Du|^{q-1} \cdot |\Delta_h Du|\,.$$

We estimate $\boxed{2}$ by the assumption (1.11)

$$\left| \int_0^1 dt \int_\Omega 2\eta g\left(\Delta_h u\right) \sum_{i=1}^n \underbrace{a^i_{x_s}}_{|\xi|^{q-1}} \eta_{x_i}\, dx \right|$$

$$\leq nM \int_0^1 dt \int_\Omega 2\eta\, |D\eta| \cdot |g\left(\Delta_h u\right)| \underbrace{\left(1 + |Du + th\Delta_h u|^2\right)^{\frac{q-1}{2}}}_{|\xi|^{q-1}} dx\,.$$

With $\boxed{\mathbf{3}}$ as in (2.23) of [51] we have

$$\left| \int_0^1 dt \int_\Omega 2\eta \underbrace{g\left(\Delta_h u\right)}_{|\xi|} \sum_{i=1}^n \sum_{j=1}^n \underbrace{a^i_{\xi_j}}_{|\xi|^{q-2}} \Delta_h u_{x_j} \eta_{x_i} dx \right| \leq$$

$$c \int_0^1 \int_\Omega \underbrace{\left(\eta^2 g' \sum_{i,j=1}^n a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j} \right)^{\frac{1}{2}}}_{|\xi|^{\frac{q-2}{2}}} \underbrace{\left(\underbrace{\frac{g^2}{g'}}_{|\xi|^2} \left(1 + |Du + th\Delta_h u|^2\right)^{\frac{q-2}{2}} |D\eta|^2 \right)^{\frac{1}{2}}}_{|\xi|^{\frac{q-2}{2}+1}} dtdx\,.$$

Finally for $\boxed{\mathbf{4}}$

$$\left| \int_\Omega b\left(x\right) \Delta_{-h} \left(\eta^2 g\left(\Delta_h u\right)\right) dx \right|$$

$$\leq \|b\|_{L^\infty(\Omega')} \int_\Omega \left\{ 2\eta \left|\eta_{x_s}\right| \cdot \left|g\left(\Delta_h u\right)\right| + \eta^2 g'\left(\Delta_h u\right) \left|\Delta_h u_{x_s}\right| \right\} dx\,.$$

Being $\left|\Delta_h u_{x_s}\right| \leq \varepsilon \left|\Delta_h u_{x_s}\right|^2 + \frac{1}{4\epsilon}$, we estimate $\boxed{\mathbf{4}}$ with the quantity

$$\leq \|b\|_{L^\infty(\Omega')} \int_\Omega \left\{ 2\eta \left|D\eta\right| \left|g\left(\Delta_h u\right)\right| + \varepsilon \eta^2 g'\left(\Delta_h u\right) \left|\Delta_h u_{x_s}\right|^2 + \frac{1}{4\epsilon} \eta^2 g'\left(\Delta_h u\right) \right\} dx\,.$$

We obtain

$$\begin{aligned} &\frac{1}{c} \int_0^1 dt \int_\Omega \eta^2 g' \cdot \left(1 + |Du + th\Delta_h u|^2\right)^{\frac{p-2}{2}} |\Delta_h Du|^2 dx \\ &\leq \int_0^1 dt \int_\Omega \eta^2 g'\left(\Delta_h u\right) \left(1 + |Du + th\Delta_h u|^2\right)^{\frac{q}{2}} dx \\ &+ \int_0^1 dt \int_\Omega \eta \left|D\eta\right| \cdot \left|g\left(\Delta_h u\right)\right| \left(1 + |Du + th\Delta_h u|^2\right)^{\frac{q-1}{2}} dx \\ &+ \int_0^1 dt \int_\Omega \underbrace{\frac{g^2}{g'} \left(1 + |Du + th\Delta_h u|^2\right)^{\frac{q-2}{2}}}_{|\xi|^q} |D\eta|^2 dx \end{aligned} \tag{4.7}$$

At this stage, with $g\left(t\right) = t$, we get $u \in W^{2,2}_{\text{loc}}\left(\Omega\right)$ and we can pass to the limit as $h \to 0$. See for instance Lemma 2.7 in [51]. Being $\Delta_h u = \frac{u(x+he_s)-u(x)}{h}$, in particular

$$\Delta_h u \to u_{x_s}\,, \quad g\left(\Delta_h u\right) \to g\left(u_{x_s}\right)\,, \quad \Delta_h Du = D\Delta_h u \to Du_{x_s}\,, \quad h\Delta_h u \to 0\,.$$

We also get a bound of u in $W^{2,2}_{\text{loc}}\left(\Omega\right)$ in terms of the data.

4.1.2. Weak solutions with second derivatives

For every $\beta \geq 0$ we consider $g(t) = t\left(1+t^2\right)^{\frac{\beta}{2}}$ and $t = u_{x_s}$. We remark that in [51] we had a similar notation with the parameter $\alpha = \beta + 2$. We can check the bound $\frac{g^2}{g'} \leq c\left(1+t^2\right)^{\frac{\beta+2}{2}}$, for instance as in Lemma 2.6(ii) of [51]. In the limit as $h \to 0$ from (4.7) we get

$$\int_\Omega \eta^2 \sum_{s=1}^n \left(1+|u_{x_s}|^2\right)^{\frac{\beta+p-2}{2}} |Du_{x_s}|^2 \, dx$$
$$\leq c(\beta+1) \int_\Omega \left(\eta^2 + |D\eta|^2\right) \sum_{s=1}^n \left(1+|u_{x_s}|^2\right)^{\frac{\beta+q}{2}} dx$$

(note that, for $\beta = 0$, $\frac{\beta+p-2}{2} = \frac{p-2}{2}$ and similarly $\frac{\beta+q}{2} = \frac{q}{2}$). Then

$$\left| D\left[\eta\left(1+|u_{x_s}|^2\right)^{\frac{\beta+p}{4}}\right]\right|^2$$
$$\leq \left(\tfrac{\beta+p}{2}\right)^2 \eta^2 \left(1+|u_{x_s}|^2\right)^{\frac{\beta+p-2}{2}} |Du_{x_s}|^2 + 2|D\eta|^2 \left(1+|u_{x_s}|^2\right)^{\frac{\beta+p}{2}},$$

which we use together with the Sobolev inequality, valid for every $s = 1, 2, \ldots, n$,

$$\left(\int_\Omega \left[\eta\left(1+|u_{x_s}|^2\right)^{\frac{\beta+p}{4}}\right]^{2^*} dx\right)^{\frac{2}{2^*}} \leq c \int_\Omega \left| D\left[\eta\left(1+|u_{x_s}|^2\right)^{\frac{\beta+p}{4}}\right]\right|^2 dx\,. \tag{4.8}$$

In a standard way we fix concentric balls B_R and B_ρ compactly contained in Ω and a test function $\eta \in C_0^1(B_R)$, $\eta \geq 0$ in B_R and $\eta = 1$ in B_ρ, with $|D\eta| \leq 2/(R-\rho)$. We have obtained

$$\left(\int_{B_\rho} \sum_{s=1}^n \left(1+|u_{x_s}|^2\right)^{2^*\frac{\beta+p}{4}} dx\right)^{\frac{2}{2^*}} \leq \frac{c(\beta+2)^3}{(R-\rho)^2} \int_{B_R} \sum_{s=1}^n \left(1+|u_{x_s}|^2\right)^{\frac{\beta+q}{2}} dx\,. \tag{4.9}$$

We observe that, for $\beta = 0$, we get the summability of u to the power

$$2^* \frac{\beta+p}{2} = \frac{2^*}{2} p \underset{\text{for } n>2}{=} \frac{2np}{n-2}\,,$$

which should be compared with the summability of u to the power q. The estimate (4.9) is relevant if $\frac{2np}{n-2} > q$. Therefore *we gain in summability* if $\frac{q}{p} < \frac{2n}{n-2}$ when $n > 2$. This in fact is satisfied since it is less strict than the assumption (4.4) $\frac{q}{p} < \frac{n}{n-2}$ in Theorem 4.1. *At this stage* there are not conditions on p, q if $n = 2$. In the sense that, when $n = 2$, we define 2^* such that $2^* > 2\frac{q}{p}$; see also Remark 4.2.

4.1.3. Moser's iteration scheme for the gradient

Starting from the estimate (4.9) we use the *Moser's iteration scheme* for the *gradient* of the solution. Precisely, we define the sequence $(\beta_k)_{k\in\mathbb{N}}$ of real numbers

$$\begin{cases} \beta_1 = 0 \\ \beta_{k+1} = \frac{2^*}{2}(\beta_k + p) - q, \quad k = 1, 2, \ldots \end{cases}. \tag{4.10}$$

Recall that now the assumption (4.4) on the exponents p, q is $\frac{q}{p} < \frac{n}{n-2} = \frac{2^*}{2}$, which is equivalent to $\frac{2^*}{2}p - q > 0$, and which implies $\beta_{k+1} > \frac{2^*}{2}\beta_k \geq \beta_k$; i.e., the sequence β_k is strictly increasing (and more! By a multiplicative factor $\frac{2^*}{2}$). We can give a representation formula for the sequence β_k defined by induction in (4.10)

$$\beta_{k+1} = \left(\frac{2^*}{2}p - q\right) \sum_{i=0}^{k-1} \left(\frac{2^*}{2}\right)^i, \quad \forall\, k = 1, 2, \dots \tag{4.11}$$

In fact, if $k = 1$ then $\beta_2 = \frac{2^*}{2}(\beta_1 + p) - q = \frac{2^*}{2}p - q$, which corresponds to the previous formula (4.11) when $k = 1$. For generic $k = 2, 3, \dots$, by the induction assumption (4.11), we obtain

$$\beta_{k+2} = \frac{2^*}{2}(\beta_{k+1} + p) - q$$

$$= \frac{2^*}{2}\left[\left(\frac{2^*}{2}p - q\right)\sum_{i=0}^{k-1}\left(\frac{2^*}{2}\right)^i + p\right] - q = \left(\frac{2^*}{2}p - q\right)\sum_{i=0}^{k-1}\left(\frac{2^*}{2}\right)^{i+1} + \left(\frac{2^*}{2}p - q\right)$$

$$= \left(\frac{2^*}{2}p - q\right)\sum_{i=1}^{k}\left(\frac{2^*}{2}\right)^i + \left(\frac{2^*}{2}p - q\right) = \left(\frac{2^*}{2}p - q\right)\sum_{i=0}^{k}\left(\frac{2^*}{2}\right)^i,$$

which corresponds to the induction thesis, when in (4.11) we change k with $k+1$. We use the well known sum of a geometrical series $\sum_{i=0}^{k-1} r^i = \frac{r^k - 1}{r - 1}$ for $r = \frac{2^*}{2}$. We also obtain the further representation formula for β_k, valid for $k = 1, 2, \dots$

$$\beta_{k+1} = \left(\frac{2^*}{2}p - q\right)\sum_{i=0}^{k-1}\left(\frac{2^*}{2}\right)^i = \frac{2^*p - 2q}{2^* - 2}\left(\left(\frac{2^*}{2}\right)^k - 1\right). \tag{4.12}$$

Recall the previous estimate (4.9)

$$\left(\int_{B_\rho} \sum_{s=1}^{n} \left(1 + |u_{x_s}|^2\right)^{2^* \frac{\beta+p}{4}} dx\right)^{\frac{2}{2^*}} \leq \frac{c\,(\beta+2)^3}{(R-\rho)^2} \int_{B_R} \sum_{s=1}^{n} \left(1 + |u_{x_s}|^2\right)^{\frac{\beta+q}{2}} dx$$

which we will now consider with $\beta = \beta_k$. By the definition (4.10) $\beta_{k+1} = \frac{2^*}{2}(\beta_k + p) - q$, we get $\beta + q = \beta_k + q$ and $2^* \frac{\beta+p}{2} = 2^* \frac{\beta_k + p}{2} = \beta_{k+1} + q$. With the aim to have a full iteration we also consider radii ρ_0 and R_0 with $0 < \rho_0 < R_0$ and

$$R_k = \rho_0 + \frac{R_0 - \rho_0}{2^{k-1}}, \quad \forall\, k = 1, 2, \dots$$

For $k = 1, 2, \dots$ we rewrite (4.9) with $\beta = \beta_k$, $\rho = R_{k+1}$, $R = R_k$. Being $R - \rho = R_{k+1} - R_k = \frac{R_0 - \rho_0}{2^k}$, we obtain

$$\left(\int_{B_{R_{k+1}}} \sum_{s=1}^{n} \left(1 + |u_{x_s}|^2\right)^{\frac{\beta_{k+1}+q}{2}} dx\right)^{\frac{2}{2^*}} \leq \frac{c4^k\,(\beta_k+2)^3}{(R_0-\rho_0)^2} \int_{B_{R_k}} \sum_{s=1}^{n} \left(1 + |u_{x_s}|^2\right)^{\frac{\beta_k+q}{2}} dx\,. \tag{4.13}$$

With the notation

$$A_k = \left(\int_{B_{R_k}} \sum_{s=1}^{n} \left(1 + |u_{x_s}|^2\right)^{\frac{\beta_k + q}{2}} dx \right)^{\frac{1}{\beta_k + q}} \tag{4.14}$$

being $\frac{2}{2^*} = \frac{\beta_k + p}{\beta_{k+1} + q}$ we finally get

$$A_{k+1} \le \left(\frac{c4^k (\beta_k + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_k + p}} (A_k)^{\frac{\beta_k + q}{\beta_k + p}} . \tag{4.15}$$

By iterating (4.15) we obtain

$$A_3 \le \left(\frac{c4^2 (\beta_2 + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_2 + p}} (A_2)^{\frac{\beta_2 + q}{\beta_2 + p}}$$

$$\le \left(\frac{c4^2 (\beta_2 + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_2 + p}} \left(\left(\frac{c4 (\beta_1 + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_1 + p}} (A_1)^{\frac{\beta_1 + q}{\beta_1 + p}} \right)^{\frac{\beta_2 + q}{\beta_2 + p}} ,$$

$$A_4 \le \left(\frac{c4^3 (\beta_3 + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_3 + p}} \prod_{i=1}^{2} \left(\frac{c4^i (\beta_i + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_i + p} \prod_{i=2}^{3} \frac{\beta_i + q}{\beta_i + p}} \cdot (A_1)^{\prod_{i=1}^{3} \frac{\beta_i + q}{\beta_i + p}} ,$$

and for generic $k = 1, 2, \ldots$,

$$A_{k+1} \le \left(\frac{c4^k (\beta_k + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_k + p}} \prod_{i=1}^{k-1} \left(\frac{c4^i (\beta_i + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_i + p} \prod_{i=2}^{k} \frac{\beta_i + q}{\beta_i + p}} \cdot (A_1)^{\prod_{i=1}^{k} \frac{\beta_i + q}{\beta_i + p}} . \tag{4.16}$$

With the aim to go to the limit as $k \to +\infty$, by using the definition (4.10) $\beta_{k+1} = \frac{2^*}{2} (\beta_k + p) - q$ we first compute the following *product* for $k \ge 2$

$$\prod_{i=1}^{k} \frac{\beta_i + q}{\beta_i + p} = \frac{\beta_1 + q}{\beta_1 + p} \cdot \frac{\beta_2 + q}{\beta_2 + p} \cdot \ldots \cdot \frac{\beta_k + q}{\beta_k + p}$$

$$= \frac{\beta_1 + q}{\beta_1 + p} \cdot \frac{\frac{2^*}{2} (\beta_1 + p)}{\beta_2 + p} \cdot \frac{\frac{2^*}{2} (\beta_2 + p)}{\beta_3 + p} \cdot \ldots \cdot \frac{\frac{2^*}{2} (\beta_{k-1} + p)}{\beta_k + p} = \frac{q}{\beta_k + p} \left(\frac{2^*}{2} \right)^{k-1}$$

and, by the expression $\beta_k = \frac{2^* p - 2q}{2^* - 2} \left(\left(\frac{2^*}{2} \right)^{k-1} - 1 \right)$ in (4.12) and $\beta_1 = 0$

$$\prod_{i=1}^{k} \frac{\beta_i + q}{\beta_i + p} = \frac{q}{\beta_k + p} \left(\frac{2^*}{2} \right)^{k-1} = \frac{q \left(\frac{2^*}{2} \right)^{k-1}}{\frac{2^* p - 2q}{2^* - 2} \left(\left(\frac{2^*}{2} \right)^{k-1} - 1 \right) + p} .$$

Since $\frac{2^*}{2} > 1$, in the limit as $k \to +\infty$ we obtain

$$\prod_{i=1}^{\infty} \frac{\beta_i + q}{\beta_i + p} = \frac{q}{\frac{2^* p - 2q}{2^* - 2}} = \frac{2^* - 2}{2^* \frac{p}{q} - 2} .$$

Recalling that $\frac{2^*}{2} p - q > 0$, the real number

Please cite this article in press as: P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124408

$$\vartheta := \frac{2^* - 2}{2^* \frac{p}{q} - 2} \tag{4.17}$$

is greater than or equal to 1, it is equal to 1 if and only if $p = q$, and it is one of the relevant exponents which will appear in the a-priori estimate. To simplify this expression, we observe that, for every $i = 1, 2, \dots,$

$$\frac{c4^i (\beta_i + 2)^3}{(R_0 - \rho_0)^2} \geq \left.\frac{c4^i (\beta_i + 2)^3}{(R_0 - \rho_0)^2}\right|_{i=1} = c\frac{4 \cdot 2^3}{(R_0 - \rho_0)^2} \geq 1$$

and we can assume that the right hand side is greater than or equal to 1 by using a greater constant c if necessary. Moreover, since

$$\prod_{i=1}^{k} \frac{\beta_i + q}{\beta_i + p} = \frac{q}{p} \prod_{i=2}^{k} \frac{\beta_i + q}{\beta_i + p} \geq \prod_{i=2}^{k} \frac{\beta_i + q}{\beta_i + p},$$

from (4.16) we also deduce

$$\begin{aligned} A_{k+1} &\leq \left(\frac{c4^k (\beta_k + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_k + p}} \prod_{i=1}^{k-1} \left(\frac{c4^i (\beta_i + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_i + p} \prod_{i=2}^{k} \frac{\beta_i + q}{\beta_i + p}} \cdot (A_1)^{\prod_{i=1}^{k} \frac{\beta_i + q}{\beta_i + p}} \\ &\leq \left(\frac{c4^k (\beta_k + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_k + p}} \prod_{i=1}^{k-1} \left(\frac{c4^i (\beta_i + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{1}{\beta_i + p}\vartheta} \cdot A_1^{\vartheta} \\ &\leq \prod_{i=1}^{k} \left(\frac{c4^i (\beta_i + 2)^3}{(R_0 - \rho_0)^2} \right)^{\frac{\vartheta}{\beta_i + p}} \cdot A_1^{\vartheta} = \frac{\prod_{i=1}^{k} \left(c4^i (\beta_i + 2) \right)^{\frac{\vartheta}{\beta_i + p}}}{(R_0 - \rho_0)^{2\vartheta \sum_{i=1}^{k} \frac{1}{\beta_i + p}}} \cdot A_1^{\vartheta}. \end{aligned} \tag{4.18}$$

Finally, since $\beta_i \simeq \left(\frac{2^*}{2}\right)^{i-1}$ as $i \to +\infty$, we have

$$\begin{aligned} \prod_{i=1}^{k} \left(c4^i (\beta_i + 2) \right)^{\frac{\vartheta}{\beta_i + p}} &= \exp\left(\sum_{i=1}^{k} \frac{\vartheta}{\beta_i + p} \log c4^i (\beta_i + 2) \right) \\ \simeq_{k \to +\infty} \exp\left(\sum_{i=1}^{k} \frac{i}{\left(\frac{2^*}{2}\right)^{i-1}} \right) &\leq \exp\left(\sum_{i=1}^{\infty} \frac{i}{\left(\frac{2^*}{2}\right)^{i-1}} \right) := c < +\infty \end{aligned} \tag{4.19}$$

and, by (4.12) being $\beta_i = \frac{2^*p - 2q}{2^* - 2} \left(\left(\frac{2^*}{2}\right)^{i-1} - 1 \right)$ for every $i = 1, 2, \dots,$ we have

$$\begin{aligned} \sum_{i=1}^{k} \frac{1}{\beta_i + p} &= \frac{2^* - 2}{2^*p - 2q} \sum_{i=1}^{k} \frac{1}{\left(\frac{2^*}{2}\right)^{i-1} - 1 + p} \\ &\leq \frac{2^* - 2}{2^*p - 2q} \sum_{i=1}^{k} \left(\frac{2}{2^*}\right)^{i-1} \leq \frac{2^* - 2}{2^*p - 2q} \cdot \frac{1}{1 - \frac{2}{2^*}} = \frac{2^*}{2^*p - 2q}. \end{aligned} \tag{4.20}$$

From (4.18), (4.19), (4.20), by the fact that $R_0 - \rho_0 \leq 1$, we get

$$A_{k+1} \leq \prod_{i=1}^{k} \left(c4^i (\beta_i + 2) \right)^{\frac{\vartheta}{\beta_i + p}} \cdot (R_0 - \rho_0)^{-2\vartheta \sum_{i=1}^{k} \frac{1}{\beta_i + p}} \cdot A_1^{\vartheta}$$

$$\leq \frac{c}{(R_0 - \rho_0)^{2\vartheta \frac{2^*}{2^* p - 2q}}} \cdot A_1^{\vartheta} .$$

With the notation (valid in the right hand side when $n > 2$)

$$\beta := 2 \frac{2^*}{2^* p - 2q} \underset{\text{for } n>2}{=} \frac{2n}{np - (n-2)\,q} \tag{4.21}$$

(again note that $2^* p - 2q > 0$) we get

$$A_{k+1} \leq c \left(\frac{A_1}{(R_0 - \rho_0)^{\beta}} \right)^{\vartheta} . \tag{4.22}$$

Recall the definition of A_k in (4.14); i.e.

$$A_k = \left(\int_{B_{R_k}} \sum_{s=1}^{n} \left(1 + |u_{x_s}|^2\right)^{\frac{\beta_k + q}{2}} dx \right)^{\frac{1}{\beta_k + q}} , \quad \forall\, k = 1, 2, \ldots , .$$

We go to the limit in (4.22) as $k \to +\infty$; since $\beta_k \simeq \left(\frac{2^*}{2}\right)^{k-1} \to +\infty$ and $B_{R_0} \supset B_{R_k} \supset B_{\rho_0}$, we obtain

$$\left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_{\rho_0})} \leq c \left(\frac{1}{(R_0 - \rho_0)^{\beta}} \left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^q(B_{R_0})} \right)^{\vartheta} . \tag{4.23}$$

By using the explicit expression of β in (4.21), ϑ in (4.17)

$$\vartheta := \frac{2^* - 2}{2^* \frac{p}{q} - 2} \underset{\text{for } n>2}{=} \frac{2q}{np - (n-2)\,q} ,$$

and the fact that $2^* - 2 = \frac{2^* 2}{n}$ when $n > 2$

$$\beta q = 2 \frac{2^*}{2^* p - 2q} q = \frac{2^* - 2}{2^* \frac{p}{q} - 2} n = \vartheta n , \tag{4.24}$$

we obtain the conclusion of the proof of Theorem 4.1

$$\left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_{\rho_0})} \leq \frac{c}{(R_0 - \rho_0)^{\vartheta\beta}} \left(\left\| \left(1 + |Du|^2\right)^{\frac{1}{2}} \right\|_{L^q(B_{R_0})} \right)^{\vartheta}$$

$$= c \left(\frac{1}{(R_0 - \rho_0)^{\beta q}} \int_{B_{R_0}} \left(1 + |Du|^2\right)^{\frac{q}{2}} dx \right)^{\frac{\vartheta}{q}} = c \left(\frac{1}{(R_0 - \rho_0)^{\vartheta n}} \int_{B_{R_0}} \left(1 + |Du|^2\right)^{\frac{q}{2}} dx \right)^{\frac{2^* - 2}{2^* p - 2q}}$$

$$\underset{\text{for } n>2}{=} c \left(\frac{1}{(R_0 - \rho_0)^{\vartheta n}} \int_{B_{R_0}} \left(1 + |Du|^2\right)^{\frac{q}{2}} dx \right)^{\frac{2}{np - (n-2) q}} .$$

Please cite this article in press as: P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124408

Remark 4.3. We make here a remark for the case $n > 2$. We can multiply by $(R_{k+1})^{2-n}$ both sides of the inequality (4.13). Since $(2-n)\,\frac{2^*}{2} = -n$ and $\rho_0 \le R_{k+1} \le R_k \le R_0$ and obtain

$$\left(\frac{1}{(R_{k+1})^n} \int\limits_{B_{R_{k+1}}} \sum_{s=1}^{n} \left(1+|u_{x_s}|^2\right)^{\frac{\beta_{k+1}+q}{2}} dx \right)^{\frac{2}{2^*}}$$

$$\le \frac{c4^k (\beta_k+2)^3 (R_{k+1})^{2-n}}{(R_0-\rho_0)^2} \int\limits_{B_{R_k}} \sum_{s=1}^{n} \left(1+|u_{x_s}|^2\right)^{\frac{\beta_k+q}{2}} dx$$

$$\le \frac{c4^k (\beta_k+2)^3 (R_0)^2 (R_k)^{-n} \left(\frac{R_{k+1}}{R_k}\right)^{-n}}{(R_0-\rho_0)^2} \int\limits_{B_{R_k}} \sum_{s=1}^{n} \left(1+|u_{x_s}|^2\right)^{\frac{\beta_k+q}{2}} dx$$

$$\le c4^k (\beta_k+2)^3 \left(\frac{R_0}{\rho_0}\right)^n \cdot \frac{1}{(R_k)^n} \int\limits_{B_{R_k}} \sum_{s=1}^{n} \left(1+|u_{x_s}|^2\right)^{\frac{\beta_k+q}{2}} dx\,. \tag{4.25}$$

If we maintain bounded the quantity $\frac{R_0}{\rho_0}$ then the constant in the estimate (4.25) remains independent of the radius of the balls B_{R_k}. Therefore with the previous analysis for $n > 2$ we obtain the more precise estimate, as stated in Theorem 4.1,

$$\left\| \left(1+|Du|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_{\rho_0})} \underset{\text{for } n>2}{\le} c \left(\frac{1}{R_0^n} \int\limits_{B_{R_0}} \left(1+|Du|^2\right)^{\frac{q}{2}} dx \right)^{\frac{2}{np-(n-2)q}}.$$

4.2. Second step: interpolation

We first emphasize the idea of the interpolation, we make a precise statement below in Theorem 4.4 and then we give its proof.

4.2.1. Description and statement of the interpolation

We make use of the standard *interpolation inequality* (which, for $v \ge 0$ in Ω, immediately follows from $\int_\Omega v^q\,dx = \int_\Omega v^{q-p} v^p\,dx \le \|v\|_{L^\infty(\Omega)}^{q-p} \int_\Omega v^p\,dx$):

$$\|v\|_{L^q} \le \|v\|_{L^p}^{\frac{p}{q}} \cdot \|v\|_{L^\infty}^{1-\frac{p}{q}}\,. \tag{4.26}$$

Then, with the notation $v(x) = \left(1+|Du(x)|^2\right)^{\frac{1}{2}}$, by the gradient estimate (4.5) we get

$$\begin{cases} \|v\|_{L^q} \le \|v\|_{L^p}^{\frac{p}{q}} \cdot \|v\|_{L^\infty}^{1-\frac{p}{q}} \\ \|v\|_{L^\infty(B_\varrho)} \le \frac{c}{(R-\varrho)^{\vartheta\beta}} \|v\|_{L^q(B_R)}^{\vartheta} \end{cases}$$

$$\Downarrow$$

$$\|v\|_{L^\infty(B_\varrho)} \le \frac{c}{(R-\varrho)^{\vartheta\beta}} \|v\|_{L^q(B_R)}^{\vartheta} \le \frac{c}{(R-\varrho)^{\vartheta\beta}} \|v\|_{L^p(B_R)}^{\vartheta\frac{p}{q}} \cdot \|v\|_{L^\infty(B_R)}^{\vartheta\left(1-\frac{p}{q}\right)}\,.$$

Therefore:

$$\|v\|_{L^\infty(B_\varrho)} \le \frac{c}{(R-\varrho)^{\vartheta\beta}} \|v\|_{L^p(B_R)}^{\vartheta \frac{p}{q}} \cdot \|v\|_{L^\infty(B_R)}^{\vartheta\left(1-\frac{p}{q}\right)} . \tag{4.27}$$

With *abuse* (not only of notations!) in the conclusion of Theorem 1.1 we "identify" ϱ and R, and we do not consider the denominator $(R-\varrho)^{\vartheta\beta}$!! Thus, formally, $\|v\|_{L^\infty}^{1-\vartheta\left(1-\frac{p}{q}\right)} \le c\,\|v\|_{L^p}^{\vartheta\frac{p}{q}}$. Here we get the condition

$$1-\vartheta\left(1-\frac{p}{q}\right) > 0 . \tag{4.28}$$

Therefore we need to use the precise expression of the exponent ϑ. By (4.17) we have $\vartheta := \frac{2^*-2}{2^*\frac{p}{q}-2}$. Note that $\vartheta = 1$ if $q = p$. If $n > 2$ then we can represent ϑ also in terms of n in the form $\vartheta = \frac{2q}{np-(n-2)q}$ and the condition (4.28) is equivalent to

$$\frac{q}{p} < \frac{n+2}{n} = 1 + \frac{2}{n} . \tag{4.29}$$

This constraint (4.29) is exactly the condition on p, q for the validity of the following Interpolation Theorem 4.4.

Theorem 4.4 *(Interpolation). Under the $p,q-$growth conditions (1.10), (1.11), if $q \ge p \ge 2$ and $\frac{q}{p} < 1+\frac{2}{n}$, then every weak solution $u \in W_{\mathrm{loc}}^{1,q}(\Omega)$ to the pde (4.1) is of class $W_{\mathrm{loc}}^{1,\infty}(\Omega)$ and there exist $c, \alpha, \beta > 0$ and $\vartheta \ge 1$ (ϑ as in (4.17)) such that, for every ϱ and R such that $0 < \rho < R \le \varrho + 1$,*

$$\left\| \left(1+|Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^q(B_\varrho)} \le \left(\frac{c}{(R-\varrho)^{\beta\left(\frac{q}{p}-1\right)}} \left\| \left(1+|Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)}^{\frac{1}{\vartheta}} \right)^{\alpha} ; \tag{4.30}$$

$$\begin{aligned} \left\| \left(1+|Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_\varrho)} &\le \left(\frac{c}{(R-\varrho)^{\frac{\beta q}{\vartheta p}}} \left\| \left(1+|Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)} \right)^{\alpha} \\ &\underset{for\ n>2}{=} \left(\frac{c}{(R-\varrho)^{\frac{\beta q}{\vartheta p}}} \left\| \left(1+|Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)} \right)^{\frac{2p}{(n+2)p-nq}} . \end{aligned} \tag{4.31}$$

Remark 4.5. In the statement of Theorem 4.4 when $n > 2$ the exponents ϑ and α have the analytic expression in terms of the dimension n

$$\vartheta := \frac{2^*-2}{2^*\frac{p}{q}-2} \underset{\text{for } n>2}{=} \frac{2q}{np-(n-2)q}, \qquad \text{note that } \frac{q}{p} < \frac{n}{n-2}; \tag{4.32}$$

$$\alpha := \frac{\vartheta\frac{p}{q}}{1-\vartheta\left(1-\frac{p}{q}\right)} \underset{\text{for } n>2}{=} \frac{2p}{(n+2)p-nq}, \qquad \text{note that } \frac{q}{p} < \frac{n+2}{n}. \tag{4.33}$$

If $n = 2$ (the argument is specific for $n = 2$ but is valid for $n > 2$ too) the exponents α and ϑ above are well defined as real positive numbers if $2^*\frac{p}{q} - 2 > 0$ and $1-\vartheta\left(1-\frac{p}{q}\right) > 0$. The first condition is satisfied also for $n = 2$ since $2^* > \frac{2q}{p}$, see Remark 4.2. The second condition is equivalent to

$$\frac{1}{1-\frac{p}{q}} > \vartheta := \frac{2^*-2}{2^*\frac{p}{q}-2} \qquad (\text{if} \quad n \ge 2)$$

and it is satisfied if

Please cite this article in press as: P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124408

$$\frac{q}{p}<2-\frac{2}{2^{*}}\quad (\text{if}\quad n\geq 2)\underset{\text{for }n\geq 3}{=}1+\frac{2}{n}\,. \tag{4.34}$$

Since the assumption (4.29) requires for $n=2$ that $\frac{q}{p}<2$, it is sufficient to fix 2^{*} large enough, precisely $2^{*}>\frac{2p}{2p-q}$, so that (4.34) holds; in this case the gradient estimate (4.31), for some $\delta>0$ and $\vartheta:=\frac{2^{*}-2}{2^{*}\frac{p}{q}-2}$, reads

$$\sup_{x\in B_{\varrho}} v\leq\left(\frac{c}{(R-\varrho)^{\delta}}\int_{B_R} v^{p}\,dx\right)^{\frac{\vartheta}{q-\vartheta(q-p)}},\quad \text{with}\ \ v=\left(1+|Du(x)|^{2}\right)^{\frac{1}{2}}.$$

Finally, in both cases $n\geq 2$ (see the formulas (4.32), (4.33) above when $n\geq 3$ too), $\alpha=\vartheta=1$ if and only if $q=p$.

A final remark: the exponent $\alpha:=\frac{\vartheta\frac{p}{q}}{1-\vartheta\left(1-\frac{p}{q}\right)}$ in the estimate (4.31) is the same as the exponent formally deduced in (4.27) by the interpolation inequality

$$\|v\|_{L^{\infty}}^{1-\vartheta\left(1-\frac{p}{q}\right)}\leq c\,\|v\|_{L^{p}}^{\vartheta\frac{p}{q}}\,,\qquad \text{with}\ \ v=\left(1+|Du(x)|^{2}\right)^{\frac{1}{2}}.$$

4.2.2. Proof of the interpolation Theorem 4.4

We make use at the same time of the interpolation inequality (4.26) for $v(x)=\left(1+|Du(x)|^{2}\right)^{\frac{1}{2}}$ and of the a-priory estimate (4.5) in Theorem 4.1

$$\begin{cases}\|v\|_{L^{q}}\leq\|v\|_{L^{p}}^{\frac{p}{q}}\cdot\|v\|_{L^{\infty}}^{1-\frac{p}{q}}\\ \|v\|_{L^{\infty}(B_{\varrho})}\leq\frac{c}{(R-\varrho)^{\vartheta\beta}}\|v\|_{L^{q}(B_R)}^{\vartheta}\end{cases}.$$

We consider here only the case when q is strictly greater than p and we obtain

$$\begin{aligned}\|v\|_{L^{q}(B_{\varrho})}&\leq\|v\|_{L^{p}(B_{\varrho})}^{\frac{p}{q}}\cdot\|v\|_{L^{\infty}(B_{\varrho})}^{1-\frac{p}{q}}\leq\|v\|_{L^{p}(B_{\varrho})}^{\frac{p}{q}}\cdot\left(\frac{c}{(R-\varrho)^{\vartheta\beta}}\|v\|_{L^{q}(B_R)}^{\vartheta}\right)^{1-\frac{p}{q}}\\ &\leq c^{1-\frac{p}{q}}\|v\|_{L^{p}(B_{\varrho})}^{\frac{p}{q}}\cdot\left(\frac{1}{(R-\varrho)^{\beta}}\|v\|_{L^{q}(B_R)}\right)^{\vartheta\left(1-\frac{p}{q}\right)}.\end{aligned}$$

Therefore

$$\|v\|_{L^{q}(B_{\varrho})}\leq c^{1-\frac{p}{q}}\|v\|_{L^{p}(B_{\varrho})}^{\frac{p}{q}}\cdot\left(\frac{1}{(R-\varrho)^{\beta}}\|v\|_{L^{q}(B_R)}\right)^{\gamma} \tag{4.35}$$

under the notation $\gamma:=\vartheta\left(1-\frac{p}{q}\right)$. Recall that $1-\vartheta\left(1-\frac{p}{q}\right)>0$; i.e., $0<\gamma<1$. Given ϱ_0 and R_0, with $0<\varrho_0<R_0\leq\varrho_0+1$, we define a decreasing sequence ϱ_k by $\varrho_k=R_0-\frac{R_0-\varrho_0}{2^{k}}$, $k=0,1,2,\ldots$ In (4.35) we pose $\varrho=\varrho_k$ and $R=\varrho_{k+1}$. Since $R-\varrho=\varrho_{k+1}-\varrho_k=\frac{R_0-\varrho_0}{2^{k+1}}$, we obtain

$$\|v\|_{L^{q}(B_{\varrho_k})}\leq c^{1-\frac{p}{q}}\|v\|_{L^{p}(B_{R_0})}^{\frac{p}{q}}\cdot\left(\frac{2^{\beta(k+1)}}{(R_0-\varrho_0)^{\beta}}\|v\|_{L^{q}\left(B_{\varrho_{k+1}}\right)}\right)^{\gamma},$$

which, under the notation $B_k=\|v\|_{L^{q}(B_{\varrho_k})}$ for $k=0,1,2,\ldots$, is equivalent to

$$B_k \le c^{1-\frac{p}{q}} \|v\|_{L^p(B_{R_0})}^{\frac{p}{q}} \cdot \frac{2^{\beta\gamma(k+1)}}{(R_0-\varrho_0)^{\beta\gamma}} B_{k+1}^{\gamma} .$$

We start to iterate with $k = 0, 1, 2, \ldots$

$$B_0 \le c^{1-\frac{p}{q}} \|v\|_{L^p(B_{R_0})}^{\frac{p}{q}} \cdot \frac{2^{\beta\gamma}}{(R_0-\varrho_0)^{\beta\gamma}} B_1^{\gamma}$$
$$\le c^{1-\frac{p}{q}} \|v\|_{L^p(B_{R_0})}^{\frac{p}{q}} \cdot \frac{2^{\beta\gamma}}{(R_0-\varrho_0)^{\beta\gamma}} \left(c^{1-\frac{p}{q}} \|v\|_{L^p(B_{R_0})}^{\frac{p}{q}} \cdot \frac{2^{\beta\gamma\cdot 2}}{(R_0-\varrho_0)^{\beta\gamma}} B_2^{\gamma} \right)^{\gamma}$$

and for general $k = 1, 2, 3, \ldots$ we have

$$B_0 \le \left(\frac{c^{1-\frac{p}{q}} \|v\|_{L^p(B_{R_0})}^{\frac{p}{q}}}{(R_0-\varrho_0)^{\beta\gamma}} \right)^{\sum_{i=0}^{k-1} \gamma^i} \left(2^{\beta}\right)^{\sum_{i=1}^{k} i\gamma^i} (B_k)^{\gamma^k} .$$

Since $0 < \gamma < 1$, the two numerical series above are finite; in particular $\sum_{i=0}^{k-1} \gamma^i = \frac{1-\gamma^k}{1-\gamma}$. Moreover the increasing sequence $B_k = \|v\|_{L^q(B_{\varrho_k})}$ is bounded by $\|v\|_{L^q(B_{R_0})}$ for $k = 0, 1, 2, \ldots$ Thus $(B_k)^{\gamma^k} = \|v\|_{L^q(B_{\varrho_k})}^{\gamma^k} \le \|v\|_{L^q(B_{R_0})}^{\gamma^k}$ and the right hand side converges to 1 as $k \to +\infty$. Therefore, in the limit as $k \to +\infty$, there exists a constant c_1 such that

$$B_0 \le c_1 \left(\frac{\|v\|_{L^p(B_{R_0})}^{\frac{p}{q}}}{(R_0-\varrho_0)^{\beta\gamma}} \right)^{\frac{1}{1-\gamma}} . \tag{4.36}$$

Recalling that $\alpha := \frac{\frac{p}{q}\vartheta}{1-\vartheta\left(1-\frac{p}{q}\right)}$ and $\gamma := \vartheta\left(1-\frac{p}{q}\right)$, it remains to compute

$$\frac{p}{q} \cdot \frac{1}{1-\gamma} = \frac{p}{q} \cdot \frac{1}{1-\vartheta\left(1-\frac{p}{q}\right)} = \frac{\alpha}{\vartheta};$$

$$\frac{\gamma}{1-\gamma} = \frac{\vartheta\left(1-\frac{p}{q}\right)}{1-\vartheta\left(1-\frac{p}{q}\right)} = \frac{\frac{p}{q}\vartheta}{1-\vartheta\left(1-\frac{p}{q}\right)} \frac{q}{p}\left(1-\frac{p}{q}\right)$$
$$= \alpha\frac{q}{p}\left(1-\frac{p}{q}\right) = \alpha\left(\frac{q}{p}-1\right).$$

Since $B_0 = \|v\|_{L^q(B_{\varrho_0})}$, from (4.36) we get the conclusion (4.30), which can be also equivalently written in the form

$$\left\| \left(1+|Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^q(B_\varrho)} \le \left(\frac{c}{(R-\varrho)^{\beta\left(\frac{q}{p}-1\right)}} \left\| \left(1+|Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)}^{\frac{1}{\vartheta}} \right)^{\alpha} . \tag{4.37}$$

The other a-priori estimate can be proved similarly, or can be deduced from the previous one. In fact, to obtain (4.31) we consider $\overline{\varrho} = \frac{R+\varrho}{2}$ and, by combining the original a-priori estimate (4.5) and (4.37), under the usual notation $v(x) = \left(1+|Du(x)|^2\right)^{\frac{1}{2}}$, since $R-\overline{\varrho} = \overline{\varrho}-\varrho$ and (see α in (4.33)) $\frac{1}{\alpha}+\frac{q}{p}-1 = \frac{q}{\vartheta p}$,

Please cite this article in press as: P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124408

$$\|v\|_{L^\infty(B_\varrho)} \leq \frac{c}{(\overline{\varrho}-\varrho)^{\vartheta\beta}} \left(\int_{B_{\overline{\varrho}}} v^q \, dx \right)^{\frac{\vartheta}{q}} \leq c \left(\frac{1}{(\overline{\varrho}-\varrho)^{\beta}} \|v\|_{L^q(B_{\overline{\varrho}})} \right)^{\vartheta}$$

$$\leq c \left(\frac{1}{(\overline{\varrho}-\varrho)^{\beta}} \left(\frac{c}{(R-\overline{\varrho})^{\beta\left(\frac{q}{p}-1\right)}} \|v\|_{L^p(B_R)}^{\frac{1}{\vartheta}} \right)^{\alpha} \right)^{\vartheta}$$

$$\leq c \left(\frac{c}{(R-\overline{\varrho})^{\frac{\beta}{\alpha}+\beta\left(\frac{q}{p}-1\right)}} \|v\|_{L^p(B_R)}^{\frac{1}{\vartheta}} \right)^{\alpha\vartheta} \leq c_1 \left(\frac{1}{(R-\varrho)^{\frac{\beta q}{\vartheta p}}} \|v\|_{L^p(B_R)} \right)^{\alpha}.$$

The conclusion (4.31) holds. When $n>2$ we observe that $\frac{\beta q}{\vartheta} = n$.

4.3. Third step: approximation

The proofs of Theorems 1.1, 1.3 and 1.4 follow from the a-priori estimate of Theorem 4.1. We consider these proofs separately.

4.3.1. Proof of Theorem 1.1

For every $\varepsilon \in (0,1]$ we consider the energy integral

$$\int_{\Omega} \left\{ \varepsilon \left| Du(x) \right|^2 + f(Du(x)) \right\} dx\,, \tag{4.38}$$

where $f:\mathbb{R}^n \to \mathbb{R}$ is a convex function of class $C^2(\mathbb{R}^n)$, satisfying the $p,q-$growth conditions (1.4) for some exponents p,q with $q \geq p \geq 2$. We fix a ball B_R compactly contained in Ω and we denote by $f^\varepsilon(\xi) = \varepsilon|\xi|^2 + f(\xi)$ and by $u^\varepsilon \in W^{1,2}(B_R)$ a minimizer of the energy integral (4.38) restricted to B_R, under some smooth boundary conditions on ∂B_R related to a local minimizer of the original energy integral $\int_\Omega f(Du(x))\,dx$ (see some details in [54]). Then $f^\varepsilon(\xi)$ satisfies the $p,q-$growth conditions

$$\left(2\varepsilon + m|\xi|^{p-2}\right)|\lambda|^2 \leq \sum_{i,j=1}^{n} \frac{\partial^2 f^\varepsilon(\xi)}{\partial \xi_i \partial \xi_j} \lambda_i \lambda_j \leq 2M \left(1+|\xi|^2\right)^{\frac{q-2}{2}} |\lambda|^2\,. \tag{4.39}$$

The growth conditions (1.11), (1.15) are satisfied with $a_i(\xi) = \frac{\partial f^\varepsilon(\xi)}{\partial \xi_i}$, since the left hand side of these inequalities is identically equal to zero. Also the first growth condition (1.10) holds under the form (4.39) and this is sufficient to conclude the analysis of the a-priori estimate in Section 4.1, since the right hand side b in the Euler's equation (4.1) in this case is also identically to zero. The term with $\varepsilon>0$ in the left hand side of (4.39) is used to prove that $u^\varepsilon \in W^{2,2}_{\mathrm{loc}}(\Omega)$ as in (4.7), but the term ε does not enter in the computations, so that the constants in the a-priori estimates remain independent of ε.

Under the assumption $\frac{q}{p} < 1 + \frac{2}{n}$ in (1.5) we can apply Theorem 4.4 and obtain the bound for the $L^\infty(B_\varrho)-$norm of the modulus of the gradient $|Du_\varepsilon|$ (for simplicity of notation we use the representation of the exponent when $n>2$)

$$\left\| \left(1+|Du_\varepsilon|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_\varrho)} \leq \left(\frac{c}{(R-\varrho)^{\frac{\beta q}{\vartheta p}}} \int_{B_R} \left(1+|Du_\varepsilon|^2\right)^{\frac{p}{2}} dx \right)^{\frac{2}{(n+2)p-nq}}. \tag{4.40}$$

Please cite this article in press as: P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124408

It is possible to bound the $W^{1,p}$–norm of u_ε uniformly with respect to $\varepsilon \in (0,1)$. We can proceed as in [54] and obtain, as $\varepsilon \to 0$, that the limit of u^ε in the weak* topology of $W^{1,\infty}_{\mathrm{loc}}(B_R)$ is the minimizer u of the energy integral $\int_\Omega f(Du(x))\,dx$ and the bound (1.6) is satisfied, in particular

$$\|Du(x)\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \le \left(\frac{c}{(R-\varrho)^{\frac{\beta q}{\vartheta p}}} \int_{B_R} \left(1+|Du|^2\right)^{\frac{p}{2}} dx \right)^{\frac{2}{(n+2)p-nq}} . \tag{4.41}$$

More details can be found in [54]. We now consider the real function $\varphi : \mathbb{R} \to [0,+\infty)$ defined by $\varphi(t) = f(t\xi)$ for $t \in \mathbb{R}$ and with fixed $\xi \in \mathbb{R}^n$. Then $\varphi \in C^2(\mathbb{R})$ and its first and second derivatives respectively hold $\varphi'(t) = (D_\xi f(t\xi), \xi)$, $\varphi''(t) = \sum_{i,j=1}^n f_{\xi_i\xi_j}(t\xi)\,\xi_i\xi_j$. By integrating by parts (this is the Taylor's formula in integral form!) and by the left hand side of the ellipticity condition (1.4) we get

$$\begin{aligned} f(\xi) - f(0) &= \varphi(1) - \varphi(0) = \int_0^1 \varphi'(s)\,ds = \left[(s-1)\varphi'(s)\right]_{s=0}^{s=1} + \int_0^1 (1-s)\,\varphi''(s)\,ds \\ &= \varphi'(0) + \int_0^1 (1-s)\sum_{i,j=1}^n f_{\xi_i\xi_j}(s\xi)\,\xi_i\xi_j\,ds \ge \varphi'(0) + m|\xi|^p \int_0^1 (1-s)\,s^{p-2}\,ds \\ &= (D_\xi f(0), \xi) + m|\xi|^p \left[\frac{s^{p-1}}{p-1} - \frac{s^p}{p}\right]_{s=0}^{s=1} \ge -|D_\xi f(0)|\,|\xi| + \frac{m}{p(p-1)}|\xi|^p . \end{aligned}$$

Therefore there exists a constant c, depending only on $|D_\xi f(0)|$, $f(0)$, m and p, such that

$$|\xi|^p \le c\,(1 + f(\xi)) . \tag{4.42}$$

By the gradient estimate (4.41) and the coercivity condition (4.42) we finally get

$$\|Du(x)\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \le \left(\frac{c'}{(R-\varrho)^{\frac{\beta q}{\vartheta p}}} \int_{B_R} \{1 + f(Du)\}\,dx \right)^{\frac{2}{(n+2)p-nq}} . \tag{4.43}$$

We observe that for $n = 2$ the proper exponent in the right hand side of (4.43), derived from the application of Theorem 4.4, is $\frac{\alpha}{p}$ with α given in (4.33) which is the correct exponent in any dimension $n \ge 2$ and it reduces to $\frac{2}{(n+2)p-nq}$ when $n > 2$.

Remark 4.6 *(The recent result by Peter Bella and Mathias Schäffner).* Giuseppe Mingione pointed out to me the recent result [7] by Peter Bella and Mathias Schäffner, where the authors prove Theorem 1.1 under a weaker assumption on the ratio $\frac{q}{p}$. Precisely, instead of $\frac{q}{p} < 1 + \frac{2}{n}$ as in (1.5), Bella and Schäffner assume

$$\frac{q}{p} < 1 + \min\left\{1; \frac{2}{n-1}\right\} \tag{4.44}$$

and they obtain the gradient bound (4.43), of course with a modified exponent: $\frac{2}{(n+1)p-(n-1)q}$ instead of $\frac{2}{(n+2)p-nq}$ (see Remark 3 and formula (41) in [7]). If $n \ge 3$ the growth condition $\frac{q}{p} < 1 + \frac{2}{n-1}$ in (4.44) is more general than the assumption $\frac{q}{p} < 1 + \frac{2}{n}$ in (1.5), although asymptotically equivalent as n grows to infinity. They apply the same method described above in this manuscript, but with a modification in the use of a smart version of the Sobolev inequality, a Sobolev inequality on spheres, i.e. surfaces of balls, instead of balls (see Lemma 3 in [7]); thus they gain in the dimensional parameter, $n-1$ instead of n.

I thank Mingione for having pointed out to me this interesting result, which I did not know; however I was already aware of this intelligent use of the Sobolev inequality already applied by Bella and Schäffner in their previous paper [6] about the local boundedness of solutions in the same spirit of [32], since I acted as referee proposing the publication of the article [6] in the *Communications on Pure and Applied Mathematics*.

4.3.2. Proof of Theorem 1.2

The regularity result in Theorem 1.2 has been proved in [44]. We have here only to fix the parameter $r = +\infty$ in the assumption (1.7) in [44]; precisely, there the summability $h \in L^r(\Omega)$ for some $r > n$ can be replaced with $h \in L^\infty(\Omega)$. Moreover the assumption there

$$|f_{\xi x}(x,\xi)| \leq h(x)\left(1+|\xi|^2\right)^{\frac{p+q-2}{4}}$$

can be reduced with the argument of Section 4.3.4 to the condition in $(1.8)_2$; that is $|f_{\xi x}(x,\xi)| \leq M\left(1+|\xi|^2\right)^{\frac{q-1}{2}}$. We have to change the bound $\frac{q}{p} < 1+\frac{2}{n}$ with the more strict condition $\frac{q}{p} < 1+\frac{1}{n}$ on the exponents p, q, as stated in (1.9), following the method of Section 4.3.4 below (see also Section 5 in [51]).

4.3.3. Proof of Theorem 1.3

We consider the Dirichlet problem (1.13) under the $p,q-$growth conditions (1.10), (1.11). For every $\varepsilon \in (0,1]$ we denote by u_ε the weak solution to the associated Dirichlet problem

$$\begin{cases} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left\{a^i(x,Du)+\varepsilon\left(1+|Du|^2\right)^{\frac{q-2}{2}}u_{x_i}\right\} = b(x), \quad x \in \Omega, \\ u = u_0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{4.45}$$

Note that $u_0 \in W^{1,p\frac{q-1}{p-1}} \subset W^{1,q}$ since $p\frac{q-1}{p-1} \geq q$. For every $\varepsilon > 0$ the differential operator in the left hand side of $(4.45)_1$ is monotone and satisfies the so-called natural growth conditions of order q; i.e. it satisfies $q,q-$*growth conditions*. By the theory of monotone operators the weak solution $u_\varepsilon \in u_0 + W_0^{1,q}(\Omega)$ to the Dirichlet problem (4.45) exists and is unique. Of course the differential operator satisfies also the original $p,q-$growth conditions (1.10), (1.11) with constants $m, 2M$ independent of $\varepsilon \in (0,1]$. Thus *for* u_ε the gradient bound (4.31) holds (for simplicity of notation we use the representation of the exponent when $n > 2$)

$$\left\|\left(1+|Du_\varepsilon|^2\right)^{\frac{1}{2}}\right\|_{L^\infty(B_\varrho)} \leq \left(\frac{c}{(R-\varrho)^{\frac{\beta q}{\vartheta p}}}\int_{B_R}\left(1+|Du_\varepsilon|^2\right)^{\frac{p}{2}}dx\right)^{\frac{2}{(n+2)p-nq}}. \tag{4.46}$$

It is also possible to see (Lemma 4.5 in [51]) that u_ε is equibounded in $W^{1,p}(\Omega)$; thus the right hand side in (4.46) is bounded with a constant independent of ε and the same holds for the left hand side. We can go to the limit as $\varepsilon \to 0$: up to a subsequence, u_ε converges in the weak and, respectively, weak* topologies of $u_0 + W_0^{1,p}(\Omega)$ and $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, to a function $u \in u_0 + W_0^{1,p}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ which satisfies the bound (1.14) in Theorem 1.3. By (4.7) u_ε is also bounded in $W^{2,2}_{\mathrm{loc}}(\Omega)$ uniformly with respect to ε. Thus by compactness we can infer the convergence of the gradient $Du_\varepsilon(x)$ to the gradient $Du(x)$ a.e. in Ω. This allows us to go to the limit in the weak/integral form of the equation and we obtain that u is a weak solution to the Dirichlet problem (1.13). Going to the limit as $\varepsilon \to 0^+$ in (4.46) we see that u satisfies the gradient bound (1.14). Some other details can be found in Section 4 of [51].

Please cite this article in press as: P. Marcellini, Growth conditions and regularity for weak solutions to nonlinear elliptic pdes, J. Math. Anal. Appl. (2020), https://doi.org/10.1016/j.jmaa.2020.124408

4.3.4. Proof of Theorem 1.4

In the previous Section 4.3.3 we studied the elliptic equation (4.1) under the $p, q-$growth conditions (1.10), (1.11) and (1.12). Here we introduce the notation

$$r = \frac{p+q}{2} \tag{4.47}$$

and we introduce the $p, r-$growth conditions

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \xi_j} \lambda_i \lambda_j \geq m \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2 , \quad \sum_{i,j=1}^{n} \left| \frac{\partial a_i}{\partial \xi_j} \right| \leq M \left(1 + |\xi|^2\right)^{\frac{r-2}{2}} , \tag{4.48}$$

$$\sum_{i,j=1}^{n} \left| \frac{\partial a_i}{\partial \xi_j} - \frac{\partial a_j}{\partial \xi_i} \right| \leq M \left(1 + |\xi|^2\right)^{\frac{r-2}{2}} , \quad \sum_{i,s=1}^{n} \left| \frac{\partial a_i}{\partial x_s} \right| \leq M \left(1 + |\xi|^2\right)^{\frac{r-1}{2}} , \tag{4.49}$$

for every $\lambda, \xi \in \mathbb{R}^n$, $x \in \Omega$ and for some positive constants m, M. Since $r \leq q$, the $p, r-$growth conditions are more strict than the $p, q-$growth ones; in fact,

$$\sum_{i,j=1}^{n} \left| \frac{\partial a_i}{\partial \xi_j} \right| \leq M \left(1 + |\xi|^2\right)^{\frac{r-2}{2}} \leq M \left(1 + |\xi|^2\right)^{\frac{q-2}{2}} ,$$

and by (4.47) the growth conditions (4.49) coincide with the growth in (1.11). Therefore the assumptions of Theorem 1.3, with p, q exponents, are satisfied and we obtain a weak solution to the Dirichlet problem (1.13)

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i \left(x, Du\left(x\right)\right) = b\left(x\right), \ \text{ in } \Omega; \ \ u = u_0 \in W^{1, p\frac{q-1}{p-1}} \text{ on } \partial\Omega , \tag{4.50}$$

when p, q are related each other by the condition (1.12) on p, q which, being $q = 2r - p$, in terms of p, r means $\frac{q}{p} = \frac{2r-p}{p} = 2\frac{r}{p} - 1 < 1 + \frac{2}{n}$, that is

$$\frac{r}{p} < 1 + \frac{1}{n} . \tag{4.51}$$

Finally we observe that the solution a-priori belongs to the Sobolev class $W^{1,p}_{u_0}(\Omega) \cap W^{1,2r-p}_{\mathrm{loc}}(\Omega)$ and, being $2r - p \geq r$, $u \in W^{1,p}_{u_0}(\Omega) \cap W^{1,r}_{\mathrm{loc}}(\Omega)$ too. The proof of Theorem 1.4 is complete when in the statement of Theorem 1.4 we change the notation according to the $p, r-$growth in (4.48), (4.49). In particular the exponent in the right hand side of (1.14), with the position $q = 2r - p$, when $n > 2$ changes into

$$\frac{2}{(n+2)\, p - nq} = \frac{1}{(n+1)\, p - nr}$$

and the gradient bound (1.14) transforms into (1.19).

Remark 4.7. If $n = 2$ it can be of interest to see how to change the exponent α in the estimate (1.19) of Theorem 4.4 under the transformation $q = 2r - p$. Being $\alpha := \frac{\vartheta \frac{p}{q}}{1-\vartheta\left(1-\frac{p}{q}\right)}$ and $\vartheta := \frac{2^*-2}{2^*\frac{p}{q}-2}$ we get

$$\alpha := \frac{\vartheta \frac{p}{q}}{1 - \vartheta\left(1 - \frac{p}{q}\right)} = \frac{(2^* - 2)\, \frac{p}{q}}{\left(2^* \frac{p}{q} - 2\right) - (2^* - 2)\left(1 - \frac{p}{q}\right)} . \tag{4.52}$$

$$= \frac{(2^* - 2)\, \frac{p}{q}}{(2^* - 1)\, \frac{2p}{q} - 2^*} .$$

The exponent α is well defined as a real positive number if $(2^* - 1)\, \frac{2p}{q} - 2^* > 0$. This is equivalent to $\frac{q}{p} < 2 - \frac{2}{2^*}$, in accord with (4.34). If we apply the transformation $q = 2r - p$ we obtain the condition $\frac{2r-p}{p} < 2 - \frac{2}{2^*}$, which gives

$$\frac{r}{p} < \frac{3}{2} - \frac{1}{2^*}. \tag{4.53}$$

Since the assumption (4.51) requires for $n = 2$ that $\frac{q}{p} < \frac{3}{2}$, it is sufficient to fix 2^* large enough, precisely $2^* > \frac{2p}{3p-2r}$, so that (4.53) holds.

In conclusion, going back to the notation p, q with $\frac{q}{p} < 1 + \frac{1}{n}$, taking into account the constraints $2^* > \frac{2q}{p}$ of Remark 4.2 and $2^* > \frac{2p}{2p-q}$ of Remark 4.5, all the procedure for regularity holds also for $n = 2$ and, in the Theorem 1.4 with the assumption $\frac{q}{p} < 1 + \frac{1}{n} = \frac{3}{2}$, it requires to fix 2^* large enough, precisely $2^* > \frac{2p}{3p-2q}$.

References

[1] V. Ambrosio, V. Rădulescu, Fractional double-phase patterns: concentration and multiplicity of solutions, J. Math. Pures Appl. (2020), in press.

[2] P. Baroni, M. Colombo, G. Mingione, Harnack inequalities for double phase functionals, Nonlinear Anal. 121 (2015) 206–222.

[3] P. Baroni, M. Colombo, G. Mingione, Non-autonomous functionals, borderline cases and related function classes, St. Petersburg Math. J. 27 (2016) 347–379.

[4] P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, Calc. Var. Partial Differ. Equ. 57 (2018), https://doi.org/10.1007/s00526-018-1332-z.

[5] L. Beck, G. Mingione, Lipschitz bounds and nonuniform ellipticity, Commun. Pure Appl. Math. 73 (2020) 944–1034.

[6] P. Bella, M. Schäffner, Local boundedness and Harnack inequality for solutions of linear non-uniformly elliptic equations, Commun. Pure Appl. Math. (2019), https://doi.org/10.1002/cpa.21876, in press.

[7] P. Bella, M. Schäffner, On the regularity of minimizers for scalar integral functionals with $(p, q)-$growth, Anal. PDE (2019), in press, https://arxiv.org/abs/1904.12279v1.

[8] M. Bildhauer, M. Fuchs, On a class of variational problems with linear growth and radial symmetry, Comment. Math. Univ. Carol. (2019), https://arxiv.org/abs/1911.10355.

[9] M. Bildhauer, M. Fuchs, Splitting type variational problems with linear growth conditions, preprint, https://arxiv.org/abs/2004.08169v2, 2020.

[10] L. Boccardo, P. Marcellini, C. Sbordone, Regularity for variational problems with sharp non standard growth conditions, Boll. Unione Mat. Ital. 4-A (1990) 219–225.

[11] V. Bögelein, B. Dacorogna, F. Duzaar, P. Marcellini, C. Scheven, Integral convexity and parabolic systems, SIAM J. Math. Anal. 52 (2020) 1489–1525.

[12] V. Bögelein, F. Duzaar, P. Marcellini, Parabolic equations with p, q-growth, J. Math. Pures Appl. 100 (2013) 535–563.

[13] V. Bögelein, F. Duzaar, P. Marcellini, C. Scheven, Doubly nonlinear equations of porous medium type, Arch. Ration. Mech. Anal. 229 (2018) 503–545.

[14] V. Bögelein, F. Duzaar, P. Marcellini, C. Scheven, A variational approach to doubly nonlinear equations, Rend. Lincei Mat. Appl. 29 (2018) 739–772.

[15] B. Botteron, P. Marcellini, A general approach to the existence of minimizers of one-dimensional non-coercive integrals of the calculus of variations, Ann. Inst. Henri Poincare (C) Non Linear Anal. 8 (1991) 197–223.

[16] P. Bousquet, L. Brasco, Lipschitz regularity for orthotropic functionals with nonstandard growth conditions, Rev. Mat. Iberoam. (2020), https://www.ems-ph.org/journals/of_article.php?jrn=rmi&doi=1189.

[17] M. Carozza, F. Giannetti, F. Leonetti, A. Passarelli di Napoli, Pointwise bounds for minimizers of some anisotropic functionals, Nonlinear Anal. 177 (2018) 254–269.

[18] M. Carozza, J. Kristensen, A. Passarelli di Napoli, On the validity of the Euler-Lagrange system, Commun. Pure Appl. Anal. 14 (2015) 51–62.

[19] M. Cencelja, V. Rădulescu, D. Repovš, Double phase problems with variable growth, Nonlinear Anal. 177 (2018) 270–287.

[20] I. Chlebicka, A pocket guide to nonlinear differential equations in Musielak–Orlicz spaces, Nonlinear Anal. 175 (1918) 1–27.

[21] I. Chlebicka, C. De Filippis, Removable sets in non-uniformly elliptic problems, Ann. Mat. Pura Appl. 199 (2020) 619–649.

[22] I. Chlebicka, C. De Filippis, L. Koch, Boundary regularity for manifold constrained $p(x)-$harmonic maps, preprint, https://arxiv.org/abs/2001.06243, 2020.

[23] M. Colombo, G. Mingione, Regularity for double phase variational problems, Arch. Ration. Mech. Anal. 215 (2015) 443–496.

[24] M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, Arch. Ration. Mech. Anal. 218 (2015) 219–273.
[25] G. Cupini, F. Giannetti, R. Giova, A. Passarelli di Napoli, Higher integrability for minimizers of asymptotically convex integrals with discontinuous coefficients, Nonlinear Anal. 54 (2017) 7–24.
[26] G. Cupini, P. Marcellini, E. Mascolo, Regularity under sharp anisotropic general growth conditions, Discrete Contin. Dyn. Syst., Ser. B 11 (2009) 66–86.
[27] G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of solutions to quasilinear elliptic systems, Manuscr. Math. 137 (2012) 287–315.
[28] G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of solutions to some anisotropic elliptic systems, in: Recent Trends in Nonlinear Partial Differential Equations. II. Stationary Problems, in: Contemp. Math., vol. 595, Amer. Math. Soc., Providence, RI, 2013, pp. 169–186.
[29] G. Cupini, P. Marcellini, E. Mascolo, Existence and regularity for elliptic equations under $p, q-$growth, Adv. Differ. Equ. 19 (2014) 693–724.
[30] G. Cupini, P. Marcellini, E. Mascolo, Local boundedness of minimizers with limit growth conditions, J. Optim. Theory Appl. 166 (2015) 1–22.
[31] G. Cupini, P. Marcellini, E. Mascolo, Regularity of minimizers under limit growth conditions, Nonlinear Anal. 153 (2017) 294–310.
[32] G. Cupini, P. Marcellini, E. Mascolo, Nonuniformly elliptic energy integrals with $p, q-$growth, Nonlinear Anal. 177 (2018) 312–324.
[33] C. De Filippis, Higher integrability for constrained minimizers of integral functionals with $(p, q)-$growth in low dimension, Nonlinear Anal. 170 (2018) 1–20.
[34] C. De Filippis, G. Mingione, On the regularity of minima of non-autonomous functionals, J. Geom. Anal. 30 (2020) 1584–1626, https://doi.org/10.1007/s12220-019-00225-z.
[35] C. De Filippis, J. Oh, Regularity for multi-phase variational problems, J. Differ. Equ. 267 (2019) 1631–1670.
[36] C. De Filippis, G. Palatucci, Hölder regularity for nonlocal double phase equations, J. Differ. Equ. 267 (2019) 547–586.
[37] E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli, Mem. Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat. 3 (1957) 25–43.
[38] T. Di Marco, P. Marcellini, A-priori gradient bound for elliptic systems under either slow or fast growth conditions, Calc. Var. Partial Differ. Equ. 59 (2020) 120, https://doi.org/10.1007/s00526-020-01769-7.
[39] G. Düzgun, P. Marcellini, V. Vespri, Space expansion for a solution of an anisotropic $p-$Laplacian equation by using a parabolic approach, Riv. Mat. Univ. Parma 5 (2014) 93–111.
[40] G. Düzgun, P. Marcellini, V. Vespri, An alternative approach to the Hölder continuity of solutions to some elliptic equations, Nonlinear Anal. 94 (2014) 133–141.
[41] M. Eleuteri, P. Marcellini, E. Mascolo, Lipschitz estimates for systems with ellipticity conditions at infinity, Ann. Mat. Pura Appl. 195 (2016) 1575–1603.
[42] M. Eleuteri, P. Marcellini, E. Mascolo, Lipschitz continuity for functionals with variable exponents, Rend. Lincei Mat. Appl. 27 (2016) 61–87.
[43] M. Eleuteri, P. Marcellini, E. Mascolo, Local Lipschitz continuity of minimizers with mild assumptions on the x-dependence, Discrete Contin. Dyn. Syst. 12 (2019) 251–265.
[44] M. Eleuteri, P. Marcellini, E. Mascolo, Regularity for scalar integrals without structure conditions, Adv. Calc. Var. 13 (2020) 279–300, https://doi.org/10.1515/acv-2017-0037.
[45] N. Fusco, P. Marcellini, A. Ornelas, Existence of minimizers for some non convex one-dimensional integrals, Port. Math. 55 (1998) 167–186.
[46] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Annals of Mathematics Studies, vol. 105, Princeton University Press, Princeton, NJ, 1983.
[47] P. Harjulehto, P. Hästö, O. Toivanen, Hölder regularity of quasiminimizers under generalized growth conditions, Calc. Var. Partial Differ. Equ. 56 (2017), 26 pp.
[48] P. Hästö, J. Ok, Maximal regularity for local minimizers of non-autonomous functionals, Preprint, https://arxiv.org/abs/1902.00261, 2019.
[49] P. Marcellini, Quasiconvex quadratic forms in two dimensions, Appl. Math. Optim. 11 (1984) 183–189.
[50] P. Marcellini, Regularity of minimizers of integrals in the calculus of variations with non standard growth conditions, Arch. Ration. Mech. Anal. 105 (1989) 267–284.
[51] P. Marcellini, Regularity and existence of solutions of elliptic equations with $p, q-$growth conditions, J. Differ. Equ. 90 (1991) 1–30.
[52] P. Marcellini, Regularity for elliptic equations with general growth conditions, J. Differ. Equ. 105 (1993) 296–333.
[53] P. Marcellini, Everywhere regularity for a class of elliptic systems without growth conditions, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 23 (1996) 1–25.
[54] P. Marcellini, Regularity for some scalar variational problems under general growth conditions, J. Optim. Theory Appl. 90 (1996) 161–181.
[55] P. Marcellini, Regularity under general and $p, q-$growth conditions, Discrete Contin. Dyn. Syst., Ser. S 13 (2020) 2009–2031.
[56] P. Marcellini, A variational approach to parabolic equations under general and $p, q-$growth conditions, Nonlinear Anal. 194 (2020), https://doi.org/10.1016/j.na.2019.02.010.
[57] P. Marcellini, Anisotropy versus inhomogeneity in the calculus of variations, J. Convex Anal. 28 (2021) 607–622, in press.
[58] P. Marcellini, G. Papi, Nonlinear elliptic systems with general growth, J. Differ. Equ. 221 (2006) 412–443.
[59] C. Mooney, Minimizers of convex functionals with small degeneracy set, Calc. Var. Partial Differ. Equ. 59 (2020), https://doi.org/10.1007/s00526-020-1723-9.

[60] C. Mooney, O. Savin, Some singular minimizers in low dimensions in the calculus of variations, Arch. Ration. Mech. Anal. 221 (2016) 1–22.
[61] T.N. Nguyen, M.P. Tran, Global Lorentz estimates for non-uniformly nonlinear elliptic equations via fractional maximal operators, preprint, https://arxiv.org/pdf/2002.00318.pdf.
[62] N.S. Papageorgiou, V. Rădulescu, D. Repovš, D. Dušan, Ground state and nodal solutions for a class of double phase problems, Z. Angew. Math. Phys. 71 (2020), 15 pp.
[63] V. Rădulescu, Isotropic and anisotropic double phase problems: old and new, Opusc. Math. 39 (2019) 259–279.
[64] V. Rădulescu, Q. Zhang, Double phase anisotropic variational problems and combined effects of reaction and absorption terms, J. Math. Pures Appl. 118 (2018) 159–203.
[65] M.A. Ragusa, A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, preprint, https://arxiv.org/abs/2005.04205v1, 2020.
[66] C. Sin, Interior gradient estimate for steady flows of electrorheological fluids, Nonlinear Anal., Real World Appl. 55 (2020), https://doi.org/10.1016/j.nonrwa.2020.103138.
[67] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems, Acta Math. 138 (1977) 219–240.